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The $f(q)$ mock theta function conjecture and partition ranks

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Abstract. In 1944, Freeman Dyson initiated the study of ranks of integer partitions. Here we solve the classical problem of obtaining formulas for $N_e(n)$ (resp. $N_o(n)$), the number of partitions of n with even (resp. odd) rank. Thanks to Rademacher's celebrated formula for the partition function, this problem is equivalent to that of obtaining a formula for the coefficients of the mock theta function $f(q)$, a problem with its own long history dating to Ramanujan's last letter to Hardy. Little was known about this problem until Dragonette in 1952 obtained asymptotic results. In 1966, G.E. Andrews refined Dragonette's results, and conjectured an exact formula for the coefficients of $f(q)$. By constructing a weak Maass-Poincaré series whose "holomorphic part" is $q^{-1}f(q^{24})$, we prove the Andrews-Dragonette conjecture, and as a consequence obtain the desired formulas for $N_e(n)$ and $N_o(n)$.

1. Introduction and statement of results

A *partition* of a positive integer n is any non-increasing sequence of positive integers whose sum is n. As usual, let $p(n)$ denote the number of partitions of n. The partition function $p(n)$ has the well known generating function

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n},$$

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Mathematics Subject Classification (2000): 11P82, 05A17 which is easily seen to coincide with $q^{\frac{1}{24}}/\eta(z)$, where

$$\eta(z) := q^{1/24} \prod_{n=1}^{\infty} (1-q^n) \qquad \left(q := e^{2\pi i z}\right)$$

is Dedekind's eta-function, a weight 1/2 modular form. Rademacher famously employed this modularity to perfect the Hardy-Ramanujan asymptotic formula

$$p(n) \sim \frac{1}{4n\sqrt{3}} \cdot e^{\pi\sqrt{2n/3}} \tag{1.1}$$

to obtain his exact formula for $p(n)$ (for example, see Chap. 14 of [22]).

To state his formula, let $I_s(x)$ be the usual I-Bessel function of order s, and let $e(x) := e^{2\pi i x}$. Furthermore, if $k \geq 1$ and n are integers, then let

$$A_k(n) := \frac{1}{2}\sqrt{\frac{k}{12}} \sum_{\substack{x \pmod{24k} \\ x^2 \equiv -24n+1 \pmod{24k}}} \chi_{12}(x) \cdot e\left(\frac{x}{12k}\right), \tag{1.2}$$

where the sum runs over the residue classes modulo $24k$, and where

$$\chi_{12}(x) := \left(\frac{12}{x}\right). \tag{1.3}$$

If n is a positive integer, then one version of Rademacher's formula reads

$$p(n) = 2\pi(24n-1)^{-\frac{3}{4}} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} \cdot I_{\frac{3}{2}}\left(\frac{\pi\sqrt{24n-1}}{6k}\right). \tag{1.4}$$

In an effort to provide a combinatorial explanation of Ramanujan's congruences

$$p(5n+4) \equiv 0 \pmod{5},$$
$$p(7n+5) \equiv 0 \pmod{7},$$
$$p(11n+6) \equiv 0 \pmod{11},$$

Dyson introduced [17] the so-called "rank" of a partition, a delightfully simple statistic. The rank of a partition is defined to be its largest part minus the number of its parts. In this famous paper [17], Dyson conjectured that ranks could be used to "explain" the congruences above with modulus 5 and 7. More precisely, he conjectured that the partitions of $5n+4$ (resp. $7n+5$) form 5 (resp. 7) groups of equal size when sorted by their ranks modulo 5 (resp. 7)[1]. He further postulated the existence of another statistic,

[1] A short calculation reveals that this phenomenon cannot hold modulo 11.

the so-called "crank"[2], which allegedly would explain all three congruences. In 1954, Atkin and Swinnerton-Dyer proved [9] Dyson's rank conjectures, consequently cementing the central role that ranks play in the theory of partitions.

