

Boundedness of pluricanonical maps of varieties of general type^{*}

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Abstract. Using the techniques of [20] and [10], we prove that certain log forms may be lifted from a divisor to the ambient variety. As a consequence of this result, following [22], we show that: For any positive integer n there exists an integer r_n such that if X is a smooth projective variety of general type and dimension n , then $\phi_{rK_X}: X \dashrightarrow \mathbb{P}(H^0(\mathcal{O}_X(rK_X)))$ is birational for all $r \geq r_n$.

1. Introduction

One of the main problems of complex projective algebraic geometry is to understand the structure of the pluricanonical maps. If X is of general type, then by definition the pluricanonical map $\phi_{rK_X}: X \dashrightarrow \mathbb{P}(H^0(\mathcal{O}_X(rK_X)))$ is birational for all sufficiently large r . It is a natural question to then ask if there is an r_n such that ϕ_{rK_X} is birational as soon as $r \geq r_n$, uniformly for any variety of general type of dimension n . When X is a curve, it is well known that ϕ_{rK_X} is birational for $r \geq 3$ and when X is a surface, Bombieri proved in [3] that ϕ_{rK_X} is birational, for $r \geq 5$.

However, starting with threefolds the problem is substantially harder, since there are threefolds of general type for which the minimal model is necessarily singular. In fact it is easy to write down examples of threefolds of general type, for which the index of K_X (the smallest multiple of K_X

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which is Cartier) on a minimal model is arbitrarily large (for example, take any hypersurface of sufficiently large degree in any weighted projective space). In this case, a priori, the degree of K_X (equivalently the volume of K_X on a smooth model) could be arbitrarily small. Even the case when the minimal model is smooth, or at least the index of K_X is bounded, has attracted considerable attention, see for example, [7], [2], [19] and [5], and there have only been partial results in the general case, see [11], [16] and [17].

Recently, Tsuji settled this problem for 3-folds and assuming the minimal model program, for all higher dimensional varieties, see [22]. Using ideas of Tsuji, we prove the following:

Theorem 1.1. *For any positive integer n , there exists an integer r_n such that if X is a smooth projective variety of general type and dimension n , then $\phi_{rK_X} : X \dashrightarrow \mathbb{P}(H^0(\mathcal{O}_X(rK_X)))$ is birational for all $r \geq r_n$.*

(1.1) has some interesting consequences:

Corollary 1.2. *Let n be a positive integer and M be a positive constant. Then the family of all smooth projective varieties of general type of dimension n and volume of K_X at most M , is birationally bounded.*

Corollary 1.3. *Let n be a positive integer. Then there is a positive constant $\eta_n > 0$ such that given any smooth projective variety X of general type of dimension n , the volume of K_X is at least η_n .*

As Maehara points out in [18] (see also [22]), one also recovers the Severi-Iitaka conjecture:

Corollary 1.4. *For any fixed variety X , there exist only finitely many dominant rational maps $\pi : X \dashrightarrow Y$ (modulo birational equivalence), where Y is a variety of general type.*

The proof of the above results closely follows Tsuji's original strategy; in fact this note grew out of our desire to understand [22]. Even though our proof is based on the ideas of this paper, it only relies on the techniques of algebraic geometry. In particular we do not use the "Analytic Zariski Decomposition" and we do not make use of the results of [23] (which we were unable to follow). The main tools employed in this paper are the algebraic techniques of the minimal model program and in particular the theory of log canonical centres and of multiplier ideals.

We now give a short informal sketch of the proof of (1.1), which closely follows the strategy of Tsuji's proof, see [22]. As one would expect, the goal is to produce a \mathbb{Q} -divisor $\Delta \sim_{\mathbb{Q}} \lambda K_X$ such that Δ has an isolated log canonical centre at a general point $x \in X$. There is a well established strategy for producing divisors with log canonical centres at a general point $x \in X$, as soon as the volume is sufficiently large, so that the rational number λ depends only on the volume of K_X .

The problem is that the smallest log canonical centre V containing x might well be of positive dimension. In order to produce an isolated log canonical centre, one proceeds by inductively cutting down the (dimension of the) log canonical centres. As x is general, a resolution W of V must have general type, and so by induction we can certainly find a \mathbb{Q} -divisor Θ on W with the required properties.

At this point, the idea is to lift sections from V to the whole of X . In principle this ought to be straightforward; indeed if K_X is nef, then we can lift Θ to the whole of X , as an easy application of Serre vanishing. In practice we cannot assume that K_X is nef, and this step of the argument forms the technical heart of the paper. In fact we generalise some results and techniques of [20] and [10] for extending pluricanonical forms from a divisor to the whole space. Instead of just lifting pluricanonical forms, we lift certain log pluricanonical forms, from a log canonical centre to the whole of X , using a variant of multiplier ideal sheaves, see Sect. 3. It is then not too hard to prove that we can then lift log canonical centres from W to X , see Sect. 4.

Then we prove, assuming (1.1) in dimension less than n , that ϕ_{rK_X} is birational, where r is a fixed linear function of $1/\text{vol}(X)^{1/n}$, which only depends on the dimension. If $\text{vol}(K_X) \geq 1$, the theorem follows easily. If $\text{vol}(K_X) < 1$, one then shows that X belongs to a birationally bounded family and the existence of r_n is then clear.

As already pointed out, the main technical point is (3.17), the result which shows that certain log forms may be lifted from a divisor to the ambient variety. We hope that it will find other applications in a variety of contexts. For example, in [6], we use this result to prove a conjecture of Shokurov which in particular implies that the fibers of a resolution of a variety with log terminal singularities are rationally chain connected.

We now raise some questions and problems, which are closely related to (1.1):

Problem 1.5. Find explicit (hopefully small) values of r_n .

Even determining r_3 would seem interesting. Indeed in the past twenty years or so, our knowledge of threefolds has increased considerably; in particular we have Reid's powerful Riemann-Roch formula for terminal threefolds and we are able to write down many explicit examples of threefolds of general type. Thus determining the value of r_3 would seem to be a reasonable problem. In fact we raise the following:

Question 1.6. Is the rational map ϕ_{27K_X} always birational, whenever X is a threefold of general type?

Note that we have little or no evidence for an affirmative answer to (1.6), apart from the examples of Fletcher in [8]. In particular for the threefold X_{46} , a hypersurface in the weighted projective space $\mathbb{P}(4, 5, 6, 7, 23)$, see [8], ϕ_{mK_X} is birational iff $m = 23$, or $m \geq 27$.

Finally it seems natural to conjecture that we can drop the condition that X is of general type in (1.1):

Conjecture 1.7. There is a positive integer r_n with the following property:

Let X be a smooth variety of non-negative Kodaira dimension, of dimension n . Then the rational map ϕ_{rK_X} is birationally equivalent to the Iitaka fibration for all $r \geq r_n$.

Unfortunately the methods of this paper don't seem to apply directly to (1.7).

2. Preliminaries

2.1. Notation and conventions. We work over the field of complex numbers \mathbb{C} . A \mathbb{Q} -Cartier divisor D on a normal variety X is nef if $D \cdot C \geq 0$ for any curve $C \subset X$. We say that two \mathbb{Q} -divisors D_1, D_2 are \mathbb{Q} -linearly equivalent ($D_1 \sim_{\mathbb{Q}} D_2$) if there exists an integer $m > 0$ such that mD_i are linearly equivalent. Given a morphism of normal varieties $f: X \rightarrow Y$, we say that two divisors D_1 and D_2 on X are f -linearly equivalent ($D_1 \sim_f D_2$) if there is a Cartier divisor B on Y such that $D_1 \sim D_2 + f^*B$. We say that D_1 and D_2 are f -numerically equivalent ($D_1 \equiv_f D_2$) if there is a \mathbb{Q} -Cartier divisor B on Y such that $D_1 \equiv D_2 + f^*B$. A **pair** (X, Δ) is a normal variety X and a \mathbb{Q} -Weil divisor Δ such that $K_X + \Delta$ is \mathbb{Q} -Cartier. We will say that a pair (X, Δ) is a **log pair**, if in addition Δ is effective. We say that (X, Δ) is a **smooth pair** if X is smooth and Δ is a \mathbb{Q} -divisor with simple normal crossings support. A projective morphism $\mu: Y \rightarrow X$ is a **log resolution** of the pair (X, Δ) if Y is smooth and $\mu^{-1}(\Delta) \cup \{\text{exceptional set of } \mu\}$ is a divisor with simple normal crossings support. We write $\mu^*(K_X + \Delta) = K_Y + \Gamma$ and $\Gamma = \sum a_i \Gamma_i$ where Γ_i are distinct reduced irreducible divisors. The pair (X, Δ) is **kawamata log terminal** (resp. **log canonical**) if there is a log resolution $\mu: Y \rightarrow X$ as above such that the coefficients of Γ are strictly less than one i.e. $a_i < 1$ for all i (resp. $a_i \leq 1$). The number $1 - a_i$ is the **log discrepancy** of Γ_i with respect to (X, Δ) . We say that a subvariety $V \subset X$ is a **log canonical centre** if it is the image of a divisor of log discrepancy at most zero. We will denote by $\text{LLC}(X, \Delta, x)$ the set of all log canonical centres containing a point $x \in X$. A **log canonical place** is a valuation corresponding to a divisor of log discrepancy at most zero. A log canonical centre is **pure** if $K_X + \Delta$ is log canonical at the generic point of V . If moreover there is a unique log canonical place lying over the generic point of V , we will say that V is an **exceptional log canonical centre**.

