Criteria for the density property
of complex manifolds[*]

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1. Introduction

The ground-breaking papers of Andersén and Lempert [1,2] established remarkable properties of the automorphism group of $\mathbb{C}^n$ ($n \geq 2$) which imply, in particular, that any local holomorphic phase flow on a Runge domain Ω in $\mathbb{C}^n$ can be approximated by global holomorphic automorphisms of $\mathbb{C}^n$ (for an exact statement see Theorem 2.1 in [9]).

The next step in the development of the Andersén–Lempert theory was made by Varolin who extended it from Euclidean spaces to a wider class of algebraic complex manifolds. He realized also that the following density property is crucial for this theory.

1.1. Definition. A complex manifold X has the density property if in the compact-open topology the Lie algebra $\mathrm{Lie}_{\mathrm{hol}}(X)$ generated by completely integrable holomorphic vector fields on X is dense in the Lie algebra $\mathrm{VF}_{\mathrm{hol}}(X)$ of all holomorphic vector fields on X. An affine algebraic manifold X has the algebraic density property if the Lie algebra $\mathrm{Lie}_{\mathrm{alg}}(X)$ generated by completely integrable algebraic vector fields on it coincides with the Lie algebra $\mathrm{VF}_{\mathrm{alg}}(X)$ of all algebraic vector fields on it (clearly, the algebraic density property implies the density property).

In this terminology the main observation of the Andersén–Lempert theory says that $\mathbb{C}^n$ ($n \geq 2$) has the algebraic density property. Varolin and Toth [16–18] established the density property for some manifolds including semi-simple complex Lie groups and some homogenous spaces of semi-simple Lie groups. Their proof relies heavily on representation theory and does not, for example, lead to an answer in the case of other linear algebraic groups.

In this paper we suggest new effective criteria for the density property. This enables us to give a trivial proof of the original Andersén–Lempert result and to establish (almost free of charge) the algebraic density property for all linear algebraic groups whose connected components are different from tori or $\mathbb{C}_+$. As another application of this approach we tackle the question (asked among others by F. Forstnerič) about the density of algebraic vector fields on Euclidean space vanishing on a codimension 2 subvariety.

Our method of establishing the algebraic density property for an affine algebraic variety X consists of two ingredients described in Sect. 2. First, we try to find a nontrivial $\mathbb{C}[X]$-module L (over the algebra $\mathbb{C}[X]$ of regular functions on X) in $\mathrm{Lie}_{\mathrm{alg}}(X)$. It turns out that this requires the existence of two commuting completely integrable algebraic vector fields on X satisfying some compatibility condition (see, Definition 2.5 below). Second, since $\mathrm{Lie}_{\mathrm{alg}}(X)$ is invariant under algebraic automorphisms of X, in the presence of some homogeneity property of X we can increase L so that it coincides with the $\mathbb{C}[X]$-module of all algebraic vector fields (in which L is contained, of course, as a submodule). In Sects. 3 and 4 we develop technique for checking this compatibility condition and apply it in the cases of linear algebraic groups and the complements to codimension 2 subvarieties in Euclidean spaces.

2. New approach to the Andersén–Lempert theory

The homogeneity property mentioned before is reflected in the following.

2.1. Definition. Let X be an algebraic manifold and $x_0 \in X$. A finite subset M of the tangent space $T_{x_0}X$ is called a generating set if the image of M under the action of the isotropy subgroup of x_0 (in the group of all algebraic automorphisms $\mathrm{Aut}\,X$ of X) generates the whole space $T_{x_0}X$.

The manifold X will be called tangentially semi-homogeneous if it is homogeneous (with respect to $\mathrm{Aut}\,X$) and admits a generating set consisting of one vector.

Theorem 1. *Let X be a homogeneous (with respect to $\mathrm{Aut}\,X$) affine algebraic manifold with algebra of regular functions $\mathbb{C}[X]$, and L be a submodule of the $\mathbb{C}[X]$-module of all vector fields such that $L \subset \mathrm{Lie}_{\mathrm{alg}}(X)$. Suppose*

that the fiber of L over some $x_0 \in X$ contains a generating set. Then X has the algebraic density property.

Proof. The $\mathbb{C}[X]$-modules TX and L generate coherent sheaves $\mathcal{T}$ and $\mathcal{L}$ on X where $\mathcal{L}$ is a subsheaf of $\mathcal{T}$. The action of $\alpha \in \text{Aut } X$ maps $\mathcal{L}$ onto another coherent subsheaf $\mathcal{L}_\alpha$ of $\mathcal{T}$. The sum of such subsheaves with α running over a finite subset of Aut X is a coherent subsheaf $\mathcal{E}$ of $\mathcal{T}$. Let $\mathfrak{m}$ be the maximal ideal for x_0. Definition 2.1 implies that $\mathcal{E}$ can be chosen so that $\mathcal{E}/\mathfrak{m}\mathcal{E}$ coincides with $T_{x_0}X$. Furthermore, since X is homogeneous we can suppose that this is true for every point in X. Thus $\mathcal{E} = \mathcal{T}$ [12, Chap. II, Exercise 5.8]. Since composition with automorphisms preserves complete integrability, all global sections of $\mathcal{E}$ are in $\text{Lie}_{\text{alg}}(X)$ which concludes the proof. $\qquad\square$

Another ingredient of our method is rooted in a new proof of the following fact.

2.2. Corollary (The main observation of the Andersén–Lempert theory). *For $n \geq 2$ the space $\mathbb{C}^n$ has the algebraic density property.*

Proof. Let $x_1, \ldots, x_n$ be a coordinate system on $\mathbb{C}^n$ and $\delta_i = \partial/\partial x_i$ be the partial derivative, i.e. Ker δ_i is the ring of polynomials independent of x_i. Hence the polynomial ring $\mathbb{C}^{[n]}$ is generated as a vector space by elements of Ker $\delta_1 \cdot$ Ker δ_2. Note also that for $f_i \in$ Ker δ_i the algebraic vector fields $f_i \delta_i$ and $x_i f_i \delta_i$ are completely integrable. Then the field

$$[f_1 \delta_1, x_1 f_2 \delta_2] - [x_1 f_1 \delta_1, f_2 \delta_2] = f_1 f_2 \delta_2$$

belongs to $\text{Lie}_{\text{alg}}(X)$ since $x_1 f_2 \in$ Ker δ_2. Thus $\text{Lie}_{\text{alg}}(X)$ contains all algebraic fields proportional to δ_2. Since $\mathbb{C}^n$ is clearly tangentially semi-homogeneous Theorem 1 implies the desired conclusion. $\qquad\square$

2.3. Remark. There is no need to use tangential semi-homogeneity in this proof since we can replace δ_2 by any other partial derivative δ_i and obtain each algebraic vector field as a sum of fields proportional to $\delta_i, i = 1, \ldots, n$. However there are no partial derivatives on affine algebraic varieties different from Euclidean spaces. Hence formalization of this argument requires Theorem 1 and some substitution for partial derivatives.

We remind that for a completely integrable algebraic vector field on an affine algebraic variety its phase flow is only a holomorphic $\mathbb{C}_+$-action that is not necessarily algebraic.

