

Tropical varieties for non-archimedean analytic spaces

Walter Gubler

Fachbereich Mathematik, Universität Dortmund, D-44221 Dortmund
(e-mail: walter.gubler@mathematik.uni-dortmund.de)

Oblatum 5-VII-2006 & 5-III-2007

Published online: 19 April 2007 – © Springer-Verlag 2007

Abstract. Generalizing the construction from tropical algebraic geometry, we associate to every (irreducible d -dimensional) closed analytic subvariety of \mathbb{G}_m^n a tropical variety in \mathbb{R}^n with respect to a complete non-archimedean place. By methods of analytic and formal geometry, we prove that the tropical variety is a totally concave locally finite union of d -dimensional polytopes. For an algebraic morphism $f : X' \rightarrow A$ to a totally degenerate abelian variety A , we give a bound for the dimension of $f(X')$ in terms of the singularities of a strictly semistable model of X' . A closed d -dimensional subvariety X of A induces a periodic tropical variety. A generalization of Mumford's construction yields models of X and A which can be handled with the theory of toric varieties. For a canonically metrized line bundle \bar{L} on A , the measures $c_1(\bar{L}|_X)^d$ are piecewise Haar measures on X . Using methods of convex geometry, we give an explicit description of these measures in terms of tropical geometry. In a subsequent paper, this is a key step in the proof of Bogomolov's conjecture for totally degenerate abelian varieties over function fields.

1. Introduction

For the whole paper, \mathbb{K} denotes an algebraically closed field endowed with a non-trivial non-archimedean complete absolute value $|\cdot|$. The corresponding valuation is $v := -\log |\cdot|$ with value group $\Gamma := v(\mathbb{K}^\times)$. The valuation ring is denoted by \mathbb{K}° . Note that the residue field $\tilde{\mathbb{K}}$ is algebraically closed. In Theorem 1.3, Sect. 8 and in the second part of Sect. 9, we start with a field K endowed with a discrete valuation and we choose \mathbb{K} to be the completion of the algebraic closure of the completion of K (see [BGR, §3.4], for these properties of \mathbb{K}).

On the torus \mathbb{G}_m^n , we always fix coordinates x_1, \dots, x_n and we consider the map

$$\text{val} : \mathbb{G}_m^n(\mathbb{K}) \longrightarrow \mathbb{R}^n, \quad \mathbf{x} \mapsto (-\log |x_1|, \dots, -\log |x_n|).$$

For an irreducible closed algebraic subvariety X of \mathbb{G}_m^n over \mathbb{K} , the closure of $\text{val}(X)$ in \mathbb{R}^n is called a tropical variety. This is the main object of study in tropical algebraic geometry. We refer to [Mi] for a survey of this relatively new area of research. Einsiedler, Kapranov and Lind [EKL] have shown that the tropical variety of X is a connected totally concave Γ -rational polyhedral set in \mathbb{R}^n of pure dimension $\dim(X)$. Here and in the following, the reader is referred to the appendix for the terminology used from convex geometry. In the present paper, the following analytic generalization is given:

Theorem 1.1. *Let X be an irreducible closed analytic subvariety of \mathbb{G}_m^n over \mathbb{K} of dimension d . Then the tropical variety associated to X is a connected totally concave locally finite union of d -dimensional Γ -rational polytopes.*

Note that the map val is continuous with respect to the Berkovich analytic structure on \mathbb{G}_m^n and therefore the tropical variety is obviously connected and compact. This makes it clear that we benefit a lot by using Berkovich analytic spaces and methods from formal geometry (see Sect. 2 for a summary). In Sect. 4, we generalize a result of Mumford to study the special fibre of the analytic subdomain $U_\Delta := \text{val}^{-1}(\Delta)$ of \mathbb{G}_m^n associated to a Γ -rational polytope in \mathbb{R}^n . This allows us to apply the theory of toric varieties to the reduction of U_Δ . In Sect. 5, we prove Theorem 1 from the corresponding local case in U_Δ .

The applications will deal with a totally degenerate abelian variety A over \mathbb{K} , i.e. $A^{\text{an}} = (\mathbb{G}_m^n)_{\mathbb{K}}^{\text{an}}/M$ for a discrete subgroup M of $\mathbb{G}_m^n(\mathbb{K})$ which is mapped isomorphically onto the complete lattice $\Lambda := \text{val}(M)$ in \mathbb{R}^n . We get a canonical map $\overline{\text{val}} : A^{\text{an}} \rightarrow \mathbb{R}^n/\Lambda$ and hence a tropical variety $\overline{\text{val}}(X^{\text{an}})$ associated to a closed analytic subvariety X of A . In Sect. 6, we will show that Theorem 1.1 holds also in this framework. This is quite obvious by lifting X to \mathbb{G}_m^n leading to a periodic tropical variety in \mathbb{R}^n . As a consequence, we obtain the following dimensionality theorem:

Theorem 1.2. *Let X' be a smooth algebraic variety over \mathbb{K} with a strictly semistable formal \mathbb{K}° -model \mathcal{X}' (see 2.10 and 3.1) and let $f : X' \rightarrow A$ be a morphism over \mathbb{K} . Then the special fibre of \mathcal{X}' has a \mathbb{K} -rational point contained in at least $1 + \dim f(X')$ irreducible components.*

If X' is projective, then we may use a strictly semistable projective \mathbb{K}° -model. If X' has good reduction at v , then f is constant. We will postpone the proof of Theorem 1.2 to the first part of Sect. 9 where it can be given very neatly and where also a generalization to arbitrary abelian varieties is given.

In non-archimedean analysis, no good definition is known for the first Chern form of a metrized line bundle. However, Chambert-Loir [Ch] has

introduced measures $c_1(\overline{L}_1) \wedge \cdots \wedge c_1(\overline{L}_d)$ on the Berkovich analytic space associated to a d -dimensional projective variety analogous to the corresponding top dimensional forms in differential geometry. In Sect. 3, we give a slightly more general approach to these measures using local heights. Our main application of the theory of tropical analytic varieties is the following result:

Theorem 1.3. *Let X be a closed subvariety of the abelian variety A over K . We assume that $A_{\mathbb{K}}^{\text{an}}$ is totally degenerate over \mathbb{K} and that X is of pure dimension d . Let $\overline{L}_1, \dots, \overline{L}_d$ be ample line bundles on A endowed with canonical metrics. Then $\mu := (\overline{\text{val}})_*(c_1(\overline{L}_1|_X) \wedge \cdots \wedge c_1(\overline{L}_d|_X))$ is a strictly positive piecewise Haar measure on the polytopal set $\overline{\text{val}}(X_{\mathbb{K}}^{\text{an}})$.*

By a *piecewise Haar measure* on the polytopal set $\overline{\text{val}}(X_{\mathbb{K}}^{\text{an}})$, we mean that $\overline{\text{val}}(X_{\mathbb{K}}^{\text{an}})$ is a union of d -dimensional polytopes $\overline{\sigma}$ such that μ is a real multiple of the relative Lebesgue measure on $\overline{\sigma}$, and *strictly positive* means that the multiple is > 0 .

The proof of Theorem 1.3 will be given in Sect. 8. It is based on our studies in Sect. 6 and Sect. 7 of Mumford's model \mathcal{A} of A associated to a Γ -rational polytopal decomposition of \mathbb{R}^n/Λ . The main idea is to work with a generic polytopal decomposition $\overline{\mathcal{C}}$ defined over a sufficiently large base extension \mathbb{K}' of \mathbb{K} . Then the irreducible components of the special fibre of the closure of X in \mathcal{A} are toric varieties by Theorem 6.10. Moreover, this holds for the sequence $\frac{1}{m}\mathcal{C}$, $m \in \mathbb{N}$, of polytopal decompositions and this allows us to compute μ by Tate's limit argument leading to an explicit expression for μ in Theorem 8.6 and proving Theorem 1.3. In Theorem 9.6, we prove that $c_1(\overline{L}_1|_X) \wedge \cdots \wedge c_1(\overline{L}_d|_X)$ itself is induced by an explicit strictly positive Haar measure on the skeleton of a strictly semistable alteration of X .

In [Gu4], Theorem 1.3 is essential to prove the following case of *Bogomolov's conjecture* over the function field $F := k(B)$. Here, B is an integral projective variety over the algebraically closed field k such that B is regular in codimension 1. The prime divisors on B are weighted by the degree with respect to a fixed ample class leading to a theory of heights.

Bogomolov conjecture ([Gu4, Theorem 1.1]). *Let A be an abelian variety over the function field F which is totally degenerate at some place v of F . Let X be a closed subvariety of A defined over the algebraic closure \overline{F} which is not a translate of an abelian subvariety by a torsion point. For every ample symmetric line bundle L on A , there is $\varepsilon > 0$ such that*

$$X(\varepsilon) := \{P \in X(\overline{F}) \mid \hat{h}_L(P) \leq \varepsilon\}$$

is not Zariski dense in X , where \hat{h}_L denotes the Néron–Tate height with respect to L .

Terminology

In $A \subset B$, A may be equal to B . The complement of A in B is denoted by $B \setminus A$ as we reserve $-$ for algebraic purposes. The zero is included in \mathbb{N} .

All occurring rings and algebras are commutative with 1. If A is such a ring, then the group of multiplicative units is denoted by A^\times . A variety over a field is a separated reduced scheme of finite type. However, a formal analytic variety is not necessarily reduced (see Sect. 2). For the degree of a map $f : X \rightarrow Y$ of irreducible varieties, we use either $\deg(f)$ or $[X : Y]$. The multiplicity of an irreducible component Y of a scheme S is denoted by $m(Y, S)$.

For $\mathbf{m} \in \mathbb{Z}^n$, let $\mathbf{x}^{\mathbf{m}} := x_1^{m_1} \cdots x_n^{m_n}$. The standard scalar product of $\mathbf{u}, \mathbf{u}' \in \mathbb{R}^n$ is denoted by $\mathbf{u} \cdot \mathbf{u}' := u_1 u'_1 + \cdots + u_n u'_n$. For the notation used from convex geometry, we refer to the Appendix (see also 6.1 for the periodic case).

Acknowledgements. The author thanks J. Eckhoff, K. Künnemann and F. Oort for precious discussions, and the referee for his suggestions. Part of the research in this paper was done during a 5 weeks stay at the CRM in Barcelona.

2. Analytic and formal geometry

In this section, we gather the results needed from Berkovich spaces and formal geometry.

2.1. The completion of $\mathbb{K}[x_1, \dots, x_n]$ with respect to the Gauss norm is called the *Tate algebra* and is denoted by $\mathbb{K}\langle x_1, \dots, x_n \rangle$. It consists of the strictly convergent power series on the closed unit ball \mathbb{B}^n in \mathbb{K}^n .

A \mathbb{K} -affinoid algebra \mathcal{A} is isomorphic to a quotient $\mathbb{K}\langle x_1, \dots, x_n \rangle / I$ and the maximal spectrum $\text{Max}(\mathcal{A})$ is equal to the zero set $Z(I) \subset \mathbb{B}^n$ of the ideal I . The supremum semi-norm of \mathcal{A} on $Z(I)$ is denoted by $| \cdot |_{\text{sup}}$. Setting

$$\mathcal{A}^\circ := \{a \in \mathcal{A} \mid |a|_{\text{sup}} \leq 1\}, \quad \mathcal{A}^\circ\circ := \{a \in \mathcal{A} \mid |a|_{\text{sup}} < 1\},$$

the *residue algebra* is defined by $\tilde{\mathcal{A}} := \mathcal{A}^\circ / \mathcal{A}^\circ\circ$. It is a finitely generated reduced $\tilde{\mathbb{K}}$ -algebra. For details about affinoid algebras, we refer to [BGR].

2.2. The *Berkovich spectrum* $\mathcal{M}(\mathcal{A})$ of a \mathbb{K} -affinoid algebra \mathcal{A} is defined as the set of semi-norms p on \mathcal{A} satisfying

$$p(ab) = p(a)p(b), \quad p(1) = 1 \quad \text{and} \quad p(a) \leq |a|_{\text{sup}}$$

for all $a, b \in \mathcal{A}$. It is endowed with the coarsest topology such that the maps $p \mapsto p(a)$ are continuous for all $a \in \mathcal{A}$.

The Berkovich spectrum is compact and every $x \in \text{Max}(\mathcal{A})$ gives rise to a semi-norm $a \mapsto |a(x)|$ such that we may view $\mathcal{M}(\mathcal{A})$ as a compactification of $\text{Max}(\mathcal{A})$. We refer to [Ber1] for proofs and more details.

The affine $\tilde{\mathbb{K}}$ -variety $\text{Spec}(\tilde{\mathcal{A}})$ is called the *reduction* of $\mathcal{M}(\mathcal{A})$ and the reduction map $p \mapsto \tilde{p} := \{p < 1\}/\mathcal{A}^{\circ\circ}$ is surjective. If \wp is a minimal prime ideal of $\tilde{\mathcal{A}}$, then there is a unique $p \in \mathcal{M}(\mathcal{A})$ with $\tilde{p} = \wp$ (see [Ber1, Proposition 2.4.4]).

2.3. An *affinoid subdomain* of $X := \mathcal{M}(\mathcal{A}) = \mathcal{M}(\mathbb{K}\langle \mathbf{x} \rangle/I)$ is characterized by an universal property (see [Ber1, 2.2] or [BGR, 7.2.2]). By a theorem of Gerritzen and Grauert [BGR, Corollary 7.3.5/3], an affinoid subdomain is a finite union of *rational domains*. The latter are defined by

$$X\left(\frac{\mathbf{f}}{g}\right) := \{x \in X \mid |f_j(x)| \leq |g(x)|, j = 1, \dots, r\}$$

where $g, f_1, \dots, f_r \in \mathcal{A}$ are without common zero. The corresponding affinoid algebra is

$$\mathcal{A}\left\langle \frac{\mathbf{f}}{g} \right\rangle := \mathbb{K}\langle \mathbf{x}, y_1, \dots, y_r \rangle / \langle I, g(\mathbf{x})y_j - f_j(\mathbf{x}) \mid j = 1, \dots, r \rangle$$

(see [BGR, Proposition 7.2.3/4]). If $g = 1$, then $X(\mathbf{f})$ is called a *Weierstrass domain* in $\mathcal{M}(\mathcal{A})$.

2.4. An *analytic space* X over \mathbb{K} is given by an atlas of affinoid subdomains $U = \mathcal{M}(\mathcal{A})$. For the precise definition, we refer to [Ber2, §1]. (Note that our definition corresponds to strictly analytic spaces in the notation of [Ber2, p. 22]). The technical difficulty in this definition arises from the fact that the charts U are not open in X but compact. The sheaf of structure \mathcal{O}_X is only defined on the Grothendieck topology of X and it is characterized by $\mathcal{O}_X(U) = \mathcal{A}$.

2.5. Let \mathcal{A} be a \mathbb{K} -affinoid algebra. A subset U of $\mathcal{M}(\mathcal{A})$ is called *formal open* if there is a open subset V of the reduction $\text{Spec}(\tilde{\mathcal{A}})$ such that $U = \pi^{-1}(V)$. The resulting quasi-compact topology on $\mathcal{M}(\mathcal{A})$ is called the *formal topology*. Together with the restriction of $\mathcal{O}_{\mathcal{M}(\mathcal{A})}$ to the formal topology, we get a ringed space called a *formal affinoid variety* over \mathbb{K} and denoted by $\text{Spf}(\mathcal{A})$. By definition, a morphism of affinoid varieties over \mathbb{K} is induced by a reverse homomorphism of the corresponding \mathbb{K} -affinoid algebras. For details, we refer to [Bo].

A *formal analytic variety* over \mathbb{K} is a \mathbb{K} -ringed space \mathfrak{X} which has a locally finite open atlas of formal affinoid varieties over \mathbb{K} . It has a *reduction* $\tilde{\mathfrak{X}}$ and a *generic fibre* \mathfrak{X}^{an} . If $\mathfrak{X} = \text{Spf}(\mathcal{A})$, then $\tilde{\mathfrak{X}} = \text{Spec}(\tilde{\mathcal{A}})$ and $\mathfrak{X}^{\text{an}} = \mathcal{M}(\mathcal{A})$. In general, the $\tilde{\mathbb{K}}$ -variety $\tilde{\mathfrak{X}}$ and the analytic space \mathfrak{X}^{an} are obtained by gluing processes (see [Bo] and [Ber3, §1]). Note that we require that an algebraic variety is reduced (see Sect. 1), but a formal analytic variety is not necessarily reduced. Since the supremum semi-norm is power-multiplicative, the reduction of a formal analytic variety is always reduced. By 2.2, there is a surjective reduction map $\mathfrak{X}^{\text{an}} \rightarrow \tilde{\mathfrak{X}}$, given locally

by $p \mapsto \tilde{p}$. For every irreducible component Y , there is a unique $\xi_Y \in \mathfrak{X}^{\text{an}}$ which reduces to the generic point of Y .

2.6. A \mathbb{K}° -algebra A is called *admissible* if it is isomorphic to $\mathbb{K}^\circ\langle x_1, \dots, x_n \rangle / I$ for an ideal I and if A has no \mathbb{K}° -torsion. An *admissible formal scheme* \mathfrak{X} over \mathbb{K}° is a formal scheme which has a locally finite atlas of open subsets isomorphic to $\text{Spf}(A)$ for admissible \mathbb{K}° -algebras A . The lack of \mathbb{K}° -torsion is equivalent to flatness over \mathbb{K}° .

These spaces are studied in detail by Bosch and Lütkebohmert [BL3, BL4] based on results of Raynaud. Note that the locally finiteness condition for the atlas in the definitions of formal analytic varieties and admissible formal schemes does not occur in [Bo] and the above quotes. We need it only to define the generic fibre as an analytic space and it could be omitted working with rigid analytic spaces (see [Gu2]).

The *special fibre* $\tilde{\mathfrak{X}}$ of an admissible formal scheme \mathfrak{X} over \mathbb{K}° is a scheme of locally finite type over $\tilde{\mathbb{K}}$ with the same underlying topological space as \mathfrak{X} and with $\mathcal{O}_{\tilde{\mathfrak{X}}} := \mathcal{O}_{\mathfrak{X}} \otimes_{\mathbb{K}^\circ} \tilde{\mathbb{K}}$. Note that the special fibre $\tilde{\mathfrak{X}}$ is not necessarily reduced. This is the main difference between admissible formal schemes and formal analytic varieties. They are closely related by the following functors:

There is a formal analytic variety $\mathfrak{X}^{\text{f-an}}$ associated to \mathfrak{X} . If $\mathfrak{X} = \text{Spf}(A)$, then $\mathcal{A} := A \otimes_{\mathbb{K}^\circ} \mathbb{K}$ is a \mathbb{K} -affinoid algebra and we set $\mathfrak{X}^{\text{f-an}} := \text{Spf}(\mathcal{A})$. In general, $\mathfrak{X}^{\text{f-an}}$ is obtained by a gluing process. The canonical morphism $(\mathfrak{X}^{\text{f-an}})^\sim \rightarrow \tilde{\mathfrak{X}}$ is finite and surjective (see [BL1, §1]).

The analytic space $\mathfrak{X}^{\text{an}} := (\mathfrak{X}^{\text{f-an}})^{\text{an}}$ is called the *generic fibre* of \mathfrak{X} . Similarly as in 2.5, there is a surjective reduction map $\mathfrak{X}^{\text{an}} \rightarrow \tilde{\mathfrak{X}}$.

If \mathfrak{X} is a formal analytic variety over \mathbb{K} , we may reverse the above process replacing locally $\text{Spf}(\mathcal{A})$ by $\text{Spf}(\mathcal{A}^\circ)$ to get a formal scheme $\mathfrak{X}^{\text{f-sch}}$ over \mathbb{K}° .

The functors $\mathfrak{X} \rightarrow \mathfrak{X}^{\text{f-an}}$ and $\mathfrak{X} \rightarrow \mathfrak{X}^{\text{f-sch}}$ give an equivalence between the category of admissible formal schemes over \mathbb{K}° with reduced special fibre and the category of reduced formal analytic varieties over \mathbb{K} . The reductions are the same, which allows us to flip from one category to the other. For details, see [BL1, §1], and [Gu1, §1].

2.7. Let X be a scheme of finite type over a subfield K of \mathbb{K} . The analytic space X^{an} over \mathbb{K} associated to X is constructed in the following way: By using a gluing process, we may assume that X is a closed subscheme of \mathbb{A}_K^n . For $r \in |\mathbb{K}^\times|$, the intersection of X^{an} with the closed ball of radius r and center $\mathbf{0}$ is defined by the same set of equations as X in \mathbb{A}_K^n . If we glue the balls for $r \rightarrow \infty$, then we get X^{an} . For more details about this functorial construction and the following GAGA theorems, we refer to [Ber1, 3.4].

X is reduced, normal, regular, smooth, d -dimensional or connected if and only if X^{an} has the same property. X is separated, resp. proper over K if and only if X^{an} is Hausdorff, resp. compact. A morphism φ of schemes

of finite type over K is flat, unramified, étale, smooth, an open immersion, a closed immersion, dominant, proper, finite if and only if this holds for φ^{an} .

Let \mathcal{X} be a flat scheme of finite type over K° with generic fibre X and let $\pi \in \mathbb{K}^\circ$. Then the associated formal scheme $\hat{\mathcal{X}}$ over \mathbb{K}° , defined locally by replacing the coordinate ring A by the π -adic completion of $A \otimes_{K^\circ} \mathbb{K}^\circ$, is admissible. Moreover, the special fibre of $\hat{\mathcal{X}}$ is isomorphic to the base change of the special fibre of \mathcal{X} to $\tilde{\mathbb{K}}$. Note that $\hat{\mathcal{X}}^{\text{an}}$ is an analytic subdomain of X^{an} with $\hat{\mathcal{X}}^{\text{an}}(\mathbb{K})$ consisting of the \mathbb{K}° -integral points of \mathcal{X} . If \mathcal{X} is proper over K° , then $\hat{\mathcal{X}}^{\text{an}} = X^{\text{an}}$. For details, we refer to [Gu2, §6]. If \mathbb{K}° is not a discrete valuation ring, one has to use [UI].

2.8. In the following, we consider an étale morphism $\varphi : \mathcal{Y} \rightarrow \mathcal{X}$ of admissible formal schemes over \mathbb{K}° , i.e. the reduction

$$\varphi_\lambda : (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}/\lambda\mathcal{O}_{\mathcal{Y}}) \longrightarrow (\mathcal{X}, \mathcal{O}_{\mathcal{X}}/\lambda\mathcal{O}_{\mathcal{X}})$$

is an étale morphism of schemes for all $\lambda \in \mathbb{K}^\circ$. Let X, Y be the generic fibres of \mathcal{X} and \mathcal{Y} .

For $\tilde{P} \in \tilde{\mathcal{X}}(\tilde{\mathbb{K}})$, the *formal fibre* $X_+(\tilde{P}) := \{x \in X \mid \tilde{x} = \tilde{P}\}$ is an open analytic subspace of X . Indeed, let $\text{Spf}(A)$ be a formal affine neighbourhood of \tilde{P} in \mathcal{X} and let $f_1, \dots, f_r \in A$ such that \tilde{P} is the only common zero of $\tilde{f}_1, \dots, \tilde{f}_r \in \tilde{A}$ in $\text{Spec}(\tilde{A})$, then

$$X_+(\tilde{P}) = \{x \in \mathcal{M}(A \otimes_{\mathbb{K}^\circ} \mathbb{K}) \mid |f_1(x)| < 1, \dots, |f_r(x)| < 1\}$$

is an open subdomain of X .

The following result is a special case of [Ber4, Lemma 4.4]. We give an elementary proof here based on the implicit function theorem.

Proposition 2.9. *Let φ be as above and let $\tilde{Q} \in \tilde{\mathcal{Y}}(\tilde{\mathbb{K}})$ with $\tilde{P} = \tilde{\varphi}(\tilde{Q})$. Then φ restricts to an isomorphism $Y_+(\tilde{Q}) \xrightarrow{\sim} X_+(\tilde{P})$ of formal fibres.*

Proof. By the local description of étale morphisms of schemes, we may assume that $\mathcal{X} = \text{Spf}(A)$ and $\mathcal{Y} = \text{Spf}(B)$, where

$$B = (A[t]/\langle p(t) \rangle)_{\langle q(t) \rangle}.$$

Here, $p(t), q(t) \in A[t]$ and the monic polynomial $p(t)$ has the property that the residue class of $\frac{d}{dt}p$ is invertible in B (see [Ber3, §2]). Note that the admissible \mathbb{K}° -algebra A has the form $\mathbb{K}^\circ\langle x_1, \dots, x_n \rangle/I$. There is $Q \in Y_+(\tilde{Q})(\mathbb{K})$ with reduction \tilde{Q} and hence $P := \varphi(Q)$ has reduction \tilde{P} (see [BGR, Theorem 7.1.5/4]).

There is $p(\mathbf{x}, t) \in \mathbb{K}^\circ\langle \mathbf{x} \rangle[t]$ with residue class $p(t) \in A[t]$. Clearly, $p(\mathbf{x}, t)$ has Gauss norm 1 and $|\frac{\partial}{\partial t}p(Q)| = 1$. By the local Eisenstein theorem [BoGu, Theorem 11.5.14], there is a unique formal power series $\xi(\mathbf{x}) \in \mathbb{K}[[\mathbf{x}]]$ with $p(\mathbf{x}, \xi(\mathbf{x})) = 0$ and $(P, \xi(P)) = Q$. Moreover, ξ is convergent on $X_+(\tilde{P})$. By Hensel's lemma, we easily deduce that $\mathbf{x} \mapsto (\mathbf{x}, \xi(\mathbf{x}))$ is the inverse of $\varphi : Y_+(\tilde{Q}) \rightarrow X_+(\tilde{P})$. \square

2.10. An admissible formal scheme \mathcal{X} over \mathbb{K}° is called *strictly semistable* if \mathcal{X} is covered by formal open subsets \mathcal{U} with an étale morphism

$$\psi : \mathcal{U} \longrightarrow \mathcal{S} := \mathrm{Spf}(\mathbb{K}^\circ \langle x_0, \dots, x_n \rangle / \langle x_0 \dots x_r = \pi \rangle)$$

for some $r \leq n$ and $\pi \in \mathbb{K}^{\circ\circ}$ (depending on \mathcal{U}).

Proposition 2.11. *Let \mathcal{X} be a strictly semistable formal scheme over \mathbb{K}° .*

- (a) *The special fibre $\tilde{\mathcal{X}}$ is reduced and hence \mathcal{X} is the formal scheme associated to a formal analytic variety.*
- (b) *For every $\tilde{P} \in \tilde{\mathcal{X}}(\tilde{\mathbb{K}})$, there is a formal open neighbourhood $\mathcal{U} = \mathrm{Spf}(B)$ in \mathcal{X} and an étale morphism ψ as in 2.10 such that $\tilde{\psi}(\tilde{P}) = \tilde{0}$ and such that every irreducible component of $\tilde{\mathcal{U}}$ passes through \tilde{P} .*
- (c) *Let \mathcal{U} as in (b) and let $\gamma_j := \psi^*(x_j)$. The irreducible components of $\tilde{\mathcal{U}}$ are equal to Y_0, \dots, Y_r , where Y_j is the zero-scheme of $\tilde{\gamma}_j$. Moreover, the group of units of the \mathbb{K} -affinoid algebra $B \otimes_{\mathbb{K}^\circ} \mathbb{K}$ is the direct product of $\mathbb{K}^\times B^\times$ and the free abelian subgroup with basis $\gamma_1, \dots, \gamma_r$.*

Proof. Property (b) is immediate from strict semistability. For (a) and (c), we may assume $\mathcal{U} = \mathcal{X}$. The special fibre $\tilde{\mathcal{S}}$ is the zero-scheme of $\tilde{x}_0 \dots \tilde{x}_r$ in $\mathbb{A}_{\tilde{\mathbb{K}}}^{n+1}$, hence it is reduced and has $r + 1$ irreducible components $\tilde{x}_j = \tilde{0}$. Since $\tilde{\psi}$ is étale, $\tilde{\mathcal{X}}$ is reduced [EGA IV, Proposition 17.5.7] and (a) follows from 2.6. Moreover, $\tilde{\psi}^{-1}\{\tilde{x}_j = \tilde{0}\}$ is smooth and every irreducible component passes through \tilde{P} , hence Y_j is irreducible. By flatness of $\tilde{\psi}$, every irreducible component of $\tilde{\mathcal{X}}$ has this form and we get the first part of (c).