2.2. Volumes.

Definition 2.1. Let X be an irreducible projective variety and let D be a big \mathbb{Q} -divisor. The **volume** of D , is

$$\text{vol}(D) = \limsup_{m \rightarrow \infty} \frac{n!h^0(X, mD)}{m^n}.$$

When D is very ample, $\text{vol}(D)$ is just the degree of the image of X in $\mathbb{P}H^0(X, \mathcal{O}_X(D))$. When D is nef, then $\text{vol}(D) = D^{\dim(X)}$. It turns out that the volume of D only depends on its numerical class and one can extend the volume function to a continuous function $\text{vol}: N_{\mathbb{R}}^1(X) \rightarrow \mathbb{R}$ (cf. [14]).

Lemma 2.2. *Let X be a projective variety, D a divisor such that ϕ_D is birational with image Z . Then, the volume of D is at least the degree of Z and hence at least 1.*

Proof. We may replace X by an appropriate birational model and in particular we may assume that $\phi := \phi_D$ is a morphism. Since ϕ is birational, its image Z is a non-degenerate subvariety of projective space \mathbb{P}^N and hence its degree is at least 1. From the inclusion

$$\phi^* \mathcal{O}_{\mathbb{P}^N}(1)|_Z \hookrightarrow \mathcal{O}_X(D),$$

one sees that $\text{vol}(D) \geq \text{vol}(\mathcal{O}_{\mathbb{P}^N}(1)|_Z) \geq 1$. □

2.3. Log canonical centres. Here we recall some well known results regarding log canonical centres. The first Lemma shows that when the volume of a \mathbb{Q} -divisor L is big, it is possible to produce a numerically equivalent \mathbb{Q} -divisor Δ_x with high multiplicity at a general point x .

Lemma 2.3. *Let V be an irreducible projective variety of dimension d , L a big \mathbb{Q} -Cartier divisor on V , and $x \in V$ a smooth point. If for a positive rational number α , one has*

$$\text{vol}(L) > \alpha^d,$$

then for any sufficiently divisible integer $k \gg 0$, there exists a divisor

$$A = A_x \in |kL| \quad \text{with} \quad \text{mult}_x(A) > k\alpha.$$

This is easily seen by a parameter count comparing the number of sections in $\mathcal{O}_V(kL)$ and the number of conditions one needs to impose for a function to vanish at a smooth point to order at least $k\alpha$. The next lemma shows that if the multiplicity of a \mathbb{Q} -divisor at a given point is sufficiently big, then the given point belongs to an appropriate log canonical centre.

Lemma 2.4. *Let (X, Δ) be a log pair, x a smooth point of X . If $\text{mult}_x(\Delta) \geq \dim X$, then $\text{LLC}(X, \Delta, x) \neq \emptyset$. If $\text{mult}_x(\Delta) < 1$, then $\text{LLC}(X, \Delta, x) = \emptyset$.*

Proof. See [14] 9.3.2 and 9.5.13. □

Assume that (X, Δ) is log canonical at x . It is often useful to assume that $Z = \text{LLC}(X, \Delta, x)$ is an irreducible variety. To this end recall (cf. [9] and [1] (3.4))

Lemma 2.5. *Let X be a normal variety and let Δ be an effective \mathbb{Q} -Cartier divisor such that $K_X + \Delta$ is \mathbb{Q} -Cartier. Assume that $x \in X$ is a kawamata log terminal point of X and that (X, Δ) is log canonical near $x \in X$. If $W_1, W_2 \in \text{LLC}(X, \Delta, x)$, and W is an irreducible component of $W_1 \cap W_2$ containing x , then W is in $\text{LLC}(X, \Delta, x)$. Therefore, if (X, Δ) is not kawamata log terminal at x , then $\text{LLC}(X, \Delta, x)$ has a unique minimal irreducible element, say V . Moreover, there exists an effective \mathbb{Q} -divisor E such that*

$$\text{LLC}(X, (1 - \epsilon)\Delta + \epsilon E, x) = \{V\}$$

for all $0 < \epsilon \ll 1$. We may also assume that there is a unique log canonical place lying above V . If X is projective, $x \in X$ is general and L is a big divisor, then one can take $E = aL$ for some positive number a .

2.4. Non-vanishing and birational maps. Recall the following well known method for producing sections of adjoint line bundles:

Lemma 2.6. *Let X be a smooth projective variety and D a big divisor on X . Let $0 < \lambda < 1$ a rational number and assume that for every general point $x \in X$, there exists $\Delta = \Delta_x \sim_{\mathbb{Q}} \lambda D$ a \mathbb{Q} -divisor such that x is a maximal (in terms of inclusion) element of $\text{LCS}(X, \Delta, x)$. Then $h^0(\mathcal{O}_X(K_X + D)) > 0$ and $h^0(\mathcal{O}_X(K_X + 2D)) \geq 2$.*

Proof. For $m \gg 0$, let G be a general divisor in $|mD|$. Then x is not contained in the support of G and so x is the only element of $\text{LLC}(X, \Delta + \frac{1-\lambda}{m}G, x)$. Let $\mu: X' \rightarrow X$ be a log resolution of $(X, \Delta + \frac{1-\lambda}{m}G)$ and let

$$\mathcal{I} := \mathcal{I}\left(X, \Delta + \frac{1-\lambda}{m}G\right) := \mu_* \mathcal{O}_{X'}\left(K_{X'/X} - \lfloor \mu^*\left(\Delta + \frac{1-\lambda}{m}G\right) \rfloor\right).$$

We may assume that $\mu^*|mD| = F + |M|$ where F has simple normal crossings and $|M|$ is base point free and hence M is nef and big. Then

$$\begin{aligned} \mu^*D - \lfloor \mu^*\left(\Delta + \frac{1-\lambda}{m}G\right) \rfloor &\sim_{\mathbb{Q}} \mu^*D - \lfloor \mu^*\left(\Delta + \frac{1-\lambda}{m}F\right) \rfloor \sim_{\mathbb{Q}} \\ &\left\{ \mu^*\left(\Delta + \frac{1-\lambda}{m}F\right) \right\} + \frac{1-\lambda}{m}M. \end{aligned}$$

So $\mu^*D - \lfloor \mu^*\left(\Delta + \frac{1-\lambda}{m}G\right) \rfloor$ is numerically equivalent to the sum of a fractional divisor with simple normal crossings and a nef and big divisor. By Kawamata-Viehweg vanishing, one sees that $R^i \mu_* \mathcal{O}_{X'}(K_{X'} + \mu^*D - \lfloor \mu^*\left(\Delta + \frac{1-\lambda}{m}G\right) \rfloor) = 0$ for $i > 0$. So there is a short exact sequence

$$\begin{aligned} 0 &\rightarrow \mathcal{O}_X(K_X + D) \otimes \mathcal{I} \rightarrow \mathcal{O}_X(K_X + D) \\ &\rightarrow \frac{\mathcal{O}_X(K_X + D)}{\mathcal{O}_X(K_X + D) \otimes \mathcal{I}} \rightarrow 0. \end{aligned}$$

Again by Kawamata-Viehweg vanishing, one has that

$$\begin{aligned} h^1(\mathcal{O}_X(K_X + D) \otimes \mathcal{I}) &= h^1\left(\mathcal{O}_{X'}\left(K_{X'} + \mu^*D - \lrcorner\mu^*\left(\Delta + \frac{1-\lambda}{m}G\right)\lrcorner\right)\right) \\ &= 0. \end{aligned}$$

Since x is the only element of $\text{LLC}(X, \Delta + \frac{1-\lambda}{m}G, x)$, one has that \mathbb{C}_x is a direct summand of $H^0(X, \frac{\mathcal{O}_X(K_X+D)}{\mathcal{O}_X(K_X+D)\otimes\mathcal{I}})$ and so $\mathcal{O}_X(K_X + D)$ is generated at x .

