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Inventiones mathematicae

Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case*

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1. Introduction

In this paper, we consider the $\dot{H}^1$ critical non-linear Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u \pm |u|^{\frac{4}{N-2}} u = 0 & (x,t) \in \mathbb{R}^N \times \mathbb{R} \\ u|_{t=0} = u_0 \in \dot{H}^1(\mathbb{R}^N). \end{cases}$$

Here the $-$ sign corresponds to the defocusing problem, while the $+$ sign corresponds to the focusing problem. The theory of the Cauchy problem (CP) for this equation was developed in [8] (Cazenave and Weissler). They show that if $\|u_0\|_{\dot{H}^1} \leq \delta$, δ small, there exists a unique solution $u \in C(\mathbb{R}; \dot{H}^1(\mathbb{R}^N))$ with the norm $\|u\|_{L_{x,t}^{\frac{2(N+2)}{N-2}}} < \infty$ (i.e. the solution scatters in $\dot{H}^1(\mathbb{R}^N)$). See Sect. 2 of this paper for a review of these results.

In the defocusing case, Bourgain [5,6] proved that, for $N = 3, 4$ and u_0 radial, this also holds for $\|u_0\|_{\dot{H}^1} < +\infty$, and that for more regular u_0, the solution preserves the smoothness for all time. (Another proof of this last fact is due to Grillakis [13] for $N = 3$). Bourgain's result was then extended to $N \geq 5$ by Tao [26], still under the assumption that u_0 is radial. Then in [9] (Colliander, Keel, Staffilani, Takaoka and Tao) the result was obtained for general u_0, when $N = 3$. This was extended to $N = 4$ in [24] (Ryckman, Visan) and finally to $N \geq 5$ in [28] (Visan).

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In the focusing case, these results do not hold. In fact, the classical virial identity (see for example Glassey in [12] and Sect. 5)

$$\frac{d^2}{dt^2}\int |x|^2 |u_0(x,t)|^2 \, dx = 8\left\{\int |\nabla u(t)|^2 - |u(t)|^{\frac{2N}{N-2}}\right\}$$

shows that if $E(u_0) = \frac{1}{2}\int |\nabla u_0|^2 - \frac{N-2}{2N}\int |u_0|^{\frac{2N}{N-2}} < 0$ and $|x|u_0 \in L^2(\mathbb{R}^N)$, the solution must break down in finite time. Moreover,

$$W(x) = W(x,t) = \frac{1}{\left(1 + \frac{|x|^2}{N(N-2)}\right)^{\frac{N-2}{2}}}$$

is in $\dot{H}^1(\mathbb{R}^N)$ and solves the elliptic equation

$$\Delta W + |W|^{\frac{4}{N-2}} W = 0,$$

so that scattering cannot always occur even for global (in time) solutions.

In this paper we initiate the detailed study of the focusing case. We show (Corollary 5.14):

Theorem 1.1. *Assume that $E(u_0) < E(W)$, $\|u_0\|_{\dot{H}^1} < \|W\|_{\dot{H}^1}$, $N = 3, 4, 5$ and u_0 is radial. Then the solution u with data u_0 at $t = 0$ is defined for all time and there exists $u_{0,+}$, $u_{0,-}$ in $\dot{H}^1$ such that*

$$\lim_{t\to+\infty} \left\|u(t) - e^{it\Delta}u_{0,+}\right\|_{\dot{H}^1} = 0, \qquad \lim_{t\to-\infty} \left\|u(t) - e^{it\Delta}u_{0,-}\right\|_{\dot{H}^1} = 0.$$

Antecedents to this kind of result can be found in the L^2 critical case, in the work of Weinstein [29] and in the H^1 subcritical case in the works of Beresticky and Cazenave [3], and Zhang [30]. In particular in [3], the authors use variational ideas and the relationship with the virial identity.

We expect that our arguments will extend to the case of radial data, for $N \geq 6$ using arguments similar to those in the appendix of [26–28] (Tao and Visan). (It remains an interesting problem to remove the radiality.) The result is optimal in that clearly the solution W does not scatter. We also show that for u_0 radial, $|x|u_0 \in L^2(\mathbb{R}^N)$, $E(u_0) < E(W)$, but $\|u_0\|_{\dot{H}^1} > \|W\|_{\dot{H}^1}$, the solution must break down in finite time.

Our proof introduces a new point of view for these problems. Using a concentration compactness argument (Sect. 4), we reduce matters to a rigidity theorem, which we prove in Sect. 5, with the aid of a localized virial identity (in the spirit of Merle [17, 18]). The radiality enters only at one point, in our proof of the rigidity theorem (see Remark 5.2). We think that the general strategy of our proof with one extra ingredient should also apply in the non-radial case. In Sect. 3, we prove some elementary variational estimates which yield the necessary coercivity for our arguments. These are automatic in the defocusing case and thus our proof gives an alternative approach to [5] and [26] for $N = 3, 4, 5$.

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2. A review of the Cauchy problem

In this section we will review the Cauchy problem

$$\text{(CP)} \qquad \begin{cases} i\partial_t u + \Delta u + |u|^{\frac{4}{N-2}} u = 0 & (x,t) \in \mathbb{R}^N \times \mathbb{R} \\ u|_{t=0} = u_0 \in \dot{H}^1(\mathbb{R}^N) \end{cases}$$

i.e., the $\dot{H}^1$ critical, focusing, Cauchy problem for NLS. We need two preliminary results.

Lemma 2.1 (Strichartz estimate [7,14]). *We say that a pair of exponents (q,r) is admissible if $\frac{2}{q} + \frac{N}{r} = \frac{N}{2}$ and $2 \le q, r \le \infty$. Then, if $2 \le r \le \frac{2N}{N-2}$ ($N \ge 3$) (or $2 \le r < \infty$, $N = 2$ and $2 \le r \le \infty$, $N = 1$) we have*

i)

$$\left\| e^{it\Delta} h \right\|_{L_t^q L_x^r} \le C \left\| h \right\|_{L^2}$$

ii)

$$\left\| \int_{-\infty}^{+\infty} e^{i(t-\tau)\Delta} g(-,\tau)\, d\tau \right\|_{L_t^q L_x^r} + \left\| \int_0^t e^{i(t-\tau)\Delta} g(-,\tau)\, d\tau \right\|_{L_t^q L_x^r} \le C \left\| g \right\|_{L_t^{q'} L_x^{r'}}$$

iii)

$$\left\| \int_{-\infty}^{+\infty} e^{it\Delta} g(-,\tau)\, d\tau \right\|_{L_x^2} \le C \left\| g \right\|_{L_t^{q'} L_x^{r'}}.$$

Lemma 2.2 (Sobolev embedding). *For $v \in C_0^\infty(\mathbb{R}^{N+1})$, we have*

$$\|v\|_{L_t^{\frac{2(N+2)}{N-2}} L_x^{\frac{2(N+2)}{N-2}}} \le C \left\| \nabla_x v \right\|_{L_t^{\frac{2(N+2)}{N-2}} L_x^{\frac{2N(N+2)}{N^2+4}}} \qquad (N \ge 3).$$

(Note that $\frac{2(N+2)}{N-2} = q$, $\frac{2N(N+2)}{N^2+4} = r$ is admissible.)

Remark 2.3. Let $f(u) = |u|^{\frac{4}{N-2}} u$, then clearly $|f(u)| \le |u|^{\frac{N+2}{N-2}}$, $|\partial_z f(u)| \le C|u|^{\frac{4}{N-2}}$, $|\partial_{\bar{z}} f(u)| \le C|u|^{\frac{4}{N-2}}$. Moreover, for $3 \le N \le 6$,

$$\left.\begin{matrix} |\partial_z f(u) - \partial_z f(v)| \\ |\partial_{\bar{z}} f(u) - \partial_{\bar{z}} f(v)| \end{matrix}\right\} \le C\,|u - v| \cdot \left\{ |u|^{\frac{6-N}{N-2}} + |v|^{\frac{6-N}{N-2}} \right\}.$$

Also, note that $(\nabla f)(u(x)) = \partial_z f(u(x)) u(x) + \partial_{\bar{z}} f(u(x)) \overline{u}(x)$, so that $|f(u) - f(v)| \le |u - v| \{ |u|^{\frac{4}{N-2}} + |v|^{\frac{4}{N-2}} \}$. Moreover,

$$\begin{aligned} \nabla_x (f(u(x))) - \nabla_x (f(v(x))) &= (\nabla f)(u(x)) \nabla u - (\nabla f)(v(x)) \nabla v \\ &= (\nabla f)(u(x)) \nabla u - (\nabla f)(u(x)) \nabla v \\ &\quad + \{ \nabla f(u(x))) - \nabla f(v(x)) \} \nabla v, \end{aligned}$$

so $|\nabla_x f(u(x)) - \nabla_x f(v(x))| \le C|u|^{\frac{4}{N-2}}|\nabla u - \nabla v| + C|\nabla v|\{|u|^{\frac{6-N}{N-2}} + |v|^{\frac{6-N}{N-2}}\}|u-v|$.

Remark 2.4. In the estimate ii) in Lemma 2.1, one can actually show: ([14])
ii')

$$\left\| \int_{-\infty}^{+\infty} e^{i(t-\tau)\Delta} g(-,\tau)\, d\tau \right\|_{L_t^q L_x^r} \le C \, \|g\|_{L_t^{m'} L_x^{n'}},$$

where (q,r), (m,n) are any pair of admissible indices as in i) of Lemma 2.1.

Let us define $S(I)$, $W(I)$ norm for an interval I by

$$\|v\|_{S(I)} = \|v\|_{L_I^{\frac{2(N+2)}{N-2}} L_x^{\frac{2(N+2)}{N-2}}} \quad \text{and} \quad \|v\|_{W(I)} = \|v\|_{L_I^{\frac{2(N+2)}{N-2}} L_x^{\frac{2N(N+2)}{N^2+4}}}.$$

Theorem 2.5 (See [8]). *Assume $u_0 \in \dot{H}^1(\mathbb{R}^N)$, $t_0 \in I$ an interval, and $\|u_0\|_{\dot{H}^1} \le A$. Then, (for $3 \le N \le 5$) there exists $\delta = \delta(A)$ such that, if $\|e^{i(t-t_0)\Delta}u_0\|_{S(I)} < \delta$, there exists a unique solution u to (CP) in $I \times \mathbb{R}^N$, with $u \in C(I; \dot{H}^1(\mathbb{R}^N))$,*

$$\left\|\nabla_x u\right\|_{W(I)} < \infty, \quad \|u\|_{S(I)} \le 2\delta.$$

Moreover, if $u_{0,k} \to u_0$ in $\dot{H}^1$ (so that, as we will see, for k large $\|e^{i(t-t_0)\Delta}u_{0,k}\|_{S(I)} < \delta$) the corresponding solutions $u_k \to u$ in $C(I; \dot{H}^1(\mathbb{R}^N))$.

Sketch of proof. Let us assume, without loss of generality, that $t_0 = 0$. (CP) is equivalent to the integral equation

$$u(t) = e^{it\Delta}u_0 + \int_0^t e^{i(t-t')\Delta} f(u)\, dt',$$

where $f(u) = |u|^{\frac{4}{N-2}}u$. We now let $B_{a,b} = \{v \text{ on } I \times \mathbb{R}^n \ : \ \|v\|_{S(I)} \le a, \|\nabla v\|_{W(I)} \le b\}$ and $\Phi_{u_0}(v) = e^{it\Delta}u_0 + \int_0^t e^{i(t-t')\Delta} f(v)\, dt'$. We will next choose δ, a, b so that $\Phi_{u_0}(v) : B_{a,b} \to B_{a,b}$ and is a contraction there: note that

$$\left\|\nabla \Phi_{u_0}(v)\right\|_{W(I)} \le CA + C \left\|\nabla_x f(v)\right\|_{L_I^2 L_x^{\frac{2N}{N+2}}}.$$

This follows, for the first term, by i) ($q = \frac{2(N+2)}{N-2}, r = \frac{2N(N+2)}{N^2+4}$) in Lemma 2.1 and by ii) in Remark 2.4, with the same q, r and $m' = 2$, $n' = \frac{2N}{N+2}$. But $\nabla_x f(u(x)) = (\nabla f)(u(x))\nabla_x u = O(|\nabla u| \cdot |u|^{\frac{4}{N-2}})$ so that, using Hölder inequality we obtain: (for $v \in B_{a,b}$)

$$\left\|\nabla \Phi_{u_0}(v)\right\|_{W(I)} \le CA + C \, \|v\|_{S(I)}^{\frac{4}{N-2}} \cdot \|\nabla v\|_{W(I)} \le CA + Ca^{\frac{4}{N-2}} b.$$

Using Lemma 2.2 for the second term in Φ_{u_0}, and the argument above together with our assumption on u_0 for the first term, we obtain:

$$\left\| \Phi_{u_0}(v) \right\|_{S(I)} \leq \delta + Ca^{\frac{4}{N-2}} b.$$

Now choose $b = 2AC$, and a so that $Ca^{\frac{4}{N-2}} \leq 1/2$. Then $\|\nabla \Phi_{u_0}(v)\|_{W(I)} \leq b$. Next, if $\delta = a/2$, and $Ca^{(\frac{4}{N-2}-1)} b \leq 1/2$ *(possible if $N < 6$)* we obtain $\|\Phi_{u_0}(v)\|_{S(I)} \leq a$, so that $\Phi_{u_0} : B_{a,b} \to B_{a,b}$. Next, for the contraction, we use the same argument in conjunction with Remark 2.3.