We denote the \mathbb{K} -affinoid algebra $B \otimes_{\mathbb{K}^\circ} \mathbb{K}$ by \mathcal{B} . Let $u \in \mathcal{B}^\times$. By Proposition 2.9, the formal fibre $X_+(\tilde{P})$ is isomorphic to the open subdomain of \mathbb{B}^n , given by $|x_j| < 1$ for $j = 1, \dots, n$ and $|x_1 \dots x_r| > |\pi|$. Hence u has the following power series development on $X_+(\tilde{P})$:

$$u|_{X_+(\tilde{P})} = \sum_{m_1=-\infty}^{\infty} \dots \sum_{m_r=-\infty}^{\infty} \sum_{m_{r+1}=0}^{\infty} \dots \sum_{m_n=0}^{\infty} a_{\mathbf{m}} \gamma_1^{m_1} \dots \gamma_n^{m_n}.$$

Since u is a unit, there is a dominant term $a_{\mathbf{m}} \gamma_1^{m_1} \dots \gamma_n^{m_n}$ with $m_{r+1} = \dots = m_n = 0$, i.e.

$$|a_{\mathbf{m}} \gamma_1^{m_1} \dots \gamma_n^{m_n}| > |a_{\mathbf{s}} \gamma_1^{s_1} \dots \gamma_n^{s_n}|$$

on $X_+(\tilde{P})$ for all $\mathbf{s} \neq \mathbf{m}$ (use [BGR, Lemma 9.7.1/1]). For $j = 0, \dots, r$, we will compute the multiplicity $m(u, Y_j)$ of u in Y_j (see [Gu2] for the theory of divisors on admissible formal schemes). By compactness of X , we easily construct a sequence $P_k \in X_+(\tilde{P})$, convergent to $P_\infty \in X$ and with $|\gamma_i(P_k)| \rightarrow 1$ for $i \in \{0, \dots, n\} \setminus \{j\}$. By [Gu3, Proposition 7.6], $m(u, Y_j) = v(u(P_\infty))$ and hence

$$m(u, Y_j) = \lim_{k \rightarrow \infty} v(u(P_k)) = v(a_{\mathbf{m}}) + m_j v(\pi).$$

We conclude that the Weil divisor of $u' := ua_{\mathfrak{m}}^{-1}\gamma_0^{-m_0} \cdots \gamma_r^{-m_r}$ is zero on X and hence $|u'(\xi_{\gamma_j})| = 1$. As this holds for all $j = 1, \dots, r$ and since $\xi_{\gamma_0}, \dots, \xi_{\gamma_r}$ is the Shilov boundary of X (see [Ber1, Proposition 2.4.4]), we conclude that u' is a unit in \mathcal{B}° . By (a) and 2.6, we have $B = \mathcal{B}^\circ$ and hence

$$\mathcal{B}^\times = \mathbb{K}^\times B^\times \gamma_1^{\mathbb{Z}} \cdots \gamma_r^{\mathbb{Z}}.$$

Restriction to the formal fibre $X_+(\tilde{P})$ shows that $\gamma_1, \dots, \gamma_r$ are multiplicatively independent. Moreover, the restriction of an element from $\mathbb{K}^\times B^\times$ to $X_+(\tilde{P})$ has constant dominant term and hence the product of $\mathbb{K}^\times B^\times$ with $\gamma_1^{\mathbb{Z}} \cdots \gamma_r^{\mathbb{Z}}$ is direct. □

3. Local heights of subvarieties

In this section, we summarize the theory of local heights of subvarieties. We use the formal and analytic geometry from the previous section. This allows larger flexibility in choosing models and metrics from which we benefit in Sect. 8. At the end, we generalize Chambert-Loir’s measures associated to metrized line bundles using a different approach through local heights. Apart from the definitions in 3.1 and 3.2, this section will be used only in Sects. 8 and 9. We consider a proper scheme X over \mathbb{K} . Note that X^{an} is compact.

3.1. A *formal \mathbb{K}° -model* of X is an admissible formal scheme with generic fibre X^{an} .

On analytic spaces, formal analytic varieties and admissible formal schemes, we may define line bundles, sections and Cartier divisors in the usual way. A *horizontal cycle* on a formal \mathbb{K}° -model \mathcal{X} of X is just a cycle on X . A *vertical cycle* on \mathcal{X} is a cycle on the special fibre $\tilde{\mathcal{X}}$ with coefficients in the value group Γ . A *cycle on \mathcal{X}* is the formal sum of a horizontal and a vertical cycle.

3.2. Let L be a line bundle on X . It induces a line bundle L^{an} on X^{an} . A *formal \mathbb{K}° -model* of L is a line bundle \mathcal{L} on a formal \mathbb{K}° -model \mathcal{X} of X with generic fibre \mathcal{L}^{an} equal to L^{an} .

A metric $\|\cdot\|$ on L^{an} is said to be a *formal metric* if there is a formal \mathbb{K}° -model \mathcal{L} of L such that for every formal trivialization \mathcal{U} of \mathcal{L} and every $s \in \Gamma(\mathcal{U}, \mathcal{L})$ corresponding to $\gamma \in \mathcal{O}_{\mathcal{X}}(\mathcal{U})$, we have $\|s(x)\| = |\gamma(x)|$ on \mathcal{U}^{an} . The formal metric is called *semipositive* if the reduction $\tilde{\mathcal{L}}$ of \mathcal{L} on $\tilde{\mathcal{X}}$ is numerically effective (see [Kl]). Every line bundle on X has a formal metric [Gu2, Corollary 7.7].

A metric on L^{an} is called a *root of a formal metric* if some positive tensor power is a formal metric. On the space of continuous metrics on L^{an} , we use the distance function

$$d(\|\cdot\|, \|\cdot\|') := \sup_{x \in X^{\text{an}}} |\log(\|\cdot\|/\|\cdot\|')(x)|$$

where $(\|\cdot\|/\|\cdot\|')(x)$ is evaluated at the section 1 of $O_X^{\text{an}} = L^{\text{an}} \otimes (L^{\text{an}})^{-1}$.

Proposition 3.3. *The roots of formal metrics are dense in the space of continuous metrics on L^{an} . In particular, the set of roots of formal metrics on O_X^{an} is embedded onto a dense subset of $C(X^{\text{an}})$ by the map $\|\cdot\| \mapsto -\log \|\cdot\|$.*

Proof. This is Theorem 7.12 in [Gu2] holding more generally for compact analytic spaces. □

3.4. A metrized pseudo-divisor \hat{D} on X is a quadruple $\hat{D} := (L, \|\cdot\|, Y, s)$ where L is a line bundle on X , $\|\cdot\|$ is a metric on L^{an} , Y is a closed subset of X and s is a nowhere vanishing section of L on $X \setminus Y$. Then $D := (L, Y, s)$ is a pseudo-divisor on X (as in [Fu1, 2.2]). The support Y is denoted by $\text{supp}(D)$ and $O(D) := L$. The most relevant example for applications is the case of an invertible meromorphic section s of a metrized line bundle $(L, \|\cdot\|)$, where the associated pseudo-divisor $\widehat{\text{div}}(s)$ is defined by choosing Y as the support of the Cartier divisor $\text{div}(s)$. Since pseudo-divisors are closed under pull-back, it is much easier to formulate the intersection theory for pseudo-divisors instead of Cartier-divisors.

On a formal \mathbb{K}° -model \mathcal{X} of X , there is a refined intersection theory of formally metrized pseudo-divisors with cycles. It has the properties expected from algebraic geometry (see [Gu2] and [Gu3]).

For a t -dimensional cycle Z on X and formally metrized pseudo-divisors $\hat{D}_0, \dots, \hat{D}_t$ with

$$(1) \quad \text{supp}(D_0) \cap \dots \cap \text{supp}(D_t) \cap \text{supp}(Z) = \emptyset,$$

there is a local height $\lambda(Z) := \lambda_{\hat{D}_0, \dots, \hat{D}_t}(Z)$ defined as the intersection number of $\hat{D}_0, \dots, \hat{D}_t$ and Z on a joint formal \mathbb{K}° -model. In case of a discrete valuation and algebraic \mathbb{K}° -models, this is the usual intersection product and hence we get the local heights used in Arakelov geometry.

Theorem 3.5. *Let $\lambda(Z) := \lambda_{\hat{D}_0, \dots, \hat{D}_t}(Z)$ be the local height of a t -dimensional cycle Z on X with respect to the formally metrized pseudo-divisors $\hat{D}_0, \dots, \hat{D}_t$ satisfying (1).*

- (a) $\lambda(Z)$ is multilinear and symmetric in the variables $\hat{D}_0, \dots, \hat{D}_t$, and linear in Z .
- (b) For a proper morphism $\varphi : X' \rightarrow X$ and a t -dimensional cycle Z' on X' , we have

$$\lambda_{\varphi^* \hat{D}_0, \dots, \varphi^* \hat{D}_t}(Z') = \lambda_{\hat{D}_0, \dots, \hat{D}_t}(\varphi_* Z').$$

- (c) If Y is a representative of $D_1 \dots D_t, Z \in CH_0(|D_1| \cap \dots \cap |D_t| \cap |Z|)$ and \hat{D}_0 is the pseudo-divisor of a rational function f on X endowed with the trivial metric, then

$$\lambda(Z) = \log |f(Y)|$$

where the right hand side is defined by linearity with respect to points.

(d) Let $\lambda'(Z)$ be the local height of Z obtained by replacing the metric $\|\cdot\|$ of \hat{D}_0 by another formal metric $\|\cdot\|'$ on $O(D_0)$. If the formal metrics of $\hat{D}_1, \dots, \hat{D}_t$ are semipositive and if Z is effective, then

$$|\lambda(Z) - \lambda'(Z)| \leq d(\|\cdot\|, \|\cdot\|') \deg_{O(D_1), \dots, O(D_t)}(Z).$$

Proof. This is proved in [Gu2, §9], in case of Cartier divisors. Using the refined intersection theory for formally metrized pseudo-divisors from [Gu3, §5], this can be proved similarly and is included in [Gu3, Theorem 10.6]. \square

3.6. Formal metrics are closed under tensor product and pull-back. However, canonical metrics of ample symmetric line bundles on an abelian variety are not formal. Hence it is desirable to extend the local heights to a larger class $\hat{\mathfrak{g}}_X$ of metrics keeping these properties and including the canonical metrics.

The tensor product induces a group law on the isometry classes of metrized line bundles on X^{an} which we denote additively by $+$. Let \mathfrak{g}_X be the group of isometry classes of formally metrized line bundles on X and let \mathfrak{g}_X^+ be the submonoid of classes with semipositive metrics.

The *completion* $\hat{\mathfrak{g}}_X^+$ of \mathfrak{g}_X^+ is the set of isometry classes of line bundles $(L, \|\cdot\|)$ on X satisfying the following property: For all $n \in \mathbb{N}$, there is a proper surjective morphism $\varphi_n : X'_n \rightarrow X$ and a root of a formal metric $\|\cdot\|_n$ on $\varphi_n^*(L^{\text{an}})$ such that $d_{X_n}(\|\cdot\|_n, \varphi_n^*\|\cdot\|) \rightarrow 0$. Moreover, $\hat{\mathfrak{g}}_X := \hat{\mathfrak{g}}_X^+ - \hat{\mathfrak{g}}_X^+$ is called the *completion* of \mathfrak{g}_X .

By the GAGA principle [Ul, Theorem 6.8], every formal metric on a projective scheme over \mathbb{K} is induced by an algebraic \mathbb{K}° -model and hence is a quotient of two very ample metrics. By Chow’s lemma, we conclude that $\mathfrak{g}_X \subset \hat{\mathfrak{g}}_X$ (see [Gu3, Proposition 10.5] for details).

Now it is a formal argument to extend the local heights uniquely to $\hat{\mathfrak{g}}_X$ -pseudo-divisors such that Theorem 3.5 still holds with $\hat{\mathfrak{g}}_X^+, \hat{\mathfrak{g}}_X$ replacing \mathfrak{g}_X^+ and \mathfrak{g}_X (see [Gu2, §1]).

It would be easier if we could just work with uniform limits of roots of semipositive metrics instead of $\hat{\mathfrak{g}}_X^+$. Indeed, this would lead to a satisfactory theory of local heights on projective schemes (see [Gu4, §2]). For proper schemes over \mathbb{K} , the coverings φ_n used in the definition of $\hat{\mathfrak{g}}_X^+$ are necessary to apply Chow’s lemma, as we have seen above.

3.7. It is not possible to extend local heights to all continuous metrics using Proposition 3.3 because the continuity property (c) in Theorem 3.5 holds only under semipositivity assumptions.

However, we can define the local height $\lambda(Z)$ with respect to a continuously metrized pseudo-divisor \hat{D}_0 and $\hat{\mathfrak{g}}_X$ -pseudo-divisors $\hat{D}_1, \dots, \hat{D}_t$ satisfying (1). Indeed, by linearity, we may assume that $\hat{D}_1, \dots, \hat{D}_t$ have $\hat{\mathfrak{g}}_X^+$ -metrics and that Z is effective. By Proposition 3.3, the metric of \hat{D}_0

is limit of formal metrics $\| \cdot \|_n$ on $O(D_0)$ with corresponding pseudo-divisors $\hat{D}_0^{(n)}$. Then

$$\lambda(Z) := \lim_{n \rightarrow \infty} \lambda_{\hat{D}_0^{(n)}, \hat{D}_1, \dots, \hat{D}_t}(Z)$$

is well-defined by the extension of Theorem 3.5 to \hat{g}_X . Obviously, Theorem 3.5 still holds for these local heights except the symmetry in (a). Then (c) is true also if \hat{D}_j and not \hat{D}_0 is induced by f , but (d) only holds if we replace the metric on $O(D_0)$ by another continuous metric.

We apply this to the case $D_0 = 0$. The generalization of Theorem 3.5(c) shows that the local height $\lambda_{\hat{D}_0, \dots, \hat{D}_t}(Z)$ depends only on \hat{D}_0 and the metrized line bundles $(O(D_j)^{\text{an}}, \| \cdot \|_{j=1, \dots, t})$, but not on the choice of the pseudo-divisors.

3.8. A continuous metric $\| \cdot \|$ on O_X^{an} is given by $\|1\| := e^{-g}$ for a continuous function g on X^{an} . We denote the metric by $\| \cdot \|_g$.

Let $\bar{L}_1, \dots, \bar{L}_d$ be \hat{g}_X -metrized line bundles on the d -dimensional proper scheme X over \mathbb{K} . For $j = 1, \dots, d$, we choose any pseudo-divisor D_j with $O(D_j) = L_j$, e.g. $D_j = (\bar{L}_j, X, 1)$. For a continuous function g on X^{an} , we consider the continuously metrized pseudo-divisor $\hat{O}^g := (O_X, \| \cdot \|_g, \emptyset, 1)$. Then we define

$$\int_{X^{\text{an}}} g c_1(\bar{L}_1) \wedge \dots \wedge c_1(\bar{L}_d) := \lambda_{\hat{O}^g, \hat{D}_1, \dots, \hat{D}_d}(X).$$

By 3.7, this is independent of the choice of D_1, \dots, D_d and the generalization of Theorem 3.5 shows that we get a continuous functional on $C(X^{\text{an}})$. By the Riesz representation theorem [Ru2, Theorem 6.19], $c_1(\bar{L}_1) \wedge \dots \wedge c_1(\bar{L}_d)$ is a regular Borel measure on X^{an} .

These measures were first introduced by Chambert-Loir (see [Ch]) using a slightly different approach and under the additional assumptions that \mathbb{K} contains a countable subfield and that X is projective.

Corollary 3.9. *For \hat{g}_X -metrized line bundles $\bar{L}_1, \dots, \bar{L}_d$ on the d -dimensional proper scheme X over \mathbb{K} , the following properties hold:*

- (a) $c_1(\bar{L}_1) \wedge \dots \wedge c_1(\bar{L}_d)$ is multilinear and symmetric in $\bar{L}_1, \dots, \bar{L}_d$.
- (b) If $\varphi : X' \rightarrow X$ is a morphism of d -dimensional integral proper schemes over \mathbb{K} , then

$$\varphi_* (c_1(\varphi^* \bar{L}_1) \wedge \dots \wedge c_1(\varphi^* \bar{L}_d)) = \deg(\varphi) c_1(\bar{L}_1) \wedge \dots \wedge c_1(\bar{L}_d).$$

- (c) If the metrics of $\bar{L}_1, \dots, \bar{L}_d$ are in \hat{g}_X^+ , then

$$\left| \int_{X^{\text{an}}} g c_1(\bar{L}_1) \wedge \dots \wedge c_1(\bar{L}_d) \right| \leq |g|_{\text{sup}} \deg_{L_1, \dots, L_d}(X)$$

for all $g \in C(X^{\text{an}})$.

Proof. These properties follow immediately from the corresponding properties of the generalization of Theorem 3.5 mentioned in 3.7. \square

Remark 3.10. Let \mathbb{K}' be an algebraically closed extension of \mathbb{K} endowed with a complete absolute value extending $|\cdot|$. Obviously, the local heights are invariant under base change to \mathbb{K}' . If $\pi : X_{\mathbb{K}'} \rightarrow X$ denotes the natural projection, then we deduce

$$\pi_*(c_1(\pi^*\bar{L}_1) \wedge \cdots \wedge c_1(\pi^*\bar{L}_d)) = c_1(\bar{L}_1) \wedge \cdots \wedge c_1(\bar{L}_d).$$

Proposition 3.11. *Let $\bar{L}_1, \dots, \bar{L}_d$ be formally metrized line bundles on the complete variety X over \mathbb{K} of dimension d . Then there is a formal \mathbb{K} -model \mathcal{X} of X with reduced special fibre and for every $j \in \{0, \dots, d\}$ a formal \mathbb{K}° -model \mathcal{L}_j of L_j on \mathcal{X} inducing the metric of \bar{L}_j . For such models, we have always*

$$c_1(\bar{L}_1) \wedge \cdots \wedge c_1(\bar{L}_d) = \sum_Y \deg_{\tilde{\mathcal{L}}_1, \dots, \tilde{\mathcal{L}}_d}(Y) \delta_{\xi_Y},$$

where Y ranges over the irreducible components of $\tilde{\mathcal{X}}$ and δ_{ξ_Y} is the Dirac measure in ξ_Y .

Proof. The existence of such formal \mathbb{K}° -models follows from [Gu2, 8.1]. To show equality of the regular Borel measures, it is enough to show that their integrals agree on the subset of $C(X^{\text{an}})$ induced by formal metrics on O_X^{an} (see Proposition 3.3). If $\|\cdot\|_g$ is such a metric with formal model denoted by $\mathcal{O}(g)$, then the section 1 of O_X^{an} induces a meromorphic section of $\mathcal{O}(g)$. By the very definition of multiplicities (see [Gu2, §3]), the corresponding divisor is vertical and has multiplicity $g(\xi_Y)$ in the irreducible component Y of $\tilde{\mathcal{X}}$. By definition of divisorial intersections on \mathcal{X} [Gu2, §4], this leads to the claim. \square

Proposition 3.12. *Let L_1, \dots, L_d be line bundles on the d -dimensional proper scheme X over \mathbb{K} . Let S_j^+ be the set of $\hat{\mathfrak{g}}_X^+$ -metrics on L_j^{an} endowed with the distance from 3.2. Then we have a continuous map from $S_1^+ \times \cdots \times S_d^+$ to the space of regular Borel measures on X^{an} endowed with the weak topology, given by $(\|\cdot\|_1, \dots, \|\cdot\|_d) \mapsto c_1(\bar{L}_1) \wedge \cdots \wedge c_1(\bar{L}_d)$. Moreover, $c_1(\bar{L}_1) \wedge \cdots \wedge c_1(\bar{L}_d)$ is positive and X^{an} has measure $\deg_{L_1, \dots, L_d}(X)$.*

Proof. Recall that the weak topology on the set of regular Borel measures of X^{an} is the coarsest topology such that the map $\mu \mapsto \int f\mu$ is continuous for every $f \in C(X^{\text{an}})$. By the Riesz representation theorem [Ru2, Theorem 6.19], the dual of the Banach space $C(X^{\text{an}})$ is isometric to the space $M_{\text{reg}}(X^{\text{an}})$ of regular Borel measures on X^{an} endowed with the variation as norm. By a standard fact of functional analysis [Ru1, Theorem 4.3], we deduce that every closed ball in $M_{\text{reg}}(X^{\text{an}})$ is compact in the weak topology.

To prove the proposition, we may assume that X is an irreducible variety. Let us consider $\hat{\mathfrak{g}}_X^+$ -metrized line bundles $\bar{L}_1 = (L_1, \|\cdot\|_1), \dots, (\bar{L}_d, \|\cdot\|_d)$.

First step. For $n \in \mathbb{N}$, let $\varphi_n : X_n \rightarrow X$ be a generically finite surjective morphism of irreducible complete varieties over \mathbb{K} . For $j = 1, \dots, d$, let $\|\cdot\|_{j,n}$ be a $\hat{\mathfrak{g}}_{X_n}^+$ -metric on $\varphi_n^* L_j$ such that $d_{X_n}(\|\cdot\|_{j,n}, \varphi_n^* \|\cdot\|_j) \rightarrow 0$ for $n \rightarrow \infty$. We assume that

$$\{\mu_n := (\varphi_n)_*(c_1(\varphi_n^* L_1, \|\cdot\|_{1,n}) \wedge \dots \wedge c_1(\varphi_n^* L_d, \|\cdot\|_{d,n})) \mid n \in \mathbb{N}\}$$

is bounded in $M_{\text{reg}}(X^{\text{an}})$. Then μ_n converges weakly to $\mu := c_1(\bar{L}_1) \wedge \dots \wedge c_1(\bar{L}_d)$.

The proof of the first step is by contradiction. By passing to a subsequence and using weak compactness of closed balls, we may assume that μ_n converges weakly to a regular Borel measure $\mu_\infty \neq \mu$. By Proposition 3.3, there is a formal metric $\|\cdot\|_g$ on O_X^{an} such that

$$(2) \quad \int g d\mu_\infty \neq \int g d\mu.$$

There is a line bundle L_0 on X with $\hat{\mathfrak{g}}_X^+$ -metrics $\|\cdot\|_\pm$ such that $\|\cdot\|_g = \|\cdot\|_+ / \|\cdot\|_-$. It is easy to construct non-zero meromorphic sections s_j of L_j ($j = 0, \dots, d$) such that $\cap_j \text{supp}(D_j) = \emptyset$ for $D_j := \text{div}(s_j)$. Let $\lambda^\pm(X)$ be the local heights with respect to D_0, \dots, D_d endowed with $\|\cdot\|_\pm, \|\cdot\|_1, \dots, \|\cdot\|_d$ and let $\lambda_n^\pm(X_n)$ be the local heights with respect to $\varphi_n^*(D_0), \dots, \varphi_n^*(D_d)$ endowed with $\varphi_n^* \|\cdot\|_\pm, \|\cdot\|_{1,n}, \dots, \|\cdot\|_{d,n}$. By Theorem 3.5 generalized to $\hat{\mathfrak{g}}_X^+$ (see 3.6), we have

$$\lim_{n \rightarrow \infty} \frac{1}{\text{deg}(\varphi_n)} \lambda_n^\pm(X_n) = \lambda^\pm(X).$$

If we subtract these two formulas, then we get a contradiction to (2). This proves the first step.

If we use Theorem 3.5(c) for a constant f (and for $\hat{\mathfrak{g}}_X^+$ -metrics as in 3.6), then we get $\mu(X^{\text{an}}) = \text{deg}_{L_1, \dots, L_d}(X)$. We claim that μ is positive. Using Proposition 3.11, this holds for roots of formal metrics. The corresponding measures have norm $\text{deg}_{L_1, \dots, L_d}(X)$. In general, the metrics used in the approximation process for $\|\cdot\|_j$ (see the definition of $\hat{\mathfrak{g}}_X^+$ in 3.6) may be chosen as in the first step (see [Gu3, Remark 10.3]). Boundedness of $\{\mu_n \mid n \in \mathbb{N}\}$ follows from the special case of roots of formal metrics. Then the first step yields positivity of μ .

Finally, we have to show that μ is continuous in $(\|\cdot\|_1, \dots, \|\cdot\|_d) \in \hat{\mathfrak{g}}_X^+$. By positivity and $\mu(X^{\text{an}}) = \text{deg}_{L_1, \dots, L_d}(X)$, every such measure has norm $\text{deg}_{L_1, \dots, L_d}(X)$ and continuity follows also from the first step. \square

3.13. Now let (L, ρ) be a rigidified line bundle on the abelian variety A over \mathbb{K} , i.e. $\rho \in L(\mathbb{K}) \setminus \{0\}$. Then there is a *canonical metric* $\|\cdot\|_\rho$

for (L, ρ) which behaves well with respect to tensor product and homomorphic pull-back (see [BoGu, Theorem 9.5.7]). If L is ample and symmetric, then $\| \cdot \|_\rho$ is given by the following variant of Tate’s limit argument [BoGu, proof of Theorem 9.5.4]: The rigidification and the theorem of the cube yield an identification $[m]^*L = L^{\otimes m^2}$ for $m \in \mathbb{Z}$. Then the canonical metric is given by

$$\| \cdot \|_{\text{can}} = \lim_{m \rightarrow \infty} ([m]^* \cdot \| \cdot \|)^{1/m^2},$$

where $\| \cdot \|$ is any continuous metric on L^{an} . In particular, we may choose $\| \cdot \|$ as a root of a semipositive formal metric and hence $\| \cdot \|_\rho$ is a $\hat{\mathfrak{g}}_A^+$ -metric.

Remark 3.14. For odd line bundles, we can’t be sure that we get $\hat{\mathfrak{g}}_A$ -metrics. Using in \mathfrak{g}_X^+ cohomologically semipositive metrics instead of the smaller class of semipositive formal metrics, we get a larger class \mathfrak{g}_X^+ with the same properties. Since canonical metrics of odd line bundles are cohomologically semipositive, the new class $\hat{\mathfrak{g}}_X$ includes all canonical metrics on abelian varieties (see [Gu3, §10]). All results of this section hold for this $\hat{\mathfrak{g}}_X$ as well.

Let X be a smooth complete variety and assume that one L_j is algebraically equivalent to 0 endowed with a canonical metric, i.e. pull-back of a canonical metric from the Picard variety. Then Theorem 3.5, applied to the above metrics, shows that the local height does not depend on the metrics of the other line bundles. In particular, we deduce

$$c_1(\bar{L}_1) \wedge \cdots \wedge c_1(\bar{L}_d) = 0.$$

3.15. Let X be a closed subvariety of A of dimension d and let $\bar{L}_1, \dots, \bar{L}_d$ be canonically metrized line bundles on A . Then $\mu := c_1(\bar{L}_1|_X) \wedge \cdots \wedge c_1(\bar{L}_d|_X)$ is called a *canonical measure* on X . Note that the canonical metric is only determined up to $|\mathbb{K}^\times|$ -multiples by the line bundle. By Theorem 3.5(c), the canonical measure μ does not depend on the choice of the canonical metrics. The same argument as in 3.14 shows that $\mu = 0$ if one line bundle is odd.

4. Polytopal domains in \mathbb{G}_m^n

Our goal is to study the formal properties of certain affinoid domains in \mathbb{G}_m^n associated to polytopes in \mathbb{R}^n . They are related to Mumford’s construction of models of totally degenerate abelian varieties discussed in Sect. 6. For the terminology used from convex geometry, the reader is referred to the Appendix.

Recall that Γ is the value group of the valuation $v := -\log | \cdot |$ on \mathbb{K} . On \mathbb{G}_m^n , we always fix coordinates x_1, \dots, x_n . Then we have a continuous map

$$\text{val} : (\mathbb{G}_m^n)_{\mathbb{K}}^{\text{an}} \longrightarrow \mathbb{R}^n, \quad p \mapsto (-\log p(x_1), \dots, -\log p(x_n)).$$

A large part of the following result is contained in [EKL, 3.1]. We give here a different proof which will be used later.

Proposition 4.1. *Let Δ be a Γ -rational polytope. Then the set of Laurent series*

$$\mathbb{K}\langle U_\Delta \rangle := \left\{ \sum_{\mathbf{m} \in \mathbb{Z}^n} a_{\mathbf{m}} x_1^{m_1} \dots x_n^{m_n} \mid \lim_{|\mathbf{m}| \rightarrow \infty} v(a_{\mathbf{m}}) + \mathbf{m} \cdot \mathbf{u} = \infty \ \forall \mathbf{u} \in \Delta \right\}$$

is the \mathbb{K} -affinoid algebra of the Weierstrass domain $U_\Delta := \text{val}^{-1}(\Delta)$ of $(\mathbb{G}_m^n)_{\mathbb{K}}^{\text{an}}$. It has supremum norm

$$(3) \quad \left| \sum_{\mathbf{m} \in \mathbb{Z}^n} a_{\mathbf{m}} \mathbf{x}^{\mathbf{m}} \right|_{\text{sup}} = \sup_{\mathbf{u} \in \Delta, \mathbf{m} \in \mathbb{Z}^n} |a_{\mathbf{m}}| e^{-\mathbf{m} \cdot \mathbf{u}} = \max_{\mathbf{u} \text{ vertex}, \mathbf{m} \in \mathbb{Z}^n} |a_{\mathbf{m}}| e^{-\mathbf{m} \cdot \mathbf{u}}.$$

Proof. For $\mathbf{u} \in \mathbb{R}^n$, the polyannulus $\text{val}^{-1}(\mathbf{u})$ has multiplicative supremum norm

$$(4) \quad \left| \sum_{\mathbf{m} \in \mathbb{Z}^n} a_{\mathbf{m}} \mathbf{x}^{\mathbf{m}} \right|_{\mathbf{u}} := \max_{\mathbf{m} \in \mathbb{Z}^n} |a_{\mathbf{m}}| e^{-\mathbf{m} \cdot \mathbf{u}}$$

which proves the claim for $\Delta = \{0\}$ (see [BGR, 6.1.4]). In general, Δ is defined by

$$\Delta = \bigcap_{\mathbf{m} \in S} \{ \mathbf{u} \in \mathbb{R}^n \mid \mathbf{m} \cdot \mathbf{u} + v(b_{\mathbf{m}}) \geq 0 \}$$

for a finite $S \subset \mathbb{Z}^n$ and suitable $b_{\mathbf{m}} \in \mathbb{K}^\times$. We conclude that U_Δ is the Weierstrass domain in $(\mathbb{G}_m^n)_{\mathbb{K}}^{\text{an}}$ given by $|b_{\mathbf{m}} \mathbf{x}^{\mathbf{m}}| \leq 1, \mathbf{m} \in S$. By [BGR, Proposition 6.1.4/2], we deduce that every analytic function f on U_Δ has a Laurent series expansion $\sum a_{\mathbf{m}} \mathbf{x}^{\mathbf{m}}$ on U_Δ . If \mathbf{u} is a vertex of Δ , then $\text{val}^{-1}(\mathbf{u})$ is a Weierstrass domain in U_Δ and we get $|f|_{\mathbf{u}} \leq |f|_{\text{sup}}$. Since $\mathbf{u} \cdot \mathbf{m} + v(a_{\mathbf{m}})$ achieves its minimum on Δ always in a vertex \mathbf{u} , we get

$$\sup_{\mathbf{u} \in \Delta} |f|_{\mathbf{u}} \leq |f|_{\text{sup}}.$$

By the ultrametric triangle inequality, we have equality and we deduce easily the claim. □

4.2. A \mathbb{G}_m^n -toric variety over $\tilde{\mathbb{K}}$ is a normal variety Y over $\tilde{\mathbb{K}}$ with an algebraic $(\mathbb{G}_m^n)_{\tilde{\mathbb{K}}}$ -action containing a dense n -dimensional orbit.