Pick a general point x_1 . Then we may find $D_1 \sim_{\mathbb{Q}} \lambda D$ with an isolated log canonical centre at x_1 . Now pick a general point x_2 , not in the support of D_1 , and pick a general divisor $D_2 \sim_{\mathbb{Q}} \lambda D$ with an isolated log canonical centre at x_2 . As x_2 and D_2 are general, D_2 does not contain x_1 . Thus x_1 and x_2 are isolated log canonical centres of $D_1 + D_2 \sim_{\mathbb{Q}} 2\lambda D$.

The rest of the proof that $h^0(\mathcal{O}_X(K_X + 2D)) \geq 2$ now follows the line above. \square

Given a line bundle with at least two sections, one has a rational map to \mathbb{P}^1 . The following lemma gives a way to produce birational maps.

Lemma 2.7. *Let X be a smooth projective variety with $h^0(X, \mathcal{O}_X(mK_X)) \geq 2$ for all $m \geq m_0$. Let $X' \rightarrow \mathbb{P}^1$ be any morphism induced by sections of $\mathcal{O}_X(m_0K_X)$ on an appropriate birational model X' of X . If F denotes the general fiber and $|sK_F|$ induces a birational morphism, then $|tK_X|$ induces a birational morphism for all $t \geq m_0(2s + 2) + s$.*

Proof. Following Theorem 4.6 of [11] and its proof, one sees that $|(m_0(2s + 1) + s)K_X|$ gives a birational map. Since mK_X is effective for all $m \geq m_0$, the assertion follows. \square

It is also possible to prove a converse to (2.6):

Lemma 2.8. *Let (X, Δ) be a log pair of dimension n , and let D be an integral Weil divisor such that the image Y of the rational map ϕ_D has dimension n .*

Then there is an open set U such that for any point $x \in U$, we may find $\Delta' \sim nD$, such that x is the only element of $\text{LLC}(X, \Delta + \Delta', x)$.

Proof. Let U be the complement of the singular locus, the support of Δ , and the locus where the map ϕ_D is not an étale morphism.

Pick $x \in X$ and let $y \in Y$ be a point of the image $\phi_D: X \rightarrow Y$. By assumption, Y sits inside projective space \mathbb{P}^N . Pick $n + 1$ general hyperplanes which contain y , H_1, H_2, \dots, H_{n+1} and $n + 1$ rational numbers a_1, a_2, \dots, a_{n+1} , such that

$$0 < a_i < 1 \quad \text{and} \quad \sum_i a_i = n.$$

Set

$$\Gamma = \sum a_i H_i.$$

Then y is the only log canonical centre of $K_Y + \Gamma$. Let W be the normalisation of the graph of $\phi = \phi_D$, so that there is a commutative diagram

$$\begin{array}{ccc} & W & \\ p \swarrow & & \searrow q \\ X & \overset{\phi}{\dashrightarrow} & Y. \end{array}$$

Let $D_i = p_* q^* H_i$, and set

$$\Delta' = \sum a_i D_i.$$

Then each D_i is linearly equivalent to D , so that Δ' is linearly equivalent to nD . On the other hand, as ϕ is étale over y , it follows that x is an isolated log canonical centre of $K_X + \Delta + \Delta'$. \square

2.5. Log additivity of the Kodaira dimension.

Lemma 2.9. *Let X and Y be smooth projective varieties and Δ an effective \mathbb{Q} -divisor on X with simple normal crossings support. Let $\pi: X \rightarrow Y$ be a morphism and suppose that $K_X + \Delta$ is log canonical on the general fiber W of π , and $\kappa(W, (K_X + \Delta)|_W) \geq 0$. Then*

$$\pi_* \mathcal{O}_X(m(K_{X/Y} + \Delta)),$$

*is weakly positive, for any m which is sufficiently divisible. In particular for any ample line bundle H on Y and any rational number $\epsilon > 0$, one has that $\kappa(K_{X/Y} + \Delta + \epsilon\pi^*H) \geq \dim Y$.*

Proof. It suffices to prove that

$$\pi_* \mathcal{O}_X(m(K_{X/Y} + \Delta_h)),$$

is weakly positive, where Δ_h consists of those components of Δ which dominate Y . In fact both [4], (4.13) and [15], (9.8) prove that this sheaf is weakly positive, as soon as $m\Delta_h$ is integral. \square

Remark 2.10. In fact both [4] and [15] prove a much stronger result, by putting an orbifold structure on the base which takes into account the multiple fibers of the morphism π . We will not need this stronger form.

The following consequences of (2.9) are well known:

Corollary 2.11. *Let X, Y, Δ, H be as in (2.9).*

(1) *If K_Y is pseudoeffective, then for all rational numbers $\epsilon > 0$*

$$\kappa(K_X + \Delta + \epsilon\pi^*H) \geq \dim Y.$$

(2) *If Y is of general type, $\kappa(K_X + \Delta) \geq \dim Y$.*

Corollary 2.12. *Let X, Y , and Δ be as in (2.9).*

Then there exists an ample line bundle A on X such that for all m sufficiently big and divisible, $h^0(X, \mathcal{O}_X(m(K_{X/Y} + \Delta)) + A) \geq 0$.

Proof. Let $n = \dim X$. Fix a very ample line bundle H on X such that $A = K_X + (n + 2)H$ is also ample. By (2.9), for all m sufficiently big and divisible, $\kappa(m(K_{X/Y} + \Delta) + H) \geq 0$. By [14] 11.2.13, one sees that $h^0(X, \mathcal{O}_X(K_X + (n + 2)H + m(K_{X/Y} + \Delta))) \geq 0$. \square

3. Lifting sections

In this section, we prove that certain log pluricanonical forms extend from a divisor to the ambient space. We follow the exposition of [10] very closely, which proves that pluricanonical forms extend, indicating those places where the definitions and proofs have to be modified.

Definition 3.1. *Let (X, Δ) be a smooth log pair, where every component of Δ has coefficient one and let D be a divisor on X . Let $\mu: W \rightarrow X$ be a birational morphism from a smooth variety W and let Θ be the support of all divisors of log discrepancy zero. Assume that there is a decomposition*

$$\mu^*D = P + M,$$

*in $\text{Div}(W) \otimes \mathbb{R}$ such that P is μ -nef and M is effective, where $\Theta \cup M \cup \{\text{the exceptional set of } \mu\}$ has normal crossing support, and Θ and M have no components in common. The **multiplier ideal sheaf** $\mathcal{I}_{\Delta, M}$ is defined by the following formula*

$$\mathcal{I}_{\Delta, M} = \mu_*\mathcal{O}_W(K_{W/X} + \Theta - \mu^*\Delta - \lfloor M \rfloor).$$

Note that any component of the strict transform of Δ is automatically a component of Θ . Equivalently, in the notation analogous to [10], one has that

$$\mu_*\mathcal{O}_W(\lceil P \rceil + K_W + \Theta) = \mathcal{I}_{\Delta, M}(D + K_X + \Delta).$$

When $\Delta = 0$, this reduces to the usual definition of \mathcal{I}_M . The notation $\lceil P \rceil$ is somewhat misleading as $\lceil P \rceil \neq \lceil P' \rceil$ for two \mathbb{Q} -linearly equivalent divisors. Therefore we replace it by $\mu^*D - \lfloor M \rfloor$.

Remark 3.2. The definition of Θ is motivated by two conflicting requirements:

(1) On the one hand, we require

$$(K_W + \Theta) - \mu^*(K_X + \Delta),$$

to be effective and exceptional. This forces us to include any divisors of log discrepancy zero into Θ .

(2) On the other hand, to be able to apply vanishing, we will need the supports of M and Θ to have no components in common. Thus we cannot allow any divisor in Θ , which belongs to M .

