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Robust rigidity for circle diffeomorphisms with singularities

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Abstract. We prove that under certain regularity conditions imposed on the renormalizations of two circle diffeomorphisms with singularities, their C^1-smooth equivalence follows from exponential convergence of those renormalizations. As an easy corollary, any two analytical critical circle maps with the same order of critical points and the same irrational rotation number are C^1-smoothly conjugate.

1 Introduction

The main result of this paper is given by the following theorem

Theorem 1. *Let T and $\tilde{T}$ be two analytical critical circle maps with the same order of critical points and the same irrational rotation number. Then they are C^1-smoothly conjugate to each other.*

This is a rigidity-type statement. J.-C. Yoccoz [14] has proved that an analytic critical circle map with irrational rotation number is topologically conjugate to a rigid rotation. Thus, any two critical circle maps with the same irrational rotation number, that is with the same combinatorial structure of trajectories, are topologically equivalent. The theorem above states that the topological equivalence implies smooth (C^1) equivalence provided that the critical points have the same order. Most importantly, the result does not depend on the Diophantine properties of rotations and C^1-rigidity holds for all irrational rotation numbers. It is well-known that similar statement is false in the case of diffeomorphisms. Arnold [2] constructed examples of analytic diffeomorphisms with irrational rotation numbers for which any conjugacy with a rigid rotation is essentially singular. The phenomenon of stronger rigidity for critical circle maps is connected with the presence of critical points which makes dynamical properties more rigid. We call this phenomenon *robust rigidity*.

Our approach is based on renormalizations. The renormalization method was used in circle dynamics for the last 20 years. The main idea is based on the following principle: two maps with the same irrational rotation number and with the same local structure of their singular points (if such points are present) belong to a same stable manifold for the renormalization operator. This means that two sequences of renormalizations constructed from these two maps converge (approach each other) with exponential rate. There exists a number of mathematical results related to the general principle formulated above. In the case of circle diffeomorphisms the corresponding statement is equivalent, in certain sense, to the Herman theory [4, 15, 9, 5]. One can show that renormalizations converge exponentially for all irrational rotation numbers which implies rigidity results under additional arithmetical conditions on the rotation numbers.

In the case of critical circle maps, that is circle homeomorphisms which are smooth everywhere except at one point where the first derivative vanishes, the convergence of renormalization was proved first by de Faria, de Melo for the rotation numbers of the bounded type. Yampolsky [13] proved that the exponential convergence of renormalizations holds for all irrational rotation numbers provided that the critical maps are analytic.

There are also results on convergence of renormalizations in the case of circle maps with break points (points with the jump of the first derivative). Such results has been proven in [7] for quadratic irrationals and were extended to the case of all irrational rotation numbers in [8].

Convergence of renormalization immediately implies rigidity for the rotation numbers of the bounded type (in fact, rigidity can be extended to a wider class of the rotation numbers) however it does not give robust rigidity. Renormalization scheme requires to consider the renormalized map f_n of the n-th step and to iterate it k_{n+1} times, where k_{n+1} is an integer in the continued fraction expansion of the rotation number ρ. For a very non-Diophantine rotation numbers k_{n+1} can be arbitrary large, so that after iteration two initially close maps f_n and $\tilde{f}_n$ are not close anymore. This is exactly what happens in the diffeomorphism case when both f_n and $\tilde{f}_n$ are close to linear maps with the slope 1. That is why rigidity in the case of diffeomorphisms holds only under additional Diophantine assumptions on the rotation numbers.

In the case of the critical circle maps, however, the maps f_n are essentially non-linear. The case of large k_{n+1} corresponds to appearance of exactly one point of almost tangency with the diagonal (almost parabolic point). Iterations of such maps are very regular and allow for a very precise asymptotic analysis. Thus the two ingredients in the proof of robust rigidity are convergence of renormalizations related to the continued-fraction expansion and regularity of iterations for maps with an almost parabolic tangency. The main difficulty here is to glue these two mechanisms together to prove a robust rigidity and that is exactly the main content of the paper. In fact, the main result is given by Theorem 2 which states that convergence of renormalizations plus regularity conditions on the renormalized maps f_n give robust rigidity. It can be shown (and we explain it in Sect. 4) that the regularity conditions always hold in the case of critical circle maps even without assuming analyticity or integer order of the critical point. However, the results on the convergence of renormalization are proven at the moment only in the case of critical circle maps with the order of critical point 3, 5, 7, ... Moreover, the full hyperbolicity of the renormalization horseshoe is rigorously proven only in the analytic case [13]. That is why our main result covers only this last case. However, the moment results on hyperbolicity of renormalization will be extended to the other cases of critical circle maps, the robust rigidity will follow immediately.

As we have mentioned above, convergence of renormalizations implies rigidity for critical circle maps with the rotation numbers of the bounded type. This was shown by de Faria and de Melo [3] who proved that any two C^∞ critical circle maps with the same order of the critical points (given by odd integer numbers) and with the same irrational rotation number of bounded type are $C^{1+\epsilon}$-smoothly conjugate to each other. It was also shown in [3] that $C^{1+\epsilon}$-rigidity cannot be extended even to the case of all Diophantine rotation numbers. In the note by Avila [1] similar result is proven in the analytic case. In fact, it follows from [1] that our result on C^1-rigidity is sharp even on a level of the modulo of continuity of the derivative of the conjugacy. It is interesting to mention that locally near the crytical point the conjugacy is $C^{1+\gamma}$-smooth, as it was shown by Khmelev and Yampolsky [10]. We shall explain in Sect. 4 that this result also follows from our analysis.

Finally, note that our approach allows to prove C^1-rigidity for a large class of rotation numbers in the case of circle diffeomorphisms with break points (see [12] for exact statements). However, it is unclear whether robust rigidity, i.e. C^1-rigidity for all irrational rotation numbers, holds in this case.

The paper has the following structure. In Sect. 2 we built up a general set-up and formulate the main results. Section 3 forms the main technical part of the paper. Here we prove Theorem 2 which gives general criteria for robust rigidity in terms of the convergence of renormalizations and regularity conditions. Theorem 1 is proven in Sect. 4. This is a rather simple and straightforward application of Theorem 2.

2 General settings and statement of Theorem 2

2.1 Circle homeomorphism and renormalizations. Let T be an orientation-preserving homeomorphism of the unit circle $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$. The particular notion of renormalization we use in this paper is directly related to the expansion of the rotation number ρ of a circle homeomorphism T in the form of *continued fraction*. The latter is defined as

$$\rho = [k_1, k_2, \dots, k_n, \dots] = \cfrac{1}{k_1 + \cfrac{1}{k_2 + \cfrac{1}{\cfrac{\cdots}{k_n + \cfrac{1}{\cdots}}}}} \tag{1}$$

where the sequence of positive integers k_n, $n \geq 1$, called *partial quotients*, can be either finite or infinite, in which two cases the right-hand side of (1) corresponds to either a rational number which can be calculated directly, or an irrational number given by a limit for the sequence of *rational convergents* (or, just *convergents*) $p_n/q_n = [k_1, k_2, \dots, k_n]$ (here p_n and q_n are mutually prime positive integers). The continued fraction expansion for rotation number

$$\rho(T) = [k_1, k_2, \dots, k_n, \dots] \tag{2}$$

is uniquely defined by (1) if we agree not to consider finite expansions with the last partial quotient equal to 1. The convergents p_n/q_n are also defined for rational $\rho(T)$, but in this case the sequence of convergents is finite, of the same length as the expansion (2). For convenience, we also define $p_0 = 0$, $q_0 = 1$ and $p_{-1} = 1$, $q_{-1} = 0$.

Given a circle homeomorphism T with irrational $\rho(T)$, one may consider the *marked trajectory* (i.e. the trajectory of the marked point) $\xi_i = T^i\xi_0 \in \mathbb{T}^1$, $i \geq 0$, and pick out of it the sequence of the *dynamical convergents* ξ_{q_n}, $n \geq 0$, indexed by the denominators of the consecutive rational convergents to $\rho(T)$. We will also conventionally use $\xi_{q_{-1}} = \xi_0 - 1$. The well-understood arithmetical properties of rational convergents and the combinatorial equivalence of all the circle homeomorphisms with a fixed irrational rotation number imply that the dynamical convergents approach the marked point, alternating their order in the following way:

$$\begin{aligned}\xi_{q_{-1}} < \xi_{q_1} < \xi_{q_3} < \cdots < \xi_{q_{2m+1}} < \cdots < \xi_0 \\ < \cdots < \xi_{q_{2m}} \quad < \cdots < \xi_{q_2} < \xi_{q_0}.\end{aligned} \tag{3}$$

We define the nth *renormalization segment* $\Delta_0^{(n)}$ as the circle arc $[\xi_0, \xi_{q_n}]$ if n is even and $[\xi_{q_n}, \xi_0]$ if n is odd. We shall also use the notations $\dot{\Delta}_0^{(n)} = \Delta_0^{(n)} \setminus \{\xi_0, \xi_{q_n}\}$, $\overline{\Delta}_0^{(n)} = \Delta_0^{(n)} \cup \Delta_0^{(n+1)}$, and $\check{\Delta}_0^{(n)} = \Delta_0^{(n)} \setminus \Delta_0^{(n+2)}$. An important addition to the property (3) can now be formulated: the first point of the marked trajectory that enters $\dot{\Delta}_0^{(n)}$, is $\xi_{q_n+q_{n+1}}$.

The iterates T^{q_n} and $T^{q_{n-1}}$ restricted to $\Delta_0^{(n-1)}$ and $\Delta_0^{(n)}$ respectively are nothing else but two continuous components of the first-return map for T on the segment $\overline{\Delta}_0^{(n-1)}$ with its endpoints being identified. The consecutive images of $\Delta_0^{(n-1)}$ and $\Delta_0^{(n)}$ until the return to $\overline{\Delta}_0^{(n-1)}$ cover the whole circle without overlapping beyond their endpoints, thus forming the nth *dynamical partition*

$$\mathcal{P}_n = \left\{T^i\Delta_0^{(n-1)}, 0 \le i < q_n\right\} \cup \left\{T^i\Delta_0^{(n)}, 0 \le i < q_{n-1}\right\}$$

of $\mathbb{T}^1$. The endpoints of the segments from $\mathcal{P}_n$ form the set

$$\Xi_n = \{\xi_i, 0 \le i < q_{n-1} + q_n\}.$$

We shall also use the extended set $\Xi_n^* = \Xi_n \cup \{\xi_{q_{n-1}+q_n}\}$ and the extended partition $\mathcal{P}_n^* = \mathcal{P}_n \cup \{T^{q_n}\Delta_0^{(n-1)}, T^{q_{n-1}}\Delta_0^{(n)}\}$.

For $n \ge 0$, the *n-th renormalization* of an orientation-preserving homeomorphism T of the unit circle $\mathbb{T}^1$ with rotation number (2) with respect to a marked point $\xi_0 \in \mathbb{T}^1$ is a function $f_n : [-1, 0] \to \mathbb{R}$ obtained from the map T^{q_n} restricted to the nth renormalization segment $\Delta_0^{(n-1)}$ by *rescaling* the coordinates:

$$f_n = \mathfrak{r}_n \circ T^{q_n} \circ \mathfrak{r}_n^{-1}$$

where $\mathfrak{r}_n$ is an affine change of coordinates that sends $\xi_{q_{n-1}}$ to -1 and ξ_0 to 0 (if we identify ξ_0 with zero, then $\mathfrak{r}_n$ is exactly multiplication by $(-1)^n/|\Delta_0^{(n-1)}|$, where $|\cdot|$ denotes length). Note the following useful equalities: $f_n(0) = |\Delta_0^{(n)}|/|\Delta_0^{(n-1)}|$ and $\mathfrak{r}_n = -f_n(0)\mathfrak{r}_{n+1}$.