$$\begin{aligned} \left\| \nabla \Phi_{u_0}(v) - \nabla \Phi_{u_0}(v') \right\|_{W(I)} &\leq C \left\| \nabla_x f(v) - \nabla_x f(v') \right\|_{L^2_I L_x^{\frac{2N}{N+2}}} \\ &\leq C \left\| |v|^{\frac{4}{N-2}} \left| \nabla v - \nabla v' \right| \right\|_{L^2_I L_x^{\frac{2N}{N+2}}} \\ &\quad + C \left\| \left| v - v' \right| |v|^{\frac{6-N}{N-2}} \left| \nabla v' \right| \right\|_{L^2_I L_x^{\frac{2N}{N+2}}} \\ &\quad + C \left\| \left| v - v' \right| \left| v' \right|^{\frac{6-N}{N-2}} \left| \nabla v' \right| \right\|_{L^2_I L_x^{\frac{2N}{N+2}}}. \end{aligned}$$

The first term is bounded as before by $C\|v\|_{S(I)}^{\frac{4}{N-2}} \|\nabla v - \nabla v'\|_{W(I)}$. For the second and third terms we use Hölder's inequality to bound them by $C\|v - v'\|_{S(I)} \|\nabla v'\|_{W(I)} \big(\|v\|_{S(I)}^{\frac{6-N}{N-2}} + \|v'\|_{S(I)}^{\frac{6-N}{N-2}} \big)$ so that

$$\begin{aligned} \left\| \nabla \Phi_{u_0}(v) - \nabla \Phi_{u_0}(v') \right\|_{W(I)} &\leq Ca^{\frac{4}{N-2}} \|\nabla v - \nabla v'\|_{W(I)} \\ &\quad + Ca^{\frac{6-N}{N-2}} b \|v - v'\|_{S(I)}. \end{aligned}$$

Lemma 2.2 gives

$$\begin{aligned} \left\| \Phi_{u_0}(v) - \Phi_{u_0}(v') \right\|_{S(I)} &\leq C \left\| \nabla \Phi_{u_0}(v) - \nabla \Phi_{u_0}(v') \right\|_{W(I)} \\ &\leq Ca^{\frac{4}{N-2}} \|\nabla v - \nabla v'\|_{W(I)} + Ca^{\frac{6-N}{N-2}} b \|v - v'\|_{S(I)} \end{aligned}$$

and thus we establish the contraction property ($N < 6$). We then find $u \in B_{a,b}$ solving $\Phi_{u_0}(u) = u$. To show that $u \in C(I; \dot{H}^1)$, note that $e^{it\Delta} u_0 \in C(I; \dot{H}^1)$ with norm bounded by A. For the term $\int_0^t e^{i(t-t')\Delta} f(u)\, dt'$, we use iii) in Lemma 2.1, with $(q', r') = (2, 2N/N+2)$. The proof of Theorem 2.5 is easily concluded from this. (The last continuity statement is an easy consequence of the fixed point argument, see also Remark 2.17.) □

Remark 2.6. Using Remark 2.4, it is easy to see that $\nabla u \in L^q_I L^r_x$ for any admissible index pair (q, r).

Remark 2.7. There exists $\widetilde{\delta}$ such that if $\|u_0\|_{\dot{H}^1} \leq \widetilde{\delta}$, the conclusion of Theorem 2.5 applies to any interval I. In fact, $\left\| e^{it\Delta} u_0 \right\|_{S(I)} \leq C \left\| \nabla e^{it\Delta} u_0 \right\|_{W(I)} \leq C\widetilde{\delta}$, by virtue of Lemma 2.1 i) and the claim follows.

Remark 2.8. Given $u_0 \in \dot{H}^1$, there exists $(0 \in)$ I such that the hypotheses of Theorem 2.5 is verified on I. This is clear because of $\left\| e^{it\Delta} u_0 \right\|_{S(I)} \le C \left\| \nabla e^{it\Delta} u_0 \right\|_{W(I)}$ and the fact that $\left\| \nabla e^{it\Delta} u_0 \right\|_{W(\mathbb{R})} < \infty$ by Lemma 2.1 i).

Remark 2.9 (Energy identity). If u is the solution constructed in Theorem 2.5, we have (with $\frac{1}{2^*} = \frac{1}{2} - \frac{1}{N}$) that

$$E(u(t)) = \int_{\mathbb{R}^N} \left\{ \frac{1}{2} |\nabla u(t,x)|^2 - \frac{1}{2^*} |u(t,x)|^{2^*} \right\} dx$$

is constant for $t \in I$. If $u_0 \in C_0^\infty(\mathbb{R}^N)$ this follows from a classical integration by parts, the general general case follows from a limiting argument.

Definition 2.10. Let $t_0 \in I$. We say that $u \in C(I; \dot{H}^1(\mathbb{R}^N)) \cap \{\nabla u \in W(I)\}$ is a solution of the (CP) if

$$u|_{t_0} = u_0, \quad \text{and} \quad u(t) = e^{i(t-t_0)\Delta} u_0 + \int_{t_0}^{t} e^{i(t-t')\Delta} f(u)\, dt',$$

with $f(u) = |u|^{\frac{4}{N-2}} u$. Note that if $u^{(1)}$, $u^{(2)}$ are solutions of (CP) on I, $u^{(1)}(t_0) = u^{(2)}(t_0)$, then $u^{(1)} \equiv u^{(2)}$ on $I \times \mathbb{R}^N$. This is because we can partition I into a finite collection of subintervals I_j, so that, with $A = \sup_{t \in I} \max_{i=1,2} \|u^{(i)}(t)\|_{\dot{H}^1}$, the $S(I_j)$ norm of $u^{(i)}$ and the $W(I_j)$ norm of $\nabla u^{(i)}$ are less than a, b, where a, b are obtained in the proof of Theorem 2.5. If j_0 is then such that $t_0 \in I_{j_0}$, the uniqueness of the fixed point in the proof of Theorem 2.5, combined with Remark 2.8 gives an interval $\widetilde{I} \ni t_0$ so that $u^{(1)}(t) = u^{(2)}(t)$, $t \in \widetilde{I}$. A continuation argument now easily gives $u^{(1)} \equiv u^{(2)}$, $t \in I$. This allows us to define a maximal interval $I(u_0) = (t_0 - T_-(u_0), t_0 + T_+(u_0))$, with $T_\pm(u_0) > 0$, where the solution is defined. If $T_1 < t_0 + T_+(u_0)$, $T_2 > t_0 - T_-(u_0)$, $T_2 < t_0 < T_1$, then u solves (CP) in $[T_2, T_1] \times \mathbb{R}^N$, so that $u \in C([T_2, T_1]; \dot{H}^1(\mathbb{R}^N))$, $\nabla u \in W([T_2, T_1])$ and $u \in S([T_2, T_1])$.

Lemma 2.11 (Standard finite blow-up criterion, see [7]). *If $T_+(u_0) < \infty$, then*

$$\|u\|_{S([t_0, t_0 + T_+(u_0)])} = +\infty.$$

A corresponding result holds for $T_-(u_0)$.

Sketch of proof. Assume $T_+(u_0) < +\infty$ and that $\|u\|_{S([t_0,t_0+T_+(u_0)])} < +\infty$. Let $M = \|u\|_{S([t_0,t_0+T_+(u_0)])}$ and, for ϵ to be chosen, find $N = N(\epsilon)$ intervals I_j, $\bigcup_{j=1}^N I_j = [t_0, t_0 + T_+(u_0)]$, such that $\|u\|_{S(I_j)} \le \epsilon$. Our first step is to show that $\|u\|_{L^\infty([t_0,t_0+T_+(u_0)];\dot{H}^1)} + \|\nabla u\|_{W([t_0,t_0+T_+(u_0)])} < \infty$. We write the integral equation on each interval I_j, to deduce (using the proof of Theorem 2.5 and iii) in Lemma 2.1) that

$$\sup_{t \in I_j} \|u(t)\|_{\dot{H}^1} + \|\nabla u\|_{W(I_j)} \le C\|u(t_j)\|_{\dot{H}^1} + C\,\|u\|_{S(I_j)}^{\frac{4}{N-2}} \cdot \|\nabla u\|_{W(I_j)},$$

where t_j is any fixed point in I_j. Our desired estimate follows inductively then, by choosing $C\epsilon^{\frac{4}{N-2}} \leq 1/2$. Once the first step is done, we then choose $t_n \uparrow t_0 + T_+(u_0)$ and show, using the integral equation once more, that $\|e^{i(t-t_n)\Delta}u(t_n)\|_{S([t_n,t_0+T_+(u_0)])} \leq \delta/2$, for n large. But then, for n large but fixed, and some $\epsilon_0 > 0$, $\|e^{i(t-t_n)\Delta}u(t_n)\|_{S([t_n,t_0+T_+(u_0)]+\epsilon_0)} \leq \delta$. Now, Theorem 2.5 applies and together with Definition 2.10 we reach a contradiction. □

Definition 2.12. Let $v_0 \in \dot{H}^1$, $v(t) = e^{it\Delta}v_0$ and let $\{t_n\}$ be a sequence, with $\lim_{n\to\infty} t_n = \bar{t} \in [-\infty, +\infty]$. We say that $u(x,t)$ is a non-linear profile associated with $(v_0, \{t_n\})$ if there exists an interval I, with $\bar{t} \in I$ (if $\bar{t} = \pm\infty$, $I = [a, +\infty)$ or $(-\infty, a]$) such that u is a solution of (CP) in I and

$$\lim_{n\to\infty} \|u(-,t_n) - v(-,t_n)\|_{\dot{H}^1} = 0.$$

Remark 2.13. There always exists a non-linear profile associated to $(v_0, \{t_n\})$. In fact, if $\bar{t} \in (-\infty, +\infty)$, this is clear by Remark 2.8, with $u_0 = v(x, \bar{t})$. If $\bar{t} = +\infty$, we solve the integral equation

$$u(t) = e^{it\Delta}v_0 + \int_t^{+\infty} e^{i(t-t')\Delta} f(u)\,dt'$$

in $(t_{n_0}, +\infty) \times \mathbb{R}^N$, for n_0 so large that $\|e^{it\Delta}v_0\|_{S((t_{n_0},\infty))} \leq \delta$, where δ is as in Theorem 2.5. Then, if n is large $u(t_n) - v(t_n) = \int_{t_n}^{+\infty} e^{i(t-t')\Delta} f(u)\,dt'$, and we have $\|f(u)\|_{L^2_{(t>t_{n_0})}L_x^{2N/N+2}} < \infty$, as in the proof of Theorem 2.5. But then, using iii) in Lemma 2.1 we obtain $\|u(t_n)-v(t_n)\|_{\dot{H}^1} \leq C\|f(u)\|_{L^2_{(t>t_n)}L_x^{2N/N+2}}$, which clearly goes to 0 as n goes infinity. A similar argument applies when $\bar{t} = -\infty$.

Note also that if $u^{(1)}$, $u^{(2)}$ are both non-linear profiles associated to $(v_0, \{t_n\})$ in an interval I with $\bar{t} \in I$, then $u^{(1)} \equiv u^{(2)}$ on I. In fact, if $\bar{t} \in (-\infty, +\infty)$, this is clear from the Definition 2.13 and the uniqueness result in Definition 2.10. If $\bar{t} = +\infty$, since $\|\nabla u^{(i)}\|_{W(I)} < \infty$, for $n \geq n_0$, we have $\|\nabla u^{(i)}\|_{W(t_n,+\infty)} \leq \widetilde{\delta}$, where $\widetilde{\delta}$ is as small as we like. By the proof of Theorem 2.5, we have (with a constant independent of u) that for $n \gg n_0$

$$\sup_{t\in(t_{n_0},t_n)} \left\|\nabla u^{(1)}(t) - \nabla u^{(2)}(t)\right\|_{L^2} \leq C\left\|\nabla u^{(1)}(t_n) - \nabla u^{(2)}(t_n)\right\|_{L^2}.$$

This easily shows that $u^{(1)} \equiv u^{(2)}$ on $(t_{n_0}, +\infty)$ and hence on I, as claimed. The case $\bar{t} = -\infty$ is similar. Because of this remark, we can always define a maximal interval I of existence for the non-linear profile associated to $(v_0, \{t_n\})$. If $\bar{t} \in (-\infty, +\infty)$, $I = (a, b)$, $I' \Subset I$, then $\sup_{t\in I'} \|u(t)\|_{\dot{H}^1} < \infty$, $\|u\|_{S(I')} < \infty$, $\|\nabla u\|_{W(I')} < \infty$, but if either a or b are finite $\|u\|_{S(I)} = +\infty$. If $\bar{t} = \pm\infty$, say $\bar{t} = +\infty$, $I = (a, +\infty)$, $I' = (\alpha, +\infty)$, $\alpha > a$, similar statements can be made. If $a > -\infty$, we can also say $\|u\|_{S(I)} = +\infty$.

Theorem 2.14 (Long-time perturbation theory, see also [27]). *Let $I \subset \mathbb{R}$ be a time interval and let $t_0 \in I$. Let $\widetilde{u}$ be defined on $I \times \mathbb{R}^N$ ($3 \le N \le 5$) and satisfy* $\sup_{t\in I} \|\widetilde{u}\|_{\dot{H}^1} \le A$, $\|\widetilde{u}\|_{S(I)} \le M$ *for some constants* $M, A > 0$. *Assume that*

$$(i\partial_t \widetilde{u} + \Delta\widetilde{u} + f(\widetilde{u})) = e \qquad (t,x) \in I \times \mathbb{R}^N$$

(in the sense of the appropriate integral equation) and that

$$\|u_0 - \widetilde{u}(t_0)\|_{\dot{H}^1} \le A', \quad \|\nabla e\|_{L_I^2 L_x^{\frac{2N}{N+2}}} \le \epsilon, \quad \left\|e^{i(t-t_0)\Delta}[u_0 - \widetilde{u}(t_0)]\right\|_{S(I)} \le \epsilon.$$

Then, there exists $\epsilon_0 = \epsilon_0(M, A, A', N)$ such that there exists a solution of (CP) with $u(t_0) = u_0$ in $I \times \mathbb{R}^N$, for $0 < \epsilon < \epsilon_0$, with $\|u\|_{S(I)} \le C(M, A, A', N)$ and $\forall t \in I$, $\|u(t) - \widetilde{u}(t)\|_{\dot{H}^1} \le C(A, A', M, N)(A' + \epsilon)$.