The theory of toric varieties will be very important in the sequel, we refer to [KKMS], [Fu2] or [Oda] for details. There are bijective correspondences between

- (a) rational polyhedral cones σ in \mathbb{R}^n which do not contain a linear subspace $\neq \{0\}$,
- (b) finitely generated saturated semigroups S in \mathbb{Z}^n which generate \mathbb{Z}^n as a group,
- (c) affine \mathbb{G}_m^n -toric varieties Y over $\tilde{\mathbb{K}}$ (up to equivariant isomorphisms).

The correspondences are given by $S = \check{\sigma} \cap \mathbb{Z}^n$ and $Y = \text{Spec}(\tilde{\mathbb{K}}[\tilde{\mathbf{x}}^S])$, where $\tilde{\mathbf{x}}^S := \{\tilde{\mathbf{x}}^{\mathbf{m}} \mid \mathbf{m} \in S\}$ for the coordinates $\tilde{\mathbf{x}}$ on $(\mathbb{G}_m^n)_{\tilde{\mathbb{K}}}$.

4.3. Let Δ be still a Γ -rational polytope in \mathbb{R}^n . For $\mathbf{m} \in \mathbb{Z}^n$, there is $b_{\mathbf{m}} \in \mathbb{K}^\times$ with $v(b_{\mathbf{m}}) = -\min_{\mathbf{u} \in \Delta} \mathbf{u} \cdot \mathbf{m}$. Note that $y_{\mathbf{m}} := b_{\mathbf{m}} \mathbf{x}^{\mathbf{m}}$ has supremum norm 1 on U_Δ . We denote by $\pi : U_\Delta \rightarrow \tilde{U}_\Delta$ the reduction map.

The affinoid torus $\mathbb{T}_1^{\text{an}} := \{p \in (\mathbb{G}_m^n)_{\mathbb{K}}^{\text{an}} \mid p(x_j) = 1, j = 1, \dots, n\}$ acts on U_Δ . Passing to reductions, we get a torus action of $(\mathbb{T}_1^{\text{an}})^\sim = (\mathbb{G}_m^n)_{\tilde{\mathbb{K}}}$ on \tilde{U}_Δ .

In the framework of algebraic geometry and for a discrete valuation, the following result is due to Mumford [Mu, §6]. We give here an analytic formulation over \mathbb{K} without assuming that the valuation is discrete.

Proposition 4.4. *The following properties hold for $U_\Delta = \text{val}^{-1}(\Delta)$:*

- (a) *The elements $(y_{\mathbf{m}})_{\mathbf{m} \in \mathbb{Z}^n}$ generate a dense subalgebra of $\mathbb{K}\langle U_\Delta \rangle^\circ$.*
- (b) *There is a bijective order reversing correspondence between torus orbits Z of \tilde{U}_Δ and open faces τ of Δ , given by $Z_\tau = \pi(\text{val}^{-1}(\tau))$ and $\tau_Z = \text{val}(\pi^{-1}(Z))$.*
- (c) *$\dim(\tau) + \dim(Z_\tau) = n$.*
- (d) *If $Y_{\mathbf{u}}$ is the irreducible component of \tilde{U}_Δ corresponding to the vertex \mathbf{u} of Δ by (b), then the natural $(\mathbb{G}_m^n)_{\tilde{\mathbb{K}}}$ -action of \tilde{U}_Δ makes $Y_{\mathbf{u}}$ into an affine toric variety. The corresponding rational polyhedral cone is generated by $\Delta - \mathbf{u}$.*
- (e) *If Δ' is also a Γ -rational polytope with $\Delta' \subset \Delta$, then the canonical morphism $U_{\Delta'} \rightarrow U_\Delta$ induces an open immersion of the reductions if and only if Δ' is a closed face of Δ .*

Proof. Property (a) follows easily from (3). For every $\mathbf{u} \in \Delta$, $|\cdot|_{\mathbf{u}}$ from (4) restricts to a multiplicative norm on $\mathbb{K}\langle U_\Delta \rangle$ which is bounded by the supremum norm. Hence it may be viewed as a point $\xi_{\mathbf{u}} \in U_\Delta$.

The *Shilov boundary* is the unique minimal set Θ of U_Δ such that every $f \in \mathbb{K}\langle U_\Delta \rangle$ has its maximum in Θ . By [Ber1, Proposition 2.4.4], Θ is the set of $\xi_Y \in U_\Delta$ corresponding to the irreducible components Y of \tilde{U}_Δ by 2.5. Using (3), we get $\Theta = \{|\cdot|_{\mathbf{u}} \mid \mathbf{u} \text{ vertex of } \Delta\}$. Note that the vertex \mathbf{u} corresponding to Y is given by $\mathbf{u} = \text{val}(\xi_Y)$. By definition of ξ_Y , we have

$$\tilde{\mathbb{K}}[Y] = \mathbb{K}\langle U_\Delta \rangle^\circ / \{|\cdot|_{\mathbf{u}} < 1\}.$$

To prove (d), it is enough to show that $\tilde{\mathbb{K}}[Y]$ is isomorphic to $\tilde{\mathbb{K}}[\tilde{\mathbf{y}}^{\sigma \cap \mathbb{Z}^n}]$ for the cone σ generated by $\Delta - \mathbf{u}$. By a change of coordinates, we may assume that $\mathbf{u} = \mathbf{0}$. For $S := \{\mathbf{m} \in \mathbb{Z}^n \mid v(b_{\mathbf{m}}) = 0\}$, (a) yields that $\tilde{\mathbb{K}}[Y]$ is generated by $(\tilde{y}_{\mathbf{m}})_{\mathbf{m} \in S}$. By construction, we have $S = \check{\sigma} \cap \mathbb{Z}^n$. It remains to show that a relation

$$\tilde{p}(\tilde{y}_{\mathbf{m}_1}, \dots, \tilde{y}_{\mathbf{m}_r}) = \tilde{0} \in \tilde{\mathbb{K}}[Y]$$

comes from a relation in $\tilde{\mathbb{K}}[\tilde{\mathbf{y}}^S]$. For simplicity, we may assume $b_{\mathbf{m}} = 1$ for all $\mathbf{m} \in S$. Then we have $|p(y_{\mathbf{m}_1}, \dots, y_{\mathbf{m}_r})| < 1$ on U_{Δ} . Replacing $y_{\mathbf{m}}$ by $\mathbf{x}^{\mathbf{m}}$ and collecting terms of the same degree, we get the desired relation in $\tilde{\mathbb{K}}[\tilde{\mathbf{y}}^S]$. This proves (d).

To prove (b), let τ be an open face of Δ . There is $I \subset \mathbb{Z}^n$ such that τ is given by

$$(5) \quad \mathbf{u} \cdot \mathbf{m} + v(b_{\mathbf{m}}) \begin{cases} = 0 & \text{if } \mathbf{m} \in I, \\ > 0 & \text{if } \mathbf{m} \in \mathbb{Z}^n \setminus I. \end{cases}$$

Then $\tilde{x} \in Z_{\tau} := \pi(\text{val}^{-1}(\tau))$ if and only if $\tilde{y}_{\mathbf{m}}(\tilde{x}) \neq \tilde{0}$ for $\mathbf{m} \in I$ and $\tilde{y}_{\mathbf{m}}(\tilde{x}) = \tilde{0}$ for $\mathbf{m} \notin I$.

We prove first that Z_{τ} is a torus orbit. We choose a vertex \mathbf{u} of $\bar{\tau}$ with associated toric variety $Y_{\mathbf{u}}$. By a change of coordinates, we may assume again that $\mathbf{u} = \mathbf{0}$. It follows from the proof of (d) that $Y_{\mathbf{u}}$ is given by the equations $\tilde{y}_{\mathbf{m}} = \tilde{0}$ for $\mathbf{m} \notin S$. Since $I \subset S$, we conclude that $Z_{\tau} \subset Y_{\mathbf{u}}$. Then Z_{τ} is given in $Y_{\mathbf{u}}$ by the equations $\tilde{y}_{\mathbf{m}} = \tilde{0}$ for $\mathbf{m} \in S \setminus I$ and the inequalities $\tilde{y}_{\mathbf{m}} \neq \tilde{0}$ for $\mathbf{m} \in I$ and hence Z_{τ} is a torus orbit (see [Fu2, 3.1]).

Since π is surjective, all torus orbits have this form. The above characterization of Z_{τ} shows that we get a bijective correspondence in (b). Moreover, $\tau \subset \text{val}(\pi^{-1}(Z_{\tau}))$ is also clear and equality follows as the torus orbits (resp. open faces) form a disjoint covering of \tilde{U}_{Δ} (resp. Δ).

Finally, (c) and (e) follow from the theory of toric varieties applied to $Y_{\mathbf{u}}$ for a vertex \mathbf{u} of $\bar{\tau}$ (resp. Δ'). □

Corollary 4.5. *For $\mathbf{u} \in \Delta$, let $\xi_{\mathbf{u}}$ be the point of U_{Δ} defined by (4). Then $\tilde{\xi}_{\mathbf{u}}$ is the generic point ζ of the torus orbit Z_{τ} associated to the unique open face τ containing \mathbf{u} . Moreover, $\xi_{\mathbf{u}}$ is the Shilov boundary of $\text{val}^{-1}(\mathbf{u})$.*

Proof. By Proposition 4.4(b), we have $\tilde{\xi}_{\mathbf{u}} \in Z_{\tau}$. Let $f = \sum a_{\mathbf{m}} \mathbf{x}^{\mathbf{m}} \in \mathbb{K}\langle U_{\Delta} \rangle^{\circ}$ with $\tilde{f}(\tilde{\xi}_{\mathbf{u}}) = \tilde{0}$. Note that (5) determines I and describes the open face τ . Using $f \in \{ |_{\mathbf{u}} < 1 \}$, we have $|a_{\mathbf{m}}| < e^{\mathbf{m} \cdot \mathbf{u}} = |b_{\mathbf{m}}|$ for all $\mathbf{m} \in I$. By the description of Z_{τ} in the proof of Proposition 4.4, \tilde{f} vanishes identically on Z_{τ} and hence $\tilde{\xi}_{\mathbf{u}} = \zeta$. We have also seen that the vertices of Δ correspond to the Shilov boundary of U_{Δ} . Using $\{\mathbf{u}\}$ instead of Δ , we get the last claim. □

4.6. We globalize our considerations. Let \mathcal{C} be a Γ -rational polytopal complex in \mathbb{R}^n . By Proposition 4.4, it is easy to deduce that $(U_{\Delta})_{\Delta \in \mathcal{C}}$ is a formal analytic atlas of $U = \cup_{\Delta \in \mathcal{C}} U_{\Delta}$. The associated admissible formal scheme \mathcal{U} over \mathbb{K}° (see 2.6) has a formal open affine atlas

$$\mathcal{U}_{\Delta} := \text{Spf}(\mathbb{K}\langle U_{\Delta} \rangle^{\circ}), \quad \Delta \in \mathcal{C}.$$

Clearly, the affinoid torus \mathbb{T}_1^{an} acts on U , $\mathbb{T}_1 := \text{Spf}(\mathbb{K}\langle \text{val}^{-1}(\mathbf{0}) \rangle^{\circ})$ acts on \mathcal{U} and $(\mathbb{G}_m^n)_{\mathbb{K}}$ acts on the special fibre $\tilde{\mathcal{U}}$. Proposition 4.4 yields the following result:

Proposition 4.7. *Under the hypothesis above, we have the following properties of \mathcal{U} :*

- (a) *There is a bijective correspondence between torus orbits of $\tilde{\mathcal{U}}$ and the set $\{\text{relint}(\Delta) \mid \Delta \in \mathcal{C}\}$ of open faces.*
- (b) *The irreducible components of $\tilde{\mathcal{U}}$ are in bijective correspondence with vertices of \mathcal{C} .*
- (c) *If $Y_{\mathbf{u}}$ is the irreducible component of $\tilde{\mathcal{U}}$ corresponding to the vertex \mathbf{u} , then $Y_{\mathbf{u}}$ is a toric variety with fan given by the cones σ in \mathbb{R}^n which are generated by $\Delta - \mathbf{u}$ for $\Delta \in \text{star}(\mathbf{u})$.*
- (d) *For $\Delta, \Delta' \in \mathcal{C}$, $\mathcal{U}_{\Delta'}$ is an open subset of \mathcal{U}_{Δ} if and only if Δ' is a closed face of Δ .*

Remark 4.8. Recall from the appendix that $\text{relint}(\Delta)$ denotes the relative interior of Δ . Every $\Delta \in \mathcal{C}$ induces a toric variety Y_{Δ} , given as the closure of the torus orbit associated to $\text{relint}(\Delta)$. Let \mathbb{L}_{Δ} be the linear space in \mathbb{R}^n generated by $\Delta - \mathbf{u}$, $\mathbf{u} \in \Delta$, and let $N_{\Delta} := \mathbb{L}_{\Delta} \cap \mathbb{Z}^n$. Then the subtorus H_{Δ} of $(\mathbb{G}_m^n)_{\tilde{\mathbb{K}}}$, given by $H_{\Delta}(\tilde{\mathbb{K}}) = N_{\Delta} \otimes_{\mathbb{Z}} \tilde{\mathbb{K}}^{\times}$, acts trivially on Y_{Δ} and hence Y_{Δ} is a toric variety with respect to the torus $T_{\Delta} = (\mathbb{G}_m^n)_{\tilde{\mathbb{K}}} / H_{\Delta}$. If we project the cones in \mathbb{R}^n generated by some $\Delta' - \mathbf{u}$, $\Delta' \in \text{star}(\Delta)$, to $\mathbb{R}^n / \mathbb{L}_{\Delta}$, then we get the fan associated to Y_{Δ} . For details, we refer to [Fu2, 3.1].

4.9. Let \mathcal{D} be also a Γ -rational polytopal complex in \mathbb{R}^n which subdivides \mathcal{C} . Then the atlas $(U_{\sigma})_{\sigma \in \mathcal{D}}$ yields a formal analytic structure on $U = \cup_{\Delta \in \mathcal{C}} U_{\Delta}$ which is finer than the formal analytic variety $\mathcal{U}^{\text{f-an}}$ associated to \mathcal{C} . Let \mathcal{U}' be the admissible formal scheme over \mathbb{K}° associated to \mathcal{D} . We get a canonical morphism $\mathcal{U}' \rightarrow \mathcal{U}$ extending the identity.

In the following proposition, we consider an étale morphism $\varphi : \mathcal{X} \rightarrow \mathcal{U}$ of admissible formal schemes over \mathbb{K}° . Obviously, the base change $\varphi' : \mathcal{X}' \rightarrow \mathcal{U}'$ of φ to \mathcal{U}' is also étale. We conclude that every irreducible component Y' of $\tilde{\mathcal{X}}'$ maps dominantly to a unique irreducible component Y of $\tilde{\mathcal{U}}$. Note that Y is a toric variety associated to a vertex \mathbf{u} of \mathcal{D} (see Proposition 4.7). We have the following converse:

Proposition 4.10. *Let Y be the irreducible component of $\tilde{\mathcal{U}}$ associated to a vertex \mathbf{u} in \mathcal{D} and let Z be the torus orbit of $\tilde{\mathcal{U}}$ corresponding to the unique open face $\text{relint}(\Delta)$, $\Delta \in \mathcal{C}$, containing \mathbf{u} (see Proposition 4.7). Then*

$$(6) \quad \sum_{Y'} [\tilde{\mathbb{K}}(Y') : \tilde{\mathbb{K}}(Y)] = \sum_{Z'} [\tilde{\mathbb{K}}(Z') : \tilde{\mathbb{K}}(Z)]$$

where Y' (resp. Z') ranges over all irreducible components of $(\tilde{\varphi}')^{-1}(Y)$ (resp. $\tilde{\varphi}^{-1}(Z)$). If $\dim(\Delta) = n$, then Z is a closed point, the number of such Y' is equal to $\text{card}(\tilde{\varphi}^{-1}(Z)) < \infty$ and $\tilde{\varphi}'$ maps Y' isomorphically onto the toric variety Y . The right hand side in (6) and the cardinality in the second claim depend only on φ and not on the subdivision \mathcal{D} .

Proof. By Corollary 4.5, the generic point $\tilde{\xi}_{\mathbf{u}}$ of Y maps to the generic point ζ of Z with respect to $\tilde{\mathcal{U}}' \rightarrow \tilde{\mathcal{U}}$. Since $\tilde{\varphi}$ is étale, the fibre $\tilde{\varphi}^{-1}(\zeta)$ is the disjoint union of the spectra for the function fields $\tilde{\mathbb{K}}(Z')$ [EGA IV, Corollaire 17.4.2] and hence the fibre is isomorphic to $\text{Spec}(\prod_{Z'} \tilde{\mathbb{K}}(Z'))$. Similarly, $(\tilde{\varphi}')^{-1}(\tilde{\xi}_{\mathbf{u}})$ is isomorphic to $\text{Spec}(\prod_{Y'} \tilde{\mathbb{K}}(Y'))$ and (6) follows from the fact that $(\tilde{\varphi}')^{-1}(\tilde{\xi}_{\mathbf{u}})$ is the base change of $\tilde{\varphi}^{-1}(\zeta)$ to $\tilde{\mathbb{K}}(\tilde{\xi}_{\mathbf{u}}) = \tilde{\mathbb{K}}(Y)$.

If $\dim(\Delta) = n$, then Proposition 4.4(c) yields that Z is a closed point of $\tilde{\mathcal{U}}$. Since $\tilde{\varphi}$ is étale, $\tilde{\varphi}^{-1}(Z)$ is the disjoint open union of its points Z' . Similarly, $(\tilde{\varphi}')^{-1}(Y)$ is the disjoint open union of its components Y' . We conclude that $(\tilde{\varphi}')^{-1}(Y)$ is the base change of $\tilde{\varphi}^{-1}(Z)$ to Y and hence there is exactly one Y' over Z' . Moreover, it is isomorphic to Y . This proves the proposition. \square

5. Tropical analytic geometry

We study the analytic analogue of tropical varieties for the polytopal domains in \mathbb{G}_m^n considered in the previous section. We generalize the basic results of Einsiedler, Kapranov and Lind [EKL] to analytic subvarieties of such a domain. In this analytic setting, the Bieri–Groves set from [EKL] (see also [BiGr]) may be strictly larger than the tropical variety and we have to use different methods from analytic and formal geometry.

5.1. Let Δ be a Γ -rational polytope in \mathbb{R}^n and let X be a closed analytic subvariety of U_Δ . By continuity of val , we conclude that $\text{val}(X)$ is a compact subset of \mathbb{R}^n . Note that X is given by an ideal I of $\mathbb{K}\langle U_\Delta \rangle$ and it is endowed with the structure of an analytic space. If X is connected, then $\text{val}(X)$ is connected.

For a closed subscheme X of \mathbb{G}_m^n defined over \mathbb{K} , the closure of $\text{val}(X(\mathbb{K}))$ is equal to $\text{val}(X^{\text{an}})$. This is clear by density, hence $\text{val}(X^{\text{an}})$ is equal to the usual definition of a tropical variety.

Proposition 5.2. *Let X be a closed analytic subvariety of U_Δ . Then $\text{val}(X)$ is a Γ -rational polytopal set in Δ .*

This is a fundamental result and follows from [Ber5, Corollary 6.2.2]. Nonetheless, we give a proof under the assumption that X has a semistable alteration. The proof in the special case is very instructive and will be used later.

Proof. In fact, we assume only that there is a dominant morphism $f : X' \rightarrow X$, where X' is the generic fibre of a quasicompact strictly semistable formal scheme \mathcal{X}' over \mathbb{K}° . This assumption is satisfied in all cases relevant for our applications. Indeed, let \mathbb{K} be the completion of the algebraic closure of a field K with $v|_K$ a complete discrete valuation. If X is a smooth compact analytic space, then Hartl proved the existence of such an f which is even étale and surjective [Ha, Corollary 1.5]. Together with de Jong’s alteration

theorem [dJ, Theorem 4.1], we conclude that such an f exists if X is an analytic subdomain of an algebraic variety.

By Proposition 2.11, \mathcal{X}' is covered by finitely many formal open affine subsets \mathcal{U}' allowing an étale morphism

$$\psi : \mathcal{U}' \longrightarrow \mathcal{S} := \text{Spf}(\mathbb{K}^\circ \langle x'_0, \dots, x'_m \rangle / \langle x'_0 \cdots x'_r - \pi \rangle)$$

such that every irreducible component of the special fibre $\tilde{\mathcal{U}}'$ passes through a closed point $\tilde{P} \in \tilde{\mathcal{U}}'$ with $\tilde{\psi}(\tilde{P}) = \tilde{\mathbf{0}}$. Moreover, for $j = 1, \dots, n$, $f^*(x_j)$ is a unit on the generic fibre U' of \mathcal{U}' and hence

$$(7) \quad f^*(x_j) = \lambda_j u_j \psi^*(x'_1)^{m_{j1}} \cdots \psi^*(x'_r)^{m_{jr}}$$

for suitable $\lambda_j \in \mathbb{K}^\times$, $u_j \in \mathcal{O}(\mathcal{U}')^\times$ and $\mathbf{m}_j \in \mathbb{Z}^r$. Let $\Sigma(r, \pi) := \{\mathbf{u}' \in \mathbb{R}_+^r \mid u'_1 + \cdots + u'_r \leq v(\pi)\}$ be the “standard simplex” in \mathbb{R}^r and let $f_{\text{aff}}^{(0)} : \mathbb{R}^r \rightarrow \mathbb{R}^n$ be the affine map given by

$$f_{\text{aff}}^{(0)}(\mathbf{u}')_j := \mathbf{m}_j \cdot \mathbf{u}' + v(\lambda_j) \quad (j = 1, \dots, n).$$

By density of $f(X')$ in X , it is enough to prove

$$(8) \quad \text{val}(f(U')) = f_{\text{aff}}^{(0)}(\Sigma(r, \pi)).$$

Since $|u_j| = 1$ on U' , the inclusion “ \subset ” is obvious. Proposition 2.9 yields $U'_+(\tilde{P}) \cong \mathcal{S}_+^{\text{an}}(\tilde{\mathbf{0}})$ for formal fibres. We conclude that

$$\text{val}(f(U'_+(\tilde{P}))) = f_{\text{aff}}^{(0)}(\text{relint}(\Sigma(r, \pi))).$$

The right hand side is dense in $f_{\text{aff}}^{(0)}(\Sigma(r, \pi))$ and we get equality in (8). \square

For the following two results, we assume that Δ is an n -dimensional Γ -rational polytope in \mathbb{R}^n . The interior of Δ is denoted by $\text{int}(\Delta)$.

Proposition 5.3. *For a closed analytic subvariety X of U_Δ , the polytopal set $\text{val}(X)$ is concave in all the points of $\text{int}(\Delta)$.*

Proof. Let \tilde{X} be the reduction of X (see 2.2). By [BGR, Theorem 6.3.4/2], the morphism $\tilde{X} \rightarrow \tilde{U}_\Delta$ is finite. By Proposition 4.4, the reduction of $\text{val}^{-1}(\text{int}(\Delta))$ in \tilde{U}_Δ is just one \mathbb{K} -rational point \tilde{P} (the zero-dimensional torus orbit). The image of $X^\circ := X \cap \text{val}^{-1}(\text{int}(\Delta))$ under the reduction map $X \rightarrow \tilde{X}$ is lying over \tilde{P} with respect to the above finite morphism. We conclude that the reduction of X° consists of finitely many closed points $\tilde{x}_1, \dots, \tilde{x}_r$ in \tilde{X} . Since the inverse image of a closed point with respect to the reduction map is open in X , the topological space X° decomposes into disjoint open and closed subsets V_j lying over \tilde{x}_j .

We have to show that $\text{val}(X)$ is concave in $\mathbf{u}_0 \in \text{val}(X^\circ)$. By Proposition 5.2 and A.1, there is a Γ -rational polytopal decomposition of $\text{val}(X)$.

Let σ be the polytope of this decomposition such that \mathbf{u}_0 is contained in the relative interior of σ . All $\mathbf{u} \in \text{relint}(\sigma)$ have the same local cone $\text{LC}_{\mathbf{u}}(\text{val}(X))$. The points with coordinates in the value group Γ are dense in $\text{relint}(\sigma)$ and hence we may assume that \mathbf{u}_0 is such a point, i.e. there is $\mathbf{z} \in U_{\Delta}$ with coordinates in \mathbb{K} and $\mathbf{u}_0 = \text{val}(\mathbf{z})$. By the coordinate transform $\mathbf{x}' := \mathbf{x}/\mathbf{z}$, we may assume that $\mathbf{u}_0 = \mathbf{0}$.

Note that $\text{LC}_{\mathbf{0}}(\text{val}(X))$ is a finite union of \mathbb{Q} -rational polyhedral cones centered at $\mathbf{0}$. By convex geometry, the convex hull of $\text{LC}_{\mathbf{0}}(\text{val}(X))$ is a finite intersection of half spaces $\{\mathbf{u} \cdot \mathbf{m} \geq 0\}$ with $\mathbf{m} \in \mathbb{Z}^n$. To show concavity in $\mathbf{0}$, we have to prove that the convex hull is a linear subspace. If $\text{LC}_{\mathbf{0}}(\text{val}(X))$ is contained in a half space $\{\mathbf{u} \cdot \mathbf{m} \geq 0\}$ as above, then it is enough to show that $\text{LC}_{\mathbf{0}}(\text{val}(X)) \subset \{\mathbf{u} \cdot \mathbf{m} = 0\}$.

By shrinking Δ , we may assume that $\text{val}(X)$ is contained in $\{\mathbf{u} \cdot \mathbf{m} \geq 0\}$ and that for $j = 1, \dots, r$, there is $v_j \in V_j$ with $\text{val}(v_j) = \mathbf{0}$. Then $|\mathbf{x}^{\mathbf{m}}|$ takes its maximum 1 in v_1, \dots, v_r . But every point in V_j has the same reduction in \tilde{X} as v_j and hence $|\mathbf{x}^{\mathbf{m}}| = 1$ on X° . We conclude that $\text{val}(X^{\circ}) \subset \{\mathbf{u} \cdot \mathbf{m} = 0\}$. This proves $\text{LC}_{\mathbf{0}}(\text{val}(X)) \subset \{\mathbf{u} \cdot \mathbf{m} = 0\}$. \square

Proposition 5.4. *Let X be a closed analytic subvariety of U_{Δ} such that $\text{val}(X) \cap \text{int}(\Delta) \neq \emptyset$. If X is of pure dimension d , then $\text{val}(X) \cap \text{int}(\Delta)$ is also of pure dimension d .*

Proof. We may assume that X is irreducible. Then $\text{val}(X)$ is connected (see 5.1). We have seen in Proposition 5.2 that $\text{val}(X)$ is a Γ -rational polytopal set. Moreover, its proof or [Ber5, Corollary 6.2.2], show that $\dim(\text{val}(X)) \leq d$ holds even without considering interior points of Δ . By subdivision of Δ , it is enough to prove that $\text{val}(X)$ is at least d -dimensional.

We proceed by induction on $N := \dim(\text{val}(X))$. If $N = 0$, then $\text{val}(X)$ is an interior point of Δ . As in the proof of Proposition 5.3, we conclude that \tilde{X} is finite and hence $\dim(X) = 0$.

Now assume $N > 0$. By shrinking Δ , we may assume that $\text{val}(X)$ is of pure dimension. By density of $X(\mathbb{K})$, there is $\mathbf{u} \in \text{int}(\Delta) \cap \text{val}(X(\mathbb{K}))$. By a change of coordinates as in the proof of Proposition 5.3, we may assume that $\mathbf{1} \in X$ and $\mathbf{u} = \mathbf{0}$. There is $\mathbf{m} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ such that the hyperplane $\{\mathbf{u} \cdot \mathbf{m} = 0\}$ intersects $\text{val}(X)$ transversally. The dimension of the closed analytic subvariety $X' := X \cap \{\mathbf{x}^{\mathbf{m}} = 1\}$ is $d - 1$. We have

$$\text{val}(X') \subset \text{val}(X) \cap \{\mathbf{u} \cdot \mathbf{m} = 0\}$$

and hence $\dim(\text{val}(X')) \leq N - 1$. By induction, we get $\dim(\text{val}(X')) \geq d - 1$. We conclude $d \leq N$ proving the claim. \square

5.5. As in 4.6, let \mathcal{U} be the admissible formal scheme over \mathbb{K}° associated to the Γ -rational polytopal complex \mathcal{C} and let $U := \mathcal{U}^{\text{an}}$. For a closed analytic subvariety X of U , the set $\text{val}(X)$ is called the *tropical variety associated to X* . We set $\Pi := \cup_{\Delta \in \mathcal{C}} \Delta$.

Theorem 5.6. *Under the assumptions in 5.5, the following properties hold:*

- (a) $\text{val}(X)$ is a locally finite union of Γ -rational polytopes in \mathbb{R}^n .
- (b) $\text{val}(X) \cap \text{int}(\Pi)$ is totally concave.
- (c) If $\text{val}(X) \cap \text{int}(\Pi)$ is non-empty and if X is of pure dimension d , then $\text{val}(X) \cap \text{int}(\Pi)$ is of pure dimension d .

Proof. Statement (a) is immediate from Proposition 5.2. Let $\mathbf{u} \in \text{int}(\Pi)$. If $\mathbf{u} \in \text{int}(\Delta)$ for some $\Delta \in \mathcal{C}$, then (b) follows locally in \mathbf{u} from Proposition 5.3. If no such Δ is available, then one may easily adjust the polytopes in \mathcal{C} without changing their union Π such that $\mathbf{u} \in \text{int}(\Delta)$ for some $\Delta \in \mathcal{C}$. Similarly, we deduce (c) from Proposition 5.4. \square

Remark 5.7. This is most useful if \mathcal{C} is a polytopal decomposition of \mathbb{R}^n . Then no boundary points occur and (b), (c) hold for $\text{val}(X)$. In particular, Theorem 5.6 holds for a closed algebraic subvariety X of \mathbb{G}_m^n over \mathbb{K} and hence implies the well known statements from tropical algebraic geometry (see [EKL, §2]). The only thing which does not hold analytically is that $\text{val}(X)$ is a finite union of polyhedrons.

Now we are able to deduce a toric version of Theorem 1.2.