To study ranks, it is natural to investigate a generating function. If $N(m,n)$ denotes the number of partitions of n with rank m, then it is well known that

$$1+\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}N(m,n)z^mq^n = 1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(zq;q)_n(z^{-1}q;q)_n},$$

where

$$(a;q)_n := (1-a)(1-aq)\cdots\left(1-aq^{n-1}\right).$$

Therefore, if $N_e(n)$ (resp. $N_o(n)$) denotes the number of partitions of n with even (resp. odd) rank, then by letting $z=-1$ we obtain

(1.5)
$$1+\sum_{n=1}^{\infty}(N_e(n)-N_o(n))q^n = 1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2}.$$

We address the following classical problem: Determine exact formulas for $N_e(n)$ and $N_o(n)$. In view of (1.4) and (1.5), since

$$p(n) = N_e(n)+N_o(n),$$

this question is equivalent to the problem of deriving exact formulas for the coefficients $\alpha(n)$ of the series

$$\begin{aligned} f(q) = 1+\sum_{n=1}^{\infty}\alpha(n)q^n &:= 1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2} \\ &= 1+q-2q^2+3q^3-3q^4+3q^5-5q^6+\cdots. \end{aligned} \tag{1.6}$$

The series $f(q)$ is one of the third order mock theta functions Ramanujan defined in his last letter to Hardy dated January 1920 (see pp. 127–131 of [23]). Surprisingly, very little is known about mock theta functions in general. For example, Ramanujan's claims about their analytic properties remain open. There is even debate concerning the rigorous definition of a mock theta function, which, of course, precedes the formulation of one's order. Despite these seemingly problematic issues, Ramanujan's mock theta functions possess many striking properties, and they have been the subject of an astonishing number of important works, (for example, see [2–5, 7,

[2] In 1988, Andrews and Garvan [8] found the crank, and they indeed confirmed Dyson's speculation that it "explains" the three Ramanujan congruences above. Recent work of Mahlburg [21] establishes that the Andrews-Dyson-Garvan crank plays an even more central role in the theory partition congruences. His work concerns partition congruences modulo arbitrary powers of all primes ≥ 5.

12–14, 16, 19, 26, 27] to name a few). This activity realizes G.N. Watson's[3] prophetic words:

"*Ramanujan's discovery of the mock theta functions makes it obvious that his skill and ingenuity did not desert him at the oncoming of his untimely end. As much as any of his earlier work, the mock theta functions are an achievement sufficient to cause his name to be held in lasting remembrance. To his students such discoveries will be a source of delight and wonder until the time shall come when we too shall make our journey to that Garden of Proserpine (a.k.a. Persephone)...*"

Returning to $f(q)$, the problem of estimating its coefficients $\alpha(n)$ has a long history, one which even precedes Dyson's definition of partition ranks. Indeed, Ramanujan's last letter to Hardy already includes the claim that

$$\alpha(n) = (-1)^{n-1} \frac{\exp\left(\pi\sqrt{\frac{n}{6} - \frac{1}{144}}\right)}{2\sqrt{n - \frac{1}{24}}} + O\left(\frac{\exp\left(\frac{1}{2}\pi\sqrt{\frac{n}{6} - \frac{1}{144}}\right)}{\sqrt{n - \frac{1}{24}}}\right).$$

Typical of his writings, Ramanujan offered no proof of this claim. Dragonette finally proved this claim in her 1951 Ph.D. thesis [16] written under the direction of Rademacher. In his 1964 Ph.D. thesis, also written under Rademacher, Andrews improved upon Dragonette's work, and he proved[4] that

$$\begin{aligned}\alpha(n) = \pi(24n-1)^{-\frac{1}{4}} \sum_{k=1}^{[\sqrt{n}]} \frac{(-1)^{\lfloor\frac{k+1}{2}\rfloor} A_{2k}\left(n - \frac{k(1+(-1)^k)}{4}\right)}{k} \\ \cdot I_{\frac{1}{2}}\left(\frac{\pi\sqrt{24n-1}}{12k}\right) + O(n^{\epsilon}).\end{aligned} \tag{1.7}$$