Remark 1. This definition is valid for all $n \ge 0$ if $\rho(T)$ is irrational. Otherwise, n needs to be less than the length of expansion (2), or can equal it providing that $\xi_{q_{n-1}} \ne \xi_0$, $n > 0$.

It is useful to notice that $q_n + q_{n-1} - q_m$, $m \le n$, is the maximal value of index l such that $\xi_l \in \Xi_n \cap \Delta_0^{(m-1)}$.

The combinatorics of dynamical partitions are summarized with the following statement.

Lemma 1. *For $m < n$ we have*

$$\Xi_n \cap \check{\Delta}_0^{(m-1)} = \bigcup_{\xi_l \in \Xi_n \cap \Delta_0^{(m)} \setminus \{\xi_{q_m}\}} \{\xi_{l+q_{m-1}+iq_m}\}_{0 \le i < k_{m+1}} \tag{4}$$

and for every $\xi_l \in \Xi_n \cap \Delta_0^{(m)} \setminus \{\xi_{q_m}\}$ we have $\xi_{l+q_{m-1}+k_{m+1}q_m} = \xi_{l+q_{m+1}} \in (\Xi_n^* \cap \overline{\Delta}_0^{(m)})$.

Proof. Follows easily from properties of continued fractions. □

Lemma 1 shows that the whole array of points of Ξ_n (and thus the segments of $\mathcal{P}_n$) contained in $\check{\Delta}_0^{(m-1)}$ can be split into 'threads' starting at appropriate points (segments) in $\Delta_0^{(m)}$ and leading back to $\Delta_0^{(m)}$. This decomposition will be used twice in inductive procedures in the sequel: firstly in Subsect. 3.1.3 (for points) and secondly in Subsects. 3.3 and 3.4 (for segments).

2.2 Statement of the results. For given $\alpha \in (0, 1)$ (this number is fixed throughout the whole paper) and a closed interval $U \subset \mathbb{R}$ consider a Banach space $C^{2+\alpha}(U)$ with a standard norm.

Diffeomorphism with singularity. An orientation-preserving homeomorphism T on $\mathbb{T}^1$ is called a *c*ircle diffeomorphism with singularity if there exists a *singularity point* $\xi_0 \in \mathbb{T}^1$ such that $T \in C^{2+\alpha}([\xi_0 - 1, \xi_0])$ and $T'(\xi) > 0$ for every $\xi \in (\xi_0 - 1, \xi_0)$. If T is a circle diffeomorphism with singularity we shall always use the singularity point ξ_0 as a marked point for renormalizations. Notice, that such renormalizations are $C^{2+\alpha}$-smooth and $f_n'(x) > 0$ for $x \in [-1, 0)$. For simplicity, we assign coordinate zero to ξ_0.

Regularity conditions. Given a vector $\mathbf{K} = (K_1, K_2, K_3, K_4)$ with positive components and a strictly increasing real function $f \in C^{2+\alpha}([-1, 0])$ such that $f(z) > z$ for each $z \in [-1, 0]$, we say that f is $\mathbf{K}$-*regular*, if the following conditions hold:

i) $\|f\|_{2+\alpha} \leq K_1$;
ii) the set $M_{f,K_2} = \{z \in [-1, 0],\ f(z) - z < K_2\}$ is either an open interval or empty (in particular, this implies $f(-1) \geq K_2 - 1$ and $f(0) \geq K_2$);
iii) $\frac{d^2 f}{dz^2}(z) > K_3$ for each $z \in M_{f,K_2}$;
iv) $\frac{df}{dz}(z) > K_4$ for each $-1 \leq z < -K_2^2$.

Note, that here K_1 has to be large enough, when K_2, K_3 and K_4 have to be small enough. We will assume that $K_1 > 1$ and $K_2, K_3, K_4 < 1$.

Refining partitions. A system of nested partitions of the circle $\mathcal{P}_n$, $n \geq 0$, (*nested* here means that each element of $\mathcal{P}_{n+1}$ is contained in an element of $\mathcal{P}_n$) is called *refining* if the maximal length of an element of $\mathcal{P}_n$ tends to zero as $n \to \infty$. That system is called *exponentially refining* if there exist constants $C_1 > 0$ and $0 < \beta < 1$ such that $|I| \leq C_1 \beta^{n-m} |J|$ for any $I \in \mathcal{P}_n$ and $J \in \mathcal{P}_m$ such that $I \subset J$.

Theorem 2. *Suppose that for two circle diffeomorphisms with singularities T and $\tilde{T}$ the following conditions hold:*

1) $\rho(T) = \rho(\tilde{T})$ *is irrational;*
2) there exists a vector $\mathbf{K}$ *such that the renormalizations f_n and $\tilde{f}_n$ are* $\mathbf{K}$*-regular uniformly with respect to n;*
3) the systems of dynamical partitions $\mathcal{P}_n$ and $\tilde{\mathcal{P}}_n$ are exponentially refining;
4) there exist constants $C_2 > 0$ and $\lambda \in (0, 1)$ such that $\|f_n - \tilde{f}_n\|_{C^2} \leq C_2 \lambda^n$.

Then there exists an orientation-preserving C^1-smooth circle diffeomorphism ϕ such that

$$\phi \circ T \circ \phi^{-1} = \tilde{T} \tag{5}$$

Remark 2. Since $f_n(0) = |\Delta_0^{(n)}|/|\Delta_0^{(n-1)}|$, **K**-regularity of f_n implies that the segments $\Delta_0^{(n)}$ and $\Delta_0^{(n-1)}$ are of the same order.

Remark 3. Condition 3) implies that T and $\tilde{T}$ are topologically conjugated. It is easy to see that in the case of circle diffeomorphisms with singularities the conjugacy ϕ can be smooth only in the case when it maps ξ_0 into $\tilde{\xi}_0$. Clearly, this condition determines ϕ uniquely. Everywhere below we discuss only this particular conjugacy and denote it by ϕ.

Here and in what follows, all the constants C_i and S_i do not depend on the renormalization step n.

2.3 Criterion of smoothness. We will use the following criterion for the smoothness of a circle homeomorphism ϕ. It is inspired by the similar statement in [3] called there the 'coherence property'.

For a segment $I \subset \mathbb{T}^1$, let us define the ratio

$$\sigma(I) = \frac{|\phi(I)|}{|I|},$$

where $|\cdot|$ is the length.

Proposition 1. *Suppose that the system of partitions $\mathcal{P}_n$ of the circle is refining, and there exist constants $C_3 > 0$ and $\mu \in (0, 1)$ such that for any two segments $I, I' \in \mathcal{P}_n$, which either are adjacent, or $I, I' \subset J$ for some $J \in \mathcal{P}_{n-1}$, the following estimate holds:*

$$|\log \sigma(I) - \log \sigma(I')| \le C_3 \mu^n. \tag{6}$$

Then $\phi \in C^1(\mathbb{T}^1)$ and $\phi' > 0$.

Proof. Denote by ϕ_n a homeomorphism of $\mathbb{T}^1$ that equals ϕ on Ξ_n and is linear on each element of $\mathcal{P}_n$ (so, ϕ_0 is an identity.) Let $\phi_n' : \mathbb{T}^1 \to (0, +\infty)$ be the right-hand derivative of ϕ_n. Then $\phi_n' = \sigma(I)$ over any $I \in \mathcal{P}_n$, excluding the right endpoint of I. It follows from (6) that $|\log \sigma(I) - \log \sigma(J)| \le C_3 \mu^n$ for any $I \in \mathcal{P}_n$ such that $I \subset J \in \mathcal{P}_{n-1}$. This implies that $\log \phi_n'(\xi)$, $n \ge 0$, is a Cauchy sequence uniformly on $\mathbb{T}^1$, so it has a limit $h(\xi)$. Now, it is easy to show that due to (6) the function h is continuous on $\mathbb{T}^1$. Taking a limit in the equality $\phi_n(\xi) = \int_0^\xi \phi_n'(\tau) d\tau$ we get $\phi(\xi) = \int_0^\xi e^{h(\tau)} d\tau$. It follows that $\phi' = e^h$ is continuous and positive on $\mathbb{T}^1$. □

Remark 4. In this proof we do not use the dynamical nature of partitions $\mathcal{P}_n$. The criterion holds for any refining system of nested partitions of the circle.

We shall also use the ratio of appropriately rescaled intervals $I \subset \mathbb{T}^1$ and $\phi(I)$:

$$\mathfrak{s}_n(I) = \frac{|\tilde{\mathfrak{r}}_n(\phi(I))|}{|\mathfrak{r}_n(I)|},$$

where $\mathfrak{r}_n(\xi) = (-1)^n\xi/|\Delta_0^{(n-1)}|$ and $\tilde{\mathfrak{r}}_n(\tilde{\xi}) = (-1)^n\tilde{\xi}/|\tilde{\Delta}_0^{(n-1)}|$ are the nth rescalings of T and $\tilde{T}$ respectively. It is obvious that $\mathfrak{s}_n(I) = \sigma(I)/\sigma(\Delta_0^{(n-1)})$ and therefore $|\log\sigma(I) - \log\sigma(J)| = |\log\mathfrak{s}_n(I) - \log\mathfrak{s}_n(J)|$ as soon as $I, J \subset \check{\Delta}_0^{(n-1)}$. In vicinity of singularity point ξ_0, we shall work with $\mathfrak{s}_n$ instead of σ.

Define also the distance between appropriately rescaled points $\xi \in \mathbb{T}^1$ and $\phi(\xi)$:

$$\mathfrak{d}_n(\xi) = |\tilde{\mathfrak{r}}_n(\phi(\xi)) - \mathfrak{r}_n(\xi)|$$

3 Proof of Theorem 2

In this section we prove Theorem 2 by showing that its conditions imply the conditions of Proposition 1. We set up our tools in Subsect. 3.1, then prove (6) on the 'core' interval $\Delta_0^{(n-2)}$ in Subsect. 3.2, and afterwards spread it onto the whole circle in two steps, which form Subsects. 3.3 and 3.4.

3.1 Preparations

3.1.1 Funnel and its center. The main difficulty is to analyse the iterates of f_m in the case when k_{m+1} is very large. In this case, the graph of f_m almost touches the identity line, but does that in a non-degenerate way determined by the regularity conditions. Informally, we call this almost-tangency a 'funnel'.

If the set $M_{f_m,K_2/2} = \{z \in [-1, 0], f_m(z) - z < K_2/2\}$ is not empty, let us define the *center of funnel* $z_*^{(m)} \in M_{f_m,K_2/2}$ by the equality $f_m'(z_*^{(m)}) = 1$, otherwise $z_*^{(m)}$ is not defined. By our definition, $z_*^{(m)}$ is the minimum point of $f_m(z) - z$ on $[0, 1]$. Similarly define $\tilde{z}_*^{(m)}$. It is obvious that if $k_{m+1} > 2/K_2$, then both $z_*^{(m)}$ and $\tilde{z}_*^{(m)}$ are defined.