Theorem 5.8. *Let \mathcal{X}' be a quasicompact strictly semistable formal scheme over \mathbb{K}° with generic fibre X' and let X be a d -dimensional closed analytic subvariety of $(\mathbb{G}_m^n)_{\mathbb{K}}^{\text{an}}$. If there is a dominant morphism $f : X' \rightarrow X$, then \mathcal{X}' has a point contained in at least $d + 1$ irreducible components.*

Proof. By Theorem 5.6, there is a d -dimensional Γ -rational polytope Δ in $\text{val}(X)$. The quasicompact set $f^{-1}(U_\Delta)$ may be covered by finitely many sets $U' := (\mathcal{U}')^{\text{an}}$ of the same form as in the proof of Proposition 5.2. The same proof shows that Δ is a finite union of simplices $f_{\text{aff}}^{(0)}(\Sigma(r, \pi))$. It follows that $r \geq d$ for at least one \mathcal{U}' proving the claim. \square

5.9. In the remaining part of this section, we consider a Γ -rational polytopal decomposition \mathcal{C} of \mathbb{R}^n . The associated admissible formal scheme over \mathbb{K}° is denoted by \mathcal{U} . Let X be a closed analytic subvariety of $(\mathbb{G}_m^n)_{\mathbb{K}}^{\text{an}} = \mathcal{U}^{\text{an}}$ of pure dimension d . In the following, we relate \mathcal{C} to the closure $\tilde{\mathcal{X}}$ of X in \mathcal{U} . The closure is the \mathbb{K}° -model \mathcal{X} of X locally defined by

$$\mathcal{U}_\Delta \cap \mathcal{X} := \text{Spf}(\mathbb{K}\langle U_\Delta \rangle^\circ / (I_\Delta(X) \cap \mathbb{K}\langle U_\Delta \rangle^\circ)),$$

where $I_\Delta(X)$ is the ideal of vanishing on U_Δ (see [Gu2, Proposition 3.3]). Note that $\tilde{\mathcal{X}}$ is a closed subvariety of $\tilde{\mathcal{U}}$ of pure dimension d .

Lemma 5.10. *Let $\Delta \in \mathcal{C}$ with $\text{codim}(\Delta, \mathbb{R}^n) = d$. We assume that $\Delta \cap \text{val}(X)$ is a non-empty finite subset of $\tau := \text{relint}(\Delta)$. Then the toric variety Y_Δ in $\tilde{\mathcal{U}}_\Delta$ (see Remark 4.8) is an irreducible component of $\tilde{\mathcal{X}}$.*

Proof. There is a Γ -rational half-space $H_+ := \{\mathbf{m} \cdot \mathbf{u} \geq c\}$ containing the finite set $\Delta \cap \text{val}(X)$ such that the boundary hyperplane intersects $\Delta \cap \text{val}(X)$

in a single point \mathbf{u} . Then $\mathbf{x}^{\mathbf{m}}$ achieves its maximum absolute value e^{-c} on $X \cap U_{\Delta}$ in every $x \in X \cap U_{\Delta}$ with $\text{val}(x) = \mathbf{u}$. The Shilov boundary of the affinoid space $X \cap U_{\Delta}$ is given by the points reducing to the generic points of

$$(X \cap U_{\Delta})^{\sim} = ((\mathcal{X} \cap \mathcal{U}_{\Delta})^{\text{f-an}})^{\sim}$$

(see [Ber1, Proposition 2.4.4]). Hence there is an irreducible component Z of $(\mathcal{X}^{\text{f-an}})^{\sim}$ with $\text{val}(\xi_Z) = \mathbf{u}$. Let Y be the image of Z under the canonical finite surjective morphism $\tilde{\tau} : (\mathcal{X}^{\text{f-an}})^{\sim} \rightarrow \tilde{\mathcal{X}}$. Note that Y is d -dimensional and has the generic point $\tilde{\tau}(\xi_Z)$. By Proposition 4.4, $\tilde{\tau}(\xi_Z)$ is also contained in the d -dimensional torus orbit Z_{τ} and hence $Y = \overline{Z}_{\tau}$. \square

Theorem 5.11. *Under the hypothesis of 5.9, we assume that \mathcal{C} is transversal to $\text{val}(X)$ (see A.2). Then there is a bijective correspondence between:*

- (a) *equivalence classes of transversal vertices of $\mathcal{C} \cap \text{val}(X)$ (see A.2),*
- (b) *irreducible components Y of $\tilde{\mathcal{X}}$.*

An equivalence class in (a) is contained in a unique $\Delta \in \mathcal{C}$ of codimension d . The corresponding irreducible component Y is the toric variety Y_{Δ} in $\tilde{\mathcal{U}}$ from Remark 4.8.

Proof. We have seen in Lemma 5.10 that Y_{Δ} is an irreducible component of $\tilde{\mathcal{X}}$. Conversely, let Y be an irreducible component of $\tilde{\mathcal{X}}$. Using the notation from the proof of Lemma 5.10, there is an irreducible component Z of $(\mathcal{X}^{\text{f-an}})^{\sim}$ with $\tilde{\tau}(Z) = Y$.

We claim that $\mathbf{u}_Z := \text{val}(\xi_Z)$ is a transversal vertex of $\mathcal{C} \cap \text{val}(X)$. To see this, let Δ be the unique polytope from \mathcal{C} with $\mathbf{u}_Z \in \tau := \text{relint}(\Delta)$. By definition of a transversal vertex, we have to prove $\text{codim}(\Delta, \mathbb{R}^n) = d$. By Proposition 4.4 and Proposition 4.7, we have $\text{codim}(\Delta, \mathbb{R}^n) = \dim(Z_{\tau})$ and Z_{τ} contains the generic point $\tilde{\tau}(\xi_Z)$ of Y , hence

$$\text{codim}(\Delta, \mathbb{R}^n) \geq \dim(Y) = d.$$

Since $\text{val}(X)$ is of pure dimension d and $\mathbf{u}_Z \in \text{val}(X) \cap \tau$, transversality yields $\text{codim}(\Delta, \mathbb{R}^n) = d$ proving that \mathbf{u}_Z is a transversal vertex of $\mathcal{C} \cap \text{val}(X)$. Moreover, we see that $Y = \overline{Z}_{\tau} = Y_{\Delta}$. This shows that the map $Y \mapsto \Delta$ is independent of the choice of Z . By construction, it is inverse to the map $\Delta \mapsto Y_{\Delta}$ from the beginning. \square

6. Mumford’s construction

We review Mumford’s construction of models \mathcal{A} of a totally degenerate abelian variety A . For a closed subscheme X of A , we study the periodic tropical variety $\text{val}(X)$ using the previous section. If we choose the polytopal decomposition for \mathcal{A} transversally to $\text{val}(X)$, then the irreducible components of the closure of X in \mathcal{A} turn out to be toric varieties.

In this section, A denotes a *totally degenerate abelian variety* over \mathbb{K} , i.e. A^{an} is isomorphic to $(\mathbb{G}_m^n)_{\mathbb{K}}^{\text{an}}/M$, where M is a subgroup of $\mathbb{G}_m^n(\mathbb{K})$ which maps isomorphically onto a complete lattice Λ of \mathbb{R}^n under the map val . Such an M is called a *lattice* of $(\mathbb{G}_m^n)_{\mathbb{K}}^{\text{an}}$.

Let $\mathbb{R}^n \rightarrow \mathbb{R}^n/\Lambda$, $\mathbf{u} \mapsto \overline{\mathbf{u}}$, be the quotient map. Clearly, the map val from Sect. 4 descends to a continuous map $\overline{\text{val}} : A^{\text{an}} \rightarrow \mathbb{R}^n/\Lambda$. First, we translate the notions of convex geometry introduced in the appendix to the torus \mathbb{R}^n/Λ .

6.1. A *polytope* $\overline{\Delta}$ in \mathbb{R}^n/Λ is given by a polytope Δ in \mathbb{R}^n such that Δ maps bijectively onto $\overline{\Delta}$. We say that $\overline{\Delta}$ is Γ -rational if Δ is Γ -rational. A (Γ -rational) *polytopal set* S in \mathbb{R}^n/Λ is a finite union of (Γ -rational) polytopes in \mathbb{R}^n/Λ .

A *polytopal decomposition* of \mathbb{R}^n/Λ is a finite family $\overline{\mathcal{C}}$ of polytopes in \mathbb{R}^n/Λ induced by a Λ -periodic polytopal decomposition \mathcal{C} of \mathbb{R}^n . It is easy to see that \mathbb{R}^n/Λ has a Γ -rational polytopal decomposition. The other notions from the appendix transfer also to the periodic situation.

6.2. Let $\overline{\mathcal{C}}$ be a Γ -rational polytopal decomposition of \mathbb{R}^n/Λ . By 4.6, the covering $(U_{\Delta})_{\Delta \in \mathcal{C}}$ is a formal analytic atlas of $(\mathbb{G}_m^n)_{\mathbb{K}}^{\text{an}}$. We may form the quotient by M leading to a formal analytic variety over \mathbb{K} . The associated formal scheme \mathcal{A} is a \mathbb{K}° -model of $A = (\mathbb{G}_m^n)_{\mathbb{K}}^{\text{an}}/M$ which has a covering by formal open affine sets $\mathcal{U}_{\overline{\Delta}}$ obtained by gluing $\mathcal{U}_{\Delta+\lambda}$ for all $\lambda \in \Lambda$.

The generic fibre of the formal torus \mathbb{T}_1 acts naturally on A and there is a unique extension to an action of \mathbb{T}_1 on \mathcal{A} . We get a torus action of $\tilde{\mathbb{T}}_1 = (\mathbb{G}_m^n)_{\tilde{\mathbb{K}}}$ on the special fibre $\tilde{\mathcal{A}}$. On $\tilde{\mathcal{U}}_{\overline{\Delta}}$, this action agrees with the action on \mathcal{U}_{Δ} defined in 4.3.

Using the Λ -periodic decomposition \mathcal{C} and passing to the quotient, we may transfer the results from Sects. 4 and 5 to A . By Proposition 4.4 and Proposition 4.7, we get:

Proposition 6.3. *For the formal \mathbb{K}° -model \mathcal{A} of A associated to $\overline{\mathcal{C}}$, we have:*

- (a) *There is a bijective order reversing correspondence between torus orbits Z of $\tilde{\mathcal{A}}$ and open faces $\overline{\tau}$ of $\overline{\mathcal{C}}$, given by*

$$\overline{\tau} = \overline{\text{val}}(\pi^{-1}(Z)), \quad Z = \pi(\overline{\text{val}}^{-1}(\overline{\tau})),$$

where $\pi : A \rightarrow \tilde{\mathcal{A}}$ is the reduction map. Moreover, we have $\dim(Z) + \dim(\overline{\tau}) = n$.

- (b) *The irreducible components Y of $\tilde{\mathcal{A}}$ are toric varieties and correspond to the vertices \mathbf{u} of $\overline{\mathcal{C}}$ by $\overline{\mathbf{u}} := \overline{\text{val}}(\xi_Y)$.*

Proposition 6.4. *Let $\overline{\mathcal{C}}$ be a Γ -rational polytopal decomposition of \mathbb{R}^n/Λ and let $m \in \mathbb{Z} \setminus \{0\}$. Then $\frac{1}{m}\overline{\mathcal{C}} := \{\frac{1}{m}\Delta \mid \Delta \in \mathcal{C}\}$ is also a Γ -rational polytopal decomposition of \mathbb{R}^n/Λ . The associated \mathbb{K}° -model \mathcal{A}_m of A has the following properties:*

- (a) The morphism $[m] : A \rightarrow A, x \mapsto mx$, has a unique extension to a morphism $\varphi_m : \mathcal{A}_m \rightarrow \mathcal{A}_1$ of admissible formal schemes over \mathbb{K}° .
- (b) The morphism $\tilde{\varphi}_m$ is finite of degree m^{2n} .
- (c) The behaviour of the reduction $\tilde{\varphi}_m$ with respect to the torus actions is given by

$$\tilde{\varphi}_m(\mathbf{t} \cdot \mathbf{z}) = \mathbf{t}^m \cdot \tilde{\varphi}_m(\mathbf{z}) \quad (\mathbf{z} \in \tilde{\mathcal{A}}_m, \mathbf{t} \in (\mathbb{G}_m^n)_{\tilde{\mathbb{K}}}).$$

- (d) The inverse image of a k -dimensional torus orbit of $\tilde{\mathcal{A}}_1$ with respect to $\tilde{\varphi}_m$ is equal to the disjoint union of m^n k -dimensional torus orbits of $\tilde{\mathcal{A}}_m$.

Proof. Obviously, $\overline{\frac{1}{m}\mathcal{C}}$ is a Γ -rational polytopal decomposition. The extension of $[m]$ is constructed locally by $\mathcal{U}_{\frac{1}{m}\Delta} \rightarrow \mathcal{U}_\Delta, \mathbf{x} \mapsto \mathbf{x}^m$. Uniqueness is clear formal analytically and hence follows from 2.6. This proves (a). By construction, we get immediately (b) and (c). Now (c) implies that the inverse image of a k -dimensional torus orbit O is the disjoint union of k -dimensional torus orbits. By Proposition 6.3, O corresponds to an open face $\bar{\tau}$ of $\bar{\mathcal{C}}$ of dimension $n - k$. Since $\{\bar{\mathbf{u}} \in \mathbb{R}^n / \Lambda \mid \overline{m\mathbf{u}} \in \bar{\tau}\}$ is the disjoint union of m^n open faces, Proposition 6.3 yields (d). □

6.5. We describe line bundles on $A = (\mathbb{G}_m^n)_{\mathbb{K}}^{\text{an}} / M$ similarly as in the complex analytic situation (see [FvdP, Chapt. VI] and [BL2, §2] for details).

Let L be a line bundle on A . The pull-back to $T := (\mathbb{G}_m^n)_{\mathbb{K}}^{\text{an}}$ with respect to the quotient morphism p is trivial and will be identified with $T \times \mathbb{K}$. It is given by a cocycle $\gamma \mapsto Z_\gamma$ of $H^1(M, \mathcal{O}(T)^\times)$ and $L = (T \times \mathbb{K}) / M$ where the quotient is with respect to the M -action

$$M \times (T \times \mathbb{K}) \longrightarrow T \times \mathbb{K}, \quad (\gamma, (\mathbf{x}, \alpha)) \mapsto (\gamma \cdot \mathbf{x}, Z_\gamma(\mathbf{x})^{-1}\alpha).$$

The cocycle has the form $Z_\gamma(\mathbf{x}) = d_\gamma \cdot \sigma_\gamma(\mathbf{x})$, where $\gamma \mapsto \sigma_\gamma$ is a homomorphism of M to the character group \check{T} and where $d_\gamma \in \mathbb{K}^\times$ satisfies

$$(9) \quad d_{\gamma\rho} \cdot d_\gamma^{-1} \cdot d_\rho^{-1} = \sigma_\rho(\gamma) \quad (\gamma, \rho \in M).$$

By the isomorphism $M \xrightarrow{\text{val}} \Lambda$, we get a unique symmetric bilinear form b on Λ characterized by

$$b(\text{val}(\gamma), \text{val}(\rho)) = v(\sigma_\rho(\gamma)).$$

Then b is positive definite on Λ if and only if L is ample. Note that the cocycle Z_γ factors over \mathbb{R}^n , i.e. for every $\lambda = \text{val}(\gamma) \in \Lambda$, there is a unique real function z_λ on \mathbb{R}^n such that

$$z_\lambda(\text{val}(\mathbf{x})) = v(Z_\gamma(\mathbf{x})) \quad (\gamma \in M, \mathbf{x} \in T).$$

The function z_λ is affine with

$$(10) \quad z_\lambda(\mathbf{u}) = z_\lambda(\mathbf{0}) + b(\mathbf{u}, \lambda) \quad (\lambda \in \Lambda, \mathbf{u} \in \mathbb{R}^n).$$

Proposition 6.6. *Let L be a line bundle on A . Repeat that L is given analytically by $L = (T \times \mathbb{K})/M$ and by a cocycle $(Z_\gamma)_{\gamma \in M}$ leading to a family $(z_\lambda)_{\lambda \in \Lambda}$ of real functions as above. Let \mathcal{A} be the formal \mathbb{K}° -model of A associated to a given Γ -rational polytopal decomposition $\bar{\mathcal{C}}$ of \mathbb{R}^n/Λ . Then there is a bijective correspondence between isomorphism classes of formal \mathbb{K}° -models \mathcal{L} of L on \mathcal{A} with trivialization $(\mathcal{U}_{\bar{\Delta}})_{\bar{\Delta} \in \bar{\mathcal{C}}}$ and continuous real functions f on \mathbb{R}^n satisfying the following two conditions:*

- (a) *For every $\Delta \in \mathcal{C}$, there are $\mathbf{m}_\Delta \in \mathbb{Z}^n$ and $c_\Delta \in \Gamma$ with $f(\mathbf{u}) = \mathbf{m}_\Delta \cdot \mathbf{u} + c_\Delta$ on Δ .*
 (b) *$f(\mathbf{u} + \lambda) = f(\mathbf{u}) + z_\lambda(\mathbf{u})$ ($\lambda \in \Lambda, \mathbf{u} \in \mathbb{R}^n$).*

If $\| \cdot \|_{\mathcal{L}}$ denotes the formal metric on L associated to \mathcal{L} (see 3.2), then the correspondence is given by $f_{\mathcal{L}} \circ \text{val} := -\log \circ p^ \| \cdot \|_{\mathcal{L}}$ on T .*

Proof. We start by describing units on polytopal domains.

First step. $g = \sum_{v \in \mathbb{Z}^n} a_v \mathbf{x}^v \in \mathbb{K}\langle \text{val}^{-1}(\Delta) \rangle$ is a unit if and only if there is a $v_0 \in \mathbb{Z}^n$ such that $|a_{v_0} \mathbf{x}^{v_0}| > |a_v \mathbf{x}^v|$ for all $\mathbf{x} \in U_\Delta$ and all $v \neq v_0$.

If g has such a dominant term, then we may assume that $v_0 = \mathbf{0}$ and $a_0 = 1$. Then we have $g := 1 - h$ with $|h|_{\text{sup}} < 1$ and $g^{-1} = \sum_{m=0}^{\infty} h^m \in \mathbb{K}\langle \text{val}^{-1}(\Delta) \rangle$. Conversely, if g has no dominant term, then there is $|a_{v_0} \mathbf{x}^{v_0}| = |a_{v_1} \mathbf{x}^{v_1}| = |g|_{\text{sup}}$ for certain $v_0 \neq v_1$ and $\mathbf{x} \in U_\Delta$. Note that $\text{val}^{-1}(\mathbf{u})$ is isomorphic to the affinoid torus \mathbb{T}_1^{an} for $\mathbf{u} := \text{val}(\mathbf{x})$. Then the restriction of g to \mathbb{T}_1^{an} has no dominant term as well and hence it is not invertible on \mathbb{T}_1^{an} [BGR, Lemma 9.7.1/1]. Since $\text{val}^{-1}(\mathbf{u})$ is an analytic subdomain of U_Δ , g is not an unit of $\mathbb{K}\langle \text{val}^{-1}(\Delta) \rangle$.

Second step. $f_{\mathcal{L}}$ is a continuous function satisfying (a) and (b).

Continuity follows from the continuity of formal metrics on analytic spaces and (b) is by construction (see 6.5). On a trivialization $U_{\bar{\Delta}}$ of \mathcal{L} , the section 1 corresponds to a unit g in $\mathbb{K}\langle U_\Delta \rangle$. By definition of formal metrics, we have $p^* \| \cdot \|_{\mathcal{L}} = |g|$ on U_Δ . By the first step, g has a dominant term $a_\Delta \mathbf{x}^{\mathbf{m}_\Delta}$ leading to (a) with $c_\Delta := v(a_\Delta)$. This proves the second step.

Now the proof is quite easy. A continuous function f on \mathbb{R}^n gives rise to a metric $\| \cdot \|'$ on $p^*(L) = T \times \mathbb{K}$ by $\| \cdot \|' = e^{-f \circ \text{val}}$. If f satisfies (b), then $\| \cdot \|'$ passes to the quotient modulo M , i.e. there is a unique metric $\| \cdot \|_f$ on L with

$$f \circ \text{val} = -\log \circ p^* \| \cdot \|_f.$$

Now we assume that f also satisfies (a). Since Γ is the value group, there is $a_\Delta \in \mathbb{K}^\times$ with $c_\Delta = v(a_\Delta)$. For every $\Delta \in \mathcal{C}$, the unit $(a_\Delta \mathbf{x}^{\mathbf{m}_\Delta})^{-1}$ gives a frame of $p^*(L) = T \times \mathbb{K}$ over U_Δ leading to a trivialization of L over $U_{\bar{\Delta}}$. This trivialization extends to the trivial line bundle over $\mathcal{U}_{\bar{\Delta}}$ and induces a metric $\| \cdot \|_\Delta$ on $L|_{U_{\bar{\Delta}}}$. A priori, $\| \cdot \|_\Delta$ depends on the choice of Δ , but

by construction and (a), we deduce that $\| \cdot \|_{\Delta}$ agrees with $\| \cdot \|_f$ over $U_{\bar{\Delta}}$. Therefore the transition functions $g_{\Delta\Delta'}$ of the above trivializations have constant absolute value 1 on $U_{\bar{\Delta}} \cap U_{\bar{\Delta}'}$. This means that they define a formal \mathbb{K}° -model \mathcal{L}_f of L on \mathcal{A} with associated metric equal to $\| \cdot \|_f$.

We have to prove that $f \mapsto \mathcal{L}_f$ is inverse to $\mathcal{L} \mapsto f_{\mathcal{L}}$. For $\mathcal{M} := \mathcal{L}_f$, we have

$$f_{\mathcal{M}} \circ \text{val} = -\log \circ p^* \|1\|_{\mathcal{M}} = -\log \circ p^* \|1\|_f = f \circ \text{val}$$

and hence $f_{\mathcal{M}} = f$. Conversely, it is clear that $\| \cdot \|_{\mathcal{L}} = \| \cdot \|_{\mathcal{L}_g}$ for given \mathcal{L} and $g := f_{\mathcal{L}}$. By considering trivializations as above or by using [Gu2, Proposition 5.5], we see that the formal \mathbb{K}° -models \mathcal{L} and \mathcal{L}_g of L are isomorphic. \square

Corollary 6.7. *In the notation of Proposition 6.6, let $\tilde{\mathcal{L}}$ be the reduction of \mathcal{L} on $\tilde{\mathcal{A}}$. Then $\tilde{\mathcal{L}}$ is ample if and only if $f_{\mathcal{L}}$ is a strongly polyhedral convex function with respect to \mathcal{C} (see Appendix A.3).*

Proof. Note that $\tilde{\mathcal{L}}$ is ample if and only if its restriction to every irreducible component Y of $\tilde{\mathcal{A}}$ is ample. By Proposition 6.3, Y is a toric variety corresponding to a vertex $\bar{\mathbf{u}}_0$ of $\bar{\mathcal{C}}$. For simplicity of notation, we may assume that $\mathbf{u}_0 = \mathbf{0}$ and $f_{\mathcal{L}}(\mathbf{0}) = 0$. For $\Delta \in \text{star}(\mathbf{u}_0)$, we choose the equation $\tilde{\mathbf{x}}^{\mathbf{m}_{\Delta}}$ on $\tilde{U}_{\bar{\Delta}}$. By Proposition 6.6(a), it is easy to see that we get a Cartier divisor D on Y of a meromorphic section of $\tilde{\mathcal{L}}|_Y$. Let $\psi_D(\mathbf{u})$ be the continuous function on the complete fan of Y centered in $\mathbf{u}_0 = \mathbf{0}$ which is equal to $-\mathbf{m}_{\Delta} \cdot \mathbf{u}$ on $\mathbb{R}\Delta$. By §3.4 of [Fu2], $-\psi_D$ is a strongly polyhedral convex function with respect to the complete fan. Note that $f_{\mathcal{L}} = -\psi_D$ on every $\Delta \in \text{star}(\mathbf{u}_0)$. We deduce easily that $f_{\mathcal{L}}$ is a strongly polyhedral convex function with respect to \mathcal{C} . \square

6.8. In the remaining part of this section, we consider a closed subscheme X of A of pure dimension d . The tropical variety $\overline{\text{val}}(X^{\text{an}})$ is well-defined in \mathbb{R}^n/Λ . The following properties are easily deduced from Theorem 5.6 and the continuity of $\overline{\text{val}}$.

Theorem 6.9. *The tropical variety $\overline{\text{val}}(X^{\text{an}})$ is a totally concave Γ -rational polytopal set in \mathbb{R}^n/Λ of pure dimension d . If X is connected, then $\overline{\text{val}}(X^{\text{an}})$ is also connected.*

Let \mathcal{A} be the formal \mathbb{K}° -model associated to the Γ -rational polytopal decomposition $\bar{\mathcal{C}}$ of \mathbb{R}^n/Λ . We denote by \mathcal{X} the closure of X^{an} in \mathcal{A} (see [Gu2, Proposition 3.3]). From Theorem 5.11, we deduce immediately:

Theorem 6.10. *Under the hypothesis above and assuming that $\bar{\mathcal{C}}$ is transversal to $\overline{\text{val}}(X^{\text{an}})$, we have a bijective correspondence between irreducible components of \mathcal{X} and equivalence classes of transversal vertices of $\bar{\mathcal{C}} \cap \overline{\text{val}}(X^{\text{an}})$. If $\bar{\mathbf{u}}$ is a transversal vertex, then there is a unique $\bar{\Delta} \in \bar{\mathcal{C}}$ with*

$\bar{\mathbf{u}} \in \bar{\Delta}$ and $\text{codim}(\Delta, \mathbb{R}^n) = d$. The corresponding irreducible component of $\tilde{\mathcal{X}}$ is the closure $Y_{\bar{\Delta}}$ of the torus orbit of $\tilde{\mathcal{A}}$ associated to the open face $\text{relint}(\bar{\Delta})$ (see Proposition 6.3). In particular, every irreducible component of $\tilde{\mathcal{X}}$ is a toric variety.

7. Semistable alterations

As in the previous section, we consider a totally degenerate abelian variety $A^{\text{an}} = (\mathbb{G}_m^n)_{\mathbb{K}}^{\text{an}}/M$ over \mathbb{K} with Mumford model \mathcal{A} over \mathbb{K}° associated to the Γ -rational polytopal decomposition $\bar{\mathcal{C}}$ of \mathbb{R}^n/Λ , where $\Lambda := \text{val}(M)$. We fix also an irreducible closed subvariety X of A with closure \mathcal{X} in \mathcal{A} . The goal of this section is to describe the multiplicity of an irreducible component Y of $\tilde{\mathcal{X}}$ using a strictly semistable alteration. Under a non-degeneracy condition for Y , this may be done in terms of convex geometry and will be used in the following section to prove the main result.

7.1. Let \mathcal{X}' be a strictly semistable formal scheme over \mathbb{K}° with generic fibre X' and let $f : X' \rightarrow A^{\text{an}}$ be a morphism over \mathbb{K} . In the first paragraphs, we will show that the polytopal decomposition $\bar{\mathcal{C}}$ endows X' with a canonical formal analytic structure \mathcal{X}' and with a morphism $\varphi : \mathcal{X}' \rightarrow \mathcal{A}^{\text{f-an}}$ of formal analytic varieties.

By Proposition 2.11, \mathcal{X}' is covered by formal open affine subsets \mathcal{U}' such that all irreducible components of $\tilde{\mathcal{U}}'$ pass through $\tilde{P} \in \tilde{\mathcal{U}}'(\tilde{\mathbb{K}})$ and with an étale morphism

$$\psi : \mathcal{U}' \longrightarrow \mathcal{S} := \text{Spf}(\mathbb{K}^\circ \langle x'_0, \dots, x'_d \rangle / \langle x'_0 \dots x'_r = \pi \rangle)$$

for suitable $\pi \in \mathbb{K}^{\circ\circ}$ such that $\tilde{\psi}^{-1}(\tilde{\mathbf{0}}) = \{\tilde{P}\}$. The simplex

$$\Delta(r, \pi) := \{\mathbf{u}' \in \mathbb{R}_+^{r+1} \mid u'_0 + \dots + u'_r = v(\pi)\}$$

is canonically associated to \mathcal{S} . To apply Sect. 5, we represent $\Delta(r, \pi)$ by the standard simplex

$$\Sigma(r, \pi) = \{\mathbf{u}' \in \mathbb{R}_+^r \mid u'_1 + \dots + u'_r \leq v(\pi)\}$$

omitting u'_0 . Then we have

$$\mathcal{S} = \text{Spf}(\mathbb{K}^\circ \langle x'_0, \dots, x'_r \rangle / \langle x'_0 \dots x'_r = \pi \rangle) \times \text{Spf}(\mathbb{K}^\circ \langle x'_{r+1}, \dots, x'_d \rangle).$$

We denote the first factor by \mathcal{S}_1 . By definition, the morphism $\phi : \mathcal{U}' \rightarrow \mathcal{S}_1$, obtained by composition with the first projection, is smooth and we have $\tilde{\phi}(\tilde{P}) = \tilde{\mathbf{0}}$. We have an isomorphism $\mathcal{S}_1 \xrightarrow{\sim} \mathcal{U}_{\Sigma(r, \pi)}$ by omitting x'_0 . By composition, we get a morphism $\phi_0 : \mathcal{U}' \rightarrow \mathcal{U}_{\Sigma(r, \pi)}$. Note that we have maps val on $\mathcal{S}_1^{\text{an}}$ and $\mathcal{U}_{\Sigma(r, \pi)}^{\text{an}}$ with images $\Delta(r, \pi)$ and $\Sigma(r, \pi)$.

7.2. The generic fibre of \mathcal{U}' is denoted by U' . We claim that $f : U' \rightarrow A^{\text{an}}$ has a lift $F : U' \rightarrow (\mathbb{G}_m^n)_{\mathbb{K}}^{\text{an}}$, unique up to M -translation. For a Hausdorff analytic space Y , we consider the cohomology group $H^1(Y, \mathbb{Z})$ of the underlying topological space. Note that it is the same as $H^1(Y_{\text{rig}}, \mathbb{Z})$ for the underlying rigid space Y_{rig} (see [Ber2, 1.6]). By [Ber4, Theorem 5.2], the generic fibre of a strictly semistable formal scheme over \mathbb{K}° is contractible to the skeleton. The skeleton of \mathcal{U}' is $\Delta(r, \pi)$ and therefore $H^1(U', \mathbb{Z}) = 0$. Since $(\mathbb{G}_m^n)_{\mathbb{K}}^{\text{an}}$ is the universal covering space of A^{an} , we get a lift F as desired (see [BL2, Theorem 1.2]).