Proof. We start the proof by showing that $\|\nabla\widetilde{u}\|_{W(I)} \le \widetilde{M}$, where $\widetilde{M} = \widetilde{M}(A, M, N)$, for $\epsilon \le \epsilon_0$. In fact, for $\eta = \eta(N)$ small, to be determined, split I into $\gamma = \gamma(M, \eta)$ interval I_j so that $\|\widetilde{u}\|_{S(I_j)} \le \eta$. Using the integral equation, we have

$$\|\nabla\widetilde{u}\|_{W(I_j)} \le A + C\,\|\widetilde{u}\|_{S(I_j)}^{\frac{4}{N-2}}\,\|\nabla\widetilde{u}\|_{W(I_j)} + C\,\|\nabla e\|_{L_I^2 L_x^{2N/N+2}},$$

as in the proof of Theorem 2.5, and the claim follows if $C\eta^{\frac{4}{N-2}} < 1/2$. Next, we write $u = \widetilde{u} + w$ and notice that

$$i\partial_t w + \Delta w + [f(\widetilde{u} + w) - f(\widetilde{u})] = e.$$

Let $I_j = [a_j, a_{j+1}]$, so that, in order to solve for w we need to solve, in I_j, the integral equation

$$w(t) = e^{i(t-a_j)\Delta}w(a_j) + \int_{a_j}^{t} e^{i(t-t')\Delta}[f(\widetilde{u}+w) - f(\widetilde{u})]\,dt' + \int_{a_j}^{t} e^{i(t-t')\Delta} e\,dt'.$$

The proof of Theorem 2.5 *(which holds for* $3 \le N \le 5$*)* now shows that, for $\eta = \eta(N)$ small enough, and $\epsilon_0 = \epsilon_0(N)$ small enough, we can solve the integral equation (assuming $t_0 = a_1$ say) in I_1 and obtain w with the bounds $\|w\|_{S(I)} \le 2\epsilon$, $\|\nabla w\|_{W(I_1)} \le C(A, A')$, $\sup_{t\in I_1}\|w(t)\|_{\dot{H}^1} \le C(A, A')(A' + \epsilon)$. We now estimate $\|e^{i(t-a_2)\Delta}w(a_2)\|_{S(I_2)}$, using the integral equation. Since $e^{i(t-a_2)\Delta}e^{i(a_2-t_0)\Delta}w(t_0) = e^{i(t-t_0)\Delta}w(t_0)$, by assumption $\|e^{i(t-a_2)\Delta}e^{i(a_2-t_0)\Delta}w(t_0)\|_{S(I_2)} \le \epsilon$. For the integral term, we use Lemma 2.1, iii) to obtain a bound for its $\dot{H}^1$ norm at a_2 by $C\|w\|_{S(I_1)}^{\frac{4}{N-2}}\|\nabla w\|_{S(I_1)} \le C(2\epsilon)^{\frac{4}{N-2}}C(A, A')$. Clearly this procedure can be iterated $\gamma = \gamma(M, N)$ times, provided ϵ_0 is small enough, yielding the theorem. □

Remark 2.15 (See [7]). If u is a solution of (CP) in $I \times \mathbb{R}^N$, $I = [a, +\infty)$ (or $I = (-\infty, a]$) there exists $u^+ \in \dot{H}^1$ such that

$$\lim_{t \to +\infty} \left\| u(t) - e^{it\Delta} u_+ \right\|_{\dot{H}^1} = 0.$$

To see this, note that $\nabla f(u) \in W(I)$ and hence $\| \int_t^\infty e^{i(t-t')\Delta} f(u)\, dt' \|_{\dot{H}^1} \to 0$ as $t \to +\infty$. Then, $u(t) = e^{i(t-a)\Delta} u_0 + \int_a^t e^{i(t-t')\Delta} f(u)\, dt'$ and hence $u^+ = e^{-ia\Delta} u_0 + \int_a^\infty e^{-it'\Delta} f(u)\, dt'$ has the desired property. In fact note that the argument used at the beginning of the proof of Theorem 2.14 shows that it suffices to assume u to be a solution of (CP) in $I' \times \mathbb{R}^N$, $I' \Subset I$, such that $\|u\|_{S(I)} < \infty$.

Remark 2.16. We recall that, since we are working in the focusing case, we have from the argument of Glassey [12] that if $\int |x|^2 |u_0|^2 < +\infty$, $E(u_0) < 0$, there exists a finite time T such that the solution cannot be extended for $t > T$. Clearly, for such a u_0, the maximal interval of existence must be finite. (See Definition 2.10.) Note that it is unknown if $\lim_{t \uparrow T} \|u(t)\|_{\dot{H}^1} = +\infty$ for a general initial data that doesn't exist for all time.

Remark 2.17. Theorem 2.14 also yields the following continuity fact, which will be used later: let $\widetilde{u}_0 \in \dot{H}^1$, $\|\widetilde{u}_0\|_{\dot{H}^1} \le A$, and let $\widetilde{u}$ be the solution of (CP), with maximal inteval of existence $(T_-(\widetilde{u}_0), T_+(\widetilde{u}_0))$ (see Definition 2.10). Let $u_{0,n} \to \widetilde{u}_0$ in $\dot{H}^1$, and let u_n be the corresponding solution of (CP), with maximal interval of existence $(T_-(u_{0,n}), T_+(u_{0,n}))$. Then, $T_-(\widetilde{u}_0) \ge \overline{\lim}_{n \to +\infty} T_-(u_{0,n})$, $T_+(\widetilde{u}_0) \le \underline{\lim}_{n \to +\infty} T_+(u_{0,n})$ and for each $t \in (T_-(\widetilde{u}_0), T_+(\widetilde{u}_0))$, $u_n(t) \to \widetilde{u}(t)$ in $\dot{H}^1$.

Indeed, let $I \subset\subset (T_-(\widetilde{u}_0), T_+(\widetilde{u}_0))$, so that $Sup_{t \in I} \|\widetilde{u}(t)\|_{\dot{H}^1} = A < +\infty$, $\|\widetilde{u}(t)\|_{S(I)} = M < +\infty$. We will show that, for n large, u_n exists on I and $\forall t \in I$, $\|u_n(t) - \widetilde{u}(t)\|_{\dot{H}^1} \le C(M, A, N) \|u_{0,n} - \widetilde{u}_0\|_{\dot{H}^1}$. This clearly yields the remark. To show this, apply Theorem 2.14, with $u = u_n$, $u_0 = u_{0,n}$. Then, if $\epsilon_0 = \epsilon_0(M, A, 2A, N)$ and n is so large that $\|u_{0,n} - \widetilde{u}_0\|_{\dot{H}^1} \le \epsilon_0$ and $\|e^{it\Delta}[u_{0,n} - \widetilde{u}_0]\|_{S(I)} \le \epsilon_0$, using the uniqueness of the solutions we obtained in Definition 2.10, the claim follows.

3. Some variational estimates

Let

$$W(x) = W(x,t) = \frac{1}{\left(1 + \frac{|x|^2}{N(N-2)}\right)^{\frac{N-2}{2}}},$$

be a stationary solution of (CP). That is, W solves the non-linear elliptic equation

$$\Delta W + |W|^{\frac{4}{N-2}} W = 0. \tag{3.1}$$

Moreover, $W \geq 0$ and it is radially symmetric and decreasing. Note that $W \in \dot{H}^1$, but W need not belong to $L^2(\mathbb{R}^N)$. By invariances of the equation, for $\theta_0 \in [-\pi, \pi]$, $\lambda_0 > 0$, $x_0 \in \mathbb{R}^N$,

$$W_{\theta_0,x_0,\lambda_0}(x) = e^{i\theta_0}\lambda_0^{\frac{N-2}{2}} W(\lambda_0(x-x_0))$$

is still a solution. By the work of Aubin [1], Talenti [25] we have the following characterization of W:

$$\forall u \in \dot{H}^1, \quad \|u\|_{L^{2^*}} \leq C_N \|\nabla u\|_{L^2}; \tag{3.2}$$

moreover,

(3.3)
If $\|u\|_{L^{2^*}} = C_N \|\nabla u\|_{L^2}$, $u \neq 0$, then $\exists(\theta_0, \lambda_0, x_0)$ such that $u = W_{\theta_0,x_0,\lambda_0}$,

where C_N is the best constant of the Sobolev inequality in dimension N.

The equation (3.1) gives $\int |\nabla W|^2 = \int |W|^{2^*}$. Also, (3.3) yields $C_N^2 \int |\nabla W|^2 = \left(\int |W|^{2^*}\right)^{N-2/N}$, so that $C_N^2 \int |\nabla W|^2 = \left(\int |\nabla W|^2\right)^{\frac{N-2}{N}}$. Hence,

$$\int |\nabla W|^2 = \frac{1}{C_N^N} \quad \text{and} \quad E(W) = \left(\frac{1}{2} - \frac{1}{2^*}\right) \int |\nabla W|^2 = \frac{1}{N}\frac{1}{C_N^N}.$$

Lemma 3.4. *Assume that*

$$\|\nabla u\|_{L^2}^2 < \|\nabla W\|_{L^2}^2.$$

Assume moreover that $E(u) \leq (1-\delta_0)E(W)$ *where* $\delta_0 > 0$. *Then, there exists* $\overline{\delta} = \overline{\delta}(\delta_0, N) > 0$ *such that*

$$\int |\nabla u|^2 \leq (1-\overline{\delta}) \int |\nabla W|^2 \tag{3.5}$$

$$\int |\nabla u|^2 - |u|^{2^*} \geq \overline{\delta} \int |\nabla u|^2 \tag{3.6}$$

$$E(u) \geq 0. \tag{3.7}$$

Proof. Consider the function $f_1(y) = \frac{1}{2}y - \frac{C_N^{2^*}}{2^*} y^{\frac{2^*}{2}}$, and let $\overline{y} = \|\nabla u\|_{L^2}^2$. Because of (3.2), $f_1(\overline{y}) \leq E(u) \leq (1-\delta_0)E(W) = (1-\delta_0)\frac{1}{N}\frac{1}{C_N^N}$. Note that $f_1(0) = 0$, $f_1'(y) = \frac{1}{2} - \frac{C_N^{2^*}}{2} y^{\frac{2^*}{2}-1}$, so that $f_1'(y) = 0$ if and only if $y = y_C$, where $y_C = \frac{1}{C_N^N} = \int |\nabla W|^2$. Note also that $f_1(y_C) = \frac{1}{NC_N^N} = E(W)$. But then, since $0 < \overline{y} < y_C$ and $f_1(\overline{y}) \leq (1-\delta_0) f_1(y_C)$ and f_1 is nonnegative and strictly increasing between 0 and y_C, $f_1''(y_C) \neq 0$, we have $0 < f_1(\overline{y})$ and $\overline{y} \leq (1-\overline{\delta}) \int |\nabla W|^2$. Thus (3.5) and (3.7) hold. To show (3.6), consider the function $g_1(y) = y - C_N^{2^*} y^{\frac{N}{N-2}}$. Because of (3.2) we have that

$\int |\nabla u|^2 - |u|^{2^*} \geq \int |\nabla u|^2 - C_N^{2^*}\left(\int |\nabla u|^2\right)^{2^*/2} = g_1(\overline{y})$. Note that $g_1(y) = 0$ if and only if $y = 0$ or $y = y_C$ and that $g_1'(0) = 1$, $g_1'(y_C) = -\frac{2}{N-2}$. We then have, for $0 < y < y_C$, $g_1(y) \geq C \min\{y, (y_C - y)\}$, and so, since $0 \leq \overline{y} < (1-\overline{\delta})y_C$ by (3.5), (3.6) follows. Note that $\overline{\delta} \simeq \delta_0{}^{\frac{1}{2}}$. □

Note that the relevance of (3.6) comes from the virial identity (see introduction).