Lemma 2. *If* $z_*^{(m)}$ *and* $\tilde{z}_*^{(m)}$ *are both defined, then* $|z_*^{(m)} - \tilde{z}_*^{(m)}| \le C_4\lambda^m$ *with* $C_4 > 0$.

Proof. Follows from Condition 4) of Theorem 2. □

3.1.2 Initial adjustment of diffeomorphisms. Conditions 2) and 4) of Theorem 2 imply that $|\log\sigma(\Delta_0^{(n-1)}) - \log\sigma(\Delta_0^{(n)})| = |\log f_n(0) - \log\tilde{f}_n(0)| \le S_1\lambda^n$, $S_1 > 0$. Therefore, the limit $s = \lim_{n\to+\infty}\log\sigma(\Delta_0^{(n)})$ exists, and the convergence is exponential. We can always achieve $s = 0$ by a C^∞-smooth initial adjustment of one of the diffeomorphisms T or $\tilde{T}$. Indeed, if $s < 0$ let us consider $\psi \circ T \circ \psi^{-1}$ instead of T where ψ is a C^∞-smooth orientation-preserving diffeomorphism of $\mathbb{T}^1$ that is affine on $\overline{\Delta_0^{(1)}}$ with factor e^s. This change of T will not affect the renormalizations f_n, $n \ge 2$, so they will stay regular uniformly w.r.t. n, but s will vanish. In the case $s > 0$, the same result is achieved by the similar change of $\tilde{T}$.

Thus, in the sequel we assume the pair of diffeomorphisms T and $\tilde{T}$ to be already adjusted as described above, and therefore there exists $C_5 > 0$

such that

$$\left|\log\sigma\left(\Delta_0^{(n)}\right)\right| \le C_5\lambda^n. \tag{7}$$

3.1.3 Closeness of rescaled points

Proposition 2. *For any* $\lambda_1 \in (\sqrt{\lambda}, 1)$ *there exist* $\nu_1 \in (0, 1)$ *and* $C_6 > 0$ *such that*

$$\mathfrak{d}_m(\xi) \le C_6\lambda_1^n \tag{8}$$

provided $(1-\nu_1)n \le m \le n$ *and* $\xi \in \Xi_n^* \cap \overline{\Delta}_0^{(m-1)}$. *(For* $m = n-1$, λ_1 *in* (8) *can be taken equal to* $\sqrt{\lambda}$.*)*

Let us first prove the following

Lemma 3. *There exist* $C_7 > 0$ *such that*

$$\max_{\xi\in\Xi_n^*\cap\overline{\Delta}_0^{(m-1)}} \mathfrak{d}_m(\xi) \le C_7\Big(\max_{\xi\in\Xi_n^*\cap\overline{\Delta}_0^{(m)}} \mathfrak{d}_{m+1}(\xi) + \lambda^{m/2}\Big) \tag{9}$$

for every $m < n$.

Proof. For fixed m and n, let us denote $d = \max_{\xi\in\Xi_n^*\cap\overline{\Delta}_0^{(m)}} \mathfrak{d}_{m+1}(\xi)$. We need to prove the estimate

$$\mathfrak{d}_m(\xi) \le C_7(d + \lambda^{m/2}) \tag{10}$$

for all $\xi \in \Xi_n^* \cap \overline{\Delta}_0^{(m-1)}$. The obvious equality $\mathfrak{d}_m(\xi) = |f_m(0)\mathfrak{r}_{m+1}(\xi) - \tilde{f}_m(0)\tilde{\mathfrak{r}}_{m+1}(\phi(\xi))|$ implies $\mathfrak{d}_m(\xi) \le K_1\mathfrak{d}_{m+1}(\xi) + K_1C_2\lambda^m$ for any $\xi \in \overline{\Delta}_0^{(m)}$, and therefore proves (10) for $\xi \in \Xi_n^* \cap \overline{\Delta}_0^{(m)}$, so we only need to prove (10) for $\xi \in \Xi_n \cap \check{\Delta}_0^{(m-1)}$.

Consider an arbitrary thread in the decomposition (4) and denote $z_i = \mathfrak{r}_m(\xi_{l+q_{m-1}+iq_m})$, $\tilde{z}_i = \tilde{\mathfrak{r}}_m(\tilde{\xi}_{l+q_{m-1}+iq_m})$ for $0 \le i \le k_{m+1}$, so that $z_{i+1} = f_m(z_i)$, $\tilde{z}_{i+1} = \tilde{f}_m(\tilde{z}_i)$. It is easy to see that $\mathfrak{d}_m(\xi_{l+q_{m-1}+iq_m}) = |z_i - \tilde{z}_i|$.

The consecutive estimates

$$\begin{aligned}
\mathfrak{d}_m(\xi_{l+q_{m-1}}) &= \left|\frac{\mathfrak{r}_{m-1}(\xi_{l+q_{m-1}})}{f_{m-1}(0)} - \frac{\tilde{\mathfrak{r}}_{m-1}(\tilde{\xi}_{l+q_{m-1}})}{\tilde{f}_{m-1}(0)}\right| \\
&\le \frac{K_1\mathfrak{d}_{m-1}(\xi_{l+q_{m-1}}) + C_2\lambda^{m-1}}{K_2^2}, \\
\mathfrak{d}_{m-1}(\xi_{l+q_{m-1}}) &= |f_{m-1}(\mathfrak{r}_{m-1}(\xi_l)) - \tilde{f}_{m-1}(\tilde{\mathfrak{r}}_{m-1}(\tilde{\xi}_l))| \\
&\le K_1\mathfrak{d}_{m-1}(\xi_l) + C_2\lambda^{m-1}, \\
\mathfrak{d}_{m-1}(\xi_l) &= |f_{m-1}(0)f_m(0)\mathfrak{r}_{m+1}(\xi_l) - \tilde{f}_{m-1}(0)\tilde{f}_m(0)\tilde{\mathfrak{r}}_{m+1}(\tilde{\xi}_l)| \\
&\le K_1^2\mathfrak{d}_{m+1}(\xi_l) + 2K_1C_1\lambda^{m-1},
\end{aligned}$$

on the one hand, and $\mathfrak{d}_m(\xi_{l+k_{m+1}}) \le K_1\mathfrak{d}_{m+1}(\xi_{l+k_{m+1}}) + K_1C_2\lambda^m$, on the other, imply

$$|z_0 - \tilde{z}_0|, |z_{k_{m+1}} - \tilde{z}_{k_{m+1}}| \le S_2(d + \lambda^m)$$

with $S_2 > 0$.

Let θ_i be a point between z_i and $\tilde{z}_i$ such that $|z_{i+1} - f_m(\tilde{z}_i)| = f_m'(\theta_i) \cdot |z_i - \tilde{z}_i|$. Then

$$|z_{i+1} - \tilde{z}_{i+1}| \le f_m'(\theta_i) \cdot |z_i - \tilde{z}_i| + C_2\lambda^m$$
$$|z_{i-1} - \tilde{z}_{i-1}| \le (f_m'(\theta_{i-1}))^{-1} \cdot \big(|z_i - \tilde{z}_i| + C_2\lambda^m\big).$$

First, we make $N = [2/K_2] + 1$ steps from both edges and obtain $|z_i - \tilde{z}_i| \le S_3(d + \lambda^m)$ with $S_3 > 0$ for $i \le N$ and for $i \ge k_{m+1} - N$. If $k_{m+1} \le 2N$ then (10) is proven, otherwise all the points $z_i, \tilde{z}_i$ for $N \le i \le k_{m+1} - N$ lie in $M_{f_m,K_2/2} \cap M_{\tilde{f}_m,K_2/2}$, and both $z_*^{(m)}$ and $\tilde{z}_*^{(m)}$ are defined (and Lemma 2 for them holds).

On the second stage, we make $l = [\lambda^{-m/2}] + 1$ steps from both edges, although stop earlier when $\max\{z_i, \tilde{z}_i\} > z_*^{(m)}$ on the motion forward and when $\min\{z_i, \tilde{z}_i\} < z_*^{(m)}$ on the motion backward. Since all the way we have $|f_m'(\theta_i)|, |(f_m'(\theta_{i-1}))^{-1}| < 1$, the inequality $|z_i - \tilde{z}_i| \le S_4(d + \lambda^{m/2})$ holds with $S_4 > 0$ for all points of this stage.

Now, one can see that if an early stop on the second stage did not occur, then for the rest of points we have $|z_i - z_*^{(m)}|, |\tilde{z}_i - z_*^{(m)}| \le S_5 l^{-1} \le S_5\lambda^{m/2}$ with $S_5 > 0$ due to the well-known asymptotic estimates for iterates under a non-degenerate tangency (see Lemma 5 below for their most precise version). If both forward and backward motions were stopped early at i_1 and i_2 respectively, then the interval between the leftmost one and the rightmost one of the four points $z_{i_1}, \tilde{z}_{i_1}, z_{i_2}, \tilde{z}_{i_2}$, has length bounded by $2S_4(d + \lambda^{m/2})$ and contains the rest of points. If the motion in one direction stopped early, and in another did not, then the two arguments can be easily combined. Thus, in all the cases (10) is proven and so is the lemma. □

Proof of Proposition 2. It is easy to verify that $\Xi_n^* \cap \overline{\Delta}_0^{(n-1)} = \{\xi_{q_{n-1}}, \xi_{q_{n-1}+q_n}, \xi_0, \xi_{q_n}\}$ and $\mathfrak{d}_n(\xi_{q_{n-1}}) = \mathfrak{d}_n(\xi_0) = 0$, $\mathfrak{d}_n(\xi_{q_{n-1}+q_n}) = |f_n(-1) - \tilde{f}_n(-1)|$, $\mathfrak{d}_n(\xi_{q_n}) = |f_n(0) - \tilde{f}_n(0)|$, so $\max_{\xi \in \Xi_n^* \cap \overline{\Delta}_0^{(n-1)}} \mathfrak{d}_n(\xi) \le C_2\lambda^n$. Starting with this estimate and using Lemma 3 inductively, it is easy to derive the formula

$$\max_{\xi \in \Xi_n^* \cap \overline{\Delta}_0^{(m-1)}} \mathfrak{d}_m(\xi) \le \sum_{j=0}^{n-m-1} C_7^{j+1}\lambda^{(m+j)/2} + C_7^{n-m}C_2\lambda^n$$
$$= C_7\lambda^{m/2}\frac{\big(C_7\lambda^{1/2}\big)^{n-m} - 1}{C_7\lambda^{1/2} - 1} + C_7^{n-m}C_2\lambda^n$$

for any $m < n$. Assuming $C_7 \ge 2\lambda^{-1/2}$ and $m \ge (1 - \nu_1)n$, we get the estimate $\mathfrak{d}_m(\xi) \le (C_7 + C_2)(C_7^{\nu_1}\sqrt{\lambda})^n$ for ξ as required. The statement of

Proposition follows after we choose $\nu_1 \in (0, 1)$ such that $C_7^{\nu_1}\sqrt{\lambda} \leq \lambda_1$. (For $m = n - 1$, we stop after the first step of the induction.) □

Remark. We can take λ_1 arbitrarily close to $\sqrt{\lambda}$ for ν_1 sufficiently small. In what follows, we assume $\nu_1 \leq 3/4$, so that $\lambda^{(1-\nu_1)n} \leq \lambda_1^{n/2}$.