7.3. By the proof of Proposition 5.2, there is a unique map $F_{\text{aff}} : \Delta(r, \pi) \rightarrow \mathbb{R}^n$ with

$$F_{\text{aff}} \circ \text{val} \circ \phi = \text{val} \circ F$$

on U' . Let f_{aff} be such an F_{aff} (without fixing F), it is determined up to Λ -translation. Then $\bar{f}_{\text{aff}} : \Delta(r, \pi) \rightarrow \mathbb{R}^n / \Lambda$ is uniquely characterized by

$$(11) \quad \bar{f}_{\text{aff}} \circ \text{val} \circ \phi = \overline{\text{val}} \circ f$$

on U' . Note that uniqueness always follows from $\text{val} \circ \phi(U') = \Delta(r, \pi)$. This was also part of the proof of Proposition 5.2, where we have considered the affine map $f_{\text{aff}}^{(0)} : \Sigma(r, \pi) \rightarrow \mathbb{R}^n$ determined by

$$f_{\text{aff}}^{(0)}(u'_1, \dots, u'_r) = f_{\text{aff}}(u'_0, \dots, u'_r)$$

for $\mathbf{u}' \in \Delta(r, \pi)$. By (7), there are $\mathbf{m}_i \in \mathbb{Z}^r$ and $\lambda_i \in \mathbb{K}^\times$ ($i = 1, \dots, n$) such that

$$(12) \quad f_{\text{aff}}^{(0)}(\mathbf{u}') = (\mathbf{m}_i \cdot \mathbf{u}' + v(\lambda_i))_{i=1, \dots, n}$$

for every $\mathbf{u}' \in \Sigma(r, \pi)$.

7.4. Now we are ready to describe the formal analytic structure \mathfrak{X}' on X' induced by $\bar{\mathcal{C}}$. It is given by the atlas $U' \cap f^{-1}(U_{\bar{\Delta}})$, where U' ranges over all formal open affinoids as in 7.1 and where $\bar{\Delta} \in \bar{\mathcal{C}}$. Note that $U' \cap f^{-1}(U_{\bar{\Delta}}) = U' \cap F^{-1}(U_{\Delta})$ is a Weierstrass domain in U' . We have unique morphisms $\iota : \mathfrak{X}' \rightarrow (\mathcal{X}')^{\text{f-an}}$ and $\varphi : \mathfrak{X}' \rightarrow \mathcal{A}^{\text{f-an}}$ extending the identity on X' and $f : X' \rightarrow A$, respectively.

Our next goal is to relate this formal analytic structure on a subset U' from 7.1 to the simplex $\Sigma(r, \pi)$. Let \mathfrak{U}' be the formal analytic variety on U' induced by \mathfrak{X}' . Note that

$$(f_{\text{aff}}^{(0)})^{-1}(\mathcal{C}) := ((f_{\text{aff}}^{(0)})^{-1}(\Delta))_{\Delta \in \mathcal{C}}$$

is a Γ -rational polytopal decomposition of $\Sigma(r, \pi)$. We denote the associated formal scheme (see 4.6) by \mathcal{T} coming with a canonical morphism $i : \mathcal{T} \rightarrow \mathcal{U}_{\Sigma(r, \pi)}$ extending the identity.

Proposition 7.5. *Under the hypothesis above, the following properties hold:*

- (a) \mathcal{U}' is given by the atlas $\phi_0^{-1}(U_\sigma)$, $\sigma \in (f_{\text{aff}}^{(0)})^{-1}(\mathcal{C})$.
- (b) There is a unique morphism $\phi'_0 : \mathcal{U}' \rightarrow \mathcal{T}^{\text{f-an}}$ with $i^{\text{f-an}} \circ \phi'_0 = \phi_0^{\text{f-an}} \circ \iota$.
- (c) If $r = d$, then every irreducible component Z of $\tilde{\mathcal{U}}'$ with $\text{val} \circ \phi_0(\xi_Z) \in \text{relint}(\Sigma(r, \pi))$ is a toric variety.

Proof. Clearly, (a) follows from (11) and hence \mathcal{U}' is obtained from $(\mathcal{U}')^{\text{f-an}}$ by base change to $\mathcal{T}^{\text{f-an}}$ proving also (b). Finally, (c) is a consequence of Proposition 4.10. \square

Remark 7.6. We assume $r = d$ and hence $\mathcal{S} \cong \mathcal{U}_{\Sigma(d, \pi)}$. The irreducible components Z in (c) are the irreducible components of $\tilde{\mathcal{X}}'$ contracting to the distinguished singularity \tilde{P} from 7.1 under the canonical morphism $\tilde{\iota} : \tilde{\mathcal{X}}' \rightarrow \tilde{\mathcal{X}}'$. Moreover, Z is isomorphic to the toric variety $Y_{\mathbf{u}'}$ in $\tilde{\mathcal{T}}$ associated to the vertex $\mathbf{u}' := \text{val} \circ \phi_0(\xi_Z)$ of $(f_{\text{aff}}^{(0)})^{-1}(\mathcal{C})$. By Proposition 4.10 again, we get a bijective correspondence between the above Z and vertices of $(f_{\text{aff}}^{(0)})^{-1}(\mathcal{C})$ contained in $\text{relint}(\Sigma(d, \pi))$. The behaviour of $\tilde{\varphi} : Z \rightarrow \tilde{\mathcal{A}}$ with respect to the torus actions is

$$(13) \quad \tilde{\varphi}(\mathbf{t} \cdot z) = (\mathbf{t}^{\mathbf{m}^1}, \dots, \mathbf{t}^{\mathbf{m}^n}) \cdot \tilde{\varphi}(z)$$

for $z \in Z$ and $\mathbf{t} \in (\mathbb{G}_m^d)_{\mathbb{K}}$. This follows from the description (7) of F and $\tilde{u}_j|_Z \equiv \tilde{u}_j(\tilde{P})$ for $u_j \in \mathcal{O}(\mathcal{U}')^\times$ (use $U'_+(\tilde{P}) \cong \mathcal{S}_+^{\text{an}}(\tilde{\mathbf{0}})$ and the proof of Proposition 2.11).

7.7. Since \mathcal{X}' is a strictly semistable formal scheme, its special fibre $\tilde{\mathcal{X}}'$ has a canonical stratification: Let $\tilde{\mathcal{X}}'(i)$ be the closed subvariety of points in $\tilde{\mathcal{X}}'$ which are contained in at least $i + 1$ irreducible components of $\tilde{\mathcal{X}}'$. Then the irreducible components of the disjoint sets $\tilde{\mathcal{X}}'(i) \setminus \tilde{\mathcal{X}}'(i + 1)$ are called the *strata* of $\tilde{\mathcal{X}}'$.

For $\tilde{P} \in \tilde{\mathcal{X}}'(\mathbb{K})$, let \mathcal{U}' be a formal neighbourhood in \mathcal{X}' as in 7.1 leading to an affine map $f_{\text{aff}} : \Delta(r, \pi) \rightarrow \mathbb{R}^n$.

Proposition 7.8. *The map \bar{f}_{aff} is determined by the stratum containing \tilde{P} up to permutation of the coordinates on $\Delta(r, \pi)$.*

Proof. We consider $\tilde{P}_1, \tilde{P}_2 \in \tilde{\mathcal{X}}'(\mathbb{K})$ in the same stratum with corresponding affine maps $f_{i, \text{aff}} : \Delta(r_i, \pi_i) \rightarrow \mathbb{R}^n$. Note that $r_i + 1$ is the number of irreducible components in $\tilde{\mathcal{X}}'$ passing through \tilde{P}_i , hence $r_1 = r_2$. By interchanging a suitable \tilde{P}_3 , we may assume $\mathcal{U}'_1 = \mathcal{U}'_2$. After a permutation, Proposition 2.11(d) yields that $\tilde{\varphi}_1^*(\tilde{x}'_j)$ is equal to $\tilde{\varphi}_2^*(\tilde{x}'_j)$ up to a unit on $\tilde{\mathcal{U}}'_1$. The latter lifts to a unit on \mathcal{U}'_1 and hence $\text{val} \circ \phi_1 = \text{val} \circ \phi_2$ on $U'_1 = U'_2$. By (11) and $\text{val} \circ \phi_i(U'_i) = \Delta(r, \pi)$, we deduce $\bar{f}_{1, \text{aff}} = \bar{f}_{2, \text{aff}}$. \square

7.9. Up to now, we assume that f is an *alteration* of X^{an} , i.e. f is a proper surjective morphism $X' \rightarrow X^{\text{an}}$ for an irreducible variety X' of dimension $d := \dim(X)$. By the GAGA-principle (see [Ber1, 3.4.7]), everything may be formulated algebraically. We assume that X' has a strictly semistable formal \mathbb{K}° -model \mathcal{X}' .

From 7.4, we get a morphism $\varphi : \mathcal{X}' \rightarrow \mathcal{X}^{\text{f-an}}$ of formal analytic varieties with $\varphi^{\text{an}} = f$, where \mathcal{X} is the closure of X in \mathcal{A} . Since f is proper, $\tilde{\varphi}$ is also proper [Gu2, Remark 3.14].

7.10. For every stratum S of $\tilde{\mathcal{X}}'$, we get a map $\bar{f}_{\text{aff}} : \Delta(r_S, \pi_S) \rightarrow \mathbb{R}/\Lambda$, determined up to permutation. For $\bar{\rho}_S := \bar{f}_{\text{aff}}(\Delta(r_S, \pi_S))$, the proof of Proposition 5.2 and Proposition 7.8 show

$$\overline{\text{val}}(X^{\text{an}}) = \bigcup_S \bar{\rho}_S.$$

Moreover, by Theorem 6.9, we may restrict S to the strata with $\dim(\rho_S) = d$.

We call $\bar{\mathbf{u}} \in \overline{\text{val}}(X^{\text{an}})$ *non-degenerate* with respect to f if $\mathbf{u} \notin \rho_S$ for all simplices ρ_S of dimension $< d$. Note that \mathbf{u} and ρ_S are only determined up to Λ -translation.

7.11. Since \mathcal{X}' is a strictly semistable formal \mathbb{K}° -model of the d -dimensional irreducible variety X' , the set $\tilde{\mathcal{X}}'(d)$ of strata introduced in 7.7 is zero-dimensional. Let $\tilde{P}_1, \dots, \tilde{P}_R$ be the points of $\tilde{\mathcal{X}}'(d)$. We denote the affine map $\Delta(d, \pi_j) \rightarrow \mathbb{R}^n$ corresponding to the stratum \tilde{P}_j by $f_{j,\text{aff}}$. The image of $f_{j,\text{aff}}$ is a simplex in \mathbb{R}^n which we denote by ρ_j . After renumbering, we may assume that $f_{j,\text{aff}}$ is one-to-one exactly for $j = 1, \dots, N$. By 7.10, we have the decomposition

$$\overline{\text{val}}(X^{\text{an}}) = \bigcup_{j=1}^N \bar{\rho}_j.$$

The lower dimensional simplices $\rho_{N+1}, \dots, \rho_R$ will play only a minor role in the following. For $j = 1, \dots, N$, the bijective projection $\Delta(d, \pi_j) \rightarrow \Sigma(d, \pi_j)$ and $f_{j,\text{aff}}$ induce $f_{j,\text{aff}}^{(0)} : \Sigma(d, \pi_j) \rightarrow \mathbb{R}^n$ (see 7.3) which extends canonically to an injective affine map $\mathbb{R}^d \rightarrow \mathbb{R}^n$ also denoted by $f_{j,\text{aff}}^{(0)}$.

7.12. We consider a polytope $\bar{\Delta} \in \bar{\mathcal{C}}$ of codimension d with relative interior $\bar{\tau}$. We assume that $\bar{\Delta} \cap \overline{\text{val}}(X^{\text{an}})$ is a non-empty finite subset of $\bar{\tau}$. We suppose also that the points of $\bar{\Delta} \cap \overline{\text{val}}(X^{\text{an}})$ are non-degenerate with respect to f .

7.13. For $j \in \{1, \dots, N\}$ with $\bar{\rho}_j \cap \bar{\Delta} \neq \emptyset$, we are going to define an index of $\bar{\Delta}$ relative to $\bar{f}_{j,\text{aff}}$. Note that $j \leq N$ means that the simplex ρ_j from 7.11 is d -dimensional and hence $\bar{\rho}_j \cap \bar{\Delta}$ is a transversal intersection. The index

will depend only on the d -codimensional linear subspace \mathbb{L}_Δ of \mathbb{R}^n with $\Delta \subset \mathbf{u} + \mathbb{L}_\Delta$ and on the injective linear map $\ell_j^{(0)} := f_{j,\text{aff}}^{(0)} - f_{j,\text{aff}}^{(0)}(\mathbf{0}) : \mathbb{R}^d \rightarrow \mathbb{R}^n$.

Note that \mathbb{L}_Δ is defined over \mathbb{Q} and hence $N_\Delta := \mathbb{L}_\Delta \cap \mathbb{Z}^n$ is a complete lattice in \mathbb{L}_Δ . Let q_Δ be the quotient map $\mathbb{R}^n \rightarrow \mathbb{R}^n / \mathbb{L}_\Delta$. Since $\bar{\Delta}$ is transversal to $\bar{\rho}_j$ and since $\ell_j^{(0)}$ is injective, $q_\Delta \circ \ell_j^{(0)}$ is also injective on \mathbb{R}^d . Using that $\ell_j^{(0)}$ is defined over \mathbb{Z} , we conclude that $(q_\Delta \circ \ell_j^{(0)})(\mathbb{Z}^d)$ is a complete lattice in $\mathbb{R}^n / \mathbb{L}_\Delta$ of finite index in \mathbb{Z}^n / N_Δ . This crucial index will be denoted by

$$\text{ind}(\bar{\Delta}, \bar{f}_{j,\text{aff}}) := [\mathbb{Z}^n / N_\Delta : (q_\Delta \circ \ell_j^{(0)})(\mathbb{Z}^d)].$$

It is important to note that $\text{ind}(\bar{\Delta}, \bar{f}_{j,\text{aff}})$ depends only on \mathbb{L}_Δ and on $\ell_j^{(0)}$. The index may be more canonically described in terms of the linear map $\ell_j := f_{j,\text{aff}} - f_{j,\text{aff}}(\mathbf{0})$ defined on the hyperplane $\{\mathbf{u}' \in \mathbb{R}^{d+1} \mid u'_0 + \dots + u'_d = 0\}$.

In the above definition of the index, not all assumptions in 7.12 are needed. In fact, $\text{ind}(\bar{\Delta}, \bar{f}_{j,\text{aff}})$ is defined for all $j \in \{1, \dots, N\}$ and every $\bar{\Delta} \in \bar{\mathcal{C}}$ of codimension d with $\mathbb{L}_\Delta \cap \mathbb{L}_{\rho_j} = \{0\}$.

Proposition 7.14. *Recall that A is a totally degenerate abelian variety over \mathbb{K} with Mumford model \mathcal{A} associated to $\bar{\mathcal{C}}$ and that X is a d -dimensional irreducible closed subvariety of A . Let $f : X' \rightarrow X$ be an alteration with strictly semistable formal \mathbb{K}° -model $\tilde{\mathcal{X}}'$ of X' . Let $\bar{\Delta}$ be a polytope from $\bar{\mathcal{C}}$ satisfying the transversality assumption 7.12. Let Y be the closure of the torus orbit of $\tilde{\mathcal{A}}$ associated to $\text{relint}(\bar{\Delta})$ (see Proposition 6.3).*

- (a) *There is a unique formal analytic structure $\tilde{\mathcal{X}}'$ on X' which is a refinement of $(\mathcal{X}')^{\text{f-an}}$ such that f extends to a morphism $\varphi : \tilde{\mathcal{X}}' \rightarrow \mathcal{A}^{\text{f-an}}$.*
- (b) *The closed set Y is an irreducible component of $\tilde{\mathcal{X}}$ for the closure \mathcal{X} of X in \mathcal{A} . Moreover, Y with its reduced induced closed subscheme structure is canonically a toric variety.*
- (c) *Let $\tilde{P}_1, \dots, \tilde{P}_N$ be the zero-dimensional strata of $\tilde{\mathcal{X}}'$ such that the associated affine maps $f_{j,\text{aff}} : \Delta(d, \pi_j) \rightarrow \mathbb{R}^n$ are one-to-one (see 7.11). Then any irreducible component Z of $\tilde{\mathcal{X}}'$ with $\tilde{\varphi}(Z) = Y$ contracts to a unique \tilde{P}_j with respect to $\tilde{\mathcal{X}}' \xrightarrow{\tilde{i}} \tilde{\mathcal{X}}'$ for some $j \in \{1, \dots, N\}$. This gives a bijective correspondence between such Z and $J := \{j \in \{1, \dots, N\} \mid \bar{\Delta} \cap \bar{f}_{j,\text{aff}}(\Delta(d, \pi_j)) \neq \emptyset\}$.*
- (d) *If Z is as in (c), then Z is also canonically a toric variety and $[Z : Y] = \text{ind}(\bar{\Delta}, \bar{f}_{j,\text{aff}})$.*
- (e) *The multiplicity of Y in $\tilde{\mathcal{X}}$ fulfills $m(Y, \tilde{\mathcal{X}}) = \frac{1}{[X':X]} \sum_{j \in J} \text{ind}(\bar{\Delta}, \bar{f}_{j,\text{aff}})$.*

Proof. Obviously, (a) is a reformulation of 7.4. The transversality assumption on $\bar{\Delta}$ yields (b) by Lemma 5.10.

Let Z be an irreducible component of $\tilde{\mathcal{X}}'$ with $\tilde{\varphi}(Z) = Y$. Then $\tilde{\varphi}(\tilde{\xi}_Z)$ is the generic point of Y , hence $(f(\xi_Z))^\sim$ is in the open dense orbit Z_τ of Y for

$\bar{\tau} := \text{relint}(\bar{\Delta})$. We conclude that $\bar{\mathbf{u}} := \overline{\text{val}}(f(\xi_Z))$ is contained in the finite set

$$\bar{\Delta} \cap \overline{\text{val}}(X^{\text{an}}) = \bar{\tau} \cap \overline{\text{val}}(X^{\text{an}}).$$

There is a formal affine open subset \mathcal{U}' of \mathcal{X}' as in 7.1 with ξ_Z contained in the generic fibre U' . The image of the corresponding affine map f_{aff} is a simplex ρ with $\bar{\mathbf{u}} \in \bar{\rho}$ by (11), hence non-degeneracy of $\bar{\Delta} \cap \overline{\text{val}}(X^{\text{an}})$ with respect to f yields that ρ is d -dimensional. By Proposition 7.8, $\tilde{P}_j \in \mathcal{U}'$ for some $j \in \{1, \dots, N\}$ and we have $f_{\text{aff}} = f_{j,\text{aff}}$.

Let $f_{j,\text{aff}}^{(0)} : \Sigma(d, \pi_j) \rightarrow \mathbb{R}^n$ be the affine map induced from $f_{j,\text{aff}}$ as in 7.3. Let $\phi_{j,0} : \mathcal{U}' \rightarrow \mathcal{U}_{\Sigma(d, \pi_j)}$ be the étale morphism from 7.1. Then (11) and non-degeneracy show that

$$\mathbf{u}' := \text{val} \circ \phi_{j,0}(\xi_Z) \in \text{relint}(\Sigma(d, \pi_j))$$

and $\bar{\mathbf{u}} = \bar{f}_{j,\text{aff}}^{(0)}(\mathbf{u}')$. For the polytopal decomposition $\mathcal{C}_j := (f_{j,\text{aff}}^{(0)})^{-1}(\mathcal{C})$ of $\Sigma(d, \pi_j)$, we have seen in Remark 7.6 that there is a bijective correspondence between irreducible components of $\tilde{\mathcal{X}}'$ contracting to \tilde{P}_j and vertices of \mathcal{C}_j contained in $\text{relint}(\Sigma(d, \pi_j))$. We conclude that Z is isomorphic to the toric variety $Y_{\mathbf{u}'}$ associated to the vertex \mathbf{u}' of \mathcal{C}_j (see Proposition 4.7) proving the first part of (d).

To prove (c), it remains to show for $j \in J$ that there is a unique irreducible component Z of $\tilde{\mathcal{X}}'$ contracting to \tilde{P}_j with $\tilde{\varphi}(Z) = Y$. The d -dimensional simplex $\rho_j := f_{j,\text{aff}}(\Delta(d, \pi_j))$ satisfies $\bar{\rho}_j \subset \overline{\text{val}}(X^{\text{an}})$ by 7.11. The assumption $\bar{\Delta} \cap \bar{\rho}_j \neq \emptyset$ leads to a lift $\Delta \subset \mathbb{R}^n$ with $\Delta \cap \rho_j \neq \emptyset$. By transversality, $\Delta \cap \rho_j = \tau \cap \rho_j$ consists of a single point \mathbf{u} . Since $f_{j,\text{aff}}^{(0)}$ is injective, there is a unique $\mathbf{u}' \in \Sigma(d, \pi_j)$ with $f_{j,\text{aff}}^{(0)}(\mathbf{u}') = \mathbf{u}$. We note that \mathbf{u}' is a vertex of \mathcal{C}_j . Since $\bar{\mathbf{u}}$ is non-degenerate with respect to f , we have $\mathbf{u}' \in \text{relint}(\Sigma(d, \pi_j))$. As we have seen above, this vertex \mathbf{u}' corresponds to a unique irreducible component Z of $\tilde{\mathcal{X}}'$ contracting to \tilde{P}_j . By $\text{val}(\phi_{j,0}(\xi_Z)) = \mathbf{u}'$ and (11), we get $\overline{\text{val}} \circ f(\xi_Z) = \bar{\mathbf{u}}$. Proposition 6.3 proves

$$\tilde{\varphi}(\tilde{\xi}_Z) = (f(\xi_Z))^\sim \in Z_{\bar{\tau}}$$

and hence $\tilde{\varphi}(Z) \subset Y$. The above application of Remark 7.6 shows more precisely that Z is isomorphic to $Y_{\mathbf{u}'}$ and therefore Z is a toric variety with respect to the induced $(\mathbb{G}_m^d)_{\tilde{\mathbb{K}}}$ -action. On the other hand, Y is a toric variety with respect to the torus T over $\tilde{\mathbb{K}}$ given by

$$T(\tilde{\mathbb{K}}) = (\mathbb{Z}^n / N_{\Delta}) \otimes_{\mathbb{Z}} \tilde{\mathbb{K}}^{\times}$$

(see Remark 4.8). Here and in the following, we use the notation from 7.13. For $i = 1, \dots, n$, there is $\mathbf{m}_i \in \mathbb{Z}^d$ and $\lambda_i \in \mathbb{K}^{\times}$ such that the i -th coordinate of $f_{j,\text{aff}}^{(0)}(\mathbf{u}'')$ is equal to $\mathbf{m}_i \cdot \mathbf{u}'' + v(\lambda_i)$ (see (12)). Let $v : (\mathbb{G}_m^d)_{\tilde{\mathbb{K}}} \rightarrow T$ be the composition of $\mathbf{t} \mapsto (\mathbf{t}^{\mathbf{m}_1}, \dots, \mathbf{t}^{\mathbf{m}_n}) \in (\mathbb{G}_m^n)_{\tilde{\mathbb{K}}}$ with the quotient

homomorphism $(\mathbb{G}_m^n)_{\mathbb{K}} \rightarrow T$. Then the homomorphism ν is induced by the linear map $q_\Delta \circ \ell_j^{(0)} : \mathbb{Z}^d \rightarrow \mathbb{Z}^n/N_\Delta$. Since $q_\Delta \circ \ell_j^{(0)}$ is one-to-one (see 7.13), we deduce that ν is a finite surjective homomorphism of degree $\text{ind}(\overline{\Delta}, \overline{f}_{j,\text{aff}})$. Now (13) yields

$$(14) \quad \tilde{\varphi}(\mathbf{t} \cdot z) = \nu(\mathbf{t})\tilde{\varphi}(z) \quad (\mathbf{t} \in (\mathbb{G}_m^d)_{\mathbb{K}}, z \in Z)$$

and we conclude that $\tilde{\varphi}(Z) = Y$. Uniqueness of Z is clear from the construction and hence we get (c). Moreover, the second part of (d) follows easily from our description of ν and (14). Finally, (e) is a consequence of (c), (d) and the projection formula [Gu2, Proposition 4.5]

$$m(Y, \tilde{\mathcal{X}}) = \frac{1}{[X' : X]} \sum_Z [Z : Y],$$

where Z ranges over all irreducible components of $\tilde{\mathcal{X}}'$ with $\tilde{\varphi}(Z) = Y$. \square

8. Canonical measures

In this section, K is a field with a discrete valuation v . The completion of the algebraic closure of the completion of K with respect to v is an algebraically closed field denoted by \mathbb{K} [BGR, Proposition 3.4.1/3]. The unique extension of v to a valuation of \mathbb{K} is also denoted by v with corresponding absolute value $|\cdot| := e^{-v}$. The value group $\Gamma = v(\mathbb{K}^\times)$ is equal to \mathbb{Q} .

Let A be an abelian variety over K which is totally degenerate over \mathbb{K} , i.e. $A^{\text{an}} = (\mathbb{G}_m^n)_{\mathbb{K}}^{\text{an}}/M$ for a lattice M isomorphic to $\Lambda := \text{val}(M)$ in \mathbb{R}^n . Let X be a geometrically integral d -dimensional closed subvariety of A .

We will show first that a generic rational polytopal decomposition $\overline{\mathcal{C}}$ of \mathbb{R}^n/Λ is transversal to $\overline{\text{val}}(X_{\mathbb{K}}^{\text{an}})$. If \mathcal{A} denotes the associated formal \mathbb{K}° -model of A and if \mathcal{X} is the closure of $X_{\mathbb{K}}^{\text{an}}$ in \mathcal{A} , then transversality allows us to handle the special fibre of \mathcal{X} by the theory of toric varieties.

By rationality, the decomposition $\frac{1}{m}\overline{\mathcal{C}} := \{\frac{1}{m}\Delta \mid \Delta \in \mathcal{C}\}$ of \mathbb{R}^n/Λ can not be transversal to $\overline{\text{val}}(X_{\mathbb{K}}^{\text{an}})$ for all $m \in \mathbb{N} \setminus \{0\}$ simultaneously. However, this may be achieved over a sufficiently large base extension \mathbb{K}' with value group Γ' by a ‘‘completely irrational’’ construction which is also suitable for extending an ample line bundle on A to a positive formal $(\mathbb{K}')^\circ$ -model on \mathcal{A} . This will be used to prove Theorem 1.3. Moreover, we will get an explicit formula for the canonical measures on $\overline{\text{val}}(X_{\mathbb{K}}^{\text{an}})$ given in Theorem 8.6.

There is no restriction of generality assuming that X is geometrically integral as we may proceed by base change and linearity to get the canonical measures in the general case.

8.1. Fix some Λ -periodic set Σ of polytopes in \mathbb{R}^n and set $\overline{\Sigma} := \{\overline{\sigma} \subset \mathbb{R}^n/\Lambda \mid \sigma \in \Sigma\}$. We assume that $\overline{\Sigma}$ is a finite set. If a polytope

is in Σ , then we require that all its closed faces are also in Σ . For a polytope $\sigma \subset \mathbb{R}^n$, \mathbb{A}_σ denotes the affine space in \mathbb{R}^n generated by σ .

The polytopal decomposition $\overline{\mathcal{C}}$ of \mathbb{R}^n/Λ is said to be *generic* with respect to $\overline{\Sigma}$ if the following conditions hold for every $\sigma \in \Sigma$, $\Delta \in \mathcal{C}$:

- (a) $\dim(\mathbb{A}_\sigma \cap \mathbb{A}_\Delta) = D$ if $D := \dim(\sigma) + \dim(\Delta) - n \geq 0$,
- (b) $\mathbb{A}_\sigma \cap \mathbb{A}_\Delta = \emptyset$ if $D < 0$.

A polytopal decomposition $\overline{\mathcal{C}}$ of \mathbb{R}^n/Λ is called $\overline{\Sigma}$ -*transversal* if $\Delta \cap \sigma$ is either empty or of dimension $\dim(\Delta) + \dim(\sigma) - n$ for all $\Delta \in \mathcal{C}$, $\sigma \in \Sigma$. If the union $\overline{\mathcal{S}}$ of all polytopes in $\overline{\Sigma}$ is pure dimensional, then a $\overline{\Sigma}$ -transversal $\overline{\mathcal{C}}$ is transversal to $\overline{\mathcal{S}}$ in the sense of A.2.

Proposition 8.2. *Every $\overline{\Sigma}$ -generic polytopal decomposition of \mathbb{R}^n/Λ is $\overline{\Sigma}$ -transversal.*

Proof. Let Δ, Σ be polytopes in \mathbb{R}^n with $\Delta \cap \Sigma \neq \emptyset$ such that all closed faces σ' of σ and Δ' of Δ satisfy (a) and (b). It is enough to show $\dim(\Delta \cap \sigma) = \dim(\Delta) + \dim(\sigma) - n$. If $\text{relint}(\Delta) \cap \text{relint}(\sigma) \neq \emptyset$, then this is obvious from (a). So we may assume $\text{relint}(\Delta) \cap \text{relint}(\sigma) = \emptyset$ which will lead to a contradiction. By symmetry and passing to closed faces if necessary, we may assume that there is a closed face σ' of codimension 1 in σ such that $\Delta \cap \sigma' = \Delta \cap \sigma$ and $\text{relint}(\Delta) \cap \text{relint}(\sigma') \neq \emptyset$. Note that $\mathbb{A}_{\sigma'}$ divides \mathbb{A}_σ into half spaces, one is containing σ . Since $\text{relint}(\Delta) \cap \text{relint}(\sigma') \neq \emptyset$, we conclude that $\mathbb{A}_\Delta \cap \mathbb{A}_\sigma$ is contained in $\mathbb{A}_{\sigma'}$. Thus $\mathbb{A}_\Delta \cap \mathbb{A}_{\sigma'} = \mathbb{A}_\Delta \cap \mathbb{A}_\sigma$ contradicts (a) and (b). □

8.3. Starting with an arbitrary rational triangulation of \mathbb{R}^n/Λ and varying the vertices a little bit in \mathbb{Q}^n , we get a rational triangulation $\overline{\mathcal{C}}$ of \mathbb{R}^n/Λ which is $\overline{\Sigma}$ -generic.