Corollary 3.8. *Assume that $u \in \dot{H}^1$ and that $\int |\nabla u|^2 < \int |\nabla W|^2$. Then $E(u) \geq 0$.*

Proof. If $E(u) \geq E(W) = \frac{1}{NC_N^N}$, this is obvious. If $E(u) < E(W)$, the claim follows from (3.7). □

Theorem 3.9 (Energy trapping). *Let u be a solution of the (CP), with $t_0 = 0$, $u|_{t=0} = u_0$ such that for $\delta_0 > 0$*

$$\int |\nabla u_0|^2 < \int |\nabla W|^2 \quad \textit{and} \quad E(u_0) < (1-\delta_0)E(W).$$

Let $I \ni 0$ be the maximal interval of existence given by Definition 2.10. Let $\overline{\delta} = \overline{\delta}(\delta_0, N)$ be as in Lemma 3.4. Then, for each $t \in I$, we have

$$\int |\nabla u(t)|^2 \leq (1-\overline{\delta}) \int |\nabla W|^2 \tag{3.10}$$

$$\int |\nabla u(t)|^2 - |u(t)|^{2^*} \geq \overline{\delta} \int |\nabla u(t)|^2 \tag{3.11}$$

$$E(u(t)) \geq 0. \tag{3.12}$$

Proof. By Remark 2.9, $E(u(t)) = E(u_0)$, $t \in I$ and the Theorem follows directly from Lemma 3.4 and a continuity argument. □

Corollary 3.13. *Let u, u_0 be as in Theorem 3.9. Then for all $t \in I$ we have $E(u(t)) \simeq \int |\nabla u(t)|^2 \simeq \int |\nabla u_0|^2$, with comparability constants which depend only on δ_0.*

Proof. $E(u(t)) \leq \int |\nabla u(t)|^2$, but by (3.11) we have

$$\begin{aligned} E(u(t)) &\geq \left(\frac{1}{2} - \frac{1}{2^*}\right) \int |\nabla u(t)|^2 + \frac{1}{2^*}\left(\int |\nabla u(t)|^2 - |u(t)|^{2^*}\right) \\ &\geq C_{\overline{\delta}} \int |\nabla u(t)|^2, \end{aligned}$$

so the first equivalence follows. For the second one note that $E(u(t)) = E(u_0) \simeq \int |\nabla u_0|^2$, by the first equivalence when $t = 0$. □

Remark 3.14. Assume that $u_0 \in \dot{H}^1$ and that $|x|u_0 \in L^2(\mathbb{R}^N)$. Assume that

$$E(u_0) < E(W), \quad \text{but} \quad \int |\nabla u_0|^2 > \int |\nabla W|^2.$$

If we choose δ_0 so that $E(u_0) < (1-\delta_0)E(W)$, arguing as in Lemma 3.4 we can conclude that $\int |\nabla u_0|^2 > (1+\overline{\delta}) \int |\nabla W|^2$, $\overline{\delta} = \overline{\delta}(\delta_0, N)$. But then, $\int |\nabla u_0|^2 - |u_0|^{2^*} = 2^* E(u_0) - \left(\frac{2}{N-2}\right) \int |\nabla u_0|^2 \leq 2^* E(W) - \frac{2}{(N-2)} \frac{1}{C_N^N} - \frac{2\overline{\delta}}{(N-2)} \frac{1}{C_N^N} = \frac{-2\overline{\delta}}{(N-2)C_N^N}$. Since $E(u(t)) = E(u_0)$, a continuity argument shows that for all $t \in I$, the maximal interval of existence, we have $\int |\nabla u(t)|^2 \geq (1+\overline{\delta}) \int |\nabla W|^2$ and $\int |\nabla u(t)|^2 - \int |u(t)|^{2^*} \leq \frac{-2\overline{\delta}}{(N-2)C_N^N}$. But, the virial identity ([12]) shows that, if $|x|u_0 \in L^2(\mathbb{R}^N)$ then $\frac{d^2}{dt^2} \int |x|^2 |u_0(x,t)|^2\, dx = 8\{\int |\nabla u(t)|^2 - |u(t)|^{2^*}\} \leq \frac{-16\overline{\delta}}{(N-2)C_N^N}$. This shows that I must be finite, i.e., the maximal interval of existence is finite. This argument is the critical analoge of the H^1 subcritical result in [3].

Note that in the case where $u_0 \in \dot{H}^1$ and $u_0 \in L^2(\mathbb{R}^N)$, the same result holds. Indeed, one can use a local version of the virial identity (See Sect. 5 for such a version) and the extra conservation law of the L^2 norm in time to control correction terms to obtain $\frac{d^2}{dt^2} \int \phi(|x|)|u_0(x,t)|^2\, dx \leq \frac{-8\overline{\delta}}{(N-2)C_N^N}$, where ϕ is a regular and compactetly supported function (See for example Ogawa and Tsutsumi [22]).

4. Existence and compactness of a critical element

Let us consider the statement:

(SC) For all $u_0 \in \dot{H}^1(\mathbb{R}^N)$, with $\int |\nabla u_0|^2 < \int |\nabla W|^2$ and $E(u_0) < E(W)$, if u is the corresponding solution to the (CP), with maximal interval of existence I (see Definition 2.10), then $I = (-\infty, +\infty)$ and $\|u\|_{S((-\infty,+\infty))} < \infty$.

We say that $(SC)(u_0)$ holds if for this particular u_0, with $\int |\nabla u_0|^2 < \int |\nabla W|^2$ and $E(u_0) < E(W)$ and u the corresponding solution to the (CP), with maximal interval of existence I, we have $I = (-\infty, +\infty)$ and $\|u\|_{S((-\infty,+\infty))} < \infty$.

Note that, because of Remark 2.7, if $\|\nabla u_0\|_{L^2} \leq \overline{\delta}$, $(SC)(u_0)$ holds. Thus, in light of Corollary 3.13, there exists $\eta_0 > 0$ such that, if u_0 is as in (SC) and $E(u_0) < \eta_0$, then $(SC)(u_0)$ holds. Moreover, for any u_0 as in (SC), $E(u_0) \geq 0$, in light of Theorem 3.9. Thus, there exists a number E_C, with $\eta_0 \leq E_C \leq E(W)$, such that, if u_0 is as in (SC) and $E(u_0) < E_C$, $(SC)(u_0)$ holds and E_C is optimal with this property. For the rest of this section we will assume that $E_C < E(W)$. We now prove that there exits a critical element $u_{0,C}$ at the critical level of energy E_C so that $(SC)(u_{0,C})$ does not hold and from the minimality, this element has a compactness property up to the symetries of this equation. This is in fact a general principle which follows from the concentration compactness ideas. More precisely,

Proposition 4.1. *There exists $u_{0,C}$ in $\dot{H}^1$, with*

$$E(u_{0,C}) = E_C < E(W), \qquad \int |\nabla u_{0,C}|^2 < \int |\nabla W|^2$$

such that, if u_C is the solution of (CP) with data $u_{0,C}$, and maximal interval of existence I, $0 \in \mathring{I}$, then $\|u_C\|_{S(I)} = +\infty$.

Proposition 4.2. *Assume u_C is as in Proposition 4.1 and that (say) $\|u_C\|_{S(I_+)} = +\infty$, where $I_+ = (0, +\infty) \cap I$. Then there exists $x(t) \in \mathbb{R}^N$ and $\lambda(t) \in \mathbb{R}^+$, for $t \in I_+$, such that*

$$K = \left\{ v(x,t) \ : \ v(x,t) = \frac{1}{\lambda(t)^{(N-2)/2}} u_C\left(\frac{x - x(t)}{\lambda(t)}, t\right)\right\}$$

has the property that $\overline{K}$ is compact in $\dot{H}^1$. A corresponding conclusion is reached if $\|u_C\|_{S(I_-)} = +\infty$, where $I_- = (-\infty, 0) \cap I$.

The main tools that we will need in order to prove Propositions 4.1 and 4.2 are the following lemmas.

Lemma 4.3 (Concentration compactness). *Let $\{v_{0,n}\} \in \dot{H}^1$, $\int |\nabla v_{0,n}|^2 \leq A$. Assume that $\|e^{it\Delta} v_{0,n}\|_{L^{2(N+2)/N-2}} \geq \delta > 0$, where $\delta = \delta(N)$ is as in Theorem 2.5. Then there exists a sequence $\{V_{0,j}\}_{j=1}^{\infty}$ in $\dot{H}^1$, a subsequence of $\{v_{0,n}\}$ (which we still call $\{v_{0,n}\}$) and a triple $(\lambda_{j,n}; x_{j,n}; t_{j,n}) \in \mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}$, with*

$$\frac{\lambda_{j,n}}{\lambda_{j',n}} + \frac{\lambda_{j',n}}{\lambda_{j,n}} + \frac{|t_{j,n} - t_{j',n}|}{\lambda_{j,n}^2} + \frac{|x_{j,n} - x_{j',n}|}{\lambda_{j,n}} \to \infty$$

as $n \to \infty$ for $j \neq j'$ (we say that $(\lambda_{j,n}; x_{j,n}; t_{j,n})$ is orthogonal if this property is verified) such that

$$\|V_{0,1}\|_{\dot{H}^1} \geq \alpha_0(A) > 0. \tag{4.4}$$

If $V_j^l(x,t) = e^{it\Delta} V_{0,j}$, then, given $\epsilon_0 > 0$, there exists $J = J(\epsilon_0)$ and

$$\{w_n\}_{n=1}^{\infty} \in \dot{H}^1, \text{ so that } v_{0,n} = \sum_{j=1}^{J} \frac{1}{\lambda_{j,n}^{(N-2)/2}} V_j^l \left(\frac{x - x_{j,n}}{\lambda_{j,n}}, \frac{-t_{j,n}}{\lambda_{j,n}^2}\right) + w_n \tag{4.5}$$

with $\|e^{it\Delta} w_n\|_{S((-\infty,+\infty))} \leq \epsilon_0$, for n large

(4.6) $$\int |\nabla v_{0,n}|^2 = \sum_{j=1}^{J} \int |\nabla V_{0,j}|^2 + \int |\nabla w_n|^2 + o(1) \text{ as } n \to \infty$$

(4.7) $$E(v_{0,n}) = \sum_{j=1}^{J} E\big(V_j^l\big(-t_{j,n}/\lambda_{j,n}^2\big)\big) + E(w_n) + o(1) \text{ as } n \to \infty.$$

Remark 4.8. Lemma 4.3 is due to Keraani [15]. It is based on the "refined Sobolev inequality" ($N = 3$)

$$\|h\|_{L^6(\mathbb{R}^3)} \le C \,\|\nabla h\|_{L^2(\mathbb{R}^3)}^{1/3} \,\|\nabla h\|_{\dot{B}^0_{2,\infty}}^{2/3},$$

where $\dot{B}^0_{2,\infty}$ is the standard Besov space [4, 11]. (4.4) is a consequence of the proof of Corollary 1.9 in [15], (here, we use the hypothesis $\|e^{it\Delta} v_{0,n}\|_{L^{2(N+2)/N-2}} \ge \delta > 0$) while (4.7) follows from the orthogonality of $(\lambda_{j,n}; x_{j,n}; t_{j,n})$ as in the proof of (4.6). The rest of the lemma is contained in the proof of Theorem 1.6 in [15]. See also [2, 10, 16, 21].

Lemma 4.9. *Let $\{z_{0,n}\} \in \dot{H}^1$, with $\int |\nabla z_{0,n}|^2 < \int |\nabla W|^2$ and $E(z_{0,n}) \to E_C$ and with $\|e^{it\Delta} z_{0,n}\|_{S((-\infty,+\infty))} \ge \delta$, with δ as in Theorem 2.5. Let $\{V_{0,j}\}$ be as in Lemma 4.3. Assume that one of the two hypothesis*

(4.10) $$\varliminf_{n\to\infty} E\big(V_1^l\big(-t_{1,n}/\lambda_{1,n}^2\big)\big) < E_C$$

or after passing to a subsequence, we have that, with $s_n = -t_{1,n}/\lambda_{1,n}^2$, $E(V_1^l(s_n)) \to E_C$, and $s_n \to s_* \in [-\infty, +\infty]$, and if U_1 is the non-linear profile (see Definition 2.12 and Remark 2.13) associated to $(V_{0,1}, \{s_n\})$ we have that the maximal interval of existence of U_1 is $I = (-\infty, +\infty)$ and $\|U_1\|_{S((-\infty,+\infty))} < \infty$ and

(4.11) $$\varliminf_{n\to\infty} E\big(V_1^l\big(-t_{1,n}/\lambda_{1,n}^2\big)\big) = E_C.$$

Then (after passing to a subsequence), for n large, if z_n is the solution of (CP) with data at $t = 0$ equal to $z_{0,n}$, then $(SC)(z_{0,n})$ holds.

Let us first assume the validity of Lemma 4.9 and use it (together with Lemma 4.3) to establish Propositions 4.1 and 4.2.

Proof of Proposition 4.1. By the definition of E_C, and the assumption that $E_C < E(W)$, we can find $u_{0,n} \in \dot{H}^1$, with $\int |\nabla u_{0,n}|^2 < \int |\nabla W|^2$, $E(u_{0,n}) \to E_C$, and such that if u_n is the solution of (CP) with data at $t = 0$, $u_{0,n}$ and maximal interval of existence $I_n = (-T_-(u_{0,n}), T_+(u_{0,n}))$, then $\|e^{it\Delta} u_{0,n}\|_{S((-\infty,+\infty))} \ge \delta = \delta(N) > 0$, where δ is as in Theorem 2.5 and $\|u_n\|_{S(I_N)} = +\infty$. (Here we are also using Lemma 2.1 and Theorem 2.5.) Note also that, since $E_C < E(W)$, there exists $\delta_0 > 0$ so that, for all n, we have $E(u_{0,n}) \le (1 - \delta_0)E(W)$. Because of Theorem 3.9, we can find

$\overline{\delta}$ so that $\int |\nabla u_n(t)|^2 \le (1-\overline{\delta}) \int |\nabla W|^2$ for all $t \in I_n$, all n. Apply now Lemma 4.3 for $\epsilon_0 > 0$ and Lemma 4.9. We then have, for $J = J(\epsilon_0)$, that

$$u_{0,n} = \sum_{j=1}^{J} \frac{1}{\lambda_{j,n}^{(N-2)/2}} V_j^l \left(\frac{x - x_{j,n}}{\lambda_{j,n}}, \frac{-t_{j,n}}{\lambda_{j,n}^2} \right) + w_n, \tag{4.12}$$

$$\int |\nabla u_{0,n}|^2 = \sum_{j=1}^{J} \int |\nabla V_{0,j}|^2 + \int |\nabla w_n|^2 + o(1), \tag{4.13}$$

$$E(u_{0,n}) = \sum_{j=1}^{J} E\left(V_j^l \left(\frac{-t_{j,n}}{\lambda_{j,n}^2} \right) \right) + E(w_n) + o(1). \tag{4.14}$$

Note that because of (4.13) we have, for all n large, that $\int |\nabla w_n|^2 \le (1-\overline{\delta}/2) \int |\nabla W|^2$ and $\int |\nabla V_{0,j}|^2 \le (1-\overline{\delta}/2) \int |\nabla W|^2$. From Corollary 3.8 it now follows that $E(V_j^l(-t_{j,n}/\lambda_{jn}^2)) \ge 0$ and $E(w_n) \ge 0$. From this and (4.14) it follows that $E(V_1^l(-t_{1,n}/\lambda_{1,n}^2)) \le E(u_{0,n}) + o(1)$ and hence $\underline{\lim}_{n\to\infty} E(V_1^l(-t_{1,n}/\lambda_{1,n}^2)) \le E_C$. If the left-hand side is strictly less than E_C, Lemma 4.9 gives us a contradiction with the choice of $u_{0,n}$, for n large (after passing to a subsequence). Hence, the left-hand side must equal E_C.