2.4. Definition. An algebraic vector field δ on X is called semi-simple if its phase flow generates an algebraic $\mathbb{C}^*$-action on X. A vector field σ is called locally nilpotent if its phase flow is an algebraic $\mathbb{C}_+$-action on X. In the last case σ can be viewed as a locally nilpotent derivation on the algebra $\mathbb{C}[X]$ of regular functions on X. That is, for every $f \in \mathbb{C}[X]$ there is $n = n(f)$ for which $\sigma^n(f) = 0$.

There is one-to-one correspondence between the set of locally nilpotent derivations on $\mathbb{C}[X]$ and the set of algebraic $\mathbb{C}_+$-actions on X (e.g., see [6] for details). We shall often use below the fact that for any locally nilpotent derivation σ and a regular function f from its kernel $\operatorname{Ker}\sigma$ (resp. regular function f of degree 1 with respect to σ, i.e. $\sigma(f) \in \operatorname{Ker}\sigma \setminus 0$) the vector field $f\sigma$ is locally nilpotent (resp. completely integrable).

2.5. Definition. Let δ_1 and δ_2 be nontrivial algebraic vector fields on an affine algebraic manifold X such that δ_1 is a locally nilpotent derivation on $\mathbb{C}[X]$, and δ_2 is either also locally nilpotent or semi-simple. That is, δ_i generates an algebraic action of H_i on X where $H_1 \simeq \mathbb{C}_+$ and H_2 is either $\mathbb{C}_+$ or $\mathbb{C}^*$. We say that δ_1 and δ_2 are compatible if

(i)   the vector space $\operatorname{Span}(\operatorname{Ker}\delta_1 \cdot \operatorname{Ker}\delta_2)$ generated by elements from $\operatorname{Ker}\delta_1 \cdot \operatorname{Ker}\delta_2$ contains a non-zero ideal in $\mathbb{C}[X]$ and

(ii)   some element $a \in \operatorname{Ker}\delta_2$ is of degree 1 with respect to δ_1, i.e. $\delta_1(a) \in \operatorname{Ker}\delta_1 \setminus \{0\}$.

2.6. Remark. Instead of Condition (ii) suppose now that δ_1 and δ_2 commute. Then by Corollary 3.5 below Condition (i) implies that the H_1-action on X generates a nontrivial algebraic $\mathbb{C}_+$-action on $X /\!/ H_2$. Taking an element of $\mathbb{C}[X /\!/ H_2]$ whose degree with respect to this action is 1 we can treat its lift-up to X as a. That is, for such commutative δ_1 and δ_2 Condition (ii) is automatic.

Theorem 2. *Let X be a smooth homogeneous affine algebraic manifold with finitely many pairs of compatible vector fields $\{\delta_1^k, \delta_2^k\}_{k=1}^m$ such that for some point $x_0 \in X$ vectors $\{\delta_2^k(x_0)\}_{k=1}^m$ form a generating set. Then $\operatorname{Lie}_{\mathrm{alg}}(X)$ contains a nontrivial $\mathbb{C}[X]$-module and X has the algebraic density property.*

Proof. Let δ_1 and δ_2 be one of our pairs. Choose an element $a \in \operatorname{Ker}\delta_2$ of degree 1 with respect to δ_1 and set $b = \delta_1(a)$. Let $f_i \in \operatorname{Ker}\delta_i$. Then $[af_1\delta_1, f_2\delta_2] - [f_1\delta_1, af_2\delta_2] = -bf_1f_2\delta_2$. The last vector field is from $\operatorname{Lie}_{\mathrm{alg}}(X)$ and since δ_1 and δ_2 are compatible, Definition 2.5 implies that sums of such vector fields include every vector field of form $I\delta_2$ where I is a non-zero ideal in $\mathbb{C}[X]$. Applying this argument to all compatible pairs we see that $\operatorname{Lie}_{\mathrm{alg}}(X)$ contains all linear combinations of δ_2^k with coefficients in some non-zero ideal $J \subset \mathbb{C}[X]$. Since under a small perturbation of x_0 the set $\{\delta_2^k(x_0)\}_{k=1}^m$ remains a generating set we can suppose that x_0 does not belong to the zero locus of J. Hence by Theorem 1 X has the algebraic density property. $\square$

2.7. Remark. If X is tangentially semi-homogenous and, furthermore, any non-zero tangent vector (at any point) is a generating set, then Theorem 2 implies that for the algebraic density property a single pair of compatible vector fields is enough. The assumption that δ_2 is locally nilpotent or semi-simple is not used in the above proof. It is sufficient that δ_2 is completely integrable.

2.8. Corollary. *Let X_1 and X_2 be homogeneous affine algebraic manifolds such that each X_i admits a finite number of completely integrable algebraic vector fields $\{\delta_i^k\}_{k=1}^{m_i}$ whose values at some point $x_i \in X_i$ form a generating set and, furthermore, in the case of X_1 these vector fields are locally nilpotent. Then $X_1 \times X_2$ has the algebraic density property.*

Proof. Note that δ_1^k and δ_2^j generate compatible vector fields on $X_1 \times X_2$ which we denote by the same symbols. Applying isotropy groups one can suppose that $\{\delta_i^k(x_i)\}$ is a basis of $T_{x_i} X_i$. In order to show that the set of vectors $M = \{0 \times \delta_2^k(x_2)\}$ form a generating set in $T_{x_1 \times x_2}(X_1 \times X_2)$ we need the following fact that is obvious in a local coordinate system.

Claim. Let X be a complex manifold and let ν be a vector field on X. Suppose that f is a holomorphic function from $\operatorname{Ker} \nu$ and $x_0 \in f^{-1}(0)$. Then phase flow induced by the vector field $f\nu$ generates a linear action on the tangent space $T_{x_0} X$ given by the formula $w \to w + df(w)\nu(x_0)$ where df is the differential and $w \in T_{x_0} X$. In particular, the span of the orbit of w under this phase flow contains vector $df(w)\nu(x_0)$.

Applying this claim for $\nu = \delta_1^j$ we see that the orbit of M under the isotropy group of $x_1 \times x_2$ contains all vectors of form $\delta_1^j(x_1) \times \delta_2^k(x_2)$. Thus M is a generating set and we are done by Theorem 2. $\square$

2.9. Remark. The reason why we use the locally nilpotent δ_1^j in this proof as ν and not (the possibly semi-simple) δ_2^j is the following: The vector field $f\delta_2^j$ with $f \in \operatorname{Ker} \delta_2^j$ may not generate an algebraic action while $f\delta_1^j$ with $f \in \operatorname{Ker} \delta_1^j$ always generates an algebraic action. It is worth mentioning if one wants to prove density property instead of algebraic density property the use of δ_2^j is permissible.

2.10. Example. (1) Let $X = \mathbb{C}^k \times (\mathbb{C}^*)^l$ with $k \geq 1$ and $k + l \geq 2$. Then X has algebraic density property by Corollary 2.8.

(2) If G is a simple Lie group then it is tangentially semi-homogeneous since the adjoint action of G generates an irreducible representation on the tangent space $\mathfrak{g}$ at the identity e (i.e., any non-zero vector in $T_e G$ is a generating set). Let X be $SL_n(\mathbb{C})$ with $n \geq 2$, i.e. X is tangentially semi-homogeneous. Then every $x \in X$ is a matrix (c_{kj}) with determinant 1. Set $\delta_1(c_{1j}) = c_{nj}$ and $\delta_1(c_{kj}) = 0$ for $k \neq 1$. Set $\delta_2(c_{nj}) = c_{1j}$ and $\delta_2(c_{kj}) = 0$ for $k \neq n$. Note that constants and functions depending on $c_{kj}, k \neq 1$ are in $\operatorname{Ker} \delta_1$ while constants and functions depending on $c_{kj}, k \neq n$ are in δ. Therefore, Condition (i) of Definition 2.5 holds. Taking c_{11} as a in Condition (ii) we see that δ_1 and δ_2 are compatible. Thus $SL_n(\mathbb{C})$ has the algebraic density property.