Up to now, we assume that Σ is rational. For the proof of Theorem 1.3, we need that $\frac{1}{m}\overline{\mathcal{C}}$ is $\overline{\Sigma}$ -generic for all non-zero $m \in \mathbb{N}$ simultaneously which is not possible for a rational $\overline{\mathcal{C}}$. Instead, we are working with an infinite dimensional \mathbb{Q} -subspace Γ' of \mathbb{R} containing \mathbb{Q} . By [Bou, Chapt. VI, n° 10, Prop. 1], there is an algebraically closed field \mathbb{K}' , complete with respect to an absolute value $|\cdot|'$ extending $|\cdot|$ such that $v'((\mathbb{K}')^\times) = \Gamma'$. Now we will see that a Γ' -rational $\overline{\mathcal{C}}$ with the desired property and which behaves well with respect to the extension of an ample line bundle can be obtained by a “completely irrational” construction. Since the lemma will be important for the sequel, we give a rather detailed proof.

Lemma 8.4. *Let L be an ample line bundle on A . Then there is a Γ' -rational polytopal decomposition $\overline{\mathcal{C}}$ of \mathbb{R}^n/Λ with the following properties:*

- (a) $\frac{1}{m}\overline{\mathcal{C}}$ is $\overline{\Sigma}$ -generic and hence $\overline{\Sigma}$ -transversal for all $m \in \mathbb{N} \setminus \{0\}$.
- (b) If \mathcal{A} denotes the formal $(\mathbb{K}')^\circ$ -model of $A_{\mathbb{K}}^{\text{an}}$ associated to $\overline{\mathcal{C}}$, then there are $N \in \mathbb{N} \setminus \{0\}$ and a formal $(\mathbb{K}')^\circ$ -model \mathcal{L} of $L^{\otimes N}$ on \mathcal{A} with $\tilde{\mathcal{L}}$ ample.

Proof. In the terminology of 6.5 and of Proposition 6.6, L induces z_λ and a bilinear form b on Λ . By 6.5, we deduce that $z_\lambda(\mathbf{0})$ is a quadratic function on Λ and therefore

$$z_\lambda(\mathbf{0}) = q(\lambda) + \ell(\lambda)$$

for the quadratic form $q(\lambda) := \frac{1}{2}b(\lambda, \lambda)$ and a linear form ℓ on Λ (see [BoGu, 8.6.5]). Both extend to corresponding forms on \mathbb{R}^n also denoted by q and ℓ . Since L is ample, q is positive definite on Λ (see 6.5) and hence its extension to \mathbb{R}^n is also positive definite (using $\Gamma = \mathbb{Q}$). We conclude that $f := q + \ell$ is a strictly convex function on \mathbb{R}^n (see A.3). Formula (10) yields

$$(15) \quad f(\mathbf{u} + \lambda) = f(\mathbf{u}) + z_\lambda(\mathbf{u}) \quad (\lambda \in \Lambda, \mathbf{u} \in \mathbb{R}^n).$$

Our goal is to construct a Γ' -rational polytopal decomposition $\bar{\mathcal{C}}$ of \mathbb{R}^n/Λ with (a) and a strongly polyhedral convex function $g : \mathbb{R}^n \rightarrow \mathbb{R}$ with respect to $\bar{\mathcal{C}}$ such that (15) holds for g and such that for every $\Delta \in \bar{\mathcal{C}}$, there are $\mathbf{m}_\Delta \in \mathbb{Q}^n, c_\Delta \in \Gamma'$ with

$$(16) \quad g(\mathbf{u}) = \mathbf{m}_\Delta \cdot \mathbf{u} + c_\Delta \quad (\mathbf{u} \in \Delta).$$

We show first that this implies the lemma. By 6.5, there is $\mathbf{m}_\lambda \in \mathbb{Z}^n$, additive in $\lambda \in \Lambda$, such that $b(\lambda, \mathbf{u}) = \mathbf{m}_\lambda \cdot \mathbf{u}$. Now (15), (16) and (10) yield

$$(17) \quad \mathbf{m}_{\Delta+\lambda} = \mathbf{m}_\Delta + \mathbf{m}_\lambda$$

and hence there is a common denominator N of all $\mathbf{m}_\Delta, \Delta \in \bar{\mathcal{C}}$. By Proposition 6.6, there is a formal \mathbb{K}° -model \mathcal{L} of $L^{\otimes N}$ on \mathcal{A} with $f_{\mathcal{L}} = Ng$. Since g is a strongly polyhedral convex function with respect to $\bar{\mathcal{C}}$, Corollary 6.7 yields that $\tilde{\mathcal{L}}$ is ample.

Before we start the construction, we note that we may assume $\ell = 0$: This corresponds to the replacement of L by $L \otimes [-1]^*(L)$. If \hat{g} is a solution for the latter, then $g := \frac{1}{2}\hat{g} + \ell$ is a solution for the original problem (use 6.5). So we may assume $f = q$.

Let $F_\Lambda := \sum_{i=1}^n [0, 1)e'_i$ be the fundamental domain of Λ induced by the basis e'_1, \dots, e'_n . We number the $r = 2^n$ points

$$\theta_1 e'_1 + \dots + \theta_n e'_n \quad \left(\theta_k \in \left\{ 0, \frac{1}{2} \right\} \right)$$

by $\mathbf{u}_1, \dots, \mathbf{u}_r$. They form the set $F_\Lambda \cap \frac{1}{2}\Lambda$. We have the affine approximation

$$(18) \quad A_i(\mathbf{u}) := b(\mathbf{u}, \mathbf{u}_i) - q(\mathbf{u}_i)$$

of q in \mathbf{u}_i . We have $A_i(\mathbf{u}_i) = q(\mathbf{u}_i)$ and $A_i(\mathbf{u}) < q(\mathbf{u})$ for $\mathbf{u} \neq \mathbf{u}_i$ by strict convexity of q . By periodicity, we extend these definitions to \mathbb{R}^n , i.e. we set for $\lambda \in \Lambda$:

$$(19) \quad \mathbf{u}_{i,\lambda} := \mathbf{u}_i + \lambda, \quad A_{i,\lambda}(\mathbf{u}) := b(\mathbf{u}, \mathbf{u}_{i,\lambda}) - q(\mathbf{u}_{i,\lambda}).$$

Then the $\mathbf{u}_{i,\lambda}$ form just $\frac{1}{2}\Lambda$ and $A_{i,\lambda}$ is the affine approximation of q in $\mathbf{u}_{i,\lambda}$. The strongly polyhedral convex function

$$(20) \quad g := \max_{i=1,\dots,r; \lambda \in \Lambda} A_{i,\lambda}$$

is a lower bound of q and satisfies (15). Moreover, g is affine on the rational polytopes

$$(21) \quad \Delta_{i,\lambda} := \left\{ \mathbf{u} \in \mathbb{R}^n \mid A_{i,\lambda}(\mathbf{u}) = g(\mathbf{u}) \right\} = \bigcap_{(h,\mu) \neq (i,\lambda)} \{A_{i,\lambda}(\mathbf{u}) \geq A_{h,\mu}(\mathbf{u})\}.$$

They are the maximal polytopes where g is affine and they are the n -dimensional polytopes of the Voronoi decomposition \mathcal{C} of $\frac{1}{2}\Lambda$ with respect to the euclidean metric q , i.e.

$$\Delta_{i,\lambda} = \left\{ \mathbf{u} \in \mathbb{R}^n \mid q(\mathbf{u} - \mathbf{u}_{i,\lambda}) \leq q(\mathbf{u} - \mathbf{u}_{h,\mu}) \ \forall h = 1, \dots, r, \ \forall \mu \in \Lambda \right\}.$$

The polytopal decomposition \mathcal{C} is $\frac{1}{2}\Lambda$ -periodic and induces a rational polytopal decomposition $\overline{\mathcal{C}}$ of \mathbb{R}^n/Λ . By construction, (15) and (16) are satisfied but $\overline{\mathcal{C}}$ does certainly not satisfy (a).

To achieve this, we modify the construction by a small perturbation. The union G_0 of the Voronoi cells $\Delta_{i,\mathbf{0}}$ ($1 \leq i \leq r$) is the closure of a fundamental domain for \mathbb{R}^n/Λ . Then

$$G_1 := \bigcup_{(i,\lambda) \in T_1} \Delta_{i,\lambda}, \quad T_1 := \{(i, \lambda) \in \{1, \dots, r\} \times \Lambda \mid \Delta_{i,\lambda} \cap G_0 \neq \emptyset\},$$

is the set of neighbours of G_0 . We approximate the gradient $\nabla q(\mathbf{u}_i)$ by $\mathbf{m}_i \in \mathbb{Q}^n$ and $-q(\mathbf{u}_i)$ by $c_i \in (\Gamma')^n$. Then we define the affine function

$$A_i(\mathbf{u}) := \mathbf{m}_i \cdot \mathbf{u} + c_i$$

which is very close to the old definition. We may still assume that g is an upper bound of A_i . With the $\mathbf{m}_\lambda \in \mathbb{Z}^n$ introduced at the beginning of the proof, we define affine functions

$$A_{i,\lambda}(\mathbf{u}) := \mathbf{m}_{i,\lambda} \cdot \mathbf{u} + c_{i,\lambda}, \quad \mathbf{m}_{i,\lambda} := \mathbf{m}_i + \mathbf{m}_\lambda \in \mathbb{Q}^n, \\ c_{i,\lambda} := c_i - q(\lambda) - \mathbf{m}_i \cdot \lambda.$$

By construction, they are very close to the old approximations, $A_{i,\mathbf{0}} = A_i$ and (10) yields

$$(22) \quad A_{h,\rho+\lambda}(\mathbf{u} + \lambda) = A_{h,\rho}(\mathbf{u}) + z_\lambda(\mathbf{u}) \quad (\lambda, \mu, \rho \in \Lambda).$$

We assume that the approximations $\mathbf{m}_{i,\lambda}, c_{i,\lambda}$ of $\nabla q(\mathbf{u}_i), -q(\mathbf{u}_i)$ satisfy the conditions:

- (c) $1, (c_i)_{i=1,\dots,r}$ are \mathbb{Q} -linearly independent in Γ' .
- (d) For all $\sigma \in \Sigma$, given by linearly independent equations $\mathbf{l}_1 \cdot \mathbf{u} = a_1, \dots, \mathbf{l}_c \cdot \mathbf{u} = a_c$, with $\mathbf{l}_k \in \mathbb{Z}^n$ and $a_k \in \mathbb{Z}$ (possible by ration-

ality of Σ), and for every $S \subset \{1, \dots, r\}$ with $\text{card}(S) \leq n - c + 1$, $\lambda : S \rightarrow \Lambda \cap (G_0 - G_0)$, the vectors

$$(\mathbf{m}_{i,\lambda(i)} - \mathbf{m}_{i_0,\lambda(i_0)})_{i \in S \setminus \{i_0\}}, \mathbf{l}_1, \dots, \mathbf{l}_c$$

are linearly independent, where i_0 is the minimal member of S .

The existence of c_i follows from $[\Gamma' : \mathbb{Q}] = \infty$ and the construction of the approximations \mathbf{m}_i is by induction where in each step, we have to omit finitely many hyperplanes which is possible in every neighbourhood of $\nabla q(\mathbf{u}_i)$.

The function g defined by (20) is a strongly polyhedral convex function with respect to \mathcal{C} which is equal to $A_{i,\lambda}$ on the Γ' -rational polytope $\Delta_{i,\lambda}$ from (21). The latter are again the n -dimensional polytopes of a Γ' -rational decomposition \mathcal{C} of \mathbb{R}^n/Λ . \mathcal{C} is very close to the Voronoi decomposition considered above in the sense that the boundary of the new $\Delta_{i,\lambda}$ is near to the boundary of the corresponding Voronoi cell. Moreover, if two cells intersect, then also the corresponding Voronoi cells intersect. By (22), \mathcal{C} is Λ -periodic and g satisfies (15). Note that (16) is clear by construction. We get a polytopal decomposition $\overline{\mathcal{C}}$ of \mathbb{R}^n/Λ .

It remains to see that $\overline{\mathcal{C}}$ satisfies (a): Let $\sigma \in \Sigma$, $m \in \mathbb{N} \setminus \{0\}$ and $\Delta \in \mathcal{C}$. We may represent Δ as a closed face of Δ_{i_0,λ_0} given by the hyperplanes

$$A_{i,\lambda}(\mathbf{u}) = A_{i_0,\lambda_0}(\mathbf{u}), \quad (i, \lambda) \in S \subset \{1, \dots, r\} \times \Lambda.$$

We may assume that every hyperplane is generated by a face of Δ_{i_0,λ_0} and that

$$(23) \quad \mathbb{A}_\Delta = \bigcap_{(i,\lambda) \in S} \{A_{i,\lambda}(\mathbf{u}) = A_{i_0,\lambda_0}(\mathbf{u})\}$$

is a transversal intersection. By Λ -periodicity, we may assume that $\Delta \subset G_1$. Since \mathcal{C} is very close to the Voronoi decomposition of $\frac{1}{2}\Lambda$ with respect to q , it is clear that for given i , there is at most one $\lambda \in \Lambda$ involved in (23). Note that such a λ is contained in $G_0 - G_0$. By (23), $\mathbb{A}_\sigma \cap \mathbb{A}_\Delta$ is the solution space of the $c + \text{codim}(\Delta)$ linear equations

$$(24) \quad \mathbf{l}_1 \cdot \mathbf{u} = a_1, \quad \dots, \quad \mathbf{l}_c \cdot \mathbf{u} = a_c$$

(in the notation borrowed from (d)) and

$$(25) \quad (\mathbf{m}_{i,\lambda} - \mathbf{m}_{i_0,\lambda_0}) \cdot \mathbf{u} = c_{i_0,\lambda_0} - c_{i,\lambda} \quad ((i, \lambda) \in S).$$

If $D := \dim(\sigma) + \dim(\Delta) - n \geq 0$, then the assumption (d) yields that the homogeneous linear equations associated to (24) and (25) are linearly independent and hence $\dim(\mathbb{A}_\sigma \cap \mathbb{A}_\Delta) = D$. If $D < 0$, then we may express $\mathbf{l}_1 \cdot \mathbf{u}$ as a \mathbb{Q} -linear combination of the left hand sides of the other equations in (24) and (25). By (c), a_1 can not be the same linear combination of

the right hand sides. This proves $\mathbb{A}_\sigma \cap \mathbb{A}_\Delta = \emptyset$ and therefore $\overline{\mathcal{C}}$ is $\overline{\Sigma}$ -generic.

If we replace Δ by $\frac{1}{m}\Delta$, then the right hand side of (25) is multiplied by $\frac{1}{m}$. This does not change the above argument and we conclude that $\frac{1}{m}\overline{\mathcal{C}}$ is $\overline{\Sigma}$ -generic. \square

8.5. Let $\overline{L}_1, \dots, \overline{L}_d$ be ample line bundles on A endowed with canonical metrics (see 3.13). Our goal is to give an explicit formula for the canonical measure

$$\mu := (\overline{\text{val}})_*(c_1(\overline{L}_1|_X) \wedge \dots \wedge c_1(\overline{L}_d|_X))$$

on the tropical variety $\overline{\text{val}}(X_{\mathbb{K}}^{\text{an}})$. By de Jong’s alteration theorem [dJ, Theorem 6.5], there is an alteration $f : X' \rightarrow X_{\mathbb{K}}^{\text{an}}$ and a strictly semistable formal \mathbb{K}° -model \mathcal{X}' of X' (see 7.9). Let $\tilde{P}_1, \dots, \tilde{P}_N$ be the zero-dimensional strata of \mathcal{X}' such that the associated affine maps $f_{j,\text{aff}} : \Delta(d, \pi_j) \rightarrow \mathbb{R}^n$ are one-to-one (see 7.11). The image of $f_{j,\text{aff}}$ is a d -dimensional simplex denoted by ρ_j . The sets $\overline{\rho}_1, \dots, \overline{\rho}_N$ cover $\overline{\text{val}}(X_{\mathbb{K}}^{\text{an}})$ (see 7.11).

For simplicity of notation, we may assume that $\overline{\rho}_1, \dots, \overline{\rho}_N$ are simplices in \mathbb{R}^n/Λ , i.e. the projection $\rho_j \rightarrow \overline{\rho}_j$ is bijective. In general, a subdivision of the ρ_j may be needed but this does not change the description of the measure μ in Theorem 8.6.

Let $\overline{\sigma}$ be an atom of the covering $\bigcup_j \overline{\rho}_j = \overline{\text{val}}(X_{\mathbb{K}}^{\text{an}})$, i.e. $\overline{\sigma}$ is the closure of $\bigcap_j \overline{T}_j$, where \overline{T}_j is either $\text{relint}(\overline{\rho}_j)$ or $\overline{\text{val}}(X_{\mathbb{K}}^{\text{an}}) \setminus \overline{\rho}_j$. We omit $\overline{\sigma} = \emptyset$. Then $\overline{\sigma}$ is a finite union of d -dimensional polytopes in \mathbb{R}^n/Λ . Moreover, the sets $\overline{\sigma}$ form a finite covering of $\overline{\text{val}}(X_{\mathbb{K}}^{\text{an}})$ with overlappings of dimension $< d$. To get polytopes in \mathbb{R}^n/Λ , a subdivision would be needed, but this is irrelevant for the description of the measure μ .

Since $\overline{\sigma} \neq \emptyset$, the set $J(\overline{\sigma}) := \{j = 1, \dots, N \mid \overline{\sigma} \subset \overline{\rho}_j\}$ is non-empty. Let \mathbb{A}_σ be the affine space generated by ρ_j for some $j \in J(\overline{\sigma})$. Up to Λ -translation, it is independent of the choice of $j \in J(\overline{\sigma})$. We may lift a measurable subset $\overline{\Omega}$ of $\overline{\sigma}$ to $\Omega \subset \rho_j$. We conclude that the relative Lebesgue measure on \mathbb{A}_σ induces a canonical measure on $\overline{\sigma}$ which we denote by $\text{vol}(\overline{\Omega})$.

Using $\Gamma = \mathbb{Q}$, we deduce that the stabilizer $\Lambda(\mathbb{A}_\sigma) := \{\lambda \in \Lambda \mid \mathbb{A}_\sigma + \lambda \subset \mathbb{A}_\sigma\}$ of \mathbb{A}_σ is a complete lattice in the linear space \mathbb{L}_σ parallel to \mathbb{A}_σ . For $j \in J(\overline{\sigma})$, let $f_{j,\text{aff}}^{(0)} : \Sigma(d, \pi_j) \rightarrow \mathbb{R}^d$ be the affine map induced from $f_{j,\text{aff}}$ as in 7.3. Then the linear map $\ell_j^{(0)} := f_{j,\text{aff}}^{(0)} - f_{j,\text{aff}}^{(0)}(\mathbf{0}) : \mathbb{R}^d \rightarrow \mathbb{R}^n$ is one-to-one, defined over \mathbb{Z} and has image \mathbb{L}_σ . We get a dual map $\hat{\ell}_j^{(0)} : \mathbb{L}_\sigma^* \rightarrow (\mathbb{R}^d)^* = \mathbb{R}^d$ which is bijective and defined over \mathbb{Z} .

Let b_j be the bilinear form associated to L_j (see 6.5). Since b_j is positive definite on \mathbb{R}^n ,

$$\Lambda(\mathbb{A}_\sigma)^{L_j} := \{b_j(\cdot, \lambda) \in \mathbb{L}_\sigma^* \mid \lambda \in \Lambda(\mathbb{A}_\sigma)\}$$

is a complete lattice in \mathbb{L}_σ^* . We will also use the dual lattice of $\Lambda(\mathbb{A}_\sigma)$, given by

$$\Lambda(\mathbb{A}_\sigma)^* := \{\ell \in \mathbb{L}_\sigma^* \mid \ell(\Lambda(\mathbb{A}_\sigma)) \subset \mathbb{Z}\}.$$

We denote by $\text{vol}(\Lambda(\mathbb{A}_\sigma))$ the volume of a fundamental domain of the lattice $\Lambda(\mathbb{A}_\sigma)$ with respect to the relative Lebesgue measure on \mathbb{L}_σ . Let $\text{vol}(\Lambda(\mathbb{A}_\sigma)^{L_1}, \dots, \Lambda(\mathbb{A}_\sigma)^{L_d})$ be the mixed volume of the corresponding fundamental domains in \mathbb{L}_σ^* (see A.6 for definition and properties). Since the mixed volume is positive, the following formula implies Theorem 1.3:

Theorem 8.6. *For a measurable subset $\overline{\Omega}$ of the atom $\overline{\sigma}$ as in 8.5, we have*

$$\mu(\overline{\Omega}) = d! \cdot \vartheta(\overline{\sigma}) \cdot \frac{\text{vol}(\Lambda(\mathbb{A}_\sigma)^{L_1}, \dots, \Lambda(\mathbb{A}_\sigma)^{L_d})}{\text{vol}(\Lambda(\mathbb{A}_\sigma)^*) \cdot \text{vol}(\Lambda(\mathbb{A}_\sigma))} \cdot \text{vol}(\overline{\Omega}),$$

where

$$\vartheta(\overline{\sigma}) := \frac{1}{[X' : X_{\mathbb{K}}^{\text{an}}]} \sum_{j \in J(\overline{\sigma})} [\mathbb{Z}^d : \hat{\ell}_j^{(0)}(\Lambda(\mathbb{A}_\sigma)^*)]$$

and the measure μ is defined by $\mu := (\overline{\text{val}})_*(c_1(\overline{L}_1|_X) \wedge \dots \wedge c_1(\overline{L}_d|_X))$.

Proof. To prove the theorem, we may assume that $\overline{\Omega}$ is a d -dimensional polytope contained in $\overline{\sigma}$ (using monotone convergence). Note that the odd part of L_j does not influence the bilinear form b_j and hence we may suppose, using 3.15, that every L_j is symmetric. By multilinearity of μ and the mixed volume, we may assume $\overline{L} := \overline{L}_1 = \dots = \overline{L}_d$. In the next paragraph, we will fix a Mumford model \mathcal{A} of A associated to a “generic” choice of a polytopal decomposition $\overline{\mathcal{C}}$. It will be crucial for the proof that this choice is as generic as possible. In particular, $\overline{\mathcal{C}}$ is only Γ' -rational for an infinite dimensional \mathbb{Q} -subspace Γ' of \mathbb{R} equal to the value group of a complete algebraically closed extension \mathbb{K}' of \mathbb{K} and hence \mathcal{A} is only defined over the valuation ring of \mathbb{K}' . By Remark 3.10 and since $\overline{\text{val}}(X_{\mathbb{K}}^{\text{an}})$ is invariant under base change, we are allowed to make the analytic calculations for μ over \mathbb{K}' .

Let $\tilde{P}_1, \dots, \tilde{P}_R$ be the zero-dimensional strata of \tilde{X}' . As in 7.11, they induce affine maps $f_{j,\text{aff}} : \Delta(d, \pi_j) \rightarrow \mathbb{R}^n$ and we denote the image simplex by ρ_j . According to 8.5, we assume that ρ_1, \dots, ρ_N are d -dimensional and that $\rho_{N+1}, \dots, \rho_R$ are lower dimensional. Let Σ be the collection of these simplices ρ_j in \mathbb{R}^n together with all their faces and their Λ -translates. We choose $\overline{\mathcal{C}}$, \mathcal{A} and \mathcal{L} as in Lemma 8.4. By multilinearity, we may assume that \mathcal{L} is a $(\mathbb{K}')^\circ$ -model of L .

We will proof Theorem 8.6 now in four steps. We give first an *outline* of the plan:

We fix a rigidification on L such that the associated canonical metric $\|\cdot\|_{\text{can}}$ is the metric of \overline{L} (see 3.13). The rigidification and the theorem of the cube yield an identification $[m]^*L = L^{\otimes m^2}$ for $m \in \mathbb{Z}$. Let $\|\cdot\|$ be the

formal metric associated to \mathcal{L} (see 3.2). By 3.13, we have

$$(26) \quad \| \|\text{can} = \lim_{m \rightarrow \infty} ([m]^* \| \|^2)^{1/m^2}.$$

Let \mathcal{A}_m be the Mumford model of A associated to $\overline{\mathcal{C}}_m := \overline{\frac{1}{m}\mathcal{C}}$ and let \mathcal{X}_m be the closure of X in \mathcal{A}_m . Using the very definition of the measure μ and (26), we will show in a first step that μ is a weak limit of a sum of Dirac measures $\overline{\text{val}}_*(\delta_Z)$, where Z ranges over the irreducible components of $(\mathcal{X}_m^{\text{f-an}})^\sim$ and where ξ_Z is the point of the Berkovich space $X_{\mathbb{K}}^{\text{an}}$ corresponding to Z (see 2.5). In a second step, we will replace the Z by the irreducible components Y of the special fibre $\tilde{\mathcal{X}}_m$. Since the reduction $(\mathcal{X}_m^{\text{f-an}})^\sim$ is a finite covering of $\tilde{\mathcal{X}}_m$, this will be a consequence of the projection formula.

After the first two steps we have $\mu(\overline{\Omega}) = \lim_{m \rightarrow \infty} S_m$, where S_m depends on the multiplicities and the degrees of the Y 's. In the third step, we transform the limit into a multiple of $\text{vol}(\overline{\Omega})$. To make this plausible, note that the multiplication by m on A extends uniquely to a morphism $\varphi_m : \mathcal{A}_m \rightarrow \mathcal{A}_1 = \mathcal{A}$ (see 6.4). Applying projection formula to $\tilde{\varphi}_m$, we will relate the degree of Y to $\text{deg}_{\tilde{\mathcal{L}}}(Y_{\mathbf{u}})$ for a vertex \mathbf{u} of $\mathcal{C} \cap (m\mathbb{A}_\sigma)$ with $\overline{\mathbf{u}} \in \overline{\Omega}$. Here, $Y_{\mathbf{u}} := Y_{\Delta(\mathbf{u})}$ is the toric variety in $\tilde{\mathcal{A}}$ associated to the unique $\Delta(\mathbf{u}) \in \mathcal{C}$ with $\mathbf{u} \in \text{relint}(\Delta(\mathbf{u}))$ (see Remark 4.8). We will use the alteration $f : X' \rightarrow X_{\mathbb{K}}^{\text{an}}$ from 8.5 and Sect. 7 to express the multiplicity of Y in terms of indices of lattices. For $m \rightarrow \infty$, we will get $\mu(\overline{\Omega}) = s \cdot \text{vol}(\overline{\Omega})$, where s is a linear combination of $\text{deg}_{\tilde{\mathcal{L}}}(Y_{\mathbf{u}})$ with \mathbf{u} ranging over the vertices of $\mathcal{C} \cap \mathbb{L}_\sigma$ modulo the stabilizer $\Lambda(\mathbb{A}_\sigma)$ from 8.5. In the theory of toric varieties, there is a formula for $\text{deg}_{\tilde{\mathcal{L}}}(Y_{\mathbf{u}})$ as the volume of a polytope associated to the fan of $Y_{\mathbf{u}}$. We apply this in the fourth step to calculate s in terms of the dual complex of $\mathcal{C} \cap \mathbb{L}_\sigma$ using the appendix on convex geometry. This will prove Theorem 8.6.

Step 1. μ is a weak limit of discrete measures related to the irreducible components of $(\mathcal{X}_m^{\text{f-an}})^\sim$.

For $m \geq 1$, we have seen in the outline that a unique morphism $\varphi_m : \mathcal{A}_m \rightarrow \mathcal{A}_1 = \mathcal{A}$ extends multiplication by m on A . Recall that $\| \|^$ is the formal metric associated to the $(\mathbb{K}')^\circ$ -model \mathcal{L} of L on \mathcal{A} . Clearly, $[m]^* \| \|^$ is the formal metric associated to $\varphi_m^*(\mathcal{L})$. The composition of the canonical finite morphism $\tilde{\iota} : (\mathcal{X}_m^{\text{f-an}})^\sim \rightarrow \tilde{\mathcal{X}}_m$ from 2.6 with $\tilde{\varphi}_m$ is denoted by $\tilde{\varphi}_m$. By (26), 3.11 and 3.12, we have the following weak limit of regular Borel measures on $\overline{\text{val}}(X_{\mathbb{K}}^{\text{an}})$:

$$(27) \quad \mu = \lim_{m \rightarrow \infty} m^{-2d} \sum_Z \text{deg}_{\tilde{\varphi}_m^*(\tilde{\mathcal{L}})}(Z) \overline{\text{val}}_*(\delta_{\xi_Z}),$$

where Z ranges over all irreducible components of $(\mathcal{X}_m^{\text{f-an}})^\sim$. For our polytope $\overline{\Omega} \subset \overline{\sigma}$, this yields

$$(28) \quad \mu(\overline{\Omega}) = \lim_{m \rightarrow \infty} m^{-2d} \sum_Z \text{deg}_{\tilde{\varphi}_m^*(\tilde{\mathcal{L}})}(Z),$$

where Z ranges over all irreducible components of $(\mathcal{X}_m^{f-\text{an}})^\sim$ satisfying $\overline{\text{val}}(\xi_Z) \in \overline{\Omega}$.

Step 2. We replace the Z 's in (28) by the irreducible components Y of $\tilde{\mathcal{X}}_m$.