Note that $\mathcal{I}_{\Delta, M}$ is a coherent sheaf of ideals of \mathcal{O}_X which is determined only by M , Δ and μ . Since $(K_W + \Theta) - \mu^*(K_X + \Delta)$ and $\mu^*\Delta - \Theta$ are effective, one has that

$$(*) \quad \mu_*\mathcal{O}_W(-\lfloor M \rfloor) \subset \mathcal{I}_{\Delta, M} \subset \mathcal{I}_M.$$

We will need the following easy fact:

Lemma 3.3. *Let $v: W' \rightarrow W$ be a birational morphism of smooth projective varieties, G a reduced divisor on W , Γ a \mathbb{Q} -divisor on W with $\lfloor \Gamma \rfloor \leq 0$ and such that $G + \Gamma$ has simple normal crossings support and where G, Γ have no components in common. Then $(W, G + \Gamma)$ is log canonical and the set of divisors in W' of log discrepancy 0 for $(W, G + \Gamma)$ and for (W, G) coincide.*

Proof. See 2.31 of [13]. □

Lemma 3.4. *Notation as in (3.1). If $v: W' \rightarrow W$ is a birational morphism, such that*

$$v^*M \cup (v \circ \mu)^*\Delta \cup \{\text{exceptional locus of } (v \circ \mu)\},$$

*has simple normal crossings, then $\mathcal{I}_{\Delta, v^*M}$ is also defined and $\mathcal{I}_{\Delta, M} = \mathcal{I}_{\Delta, v^*M}$.*

Proof. Let Θ' be the sum of all the divisors in W' of log discrepancy zero for (X, Δ) . Then Θ' is also the sum of all divisors in W' of log discrepancy zero for $(W, \mu^*\Delta - K_{W/X})$ and so for (W, Θ) (as one has $\mu^*\Delta - K_{W/X} = \Theta - F$ where $F \geq 0$ has no component in common with Θ and $\Theta + F$ has simple normal crossings support, and so (3.3) applies). Since M and Θ have no common components and $M + \Theta$ has simple normal crossings, one sees that M contains no log canonical centres of (W, Θ) . Thus v^*M has no components in common with Θ' and so $\mathcal{I}_{\Delta, v^*M}$ is defined. Note that

$$\begin{aligned} M &= \lfloor M \rfloor + \{M\} && \text{so that,} \\ v^*M &= v^*\lfloor M \rfloor + v^*\{M\} && \text{and so,} \\ \lfloor v^*M \rfloor &= v^*\lfloor M \rfloor + \lfloor v^*\{M\} \rfloor. \end{aligned}$$

By definition

$$\begin{aligned} \mathcal{I}_{\Delta, v^*M} &= (v \circ \mu)_* \mathcal{O}_{W'}(K_{W'/X} + \Theta' - (v \circ \mu)^* \Delta - \lfloor v^*M \rfloor) \\ &= \mu_* (\mathcal{O}_W(K_{W/X} + \Theta - \mu^* \Delta - \lfloor M \rfloor) \\ &\quad \otimes v_* \mathcal{O}_{W'}(K_{W'/W} + \Theta' - v^* \Theta - \lfloor v^* \{M\} \rfloor)). \end{aligned}$$

Suppose that

$$K_{W'/W} + \Theta' - v^* \Theta - \lfloor v^* \{M\} \rfloor,$$

is effective and exceptional for v . Then

$$v_* \mathcal{O}_{W'}(K_{W'/W} + \Theta' - v^* \Theta - \lfloor v^* \{M\} \rfloor) = \mathcal{O}_W,$$

and the lemma follows. We may write

$$K_{W'} + \Theta^* = v^*(K_W + \Theta + \{M\}) + E,$$

where Θ^* is the reduced divisor whose components are the divisors on W' of log discrepancy zero for $(W, \Theta + \{M\})$ and hence for (W, Θ) . By (3.3), $\Theta^* = \Theta'$ and so

$$K_{W'/W} + \Theta' - v^* \Theta - \lfloor v^* \{M\} \rfloor = \lceil E \rceil,$$

is effective and exceptional as desired. \square

It is important to be able to compare multiplier ideal sheaves. Here is the first basic result along these lines:

Lemma 3.5. *Let (X, Δ) be a smooth log pair, where every component of Δ has coefficient one and let D be a divisor on X . Assume that there is a birational map $\mu: W \rightarrow X$ with the properties of (3.1). Let $\Delta' \leq \Delta$ be another divisor, where every component of Δ' has coefficient one.*

Then

$$\mathcal{I}_{\Delta, M} \subset \mathcal{I}_{\Delta', M}.$$

Proof. Let F be an exceptional divisor extracted by μ . Then the coefficient of F in

$$(K_W + \Theta) - \mu^*(K_X + \Delta),$$

is either the log discrepancy minus one, whenever the log discrepancy is at least one, or it is zero. In particular the coefficient of F is an increasing function of the log discrepancy of F . It follows easily that

$$(K_W + \Theta) - \mu^*(K_X + \Delta) \leq (K_W + \Theta') - \mu^*(K_X + \Delta'),$$

which in turn gives the inclusion of ideal sheaves. \square

Theorem 3.6. *Let (X, Δ) be a smooth log pair, where every component of Δ has coefficient one and let $\pi: X \rightarrow S$ be a projective morphism of algebraic varieties. Let M be an integral divisor such that $M \equiv_{\pi} L + G$ where*

- (1) L is π -nef and there is an effective \mathbb{Q} -divisor B such that $A = L - B$ is π -ample,
- (2) Δ and $G \cup B$ have no components in common, $\lfloor G \rfloor = 0$ and
- (3) the support of $\Delta \cup B \cup G$ has simple normal crossings.

Then $R^p \pi_* \mathcal{O}_X(K_X + \Delta + M) = 0$, for all $p > 0$.

Proof. As L is π -nef, $L - \delta B$ is π -ample, for any $\delta > 0$. Pick $\epsilon > 0$ such that $\epsilon \Delta + L - \delta B$ is π -ample. As Δ and $G \cup B$ have no components in common and every component of Δ has coefficient one, we see that $(X, (1 - \epsilon)\Delta + G + \delta B)$ is kawamata log terminal for any δ sufficiently small. But then, since

$$\Delta + M \equiv_{\pi} (\epsilon \Delta + L - \delta B) + (1 - \epsilon)\Delta + G + \delta B,$$

we may apply Kawamata-Viehweg vanishing. □

We fix some notation:

- (1) (Y, Γ) is a smooth log pair,
- (2) the divisor $X \subset Y$ is a component of Γ of coefficient one,
- (3) (Y, Γ) is log canonical,
- (4) S is the germ of an algebraic variety,
- (5) $\pi: Y \rightarrow S$ is a projective morphism,
- (6) Δ is the restriction of $\Gamma - X$ to X .

Note that we make no requirement about how X sits in Y , with respect to π . We will also assume that every coefficient of Γ has coefficient one, until (3.16).

Definition 3.7. *A divisor D on Y will be called π -effective for the pair (Y, X) if the natural homomorphism*

$$\pi_* \mathcal{O}_Y(D) \rightarrow \pi_* \mathcal{O}_X(D),$$

is not zero. It is called π - \mathbb{Q} -effective for the pair (Y, X) if mD is π -effective for the pair (Y, X) for some positive integer m . D is said to be π -big for the pair (Y, X) if we can write $mD = A + B$ for a positive integer m , a π -ample divisor A and a π -effective divisor B for the pair (Y, X) .

A divisor D on X will be called π -transverse for (X, Δ) if the sheaf $\mathcal{O}_X(D)$ is π -generated at the generic point of every log canonical centre of $K_X + \Delta$. D will be called π - \mathbb{Q} -transverse for (X, Δ) if mD is π -transverse for some $m > 0$.

Now we turn to the definition of sheaves $\mathcal{F}_{\Delta, D}^0, \mathcal{I}_{\Delta, D}^0, \mathcal{F}_{\Delta, D}^1, \mathcal{I}_{\Delta, D}^1$, which generalize the sheaves $\mathcal{F}_D^0, \mathcal{I}_D^0, \mathcal{F}_D^1, \mathcal{I}_D^1$ of Definition 2.10 of [10].

We will need a:

Definition 3.8. *Let (X, Δ) be a log pair and let $\mu: W \rightarrow X$ be a birational morphism. We say that μ is **canonical** if every exceptional divisor extracted by μ has log discrepancy at least one.*

If (X, Δ) is a pair consisting of a smooth variety and a divisor with simple normal crossings support, then note that μ is canonical provided it is an isomorphism in a neighbourhood of the generic point of any irreducible component of the intersection of the components of Δ , that is provided it is an isomorphism in a neighbourhood of any log canonical centre. Note also that if μ is canonical, then Θ is the strict transform of Δ .