3. Density of affine algebraic groups with connected components different from tori or $\mathbb{C}_+$

We start with a digest of the notion of categorical (algebraic) quotient and its properties which will be used extensively in the rest of this section.

3.1. Definition. Let G be a linear algebraic group acting algebraically on an affine algebraic variety X and, therefore, on its algebra $\mathbb{C}[X]$ of regular functions (we are going to use this notation for the algebra of regular functions further for any variety X, not necessarily affine). Consider the subalgebra $\mathbb{C}[X]^G$ of G-invariant functions. Its spectrum is called the categorical quotient of this action and it is denoted by $X /\!/ G$. The monomorphism $\mathbb{C}[X]^G \hookrightarrow \mathbb{C}[X]$ generates a dominant (but not necessarily surjective) morphism $\rho : X \to X /\!/ G$ which is called the quotient morphism. The universal property of categorical quotients says that any morphism from X that is constant on orbits of G factors through ρ.

3.2. Remark. For a reductive G the subalgebra $\mathbb{C}[X]^G$ is always finitely generated by Nagata's theorem and, therefore, $X /\!/ G$ can be viewed as an affine algebraic variety. Furthermore, ρ is surjective in this case, the points of $X /\!/ G$ are in one-to-one correspondence with closed orbits of G in X, and every fiber of ρ is the union of those orbits whose closure contains the corresponding closed orbit (e.g., see [15]). In particular, if each orbit is closed then the categorical quotient coincides with the usual geometric quotient (this happens, say, when a reductive subgroup acts on a linear algebraic group by multiplication). If G is not reductive then $\mathbb{C}[X]^G$ is not finitely generated in general (by Nagata's counterexample to the fourteenth Hilbert problem). However, $X /\!/ G$ can be viewed as a quasi-affine algebraic variety and $\mathbb{C}[X]^G$ as its algebra of regular functions [21]. We will work mostly with $G \simeq \mathbb{C}_+$. In this case general fibers of ρ are always orbits of the $\mathbb{C}_+$-action (i.e. $\dim X /\!/ G = \dim X - 1$) and $\mathbb{C}[X]^G$ coincides with the kernel of the corresponding locally nilpotent derivation (e.g., see [6]).

3.3. Notation. In this section H_1 is isomorphic to $\mathbb{C}_+$ and H_2 is isomorphic either to $\mathbb{C}_+$ or $\mathbb{C}^*$. We suppose also that X is a normal affine algebraic variety equipped with nontrivial algebraic H_i-actions where $i = 1, 2$ (in particular, each H_i generates an algebraic vector field δ_i on X). The categorical quotients will be denoted $X_i = X /\!/ H_i$ and the quotient morphisms by $\rho_i : X \to X_i$.

We start with a geometric reformulation of Definition 2.5.

3.4. Proposition. *Set $\rho = (\rho_1, \rho_2) : X \to Y := X_1 \times X_2$ and Z equal to the closure of $\rho(X)$ in Y. Then $\mathrm{Span}(\mathrm{Ker}\, \delta_1 \cdot \mathrm{Ker}\, \delta_2)$ contains a non-zero ideal of $\mathbb{C}[X]$ iff $\rho : X \to Z$ is a finite birational morphism.*

Proof. Every non-zero element of $\mathrm{Span}(\mathrm{Ker}\, \delta_1 \cdot \mathrm{Ker}\, \delta_2)$ is of the form $g \circ \rho$ where $g \in \mathbb{C}[Z] = \mathbb{C}[Y]|_Z$. Thus $\mathrm{Span}(\mathrm{Ker}\, \delta_1 \cdot \mathrm{Ker}\, \delta_2)$ coincides with the subalgebra $\rho^*(\mathbb{C}[Z]) \subset \mathbb{C}[X]$ and we need to establish when $\rho^*(\mathbb{C}[Z])$ contains a nontrivial ideal of $\mathbb{C}[X]$. Note that functions from any nontrivial ideal separate general points of X while functions from $\rho^*(\mathbb{C}[Z])$ do not separate points of $\rho^{-1}(z)$ for any $z \in Z$. Hence $\rho : X \to Z$ must be birational if we want δ_1 and δ_2 to be compatible.

Assume now that the closure of $Z \setminus \rho(X)$ contains a divisor $D \subset Z$. There is a rational function f on Z so that it has poles on D and nowhere else. Multiplying f by $h \in \mathbb{C}[Z]$ such that h is not identically zero on D but vanishes on $D \cap \rho(X)$ with sufficient multiplicity, one can suppose that $f \circ \rho$ is regular on X. On the other hand for n sufficiently large and g as before gf^n has poles on D and cannot be a regular function on Z. Thus $(gf^n) \circ \rho \notin \mathrm{Span}(\mathrm{Ker}\,\delta_1 \cdot \mathrm{Ker}\,\delta_2)$ and the last vector space cannot contain a non-zero ideal in this case.

Let $\rho = \nu \circ \rho_0$ where $\nu : Z_0 \to Z$ is a normalization of Z and $\rho_0 : X \to Z_0$ is the induced morphism of normal varieties. Then ρ_0 is birational and $Z_0 \setminus \rho_0(X)$ is of codimension at least 2 since otherwise even $\rho_0^*(\mathbb{C}[Z_0])$ does not contain a nontrivial ideal of $\mathbb{C}[X]$. The indeterminacy set $V \subset \rho_0(X)$ of the rational map ρ_0^{-1} is of codimension at least 2. Hence any regular function on $\rho_0(X) \setminus V$ extends to Z_0 by the Hartogs theorem. This implies that ρ_0^{-1} is regular and, therefore, ρ_0 is an isomorphism, i.e. ρ is finite birational. Since ν is finite $\mathbb{C}[Z_0]$ is generated over $\mathbb{C}[Z]$ by a finite number of functions of form f_i/g_i, $i = 1, 2, \ldots, n$ where f_i and g_i are regular on Z. Treat $\mathbb{C}[Z] \simeq \nu^*(\mathbb{C}[Z])$ as a subalgebra of $\mathbb{C}[Z_0]$ and consider the principal ideal J in $\mathbb{C}[Z_0]$ generated by $\prod_{i=1}^{n} g_i$. By construction, $J \subset \mathbb{C}[Z]$. Hence $\rho^*(\mathbb{C}[Z])$ contains a non-zero ideal of $\mathbb{C}[X] \simeq \mathbb{C}[Z_0]$ which is the desired conclusion. $\qquad\square$

Note that for every (resp. a general) $(x_1, x_2) \in Z$ the set $\rho^{-1}(x_1, x_2) = \rho_1^{-1}(x_1) \cap \rho_2^{-1}(x_2)$ is finite (resp. a singleton) in this Proposition. Hence a non-constant orbit of H_1 cannot be contained in a fiber of ρ_2 and we have the following.