Since the morphism $\tilde{\iota}$ is finite and surjective, the set of irreducible components of $(\mathcal{X}_m^{f-\text{an}})^\sim$ is mapped onto the set of irreducible components of $\tilde{\mathcal{X}}_m$. By our choice of $\overline{\mathcal{C}}$ from Lemma 8.4, we note that $\overline{\mathcal{C}}_m$ is transversal to $\overline{\text{val}}(X_{\mathbb{K}}^{\text{an}})$ and hence every irreducible component Y of $\tilde{\mathcal{X}}_m$ corresponds to an equivalence class $\overline{\Delta} \cap \overline{\text{val}}(X_{\mathbb{K}}^{\text{an}})$ of transversal vertices of $\overline{\mathcal{C}}_m \cap \overline{\text{val}}(X_{\mathbb{K}}^{\text{an}})$ for a unique d -codimensional polytope $\overline{\Delta} \in \overline{\mathcal{C}}_m$ (see Theorem 6.10). Since $\overline{\mathcal{C}}_m$ is $\overline{\Sigma}$ -generic, it is clear that $\overline{\Delta}$ intersects the d -dimensional atom $\overline{\sigma}$ in at most one point. If a Z from (28) is lying over Y , then the proof of Theorem 5.11 shows that $\overline{\text{val}}(\xi_Z) \in \overline{\Delta} \cap \overline{\text{val}}(X_{\mathbb{K}}^{\text{an}})$, i.e. we have a transversal vertex corresponding to Y which is contained in $\overline{\Omega}$. We say that Y is $\overline{\Omega}$ -inner if

$$\overline{\Delta} \cap \overline{\text{val}}(X_{\mathbb{K}}^{\text{an}}) = \overline{\Delta} \cap \overline{\Omega}.$$

If Y is not $\overline{\Omega}$ -inner, then the corresponding $\overline{\Delta}$ intersects also an atom $\overline{\sigma}' \neq \overline{\sigma}$. This $\overline{\Delta}$ has to be a face of an n -dimensional polytope in $\overline{\mathcal{C}}$ intersecting the boundary of $\overline{\sigma}$. By an easy argument covering the boundary, we conclude that the number of such Y is of order $O(m^{d-1})$. We will use this later to show that the Z lying over non- $\overline{\Omega}$ -inner Y may be neglected in (28).

We consider now an $\overline{\Omega}$ -inner Y . Since $\overline{\Omega}$ is a polytope contained in $\overline{\sigma}$, we conclude that $\overline{\Delta} \cap \overline{\Omega}$ is just a point $\overline{\mathbf{u}}$. For an irreducible component Z of $(\mathcal{X}_m^{f-\text{an}})^\sim$ lying over Y , we have seen above that $\overline{\text{val}}(\xi_Z) \in \overline{\Delta} \cap \overline{\text{val}}(X_{\mathbb{K}}^{\text{an}})$ and hence $\overline{\text{val}}(\xi_Z) = \overline{\mathbf{u}} \in \overline{\Omega}$. Since $(\mathcal{X}_m^{f-\text{an}})^\sim$ is reduced, we have

$$\sum_Z \text{deg}_{\tilde{\varphi}_m^*(\tilde{\mathcal{L}})}(Z) = \text{deg}_{\tilde{\varphi}_m^*(\tilde{\mathcal{L}})}((\mathcal{X}_m^{f-\text{an}})^\sim),$$

where Z ranges over all irreducible components of $(\mathcal{X}_m^{f-\text{an}})^\sim$. Since $\mathcal{X}_m^{f-\text{an}}$ and \mathcal{X}_m have the same generic fibre, the projection formula [Gu2, Proposition 4.5] shows that $\tilde{\iota}_*((\mathcal{X}_m^{f-\text{an}})^\sim)$ is equal to the cycle associated to $\tilde{\mathcal{X}}_m$. For the multiplicity $m(Y, \tilde{\mathcal{X}}_m)$, projection formula yields

$$(29) \quad \sum_{\tilde{\iota}(Z)=Y} \text{deg}_{\tilde{\varphi}_m^*(\tilde{\mathcal{L}})}(Z) = m(Y, \tilde{\mathcal{X}}_m) \text{deg}_{\tilde{\varphi}_m^*(\tilde{\mathcal{L}})}(Y).$$

This is true for any irreducible component Y of $\tilde{\mathcal{X}}_m$. First, we apply (29) on the right hand side of (28) to transform the contribution of all Z lying over an $\overline{\Omega}$ -inner Y into the sum

$$(30) \quad S_m := m^{-2d} \sum_Y m(Y, \tilde{\mathcal{X}}_m) \text{deg}_{\tilde{\varphi}_m^*(\tilde{\mathcal{L}})}(Y),$$

where Y ranges over all $\overline{\Omega}$ -inner irreducible components of $\tilde{\mathcal{X}}_m$. If Y is an irreducible component of $\tilde{\mathcal{X}}_m$ which is not $\overline{\Omega}$ -inner, then it is possible that there are irreducible components Z, Z' of $(\mathcal{X}_m^{f\text{-an}})^\sim$ lying over Y with $\overline{\text{val}}(\xi_Z) \in \overline{\Omega}$ and $\overline{\text{val}}(\xi_{Z'}) \notin \overline{\Omega}$. Then (30) is just an upper bound for the contribution of those Z in (28) lying over such a Y .

Step 3. Transformation of the limit in (28) into a multiple of $\text{vol}(\overline{\Omega})$.

Let Y be an irreducible component of $\tilde{\mathcal{X}}_m$. By transversality of $\overline{\mathcal{C}}_m$ and Theorem 6.10, there is a unique d -codimensional polytope $\overline{\Delta} \in \overline{\mathcal{C}}_m$ such that Y is the toric variety given as the closure of the torus orbit associated to $\text{relint}(\overline{\Delta})$.

Now we assume that Y is $\overline{\Omega}$ -inner. As we have seen in Step 2, $\overline{\Delta} \cap \overline{\text{val}}(X_{\mathbb{K}}^{\text{an}})$ is a transversal vertex $\overline{\mathbf{u}}$ of $\overline{\mathcal{C}}_m \cap \overline{\text{val}}(X_{\mathbb{K}}^{\text{an}})$. Since $\overline{\mathbf{u}} \in \overline{\Omega}$, there is a unique lift \mathbf{u} to the affine space \mathbb{A}_σ from 8.5. We conclude that $m\mathbf{u}$ is a vertex of $\mathcal{C} \cap (m\mathbb{A}_\sigma)$.

Every $\mathbf{u}' \in \mathbb{R}^n$ is contained in the relative interior of a unique $\Delta(\mathbf{u}') \in \mathcal{C}$. Let $Y_{\mathbf{u}'} := Y_{\Delta(\mathbf{u}')}$ be the associated toric variety in $\tilde{\mathcal{A}}$ (see Remark 4.8 and Proposition 6.3). Note that $Y_{\mathbf{u}'+\lambda} = Y_{\mathbf{u}'}$ for $\lambda \in \Lambda$. In the situation above, we have $m\Delta = \Delta(m\mathbf{u})$ and $\tilde{\varphi}_m(Y) = Y_{m\mathbf{u}}$. Applying projection formula to the morphism $\tilde{\varphi}_m : \tilde{\mathcal{A}}_m \rightarrow \tilde{\mathcal{A}}$ and using Proposition 6.4, we get

$$(31) \quad \text{deg}_{\mathcal{G}_{\tilde{\varphi}_m^*(\tilde{\mathcal{L}})}}(Y) = m^d \text{deg}_{\tilde{\mathcal{L}}}(Y_{m\mathbf{u}}).$$

Now we use the alteration $f : X' \rightarrow X_{\mathbb{K}}^{\text{an}}$ and the associated affine maps $f_{j,\text{aff}} : \Delta(d, \pi_j) \rightarrow \mathbb{R}^n$ with d -dimensional images $\overline{\rho}_j, j = 1, \dots, N$, from 8.5. Since $\overline{\mathcal{C}}$ is $\overline{\Sigma}$ -generic, the final remark in 7.13 shows that $\text{ind}(\overline{\Delta}', \overline{f}_{j,\text{aff}})$ is well-defined for all d -codimensional $\overline{\Delta}' \in \overline{\mathcal{C}}$ and depends only on the linear space $\mathbb{L}_{\Delta'}$ for $j = 1, \dots, N$. We consider the multiplicity

$$\vartheta(\overline{\Delta}', \overline{\sigma}) := \frac{1}{[X' : X_{\mathbb{K}}^{\text{an}}]} \sum_{j \in J(\overline{\sigma})} \text{ind}(\overline{\Delta}', \overline{f}_{j,\text{aff}})$$

of $\overline{\Delta}'$ relative to the atom $\overline{\sigma}$, where $J(\overline{\sigma}) := \{j = 1, \dots, N \mid \overline{\sigma} \subset \overline{\rho}_j\}$. Proposition 7.14 yields

$$(32) \quad m(Y, \tilde{\mathcal{X}}_m) = \frac{1}{[X' : X_{\mathbb{K}}^{\text{an}}]} \sum_{j \in J} \text{ind}(\overline{\Delta}, \overline{f}_{j,\text{aff}}),$$

where $J := \{j = 1, \dots, N \mid \overline{\Delta} \cap \overline{\rho}_j \neq \emptyset\}$. Since $\overline{\mathcal{C}}_m$ is $\overline{\Sigma}$ -transversal, we note that $\overline{\mathbf{u}}$ is an interior point of the atom $\overline{\sigma}$. By $\overline{\Delta} \cap \overline{\text{val}}(X_{\mathbb{K}}^{\text{an}}) = \{\overline{\mathbf{u}}\}$, we deduce that $j \in J$ if and only if $\overline{\Delta} \cap \overline{\rho}_j = \{\overline{\mathbf{u}}\}$ and this is equivalent to $\overline{\sigma} \subset \overline{\rho}_j$. This means $J = J(\overline{\sigma})$. We have $\mathbb{L}_{m\Delta} = \mathbb{L}_\Delta$ and $N_{m\Delta} = N_\Delta$, hence $\text{ind}(\overline{m\Delta}, \overline{f}_{j,\text{aff}}) = \text{ind}(\overline{\Delta}, \overline{f}_{j,\text{aff}})$. Now (32) yields

$$(33) \quad m(Y, \tilde{\mathcal{X}}_m) = \vartheta(\overline{\Delta}(m\mathbf{u}), \overline{\sigma}).$$

In (31) and (33), we have related the degree and the multiplicity of Y (or equivalently an $\overline{\Omega}$ -inner transversal vertex of $\overline{\mathcal{C}}_m \cap \overline{\text{val}}(X_{\mathbb{K}}^{\text{an}})$ as we have seen in Step 2) to the vertex $m\mathbf{u}$ of $\mathcal{C} \cap (m\mathbb{A}_\sigma)$. Both formulas are Λ -periodic in $m\mathbf{u}$. Note that the atom $\overline{\sigma}$ of $\overline{\Sigma}$ is rational and hence there is $m_0 \in \mathbb{N}$ such that $m_0\mathbb{A}_\sigma = \lambda_0 + \mathbb{L}_\sigma$ for some $\lambda_0 \in \Lambda$. Up to now, we assume that $m \in \mathbb{N}m_0$. By periodicity, we may express (31) and (33) in terms of the vertex $\mathbf{u}' := m\mathbf{u} - \frac{m}{m_0}\lambda_0$ of $\mathcal{C} \cap \mathbb{L}_\sigma$. Note that $\mathcal{C} \cap \mathbb{L}_\sigma$ is $\Lambda(\mathbb{A}_\sigma)$ -periodic. The fundamental domain of $\Lambda(\mathbb{A}_\sigma)$ in \mathbb{L}_σ is denoted by F . For m sufficiently large, the number of $\frac{1}{m}\Lambda(\mathbb{A}_\sigma)$ -translates of $\frac{1}{m}F$ contained in the lift Ω of $\overline{\Omega}$ to \mathbb{A}_σ is $m^d \text{vol}(\Omega) / \text{vol}(F) + O(m^{d-1})$. By (31) and (33) inserted in (30), we get

$$(34) \quad S_m = \frac{\text{vol}(\overline{\Omega})}{\text{vol}(F)} \cdot \sum_{\mathbf{u}'} \vartheta(\overline{\Delta}(\mathbf{u}'), \overline{\sigma}) \deg_{\tilde{\mathcal{L}}}(Y_{\mathbf{u}'}) + O\left(\frac{1}{m}\right),$$

where \mathbf{u}' ranges over the vertices of $\mathcal{C} \cap \mathbb{L}_\sigma$ contained in F . We claim that

$$(35) \quad \mu(\overline{\Omega}) = \frac{\text{vol}(\overline{\Omega})}{\text{vol}(\Lambda(\mathbb{A}_\sigma))} \cdot \sum_{\mathbf{u}'} \vartheta(\overline{\Delta}(\mathbf{u}'), \overline{\sigma}) \deg_{\tilde{\mathcal{L}}}(Y_{\mathbf{u}'}),$$

where \mathbf{u}' ranges over all vertices of $\mathcal{C} \cap \mathbb{L}_\sigma$ modulo $\Lambda(\mathbb{A}_\sigma)$. By (28), (30) and (34), it remains to show that the Z in (28) lying over non- $\overline{\Omega}$ -inner Y may be neglected. We have seen in Step 2 that the number of such Y is $O(m^{d-1})$. Formula (32) holds also for non- $\overline{\Omega}$ -inner Y and proves

$$m(Y, \tilde{\mathcal{X}}_m) \leq \frac{1}{[X' : X_{\mathbb{K}}^{\text{an}}]} \sum_{j=1}^N \text{ind}(\overline{\Delta}, \overline{f}_{j,\text{aff}}),$$

where $\overline{\Delta}$ is the d -codimensional polytope in $\frac{1}{m}\overline{\mathcal{C}}$ corresponding to Y . We have

$$\text{ind}(\overline{\Delta}, \overline{f}_{j,\text{aff}}) \leq \max_{\overline{\Delta}' } (\text{ind}(\overline{\Delta}', \overline{f}_{j,\text{aff}})),$$

where $\overline{\Delta}'$ is ranging over all d -codimensional simplices in $\overline{\mathcal{C}}$. This leads to a bound of $m(Y, \tilde{\mathcal{X}}_m)$ independent of Y and m . There is also such a bound for $m^{-d} \deg_{\tilde{\varphi}_m^*(\tilde{\mathcal{L}})}(Y)$. Indeed, this follows from projection formula as in (31) replacing $Y_{m\mathbf{u}}$ by the closure of a d -dimensional torus orbit of $\tilde{\mathcal{A}}$. Using the final remark in Step 2, the contribution of the Z in (28) lying over non- $\overline{\Omega}$ -inner Y is bounded by $O(\frac{1}{m})$ and therefore may be neglected in (28). This proves (35).

Step 4. Calculation of the sum in (35) in terms of the dual complex of $\mathcal{C} \cap \mathbb{L}_\sigma$.

We have chosen $\overline{\mathcal{C}}$ and the $(\mathbb{K}')^\circ$ -model \mathcal{L} of L from Lemma 8.4. Recall from its proof that \mathcal{L} was constructed by a strongly polyhedral convex

function $f_{\mathcal{L}}$ with respect to \mathcal{C} using Proposition 6.6 and Corollary 6.7. In particular, $f_{\mathcal{L}}(\mathbf{u}) = \mathbf{m}_{\Delta} \cdot \mathbf{u} + c_{\Delta}$ on $\Delta \in \mathcal{C}$ with $\mathbf{m}_{\Delta} \in \mathbb{Z}^n$. By A.4, $f_{\mathcal{L}}$ induces a dual complex $\mathcal{C}^{f_{\mathcal{L}}}$ in \mathbb{R}^n where the vertices (called pegs) are given by \mathbf{m}_{Δ} with Δ ranging over all n -dimensional polytopes of \mathcal{C} . Let \mathbf{u}' be a vertex of $\mathcal{C} \cap \mathbb{L}_{\sigma}$. Since $\overline{\mathcal{C}}$ is $\overline{\Sigma}$ -generic, the unique polytope $\Delta(\mathbf{u}')$ of \mathcal{C} containing \mathbf{u}' in its relative interior is d -codimensional. It follows that the dual polytope $\Delta(\mathbf{u}')^{f_{\mathcal{L}}}$, given by the vertices $\{\mathbf{m}_{\tau} \mid \tau \in \text{star}_n(\Delta(\mathbf{u}'))\}$, is d -dimensional (see A.4 and A.5). We claim that

$$(36) \quad \text{deg}_{\tilde{\mathcal{L}}}(Y_{\mathbf{u}'}) = d! \cdot \text{vol}(\Delta(\mathbf{u}')^{f_{\mathcal{L}}}) \cdot \text{vol}(\mathbb{Z}^n \cap \Delta(\mathbf{u}')^{\perp})^{-1},$$

where $\Delta(\mathbf{u}')^{\perp}$ denotes the orthogonal complement of $\Delta(\mathbf{u}')$. We would like to deduce it from the corollary on p. 112 in [Fu2, §5.3]. We choose a vertex \mathbf{u}_0 of $\Delta(\mathbf{u}')$. For simplicity of notation, we may assume $\mathbf{u}_0 = \mathbf{0}$ and $f_{\mathcal{L}}(\mathbf{0}) = 0$. The general case is obtained by a transformation of the \mathbf{x} -coordinates which yields a translation of the \mathbf{u} -coordinates. Let \mathcal{F} be the complete fan of cones generated by the polytopes of \mathcal{C} with vertex $\mathbf{0}$. Let Y_0 be the irreducible component of $\tilde{\mathcal{A}}$ corresponding to the vertex $\mathbf{u}_0 = \mathbf{0}$ (see Proposition 6.3). It follows from Proposition 4.7 that Y_0 is a toric variety with fan \mathcal{F} . The restriction of the ample line bundle $\tilde{\mathcal{L}}$ to Y_0 is isomorphic to $\mathcal{O}(D)$ for a Cartier divisor D on Y_0 which is equivariant with respect to the torus action. In the theory of toric varieties, such an ample divisor defines a continuous piecewise linear function ψ_D on the fan \mathcal{F} and a polytope $P_D := \{\omega \in \mathbb{R}^n \mid \omega \cdot \mathbf{u} \geq \psi_D(\mathbf{u}) \ \forall \mathbf{u} \in \mathbb{R}^n\}$ (see [Fu2, §3.4]). The proof of Corollary 6.7 gives an explicit choice of D and shows that $f_{\mathcal{L}} = -\psi_D$ on every $\Delta \in \text{star}_n(\mathbf{u}_0)$. By formula (45) in A.5, we conclude that $-P_D = \{\mathbf{u}_0\}^{f_{\mathcal{L}}}$. The polytope considered in the corollary on p. 112 of [Fu2] is equal to $P_D \cap (-\mathbf{m}_{\Delta} + \Delta(\mathbf{u}')^{\perp})$ for any $\Delta \in \text{star}_n(\Delta(\mathbf{u}'))$ and formula (46) in A.5 shows that this polytope is equal to $-\Delta(\mathbf{u}')^{f_{\mathcal{L}}}$. So we may apply this corollary and (36) follows.

We may also consider the complex $\mathcal{C} \cap \mathbb{L}_{\sigma}$ in \mathbb{L}_{σ} . Clearly, $f_{\mathcal{L}}$ restricts to a strongly polyhedral convex function g on \mathbb{L}_{σ} and we may form the dual complex $(\mathcal{C} \cap \mathbb{L}_{\sigma})^g$ in \mathbb{L}_{σ}^* which is $\Lambda(\mathbb{A}_{\sigma})^L$ -periodic and hence covers \mathbb{L}_{σ}^* (use [McM, Theorem 3.1]). Let P be the dual map of $\mathbb{L}_{\sigma} \hookrightarrow \mathbb{R}^n$. Since \mathcal{C} is transversal to \mathbb{L}_{σ} , the pegs of $\mathcal{C} \cap \mathbb{L}_{\sigma}$ are equal to $\{P(\mathbf{m}_{\tau}) \mid \tau \in \mathcal{C}, \dim(\tau) = n, \tau \cap \mathbb{L}_{\sigma} \neq \emptyset\}$. By A.4, the dual polytope $\{\mathbf{u}'\}^g$ in $(\mathcal{C} \cap \mathbb{L}_{\sigma})^g$ of the vertex \mathbf{u}' of $\mathcal{C} \cap \mathbb{L}_{\sigma}$ is given by the vertices $\{P(\mathbf{m}_{\tau}) \mid \tau \in \text{star}_n(\mathbf{u}')\}$ and hence we have

$$(37) \quad P(\Delta(\mathbf{u}')^{f_{\mathcal{L}}}) = \{\mathbf{u}'\}^g.$$

By A.5, $\Delta(\mathbf{u}')^{f_{\mathcal{L}}}$ is a polytope of full dimension in $\mathbf{m}_{\Delta} + \Delta(\mathbf{u}')^{\perp}$ and hence the change of volumes in (37) is given by the factor $\text{vol}(P(\mathbb{Z}^n \cap \Delta(\mathbf{u}')^{\perp})) \cdot \text{vol}(\mathbb{Z}^n \cap \Delta(\mathbf{u}')^{\perp})^{-1}$. Now (36) yields

$$(38) \quad \text{deg}_{\tilde{\mathcal{L}}}(Y_{\mathbf{u}'}) = d! \cdot \text{vol}(\{\mathbf{u}'\}^g) \cdot \text{vol}(P(\mathbb{Z}^n \cap \Delta(\mathbf{u}')^{\perp}))^{-1}.$$

Duality of lattices and the definitions of the multiplicity $\vartheta(\bar{\Delta}, \bar{\sigma})$ and the index in 7.13 imply

$$\begin{aligned} \vartheta(\bar{\Delta}(\mathbf{u}'), \bar{\sigma}) &= \frac{1}{[X' : X_{\mathbb{K}}^{\text{an}}]} \sum_{j \in J(\bar{\sigma})} [Z^d : \hat{\ell}_j^{(0)}(P(\mathbb{Z}^n \cap \Delta(\mathbf{u}')^\perp))] \\ &= \frac{1}{[X' : X_{\mathbb{K}}^{\text{an}}]} \sum_{j \in J(\bar{\sigma})} [Z^d : \hat{\ell}_j^{(0)}(\Lambda(\mathbb{A}_\sigma)^*)] \cdot \frac{\text{vol}(P(\mathbb{Z}^n \cap \Delta(\mathbf{u}')^\perp))}{\text{vol}(\Lambda(\mathbb{A}_\sigma)^*)} \\ &= \vartheta(\bar{\sigma}) \cdot \frac{\text{vol}(P(\mathbb{Z}^n \cap \Delta(\mathbf{u}')^\perp))}{\text{vol}(\Lambda(\mathbb{A}_\sigma)^*)}. \end{aligned}$$

Using this and (38) in (35), we get

$$\mu(\bar{\Omega}) = d! \cdot \vartheta(\bar{\sigma}) \cdot \sum_{\mathbf{u}'} \text{vol}(\{\mathbf{u}'\}^g) \cdot \frac{\text{vol}(\bar{\Omega})}{\text{vol}(\Lambda(\mathbb{A}_\sigma)^*) \text{vol}(\Lambda(\mathbb{A}_\sigma))},$$

where \mathbf{u}' ranges over the vertices of $\mathcal{C} \cap \mathbb{L}_\sigma$ modulo $\Lambda(\mathbb{A}_\sigma)$. The theorem follows now from the $\Lambda(\mathbb{A}_\sigma)^L$ -periodicity of the covering $(\mathcal{C} \cap \mathbb{L}_\sigma)^g$ of \mathbb{L}_σ^* . \square

Remark 8.7. For arbitrary line bundles $\bar{L}_1, \dots, \bar{L}_d$ on A endowed with canonical metrics, Theorem 1.3 yields that μ is a piecewise Haar measure on $\text{val}(X_{\mathbb{K}}^{\text{an}})$. Indeed, every line bundle on A is isomorphic to the “difference” of two ample line bundles and multilinearity yields the claim.

9. Generalizations

First, we relate tropical varieties to the skeleton of a strictly semistable \mathbb{K}° -model. We use it to prove Theorem 1.2. Then we describe the canonical measures on a closed subvariety of a totally degenerate abelian variety generalizing Theorem 8.6.

9.1. Berkovich has shown in [Ber5, §5], that a strongly non-degenerate polystable fibration over \mathbb{K}° has a canonical polytopal subset called the skeleton. For simplicity, we restrict its description to the case of a strictly semistable formal scheme \mathcal{X}' over \mathbb{K}° :

By Proposition 2.11, every irreducible component Y of $\tilde{\mathcal{X}}'$ induces a Cartier divisor D_Y on \mathcal{X}' with $\text{cyc}(D_Y) = v(\pi)Y$ just by lifting the local equations $\tilde{\gamma}$ for Y in $\tilde{\mathcal{X}}'$, where $\pi \in K^{\circ\circ}$ is from 2.10. If $X' := (\mathcal{X}')^{\text{an}}$ is connected, then we may choose π independent of Y . For simplicity of notation, this will be assumed in the following.

By 3.2 and 3.4, there is a global section s_Y of $O_{X'}$ and a formal metric $\|\cdot\|_Y$ on $O_{X'}$ such that $D_Y = \widehat{\text{div}}(s_Y)$. Let I be the set of irreducible components of \mathcal{X}' . Then we get an analogue of tropical geometry by considering

$$\text{val} : X' \longrightarrow \mathbb{R}^I, \quad x' \mapsto (-\log \|s_Y(x')\|)_{Y \in I}.$$

We have seen in Proposition 7.8 that every stratum S of $\tilde{\mathcal{X}}'$ gives rise to a canonical simplex

$$\Delta_S := \left\{ \mathbf{u} = (u_Y)_{Y \in I} \in \mathbb{R}^I \mid u_Y = 0 \text{ if } S \cap Y = \emptyset, \sum_{Y \cap S \neq \emptyset} u_Y = v(\pi) \right\}.$$

As in the proof of Proposition 5.2, we deduce that $\text{val}(X')$ is covered by Δ_S , where S is ranging over the set $\text{str}(\tilde{\mathcal{X}}')$ of strata. The strata of dimension $d := \dim(X')$ are in one-to-one correspondence with I and hence with the vertices $(0, \dots, v(\pi), \dots, 0)$, where just the Y -th entry is non-zero. Note however that different lower dimensional strata may induce the same canonical simplex. To omit this, we define the abstract metrized polytopal set

$$\mathbb{D}(\mathcal{X}') := \left(\coprod_{S \in \text{str}(\tilde{\mathcal{X}}')} (\Delta_S, S) \right) / \sim .$$

Here, (Δ_S, S) is isometrically identified with Δ_S and $(\mathbf{u}_1, S_1) \sim (\mathbf{u}_2, S_2)$ if there is $S_3 \in \text{str}(\tilde{\mathcal{X}}')$ such that the closure of S_3 contains $S_1 \cup S_2$ and if $\mathbf{u}_1 = \mathbf{u}_2 \in \Delta_{S_3}$. The set $\mathbb{D}(\mathcal{X}')$ reflects the incidence of strata closures as there is a bijective correspondence between $\text{str}(\tilde{\mathcal{X}}')$ and the simplices (Δ_S, S) of $\mathbb{D}(\mathcal{X}')$. Note that we may lift val to a continuous surjective map

$$\text{Val} : X' \longrightarrow \mathbb{D}(\mathcal{X}'), \quad x' \mapsto (\text{val}(x'), S(\tilde{x}')),$$

where $S(\tilde{x}')$ is the stratum containing the reduction \tilde{x}' of x' .

Berkovich introduces a partial ordering \leq on X' (depending on \mathcal{X}') by $x' \leq y'$ if, for every affinoid algebra \mathcal{A} with étale morphism $\varphi : \mathcal{X} = \text{Spf}(\mathcal{A}^\circ) \rightarrow \mathcal{X}'$ and $x \in (\varphi^{\text{an}})^{-1}(x')$, there is $y \in (\varphi^{\text{an}})^{-1}(y')$ with $|f(x)| \leq |f(y)|$ for all $f \in \mathcal{A}$. The set of maximal points with respect to this ordering is called the *skeleton* $S(\mathcal{X}')$. By [Ber5, Theorem 5.1.1], the map Val restricts to a homeomorphism of $S(\mathcal{X}')$ onto $\mathbb{D}(\mathcal{X}')$. We use it to endow $S(\mathcal{X}')$ with the structure of a metrized polytopal set, i.e. we identify $S(\mathcal{X}')$ with $\mathbb{D}(\mathcal{X}')$.

Proposition 9.2. *Let \mathcal{X}' be a strictly semistable formal scheme over \mathbb{K}° with generic fibre X' and let A be a totally degenerate abelian variety over \mathbb{K} , i.e. $A^{\text{an}} = (\mathbb{G}_m^n)_{\mathbb{K}}^{\text{an}}/M$ for a lattice M . For a morphism $f : X' \rightarrow A$ and $\Lambda := \text{val}(M)$, there is a unique continuous map $\bar{f}_{\text{aff}} : S(\mathcal{X}') \rightarrow \mathbb{R}^n/\Lambda$ with*

$$(39) \quad \bar{f}_{\text{aff}} \circ \text{Val} = \overline{\text{val}} \circ f.$$

Moreover, f_{aff} is an affine map on every simplex (Δ_S, S) , $S \in \text{str}(\tilde{\mathcal{X}}')$.

Proof. By 7.3 and Proposition 7.8, we get affine maps satisfying (39) on every simplex (Δ_S, S) of $S(\mathcal{X}')$. They fit to define \bar{f}_{aff} on $S(\mathcal{X}')$. Uniqueness follows from surjectivity of Val . \square

Proof of Theorem 1.2. We may assume that X' is irreducible. By Chevalley's theorem, $f(X')$ contains an open dense subset of a d -dimensional closed subvariety X of A . Since $\overline{\text{val}}$ is continuous, we conclude that

$$\overline{\text{val}}(X^{\text{an}}) = \overline{\text{val}}(f(X')^{\text{an}}).$$

By Proposition 9.2, we have $\overline{\text{val}}(X^{\text{an}}) = \overline{f}_{\text{aff}}(S(\mathcal{X}'))$. The tropical variety $\overline{\text{val}}(X^{\text{an}})$ is d -dimensional (Theorem 6.9) and hence $S(\mathcal{X}')$ contains a simplex (Δ_S, S) of dimension at least d . The vertices of Δ_S correspond to irreducible components of $\tilde{\mathcal{X}}'$ containing S . □

Remark 9.3. Theorem 1.2 is also true analytically in the style of Theorem 5.8 with A replacing \mathbb{G}_m^n . It wouldn't be difficult to deduce Theorem 1.2 directly from Theorem 5.8.