3.1.4 Rescaled ratio distortion

Proposition 3. *There exist $C_8 > 0$ such that*

$$\left|\log \mathfrak{s}_m(T^{q_m} I) - \log \mathfrak{s}_m(I)\right| \leq C_8 \lambda_1^{n/2} \tag{11}$$

for any $(1 - \nu_1)n \leq m < n$ and $I \in \mathcal{P}_n$ such that $I \subset \check{\Delta}_0^{(m-1)}$. (For $m = n - 1$, λ_1 in (11) *can be taken equal to $\sqrt{\lambda}$.)*

Proof. This statement follows from Proposition 2, though differently in two cases. Let $\xi, \eta \in \Xi_n$ be the endpoints of I.

If $|\mathfrak{r}_m(\xi) - \mathfrak{r}_m(\eta)| \geq \lambda_1^{n/2}$, then $|\mathfrak{s}_m(I) - 1| \leq \frac{\mathfrak{d}_m(\xi)+\mathfrak{d}_m(\eta)}{|\mathfrak{r}_m(\xi)-\mathfrak{r}_m(\eta)|} \leq 2C_6\lambda_1^{n/2}$, hence $|\log \mathfrak{s}_m(I)| \leq S_6\lambda_1^{n/2}$ with some $S_6 > 0$. By the regularity conditions, we have $|\mathfrak{r}_m(T^{q_m}\xi) - \mathfrak{r}_m(T^{q_m}\eta)| = |f_m(\mathfrak{r}_m(\xi)) - f_m(\mathfrak{r}_m(\eta))| \geq K_4\lambda_1^{n/2}$, hence $|\log \mathfrak{s}_m(T^{q_m} I)| \leq S_7\lambda_1^{n/2}$ with some $S_7 > 0$, and (11) follows.

In the opposite case, i.e. $|\mathfrak{r}_m(\xi) - \mathfrak{r}_m(\eta)| \leq \lambda_1^{n/2}$, note that $|\log \mathfrak{s}_m(T^{q_m} I) - \log \mathfrak{s}_m(I)| = |\log \tilde{f}_m'(\tilde{\theta}) - \log f_m'(\theta)| \leq K_1K_4^{-1}|\theta - \tilde{\theta}| + C_2K_4^{-1}\lambda^m$ with $\theta \in \mathfrak{r}_m(I)$, $\tilde{\theta} \in \tilde{\mathfrak{r}}_m(\phi(I))$. This estimate implies (11) since $|\theta - \tilde{\theta}| \leq |\mathfrak{r}_m(\xi) - \mathfrak{r}_m(\eta)| + \mathfrak{d}_m(\xi) + \mathfrak{d}_m(\eta)$ and $\lambda^m \leq \lambda_1^{n/2}$. □

3.2 The core interval $\overline{\Delta}_0^{(n-2)}$. In this subsection we prove the following

Proposition 4. *There exist $C_9 > 0$ such that for any segment $I \in \mathcal{P}_n^*$, $I \subset \overline{\Delta}_0^{(n-2)}$, the following estimate holds:*

$$|\log \sigma(I)| \leq C_9\lambda_2^n, \tag{12}$$

where $\lambda_2 = \lambda^{\frac{(1+\alpha)\alpha}{8(2+\alpha)}}$.

3.2.1 Settings on the core interval. The segment $\overline{\Delta}_0^{(n-2)}$ is a union of $k_n + 1$ elements of $\mathcal{P}_n$, namely $\Delta_0^{(n-1)}$, $\Delta_0^{(n)}$ and $I_i = T^{q_{n-2}+iq_{n-1}}\Delta_0^{(n-1)}$, $0 \leq i \leq k_n - 1$. We shall also consider the segment $I_{k_n} = T^{q_n}\Delta_0^{(n-1)} \in \mathcal{P}_n^*$. Due to the initial adjustment of diffeomorphisms we have made, the estimate (7) holds for the segments $\Delta_0^{(n-1)}$ and $\Delta_0^{(n)}$, as well as for $\Delta_0^{(n-2)}$, so if we prove that

$$|\log \mathfrak{s}_{n-1}(I_i)| \leq S_8\lambda_2^n \tag{13}$$

for $0 \leq i \leq k_n$ with $S_8 > 0$, then the statement of Proposition 4 will follow since $\sigma(I) = \mathfrak{s}_{n-1}(I)\sigma(\Delta^{(n-2)})$ and $\lambda_2 > \lambda$. To prove (12) for

$I = T^{q_{n-1}}\Delta_0^{(n)} \in \mathcal{P}_n^*\setminus\mathcal{P}_n$, we also use Proposition 2 and the bound $|\mathfrak{r}_n(T^{q_{n-1}}\Delta_0^{(n)})| \geq K_2$.

In the renormalized coordinates, we have $\mathfrak{s}_{n-1}(I_i) = \Delta z_i/\Delta\tilde{z}_i$ with $\Delta z_i = z_{i+1} - z_i$, where $z_i = f_{n-1}^i(-1)$; and corresponding notations for $\tilde{T}$ with tildes over z and f.

The easy case when k_n is bounded by some constant will be considered in the proof of Proposition 4 (see Subsect. 3.2.4 below). Now, let us consider the case $k_n > 2/K_2$, in which the funnel $M_{f_{n-1},K_2/2}$ is non-empty, and its center $z_*^{(n-1)}$ is defined. The affine orientation-preserving change of variables

$$x = \varphi(z) = \frac{1}{2}f''_{n-1}(z_*^{(n-1)})(z - z_*^{(n-1)})$$

sends the center of funnel into the origin and normalizes the second derivative of f_{n-1} there, thus transforming f_{n-1} into $g = \varphi f_{n-1}\varphi^{-1}$ such that $g'(0) = 1$, $g''(0) = 2$. The value $\varepsilon = g(0) = \min_x\{g(x) - x\}$ we call the *size of funnel*. Since $f_{n-1}(z_*^{(n-1)}) - z_*^{(n-1)} \leq k_n^{-1}$, we have

$$0 < \varepsilon \leq \frac{2}{K_2}k_n^{-1}. \tag{14}$$

It follows from our construction that

$$|g(x) - (\varepsilon + x + x^2)| \leq C_{10}|x|^{2+\alpha}, \quad x \in \varphi[-1, 0] \tag{15}$$

with $C_{10} > 0$. Denote $x_i = \varphi z_i$, so that $x_i = g^i(\varphi(-1))$, and $\Delta x_i = x_{i+1} - x_i$.

Finally, let us make the corresponding change of variables for $\tilde{T}$ with tildes over x, φ, z, f, g and ε and look at values $\mathfrak{s}_i = \Delta\tilde{x}_i/\Delta x_i$. The following simple lemma implies that (13) is equivalent to the similar estimate on $\mathfrak{s}_i$.

Lemma 4. $|\log\mathfrak{s}_{n-1}(I_i) - \log\mathfrak{s}_i| \leq C_{11}\lambda^n$ *for* $0 \leq i \leq k_n$ *with* $C_{11} > 0$.

Proof. Notice that

$$|\log\mathfrak{s}_{n-1}(I_i) - \log\mathfrak{s}_i| = \left|\log f''_{n-1}(z_*^{(n-2)}) - \log\tilde{f}''_{n-1}(\tilde{z}_*^{(n-2)})\right|$$

and use Lemma 2. □

Thus to prove Proposition 4 in the case of large k_n it is enough to show that $|\log\mathfrak{s}_i| \leq C_{12}\lambda^n$ for $0 \leq i \leq k_n$ with $C_{12} > 0$.

3.2.2 Funnel and tunnel asymptotics. According to (15), for $|x|^{2+\alpha} >$ const $\cdot\,\varepsilon$ the ε term does not affect the asymptotics of g, while in the opposite case it does. Thus we need two different asymptotic formulas: one for $|x| >$ const $\varepsilon^{\frac{1}{2+\alpha}}$ and another for $|x| <$ const $\varepsilon^{\frac{1}{2+\alpha}}$ (the latter part of the funnel we call the 'tunnel'). We shall make use of the following lemmas presented here in the most general terms. Their proofs can be found in Appendix.

Lemma 5 (Funnel). *Suppose that for a sequence of real numbers* $\{s_i\}_{i\geq 0}$ *there exist* $C_{13} > 0$ *and* $\alpha \in (0, 1)$ *such that* $|s_{i+1} - (s_i - s_i^2)| \leq C_{13}|s_i|^{2+\alpha}$ *for every* $i \geq 0$. *Then there exist constants* $D_1 = D_1(\alpha, C_{13}) > 0$ *and* $d_1 = d_1(\alpha, C_{13}) \in (0, 1)$ *such that as long as* $s_0 \in (0, d_1]$, *the estimate*

$$\left| s_i - \frac{1}{i + s_0^{-1}} \right| \leq \frac{D_1}{\left(i + s_0^{-1}\right)^{1+\alpha}} \tag{16}$$

holds for every $i \geq 0$. *Moreover, there exists* $D_2 = D_2(\alpha, C_{13}) > 0$ *such that*

$$s_i - s_{i+1} = \frac{1}{\left(i + s_0^{-1}\right)^2}(1 + \delta_i), \tag{17}$$

where $|\delta_i| \leq D_2 s_0^\alpha$ *for all* $i \geq 0$ *as long as* $s_0 \in (0, d_1]$.

Lemma 6 (Tunnel). *Suppose that for a sequence of real numbers* $\{s_i\}_{i\geq 0}$ *there exist* $C_{14}, C_{15} > 0$ *and* $\varepsilon, \alpha \in (0, 1)$ *such that*

1) $|s_0| \leq C_{14}\varepsilon$,
2) $|s_{i+1} - (\varepsilon + s_i + s_i^2)| \leq C_{15}|s_i|^{2+\alpha}$ *for every* $i \geq 0$.

Fix arbitrary $C_{16} > 0$ *and define* $N = N(C_{16}, \varepsilon) = \varepsilon^{-\frac{1}{2}} \tan^{-1}(C_{16}\varepsilon^{-\frac{\alpha}{2(2+\alpha)}})$. *Then there exist constants* $D_3 = D_3(\alpha, C_{14}, C_{15}, C_{16}) > 0$ *and* $d_2 = d_2(\alpha, C_{14}, C_{15}, C_{16}) \in (0, 1)$ *such that as long as* $\varepsilon \in (0, d_2]$, *the following estimate holds for every* $0 \leq i \leq N$:

$$|s_i - \sqrt{\varepsilon}\tan(\sqrt{\varepsilon} i + a_0)| \leq D_3(\sqrt{\varepsilon}\tan\sqrt{\varepsilon} i)^{1+\frac{\alpha(\alpha+1)}{2}}, \tag{18}$$

where $a_0 = \tan^{-1}(s_0/\sqrt{\varepsilon})$. *Moreover, there exists* $D_4 = D_4(\alpha, C_{14}, C_{15}, C_{16}) > 0$ *such that*

$$s_{i+1} - s_i = \frac{\varepsilon}{(\cos\sqrt{\varepsilon} i)^2}(1 + \delta_i), \tag{19}$$

where $|\delta_i| \leq D_4 \varepsilon^{\frac{\alpha(\alpha+1)}{2(2+\alpha)}}$ *for all* $0 \leq i < N$ *as long as* $\varepsilon \in (0, d_2]$.