3.5. Corollary. *In the case of $[\delta_1, \delta_2] = 0$ the H_1-action on X generates a nontrivial $\mathbb{C}_+$-action on X_2.*

3.6. Lemma. *Suppose that X, H_i, X_i, δ_i, and ρ_i are as in Notation 3.3, and either (i) $[\delta_1, \delta_2] = 0$; or (ii) δ_1 and δ_2 are both locally nilpotent and generate a Lie algebra $\mathfrak{sl}_2$ that induces an algebraic action of $SL_2(\mathbb{C})$ on X. Set $\Gamma = H_1 \times H_2$ in Case (i), and $\Gamma = SL_2(\mathbb{C})$ in Case (ii). Suppose that X' is a normal affine algebraic variety equipped with a non-degenerate Γ-action and $p : X \to X'$ is a finite Γ-equivariant morphism (for each $i = 1, 2$), i.e. we have commutative diagrams*

$$
\begin{array}{ccc}
X & \xrightarrow{\;\rho_i\;} & X_i \\
{\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle q} \\
X' & \xrightarrow{\;\rho_i'\;} & X_i'
\end{array}
$$

where $\rho_i' : X' \to X_i' = X'/\!/H_i$ is the quotient morphism of the H_i-action on X' (i.e., we treat $\mathbb{C}[X_i']$ as a subalgebra of $\mathbb{C}[X']$). Let $\mathrm{Span}(\mathbb{C}[X_1] \cdot \mathbb{C}[X_2])$ contain a non-zero ideal of $\mathbb{C}[X]$. Then $\mathrm{Span}\,(\mathbb{C}[X_1'] \cdot \mathbb{C}[X_2'])$ contains a non-zero ideal of $\mathbb{C}[X']$.

Proof. Since p is finite, every $f \in \mathbb{C}[X_i] \subset \mathbb{C}[X]$ is a root of a minimal monic polynomial with coefficients in $\mathbb{C}[X']$ that are constant on H_i-orbits (since otherwise f is not constant on these orbits). By the universal property these coefficients are regular on X_i', i.e. f is integral over $\mathbb{C}[X_i']$ and q_i is finite. Consider the commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\ \rho\ } & X_1 \times X_2 \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\
X' & \xrightarrow{\ \rho'\ } & X_1' \times X_2'
\end{array}
$$

where $\rho = (\rho_1, \rho_2)$, $\rho' = (\rho_1', \rho_2')$, and $q = (q_1, q_2)$. Let Z (resp. Z') be the closure of $\rho(X)$ in $X_1 \times X_2$ (resp. $\rho'(X')$ in $X_1' \times X_2'$). By Proposition 3.4 $\rho(X) = Z$ and, therefore, (since q is finite) $q(\rho(X)) = \rho'(X') = Z'$. Let $\nu : Z_0 \to Z$ be a normalization, i.e. X is naturally isomorphic to Z_0. Since $q \circ \nu : Z_0 \to Z'$ is finite it generates a finite morphism $X \simeq Z_0 \to Z_0'$ onto a normalization Z_0' of Z'. The commutativity of our diagram implies that ρ' generates a finite morphism $\rho_0' : X' \to Z_0'$. Thus it suffices to prove the following.

Claim. In the last commutative diagram of Γ-equivariant morphisms the fact that $\rho : X \to Z$ is birational and p (and, therefore, q) is finite implies that morphism $\rho' : X' \to Z'$ is birational.

For any $x \in X$ we set $x' = p(x)$, $x_j = \rho_j(x)$, and $x_j' = q_j(x_j) = \rho_j'(x')$. Assume that x is a general point of X and $y \in X$ is such that $\rho'(x') = (x_1', x_2') = (y_1', y_2') = \rho'(y')$. Hence $q_j^{-1}(x_j') = q_j^{-1}(y_j')$ for $j = 1, 2$. Since p is finite and $x' \in X'$ is a general point we have $\rho_j(p^{-1}(x')) = q_j^{-1}(x_j')$ (otherwise ρ_j is not dominant). Replacing x, if necessary, by another point from $p^{-1}(x')$, we can suppose that $x_1 = y_1$ and that $y_2 = z_2$ for some $z \in p^{-1}(x)$ (this means that y and x belong to the same orbit O of H_1 because x is general, see Remark 3.2). Since $x' = z'$ we have $x_i' = y_i' = z_i'$ which implies that q_2 sends x_2 and z_2 to the same point. By the assumption q_2 is H_1-equivariant in Case (i). In particular, it sends the orbit $O_2 = \rho_2(O) \subset X_2$ into an H_1-orbit $O_2' \subset X_2'$. Both orbits are isomorphic to $H_1 \simeq \mathbb{C}_+$, i.e. the H_1-equivariant morphism $q_2|_{O_2} : O_2 \to O_2'$ must be an isomorphism. That is, $x_2 = z_2 = y_2$ and, therefore, $\rho(x) = (x_1, x_2) = (y_1, y_2) = \rho(y)$. Since $\rho : X \to Z$ is birational and x is general we have $x = y$. Hence ρ' is an embedding in a neighborhood of a general point x' which implies the desired conclusion for (i).

In Case (ii) the general Γ-orbit U in X containing O (resp. Γ-orbit $U' = p(U) \subset X'$ containing $O' = p(O)$) is the set of left cosets $SL_2(\mathbb{C})/K$ of a finite subgroup K (resp. K') in $SL_2(\mathbb{C})$. The $SL_2(\mathbb{C})$-action is generated by multiplication on the left while the K-action on $SL_2(\mathbb{C})$ is given by multiplication on the right. Hence the action of $\mathbb{C}_+ \simeq H_i < SL_2(\mathbb{C})$ commutes with the K-action. This implies that each nonidentical element of K sends

any H_i-orbit isomorphically into a different orbit because among finite group actions only the trivial one commutes with the translation on $\mathbb{C}$. Thus the quotient morphism $SL_2(\mathbb{C}) \to U$ (resp. $SL_2(\mathbb{C}) \to U'$) maps any H_i-orbit into a similar orbit isomorphically which implies that $p|_O : O \to O'$ is an isomorphism.

If K is trivial one can suppose that the restrictions of δ_1 and δ_2 to $U \simeq SL_2(\mathbb{C})$ are as in Example 2.10(2). Then the quotient morphism $\varrho : SL_2(\mathbb{C}) \to SL_2(\mathbb{C})//H_2 \simeq \mathbb{C}^2$ sends matrix $(c_{ij}) \in SL_2(\mathbb{C})$ into vector (c_{11}, c_{12}), and $\rho_2|_U = \nu \circ \varrho$ by the universal property of quotient morphisms. Since general fibers of ϱ and ρ_2 are H_2-orbits, morphism $\nu : SL_2(\mathbb{C})//H_2 \to \rho_2(U)$ must be birational. The H_1-orbit O of $x = (c_{ij}^o) \in SL_2(\mathbb{C})$ consists of matrices $(c_{ij}(t))$ with $c_{1j}(t) = c_{1j}^o + tc_{2j}^o$ and $c_{2j}(t) = c_{2j}^o$. Thus $\varrho|_O : O \to \varrho(O)$ is an isomorphism, and x and y belong to a Zariski open subset of O on which $\rho_2|_O$ is an embedding, because these points are general (in X and, therefore, in O) and ν is birational. Similarly, if K' is trivial point x' belongs to a Zariski open subset of O' on which $\rho_2'|_{O'}$ is an embedding. Taking into account the commutative diagram and the fact that $p|_O$ is an isomorphism we see that x_2 and $z_2 = y_2 = \rho_2(y)$ are contained a Zariski open subset of $O_2 = \rho_2(O)$ where $q_2|_{O_2}$ is an embedding. The same argument as in Case (i) implies now that ρ' is birational.