Definition 3.9. *Let D be a π - \mathbb{Q} -transverse divisor for (X, Δ) . For each positive integer m , such that mD is π -transverse, pick a proper birational canonical morphism $\mu_m: W_m \rightarrow X$ from a smooth variety W_m , such that*

- (1) *there is a decomposition $\mu_m^*(mD) = P_m + M_m$ in $\text{Div}(W_m)$ such that the natural homomorphism*

$$(\pi \circ \mu_m)_* \mathcal{O}_{W_m}(P_m) \rightarrow \pi_* \mathcal{O}_X(mD),$$

is an isomorphism,

- (2) *P_m is $\pi \circ \mu_m$ -free, and*
 (3) *$M_m \cup \Theta_m \cup \{\text{the exceptional set of } \mu_m\}$ is effective, with normal crossing support, where Θ_m is defined as in (3.1).*

We define

$$\mathcal{I}_{\Delta, D}^0 = \bigcup \mathcal{I}_{\Delta, \frac{1}{m}M_m},$$

where the union is taken over all positive integers m such that mD is π -transverse. If D itself is π -transverse then we set $\mathcal{I}_{\Delta, D}^0 = \mathcal{I}_{\Delta, M_1}$.

Now suppose that D is a π - \mathbb{Q} -transverse divisor on Y for (Y, Γ) . For every positive integer m , such that mD is a π -transverse divisor for (Y, Γ) , pick a birational morphism $\mu_m: W_m \rightarrow Y$ from a smooth variety W_m , such that

- (1) *there is a decomposition $\mu_m^*(mD) = Q_m + N_m$ in $\text{Div}(W_m)$ such that the natural homomorphism*

$$(\pi \circ \mu_m)_* \mathcal{O}_{W_m}(Q_m) \rightarrow \pi_* \mathcal{O}_Y(mD),$$

is an isomorphism,

- (2) *Q_m is $\pi \circ \mu_m$ -free, and*
 (3) *N_m is effective, and $X_m + N_m + \Theta_m + \{\text{the exceptional set of } \mu_m\}$ has normal crossing support, where X_m is the strict transform of X and Θ_m is defined as in (3.1).*

By assumption X_m is not contained in the support of N_m . We define

$$\mathcal{F}_{\Delta, D}^1 = \bigcup \mathcal{I}_{\Delta, \frac{1}{m}N_m|_{X_m}},$$

where the union is taken over all positive integers m such that mD is π -transverse for the pair (Y, Γ) . If D itself is π -transverse for (Y, Γ) , then we set $\mathcal{I}_{\Delta, D}^1 = \mathcal{I}_{\Delta, N_1|_{X_1}}$.

Remark 3.10. We will not make use of the sheaves $\mathcal{I}_{\Delta, D}^i$, for $i = 0$ and 1 . We have included their definition for the sake of completeness.

Lemma 3.11. *Let D be a π - \mathbb{Q} -transverse divisor for (Y, Γ) (respectively for (X, Δ)). Then the sheaf $\mathcal{F}_{\Delta, D}^i$ is defined, it does not depend on the resolution μ_m and we may choose μ_m to be canonical for any m sufficiently divisible, where $i = 1$ (respectively $i = 0$). Further, we may assume that there is a μ_m -ample divisor whose support is exceptional.*

Proof. This is an easy consequence of (3.4) and of the Resolution Lemma of [21]. \square

Lemma 3.12. *Let D be a π - \mathbb{Q} -transverse divisor for (Y, Γ) .*

- (1) $\mathcal{F}_{\Delta, D}^1 \subset \mathcal{F}_{\Delta, D|_X}^0$
- (2) *If B is an effective integral divisor, which does not contain X and $D+B$ is π - \mathbb{Q} -transverse for (Y, Γ) , then*

$$\mathcal{F}_{\Delta, D}^i(-B) \subset \mathcal{F}_{\Delta, D+B}^i, \quad i = 0, 1.$$

- (3) *There is a positive integer m such that*

$$\mathcal{F}_{\Delta, D}^0 = \mathcal{I}_{\Delta, \frac{1}{m}M_m} \quad \text{and} \quad \mathcal{F}_{\Delta, D}^1 = \mathcal{I}_{\Delta, \frac{1}{m}N_m|_{X_m}}.$$

Proof. (1) follows, since $N_m|_{X_m} \geq M_m$ as divisors on X_m . (2) is clear. (3) follows as X is Noetherian. \square

Lemma 3.13. *Let D be a π - \mathbb{Q} -transverse divisor for (Y, Γ) . Then*

- (1) $\mathcal{F}_{\Delta, \alpha D}^i \subset \mathcal{F}_{\Delta, D}^i$, for any rational number $\alpha \geq 1$ and $i = 0, 1$.
- (2) $\mathcal{F}_{\Delta, D}^i \subset \mathcal{F}_{\Delta, D+L}^i$, for any π -free divisor L , and $i = 0, 1$.
- (3) $\text{Im}(\pi_*\mathcal{O}_Y(D) \longrightarrow \pi_*\mathcal{O}_X(D)) \subset \pi_*\mathcal{F}_{\Delta, D}^1(D)$.

Proof. First of all, notice that Definition (3.9) also makes sense for \mathbb{Q} -divisors. (1) and (2) are clear. We may assume that μ_m is chosen so that $\mu_m^*D = P + M$ in $\text{Div}(W_m)$ and

$$(\pi \circ \mu_m)_*\mathcal{O}_{W_m}(P) \longrightarrow \pi_*\mathcal{O}_Y(D),$$

is an isomorphism. Since $M > M_m/m$ one sees that there are inclusions

$$\mathcal{O}_{W_m}(\mu_m^*D - M) \subset \mathcal{O}_{W_m}(\mu_m^*D - \lfloor M_m/m \rfloor) \subset \mathcal{O}_{W_m}(\mu_m^*D).$$

We now push forward via $(\pi \circ \mu_m)$. Since $P = \mu_m^*D - M$, both the left and right hand sides push forward to $\pi_*\mathcal{O}_Y(D)$. So the image of $\pi_*\mathcal{O}_Y(D)$

is contained in the image of $(\pi \circ \mu_m)_* \mathcal{O}_{W_m}(\mu_m^* D - \lfloor M_m/m \rfloor)$ and hence in $\pi_* \mathcal{F}_{\Delta, D}^1(D)$ (cf. equation (*), after (3.2)). \square

The following lemma is the key result that will allow us to generalize the results of Siu and Kawamata to the setting of log-forms:

Lemma 3.14. *Suppose that D is a divisor on Y and suppose that there is an effective divisor B on Y such that*

1. $A = D - B$ is π -ample, and
2. B is π - \mathbb{Q} -transverse for (Y, Γ) .

Then

$$\begin{aligned} & \pi_* \mathcal{F}_{\Delta, D}^1(D + K_X + \Delta) \\ & \subset \text{Im}(\pi_* \mathcal{O}_Y(D + K_Y + \Gamma) \longrightarrow \pi_* \mathcal{O}_X(D + K_X + \Delta)). \end{aligned}$$

Proof. Notation as in (3.9). There is an exact sequence

$$\begin{aligned} 0 & \longrightarrow \mathcal{O}_{W_m} \left(K_{W_m} + \Theta_m - X_m + \mu_m^* D - \lfloor \frac{N_m}{m} \rfloor \right) \\ & \longrightarrow \mathcal{O}_{W_m} \left(K_{W_m} + \Theta_m + \mu_m^* D - \lfloor \frac{N_m}{m} \rfloor \right) \\ & \longrightarrow \mathcal{O}_{X_m} \left(K_{X_m} + (\Theta_m - X_m)|_{X_m} + \left(\mu_m^* D - \lfloor \frac{N_m}{m} \rfloor \right) \Big|_{X_m} \right) \longrightarrow 0. \end{aligned}$$

Claim 3.15. If m is sufficiently large then

$$R^1(\pi \circ \mu_m)_*(\mathcal{O}_{W_m}(K_{W_m} + \Theta_m - X_m + \mu_m^* D - \lfloor \frac{N_m}{m} \rfloor)) = 0.$$