This result falls short of the problem of obtaining an exact formula for $\alpha(n)$, and as a consequence represents the obstruction to obtaining formulas for $N_e(n)$ and $N_o(n)$. In his plenary address "Partitions: At the interface of q-series and modular forms", delivered at the Millenial Number Theory Conference at the University of Illinois in 2000, Andrews highlighted this classical problem by promoting his conjecture[5] of 1966 (see p. 456 of [2], and Sect. 5 of [4]) for the coefficients $\alpha(n)$.

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[3] This quote is taken from Watson's 1936 Presidential Address to the London Mathematical Society entitled "The final problem: An account of the mock theta functions" (see p. 80 of [26]).

[4] This is a reformulation of Theorem 5.1 of [2] using the identity $I_{\frac{1}{2}}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \cdot \sinh(z)$.

[5] This conjecture is suggested as a speculation by Dragonette in [16].

Conjecture (Andrews-Dragonette). *If n is a positive integer, then*

$$\alpha(n) = \pi(24n-1)^{-\frac{1}{4}} \sum_{k=1}^{\infty} \frac{(-1)^{\lfloor \frac{k+1}{2} \rfloor} A_{2k}\left(n - \frac{k(1+(-1)^k)}{4}\right)}{k} \cdot I_{\frac{1}{2}}\left(\frac{\pi\sqrt{24n-1}}{12k}\right). \tag{1.8}$$

The following theorem gives the first exact formulas for the coefficients of a mock theta function.

Theorem 1.1. *The Andrews-Dragonette Conjecture is true.*

Remark. Since $N_e(n) = (p(n) + \alpha(n))/2$ and $N_o(n) = (p(n) - \alpha(n))/2$, Theorem 1.1, combined with (1.4), provides the desired formulas for $N_e(n)$ and $N_o(n)$.

To prove Theorem 1.1, we use recent work of Zwegers [27] which nicely packages Watson's transformation properties of $f(q)$ in terms of real analytic vector valued modular forms. Loosely speaking, Zwegers "completes" $q^{-1/24} f(q)$ to obtain a three dimensional real analytic vector valued modular form of weight 1/2. We recall his results in Sect. 2. To prove Theorem 1.1, we realize the q-series, whose coefficients are given by the infinite series expansions in (1.8), as the "holomorphic part" of a weak Maass form. This form is defined in Sect. 3.1 as a specialization of a Poincaré series, and in Sect. 3.2 we confirm that the coefficients of its holomorphic part are indeed in agreement with the expansions in (1.8). To complete the proof of Theorem 1.1, it then suffices to establish a suitable identity relating this weak Maass form to Zwegers' form. We achieve this in Sect. 5 by analyzing the image of these forms under the differential operator $\xi_{\frac{1}{2}}$ (defined in Sect. 5). This task requires the Serre-Stark Basis Theorem for weight 1/2 holomorphic modular forms, and estimates on sums of the A_{2k}-sums derived in Sect. 4.

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2. Modular transformation properties of $q^{-\frac{1}{24}} f(q)$

Here we recall what is known about $q^{-1/24} f(q)$ and its modular transformation properties. An important first step was already achieved by G.N. Watson in [26]. Although $f(q)$ is not the Fourier expansion of a usual meromorphic modular form, in this classic paper Watson determined modular transformation properties which strongly suggested that $f(q)$ is a "piece" of a real analytic modular form, as opposed to a classical meromorphic modular form.

Watson's modular transformation formulas are very complicated, and are difficult to grasp at first glance. In particular, the collection of these formulas involve another third order mock theta function, as well as terms arising from Mordell integrals. Recent work of Zwegers [27] nicely packages Watson's results in the modern language of real analytic vector valued modular forms. We recall some of his results as they pertain to $f(q)$.