Let then U_1 be the non-linear profile associated to $(V_1^l, \{s_n\})$, with $s_n = -t_{1,n}/\lambda_{1,n}^2$ (after passing to a subsequence). We first note that we must have $J = 1$. This is because (4.14) and $E(u_{0,n}) \to E_C$, $E(V_1^l(-s_n)) \to E_C$ now imply that $E(w_n) \to 0$ and $E(V_j^l(-t_{j,n}/\lambda_{j,n}^2)) \to 0$, $j = 2, ..., J$. Using (3.6) and the argument in the proof of Corollary 3.13, we have $\sum_{j=2}^{J} \int |\nabla V_j^l(-t_{j,n}/\lambda_{j,n}^2)|^2 + \int |\nabla w_n|^2 \to 0$. We then have, since $\int |\nabla V_j^l(-t_{j,n}/\lambda_{j,n}^2)|^2 = \int |\nabla V_{0,j}|^2$ that $V_{0,j} = 0$, $j = 2, \ldots, J$ and $\int |\nabla w_n|^2 \to 0$. Hence (4.12) becomes $u_{0,n} = \frac{1}{\lambda_{1,n}^{(N-2)/2}} V_1^l\big(\frac{x - x_{1,n}}{\lambda_{1,n}}, s_n\big) + w_n$. Let $v_{0,n} = \lambda_{1,n}^{(N-2)/2} u_{0,n}(\lambda_{1,n}(x + x_{1,n}))$ and note that scaling gives us that $v_{0,n}$ verifies the same hypothesis as $u_{0,n}$. Moreover, $\widetilde{w}_n = \lambda_{1,n}^{(N-2)/2} w_n(\lambda_{1,n}(x + x_{1,n}))$ still verifies $\int |\nabla \widetilde{w}_n|^2 \to 0$. Thus

$$v_{0,n} = V_1^l(s_n) + \widetilde{w}_n, \qquad \int |\nabla \widetilde{w}_n|^2 \to 0.$$

Let us return to U_1, the non-linear profile associated to $(V_{0,1}, \{s_n\})$ and let $I_1 = (T_-(U_1), T_+(U_1))$ be its maximal interval of existence (see Remark 2.13). Note that, by definition of non-linear profile, we have $\int |\nabla U_1(s_n)|^2 = \int |\nabla V_1^l(s_n)|^2 + o(1)$ and $E(U_1(s_n)) = E(V_1^l(s_n)) + o(1)$. Note that in this case $E(V_1^l(s_n)) = E_C + o(1)$ and that $\int |\nabla V_1^l(s_n)|^2 = \int |\nabla V_{0,1}|^2 = \int |\nabla u_{0,n}|^2 + o(1) < \int |\nabla W|^2$ for n large by Theorem 3.9. Let's fix $\overline{s} \in I_1$. Then $E(U_1(s_n)) = E(U_1(\overline{s}))$, so that

$$E(U_1(\overline{s})) = E_C.$$

Moreover, $\int |\nabla U_1(s_n)|^2 < \int |\nabla W|^2$ for n large and hence by (3.10) $\int |\nabla U_1(\overline{s})|^2 < \int |\nabla W|^2$. If $\|U_1\|_{S(I_1)} < +\infty$, Lemma 2.11 gives us that $I_1 = (-\infty, +\infty)$ and we then obtain a contradiction from Lemma 4.9. Thus,

$$\|U_1\|_{S(I_1)} = +\infty$$

and we then set $u_C = U_1$ (after a translation in time to make $\overline{s} = 0$). □

Proof of Proposition 4.2. We argue by contradiction. For brevity of notation, let us set $u(x,t) = u_C(x,t)$. If not, there exists $\eta_0 > 0$ and a sequence $\{t_n\}_{n=1}^{\infty}$, $t_n \geq 0$ such that, for all $\lambda_0 \in \mathbb{R}^+$, $x_0 \in \mathbb{R}^N$, we have

$$\left\| \frac{1}{\lambda_0^{(N-2)/2}} u\left(\frac{x - x_0}{\lambda_0}, t_n\right) - u(x, t_{n'}) \right\|_{\dot{H}^1_x} \geq \eta_0, \quad \text{for } n \neq n'. \tag{4.15}$$

Note that (after passing to a subsequence, so that $t_n \to \overline{t} \in [0, T_+(u_0)]$), we must have $\overline{t} = T_+(u_0)$, in view of the continuity of the flow in $\dot{H}^1$, as guaranteed by Theorem 2.5. Note that, in view of Theorem 2.5 we must also have $\|e^{it\Delta}u(t_n)\|_{S((0,+\infty))} \geq \delta$.

Let us apply Lemma 4.3 to $v_{0,n} = u(t_n)$ with $\epsilon_0 > 0$. We next prove that $J = 1$. In fact, if $\underline{\lim}_{n\to\infty} E(V_1^l(-t_{1,n}/\lambda_{1,n}^2)) < E_C$, since $\int |\nabla u(t)|^2 \leq (1 - \overline{\delta}) \int |\nabla W|^2$ by Theorem 3.9, for all $t \in I_+$ and $E(u(t)) = E(u_0) = E_C < E(W)$, by Lemma 4.9 we obtain a contradiction. Hence, we must have $\underline{\lim}_{n\to\infty} E(V_1^l(-t_{1,n}/\lambda_{1,n}^2)) = E_C$. The argument used in the proof of Proposition 4.1 now applies and gives $J = 1$, $\int |\nabla w_n|^2 \to 0$. Thus, we have

$$u(t_n) = \frac{1}{\lambda_{1,n}^{(N-2)/2}} V_1^l\left(\frac{x - x_{1,n}}{\lambda_{1,n}}, \frac{-t_{1,n}}{\lambda_{1,n}^2}\right) + w_n, \quad \int |\nabla w_n|^2 \to 0. \tag{4.16}$$

Our next step is to show that $s_n = \frac{-t_{1,n}}{\lambda_{1,n}^2}$ must be bounded. To see this note that

$$e^{it\Delta}u(t_n) = \lambda_{1,n}^{-(N-2)/2} V_1^l\left(\frac{x - x_{1,n}}{\lambda_{1,n}}, \frac{t - t_{1,n}}{(\lambda_{1,n})^2}\right) + e^{it\Delta}w_n.$$

Assume that $t_{1,n}/\lambda_{1,n}^2 \leq -C_0$, C_0 a large positive constant. Then, since $\|e^{it\Delta}w_n\|_{S((-\infty,+\infty))} < \delta/2$ for n large, and

$$\left\| \lambda_{1,n}^{-(N-2)/2} V_1^l\left(\frac{x - x_{1,n}}{\lambda_{1,n}}, \frac{t - t_{1,n}}{(\lambda_{1,n})^2}\right) \right\|_{S((0,+\infty))} \leq \left\| V_1^l(y, s) \right\|_{S((C_0,+\infty))} \leq \delta/2,$$

for C_0 large, we get a contradiction.

If, on the other hand, $\frac{t_{1,n}}{\lambda_{1,n}^2} \geq C_0$, for a large positive constant C_0, n large, we have

$$\left\| \lambda_{1,n}^{-(N-2)/2} V_1^l \left(\frac{x - x_{1,n}}{\lambda_{1,n}}, \frac{t - t_{1,n}}{(\lambda_{1,n})^2} \right) \right\|_{S((-\infty,0))} \leq \left\| V_1^l(y, s) \right\|_{S((-\infty,-C_0))} \leq \delta/2,$$

for C_0 large. Hence, $\|e^{it\Delta}u(t_n)\|_{S((-\infty,0))} \leq \delta$, for n large and hence, Theorem 2.5 now gives $\|u\|_{S((-\infty,t_n))} \leq 2\delta$, which, since $t_n \to T_+(u_0)$ gives us a contradiction. Thus $|t_{1,n}/\lambda_{1,n}^2| \leq C_0$ and after passing to a subsequence,

$$t_{1,n}/\lambda_{1,n}^2 \to t_0 \in (-\infty, +\infty).$$

But then, since (4.15) and (4.16) imply that, for $n \neq n'$ large (independently of λ_0, x_0) we have

$$\Bigg\| \frac{1}{(\lambda_0)^{(N-2)/2}} \frac{1}{(\lambda_{1,n})^{(N-2)/2}} V_1^l \left(\frac{\frac{x-x_0}{\lambda_0} - x_{1,n}}{\lambda_{1,n}}, -t_{1,n}/(\lambda_{1,n})^2 \right) - \frac{1}{(\lambda_{1,n'})^{(N-2)/2}} V_1^l \left(\frac{x - x_{1,n'}}{\lambda_{1,n'}}, -t_{1,n'}/(\lambda_{1,n'})^2 \right) \Bigg\|_{\dot{H}^1} \geq \eta_0/2$$

or

$$\Bigg\| \left(\frac{\lambda_{1,n'}}{\lambda_{1,n}\lambda_0} \right)^{(N-2)/2} V_1^l \left(\frac{y\lambda_{1,n'}}{\lambda_0\lambda_{1,n}} + \widetilde{x}_{n,n'} - \widetilde{x}_0, -t_{1,n}/(\lambda_{1,n})^2 \right) - V_1^l \left(y, -t_{1,n'}/\lambda_{1,n'}^2 \right) \Bigg\|_{\dot{H}^1} \geq \eta_0/2,$$

where $\widetilde{x}_{n,n'}$ is a suitable point in $\mathbb{R}^N$ and $\lambda_0, \widetilde{x}_0$ are arbitrary. But if we choose $\lambda_0 = \lambda_{1,n'}/\lambda_{1,n}$, $\widetilde{x}_0 = x_{n,n'}$, we reach a contradiction since $-t_{1,n}/(\lambda_{1,n})^2 \to -t_0$ and $-t_{1,n'}/(\lambda_{1,n'})^2 \to -t_0$. □

Thus, to complete the proofs of Propositions 4.1 and 4.2 we only need to provide the proof of Lemma 4.9.

Proof of Lemma 4.9. Let us assume first that (4.11) holds and set $A = \int |\nabla W|^2$, $A' = \int |\nabla W|^2$, $M = \|U_1\|_{S((-\infty,+\infty))}$. Arguing (for some $\epsilon_0 > 0$ in Lemma 4.3) as in the proof of Proposition 4.1, we see that $\underline{\lim}_{n\to\infty} E(V_1^l(-t_{1,n}/\lambda_{1,n}^2)) = E_C$ and $E_C < E(W)$, imply that $J = 1$, $\int |\nabla w_n|^2 \to 0$. Moreover, if

$$v_{0,n} = \lambda_{1,n}^{(N-2)/2} z_{0,n}(\lambda_{1,n}(x + x_{1,n})), \ \widetilde{w}_n = \lambda_{1,n}^{(N-2)/2} w_n(\lambda_{1,n}(x + x_{1,n})),$$
$$s_n = -\frac{t_{1,n}}{\lambda_{1,n}^2},$$

we have $\int |\nabla \widetilde{w}_n|^2 \to 0$ and $v_{0,n} = V_1^l(s_n) + \widetilde{w}_n$, while $\|e^{it\Delta} v_{0,n}\|_{S((-\infty,+\infty))} \geq \delta$, $\int |\nabla v_{0,n}|^2 < \int |\nabla W|^2$, $E(v_{0,n}) \to E_C$. Note now that $\int |\nabla \widetilde{V}_1^l(s_n) - U_1(s_n)|^2 = o(1)$ by definition of non-linear profile. We then have

$$v_{0,n} = U_1(s_n) + \widetilde{\widetilde{w}}_n, \quad \int \left|\nabla \widetilde{\widetilde{w}}_n\right|^2 \to 0.$$

Moreover, as in the proof of Proposition 4.1, $E(U_1(0)) = E_C$ and $\int |\nabla U_1(t)|^2 < \int |\nabla W|^2$ for all t. We now apply Theorem 2.14, with $\epsilon_0 < \epsilon_0(M, A, A', N)$ and n large, with $\widetilde{u} = U_1$, $e \equiv 0$, $t_0 = 0$, $u_0 = v_{0,n}$. This case now follows.

Assume next that (4.10) holds. The first claim is that for $j \geq 2$ we also have $\underline{\lim}_{n\to\infty} E(V_j^l(-t_{j,n}/\lambda_{j,n}^2)) < E_C$. In fact, after passing to a subsequence, assume $\lim_{n\to\infty} E(V_1^l(-t_{1,n}/\lambda_{1,n})) < E_C$. Because of (4.6) we have

$$\int |\nabla z_{0,n}|^2 \geq \sum_{j=1}^{J} \int |\nabla V_{0,j}|^2 + o(1),$$

and since $E_C < E(W)$, for n large we have $E(z_{0,n}) \leq (1 - \delta_0)E(W)$, by Lemma 3.4, $\int |\nabla z_{0,n}|^2 \leq (1 - \overline{\delta}) \int |\nabla W|^2$ and hence $\int |\nabla V_{0,j}|^2 \leq (1-\overline{\delta}) \int |\nabla W|^2$. Similarly, $\int |\nabla w_n|^2 \leq (1-\overline{\delta}) \int |\nabla W|^2$. By Corollary 3.8, we have $E(V_j^l(-t_{j,n}/\lambda_{j,n}^2)) \geq 0$, $E(w_n) \geq 0$. Also, from (4.4) and the proof of Corollary 3.13, we have, for n large, that $E(V_1^l(-t_{1,n}/\lambda_{1,n}^2)) \geq C \int |\nabla V_{0,1}|^2 \geq c\alpha_0 = \overline{\alpha_0} > 0$, so that, from (4.7) we obtain, for n large

$$E(z_{0,n}) \geq \overline{\alpha_0} + \sum_{j=2}^{J} E\big(V_j^l\big(- t_{j,n}/\lambda_{j,n}^2\big)\big) + o(1),$$

so that the claim follows from $E(z_{0,n}) \to E_C$.