Proof of Claim 3.15. By (3.11), we may assume that μ_m is canonical and there is an effective divisor E supported on the exceptional locus such that $\mu_m^* A - E$ is $\pi \circ \mu_m$ -ample. As μ_m is canonical, $E + \{N_m/m\}$ and $\Theta_m - X_m$ have no common component. Since B is π - \mathbb{Q} -transverse, we may assume that $\mu_m^*(mB) \sim_{\pi, \mathbb{Q}} N'$ where N' and $\Theta_m - X_m$ have no common component and $N_m + N' + \Theta_m + \{\text{exceptional locus of } \mu_m\}$ has simple normal crossings support. Notice that since $D - B$ is π -ample, we have that D is π - \mathbb{Q} -transverse and that $N_m \leq N'$. We now apply (3.6) to $\Delta = \Theta_m - X_m$, $M = \mu_m^* D - \lfloor N_m/m \rfloor$, $G = \{N_m/m\}$ and $L = Q_m/m$. Notice that $L - (\frac{N' - N_m}{m} + E) \sim_{\pi \circ \mu_m} \mu_m^* A - E$ is $\pi \circ \mu_m$ -ample and $\frac{N' - N_m}{m} + E$ is effective. The hypothesis of (3.6) are now verified and so

$$R^p(\pi \circ \mu_m)_* \mathcal{O}_{W_m} \left(K_{W_m} + \Theta_m - X_m + \mu_m^* D - \lfloor \frac{N_m}{m} \rfloor \right)$$

vanishes for $p > 0$. \square

Thus there is a surjective homomorphism

$$\begin{aligned} \pi_* \mathcal{O}_Y(D + K_Y + \Gamma) &\supset (\pi \circ \mu_m)_* \mathcal{O}_{W_m} \left(K_{W_m} + \Theta_m + \mu_m^* D - \lfloor \frac{N_m}{m} \rfloor \right) \\ &\longrightarrow (\pi \circ \mu_m)_* \mathcal{O}_{X_m} \left(K_{X_m} + \Theta_m - X_m + \mu_m^* D - \lfloor \frac{N_m}{m} \rfloor \right) \\ &= \mathcal{F}_{\Delta, D}^1(D + K_X + \Delta), \end{aligned}$$

hence the assertion. \square

Theorem 3.16. *Let $X \subset Y$ be a smooth divisor in a smooth variety. Fix a positive integer e . Let H be a sufficiently π -very ample divisor and set $A = (\dim X + 1)H$. Assume that*

- (1) Γ is a \mathbb{Q} -divisor with simple normal crossings support such that Γ contains X with coefficient one and (Y, Γ) is log canonical;
- (2) k is a positive integer such that $k\Gamma$ is integral;
- (3) $C \geq 0$ is an integral divisor not containing X ;
- (4) Given $D = k(K_Y + \Gamma)$, $D' = D|_X = k(K_X + \Delta)$ and $G = D + C$, then D' is π - \mathbb{Q} -pseudo-effective (i.e. $D' + \epsilon H'$ is π - \mathbb{Q} -effective for any $\epsilon > 0$) and that G is π - \mathbb{Q} -transverse for $(Y, \lceil \Gamma \rceil)$.

Then

$$\mathcal{F}_{mD' + \epsilon H'}^0(-mC) \subset \mathcal{F}_{mG + \epsilon H + A}^1, \quad \forall m > 0,$$

where $H' := H|_X$.

Proof. It suffices to prove the stronger inclusion

$$\mathcal{F}_{mD' + \epsilon H'}^0(-mC) \subset \mathcal{F}_{\lceil \Gamma \rceil, mG + \epsilon H + A}^1.$$

This is a modification of the proof of Lemma 3.7 of [10], which we follow quite closely. We prove the inclusion by induction on m , in the case $e = 1$, the general case being left to the reader.

By assumption $mD' + H'$ is π -big, C does not contain X and $mG + H + A$ is π - \mathbb{Q} -transverse for $(Y, \lceil \Gamma \rceil)$ for any non-negative integer m . In particular, both sides of the proposed inclusion are well-defined.

If $m = 0$ the result is clear. Assume the result for m . Suppose that

$$\Gamma = \sum_j \gamma_j \Gamma(j), \quad \text{where} \quad 0 < \gamma_j := \frac{g_j}{k} \leq 1.$$

Then set

$$\Gamma_i = \sum_j \gamma_i(j) \Gamma(j) \quad \text{where} \quad \gamma_i(j) = \begin{cases} 0 & \text{if } 1 \leq i \leq k - g_j, \\ 1 & \text{if } k - g_j < i \leq k + 1. \end{cases}$$

With this choice of Γ_i , we have

$$X \leq \Gamma_1 \leq \Gamma_2 \leq \Gamma_3 \leq \cdots \leq \Gamma_k = \Gamma_{k+1} = \lceil \Gamma \rceil,$$

and each Γ_i is integral. Let Δ_i be the restriction of $\Gamma_i - X$ to X . Set

$$D_i = K_Y + \Gamma_i \quad \text{and} \quad D_{\leq i} = \sum_{j \leq i} D_j.$$

With this choice of D_i , we have $D = D_{\leq k}$. Possibly replacing H by a multiple, we may assume that $H_i = D_{\leq i-1} + \Gamma_i - X + H$ and $A_i = H + D_{\leq i} + A$ are π -ample, for $1 \leq i \leq k$. We are now going to prove, by induction on i , that

$$\mathcal{F}_{mD'+H'}^0(-mC) \subset \mathcal{F}_{\Delta_{i+1}, mG+D_{\leq i}+H+A}^1 \quad \text{where } 0 \leq i \leq k,$$

where we adopt the convention that $D_{\leq 0} = 0$.

Since $\Delta_1 \leq \lceil \Delta \rceil$, it follows by (3.5)

$$\mathcal{F}_{\lceil \Delta \rceil, mG+H+A}^1 \subset \mathcal{F}_{\Delta_1, mG+H+A}^1,$$

and so the case $i = 0$ follows from the inclusion

$$\mathcal{F}_{mD'+H'}^0(-mC) \subset \mathcal{F}_{\lceil \Delta \rceil, mG+H+A}^1,$$

which we are assuming by induction on m . Assume the result up to $i - 1$. Then

$$\begin{aligned} & \pi_* \mathcal{F}_{mD'+H'}^0(mD + D_{\leq i} + H + A) \\ &= \pi_* \mathcal{F}_{mD'+H'}^0(-mC)(mG + D_{\leq i} + H + A) \\ &\subset \pi_* \mathcal{F}_{\Delta_i, mG+D_{\leq i-1}+H+A}^1(mG + D_{\leq i} + H + A) \\ &= \pi_* \mathcal{F}_{\Delta_i, mG+D_{\leq i-1}+H+A}^1(mG + D_{\leq i-1} + H + A + K_Y + \Gamma_i) \\ &\subset \text{Im}(\pi_* \mathcal{O}_Y(mG + D_{\leq i} + H + A) \longrightarrow \pi_* \mathcal{O}_X(mG + D_{\leq i} + H + A)) \\ &\subset \pi_* \mathcal{F}_{\Delta_{i+1}, mG+D_{\leq i}+H+A}^1(mG + D_{\leq i} + H + A), \end{aligned}$$

where we use induction to get the inclusion of the second line in the third line, (3.14) applied to the basic identity

$$mG + D_{\leq i-1} + H + A = A_{i-1} + B,$$

to get the inclusion of the fourth line in the fifth line, and (3) of (3.13) to get the inclusion of the fifth line in the sixth line. As

$$\begin{aligned} & \mathcal{F}_{mD'+H'}^0(mD + D_{\leq i} + H + A) \\ &= \mathcal{F}_{mD'+H'}^0((mD + H) + H_i + (\dim X)H + K_X), \end{aligned}$$

is π -generated by (a trivial generalization of) (3.2) of [10], it follows that

$$\mathcal{F}_{mD'+H'}^0(-mC) \subset \mathcal{F}_{\Delta_{i+1}, mG+D_{\leq i}+H+A}^1.$$

This completes the induction on i . It follows that

$$\mathcal{F}_{mD'+H'}^0(-mC) \subset \mathcal{F}_{\Delta_{k+1}, mG+D_{\leq k}+H+A}^1,$$

so that, by (2) of (3.12),

$$\mathcal{F}_{mD'+H'}^0(-(m+1)C) \subset \mathcal{F}_{\Gamma_{\Delta^\nabla}, (m+1)G+H+A}^1.$$

But by (1) and (2) of (3.13),

$$\mathcal{F}_{(m+1)D'+H'}^0 \subset \mathcal{F}_{mD'+H'}^0.$$

Thus

$$\mathcal{F}_{(m+1)D'+H'}^0(-(m+1)C) \subset \mathcal{F}_{\Gamma_{\Delta^\nabla}, (m+1)G+H+A}^1,$$

and this completes the induction on m and the proof. \square

Corollary 3.17. *Let $X \subset Y$ be a smooth divisor in a smooth variety. Let H be a sufficiently π -very ample divisor and set $A = (\dim X + 1)H$. Assume that*

- (1) Γ is a \mathbb{Q} -divisor with simple normal crossings support such that Γ contains X with coefficient one and (Y, Γ) is log canonical;
- (2) $C \geq 0$ is a \mathbb{Q} -divisor not containing X ;
- (3) $K_X + \Delta$ is π - \mathbb{Q} -pseudo-effective, where $\Delta = (\Gamma - X)|_X$ and $K_Y + \Gamma + C$ is π - \mathbb{Q} -transverse for $(Y, \Gamma_{\Delta^\nabla})$.