We begin by fixing notation. Let $\omega(q)$ be the third order mock theta function

$$\begin{aligned}\omega(q) &:= \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q;q^2)_{n+1}^2} \\ &= \frac{1}{(1-q)^2} + \frac{q^4}{(1-q)^2(1-q^3)^2} \\ &\quad + \frac{q^{12}}{(1-q)^2(1-q^3)^2(1-q^5)^2} + \cdots . \end{aligned} \tag{2.1}$$

If $q := e^{2\pi i z}$, where $z \in \mathbb{H}$, then define the vector valued function $F(z)$ by

$$\begin{aligned} F(z) &= (F_0(z),\ F_1(z),\ F_2(z))^T \\ &:= (q^{-\frac{1}{24}} f(q),\ 2q^{\frac{1}{3}}\omega(q^{\frac{1}{2}}),\ 2q^{\frac{1}{3}}\omega(-q^{\frac{1}{2}}))^T . \end{aligned} \tag{2.2}$$

Similarly, let $G(z)$ be the vector valued non-holomorphic function defined by

$$\begin{aligned} G(z) &= (G_0(z),\ G_1(z),\ G_2(z))^T \\ &:= 2i\sqrt{3} \int_{-\bar{z}}^{i\infty} \frac{(g_1(\tau),\ g_0(\tau),\ -g_2(\tau))^T}{\sqrt{-i(\tau+z)}}\, d\tau, \end{aligned} \tag{2.3}$$

where the $g_i(\tau)$ are the cuspidal weight $3/2$ theta functions

$$\begin{aligned} g_0(\tau) &:= \sum_{n=-\infty}^{\infty} (-1)^n \left(n+\frac{1}{3}\right) e^{3\pi i \left(n+\frac{1}{3}\right)^2 \tau}, \\ g_1(\tau) &:= -\sum_{n=-\infty}^{\infty} \left(n+\frac{1}{6}\right) e^{3\pi i \left(n+\frac{1}{6}\right)^2 \tau}, \\ g_2(\tau) &:= \sum_{n=-\infty}^{\infty} \left(n+\frac{1}{3}\right) e^{3\pi i \left(n+\frac{1}{3}\right)^2 \tau}. \end{aligned} \tag{2.4}$$

Using these vector valued functions, Zwegers defines $H(z)$ by

$$H(z) := F(z) - G(z). \tag{2.5}$$

The following description of $H(z)$ is the main result of [27].

Theorem 2.1 (Zwegers). *The function $H(z)$ is a vector valued real analytic modular form of weight $1/2$ satisfying*

$$H(z+1) = \begin{pmatrix} \zeta_{24}^{-1} & 0 & 0 \\ 0 & 0 & \zeta_3 \\ 0 & \zeta_3 & 0 \end{pmatrix} H(z),$$

$$H(-1/z) = \sqrt{-iz} \cdot \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} H(z),$$

where $\zeta_n := e^{2\pi i/n}$. Furthermore, $H(z)$ is an eigenfunction of the Casimir operator $\Omega_{\frac{1}{2}} := -4y^2 \frac{\partial^2}{\partial z \partial \overline{z}} + iy\frac{\partial}{\partial \overline{z}} + \frac{3}{16}$ with eigenvalue $\frac{3}{16}$, where $z = x+iy$, $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$, and $\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$.