In the general case multiplication of $SL_2(\mathbb{C})$ by elements of K on the right generates a linear K-action on $\mathbb{C}^2 \simeq \varrho(SL_2(\mathbb{C}))$ and we have morphism $\tau : U = SL_2(\mathbb{C})/K \to \mathbb{C}^2/K$ generated by ϱ. Since $\varrho|_O$ is an isomorphism and $\varrho(O)$ is a general line in $\mathbb{C}^2$ morphism $\tau|_O : O \to \tau(O)$ is finite birational. As in the case of trivial K we have $\rho_2|_U = \mu \circ \tau$ where $\mu : \mathbb{C}^2/K \to \rho_2(U)$ is also birational. Thus the general points x and y belong again to a Zariski open subset of O on which $\rho_2|_O$ is an embedding. Similarly, point x' belongs to a Zariski open subset of O' on which $\rho_2'|_{O'}$ is an embedding. Now the same argument as before concludes the proof. □

Though Lemma 3.6 (together with a weak version of Corollary 3.10 that follows from it) enables us to go directly to the proof of Theorem 3, we include some other results to provide a stronger tool for establishing compatibility condition.

3.7. Lemma. *Let the assumption of Lemma 3.6 hold with one exception: instead of the finiteness of p we suppose that there are a surjective quasi-finite morphism $r : S \to S'$ of normal affine algebraic varieties equipped with trivial Γ-actions and a surjective Γ-equivariant morphism $\varrho' : X' \to S'$ such that X is isomorphic to fibred product $X' \times_{S'} S$ with $p : X \to X'$ being the natural projection (i.e. p is surjective quasi-finite). Then the conclusion of Lemma 3.6 remains valid.*

Proof. By the Noether normalization theorem, taking the spectrum of the integral closure of $\mathbb{C}[S']$ in the field of rational functions on S, we obtain a normal affine algebraic variety $\tilde{S} \supset S$ with a finite morphism $\tilde{r} : \tilde{S} \to S'$ extending $r : S \to S'$. Set $\tilde{X} = X' \times_{S'} \tilde{S}$ and denote by $\tilde{p} : \tilde{X} \to X'$ the

natural projection. Then $\tilde{X}$ contains X as a Zariski dense open subset, $\tilde{p}$ extends p, and the Γ-action can be extended to $\tilde{X}$. Let $\tilde{\rho}_i : \tilde{X} \to \tilde{X}_i$ be the quotient morphism of the H_i-action on $\tilde{X}$. For any nontrivial $f \in \mathbb{C}[\tilde{S}]$ whose zero locus contains $\tilde{S} \setminus S$ the f-localizations (i.e. the localizations with respect to the multiplicative system generated by f) of algebras $\mathbb{C}[\tilde{X}]$ and $\mathbb{C}[X]$ are isomorphic. Hence the similar localizations of $\mathbb{C}[\tilde{X}_i]$ and $\mathbb{C}[X_i]$ are isomorphic. This implies that the natural morphism $X_i \to \tilde{X}_i$ is an embedding (over $S \hookrightarrow \tilde{S}$) and we have the following commutative diagram

$$
\begin{array}{ccccccc}
X & \hookrightarrow & \tilde{X} & \xrightarrow{\tilde{\rho}} & \tilde{X}_1 \times \tilde{X}_2 & \xrightarrow{\tilde{\tau}} & \tilde{S} \\
\downarrow{p} & & \downarrow{\tilde{p}} & & \downarrow{\tilde{q}} & & \downarrow{\tilde{r}} \\
X' & = & X' & \xrightarrow{\rho'} & X'_1 \times X'_2 & \xrightarrow{\tau'} & S'
\end{array}
$$

where $\tilde{\rho} = (\tilde{\rho}_1, \tilde{\rho}_2)$ extends $\rho = (\rho_1, \rho_2) : X \to X_1 \times X_2$, $\varrho' = \tau' \circ \rho'$, and the morphisms $\tilde{\rho}, \tilde{q}, \tilde{r}$ are finite.

Set Z (resp. Z', resp. $\tilde{Z}$) equal to the closure of $\rho(X)$ in $X_1 \times X_2$ (resp. $\rho'(X')$ in $X'_1 \times X'_2$, resp. $\tilde{\rho}(\tilde{X})$ in $\tilde{X}_1 \times \tilde{X}_2$). By Proposition 3.4 $\rho : X \to Z$ is birational and hence $\tilde{\rho} : \tilde{X} \to \tilde{Z}$ is birational being the extension of ρ. By the claim in the proof of Lemma 3.6 morphism $\rho' : X' \to Z'$ is birational. Note also that $\rho' : X' \to Z'$ is quasi-finite (indeed, otherwise the commutative diagram implies that contrary to Proposition 3.4 ρ would not be quasi-finite because $\tilde{q}$ is finite and p is surjective).

Suppose that $z' \in Z'$ and $s' = \tau'(z')$. Since r is surjective one can choose $s \in S \subset \tilde{S}$ with $\tilde{r}(s) = s'$. Take $z \in \tilde{q}^{-1}(z') \cap \rho(X)$ so that $\tilde{\tau}(z) = s$ (we can do this because the natural projection $X \to S$ is surjective). Hence $z' \in \rho'(X')$, i.e. ρ' is surjective. Furthermore, for any sequence $\{x'_i\}$ of points in X' such that $\rho'(x'_i) \to z'$ we can choose $x_i \in p^{-1}(x'_i)$ so that $\rho(x_i) \to z$. Since morphism $\rho : X \to Z$ is finite by Proposition 3.4, one can suppose that the sequence $\{x_i\}$ is convergent to a point $x \in X$. Hence the sequence $\{x'_i\}$ is convergent to $x' = p(x)$ which means that $\rho' : X' \to Z'$ is proper. Being also quasi-finite, this morphism is finite by Grothendieck's theorem. Now Proposition 3.4 yields the desired conclusion. $\qquad\square$

3.8. Remark. We do not know whether the assumption, that ϱ' and r are surjective, is essential. Without this assumption the statement of Lemma 3.7 says only that p is quasi-finite since one can put $S = X /\!/ \Gamma$ and $S' = X' /\!/ \Gamma$. However the surjectivity of p may be sufficient for our purposes. Indeed, our aim is to check preservation of the algebraic density property under quasi-finite morphisms and there are examples of affine algebraic manifolds that are not homogeneous (and, therefore, have no algebraic density property [19]) but contain Zariski dense affine algebraic subvarieties with the algebraic density property. For instance, the hypersur-

face in $\mathbb{C}^3_{x,y,z}$ given by $xy = z^2 - 1$ has the algebraic density [13] and it is not difficult to show that it is isomorphic to the complement to the line $x = z = 1$ in the hypersurface in $\mathbb{C}^3$ given by $x(x - 1)y = z^2 - 1$. The dual graph of a simple normal crossing completion of the latter hypersurface cannot be contracted to a zigzag in the terminology of [10] and, hence, this hypersurface is not even quasi-homogeneous by Gizatullin's theorem.