Theorem 9.4. *If A is an arbitrary abelian variety over \mathbb{K} with t -dimensional formal abelian scheme \mathcal{B} in the Raynaud extension E of A (see [BL2, §1]), then Theorem 1.2 holds with $1 - t + \dim f(X')$ replacing $1 + \dim f(X')$.*

Proof. We define val on E as in [BL2, p. 656]. Note that E is locally trivial over \mathcal{B}^{an} and hence we deduce easily from Theorem 6.9 that $\overline{\text{val}}(X)$ is at least of dimension $\dim(X) - t$. Since Proposition 9.2 generalizes to this context, we can follow the proof of Theorem 1.2 to get the claim. □

9.5. For the remaining part of this section, we consider a field K with a discrete valuation v and we assume that \mathbb{K} is the completion of the algebraic closure of the completion of K with respect to v . The unique extension of v to a valuation of \mathbb{K} is also denoted by v with corresponding absolute value $|\cdot| := e^{-v}$.

Let A be an abelian variety over K which is totally degenerate over \mathbb{K} , i.e. $A_{\mathbb{K}}^{\text{an}} = (\mathbb{G}_m^n)_{\mathbb{K}}^{\text{an}}/M$ for a lattice M . Again, let $\Lambda := \text{val}(M)$ be the corresponding complete lattice in \mathbb{R}^n . Let X be a closed geometrically integral d -dimensional closed subvariety of A .

For canonically metrized ample line bundles $\overline{L}_1, \dots, \overline{L}_d$ on A , we want to describe the canonical measure $c_1(\overline{L}_1|_X) \wedge \dots \wedge c_1(\overline{L}_d|_X)$ on X^{an} . By de Jong's alteration theorem [dJ, Theorem 6.5], there is an alteration $f : X' \rightarrow X_{\mathbb{K}}^{\text{an}}$ and a strictly semistable formal \mathbb{K}° -model \mathcal{X}' of X' .

A d -dimensional simplex (Δ_S, S) of $S(\mathcal{X}')$, $S \in \text{str}(\tilde{\mathcal{X}}')$, is called *non-degenerate* with respect to f if $f_{\text{aff}}(\Delta_S)$ is also d -dimensional. Then S is a \mathbb{K} -rational point contained in $d + 1$ -irreducible components Y_0, \dots, Y_d of $\tilde{\mathcal{X}}'$. If u_0, \dots, u_d denote the corresponding coordinates, then Δ_S is given by $\{u_0 + \dots + u_d = v(\pi)\}$. We consider the d -dimensional standard simplex

$$\Sigma_S := \{\mathbf{u} \in \mathbb{R}_+^d \mid u_1 + \dots + u_d \leq v(\pi)\}$$

and the affine map $f_{\text{aff}}^{(0)} : \Sigma_S \rightarrow \mathbb{R}^n$ given by

$$f_{\text{aff}}^{(0)}(u_1, \dots, u_d) = f_{\text{aff}}(u_0, \dots, u_d), \quad (u_0, \dots, u_d) \in \Delta_S.$$

If we extend $f_{\text{aff}}^{(0)} - f_{\text{aff}}^{(0)}(\mathbf{0})$, then we get an associated injective linear map $\ell_S^{(0)} : \mathbb{R}^d \rightarrow \mathbb{R}^n$ as in 7.13. It is defined over \mathbb{Z} and hence $\Lambda_S := (\ell_S^{(0)})^{-1}(\Lambda)$ is a complete rational lattice in \mathbb{R}^d . The positive definite bilinear form b_j associated to L_j induces a complete lattice

$$\Lambda_S^{L_j} := \{b_j(\ell_S^{(0)}(\cdot), \lambda) \mid \lambda \in \Lambda\}$$

on $(\mathbb{R}^d)^* = \mathbb{R}^d$. We denote by vol the (mixed) volume with respect to the Lebesgue measure on \mathbb{R}^d . By Corollary 3.9, the following result describes the canonical measure on $X_{\mathbb{K}}^{\text{an}}$:

Theorem 9.6. *Under the hypothesis of 9.5, the measure $\mu := c_1(f^*(\bar{L}_1)) \wedge \dots \wedge c_1(f^*(\bar{L}_d))$ is supported in the union of the non-degenerated simplices (Δ_S, S) of $S(\mathcal{X}')$. For a measurable subset Ω of such a simplex, we have*

$$\mu(\Omega) = d! \cdot \frac{\text{vol}(\Lambda_S^{L_1}, \dots, \Lambda_S^{L_d})}{\text{vol}(\Lambda_S)} \cdot \text{vol}(\Omega).$$

Proof. We follow the steps of the proof of Theorem 8.6. We may assume that $\bar{L} := \bar{L}_1 = \dots = \bar{L}_d$ and L symmetric. We use the same $\bar{\Sigma}, \bar{\mathcal{C}}, \mathcal{A}, \mathcal{L}, \mathcal{A}_m$ and \mathcal{X}_m . The main difference is that the role of $\mathcal{X}_m^{\text{f-an}}$ is replaced by the minimal formal analytic structure \mathcal{X}'_m on X' which refines $(\mathcal{X}')^{\text{f-an}}$ such that f extends to a morphism $\phi_m : \mathcal{X}'_m \rightarrow \mathcal{A}_m^{\text{f-an}}$. Note that we obtain \mathcal{X}'_m and ϕ_m by applying 7.4 to $\mathcal{C}_m := \frac{1}{m}\mathcal{C}$ instead of \mathcal{C} . In this sense, we may use in the following the description and the properties of \mathcal{X}'_m from Sect. 7. Similarly as in Step 1, we have the following weak limit of regular Borel measures on X' :

$$(40) \quad \mu = \lim_{m \rightarrow \infty} m^{-2d} \sum_Z \text{deg}_{\tilde{\phi}_m^*(\tilde{\mathcal{L}})}(Z) \delta_{\xi_Z},$$

where Z ranges over all irreducible components of $\tilde{\mathcal{X}}'_m$. Note that $\text{Val}(\xi_Z)$ is in the relative interior of a simplex $\Delta := (\Delta_S, S)$ of $S(\mathcal{X}')$ for a unique $S \in \text{str}(\tilde{\mathcal{X}}')$. If Δ is degenerate with respect to f , then we claim

$$(41) \quad \text{deg}_{\tilde{\phi}_m^*(\tilde{\mathcal{L}})}(Z) = 0.$$

By definition, the simplex $\rho := f_{\text{aff}}(\Delta)$ is contained in Σ and has dimension $< d$. Formula (39) yields $\text{val}(f(\xi_Z)) \in \bar{\rho}$. By projection formula, it is enough to show that $Y := \tilde{\phi}_m(Z)$ is no irreducible component of $\tilde{\mathcal{X}}_m$. We argue by contradiction. We apply Theorem 6.10 to the irreducible component Y using that $\bar{\mathcal{C}}_m$ is transversal to $\text{val}(X_{\mathbb{K}}^{\text{an}})$. We conclude that Y corresponds to an equivalence class of transversal vertices, i.e. there is a unique d -codimensional $\bar{\Delta}_m \in \bar{\mathcal{C}}_m$ such that the torus orbit in $\tilde{\mathcal{A}}_m$ associated to $\text{relint}(\bar{\Delta}_m)$ is dense in Y . Since $\bar{\mathcal{C}}_m$ is $\bar{\Sigma}$ -generic, we have $\bar{\rho} \cap \bar{\Delta}_m = \emptyset$. But $(f(\xi_Z))^\sim = \tilde{\phi}_m(\xi_Z)$ is the generic point of Y and hence contained in

the above open dense torus orbit. This means $\overline{\text{val}}(f(\xi_Z)) \in \text{relint}(\overline{\Delta}_m)$ (see Proposition 6.3) leading to a contradiction and proving (41).

Let $\Delta = (\Delta_S, S)$ be a canonical simplex of $S(\mathcal{X}')$ which is non-degenerate with respect to f . We have seen in 9.5 that Δ_S is d -dimensional and that S is a \mathbb{K} -rational point contained in $d + 1$ irreducible components of $\tilde{\mathcal{X}}'$. Now we use Sect. 7, with Δ_S, Σ_S playing the role of $\Delta(d, \pi)$ and $\Sigma(d, \pi)$, to express properties of $\tilde{\mathcal{X}}'_m$ in terms of the polytopal decomposition $\mathcal{D}_m := (f_{\text{aff}}^{(0)})^{-1}(\mathcal{C}_m)$ of Σ_S (see 9.5). By Remark 7.6, the irreducible components Z of $\tilde{\mathcal{X}}'_m$ with $\text{Val}(\xi_Z) \in \text{relint}(\Delta)$ correspond bijectively to the vertices \mathbf{u}' of \mathcal{D}_m contained in $\text{relint}(\Sigma_S)$. For such a Z , the corresponding \mathbf{u}' is given by omitting the coordinate u_0 of $\mathbf{u} := \text{Val}(\xi_Z) \in \text{relint}(\Delta_S)$. Moreover, Z is isomorphic to the toric variety $Y_{\mathbf{u}'}$ associated to the vertex \mathbf{u}' and we will identify them later. This replaces Step 2.

Note that the point $\xi_{\mathbf{u}'}$ of Corollary 4.5 is in the skeleton of the formal scheme \mathcal{U}_{Σ_S} associated to the standard simplex Σ_S (see [Ber5, 4.2 and Theorem 4.3.1]). This skeleton consists of Σ_S itself. We have seen in Remark 7.6 that there is a formal affine neighbourhood \mathcal{U}' of S in \mathcal{X}' and an étale morphism $\phi_0 : \mathcal{U}' \rightarrow \mathcal{U}_{\Sigma_S}$ with $\mathbf{u}' = \text{val}(\phi_0(\xi_Z))$. This proves $\phi_0(\xi_Z) = \xi_{\mathbf{u}'}$ and hence $\xi_Z \in S(\mathcal{X}')$ (see [Ber5, Corollary 4.3.2]). By the identification in 9.1, Val is the identity on $S(\mathcal{X}')$ and hence $\xi_Z \in \Delta$. By (40) and (41), we conclude that the support of μ is contained in the union of the non-degenerate simplices with respect to f .

To prove the remaining formula, we may assume that Ω is a polytope contained in such a $\Delta = (\Delta_S, S)$. Note that $f_{\text{aff}}^{(0)}$ from 9.5 extends to an affine map $f_0 : \mathbb{R}^d \rightarrow \mathbb{R}^n$ which is also one-to-one. The polytopal decomposition $\mathcal{D} := f_0^{-1}(\mathcal{C})$ of \mathbb{R}^d is periodic with respect to the lattice Λ_S from 9.5. Similarly as in Step 3, we deduce from (40) the formula

$$(42) \quad \mu(\Omega) = \frac{\text{vol}(\Omega)}{\text{vol}(\Lambda_S)} \cdot \sum_{\mathbf{u}'} \text{deg}_{\tilde{\phi}^*(\tilde{\mathcal{L}})}(Y_{\mathbf{u}'}),$$

where \mathbf{u}' ranges over the vertices of \mathcal{D} modulo Λ_S . Since no multiplicities occur, the argument is easier here and will be omitted. We have seen in Step 4 that $f_{\mathcal{L}}$ is a strongly polyhedral convex function with respect to \mathcal{C} (Corollary 6.7) and hence $g := f_{\mathcal{L}} \circ f_0$ is a strongly polyhedral convex function with respect to \mathcal{D} . As in (36), we conclude that

$$(43) \quad \text{deg}_{\tilde{\phi}^*(\tilde{\mathcal{L}})}(Y_{\mathbf{u}'}) = d! \cdot \text{vol}(\{\mathbf{u}'\}^g).$$

If \mathbf{u}' ranges over the vertices of \mathcal{D} , the rational polytopes $\{\mathbf{u}'\}^g$ are the d -dimensional polytopes of the dual polytopal decomposition \mathcal{D}^g of \mathbb{R}^d . Since \mathcal{D}^g is Λ_S^{\perp} -periodic, the formula in the claim follows from (42) and (43). \square

Corollary 9.7. *If we do not require that L_0, \dots, L_d are ample in Theorem 9.6, then μ is still supported in the union of the non-degenerate*

simplices (Δ_S, S) of $S(\mathcal{X}')$ and the restriction of μ to such a simplex is still a Haar measure.

Proof. This follows from Theorem 9.6 by multilinearity as in Remark 8.7. □

Example 9.8. We consider the special case $X = A$ in Theorem 9.6. Using $A_{\mathbb{K}}^{\text{an}} = (\mathbb{G}_m^n)_{\mathbb{K}}^{\text{an}}/M$, the points $\xi_{\mathbf{u}}$ from Corollary 4.5 form a canonical subset $S(A)$ of $A_{\mathbb{K}}^{\text{an}}$ which we call the *skeleton* of A . By [Ber1, Example 5.2.12 and Theorem 6.5.1], this is a closed subset of $A_{\mathbb{K}}^{\text{an}}$ and $\overline{\text{val}}$ restricts to a homeomorphism from $S(A)$ onto \mathbb{R}^n/Λ which we use for identification.

By a combinatorial result of Knudson and Mumford [KKMS, Chapt. III], there is a rational triangulation $\overline{\mathcal{C}}$ of \mathbb{R}^n/Λ (even refining any given rational polytopal decomposition) and $m \in \mathbb{N} \setminus \{0\}$ such that for every $\Delta \in \mathcal{C}$, the simplex $m\Delta$ is $\text{GL}(n, \mathbb{Z})$ -isomorphic to a \mathbb{Z}^n -translate of the standard simplex $\{\mathbf{u} \in \mathbb{R}_+^n \mid u_1 + \dots + u_n \leq 1\}$. Then the formal \mathbb{K}° -model \mathcal{A} of A associated to $\overline{\mathcal{C}}$ is strictly semistable. By the way, Künnemann generalized this construction to prove the existence of projective strictly semistable \mathbb{K}° -models for arbitrary abelian varieties (see [Ku1] and also the erratum in [Ku2, 5.8]). By the second step in the proof of Theorem 9.6, the skeleton of \mathcal{A} agrees with $S(A)$. We get a triangulation of $S(A)$ corresponding to $\overline{\mathcal{C}}$.

We apply Theorem 9.6 with $X' = X = A$ and $\mathcal{X}' = \mathcal{A}$. Note that the non-degenerate simplices (Δ_S, S) correspond to the n -dimensional simplices of $\overline{\mathcal{C}}$ and hence the lattice $\Lambda_S^{L_j}$ does not depend on the choice of the stratum S . We conclude that the measure μ from Theorem 9.6 is supported in $S(A)$ and corresponds to a Haar measure on \mathbb{R}^n/Λ . By Proposition 3.12, it has total measure $\text{deg}_{L_1, \dots, L_n}(A)$. Using multilinearity for non-ample line bundles, this proves the following result:

Corollary 9.9. *Let $\overline{L}_1, \dots, \overline{L}_n$ be canonically metrized line bundles on the totally degenerate abelian variety A from above. Then $c_1(\overline{L}_1) \wedge \dots \wedge c_1(\overline{L}_n)$ is supported in the skeleton $S(A)$ and corresponds to the Haar measure on \mathbb{R}^n/Λ with total measure $\text{deg}_{L_1, \dots, L_n}(A)$.*

A. Convex geometry

In this appendix, we gather notions and results from convex geometry.

A.1. A *polyhedron* Δ in \mathbb{R}^n is an intersection of finitely many closed half-spaces $\{\mathbf{u} \in \mathbb{R}^n \mid \mathbf{m}_i \cdot \mathbf{u} \geq c_i\}$. We say that Δ is Γ -rational if we may choose all $\mathbf{m}_i \in \mathbb{Z}^n$ and all $c_i \in \Gamma$. A *closed face* of Δ is either Δ itself or has the form $H \cap \Delta$ where H is the boundary of a closed half-space containing Δ . An *open face* of Δ is a closed face without all its properly contained closed faces. We denote by $\text{int}(\Delta)$ the topological interior of Δ in \mathbb{R}^n and by $\text{relint}(\Delta)$ the unique open face of Δ which is dense in Δ .

A bounded polyhedron is called a *polytope*. By linear algebra, a polytope is Γ -rational if and only if all vertices are in Γ^n and the edges have rational slopes. A (Γ -rational) *polytopal set* S in \mathbb{R}^n is a finite union of (Γ -rational) polytopes in \mathbb{R}^n . S is said to have *pure dimension* d if all maximal polytopes of S have dimension d .

A *polytopal complex* \mathcal{C} in \mathbb{R}^n is a locally finite set of polytopes such that

- (a) $\Delta \in \mathcal{C} \Rightarrow$ all closed faces of Δ are in \mathcal{C} ;
- (b) $\Delta, \sigma \in \mathcal{C} \Rightarrow \Delta \cap \sigma$ is either empty or a closed face of Δ and σ .

The polytopal complex is called *Γ -rational* if every $\Delta \in \mathcal{C}$ is Γ -rational. A *polytopal decomposition* of $S \subset \mathbb{R}^n$ is a polytopal complex with $S = \bigcup_{\Delta \in \mathcal{C}} \Delta$. It is easy to see that every Γ -rational polytopal set has a finite Γ -rational polytopal decomposition. A *triangulation* of S is a polytopal decomposition consisting only of simplices. A polytopal complex \mathcal{D} *subdivides* \mathcal{C} if every polytope Δ in \mathcal{C} has a polytopal decomposition in \mathcal{D} .

A *cone* σ in \mathbb{R}^n is centered at $\mathbf{0}$, i.e. it is characterized by $\mathbb{R}_+\sigma = \sigma$. Its *dual* is defined by

$$\check{\sigma} := \{\mathbf{u}' \in \mathbb{R}^n \mid \mathbf{u} \cdot \mathbf{u}' \geq 0 \forall \mathbf{u} \in \sigma\}.$$

A.2. Let S be a locally finite union of polytopes in \mathbb{R}^n . The *local cone* $\text{LC}_{\mathbf{u}}(S)$ at \mathbf{u} is defined by

$$\text{LC}_{\mathbf{u}}(S) := \{\mathbf{w} + \mathbf{u} \mid \mathbf{w} \in \mathbb{R}^n, [0, \varepsilon)\mathbf{w} + \mathbf{u} \subset S \text{ for some } \varepsilon > 0\}.$$

Then S is said to be *concave in* $\mathbf{u} \in S$ if the convex hull of $\text{LC}_{\mathbf{u}}(S)$ is an affine subspace of \mathbb{R}^n . The set S is called *totally concave* if it is concave in all $\mathbf{u} \in S$.

Let S be a locally finite union of d -dimensional polytopes in \mathbb{R}^n . A polytopal decomposition \mathcal{C} of \mathbb{R}^n is said to be *transversal* to S if the polytopal set $\Delta \cap S$ is either empty or of pure dimension $d - \text{codim}(\Delta)$ for every $\Delta \in \mathcal{C}$.

If \mathcal{C} is transversal to S and $\Delta \in \mathcal{C}$ is of codimension d , then $\Delta \cap S$ consists of finitely many points. Such points are called *transversal vertices* of $\mathcal{C} \cap S$. Two transversal vertices are called *equivalent* if they are contained in the same open face of \mathcal{C} .

A.3. Let \mathcal{C} be a polytopal decomposition of \mathbb{R}^n . A *strongly polyhedral convex function* f with respect to \mathcal{C} is a convex function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ such that the n -dimensional $\Delta \in \mathcal{C}$ are the maximal subsets of \mathbb{R}^n where f is affine, i.e. there are $\mathbf{m}_{\Delta} \in \mathbb{R}^n, c_{\Delta} \in \mathbb{R}$ with

$$f(\mathbf{u}) = \mathbf{m}_{\Delta} \cdot \mathbf{u} + c_{\Delta}$$

for every $\mathbf{u} \in \Delta$. The vector \mathbf{m}_{Δ} is called the *peg* of Δ . Recall that f is *convex* if

$$(44) \quad f(rx + sy) \leq rf(x) + sf(y)$$

for $x, y \in \mathbb{R}^n$ and $r, s \in [0, 1]$ with $r + s = 1$. **Warning:** In the theory of toric varieties, convex functions are defined the opposite way! Here, we follow the terminology from analysis and we call a convex function f *strictly convex* if we have $<$ in (44) for $x \neq y$ and $0 < r < 1$.

A.4. There is a *dual complex* \mathcal{C}^f of \mathcal{C} realized in \mathbb{R}^n with respect to a strongly polyhedral convex function f : For $\sigma \in \mathcal{C}$, let

$$\text{star}(\sigma) := \{\Delta \in \mathcal{C} \mid \sigma \subset \Delta\}, \quad \text{star}_n(\sigma) := \{\Delta \in \mathcal{C} \mid \sigma \subset \Delta, \dim(\Delta) = n\}.$$

There is a dual polytope σ^f in \mathbb{R}^n with vertices equal to $\{\mathbf{m}_\Delta \mid \Delta \in \text{star}_n(\sigma)\}$. These polytopes form the dual complex \mathcal{C}^f [McM, Theorem 3.1]. It may happen that \mathcal{C}^f does not cover \mathbb{R}^n . For more details and biduality, we refer to [McM].

A.5. For a vertex \mathbf{u}_0 of \mathcal{C} , the polytope $\{\mathbf{u}_0\}^f$ depends only on the local cones $\text{LC}_{\mathbf{u}_0}(\Delta)$, $\Delta \in \text{star}_n(\mathbf{u}_0)$, and the pegs on $\text{star}_n(\mathbf{u}_0)$. We have

$$(45) \quad \{\mathbf{u}_0\}^f = \{\omega \in \mathbb{R}^n \mid \mathbf{u} \in \Delta \in \text{star}_n(\mathbf{u}_0) \Rightarrow \omega \cdot (\mathbf{u} - \mathbf{u}_0) \leq \mathbf{m}_\Delta \cdot (\mathbf{u} - \mathbf{u}_0)\}.$$

This follows from [Oda, A.3 and Lemma 2.12]. For a d -dimensional polytope σ with vertex \mathbf{u}_0 , we have

$$(46) \quad \sigma^f = \{\mathbf{u}_0\}^f \cap (\mathbf{m}_\Delta + \sigma^\perp), \quad \dim(\sigma) + \dim(\sigma^f) = n,$$

where Δ is any element of $\text{star}_n(\sigma)$ (see [Oda, Corollary A.19]).

A.6. For compact convex subsets P and Q of \mathbb{R}^n , we have the *Minkowski sum*

$$P + Q := \{\mathbf{u} + \mathbf{u}' \mid \mathbf{u} \in P, \mathbf{u}' \in Q\}.$$

This is again a compact convex set. Similarly or by associativity, we define the Minkowski sum for more than two summands. For a non-negative real number λ , we use $\lambda P := \{\lambda \mathbf{u} \mid \mathbf{u} \in P\}$. There is a unique symmetric real function $V(P_1, \dots, P_n)$ on the set of compact convex subsets of \mathbb{R}^n which is multilinear with respect to the above operations and which satisfies

$$V(P, \dots, P) = \text{vol}(P).$$

The number $V(P_1, \dots, P_n)$ is called the *mixed volume* of P_1, \dots, P_n . The mixed volume is monotone increasing with respect to inclusion and hence it is non-negative. Moreover, it follows from translation invariance that $V(P_1, \dots, P_n) > 0$ if all the P_j have non-empty interiors. For proofs and more details, we refer to [BZ, Chapt. 4].

References

- [Ber1] Berkovich, V.G.: Spectral theory and analytic geometry over non-archimedean fields. *Math. Surv. Monogr.*, vol. 33. Am. Math. Soc., Providence, RI (1990)
- [Ber2] Berkovich, V.G.: Étale cohomology for non-archimedean analytic spaces. *Publ. Math., Inst. Hautes Étud. Sci.* **78**, 5–161 (1993)
- [Ber3] Berkovich, V.G.: Vanishing cycles for formal schemes. *Invent. Math.* **115**(3), 539–571 (1994)
- [Ber4] Berkovich, V.G.: Smooth p -adic analytic spaces are locally contractible. *Invent. Math.* **137**(1), 1–84 (1999)
- [Ber5] Berkovich, V.G.: Smooth p -adic analytic spaces are locally contractible. II. In: Adolphson, A. et al. (eds.) *Geometric aspects of Dwork theory*, vol. I, pp. 293–370. de Gruyter, Berlin (2004)
- [BiGr] Bieri, R., Groves, J.R.J.: The geometry of the set of characters induced by valuations. *J. Reine Angew. Math.* **347**, 168–195 (1984)
- [BoGu] Bombieri, E., Gubler, W.: *Heights in Diophantine geometry*. New Mathematical Monographs, vol. 4. Cambridge University Press, Cambridge (2006)
- [Bo] Bosch, S.: Zur Kohomologietheorie rigid analytischer Räume. *Manuscr. Math.* **20**, 1–27 (1977)
- [BGR] Bosch, S., Güntzer, U., Remmert, R.: *Non-Archimedean analysis. A systematic approach to rigid analytic geometry*. Grundlehren Math. Wiss., vol. 261. Springer, Berlin (1984)
- [BL1] Bosch, S., Lütkebohmert, W.: Néron models from the rigid analytic viewpoint. *J. Reine Angew. Math.* **364**, 69–84 (1986)
- [BL2] Bosch, S., Lütkebohmert, W.: Degenerating abelian varieties. *Topology* **30**(4), 653–698 (1991)
- [BL3] Bosch, S., Lütkebohmert, W.: Formal and rigid geometry. I: Rigid spaces. *Math. Ann.* **295**(2), 291–317 (1993)
- [BL4] Bosch, S., Lütkebohmert, W.: Formal and rigid geometry. II: Flattening techniques. *Math. Ann.* **296**(3), 403–429 (1993)
- [Bou] Bourbaki, N.: *Éléments de Mathématique*. Fasc. XXX. Algèbre commutative. Actual. Sci. Ind., vol. 1308, Chapt. 5: Entiers, Chapt. 6: Valuations. Hermann, Paris (1964)
- [BZ] Burago, Y.D., Zalgaller, V.A.: *Geometric inequalities*. Translated from Russian by Sossinsky, A.B. Grundlehren Math. Wiss., vol. 285. Springer Series in Soviet Mathematics. Springer, Berlin (1988)
- [Ch] Chambert-Loir, A.: Measure et équidistribution sur les espaces de Berkovich. *J. Reine Angew. Math.* **595**, 215–235 (2006)
- [EKL] Einsiedler, M., Kapranov, M., Lind, D.: Non-archimedean amoebas and tropical varieties. *J. Reine Angew. Math.* **601**, 139–157 (2006)
- [FvdP] Fresnel, J., van der Put, M.: *Rigid analytic geometry and its applications*. *Prog. Math.*, vol. 218. Birkhäuser, Boston, MA (2004)
- [Fu1] Fulton, W.: *Intersection theory*. *Ergeb. Math. Grenzgeb.*, 3. Folge, vol. 2. Springer, Berlin (1984)
- [Fu2] Fulton, W.: *Introduction to toric varieties*. The 1989 W.H. Roever Lectures in Geometry. *Ann. Math. Stud.*, vol. 131. Princeton University Press, Princeton, NJ (1993)
- [EGA IV] Grothendieck, A., Dieudonné, J.: *Éléments de géométrie algébrique*. IV: Étude locale des schémas et des morphismes de schémas (Quatrième partie). *Publ. Math., Inst. Hautes Étud. Sci.* **32**, 1–361 (1967)
- [Gu1] Gubler, W.: Heights of subvarieties over M -fields. In: Catanese, F. (ed.) *Arithmetic geometry*. Proceedings of a symposium, Cortona, 1994. *Symp. Math.*, vol. 37, pp. 190–227. Cambridge University Press, Cambridge (1997)
- [Gu2] Gubler, W.: Local heights of subvarieties over non-archimedean fields. *J. Reine Angew. Math.* **498**, 61–113 (1998)

- [Gu3] Gubler, W.: Local and canonical heights of subvarieties. *Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5)* **2**(4), 711–760 (2003)
- [Gu4] Gubler, W.: The Bogomolov conjecture for totally degenerate abelian varieties. *Invent. Math.* (2007). DOI: 10.1007/s00222-007-0049-y
- [Ha] Hartl, U.: Semi-stable models for rigid-analytic spaces. *Manuscr. Math.* **110**(3), 365–380 (2003)
- [dJ] de Jong, A.J.: Smoothness, semi-stability and alterations. *Publ. Math., Inst. Hautes Étud. Sci.* **83**, 51–93 (1996)
- [KKMS] Kempf, G., Knudsen, F., Mumford, D., Saint-Donat, B.: Toroidal embeddings. I. *Lect. Notes Math.*, vol. 339. Springer, Berlin-New York (1973)
- [Ku1] Künnemann, K.: Projective regular models for abelian varieties, semistable reduction, and the height pairing. *Duke Math. J.* **95**(1), 161–212 (1998)
- [Ku2] Künnemann, K.: Height pairings for algebraic cycles on abelian varieties. *Ann. Sci. Éc. Norm. Supér., IV. Ser.* **34**(4), 503–523 (2001)
- [Kl] Kleiman, S.L.: Toward a numerical theory of ampleness. *Ann. Math. (2)* **84**, 293–344 (1966)
- [McM] P. McMullen: Duality, sections and projections of certain Euclidean tilings. *Geom. Dedicata* **49**(2), 183–202 (1994)
- [Mi] Mikhalkin, G.: Amoebas of algebraic varieties and tropical geometry. In: Donaldson, S. et al. (eds.) *Different faces of geometry*. *Int. Math. Ser.*, vol. 3, pp. 257–300. Kluwer Academic/Plenum Publishers, New York, NY (2004)
- [Mu] Mumford, D.: An analytic construction of degenerating abelian varieties over complete rings. *Compos. Math.* **24**, 239–272 (1972)
- [Oda] Oda, T.: Convex bodies and algebraic geometry. An introduction to the theory of toric varieties. *Ergeb. Math. Grenzgeb., 3. Folge*, vol. 15. Springer, Berlin (1988)
- [Ru1] Rudin, W.: *Functional analysis*. McGraw-Hill Series in Higher Mathematics. McGraw-Hill, New York-Düsseldorf-Johannesburg (1973)
- [Ru2] Rudin, W.: *Real and complex analysis*, 3rd. edn. McGraw-Hill, New York, NY (1987)
- [Ul] Ullrich, P.: The direct image theorem in formal and rigid geometry. *Math. Ann.* **301**(1), 69–104 (1995)