We give a consequence of Zwegers' result in terms of weak Maass forms of half-integral weight. To make this precise, suppose that $k \in \frac{1}{2} + \mathbb{Z}$. If v is odd, then define ϵ_v by

$$\epsilon_v := \begin{cases} 1 & \text{if } v \equiv 1 \pmod 4, \\ i & \text{if } v \equiv 3 \pmod 4. \end{cases} \tag{2.6}$$

The weight k Casimir operator is defined by

$$\Omega_k := -4y^2 \frac{\partial^2}{\partial z \partial \overline{z}} + 2iky\frac{\partial}{\partial \overline{z}} + \frac{2k - k^2}{4}. \tag{2.7}$$

Notice that the weight k hyperbolic Laplacian

$$\Delta_k := -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \tag{2.8}$$

is related to the Casimir operator Ω_k by the simple identity

$$\Omega_k = \Delta_k + \frac{2k - k^2}{4},$$

where $z = x + iy$ with $x, y \in \mathbb{R}$.

Following Bruinier and Funke, we now recall the notion [11] of a weak Maass form of half-integral weight.

Definition 2.2. *Suppose that $k \in \frac{1}{2} + \mathbb{Z}$, N is a positive integer, and that ψ is a Dirichlet character with modulus $4N$. A weak Maass form of weight k on $\Gamma_0(4N)$ with Nebentypus character ψ is any smooth function $f : \mathbb{H} \to \mathbb{C}$ satisfying the following:*

(1) *For all* $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4N)$ *and all* $z \in \mathbb{H}$, *we have*[6]

$$f(Az) = \psi(d) \left(\frac{c}{d}\right)^{2k} \epsilon_d^{-2k} \, (cz+d)^k \, f(z).$$

(2) *We have that* $\Delta_k f = 0$.
(3) *The function* $f(z)$ *has at most linear exponential growth at all the cusps of* $\Gamma_0(4N)$.

Before we state a useful corollary to Theorem 2.1, we recall certain facts about Dedekind sums and their role in describing the modular transformation properties of Dedekind's eta-function. If $x \in \mathbb{R}$, then let

$$((x)) := \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & \text{for } x \in \mathbb{R} \setminus \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

For coprime integers c and d, let $s(d,c)$ be the usual Dedekind sum

$$s(d,c) := \sum_{\mu \pmod{c}} \left(\left(\frac{\mu}{c}\right)\right) \left(\left(\frac{d\mu}{c}\right)\right).$$

In terms of these sums, we define $\omega_{d,c}$ by

$$\omega_{d,c} := e^{\pi i s(d,c)}. \tag{2.9}$$

Using this notation, if $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, with $c > 0$, then we have[7]

$$\eta\left(\frac{az+b}{cz+d}\right) = i^{-\frac{1}{2}} \cdot \omega_{-d,c} \cdot \exp\left(\frac{\pi i(a+d)}{12c}\right) \cdot (cz+d)^{\frac{1}{2}} \cdot \eta(z). \tag{2.10}$$

Remark. The exponential sums defined by (1.2) may also be described in terms of Dedekind sums. In particular, if $k \geq 1$ and n are integers, then $A_k(n)$ is also given by (see (120.5) on p. 272 of [22])

$$A_k(n) = \sum_{x \pmod{k}^*} \omega_{-x,k} \cdot e\left(\frac{nx}{k}\right), \tag{2.11}$$

where the sum runs over the primitive residue classes x modulo k.

Theorem 2.1 implies the following convenient corollary.

Corollary 2.3. *The function* $M(z) := F_0(24z) - G_0(24z)$ *is a weak Maass form of weight* $1/2$ *on* $\Gamma_0(144)$ *with Nebentypus character* χ_{12}.

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[6] This transformation law agrees with Shimura's notion of half-integral weight modular forms [25].
[7] This formula is easily derived from the formulas appearing in Chap. 9 of [22].