Recall that an étale neighborhood of a point y of an algebraic variety Y is an étale morphism $g : W \to Y$ whose image contains y.

3.9. Proposition. *Let Y be a normal affine algebraic variety equipped with a trivial Γ-action (where Γ is from Lemma 3.6) and $r : X \to Y$ be a surjective Γ-equivariant morphism. Suppose that for any $y \in Y$ there exists an étale neighborhood $g : W \to Y$ such that the vector fields induced by δ_1 and δ_2 on the fibred product $X \times_Y W$ are compatible. Then δ_1 and δ_2 are compatible.*

Proof. Set $Y_1 = g(W)$. Then the restrictions δ_1^1 and δ_2^1 of δ_1 and δ_2 to $X_1 = r^{-1}(Y_1)$ are compatible by Lemma 3.7. Suppose that $\{Y_i\}$ is a finite cover of Y by open sets similar to Y_1 and notation X_i, δ_1^i, δ_2^i have also the similar meaning. Without loss of generality we can assume that $Y_i = Y \setminus f_i^{-1}(0)$ for some $f_i \in \mathbb{C}[Y_i] \subset \mathbb{C}[X_i]$. Let $I_i \subset \mathbb{C}[X_i]$ be the largest ideal contained in $\mathrm{Span}(\mathrm{Ker}\,\delta_1^i \cdot \mathrm{Ker}\,\delta_2^i)$ and I be the largest ideal in $\mathbb{C}[X]$ whose f_i-localization is contained in I_i for every i. In particular, I is non-zero since each I_i is such. Show that $I \subset \mathrm{Span}(\mathrm{Ker}\,\delta_1 \cdot \mathrm{Ker}\,\delta_2)$.

Indeed, $f_i \in \mathbb{C}[Y] \subset \mathrm{Ker}\,\delta_j$, $j = 1, 2$. Hence for every $a \in I$ there exists k_i such that $a f_i^{k_i}$ is in $\mathrm{Span}(\mathrm{Ker}\,\delta_1 \cdot \mathrm{Ker}\,\delta_2)$. By Hilbert's Nullstellensatz there are regular functions g_i on Y such that $\sum_i f_i^{k_i} g_i \equiv 1$. Since g_i is in the kernel of δ_1 we see that $a \in \mathrm{Span}(\mathrm{Ker}\,\delta_1 \cdot \mathrm{Ker}\,\delta_2)$ which concludes the proof. $\square$

3.10. Corollary. *Let a linear algebraic group G act algebraically on X so that $X/\!/G$ is affine, the quotient morphism $X \to X/\!/G$ is surjective (which is always true when G is reductive) and makes X an étale G-principal bundle over $X/\!/G$. Suppose that Γ (from Lemma 3.6) is an algebraic subgroup of G and the actions of H_i, $i = 1, 2$ on G induced by left multiplication generate compatible derivations on $\mathbb{C}[G]$. Let the induced H_i-actions on X correspond to derivations δ_i on $\mathbb{C}[X]$. Then δ_1 and δ_2 are compatible.*

Theorem 3. *Let G be a linear algebraic group whose connected component is different from a torus or $\mathbb{C}_+$. Then G has the algebraic density property.*

Proof. Since all components of G are isomorphic as varieties we can suppose that G is connected. Recall that the unipotent radical R of G is an algebraic subgroup of G [4, p. 183]. By Mostow's theorem [14] (see also [4, p. 181]) G contains a (Levi) maximal closed reductive algebraic subgroup L (which is, in particular, affine) such that G is the semi-direct product

of L and R, i.e. G is isomorphic as affine variety to the product $R \times L$. In case L is trivial $G = R \simeq \mathbb{C}^n$, $n \geq 2$ and we are done by Corollary 2.2. In the case of both R and L being nontrivial we are done by Corollary 2.8 with R playing the role of X_1 and L of X_2.

Thus it remains to cope with reductive groups G. Let $Z \simeq (\mathbb{C}^*)^n$ denote the center of G and S its semisimple part. First we suppose that Z is nontrivial. The case when G is isomorphic as group to the direct product $S \times Z$ can be handled as above by Corollary 2.8 with S playing the role of X_1 and Z of X_2. In particular, we have a finite set of pairs of compatible vector fields $\{\delta_1^k, \delta_2^k\}$ as in Theorem 2. Furthermore, one can suppose that the fields δ_1^k correspond to one parameter subgroups of S isomorphic to $\mathbb{C}_+$ and δ_2^k to one parameter subgroups of Z isomorphic to $\mathbb{C}^*$. In the general case G is the factor group of $S \times Z$ by a finite (central) normal subgroup Λ. Since Λ is central the fields δ_1^k, δ_2^k induce completely integrable vector fields $\tilde{\delta}_1^k$, $\tilde{\delta}_2^k$ on G while $\tilde{\delta}_2^k(x_0)$ is a generating set for some $x_0 \in G$. By Lemma 3.6 the pairs $\{\tilde{\delta}_1^k, \tilde{\delta}_2^k\}$ are compatible and the density property for G follows again from Theorem 2.

It remains now to consider a semi-simple G which can be assumed simply connected by Lemma 3.6. That is, it is a product of simple Lie groups and by Corollary 2.8 it suffices to consider the case when G is simple. Such G contains $SL_2(\mathbb{C})$ as a subgroup. The existence of two compatible vector fields δ_1 and δ_2 on $SL_2(\mathbb{C})$ implies their existence on G by Corollary 3.10. Since a simple Lie group is tangentially semi-homogenous (see Example 2.10) the algebraic density property for G follows again from Theorem 2. $\square$

4. Codimension 2 case

Motivation and notation. In this section X will be a closed affine algebraic subvariety of $\mathbb{C}^n$ whose codimension $n - k$ is at least 2. By the Hartogs theorem any completely integrable algebraic (or holomorphic) vector field on $\mathbb{C}^n \setminus X$ extends to a similar vector field on $\mathbb{C}^n$ tangent to X. In particular, the Lie algebra generated by completely integrable algebraic (or holomorphic) vector fields contains only vector fields tangent to X, i.e. there is no density property for $\mathbb{C}^n \setminus X$. In general there is no also hope that this Lie algebra coincides with the Lie algebra of all algebraic vector fields tangent to X, since this would imply density property for X (and our X maybe even not smooth!). Therefore, it is natural to study the Lie algebra $\mathrm{Lie}_{\mathrm{alg}}(\mathbb{C}^n, X)$ generated by completely integrable algebraic vector fields on $\mathbb{C}^n$ that vanishes on X. According to Forstnerič the best possible result to expect is that $\mathrm{Lie}_{\mathrm{alg}}(\mathbb{C}^n, X)$ is equal to the Lie algebra of all algebraic vector fields vanishing on X. We use notation $\mathrm{AVF}_I(\mathbb{C}^n)$ for the latter algebra where $I \subset \mathbb{C}^{[n]}$ is the defining ideal of X (more generally, for any affine algebraic variety Y and an ideal $L \subset \mathbb{C}[Y]$ we denote by $\mathrm{AVF}_L(Y)$ the Lie algebra of vector fields whose coordinate functions are from L). If the

above property holds then the geometric structure of algebraic vector fields vanishing on X has the algebraic density property in the terminology of Varolin [19].