We next claim that (after passing to a subsequence so that, for each j, $\lim_n E(V_j^l(-t_{j,n}/\lambda_{j,n}^2))$ exists and $\lim_n(-t_{j,n}/\lambda_{j,n}^2) = \overline{s_j} \in [-\infty, +\infty]$ exists) if U_j is the non-linear profile associated to $(V_j^l, \{-t_{j,n}/\lambda_{j,n}^2\})$, then U_j verifies (SC). In fact, by definition of non-linear profile, $E(U_j) < E_C$, since $\lim_n E(V_j^l(-t_{j,n}/\lambda_{j,n}^2)) < E_C$. Moreover, since $\int |\nabla V_j^l(-t_{j,n}/\lambda_{j,n}^2)|^2 \leq (1 - \overline{\delta}) \int |\nabla W|^2$, by the definition of non-linear profile and Theorem 3.9, if $\overline{t} \in I_j$, the maximal interval for U_j, $\int |\nabla U_j(\overline{t})|^2 < \int |\nabla W|^2$ so that, by the definition of E_C our claim follows. Note that the argument in the proof of Theorem 2.14 also gives that $\|\nabla U_j\|_{W((-\infty,+\infty))} < +\infty$.

Our final claim is that there exists j_0 so that, for $j \geq j_0$ we have

$$\|U_j\|_{S((-\infty,+\infty))}^{2(N+2)/(N-2)} \leq C\Big(\int |\nabla V_{0,j}|^2\Big)^{N+2/N-2}. \tag{4.17}$$

In fact, from (4.6), for fixed J we see that (choosing n large) $\sum_{j=1}^{J}\int|\nabla V_{0,j}|^2 \leq \int|\nabla z_{0,n}|^2 + o(1) \leq 2\int|\nabla W|^2$. Thus, for $j \geq j_0$, we have $\int|\nabla V_{0,j}|^2 \leq \widetilde{\delta}$, where $\widetilde{\delta}$ is so small that $\|e^{it\Delta}V_{0,j}\|_{S((-\infty,+\infty))} \leq \delta$, with δ as in Theorem 2.5. From Remark 2.13 it then follows that $\|U_j\|_{S((-\infty,+\infty))} \leq 2\delta$, and using the integral equation in Remark 2.13, that $\|U_j(0)\|_{\dot{H}^1} \leq C\|V_{0,j}\|_{\dot{H}^1}$ and $\|\nabla U_j\|_{W((-\infty,+\infty))} \leq C\|V_{0,j}\|_{\dot{H}^1}$, which gives (4.17).

For $\epsilon_0 > 0$, to be chosen, define now

$$H_{n,\epsilon_0} = \sum_{j=1}^{J(\epsilon_0)} \frac{1}{\lambda_{j,n}^{(N-2)/2}} U_j\left(\frac{x - x_{j,n}}{\lambda_{j,n}}, \frac{t - t_{j,n}}{\lambda_{j,n}^2}\right). \tag{4.18}$$

We then have:

$$\|H_{n,\epsilon_0}\|_{S((-\infty,+\infty))} \leq C_0, \tag{4.19}$$

uniformly in ϵ_0, for $n \geq n(\epsilon_0)$. In fact,

$$\begin{aligned}
\|H_{n,\epsilon_0}\|_{S((-\infty,+\infty))}^{2(N+2)/(N-2)} &= \iint\left[\sum_{j=1}^{J(\epsilon_0)} \frac{1}{\lambda_{j,n}^{(N-2)/2}} U_j\left(\frac{x - x_{j,n}}{\lambda_{j,n}}, \frac{t - t_{j,n}}{\lambda_{j,n}^2}\right)\right]^{\frac{2(N+2)}{N-2}} \\
&\leq \sum_{j=1}^{J(\epsilon_0)} \iint\left|\frac{1}{\lambda_{j,n}^{(N-2)/2}} U_j\left(\frac{x - x_{j,n}}{\lambda_{j,n}}, \frac{t - t_{j,n}}{\lambda_{j,n}^2}\right)\right|^{\frac{2(N+2)}{N-2}} \\
&\quad + C_{J(\epsilon_0)} \sum_{j' \neq j} \iint\left|\frac{1}{\lambda_{j,n}^{(N-2)/2}} U_j\left(\frac{x - x_{j,n}}{\lambda_{j,n}}, \frac{t - t_{j,n}}{\lambda_{j,n}^2}\right)\right| \\
&\quad \left|\frac{1}{\lambda_{j',n}^{(N-2)/2}} U_{j'}\left(\frac{x - x_{j',n}}{\lambda_{j',n}}, \frac{t - t_{j',n}}{\lambda_{j',n}^2}\right)\right|^{\frac{N+6}{N-2}} = \mathrm{I} + \mathrm{II}.
\end{aligned}$$

For n large, $\mathrm{II} \to 0$, by the orthogonality of $(\lambda_{j,n}; x_{j,n}; t_{j,n})$ (see Keraani [15], Lemma 2.7, (2.95), (2.96), etc.) Hence, for n large we have $\mathrm{II} \leq \mathrm{I}$. But (with j_0 as in (4.17)),

$$\begin{aligned}
I &\leq \sum_{j=1}^{j_0} \|U_j\|_{S((-\infty,+\infty))}^{2(N+2)/(N-2)} + \sum_{j=j_0}^{J(\epsilon_0)} \|U_j\|_{S((-\infty,+\infty))}^{2(N+2)/(N-2)} \\
&\leq \sum_{j=1}^{j_0} \|U_j\|_{S((-\infty,+\infty))}^{2(N+2)/(N-2)} + C\sum_{j=j_0}^{J(\epsilon_0)} \left(\int|\nabla V_{0,j}|^2\right)^{N+2/(N-2)} \leq C_0/2
\end{aligned}$$

because of (4.6).

For $\epsilon_0 > 0$, to be chosen, define

(4.20)

$$R_{n,\epsilon_0} = |H_{n,\epsilon_0}|^{\frac{4}{N-2}} H_{n,\epsilon_0} - \sum_{j=1}^{J(\epsilon_0)} \left| \frac{1}{\lambda_{j,n}^{(N-2)/2}} U_j \left(\frac{x - x_{j,n}}{\lambda_{j,n}}, \frac{t - t_{j,n}}{\lambda_{j,n}^2} \right) \right|^{\frac{4}{N-2}} \times \frac{1}{\lambda_{j,n}^{(N-2)/2}} U_j \left(\frac{x - x_{j,n}}{\lambda_{j,n}}, \frac{t - t_{j,n}}{\lambda_{j,n}^2} \right).$$

We then have

(4.21) For $n = n(\epsilon_0)$ large, $\|\nabla R_{n,\epsilon_0}\|_{L_t^2 L_x^{2N/N+2}} \to 0$ as $n \to \infty$.

This follows from the orthogonality of $(\lambda_{j,n}; x_{j,n}; t_{j,n})$, the fact that $\|U_j\|_{S((-\infty,+\infty))} < \infty$, $\|\nabla U_j\|_{W((-\infty,+\infty))} < \infty$, and arguments of Keraani [15] (see in particular (2.95), (2.96)).

We now will apply Theorem 2.14. Let $\widetilde{u} = H_{n,\epsilon_0}$, $e = R_{n,\epsilon_0}$, where ϵ_0 is still to be determined. Recall that $z_{0,n} = \sum_{j=1}^{J(\epsilon_0)} \frac{1}{\lambda_{j,n}^{(N-2)/2}} V_j^l \left(\frac{x - x_{j,n}}{\lambda_{j,n}}, \frac{-t_{j,n}}{\lambda_{j,n}^2} \right)$ $+ w_n$, where $\|e^{it\Delta} w_n\|_{S((-\infty,+\infty))} \le \epsilon_0$. By the definition of non-linear profile, we now have

(4.22) $$z_{0,n}(x) = H_{n,\epsilon_0}(x, 0) + \widetilde{w}_n(x),$$

where, for n large $\|e^{it\Delta} \widetilde{w}_n\|_{S((-\infty,+\infty))} \le 2\epsilon_0$.

Notice also that, because of the orthogonality of $(\lambda_{j,n}; x_{j,n}; t_{j,n})$, for $n = n(\epsilon_0)$ large, we have (using also Corollary 3.13), that $\int |\nabla H_{n,\epsilon_0}(t)|^2 \le 2 \sum_{j=1}^{J(\epsilon_0)} E_0\left(U_j \left(\frac{t - t_{j,n}}{\lambda_{j,n}^2} \right) \right) \le 4C \sum_{j=1}^{J(\epsilon_0)} \int |\nabla V_{0,j}|^2$, and $\sum_{j=1}^{J(\epsilon_0)} \int |\nabla V_{0,j}|^2$ $\le \int |\nabla z_{0,n}|^2 \int |\nabla z_{0,n}|^2 + o(1) \le 2 \int |\nabla W|^2$. Let now $M = C_0$, with C_0 as in (4.19), $A = \widetilde{C} \int |\nabla W|^2$, $A' = A + \int |\nabla W|^2$, $\epsilon_0 < \epsilon_0(M, A, A', N)/2$, where $\epsilon_0(M, A, A', N)$ is as in Theorem 2.14. Fix ϵ_0 and choose n so large that $\|\nabla R_{n,\epsilon_0}\|_{L_t^2 L_x^{2N/N+2}} < \epsilon_0$ and so that all the above properties hold. Then Theorem 2.14 gives the conclusion of Lemma 4.9 in the case when (4.10) holds. □

Remark 4.23. Assume that $\{z_{0,n}\}$ in Lemma 4.3 are all radial. Then $V_{0,j}$, w_n can be chosen to be radial and we can choose $x_{j,n} \equiv 0$. This follows directly from Keraani's proof [15]. If we then define (SC) and E_C by restricting only to radial functions, we obtain a u_C as in Proposition 4.1 which is radial, and we can establish Proposition 4.2 with $x(t) \equiv 0$.

5. Rigidity theorem

In this section we will prove the following:

Theorem 5.1. *Assume that $u_0 \in \dot{H}^1$ is such that*

$$E(u_0) < E(W), \quad \int |\nabla u_0|^2 < \int |\nabla W|^2.$$

Let u be the solution of (CP) with $u|_{t=0} = u_0$, with maximal interval of existence $(-T_-(u_0), T_+(u_0))$ (see Definition 2.10). Assume that there exists $\lambda(t) > 0$, for $t \in [0, T_+(u_0))$, with the property that

$$K = \left\{ v(x,t) = \frac{1}{\lambda(t)^{(N-2)/2}} u\left(\frac{x}{\lambda(t)}, t\right) \; : \; t \in [0, T_+(u_0)) \right\}$$

is such that $\overline{K}$ is compact in $\dot{H}^1$. Then $T_+(u_0) = +\infty$, $u_0 \equiv 0$.

Remark 5.2. We conjecture that Theorem 5.1 remains true if $v(x,t) = \frac{1}{\lambda(t)^{(N-2)/2}} u\left(\frac{x-x(t)}{\lambda(t)}, t\right)$, with $x(t) \in \mathbb{R}^N$, $t \in [0, T_+(u_0))$. In other words, for "energy subcritical" initial data, compactness up to the invariances of the equation, for solutions, is only true for $u \equiv 0$.

We start out with a special case of the strengthened form of Theorem 5.1, namely:

Proposition 5.3. *Assume that u, v, $\lambda(t)$, $x(t)$ are as in Remark 5.2, that $|x(t)| \leq C_0$ and that $\lambda(t) \geq A_0 > 0$. Then the conclusion of Theorem 5.1 holds. Moreover, if $T_+(u_0) < +\infty$, the hypothesis $|x(t)| \leq C_0$ is not needed.*

Remark 5.4. Because of the continuity of $u(t)$ in $\dot{H}^1$, it is clear that in proving Proposition 5.3 we can assume that $\lambda(t), x(t) \in C^\infty([0, T_+(u_0)))$ and that $\lambda(t) > 0$ for each $t \geq 0$. Indeed, first by the compactness of $\overline{K}$ and the theory of (CP), we construct piecewise contant (with small jumps) $\lambda_1(t), x_1(t)$ such that the corresponding set K_1 is included in $\tilde{K}_1 = \{w(t)$ solution of (CP) with initial data in $\overline{K}, t \in [0, t_0]\}$, t_0 small. Then we can contruct regular $\lambda_2(t), x_2(t)$ such that K_2 is included in the precompact set $\left\{ \lambda_0^{-\frac{(N-2)}{2}} w((x - x_0)\lambda_0^{-1}), \text{ for } w \in \tilde{K}_1, 1/2 \leq \lambda_0 \leq 2, |x_0| \leq 1 \right\}$. The continuity of $\lambda(t), x(t)$ will not be used in our proof.