Remark. The choice of $N(C_{16}, \varepsilon)$ in this lemma implies the bound

$$\sqrt{\varepsilon}\tan\sqrt{\varepsilon} i \leq C_{16}\varepsilon^{\frac{1}{2+\alpha}}$$

for all i under consideration. Combining the relation $\tan^{-1}\frac{1}{t} = \frac{\pi}{2} - \tan^{-1} t$ with the asymptotic formula $\tan^{-1} t = t + O(t^3)$, $t \to 0$, it is easy to derive

$$\varepsilon^{-\frac{1}{2}} \tan^{-1} \varepsilon^{-\frac{\alpha}{2(2+\alpha)}} = \frac{\pi}{2}\varepsilon^{-\frac{1}{2}} - \varepsilon^{-\frac{1}{2+\alpha}} + O(\varepsilon^{\frac{-1+\alpha}{2+\alpha}}), \quad \varepsilon \to 0 \tag{20}$$

3.2.3 Estimates for parameters of the funnel. The next four lemmas estimate (in terms of ε) the most important parameters of the funnel as well as their closeness for T and $\tilde{T}$. Note that each of those lemmas works for small enough ε and therefore imposes some lower bound on k_n (due to (14), such a bound implies a corresponding upper bound on ε). We will incorporate all of them in Subsect. 3.2.4.

Since $[0, \varepsilon]$ is a fundamental interval for g, there exists a unique number $0 < i_c < k_n$ such that $x_{i_c} \in [0, \varepsilon)$. Denote $i_l = i_c - [\varepsilon^{-\frac{1}{2}} \tan^{-1} \varepsilon^{-\frac{\alpha}{2(2+\alpha)}}]$ and $i_r = i_c + [\varepsilon^{-\frac{1}{2}} \tan^{-1} \varepsilon^{-\frac{\alpha}{2(2+\alpha)}}]$. (Here the indices 'c', 'l' and 'r' refer to the central, leftmost and rightmost points of the trajectory in the tunnel respectively.) For $\tilde{g}$, we similarly define $\tilde{i}_c, \tilde{i}_l$ and $\tilde{i}_r$.

Lemma 7. *There exist constants $C_{17}, C_{18} > 0$ such that the inequality*

$$\left|k_n - \pi\varepsilon^{-\frac{1}{2}}\right| \le C_{17}\varepsilon^{\frac{-1+\alpha}{2}}$$

holds if $k_n \ge C_{18}$.

Proof. It follows from Lemma 6 that for small enough ε

$$\left|x_{i_r} - \varepsilon^{\frac{1}{2+\alpha}}\right| \le S_9\varepsilon^{\frac{1}{2+\alpha}+\frac{\alpha(\alpha+1)}{2(2+\alpha)}}, \tag{21}$$

$$\left|x_{i_l} + \varepsilon^{\frac{1}{2+\alpha}}\right| \le S_9\varepsilon^{\frac{1}{2+\alpha}+\frac{\alpha(\alpha+1)}{2(2+\alpha)}}, \tag{22}$$

with $S_9 > 0$.

Inequality (15) and Lemma 5 imply that there exist $S_{10} > 0$, small enough $S_{11}, S_{12} \in (0, 1)$, $S_{12} > S_{11}$, and an integer $i_0 \ge 1$ such that $x_{k_n - i_0} \in (S_{11}, S_{12})$, $x_{i_0} \in (-S_{12}, -S_{11})$ and

$$\left|x_{k_n-i_0-i} - \frac{1}{i + x^{-1}_{k_n-i_0}}\right| \le \frac{S_{10}}{\left(i + x^{-1}_{k_n-i_0}\right)^{1+\alpha}}, \quad 0 \le i \le k_n - i_0 - i_r, \tag{23}$$

$$\left|x_{i_0+i} + \frac{1}{i - x^{-1}_{i_0}}\right| \le \frac{S_{10}}{\left(i - x^{-1}_{i_0}\right)^{1+\alpha}}, \quad 0 \le i \le i_l - i_0 \tag{24}$$

From (21)–(24) it follows that for small enough ε

$$\left|\left(k_n - i_0 - i_r + x^{-1}_{k_n-i_0}\right) - \varepsilon^{-\frac{1}{2+\alpha}}\right| \le S_{13}\varepsilon^{\frac{-1+\alpha}{2}}, \tag{25}$$

$$\left|\left(i_l - i_0 - x^{-1}_{i_0}\right) - \varepsilon^{-\frac{1}{2+\alpha}}\right| \le S_{13}\varepsilon^{\frac{-1+\alpha}{2}} \tag{26}$$

with $S_{13} > 0$. Now recall the asymptotics (20). Since $k_n = (k_n - i_0 - i_r) + (i_r - i_c) + (i_c - i_l) + (i_l - i_0) + 2i_0$ and $\varepsilon^{\frac{-1+\alpha}{2}} > \varepsilon^{\frac{-1+\alpha}{2+\alpha}}$, the obtained estimates prove the lemma. □

Lemma 8. *There exist constants $C_{19}, C_{20} > 0$ such that*

$$\left|\frac{\tilde{\varepsilon}}{\varepsilon} - 1\right| \le C_{19}\varepsilon^{\frac{\alpha}{2}}$$

if $k_n \ge C_{20}$.

Proof. Apply Lemma 7 first to ε and then to $\tilde{\varepsilon}$. □

Lemma 9. *There exist constants $C_{21}, C_{22} > 0$ such that the inequality*

$$\left| i_{\mathrm{c}} - \frac{k_n}{2} \right| \leq C_{21} \varepsilon^{\frac{-1+\alpha}{2}} \tag{27}$$

holds if $k_n \geq C_{22}$.

Proof. Since $(i_{\mathrm{l}} - i_0) - (k_n - i_0 - i_{\mathrm{r}}) = 2i_{\mathrm{c}} - k_n$, and both $x^{-1}_{k_n - i_0}$ and $x^{-1}_{i_0}$ are bounded (see the proof of Lemma 7), the inequalities (25), (26) imply (27). □

Lemma 10. *There exist constants $C_{23}, C_{24} > 0$ such that*

$$|i_{\mathrm{r}} - \tilde{i}_{\mathrm{r}}|, |i_{\mathrm{l}} - \tilde{i}_{\mathrm{l}}|, |i_{\mathrm{c}} - \tilde{i}_{\mathrm{c}}| \leq C_{23} \varepsilon^{\frac{-1+\alpha}{2}}$$

if $k_n \geq C_{24}$.

Proof. Follows from Lemma 9, Lemma 8 and the formula (20). □

3.2.4 Proof of Proposition 4. Due to the regularity conditions, the segments $\Delta_0^{(n-1)}$, $\Delta_0^{(n-2)}$ and $\Delta_0^{(n-3)}$ are of the same order. Now notice that $I_0 = T^{q_{n-2}} \Delta_0^{(n-1)}$ and $I_{k_n} = T^{q_n} \Delta_0^{(n-1)}$. Thus, due to (7) and the regularity conditions on f_{n-2}, $\tilde{f}_{n-2}$ and f_n, $\tilde{f}_n$, there exists $S_{14} > 0$ such that

$$|\log \mathfrak{s}_{n-1}(I_i)| \leq S_{14} \lambda^n$$

for $i = 0$ and $i = k_n$.

For any chosen integer $N_0 > 0$, it follows from Proposition 3 that

$$|\log \mathfrak{s}_{n-1}(I_i)| \leq S_{14} \lambda^n + N_0 C_8 \lambda^{n/4} \tag{28}$$

for $0 \leq i \leq \min\{N_0, k_n\}$ and for $\max\{k_n - N_0, 0\} \leq i \leq k_n$. In particular,

$$|\log \mathfrak{s}_{n-1}(I_i)| \leq S_{14} \lambda^n + C_8 \lambda^{n/8} \tag{29}$$

for $0 \leq i \leq \min\{\lambda^{-n/8}, k_n\}$ and for $\max\{k_n - \lambda^{-n/8}, 0\} \leq i \leq k_n$.

Consider $N_0 = \max_{1 \leq m \leq n_0}\{k_m\}$ and $n_0 \geq 1$ such that

$$\lambda^{-n_0/8} \geq D, \tag{30}$$

where $D > 0$ will be determined at the end of the proof to incorporate all lower bounds on k_n. If $k_n \leq 2\max\{\lambda^{-n/8}, N_0\}$, then (28) and (29) prove the proposition. Now, let us assume that

$$k_n \geq 2 \max\left\{\lambda^{-n/8}, N_0\right\}. \tag{31}$$

In particular, the center of the funnel is defined, so Lemma 7 implies that

$$\varepsilon \leq \frac{\pi^2}{2} \lambda^{n/4}. \tag{32}$$

Hence, we enter the funnel with the estimate (29). Next we show that there exists $S_{15} > 0$ such that

$$|\log \mathfrak{s}_i| \le S_{15}\lambda^{n\alpha/8} \tag{33}$$

for $0 \le i \le \min\{i_{\mathrm{l}}, \tilde{i}_{\mathrm{l}}\}$ and for $\max\{i_{\mathrm{r}}, \tilde{i}_{\mathrm{r}}\} \le i \le k_n$. Indeed, if $i_{\mathrm{l}} < \lambda^{-n/8}$, then (33) holds for $0 \le i \le i_{\mathrm{l}}$ due to (29) and Lemma 4. Otherwise (33) is satisfied for $i \le \lambda^{-n/8}$, and using Lemma 5 we can spread this property from $i = [\lambda^{-n/8}]$ onto the set $[\lambda^{-n/8}] \le i \le \min\{i_{\mathrm{l}}, \tilde{i}_{\mathrm{l}}\}$ as follows. Lemma 5 implies for $[\lambda^{-n/8}] \le i \le \min\{i_{\mathrm{l}}, \tilde{i}_{\mathrm{l}}\}$

$$\mathfrak{s}_i = \frac{\left(i - [\lambda^{-n/8}] + x^{-1}_{[\lambda^{-n/8}]}\right)^2}{\left(i - [\lambda^{-n/8}] + \tilde{x}^{-1}_{[\lambda^{-n/8}]}\right)^2} \cdot \frac{1 + \tilde{\delta}_i}{1 + \delta_i},$$

where $|\delta_i|$, $|\tilde{\delta}_i| \le S_{16}\lambda^{n\alpha/8}$ with $S_{16} > 0$. Now, using the inequality

$$\left|\log \frac{a+1}{b+1}\right| \le \left|\log \frac{a}{b}\right|$$

that holds for any $a, b > 0$, we get

$$\left|\log\left(\frac{1+\delta_i}{1+\tilde{\delta}_i}\mathfrak{s}_i\right)\right| \le \left|\log\left(\frac{1+\delta_{[\lambda^{-n/8}]}}{1+\tilde{\delta}_{[\lambda^{-n/8}]}}\mathfrak{s}_{[\lambda^{-n/8}]}\right)\right|,$$

which implies (33) for $[\lambda^{-n/8}] \le i \le \min\{i_{\mathrm{l}}, \tilde{i}_{\mathrm{l}}\}$. A similar argument for $\max\{i_{\mathrm{r}}, \tilde{i}_{\mathrm{r}}\} \le i \le k_n$ is valid.

So, the estimate (33) is satisfied when we reach the tunnel, i.e. the zone described by Lemma 6. Note that, generally speaking, $\tilde{i}_{\mathrm{c}} \ne i_{\mathrm{c}}$, and that is why we need to show an estimate

$$\left|\log \mathfrak{s}_i - \log \frac{\Delta\tilde{x}_{i+\tilde{i}_{\mathrm{c}}-i_{\mathrm{c}}}}{\Delta x_i}\right| \le S_{17}\lambda^{n\frac{\alpha(1+\alpha)}{8(2+\alpha)}} \tag{34}$$

with $S_{17} > 0$ for $\min\{i_{\mathrm{l}}, \tilde{i}_{\mathrm{l}}\} \le i \le \max\{i_{\mathrm{r}}, \tilde{i}_{\mathrm{r}}\}$. The estimate (34) follows from Lemma 10, (32) and the estimate

$$|\log \Delta\tilde{x}_i - \log \Delta\tilde{x}_{i+1}| = \left|\log \frac{d\tilde{g}}{d\tilde{x}}(\theta)\right| \le S_{18}\varepsilon^{\frac{1}{2+\alpha}},$$

where $\theta \in [\tilde{x}_i, \tilde{x}_{i+1}]$ and $S_{18} > 0$.