We will prove this property under some weak additional assumption in Theorem 6 and a very close result without any additional assumption in Theorem 4. Both results lead to a generalization of the main theorem of the Andersén–Lempert theory, allowing now to construct holomorphic automorphisms of $\mathbb{C}^n$ not only with control on compacts but with additional control on algebraic subvarieties of codimension at least 2.

4.1. Lemma. *The group* $\mathrm{Aut}(\mathbb{C}^n, X)$ *of algebraic automorphisms of* $\mathbb{C}^n$ *identical on* X *acts transitively on* $Z = \mathbb{C}^n \setminus X$ *and, furthermore, for any* $z \in Z$ *the image of any vector* $v \in T_z Z$ *under the isotropy group* $\mathrm{Aut}(\mathbb{C}^n, X)_z$ *generates* $T_z Z$ *(compare with Definition 2.1).*

Proof. By a theorem of Gromov [11] and Winkelmann [20] Z is homogenous. We will use the idea of their proof. More precisely, consider a general linear projection $p : \mathbb{C}^n \to \mathcal{H} \simeq \mathbb{C}^{n-1}$ and a non-zero constant vector field v such that $p_*(v) = 0$. Then $p(X)$ is a subvariety of codimension at least 1 in $\mathcal{H}$. For every regular function h on $\mathcal{H}$ that vanishes on $p(X)$ the vector field hv generates a $\mathbb{C}_+$-action on Z. Changing $\mathcal{H}$ we get a transitive action.

Consider a general point $z \in Z$ whose projection $z_0 \in \mathcal{H}$ is not in $p(X)$. Suppose that h has a simple zero at z_0. By the claim in the proof of Corollary 2.8 the $\mathbb{C}_+$-action generated by hv acts on $T_z Z$ by the formula $w \to w + dh(w)v(w)$ where dh is the differential of h and $w \in T_z Z$. Since v may be chosen as a general constant vector field on $\mathbb{C}^n$ we see that $\mathrm{Aut}(\mathbb{C}^n, X)_z$ induces an irreducible representation on $T_z Z$ which implies the second statement. $\square$

Theorem 4. *There is an ideal* $L \subset \mathbb{C}^{[n]}$ *whose radical is* I *such that* $\mathrm{Lie}_{\mathrm{alg}}(\mathbb{C}^n, X)$ *contains* $AVF_L(\mathbb{C}^n)$.

Proof. Suppose that $x_1, \ldots, x_n$ is a coordinate system, $p_i : \mathbb{C}^n \to \mathbb{C}^{n-1}$ is a projection to the coordinate hyperplane $\mathcal{H}_i = \{x_i = 0\}$, and h_i is a non-zero function on $\mathcal{H}_i$ that vanishes on $p_i(X)$. Set $\delta_i = \partial/\partial x_i$ and choose $f_i \in \mathrm{Ker}\, \delta_i$. Then $f_i h_i \delta_i$ is a completely integrable algebraic vector field on $\mathbb{C}^n$ that vanishes on X, i.e. it generates a $\mathbb{C}_+$-action on Z (since the elements of this action are from $\mathrm{Aut}(\mathbb{C}^n, X)$). Then

$$[f_1 h_1 \delta_1, x_1 f_2 h_2 \delta_2] - [x_1 f_1 h_1 \delta_1, f_2 h_2 \delta_2] = f_1 f_2 h_1 h_2 \delta_2$$

belongs to $\mathrm{Lie}_{\mathrm{alg}}(\mathbb{C}^n, X)$. Since $\mathrm{Ker}\, \delta_1 \cdot \mathrm{Ker}\, \delta_2$ generates the ring of polynomials $\mathbb{C}^{[n]}$ as a vector space we see that $\mathrm{Lie}_{\mathrm{alg}}(\mathbb{C}^n, X)$ contains all algebraic fields proportional to δ_2 with coordinate functions in the principal ideal generated by $h_1 h_2$. Since one can perturb x_2 (as a linear function) $\mathrm{Lie}_{\mathrm{alg}}(\mathbb{C}^n, X)$ contains all algebraic vector fields whose coordinates are in some (non-zero) ideal L. Since Z is homogenous under $\mathrm{Aut}(\mathbb{C}^n, X)$ arguing as in the proof of Theorem 1 one can suppose that the radical of L is I. $\square$

Though Theorem 4 does not give the algebraic density of the Lie algebra of algebraic vector fields vanishing on X it implies already a strong approximation result generalizing the Andersén–Lempert theorem. We omit its proof since it repeats the arguments in [9] with minor modifications.

Theorem 5. *Let X be an algebraic subvariety of $\mathbb{C}^n$ of codimension at least 2 and Ω be an open set in $\mathbb{C}^n$ ($n \geq 2$). Let $\Phi : [0, 1] \times \Omega \to \mathbb{C}^n \setminus X$ be a $\mathcal{C}^2$-map such that for every $t \in [0, 1]$ the restriction $\Phi_t = \Phi|_{(t \times \Omega)}$ is an injective holomorphic map onto a Runge domain $\Phi_t(\Omega)$ in $\mathbb{C}^n$. If Φ_0 can be approximated on Ω by holomorphic automorphisms of $\mathbb{C}^n$ identical on X, then for every $t \in [0, 1]$ the map Φ_t can be approximated on Ω by such automorphisms.*

As a consequence we recover the result of Buzzard and Hubbard [3] answering Siu's question (see also [8], Prop. 1.4.).

4.2. Corollary. *Any point z in the complement of an algebraic subset X of $\mathbb{C}^n$ of codimension at least 2 has a neighborhood U in $\mathbb{C}^n \setminus X$ that is biholomorphic to $\mathbb{C}^n$ (such U is called a Fatou–Bieberbach domain).*

Proof. Following the standard scheme of Rudin and Rosay, choose a ball $\Omega \subset \mathbb{C}^n \setminus X$ around z and consider Φ_t contracting this ball radially towards z. The resulting automorphism approximating Φ_1 from Theorem 5 has an attracting fixed point near z and z is contained in the basin of attraction. This basin is a Fatou–Bieberbach domain and it does not meet X since the automorphism is identical on X. $\square$

Though Theorem 4 is sufficient for the approximation purposes, let us be accurate and establish the algebraic density for algebraic vector fields vanishing on X under an additional assumption.

4.3. Convention. We suppose further in this section that the dimension of the Zariski tangent space $T_x X$ is at most $n - 1$ for every point $x \in X$.

4.4. Lemma. *Lie algebra $\mathrm{Lie}_{\mathrm{alg}}(\mathbb{C}^n, X)$ contains $\mathrm{AVF}_{I^2}(\mathbb{C}^n)$.*

Proof. It suffices to show that for every point $o \in \mathbb{C}^n$ there exists a Zariski neighborhood V and a submodule M_V from $\mathrm{Lie}_{\mathrm{alg}}(\mathbb{C}^n, X)$ such that its localization to V coincides with the localization of $\mathrm{AVF}_{I^2}(\mathbb{C}^n)$ to V. Indeed, because of quasi-compactness we can find a finite number of such open sets V_i that cover $\mathbb{C}^n$. Hence the coherent sheaves generated by $\mathrm{AVF}_{I^2}(\mathbb{C}^n)$ and $\sum_i M_{V_i}$ coincide locally which implies that they have the same global sections over affine varieties by Serre's theorem B. In fact, it suffices to show that the localization of M_V to V contains all fields from the localization of $\mathrm{AVF}_{I^2}(\mathbb{C}^n)$ to V that are proportional to some general constant vector field δ which is *our aim now*. By Theorem 4 it is also enough to consider $o \in X$ only. The construction of the desired neighborhood V of o starts with the following.