In the next lemma we will collect some useful facts:

Lemma 5.5. *Let u, v be as in Proposition 5.3.*

i) Let $\delta_0 > 0$ be such that $E(u_0) \leq (1-\delta_0)E(W)$. Then for all $t \in [0, T_+(u_0))$, we have

$$\int |\nabla u(t)|^2 \leq (1-\overline{\delta}) \int |\nabla W|^2$$

$$\int |\nabla u(t)|^2 - |u(t)|^{2^*} \geq \overline{\delta} \int |\nabla u(t)|^2$$

$$C_{1,\delta_0} \int |\nabla u_0|^2 \leq E(u_0) \leq C_2 \int |\nabla u_0|^2$$

$$E(u(t)) = E(u_0)$$

$$C_{1,\delta_0} \int |\nabla u_0|^2 \leq \int |\nabla u(t)|^2 \leq C_2 \int |\nabla u_0|^2.$$

ii)

$$\int |\nabla v(t)|^2 \leq C_2 \int |\nabla W|^2$$

$$\|v(t)\|^2_{L_x^{2^*}} \leq C_3 \int |\nabla W|^2.$$

iii) For all $x_0 \in \mathbb{R}^N$

$$\int \frac{|v(x,t)|^2}{|x-x_0|^2} \leq C_4 \int |\nabla W|^2.$$

vi) For each $\epsilon > 0$, there exists $R(\epsilon_0) > 0$, such that, for $0 \leq t < T_+(u_0)$, we have

$$\int_{|x| \geq R(\epsilon_0)} |\nabla v|^2 + |v|^{2^*} + \frac{|v|^2}{|x|^2} \leq \epsilon_0.$$

Proof. i) follows from Theorem 3.9 and Corollary 3.13. ii) follows from i) by Sobolev embedding, while iii) follows from i) by Hardy's inequality. iv) follows (using Sobolev embedding and the Hardy inequality) from the compactness of $\overline{K}$. □

The next lemma is a localized virial identity, in the spirit of Merle [17], Lemma 3.6.

Lemma 5.6. *Let $\varphi \in C_0^\infty(\mathbb{R}^N)$, $t \in [0, T_+(u_0))$. Then:*

i)

$$\frac{d}{dt}\int |u|^2 \varphi\, dx = 2\mathrm{Im}\int \overline{u}\nabla u \nabla\varphi\, dx$$

ii)

$$\frac{d^2}{dt^2}\int |u|^2 \varphi\, dx = 4\sum_{l,j} \mathrm{Re}\int \partial_{x_l}\partial_{x_j}\varphi \cdot \partial_{x_l}u \cdot \partial_{x_j}\overline{u} - \int \Delta^2\varphi\, |u|^2 - \frac{4}{N}\int \Delta\varphi\, |u|^{2^*}.$$

The proof of Lemma 5.6 is standard, see [17] and Glassey [12].

Proof of Proposition 5.3. The proof splits in two cases, the finite time blow-up case for u and the infinite time of existence for u.

Case 1: $T_+(u_0) < \infty$. (In this case we don't need the assumption $|x(t)| < C_0$ or the energy constraints on u, only $sup_{t\in[0,T_+(u_0))} \int |\nabla u(t)|^2 < \infty$ is needed. Note that this rules out the existence of self similar solutions in $\dot{H}^1$, i.e. solutions for which $\lambda(t) \sim (T-t)^{-1/2}$.)

Note first that $\lambda(t) \to \infty$ as $t \to T_+(u_0)$. If not, there exists $t_i \uparrow T_+(u_0)$, with $\lambda(t_i) \to \lambda_0 < +\infty$. Let $v_i(x) = \frac{1}{\lambda(t_i)^{(N-2)/2}} u\big(\frac{x - x(t_i)}{\lambda(t_i)}, t_i\big)$ and let $v(x) \in \dot{H}^1$ be such that $v_i \to v$ in $\dot{H}^1$ (from the compactness of $\overline{K}$). Hence, $u\big(x - \frac{x(t_i)}{\lambda(t_i)}, t_i\big) = \lambda(t_i)^{(N-2)/2} v_i(\lambda(t_i)x) \to \lambda_0^{(N-2)/2} v(\lambda_0 x)$ in $\dot{H}^1$ (since $\lambda(t_i) \geq A_0$, $\lambda_0 \geq A_0$). Let now $h(x,t)$ be the solution of (CP), given by Remark 2.8 with data $\lambda_0^{(N-2)/2} v(\lambda_0 x)$ at time $T_+(u_0)$, in an interval $(T_+(u_0) - \delta, T_+(u_0) + \delta)$, with $\|h\|_{S((T_+(u_0)-\delta, T_+(u_0)+\delta))} < \infty$. Let $h_i(x,t)$ be the solution with data at $T_+(u_0)$ equal to $u\big(x - \frac{x(t_i)}{\lambda(t_i)}, t_i\big)$. Then, the (CP) theory guarantees that

$$\sup_i \|h_i\|_{S((T_+(u_0)-\frac{\delta}{2}, T_+(u_0)+\frac{\delta}{2}))} < \infty.$$

But, $u\big(x - \frac{x(t_i)}{\lambda(t_i)}, t + t_i - T_+(u_0)\big) = h_i(x,t)$, contradicting Lemma 2.11, since $T_+(u_0) < \infty$.

Let us prove now a decay result for u from the concentration properties in L^{2^*} of u at $T_+(u_0)$. Let us now fix $\varphi \in C_0^\infty(\mathbb{R}^N)$, φ radial, $\varphi \equiv 1$ for $|x| \leq 1$, $\varphi \equiv 0$ for $|x| \geq 2$ and set $\varphi_R(x) = \varphi(x/R)$. Define

$$y_R(t) = \int |u(x,t)|^2 \varphi_R(x)\, dx, \quad t \in [0, T_+(u_0)). \tag{5.7}$$

We then have:

$$|y_R'| \leq C_N \int |\nabla W|^2. \tag{5.8}$$

In fact, by Lemma 4.6, i)

$$
\begin{aligned}
|y_R'| &\leq \frac{2}{R}\left|\operatorname{Im}\int \overline{u}\nabla u \nabla\varphi(x/R)\,dx\right| \\
&\leq C_N\left(\int |\nabla u|^2\right)^{1/2}\cdot\left(\int \frac{|u|^2}{|x|^2}\right)^{1/2} \leq C_N\int |\nabla W|^2,
\end{aligned}
$$

by ii) in Lemma 5.5.

We also have:

$$
\text{(5.9)}\qquad \text{For all } R>0,\quad \int_{|x|<R}\int |u(x,t)|^2\,dx \to 0 \quad \text{as } t\to T_+(u_0).
$$

In fact, $u(y,t) = \lambda(t)^{(N-2)/2}v(\lambda(t)y + x(t), t)$, so that

$$
\begin{aligned}
\int_{|x|<R} |u(x,t)|^2\,dx &= \lambda(t)^{-2}\int_{|y|<R\lambda(t)} |v(y+x(t),t)|^2\,dy \\
&= \lambda(t)^{-2}\int_{B(x(t),R\lambda(t))} |v(z,t)|^2\,dz \\
&= \lambda(t)^{-2}\int_{B(x(t),R\lambda(t))\cap B(0,\epsilon R\lambda(t))} |v(z,t)|^2\,dz \\
&\quad + \lambda(t)^{-2}\int_{B(x(t),R\lambda(t))\setminus B(0,\epsilon R\lambda(t))} |v(z,t)|^2\,dz \\
&= A + B.
\end{aligned}
$$

By Hölder's inequality, $A \leq \lambda(t)^{-2}(\epsilon R\lambda(t))^{N\frac{2}{N}}\|v\|^2_{L^{2^*}} \leq \epsilon^2 R^2 C_3 \int |\nabla W|^2$, which is small with ϵ.

$$
\begin{aligned}
B &\leq \lambda(t)^{-2}(R\lambda(t))^{N\frac{2}{N}}\,\|v\|^2_{L^{2^*}(|x|\geq\epsilon R\lambda(t))} \\
&= R^2\,\|v\|^2_{L^{2^*}(|x|\geq\epsilon R\lambda(t))} \to 0 \quad \text{as } t\to T_+(u_0),
\end{aligned}
$$

by iv) in Lemma 5.5, since $\lambda(t)\to+\infty$ as $t\to T_+(u_0)$.

From (5.9) and (5.8), we have:

$$
y_R(0) \leq y_R(T_+(u_0)) + C_N T_+(u_0)\int |\nabla W|^2 = C_N T_+(u_0)\int |\nabla W|^2.
$$

Thus, letting $R\to+\infty$ we obtain

$$
u_0 \in L^2(\mathbb{R}^N).
$$

Arguing as before, $|y_R(t) - y_R(T_+(u_0))| \leq C_N(T_+(u_0)-t)\int|\nabla W|^2$ so that $y_R(t) \leq C_N(T_+(u_0)-t)\int|\nabla W|^2$. Letting $R\to\infty$, we see that $\|u(t)\|^2_{L^2} \leq C_N(T_+(u_0)-t)\int|\nabla W|^2$ and so by the conservation of the L^2 norm $\|u(T_+(u_0))\|_{L^2} = \|u_0\|_{L^2} = 0$. But then $u\equiv 0$, contradicting $T_+(u_0)<\infty$.

Case 2: $T_+(u_0) = +\infty$.
In this case we assume, in addition, that $|x(t)| \le C_0$. We first note that

(5.10) For each $\epsilon > 0$, there exists $R(\epsilon) > 0$ such that, for all $t \in [0, \infty)$, we have:

$$\int_{|x|>R(\epsilon)} |\nabla u|^2 + |u|^{2^*} + \frac{|u|^2}{|x|^2} \le \epsilon.$$

In fact, $u(y,t) = \lambda(t)^{(N-2)/2} v(\lambda(t)y + x(t), t)$, so that

$$\begin{aligned}\int_{|y|>R(\epsilon)} |\nabla u(y,t)|^2 \, dy &= \int_{|y|>R(\epsilon)} \lambda(t)^N \, |\nabla v(\lambda(t)y + x(t), t)|^2 \, dy \\ &= \int_{|z|>R(\epsilon)\lambda(t)} |\nabla v(z + x(t), t)|^2 \, dz \\ &\le \int_{|z|\ge R(\epsilon)A_0} |\nabla v(z + x(t), t)|^2 \, dz \\ &\le \int_{|\alpha|\ge R(\epsilon)A_0 - C_0} |\nabla v(\alpha, t)|^2 \, d\alpha,\end{aligned}$$

and the statement for this term now follows from Lemma 5.5 iv). The other terms are handled similarly.

(5.11) There exists $R_0 > 0$ such that, for all $t \in [0, +\infty)$, we have

$$8 \int_{|x|\le R_0} |\nabla u|^2 - 8 \int_{|x|\le R_0} |u|^{2^*} \ge C_{\delta_0} \int |\nabla u_0|^2.$$

In fact, (3.11) combined with Lemma 5.5 i) yields $8 \int |\nabla u|^2 - 8 \int |u|^{2^*} \ge \widetilde{C}_{\delta_0} \int |\nabla u_0|^2$. Now combine this with (5.10), with $\epsilon = \epsilon_0 \int |\nabla u_0|^2$ to obtain (5.11).

To prove Case 2, we choose $\varphi \in C_0^\infty(\mathbb{R}^N)$, radial, with $\varphi(x) = |x|^2$ for $|x| \le 1$, $\varphi(x) \equiv 0$ for $|x| \ge 2$. Define $z_R(t) = \int |u(x,t)|^2 R^2 \varphi\left(\frac{x}{R}\right) dx$. We then have:

$$\text{for } \ t > 0, \quad |z_R'(t)| \le C_{N,\delta_0} \int |\nabla u_0|^2 \, R^2$$

$$\text{for } R \text{ large enough, } t > 0, \quad z_R'' \ge C_{N,\delta_0} \int |\nabla u_0|^2.$$

In fact, from Lemma 5.6, i),

$$\begin{aligned}\left|z_R'(t)\right| &\le 2R \left| \operatorname{Im} \int \overline{u} \nabla u \nabla \varphi\left(\frac{x}{R}\right) dx \right| \le C_N R \int_{0\le|x|\le 2R} \frac{|x|}{|x|} \, |\nabla u| \, |u| \\ &\le C_N R^2 \left(\int |\nabla u|^2 \right)^{1/2} \left(\int \frac{|u|^2}{|x|^2} \right)^{1/2} \le C_N R^2 \int |\nabla u_0|^2,\end{aligned}$$

because of Lemma 5.5 i), while from Lemma 5.6, ii),

$$z_R''(t) = 4\sum_{l,j} \mathrm{Re} \int \partial_{x_l}\partial_{x_j}\varphi\left(\frac{x}{R}\right)\partial_{x_l}u \cdot \partial_{x_j}\overline{u} - \int \Delta^2\varphi\left(\frac{x}{R}\right)\frac{|u|^2}{R^2}$$
$$- \frac{4}{N}\int \Delta\varphi\, |u|^{2^*} \geq 8\left[\int_{|x|\leq R} |\nabla u|^2 - |u|^{2^*}\right]$$
$$- C_N \int_{R\leq |x|\leq 2R}\left[|\nabla u|^2 + \frac{|u|^2}{|x|^2} + |u|^{2^*}\right] \geq C_{N,\delta_0}\int |\nabla u_0|^2$$

for R large, in view of (5.11) and (5.10).