Lemma 6 implies that for $\min\{i_{\mathrm{l}}, \tilde{i}_{\mathrm{l}}\} \le i \le \max\{i_{\mathrm{r}}, \tilde{i}_{\mathrm{r}}\}$

$$\frac{\Delta x_i}{\Delta\tilde{x}_{i+\tilde{i}_{\mathrm{c}}-i_{\mathrm{c}}}} = \frac{\varepsilon}{\tilde{\varepsilon}} \cdot \frac{(\cos\sqrt{\tilde{\varepsilon}}(i - i_{\mathrm{c}}))^2}{(\cos\sqrt{\varepsilon}(i - i_{\mathrm{c}}))^2} \cdot \frac{1+\delta_i}{1+\tilde{\delta}_i}, \tag{35}$$

where $|\delta_i|$, $|\tilde{\delta}_i| \le S_{19}\lambda^{n\frac{\alpha(1+\alpha)}{8(2+\alpha)}}$ and $S_{19} > 0$. It is assumed here that $k_n \ge S_{20} > 0$.

It follows from (34) and (35) that for $\min\{i_l, \tilde{i}_l\} \le i \le \max\{i_r, \tilde{i}_r\}$

$$\left|\log \mathfrak{s}_i - \log \frac{(\cos\sqrt{\varepsilon}(i-i_c))^2}{(\cos\sqrt{\tilde{\varepsilon}}(i-i_c))^2}\right| \le S_{21}\lambda^{n\frac{\alpha(1+\alpha)}{8(2+\alpha)}}$$

with $S_{21} > 0$.

Now, using the bound (33) for $i = \min\{i_l, \tilde{i}_l\}$ and for $i = \max\{i_r, \tilde{i}_r\}$, and the elementary inequalities

$$\left|\log \frac{(\cos\sqrt{\varepsilon}(i-1-i_c))}{(\cos\sqrt{\tilde{\varepsilon}}(i-1-i_c))}\right| \le \left|\log \frac{(\cos\sqrt{\varepsilon}(i-i_c))}{(\cos\sqrt{\tilde{\varepsilon}}(i-i_c))}\right|$$

for $i_c < i \le \max\{i_r, \tilde{i}_r\}$ and

$$\left|\log \frac{(\cos\sqrt{\varepsilon}(i+1-i_c))}{(\cos\sqrt{\tilde{\varepsilon}}(i+1-i_c))}\right| \le \left|\log \frac{(\cos\sqrt{\varepsilon}(i-i_c))}{(\cos\sqrt{\tilde{\varepsilon}}(i-i_c))}\right|$$

for $\min\{i_l, \tilde{i}_l\} \le i < i_c$ we obtain

$$|\log \mathfrak{s}_i| \le S_{22}\lambda^{n\frac{\alpha(1+\alpha)}{8(2+\alpha)}} \tag{36}$$

for all $0 \le i \le k_n$ with $S_{22} > 0$.

Finally, we put $D = \max\{2/K_2, C_{18}, C_{20}, C_{22}, C_{24}, i_0, S_{20}\}$ (so D does not depend on n). Proposition 4 is now proven with $\lambda_2 = \lambda^{\frac{\alpha(1+\alpha)}{8(2+\alpha)}}$ and $C_9 = \max\{S_{14} + N_0 C_8, C_{11} + S_{22}\}$. □

3.3 Spreading onto $\overline{\Delta}_0^{(m)}$ with m a fixed fraction of n

Proposition 5. *For any $\lambda_3 \in (\lambda_2, 1)$ there exist $\nu_2 \in (0, \nu_1)$ and $C_{25} > 0$ such that*

$$|\log \sigma(I)| \le C_{25}\lambda_3^n \tag{37}$$

for any segment $I \in \mathcal{P}_n$ such that $I \subset \overline{\Delta}_0^{(n-[\nu_2 n])}$.

First, we prove

Lemma 11. *There exists $C_{26} > 0$ such that*

$$|\log \mathfrak{s}_m(I)| \le C_{26}\Big(\max_{I \in \mathcal{P}^*_{n-1}, I \subset \overline{\Delta}_0^{(m-1)}} |\log \mathfrak{s}_m(I)| + \max_{I \in \mathcal{P}^*_n, I \subset \overline{\Delta}_0^{(m)}} |\log \mathfrak{s}_{m+1}(I)| + \lambda_1^{m/4}\Big) \tag{38}$$

for any $n - \nu_1 n \le m < n - 1$ and $I \in \mathcal{P}^*_n$ such that $I \subset \overline{\Delta}_0^{(m-1)}$.

Proof. For $I \in \mathcal{P}_n^*$ such that $I \subset \overline{\Delta}_0^{(m)}$, the estimate (38) holds with $C_{26} = C_5(1+\lambda)$ due to (7). As Lemma 1 implies, the rest of the segments, for which we have to prove (38), are aligned in threads $I_i = T^{iq_m}(I_0)$, $0 \le i < k_{m+1}$, with $\mathcal{P}_n^* \ni I_{k_{m+1}} = T^{k_{m+1}q_m}(I_0) \subset \overline{\Delta}_0^{(m)}$. Let us consider any such thread. If $k_{m+1} \le \max\{2\lambda_1^{-m/4}, 2/K_2\}$, then (38) follows from Proposition 3 with $C_{26} = \max\{C_5(1+\lambda), 2C_8, 2/K_2\}$. In the opposite case, we have

$$|\log \mathfrak{s}_m(I_{k_{m+1}-i-1})| \le |\log \mathfrak{s}_m(I_{k_{m+1}})| + C_8\lambda_1^{m/4} \tag{39}$$

$$|\log \mathfrak{s}_m(I_i)| \le |\log \mathfrak{s}_m(I_{[\lambda_1^{-m/4}]})| + C_8\lambda_1^{m/4} \tag{40}$$

for $0 \le i < [\lambda_1^{-m/4}]$.

Note now, that for $[\lambda_1^{-m/4}] \le i \le k_{m+1} - [\lambda_1^{-m/4}]$ there exists a unique segment $J_i \in \mathcal{P}_{n-1}$ such that $I_i \subset J_i \subset T^{q_{m-1}+iq_m}\Delta_0^{(m)}$ (possibly, $J_i = I_i$). Since $J_{i+1} = T^{q_m}J_i$, there exist $\theta, \dot{\theta}, \ddot{\theta} \in \mathfrak{r}_m(J_i)$ such that $|\log \mathfrak{r}_m(I_{i+1}) - \log \mathfrak{r}_m(I_i) - \log \mathfrak{r}_m(J_{i+1}) + \log \mathfrak{r}_m(J_i)| = |\log f_m'(\theta) - \log f_m'(\dot{\theta})| = |(\log f_m')'(\ddot{\theta})| \cdot |\theta - \dot{\theta}| \le \frac{K_1}{K_4}|\mathfrak{r}_m(J_i)| \le \frac{K_1}{K_4}(z_{i+1}^{(m)} - z_i^{(m)})$, where $z_i^{(m)} = \mathfrak{r}_m(\xi_{q_{m-1}+iq_m}) = f_m^i(-1)$. Due to Lemma 5, we have $z^{(m)}_{k_{m+1}-[\lambda_1^{-m/4}]} - z^{(m)}_{[\lambda_1^{-m/4}]} \le S_{23}\lambda_1^{m/4}$ with $S_{23} > 0$. Hence,

$$\begin{aligned} &|\log \mathfrak{s}_m(J_i) - \log \mathfrak{s}_m(I_i)| \\ &\quad \le |\log \mathfrak{s}_m(J_{k_{m+1}-[\lambda_1^{-m/4}]}) - \log \mathfrak{s}_m(I_{k_{m+1}-[\lambda_1^{-m/4}]})| + 2\frac{K_1}{K_4}S_{23}\lambda_1^{m/4} \end{aligned} \tag{41}$$

for any $[\lambda_1^{-m/4}] \le i < k_{m+1} - [\lambda_1^{-m/4}]$.

The bounds (39), (41) and (40) together imply (38) for all I_i, $0 \le i < k_{m+1}$. □

Proof of Proposition 5. Let us assume that $\lambda_1 \le \lambda_2^4$ (by Proposition 2 it can be chosen arbitrary close to $\sqrt{\lambda} < \lambda_2^4 = \lambda^{\frac{\alpha(1+\alpha)}{2(2+\alpha)}}$). It is easy to derive from (38) by induction, using Proposition 4 as its base, that for all I as in Lemma 11 we have

$$|\log \mathfrak{s}_m(I)| \le S_{24}(3C_{26})^{n-m}\lambda_2^m$$

with $S_{24} > 0$. The statement of Proposition follows after we choose $\nu_2 \in (0, \nu_1)$ such that $(3C_{26})^{\nu_2}\lambda_2^{1-\nu_2} \le \lambda_3$. □

3.4 Spreading onto $\mathbb{T}^1$. In this subsection we finish the proof of Theorem 2. In the case of critical circle maps we could have used here the standard small distortion argument. However, since we are proving Theorem 2 in full generality, that is without specifying the type of singularity, we present a different proof. It relies only on the regularity conditions and the exponential refinement property.

Proof of Theorem 2. Note that we did not use the Condition 3) of the theorem before. Spreading (6) from $\overline{\Delta}_0^{(n-[\nu_2 n])}$ onto the whole circle $\mathbb{T}^1$ is the only step on which we need that condition. Let C_1 and β be constants of exponential refinement for the dynamical partitions of both T and $\tilde{T}$ (see the definition in Subsect. 2.2).

It follows from Proposition 5 that the estimate

$$|\log\sigma(I) - \log\sigma(I')| \le 2C_{25}\lambda_3^n \tag{42}$$

holds for all pairs of segments (I, I') as in Proposition 1 such that both I and I' are contained in $\overline{\Delta}_0^{(n-[\nu_2 n])}$. We are about to spread such estimate further using an inductive argument. On mth step, starting from $m = n - [\nu_2 n] - 1$ and counting down to 0, we shall spread it from $\overline{\Delta}_0^{(m+1)}$ to $\overline{\Delta}_0^{(m)}$.