For any positive integer m , such that $m(K_Y + \Gamma)$ is integral, the image of the natural homomorphism

$$\pi_* \mathcal{O}_Y(m(K_Y + \Gamma + C) + H + A) \longrightarrow \pi_* \mathcal{O}_X(m(K_X + \Delta + C) + H + A),$$

contains the image of the sheaf $\pi_ \mathcal{O}_X(m(K_X + \Delta) + H)$ considered as a subsheaf of $\pi_* \mathcal{O}_X(m(K_X + \Delta + C) + H + A)$ by the inclusion induced by any divisor in $mC + |A|$ not containing X .*

Proof. Let $H' = H|_X$. Fixing a section $\sigma \in H^0(Y, \mathcal{O}_Y(A))$ not vanishing on X , we can view

$$\mathcal{F}_{m(K_X+\Delta)+H'}^0(m(K_Y + \Gamma) + H),$$

as a subsheaf of

$$\mathcal{F}_{m(K_X+\Delta)+H'}^0(m(K_Y + \Gamma) + H + A),$$

via the map induced by multiplication by σ . As we have seen in the proof of (3.16),

$$\begin{aligned} \pi_* \mathcal{F}_{m(K_X + \Delta) + H'}^0(m(K_Y + \Gamma) + H + A) = \\ \pi_* \mathcal{F}_{m(K_X + \Delta) + H'}^0(-mC)(m(K_Y + \Gamma + C) + H + A), \end{aligned}$$

is contained in the image of $\pi_* \mathcal{O}_Y(m(K_Y + \Gamma + C) + H + A)$. Since

$$\pi_* \mathcal{O}_X(m(K_X + \Delta) + H) = \pi_* \mathcal{F}_{m(K_X + \Delta) + H'}^0(m(K_Y + \Gamma) + H),$$

the assertion now follows easily. \square

4. Lifting log canonical centres

We fix some notation for this section. Let (X, Δ) be a log pair, where X is \mathbb{Q} -factorial, and Δ is an effective \mathbb{Q} -divisor. Let V be an exceptional log canonical centre of $K_X + \Delta$. Let $f: W \rightarrow V$ be a resolution of V . Let Θ be an effective \mathbb{Q} -divisor on W . Suppose that there are positive rational numbers λ and μ such that $\Delta \sim \lambda K_X$ and $\Theta \sim \mu K_W$. Let $\nu = (\lambda + 1)(\mu + 1) - 1$. Suppose that W is of general type.

The main result of this section is:

Theorem 4.1. *There is a very general subset U of V with the following property:*

Suppose that $W' \subset W$ is a pure log canonical centre of $K_W + \Theta$, whose image $V' \subset V$ intersects U .

Then, for every positive rational number δ , we may find an effective divisor Δ' on X such that V' is a pure log canonical centre of $K_X + \Delta'$, where $\Delta' \sim (\nu + \delta)K_X$. Now suppose that we may write $K_X \sim A + E$, where A is ample, E is effective and V is not contained in E . Then we may assume that V' is an exceptional log canonical centre of $K_X + \Delta'$.

(4.1) is the main step in the proof of (1.1). In fact (1.1) is a standard consequence of (4.1), see Sect. 6.

The idea is to lift Θ as a \mathbb{Q} -divisor, using (3.17). However this is more delicate than it might first appear, as we cannot lift Θ directly to X . Instead, we are able to find a sequence of successively better approximations Θ_m . In particular we have to pass to a resolution of X , and to this end, we introduce some more notation.

Let $\pi: Y \rightarrow X$ be a log resolution of (X, Δ) . Let E be the unique exceptional divisor of log discrepancy zero with centre V . Possibly blowing up further, we may assume that the restriction $p: E \rightarrow V$ of π to E factors

$g: E \rightarrow W$ through f . Thus we have a commutative diagram,

$$\begin{array}{ccccc}
 E_\xi & \subset & E & \hookrightarrow & Y \\
 & \nearrow g & \downarrow p & & \downarrow \pi \\
 W & & V & \hookrightarrow & X \\
 & \searrow f & & & \\
 \xi & \in & & &
 \end{array}$$

Let ξ be the generic point of V and let E_ξ be the generic fibre of p .

Lemma 4.2. *We may find a positive integer m_0 and a positive integer s , with the following properties:*

For every sufficiently large and divisible integer m , there is an effective divisor G_m on Y such that

- (1) G_m does not contain E ,
- (2) $G_m + sE \sim (m_0 + m(1 + \lambda))\pi^*K_X$,
- (3) $G_m|_E = mg^*\Theta + B_m$, where the restriction of B_m to the inverse image of a fixed open subset of V is effective, and
- (4) $B_m|_{E_\xi}$ belongs to a fixed linear system.

Proof. We may assume that

$$K_Y + \Gamma = \pi^*(K_X + \Delta) + F,$$

where Γ and F are effective, with no common components and F is exceptional. We may write $\Gamma = E + \Gamma^h + \Gamma^v$, $F = F^h + F^v$ where the components of Γ^h and F^h are the components of $\Gamma - E$ and F whose images contain V . As V is an exceptional log canonical centre for (X, Δ) , $\{\Gamma^h\} = \Gamma^h$ and $\Gamma^h|_{E_\xi} = (\Gamma - E)|_{E_\xi}$ and so,

$$(K_E + \Gamma^h)|_{E_\xi} = (K_Y + \Gamma)|_{E_\xi} = F|_{E_\xi} \geq 0.$$

By Corollary (2.12), we may assume that

$$h^0(E, \mathcal{O}_E(m(K_{E/W} + \Gamma^h) + H)) > 0,$$

where H is a sufficiently ample divisor on Y . Therefore, there is an injection

$$g^*|mK_W| \rightarrow |m(K_E + \Gamma^h) + H|,$$

induced by a choice of a divisor in $|m(K_{E/W} + \Gamma^h) + H|$.

We now apply (3.17). Possibly blowing up further, we may assume that the image of every element of $\text{LLC}(Y, E + \lceil \Gamma^h \rceil)$ contains V . As the divisors $E + \lceil \Gamma^h \rceil$ and F have no common components, it follows that $K_Y + \Gamma = \pi^*(K_X + \Delta) + F$ is π - \mathbb{Q} -transverse for $(Y, E + \lceil \Gamma^h \rceil)$. Notice also that as W is of general type, $(K_Y + E + \Gamma^h)|_E = (K_E + \Gamma^h|_E)$ is

\mathbb{Q} -effective, by (2.11). By Corollary (3.17), with $S = \text{Spec}(\mathbb{C})$, it follows that elements of

$$|m(K_E + \Gamma^h|_E) + H|_E| \hookrightarrow |m(K_E + (\Gamma - E)|_E) + H|_E + A|_E|,$$

can be lifted to elements of

$$|m(K_Y + \Gamma) + H + A| = |m(\pi^*(K_X + \Delta) + F) + H + A|.$$

Denote by S_m the lift of $mg^*\Theta$ to $|m\mu(\pi^*(K_X + \Delta) + F) + H + A|$. As K_X is big, there is a fixed positive integer m_0 and an effective divisor

$$S_0 \in |m_0\pi^*K_X - H - A|.$$

Then

$$\begin{aligned} S_m + S_0 &\in |(m_0 + m(1 + \lambda)\mu)\pi^*K_X + m\mu F| \\ &= |(m_0 + m(1 + \lambda)\mu)\pi^*K_X| + m\mu F \end{aligned}$$

so that we may write $S_m + S_0 = G'_m + m\mu F$ for some

$$G'_m \in |(m_0 + m(1 + \lambda)\mu)\pi^*K_X|.$$

We may write

$$G'_m = G_m + sE,$$

where G_m does not contain E and s is a positive integer. Since S_m does not contain E and S_0 is a fixed divisor, it is clear that s is fixed. Since $G_m + m\mu F$ contains S_m , it follows that we may write

$$G_m|_E = mg^*\Theta + B_m,$$

where the only components of B_m with negative coefficients, are supported on $g^*\Theta$ and contained in $E \cap F$. Thus (3) holds, and when we restrict to E_ξ we may ignore $g^*\Theta$. Thus

$$B_m|_{E_\xi} \sim ((m_0 + m(1 + \lambda)\mu)\pi^*K_X - sE)|_{E_\xi} = (-sE)|_{E_\xi}. \quad \square$$

Proof of (4.1). Let U be any very general subset of V , such that

- (1) the morphism p restricted to $p^{-1}(U)$ is smooth,
- (2) every exceptional divisor whose centre intersects U , contains V ,
- (3) every component of Δ which intersects U , contains V , and
- (4) every component of B_m whose image intersects U , dominates V , for every sufficiently large and divisible integer m , as in (4.2).