If we now integrate in t, we have $z_R'(t) - z_R'(0) \geq C_{N,\delta_0} t \int |\nabla u_0|^2$, but we also have $|z_R'(t) - z_R'(0)| \leq 2C_N R^2 \int |\nabla u_0|^2$, a contradiction for t large, unless $\int |\nabla u_0|^2 = 0$. □

Proof of Theorem 5.1. (See [19] for similar proof) Assume that $u_0 \not\equiv 0$ so that $\int |\nabla u_0|^2 > 0$ and because of Lemma 5.5 i) (which is still valid here), $E(u_0) \geq C_{1,\delta_0}\int |\nabla u_0|^2$ and hence $E(u_0) > 0$. Because of Proposition 5.3, we only need to treat the case where there exists $\{t_n\}_{n=1}^{\infty}$, $t_n \geq 0$, $t_n \uparrow T_+(u_0)$, so that

$$\lambda(t_n) \to 0.$$

(If $t_n \to t_0 \in [0, T_+(u_0))$, we obtain for all $R > 0$, $\int_{|x|\geq R} |v(t_0)|^{2*} = 0$ but $\int |\nabla v(t_0)|^2 > 0$). After possibly redefining $\{t_n\}_{n=1}^{\infty}$ we can assume that

$$\lambda(t_n) \leq 2inf_{t\in[0,t_n]}\lambda(t)$$

and from our hypothesis

$$w_n(x) = \frac{1}{\lambda(t_n)^{(N-2)/2}} u\left(\frac{x}{\lambda(t_n)}, t_n\right) \to w_0 \text{ in } \dot{H}^1.$$

By Theorem 3.9 we have $E(W) > E(w_0) = E(u_0) > 0$, $\int |\nabla u(t)|^2 \leq (1-\overline{\delta})\int |\nabla W|^2$ so that $\int |\nabla w_0|^2 < \int |\nabla W|^2$. Thus $w_0 \not\equiv 0$. Let us now consider solutions of (CP), $w_n(x, \tau)$, $w_0(x, \tau)$ with data $w_n(-, 0)$, $w_0(-, 0)$ at $\tau = 0$, defined in maximal intervals $\tau \in (-T_-(w_n), 0]$, $\tau \in (-T_-(w_0), 0]$ respectively.

Since $w_n \to w_0$ in $\dot{H}^1$, $\underline{\lim}_{n\to\infty} T_-(w_n) \geq T_-(w_0)$ and

for each $\tau \in (-T_-(w_0), 0]$, $w_n(x, \tau) \to w_0(x, \tau)$ in $\dot{H}^1$.
(See Remark 2.17)

Note that by uniqueness in (CP) (see Definition 2.10), for $0 \leq t_n + \tau/\lambda(t_n)^2$, $w_n(x, \tau) = \frac{1}{\lambda(t_n)^{(N-2)/2}} u\big(\frac{x}{\lambda(t_n)}, t_n + \frac{\tau}{\lambda(t_n)^2}\big)$. Remark that $\underline{\lim}_{n\to\infty}\tau_n = \underline{\lim}_{n\to\infty} t_n\lambda(t_n)^2 \geq T_-(w_0)$ and thus for all $\tau \in (-T_-(w_0), 0]$ for n large,

$0 \le t_n + \tau/\lambda(t_n)^2 \le t_n$. Indeed,if $\tau_n \to \tau_0 < T_-(w_0)$, then $w_n(x, -\tau_n) = \frac{1}{\lambda(t_n)^{(N-2)/2}} u_0(\frac{x}{\lambda(t_n)}) \to w_0(x, -\tau_0)$ in $\dot{H}^1$ with $\lambda(t_n) \to 0$ which is a contradiction from $u_0 \not\equiv 0$, $w_0 \not\equiv 0$.

Fix now $\tau \in (-T_-(w_0), 0]$, for n sufficiently large $v(x, t_n + \tau/\lambda(t_n)^2)$, $\lambda(t_n + \tau/\lambda(t_n)^2)$ are defined and we have

$$(5.12)\quad \begin{aligned} &v(x, t_n + \tau/\lambda(t_n)^2) \\ &= \frac{1}{\lambda(t_n + \tau/\lambda(t_n)^2)^{(N-2)/2}} u\left(\frac{x}{\lambda(t_n + \tau/\lambda(t_n)^2)}, t_n + \tau/\lambda(t_n)^2\right) \\ &= \frac{1}{\widetilde{\lambda}_n(\tau)^{\frac{N-2}{2}}} w_n\left(\frac{x}{\widetilde{\lambda}_n(\tau)}, \tau\right), \end{aligned}$$

with

$$(5.13)\qquad \widetilde{\lambda}_n(\tau) = \frac{\lambda(t_n + \tau/\lambda(t_n)^2)}{\lambda(t_n)} \ge \frac{1}{2}$$

(because of the fact $\lambda(t_n) \le 2inf_{t\in[0,t_n]}\lambda(t)$.) One can assume after passing to a subsequence that $\widetilde{\lambda}_n(t_n + \tau/\lambda(t_n)^2) \to \widetilde{\lambda}_0(\tau)$ with $\frac{1}{2} \le \widetilde{\lambda}_0(\tau) \le +\infty$ and $v(x, t_n + \tau/\lambda(t_n)^2) \to v_0(x, \tau)$ in $\dot{H}^1$, as $n \to \infty$. Remark that $\widetilde{\lambda}_0(\tau) < +\infty$. If not, we will have $\frac{1}{\widetilde{\lambda}_n(\tau)^{(N-2)/2}} w_0\big(\frac{x}{\widetilde{\lambda}_n(\tau)}, \tau\big) \to v_0(x, \tau)$ which implies $w_0(x, \tau) = 0$ which contradicts $E(w_0) = E(u_0) > 0$. Thus $\widetilde{\lambda}_0(\tau) < +\infty$ and $v_0(x, \tau) = \frac{1}{\widetilde{\lambda}_0(\tau)^{(N-2)/2}} w_0\big(\frac{x}{\widetilde{\lambda}_0(\tau)}, \tau\big)$ where $v_0(\tau) \in \overline{K}$. We thus obtain a contradiction from Proposition 5.3. Note that the same proof applies in the nonradial situation with the extra parameter $x(t_n)$. □

Corollary 5.14. *Assume that $E(u_0) < E(W)$, $\int |\nabla u_0|^2 < \int |\nabla W|^2$ and u_0 is radial. Then the solution u of the Cauchy problem (CP) with data u_0 at $t = 0$ has time interval of existence $I = (-\infty, +\infty)$, $\|u\|_{S((-\infty,+\infty))} < +\infty$ and there exists $u_{0,+}$, $u_{0,-}$ in $\dot{H}^1$ such that*

$$\lim_{t\to+\infty} \left\|u(t) - e^{it\Delta}u_{0,+}\right\|_{\dot{H}^1} = 0, \qquad \lim_{t\to-\infty} \left\|u(t) - e^{it\Delta}u_{0,-}\right\|_{\dot{H}^1} = 0.$$

Moreover, if we define δ_0 so that $E(u_0) \le (1 - \delta_0)E(W)$, there exists a function $M(\delta_0)$ so that

$$\|u\|_{S((-\infty,+\infty))} \le M(\delta_0).$$

Proof. From the integral equation in Theorem 2.5, it is clear that $u(t)$ is radial for each $t \in I$. Using Remark (4.23) and Theorem 5.1 we obtain (SC) or $I = (-\infty, +\infty)$, $\|u\|_{S((-\infty,+\infty))} < +\infty$. Now Remark 2.15 finishes the proof of the first statement.

For the last statement, let

$$D_{\delta_0} = \left\{ u_0 \in \dot{H}^1 \text{ radial}, \int |\nabla u_0|^2 < \int |\nabla W|^2 \text{ and } E(u_0) \leq (1-\delta_0) E(W) \right\}$$

$$M(\delta_0) = sup_{u \in D_{\delta_0}} \|u\|_{S((-\infty,+\infty))}.$$

We need to show $M(\delta_0) < +\infty$. If not there is a sequence $u_{0,n}$ in D_{δ_0} and the corresponding solutions u_n such that $\|u_n\|_{S((-\infty,+\infty))} \to +\infty$ as $n \to +\infty$. Note that we can assume that $\|e^{it\Delta}u_{0,n}\|_{S((-\infty,+\infty))} \geq \delta$, with δ as in Theorem 2.5. Arguing as in the proof of Proposition 4.1, we would conclude that first $J = 1$ in the decomposition given in Lemma 4.3 and then since $\|U_1\|_{S((-\infty,+\infty))} < +\infty$ we reach a contradiction (See also [15], Corollary 1.14). □

Remark 5.15. Note that Corollary 5.14 is sharp. In fact, $W(x)$ is radial and clearly $\|W\|_{S((-\infty,+\infty))} = +\infty$. Moreover, Remark 3.14 shows that if $u_0 \in H^1$ radial, $E(u_0) < E(W)$, but $\int |\nabla u_0|^2 > \int |\nabla W|^2$, we have that I, the maximal interval of existence, is finite.

Let us remark that we have, in fact, proved a slightly stronger result:

Corollary 5.16. *Let $u_0 \in \dot{H}^1$ be radially symetric and assume that for all $t \in (-T_-(u_0), T_+(u_0))$ we have $\int |\nabla u(t)|^2 \leq \int |\nabla W|^2 - \delta_0$, for $\delta_0 > 0$. Then the solution u of the Cauchy problem (CP) with data u_0 at $t = 0$ has time interval of existence $I = (-\infty, +\infty)$, $\|u\|_{S((-\infty,+\infty))} < +\infty$.*

Proof. Note first that if $E(u_0) < E(W)$, Corollary 5.14 yields the result, so that we can assume that $E(u_0) \geq E(W)$. Observe that the end of the proof of Lemma 3.4 gives us that, for $t \in (-T_-(u_0), T_+(u_0))$,

$$\int |\nabla u(t)|^2 - \int |u(t)|^{2^*} \geq C_{\delta_0} \int |\nabla u(t)|^2 \tag{5.17}$$

and that energy conservation, the assumption that $E(u_0) \geq E(W)$ yields

$$inf_{t \in (-T_-(u_0), T_+(u_0))} \int |\nabla u(t)|^2 \geq C.$$

From the Remark 2.7, if δ_o is close to $\int |\nabla W|^2$, our conclusion holds. We can then find $0 \leq \delta_c < \int |\nabla W|^2$, so that if for $t \in (-T_-(u_0), T_+(u_0))$, $\int |\nabla u(t)|^2 \leq \int |\nabla W|^2 - \delta_0$, $\delta_0 > \delta_c$, our desired conclusion holds and δ_c is optimal with this property. We assume $\delta_c > 0$ and reach a contradiction by establishing the analogs of Propositions 4.1, 4.2 and using (5.17). In order to do so, we just need to modify the statement of Lemma 4.9, by replacing (4.10), say, by for $t \in I$, $\int |\nabla U_1(t)|^2 \leq \int |\nabla W|^2 - \delta_c$, where $U_1(t)$ is the non-linear profile in Lemma 4.9 and I its maximal interval of existence as in Remark 2.13. (4.11) is replaced analogously. The proofs are then identical to those given in Sect. 4 and that of Corollary 5.14. □

As a consequence of Corollaries 5.14 and 5.16, we obtain the following concentration phenomenon for all radial type II finite blow-up solutions. Here by type II finite blow-up solution, we mean a solution u whose maximal interval of existence I is finite and for which there is a C, such that for all $t \in I$, $\int |\nabla u(t)|^2 < C$. On the other hand, type I finite blow-up solution is such that what the time of existence is finite but the $\dot{H}^1$ norm blows up.

Corollary 5.18. *Let $u_0 \in \dot{H}^1$ (no size restrictions) be radially symetric and assume that $T_+(u_0) < +\infty$, and that $\forall t \in [0, T_+(u_0))$, $\int |\nabla u(t)|^2 \leq C_0$. Then, for all $R > 0$, we have*

$$\underline{\lim}_{t \to T_+(u_0)} \int_{|x| \leq R} |\nabla u(t)|^2 \geq \frac{2}{N} \int |\nabla W|^2$$
$$\overline{\lim}_{t \to T_+(u_0)} \int_{|x| \leq R} |\nabla u(t)|^2 \geq \int |\nabla W|^2 .$$

Proof. Consider $t_n \to T_+(u_0)$ and apply Lemma 4.3 to the sequence $u(t_n)$. Arguing in an analogous manner to the proof of Theorem 2.14, we must have $\lambda_{j,n} \to +\infty$ for some j and the corresponding non-linear profile U_j has $\|U_j\|_{S((0,T_+(U_j))} = +\infty$. If the first inequality does not hold, we can find a sequence $\{t_n\}$ as before and $R_0 > 0$, $\eta_0 > 0$ so that

$$\int_{|x| \leq R_0} |\nabla u(t_n)|^2 \leq \frac{2}{N} \int |\nabla W|^2 - \eta_0.$$

In addition,we must have (since $\lambda_{j,n} \to +\infty$) that

$$\int \left|\nabla U_j\big(-t_{j,n}/\lambda_{j,n}^2\big)\right|^2 \leq \frac{2}{N} \int |\nabla W|^2 - \eta_0 < \frac{2}{N} \int |\nabla W|^2 < \int |\nabla W|^2 .$$

Thus $E(U_j) < E(W) = \frac{1}{N} \int |\nabla W|^2$ and Corollary 5.14 gives a contradiction.

If the second inequality does not hold, we can find $R_0 > 0$, $\eta_0 > 0$ so that for all $t \in I$, $\int_{|x| \leq R_0} |\nabla u(t)|^2 \leq \int |\nabla W|^2 - \eta_0$. By the argument at the beginning of the proof of case 1 of Proposition 5.13, we must have $-t_{j,n}/\lambda_{j,n}^2 < C$. Thus, we obtain, for $t > M$, that $\int |\nabla U_j(t)|^2 \leq \int |\nabla W|^2 - \eta_0$, so that Corollary 5.16 concludes the proof. □

Remark 5.19. Note that we have not yet shown that u_0 as in Lemma 5.18 exist, but we expect that this is the case. We also expect to have data u_0 for which type I blow-up occurs.

Remark 5.20. In the case $N \geq 4$, consider now $u_0 \in H^1$ radial as in Corollary 5.18 (but not type II), then using the L^2 conservation and energy laws, estimates as in [20] yield for any sequence t_n such that $\int |\nabla u(t_n)|^2 \to +\infty$ that for all $R > 0$, we have $\int_{|x| \leq R} |\nabla u(t_n)|^2 \to +\infty$ which leads to the same conclusions as in Corollary 5.18. Note that when $N = 3$, one expects that the conclusion in this remark is false in light of examples analogous to the ones constructed by Raphael in [23] which give a radial solution blowing up exactly on a sphere.

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