On each step the new pairs, for which (6) needs to be demonstrated, come in threads $I_i = T^{iq_m} I_0$, $I'_i = T^{iq_m} I'_0$, $0 \le i < k_{m+1}$. We fix the order in pairs in such a way that I'_0 lies closer to ξ_0 than I_0. This implies that $I_0 \subset T^{q_{m-1}}(\Delta_0^{(m)})$, and I'_0 either belongs to $T^{q_{m-1}}(\Delta_0^{(m)})$ as well or is adjacent to it. We also consider the pair of segments $I_{k_{m+1}} = T^{k_{m+1}q_m} I_0$ and $I'_{k_{m+1}} = T^{k_{m+1}q_m} I'_0$, which are contained in $\overline{\Delta}_0^{(m+1)}$ and belong to $\mathscr{P}_n^*$. We shall prove that there exists $S_{25} > 0$ such that for any thread we have

$$\begin{aligned} &\left|\log\mathfrak{s}_m(I_i) - \log\mathfrak{s}_m(I'_i)\right| \\ &\quad \le \left|\log\mathfrak{s}_m(I_{k_{m+1}}) - \log\mathfrak{s}_m(I'_{k_{m+1}})\right| + S_{25}\beta^{n-m} \end{aligned} \tag{43}$$

for any $0 \le i < k_{m+1}$. There are three possible types of threads, which we specifically describe below. We shall prove (43) for each one of them. Denote

$$a_i = \Big|\log|\mathfrak{r}_m(I_{i+1})| - \log|\mathfrak{r}_m(I_i)| - \log|\mathfrak{r}_m(I'_{i+1})| + \log|\mathfrak{r}_m(I'_i)|\Big|.$$

We have then $a_i = |f''_m(\ddot{\theta}_i)/f'_m(\ddot{\theta}_i)| \cdot |\dot{\theta}_i - \theta_i|$ for some $\theta_i \in \mathfrak{r}_m(I_i)$, $\dot{\theta}_i \in \mathfrak{r}_m(I'_i)$ and $\ddot{\theta}_i \in (\theta_i, \dot{\theta}_i)$. The inequality (43) will follow from bounds on a_j, $i \le j < k_{m+1}$, which we prove below, and similar bounds on $\tilde{a}_j$ for $\tilde{T}$.

Type 1. I_0 and I'_0 belong to the same element J_0 of $\mathscr{P}_{n-1}$. Than there exists a thread $J_i = T^{iq_m} J_0 \in \mathscr{P}_{n-1}$ such that $I_i \cup I'_i \subset J_i \subset T^{q_{m-1}+iq_m}\Delta_0^{(m)}$, $0 \le i < k_{m+1}$. In this case, $a_i \le \frac{K_1}{K_4}|\mathfrak{r}_m(J_i)| \le \frac{K_1}{K_4} C_1 \beta^{n-m}(z_{i+1}^{(m)} - z_i^{(m)})$. Since the sum of the differences $z_{i+1}^{(m)} - z_i^{(m)}$ does not exceed 1, the bound (43) follows.

Type 2. The segments I_0 and I'_0 are adjacent, contained in different elements of $\mathscr{P}_{n-1}$, and $I_0 \cup I'_0 \subset T^{q_{m-1}}\Delta_0^{(m)}$. In this case, $a_i \le \frac{K_1}{K_4}(|\mathfrak{r}_m(I_i \cup I'_i)|) \le \frac{K_1}{K_4} \cdot 2C_1\beta^{n-m+1}(z_{i+1}^{(m)} - z_i^{(m)})$ for $0 \le i < k_{m+1}$, which similarly implies (43).

Type 3. $\xi_{q_{m-1}+q_m}$ is the common endpoint of I_0 and I'_0. Again we have $a_i \le \frac{K_1}{K_4}(|\mathfrak{r}_m(I_i \cup I'_i)|)$ for $0 \le i \le k_{m+1} - 2$. But $I'_{k_{m+1}-1}$ is contained in $\Delta_0^{(m+1)}$, so we cannot guarantee that $f'_m(\ddot{\theta}_{k_{m+1}-1}) > K_4$. However, it is adjacent to the point $\xi_{q_{m+1}}$ and $\mathfrak{r}_m(I'_{k_{m+1}-1}) \le C_1\beta^{n-m+1}(0 - z^{(m)}_{k_{m+1}})$ gets arbitrary small as n grows, therefore for large enough n we have $f'_m(\ddot{\theta}_{k_{m+1}-1}) > K_4/2$ and $a_{k_{m+1}-1} \le 2\frac{K_1}{K_4}(|\mathfrak{r}_m(I_{k_{m+1}-1} \cup I'_{k_{m+1}-1})|)$ which yields (43) as well.

By induction in m (from $m = n - [\nu_2 n] - 1$ to 0), using (42) as a base and (43) on each step, it is now easy to derive – at last for all pairs (I, I') as in Proposition 1 – that

$$\left|\log\sigma(I) - \log\sigma(I')\right| \le 2C_{25}\lambda_3^n + S_{25}\sum_{m=0}^{n-[\nu_2 n]-1}\beta^{n-m} \le C_{27}\left(\lambda_3^n + \beta^{\nu_2 n}\right),$$

with $C_{27} > 0$. Hence, (6) holds true with $\mu = \max\{\lambda_3, \beta^{\nu_2}\}$, and Theorem 2 is proven. □

4 Proof of Theorem 1

In order to prove Theorem 1 we have to check that Conditions 2)–4) of Theorem 2 are satisfied. Exponential convergence of renormalizations (Condition 4)) follows from [13]. To check Conditions 2) and 3) we shall use real a priori bounds. Real a priori bounds form an important step in analysis of renormalizations for unimodal maps and critical circle maps. By now real a priori bounds are well understood and there exist several approaches leading to their derivation (see, for example, [3,6]).

Below we formulate four properties which hold for critical circle maps. We shall use this properties in order to establish Conditions 2) and 3). All the properties either have been directly proved in [3] or follow immediately from the estimates there. Let T be a C^3-smooth critical circle map with an irrational rotation number.

1. There exist constants $0 < \gamma_1, \gamma_2 < 1$ such that
$$\gamma_1 \le \frac{\left|\Delta_i^{(n)}\right|}{\left|\Delta_i^{(n-2)}\right|} \le \gamma_2, \quad 0 \le i < q_{n-1}.$$
2. There exists a constant $M_1 > 0$ such that $\|f_n\|_{C^3} \le M_1$.
3. There exists a constant $M_2 > 0$ such that $f'(x) \ge M_2\epsilon^2,\ x \in [-1, -\epsilon]$.
4. There exists a constant $M_3 > 0$ such that $Sf_n(x) \le -M_3,\ x \in [-1, 0)$, where $Sf = f'''/f' - 3/2(f''/f')^2$ is the Schwarz derivative of f.

In fact, the constants $\gamma_1, \gamma_2, M_1, M_2, M_3$ are universal and do not depend on T for large enough n, but only on the order of its critical point. Note, that the first estimate (with non-universal constants) follows basically from Swiatek's estimates [11].

Now, Condition 3) of Theorem 2 (exponential refinement) follows immediately from the Property 1 above. We finish by showing that the regularity conditions are satisfied.

Lemma 12. *There exists a vector* $\mathbf{K}$ *such that* f_n *is* $\mathbf{K}$*-regular.*

Proof. Take K_2 very small and suppose the set M_{f_n,K_2} is not empty. Notice that two finite-size intervals near the points -1 and 0 do not belong to M_{f_n,K_2} if K_2 is small enough. This follows from the Property 1. It is easy to see that the first derivative $f_n'(x)$ must be close to 1 for $x \in M_{f_n,K_2}$. Otherwise the graph of $f_n(x)$ will cross the diagonal, which is impossible. Since the graph of f_n is above the diagonal, it follows that $f_n''(x)$ is greater than some positive constant K_3 for all $x \in M_{f_n,K_2}$. Indeed, if the second derivative is of the order of constant and negative then the graph of $f_n(x)$ would cross the diagonal. On the other hand, if the second derivative is small in absolute value, then the third derivative must be negative and not small (otherwise $Sf_n > -M_3$) and again the graph of $f_n(x)$ would cross the diagonal. We next show that M_{f_n,K_2} consists of just one open interval if K_2 is small enough. Indeed, if there are two disconnected components then there exists a point y in between such $f_n''(y) = 0$ and the second derivative changes its sign at point y from minus to plus. It follows that $f_n'''(y) \geq 0$ which implies $Sf_n(y) \geq 0$ in contradiction with the Property 4.

Finally, existence of constants K_1 and K_4 follow from the Properties 2 and 3 respectively. □

Remark 5. It is easy to see from the proof of Proposition 1 that for $\xi \in \check{\Delta}_0^{(n)}$ we have $|\phi'(\xi_0) - \phi'(\xi)| \leq S_{26}\mu^n$ with some $S_{26} > 0$. On the other hand, Condition 2) of Theorem 2 implies $|\xi_0 - \xi| \geq |\Delta_0^{(n+2)}| \geq S_{27}K_2^n$ with $S_{27} > 0$. Together, the latter two estimates prove $C^{1+\gamma}$-smoothness of ϕ at ξ_0, where $\gamma = \log_{K_2} \mu$. Thus, our results imply local $C^{1+\gamma}$-smoothness of the conjugacy at the critical point, which was earlier proven in [10].

Appendix

Proof of Lemma 5. Let us denote $y_i = s_i - 1/(i + s_0^{-1})$. Then $y_0 = 0$ and

$$y_{i+1} = y_i + A_i y_i + B_i \tag{44}$$

for all $i \geq 0$, where $A_i = -2/(i + s_0^{-1})$ and

$$B_i = -\frac{1}{\left(i + s_0^{-1}\right)^2\left(i + 1 + s_0^{-1}\right)} - y_i^2 + B_i^*$$

with $|B_i^*| = |s_{i+1} - (s_i - s_i^2)| \leq C_{13}|y_i + 1/(i + s_0^{-1})|^{2+\alpha}$.

We will prove (16) by an inductive procedure. Fix some $i > 0$ and suppose we have proven that

$$|y_j| \leq \frac{D_1}{\left(j + s_0^{-1}\right)^{1+\alpha}}$$

for all $0 \le j < i$ as long as $s_0 \in (0, d_1]$, with some $D_1 > 0$ and $d_1 \in (0, 1)$ (note, that for $i = 1$ this is true for any D_1 and d_1 since $y_0 = 0$). Then there exists $S_{28} = S_{28}(\alpha, D_1) \in (0, 1)$ such that

$$|y_j| \le \frac{1}{\left(j + s_0^{-1}\right)^{1+\alpha/2}}$$

for all $0 \le j < i$ as long as $s_0 \in (0, \min\{d_1, S_{28}\}]$. Therefore, there exists $S_{29} = S_{29}(\alpha, C_{13}) > 0$ such that

$$|B_j| \le \frac{S_{29}}{\left(j + s_0^{-1}\right)^{2+\alpha}} \tag{45}$$

for all $0 \le j < i$ as long as $s_0 \in (0, \min\{d_1, S_{28}\}]$.

The solution of the difference equation (44) may be written as

$$y_i = \sum_{j=0}^{i-1} B_j \prod_{k=j+1}^{i-1} (1 + A_k). \tag{46}$$

Taking into account that

$$0 < \prod_{k=k+1}^{i-1} \left(1 - \frac{2}{k + s_0^{-1}}\right) \le \exp\left\{-2 \int_{j+1}^{i} \frac{dt}{t + s_0^{-1}}\right\} = \frac{\left(j + 1 + s_0^{-1}\right)^2}{\left(i + s_0^{-1}\right)^2},$$

it is easy to derive from (46) and (45) that

$$|y_i| \le \frac{S_{30}}{\left(i + s_0^{-1}\right)^{1+\alpha}}$$

with some $S_{30} = S_{30}(\alpha, C_{13}) > 0$. Now we assign $D_1 = S_{30}$ and then $d_1 = S_{28}$, closing the induction and completing the proof of the estimate (16).