Claim. For any point $o \in X, l \geq \max(k+1, \dim T_o X)$ (where $\dim X = k$), and a general linear projection $p : \mathbb{C}^n \to \mathcal{H} \simeq \mathbb{C}^l$ one can choose a projection $p_0 : \mathbb{C}^n \to \mathcal{H}_0 \simeq \mathbb{C}^{l-1}$ for which

(i) $p_0 = \varrho \circ p$ where $\varrho : \mathcal{H} \to \mathcal{H}_0$ is a general linear projection, and
(ii) there exists $h \in \mathbb{C}[\mathcal{H}_0] \simeq \mathbb{C}^{[l-1]} \simeq \varrho^*(\mathbb{C}^{[l-1]}) \subset \mathbb{C}^{[l]}$ such that h does not vanish at $p_0(o)$ and $p|_{X \setminus (h \circ p)^{-1}(0)} : X \setminus (h \circ p)^{-1}(0) \to p(X) \setminus h^{-1}(0)$ is an isomorphism.

Since p is general the condition on l implies that p is a local isomorphism in a neighborhood of o and, furthermore, since ϱ is also general then by Bertini's theorem $p_0^{-1}(p_0(o))$ contains only smooth points of X except, may be, for o, i.e. p is a local isomorphism in a neighborhood of each of these points which implies the claim.

From now on let $l = n - 1$. Choose a general coordinate system $\bar{x} = (x_1, \ldots, x_n)$ on $\mathbb{C}^n$ such that $p(\bar{x}) = (x_2, \ldots, x_n)$ and $p_0(\bar{x}) = (x_3, \ldots, x_n)$, i.e. $h = h(x_3, \ldots, x_n)$. Set $V = \mathbb{C}^n \setminus h^{-1}(0)$.

Since $p(X) \cap V \simeq X \cap V$ we have $x_1 = r/h^s$ where r is a polynomial in $x_2, \ldots, x_n$ and $s \geq 0$. Set $v_i = \partial/\partial x_i$ for $i \neq 2$, and $v_2 = h^s \partial/\partial x_2 + (\partial r/\partial x_2)\partial/\partial x_1$. Then each v_i is a locally nilpotent derivation and $\text{Ker } v_1$ contains the defining ideal I_p of $p(X)$ in $\mathbb{C}[\mathcal{H}] \simeq \mathbb{C}^{[n-1]} \simeq p^*(\mathbb{C}^{[n-1]}) \subset \mathbb{C}^{[n]}$. Furthermore, for $\xi = h^s x_1 - r$ we have $\xi \in \text{Ker } v_2$, and ξ (resp. x_2) is of degree 1 with respect to v_1 (resp. v_2). This implies that for $f, g \in I_p$ the vector fields that appear in the Lie brackets below are completely integrable and vanish on X:

$$[fv_1, \xi g v_1] = h^s fg v_1, \quad [\xi v_2, x_2 \xi v_1] - [x_2 \xi v_2, \xi v_1] = h^s \xi^2 v_1,$$
$$\text{and } [\xi v_2, x_2 fv_1] - [x_2 \xi v_2, fv_1] = h^s \xi f v_1.$$

The defining ideal of $X \cap V$ is generated by ξ and elements of I_p. Since h is invertible on V and v_1 is a general constant vector field from the formulas before we see that the localization of $\text{Lie}_{\text{alg}}(Z)$ to V contains the localization of $\text{AVF}_{I^2}(\mathbb{C}^n)$ which is the desired conclusion. $\qquad \square$

Theorem 6. *Let X be a closed algebraic subset of $\mathbb{C}^n$ of codimension at least 2 such that the Zariski tangent space $T_x X$ has dimension at most $n - 1$ for any point $x \in X$. Then $\text{Lie}_{\text{alg}}(\mathbb{C}^n, X)$ is equal to $\text{AVF}_I(\mathbb{C}^n)$, i.e. the vector fields vanishing on X have algebraic density property.*

Proof. Similarly to the proof of Lemma 4.4, it suffices to show that for every point $o \in \mathbb{C}^n$ there exists a Zariski neighborhood V and a submodule from $\text{Lie}_{\text{alg}}(\mathbb{C}^n, X)$ such that its localization M to V coincides with localization of $\text{AVF}_I(\mathbb{C}^n)$ to V. By Theorem 4 it is enough to consider $o \in X$ and, furthermore, it suffices to show that this localization M contains all elements of $\text{AVF}_I(\mathbb{C}^n)$ proportional to some general constant vector field.

Let v_i, p, I_p, and ξ have the same meaning as in the proof of Lemma 4.4. Choose v_1 as this constant vector field. Since I is generated by ξ and I_p one needs to show that all fields of the form $\mu = (\xi g_0 + \sum g_i f_i)v_1$ are

contained in M where g_0, g_i are regular on V and $f_i \in I_p$. Since p yields an isomorphism between $p(X) \cap V$ and $X \cap V$ there are functions e_0, e_i that do not depend on x_1 and such that $e_0|_X = g_0|_X$ and $e_i|_X = g_i|_X$. Then $\mu = (\xi e_0 + \sum e_i f_i) v_1 + a v_1$ where a belongs to the localization of I^2 to V (e.g. $a = \xi(g_0 - e_0) + \sum(g_i - e_i) f_i$). Since the first summand in the last formula for μ is completely integrable we have the desired conclusion from Lemma 4.4. $\square$

4.5. Remark. (1) The authors believe that the condition $\dim T_x X \leq n - 1$ in Theorem 6 is essential. As a potential counterexample one may try to take X equal to polynomial curve in $\mathbb{C}^3$ with one singular point whose Zariski tangent space is 3-dimensional. More precisely, let L be the Lie algebra generated by vector fields that vanish on X and have form $f\sigma$ where $f \in \operatorname{Ker} \sigma$ and σ is either locally nilpotent or semi-simple. Then we can show that L does not coincide with $\mathrm{AVF}_I(\mathbb{C}^3)$ but we do not know whether $L = \operatorname{Lie}_{\mathrm{alg}}(\mathbb{C}^3, X)$.

(2) In view of Theorem 6 the assumptions of Theorem 5 can be weakened in case of $\dim T_x X \leq n - 1$ to the following extend: the assumption $\Phi_t(\Omega) \cap X = \varnothing$ can be replaced by the assumption that Φ_t is identical on $\Phi_t(\Omega) \cap X$ for all t.

(3) The assumption of codimension at least 2 for X cannot be removed, since the complement to a hypersurface in $\mathbb{C}^n$ can be of general type or even Kobayashi hyperbolic and, therefore, there is no nontrivial completely integrable holomorphic vector field vanishing on X. Also the assumption that X is not just a holomorphic but an algebraic subvariety of $\mathbb{C}^n$ cannot be weakened. This follows from the fact that there are holomorphic embeddings of $\mathbb{C}$ into $\mathbb{C}^n$ (for any n) such that the group of holomorphic automorphisms of $\mathbb{C}^n$ identical on the image is trivial (e.g., see [5]).

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