Let G_m be the sequence of divisors whose existence is guaranteed by (4.2). Set

$$\Gamma_m = G'_m/m, \quad \Theta_m = \pi_*\Gamma_m \quad \text{and} \quad \Delta_m = a_m\Delta + \Theta_m,$$

where $a_m \leq 1$ is chosen so that the log discrepancy of E with respect to $K_X + \Delta_m$ is zero.

Claim 4.3. V' is a pure log canonical centre of $K_X + \Delta_m$.

Proof of Claim 4.3. Since this result is local about the generic point of V , passing to an open subset of X , we may assume that properties (1)–(4) hold, and that B_m is effective. In this case

$$\begin{aligned} K_Y + E + F_m + G_m/m &= \pi^*(K_X + \Delta_m) && \text{where} \\ K_Y + c_m E + F_m &= \pi^*(K_X + a_m \Delta), \end{aligned}$$

F_m is a divisor, which does not contain E and $c_m = 1 - s/m < 1$ is a positive number. Restricting to E , we get

$$K_E + F_m|_E + g^*\Theta + B_m/m.$$

Let E' be the inverse image of W' . Since the morphism p is smooth, E' is a pure log canonical centre of $K_E + g^*\Theta$. Since V is an exceptional log canonical centre, we have $\lfloor \Gamma^h \rfloor = 0$. Thus $K_E + \Gamma^h$ is kawamata log terminal. Since $B_m|_{E_\xi}$ belongs to a fixed linear system and p is smooth over U , it follows that the restriction of B_m to any fibre of p over U , belongs to a fixed linear system. Thus $K_E + \Gamma^h|_E + B_m/m$ is kawamata log terminal, for m large enough. As $F_m \leq \Gamma^h$, it follows that $K_E + F_m|_E + B_m/m$ is also kawamata log terminal, for m large enough. It follows that E' is a pure log canonical centre of $K_E + F_m|_E + g^*\Theta + B_m/m$.

We now apply inversion of adjunction, see (17.7) of [12]. (17.7) is only stated for effective divisors. However it is easy to see that (17.6) and (17.7) of [12], extend to the case when the components of negative coefficient are contracted by a birational map, since this is the case for (17.4) of [12]. \square

Clearly $0 \leq a_m \leq 1$. Moreover, a_m approaches 1 as m approaches infinity, as the coefficient s/m of E in Γ_m approaches zero. Pick m so that $m_0/m < \delta/2$. Then, using the decomposition $K_X \sim A + E$, there is a perturbation $\Delta' \sim (\nu + \delta)K_X$ of Δ_m , such that V' is an exceptional log canonical centre of $K_X + \Delta'$. \square

5. Effective birational freeness

In this section we give an effective bound for the r th pluricanonical map of a variety of general type, which only depends on the volume of K_X and the value of this constant for general subvarieties of X :

Proposition 5.1. *Let X be a smooth variety of general type of dimension n . Let s' be a positive integer such that for every subvariety V of X which contains a very general point of X , the map $\phi_{s'K_V}$ is birational. Set*

$$s = \left(\lfloor \frac{n}{\text{vol}(K_X)^{1/n}} \rfloor + 2 \right) (ns' + 1).$$

Then ϕ_{rK_X} is birational, for all $r \geq 4s(s' + 1) + s'$.

Proof. By (2.3) and (2.4), for every very general point $x \in X$, we may find a \mathbb{Q} -divisor $\Delta \sim_{\mathbb{Q}} \lambda K_X$ with

$$\lambda < \lfloor \frac{n}{\text{vol}(K_X)^{1/n}} \rfloor + 1,$$

such that $\text{LLC}(X, \Delta, x) \neq \emptyset$. By (2.5), we may assume that (X, Δ) is log canonical near x , that $\text{LLC}(X, \Delta, x)$ contains a unique element V and that there is a unique place of log canonical singularities dominating V . Let $f: W \rightarrow V$ be a resolution of V . Then, by (2.8), for a general point $x' \in V$, we may find a divisor Θ on W , such that x' is an isolated log canonical centre of $K_W + \Theta$, where $\Theta \sim_{\mathbb{Q}} ns'K_W$. By (4.1) it follows that we may find $\Delta' \sim (s-1)K_X$ such that x is an isolated point of $K_X + \Delta'$. By (2.6) it follows that $h^0(2sK_X) \geq 2$, and so the result follows from (2.7). \square

Corollary 5.2. *Let X be a smooth variety of general type of dimension n . Let s' be a positive integer such that for every subvariety V of X which contains a very general point of X , the map $\phi_{s'K_V}$ is birational.*

If the volume of K_X is at most v , then X is birational to a subvariety of degree at most

$$(8(ns' + 1)(s' + 1))^n \max(n^n, 3^n v).$$

Proof. By (5.1), ϕ_{rK_X} is birational, where

$$r = 4s(s' + 1) + s' \quad \text{and} \quad s = \left(\lfloor \frac{n}{\text{vol}(K_X)^{1/n}} \rfloor + 2 \right) (ns' + 1).$$

Now

$$\begin{aligned} \text{vol}(rK_X) &= r^n \text{vol}(K_X) \\ &= (4s(s' + 1) + s')^n \text{vol}(K_X) \\ &\leq ((4s + 1)(s' + 1))^n \text{vol}(K_X) \\ &\leq \left(\lfloor \frac{n}{\text{vol}(K_X)^{1/n}} \rfloor + 3 \right)^n (4(ns' + 1)(s' + 1))^n \text{vol}(K_X) \\ &\leq \max \left(\frac{n^n}{\text{vol}(K_X)}, 3^n \right) (8(ns' + 1)(s' + 1))^n \text{vol}(K_X) \\ &\leq (8(ns' + 1)(s' + 1))^n \max(n^n, 3^n v), \end{aligned}$$

and so we are done by (2.2). \square

6. Proof of (1.1)

Lemma 6.1. *Let $\pi: X \rightarrow B$ be a bounded family of projective varieties of general type. Then there is a positive integer R such that if Y is a resolution of any fiber of π , then ϕ_{rK_Y} is birational for all $r \geq R$.*

Proof. Let Y be a fiber over the generic point ξ of a component of B . If we pick a projective resolution of Y , then we may extend this to an open neighbourhood of ξ . Thus, by Noetherian induction, possibly replacing B by a union of locally closed subsets, we may assume that π is projective and smooth. If ϕ_{rK_Y} is birational, then this map remains birational over an open neighbourhood of ξ , and we are done by Noetherian induction. \square

Proof of (1.1). We proceed by induction on the dimension. Assume that (1.1) holds up to dimension $n - 1$. By assumption there is a positive integer r_{n-1} such that ϕ_{rK_W} is birational, for any smooth variety W of general type of dimension at most $n - 1$, and for any $r \geq r_{n-1}$.

Now if x is a very general point of X , and V is any subvariety containing x , then any resolution W of V is of general type. Hence if $\text{vol}(K_X) \geq 1$, then we are done by (5.1). Thus we may assume that $\text{vol}(K_X) < 1$. By (5.2), it follows that X is birational to a subvariety of projective space of bounded degree. But then the set of all varieties of general type such that $\text{vol}(K_X) < 1$ is birationally bounded, and we are done by (6.1). \square

Proof of (1.2). By (1.1) and (2.2) it follows that any smooth variety of general type of dimension n , where $\text{vol}(K_X) \leq M$, is birational to a subvariety of projective space of degree at most $(r_n)^n M$, and the result follows. \square

Proof of (1.3). If $\text{vol}(K_X) \geq 1$ there is nothing to prove, otherwise we apply (1.2) and (6.1). \square

Proof of (1.4). This is an immediate consequence of the main result of [18]. \square

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