It follows easily from (16) that

$$\begin{aligned}\left|(s_i - s_{i+1}) - \frac{1}{\left(i + s_0^{-1}\right)^2}\right| &\le \left|s_i^2 - \frac{1}{\left(i + s_0^{-1}\right)^2}\right| + C_{13}|s_i|^{2+\alpha} \\ &\le \frac{D_2}{\left(i + s_0^{-1}\right)^{2+\alpha}} \le \frac{D_2 s_0^{\alpha}}{\left(i + s_0^{-1}\right)^2}\end{aligned}$$

with $D_2 > 0$ as required. □

Proof of Lemma 6. Let us denote $y_i = s_i - \sqrt{\varepsilon} \tan(\sqrt{\varepsilon} i + a_0)$. Then the Condition 2) implies that $y_0 = 0$ and

$$y_{i+1} = y_i + A_i y_i + B_i \tag{47}$$

for all $i \geq 0$ (the same as (44)), where $A_i = 2\sqrt{\varepsilon}\tan\sqrt{\varepsilon}i$ and

$$B_i = 2y_i(\sqrt{\varepsilon}\tan(\sqrt{\varepsilon}i + a_0) - \sqrt{\varepsilon}\tan\sqrt{\varepsilon}i) - \frac{\varepsilon}{(\cos(\sqrt{\varepsilon}i + a_0))^2}$$
$$\cdot\frac{(\sqrt{\varepsilon})^{-1}\tan\sqrt{\varepsilon} - 1 + (\tan\sqrt{\varepsilon})\tan(\sqrt{\varepsilon}i + a_0)}{1 - (\tan\sqrt{\varepsilon})\tan(\sqrt{\varepsilon}i + a_0)} + y_i^2 + B_i^* \qquad (48)$$

with $|B_i^*| = |s_{i+1} - (\varepsilon + s_i + s_i^2)| \leq C_{15}|y_i + \sqrt{\varepsilon}\tan(\sqrt{\varepsilon}i + a_0)|^{2+\alpha}$.

An easy calculation shows that there exists $S_{31} = S_{31}(\alpha, C_{14}, C_{16}) \in (0, 1)$ such that

$$|\sqrt{\varepsilon}\tan\sqrt{\varepsilon}i - \sqrt{\varepsilon}\tan(\sqrt{\varepsilon}i + a_0)| \leq 2C_{14}\frac{\varepsilon}{(\cos\sqrt{\varepsilon}i)^2} \qquad (49)$$

for all $0 \leq i \leq N$ as long as $\varepsilon \in (0, S_{31}]$. The inequality (49) implies that for $i \neq 0$ we have

$$|\sqrt{\varepsilon}\tan(\sqrt{\varepsilon}i + a_0)| \leq 2(C_{14} + 1)\sqrt{\varepsilon}\tan\sqrt{\varepsilon}i. \qquad (50)$$

It also follows from (49) that there exists $S_{32} = S_{32}(\alpha, C_{14}, C_{16}) > 0$ such that

$$\frac{1}{(\cos(\sqrt{\varepsilon}i + a_0))^2} \leq \frac{S_{32}}{(\cos\sqrt{\varepsilon}i)^2} \qquad (51)$$

for $0 \leq i \leq N$, $\varepsilon \in (0, S_{31}]$.

Again, we are using an inductive procedure in order to prove this lemma. Let us fix some $0 < i \leq N$ and suppose we have proven that

$$|y_j| \leq D_3(\sqrt{\varepsilon}\tan\sqrt{\varepsilon}j)^{1+\frac{\alpha(\alpha+1)}{2}} \qquad (52)$$

for all $0 \leq j < i$ and $\varepsilon \in (0, d_2]$ with some $D_3 > 0$ and $d_2 \in (0, 1)$ (note, that for $i = 1$ this is true for any D_3 and d_2 since $y_0 = 0$). Then there exists $S_{33} = S_{33}(\alpha, D_3, C_{16}) \in (0, 1)$ such that

$$|y_j| \leq (\sqrt{\varepsilon}\tan\sqrt{\varepsilon}j)^{1+\frac{\alpha}{2}} \qquad (53)$$

for all $0 \leq j < i$ as long as $\varepsilon \in (0, \min\{d_2, S_{33}\}]$. Using the asymptotic formula $\tan t = t + O(t^3)$, $t \to 0$, together with the inequalities (49), (50), (51) and (53), we obtain that there exist $S_{34} = S_{34}(\alpha, C_{14}, C_{15}, C_{16}) > 0$ and $S_{35} = S_{35}(\alpha, D_3, C_{14}, C_{16}) \in (0, 1)$ such that $|B_0| \leq S_{34}\varepsilon^2$ and

$$|B_j| \leq \frac{S_{34}\varepsilon}{(\cos\sqrt{\varepsilon}j)^2}(\sqrt{\varepsilon}\tan\sqrt{\varepsilon}j)^\alpha \qquad (54)$$

for all $0 < j < i$ as long as $\varepsilon \in (0, \min\{d_2, S_{35}\}]$.

The solution for the difference equation (47) may be written again as

$$y_i = \sum_{j=0}^{i-1} B_j \prod_{k=j+1}^{i-1}(1 + A_k). \qquad (55)$$

Taking into account that

$$0 < \prod_{k=j+1}^{i-1} (1 + 2\sqrt{\varepsilon} \tan \sqrt{\varepsilon} k) \le \exp\left\{ 2\int_j^i \sqrt{\varepsilon} \tan \sqrt{\varepsilon} t dt \right\} = \frac{(\cos \sqrt{\varepsilon} j)^2}{(\cos \sqrt{\varepsilon} i)^2},$$

we obtain due to (55) and (54) that

$$\begin{aligned} |y_i| &\le \frac{S_{34}\varepsilon}{(\cos \sqrt{\varepsilon} i)^2} \left(\varepsilon + \sum_{j=1}^{i-1} (\sqrt{\varepsilon} \tan \sqrt{\varepsilon} j)^\alpha \right) \\ &\le \frac{2S_{34}\varepsilon}{(\cos \sqrt{\varepsilon} i)^2} \int_0^i (\sqrt{\varepsilon} \tan \sqrt{\varepsilon} t)^\alpha dt \end{aligned} \tag{56}$$

as long as $\varepsilon \in (0, \min\{d_2, S_{35}\}]$.

Let us compare two functions

$$\phi(t) = \int_0^t (\sqrt{\varepsilon} \tan \sqrt{\varepsilon}\tau)^\alpha d\tau \quad \text{and} \quad \psi(t) = \frac{(\cos \sqrt{\varepsilon} t)^2}{\varepsilon} (\sqrt{\varepsilon} \tan \sqrt{\varepsilon} t)^{1+\frac{\alpha(\alpha+1)}{2}}$$

on the segment $[0, \frac{\pi}{2\sqrt{\varepsilon}})$ for arbitrary $\varepsilon \in (0, 1)$. We have

$$\psi'(t) = \left(\frac{\alpha(\alpha+1)}{2} + 1 - 2(\sin \sqrt{\varepsilon} t)^2 \right) \cdot (\sqrt{\varepsilon} \tan \sqrt{\varepsilon} t)^{\frac{\alpha(\alpha+1)}{2}},$$

so for $t \le \frac{\pi}{4\sqrt{\varepsilon}}$ the inequality $\psi'(t) \ge \frac{\alpha(\alpha+1)}{2}\phi'(t)$ holds, and the equality $\phi(0) = \psi(0) = 0$ implies $\psi \ge \frac{\alpha(\alpha+1)}{2}\phi$ on $[0, \frac{\pi}{4\sqrt{\varepsilon}}]$. The change of variables $\kappa = \sqrt{\varepsilon} \tan \sqrt{\varepsilon}\tau$ in the integral leads to the estimate

$$\begin{aligned} \phi(t) = \int_0^{\sqrt{\varepsilon} \tan \sqrt{\varepsilon} t} \frac{\kappa^\alpha d\kappa}{\kappa^2 + \varepsilon} &\le \int_0^{+\infty} \frac{\kappa^\alpha d\kappa}{\kappa^2 + \varepsilon} \\ &\le \int_0^{\sqrt{\varepsilon}} \frac{\kappa^\alpha d\kappa}{\varepsilon} + \int_{\sqrt{\varepsilon}}^{+\infty} \frac{\kappa^\alpha d\kappa}{\kappa^2} = \frac{2}{1-\alpha^2} \varepsilon^{\frac{\alpha-1}{2}}, \end{aligned}$$

so for $t \ge \frac{\pi}{4\sqrt{\varepsilon}}$ we have $\psi(t) = (\sin \sqrt{\varepsilon} t)^2 (\sqrt{\varepsilon} \tan \sqrt{\varepsilon} t)^{\frac{\alpha(\alpha+1)}{2} - 1} \ge \frac{1}{2}(\sqrt{\varepsilon})^{\frac{\alpha(\alpha+1)}{2} - 1} \ge \frac{1-\alpha^2}{4}\phi(t)$.

Now, it follows from (56) that

$$|y_i| \le S_{36} (\sqrt{\varepsilon} \tan \sqrt{\varepsilon} i)^{1+\frac{\alpha(\alpha+1)}{2}},$$

where $S_{36} = 2S_{34} \max\{\frac{2}{\alpha(\alpha+1)}, \frac{4}{1-\alpha^2}\}$. The assignments $D_3 = S_{36}$ and then $d_2 = S_{35}$ close the induction and prove the estimate (18).

It follows from the Condition 2) and (18) that

$$\begin{aligned}
&\left|(s_{i+1}-s_i)-\frac{\varepsilon}{(\cos\sqrt{\varepsilon}i)^2}\right| \\
&\quad\leq C_{15}|s_i|^{2+\alpha}+\left|\frac{\varepsilon}{(\cos(\sqrt{\varepsilon}i+a_0))^2}-\frac{\varepsilon}{(\cos\sqrt{\varepsilon}i)^2}\right| \\
&\quad\quad+2(\sqrt{\varepsilon}\tan(\sqrt{\varepsilon}i+a_0))\cdot D_3(\sqrt{\varepsilon}\tan\sqrt{\varepsilon}i)^{1+\frac{\alpha(\alpha+1)}{2}} \\
&\quad\quad+D_3^2(\sqrt{\varepsilon}\tan\sqrt{\varepsilon}i)^{2+\alpha(\alpha+1)}
\end{aligned}$$

for all $0\leq i<N$ and $\varepsilon\in(0,d_2]$. Applying the inequalities (49) and (50) to the right-hand side, it is not hard to derive that there exist $S_{37}=S_{37}(\alpha,C_{14},C_{15},C_{16})>0$ and $d_3\in(0,d_2]$ such that for $i\neq 0$ we have

$$\left|(s_{i+1}-s_i)-\frac{\varepsilon}{(\cos\sqrt{\varepsilon}i)^2}\right|\leq\frac{S_{37}\varepsilon}{(\cos\sqrt{\varepsilon}i)^2}(\sqrt{\varepsilon}\tan\sqrt{\varepsilon}i)^{\frac{\alpha(\alpha+1)}{2}} \tag{57}$$

as long as $\varepsilon\in(0,d_3]$. It also follows from 1) and 2) that $|(s_1-s_0)-\varepsilon|\leq S_{38}\varepsilon^2\leq S_{38}\varepsilon\cdot\varepsilon^{\alpha(\alpha+1)/2}$ with some $S_{38}=S_{38}(\alpha,C_{14},C_{15})>0$, so the last statement of the lemma holds with $D_4=\max\{S_{37}C_{16}^{\alpha(\alpha+1)/2},S_{38}\}$ after we reassign d_2 to be equal to d_3. □

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