Sketch of the proof. It is well known that $\eta(24z)$ is a cusp form of weight $1/2$ for the group $\Gamma_0(576)$ with Nebentypus χ_{12}. For integers n, we obviously have

$$\widetilde{M}(z+n) = e(-n/24)\cdot \widetilde{M}(z),$$

where $\widetilde{M}(z) = F_0(z) - G_0(z)$. Therefore, to prove the claim it suffices to compare the automorphy factors of $M(z)$ with those appearing in (2.10) when interpreted for $\eta(24z)$. By Theorem 2.1 (see also Theorem[8] 2.2 of [2]), if $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)$, with $c > 0$, then

$$\widetilde{M}\left(\frac{az+b}{cz+d}\right) = i^{-\frac{1}{2}} \cdot \omega_{-d,c}^{-1} \cdot (-1)^{\frac{c+1+ad}{2}} \cdot e\left(-\frac{a+d}{24c} - \frac{a}{4} + \frac{3dc}{8}\right)(cz+d)^{\frac{1}{2}} \cdot \widetilde{M}(z).$$

In view of these formulas, it is then straightforward to verify that the automorphy factors above agree with those for $\eta(24z)$ when restricted to $\Gamma_0(576)$. Consequently, it then follows that $M(z)$ is also a weak Maass form of weight $1/2$ on $\Gamma_0(576)$ with Nebentypus χ_{12}.

In order to verify that $M(z)$ satisfies the desired transformation law under $\Gamma_0(144)$, it suffices to check that its images under the representatives for the non-trivial classes in $\Gamma_0(144)/\Gamma_0(576)$ behave properly. For example, if $H(z) = (H_0(z), H_1(z), H_2(z))^T$, then Theorem 2.1 gives

$$\begin{aligned} M\left(\frac{z}{288z+1}\right) &= H_0\left(\frac{24z}{12(24z)+1}\right) \\ &= \left(-i\left(\frac{12(24z)+1}{-24z}\right)\right)^{\frac{1}{2}} \cdot H_1\left(\frac{12(24z)+1}{-24z}\right) \\ &= \left(-i\left(\frac{12(24z)+1}{-24z}\right)\right)^{\frac{1}{2}} \cdot H_1\left(-\frac{1}{24z}\right) \\ &= (288z+1)^{\frac{1}{2}} \cdot H_0(24z) = (288z+1)^{\frac{1}{2}} \cdot M(z). \end{aligned}$$

This is the desired transformation law under $z \to \frac{1}{288z+1}$. The analogous computation for the remaining representatives completes the proof. □

Remark. Let $\psi \pmod 6$ be the Dirichlet character

$$\psi(n) := \begin{cases} 1 & \text{if } n \equiv 1 \pmod 6, \\ -1 & \text{if } n \equiv 5 \pmod 6. \end{cases}$$

The theta-function

$$\vartheta(\psi; z) := \sum_{n=1}^{\infty} \psi(n)\, n\, q^{n^2}$$

[8] There is a minor typo in the displayed formula which is easily found when reading the proof.

is well known to be a cusp form of weight $3/2$ on $\Gamma_0(144)$ with Nebentypus χ_{12} (for example, see [25]). One easily sees that

$$g_1(24z) = -\frac{1}{6} \cdot \vartheta(\psi; z).$$

In fact, one could also use this fact to deduce Corollary 2.3.

3. The Poincaré series $P_k(s; z)$

Here we construct a Poincaré series which has the property that the Fourier coefficients of its "holomorphic part", when $s = 3/4$ and $k = 1/2$, are given by the infinite series expansions appearing in (1.8). In Sect. 3.1, we begin by defining this series as a trace over Möbius transformations, and in Sect. 3.2 we compute its Fourier expansion. The main result of this calculation is the reproduction of the infinite series formulas in (1.8) as coefficients of the holomorphic part of a weak Maass form of weight $1/2$ on $\Gamma_0(144)$ with Nebentypus character χ_{12}.

3.1. The construction. Suppose that $k \in \frac{1}{2} + \mathbb{Z}$. We now define an important class of Poincaré series $P_k(s; z)$. For matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)$, with $c \geq 0$, define the character $\chi(\cdot)$ by

(3.1)
$$\chi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) := \begin{cases} e\left(-\frac{b}{24}\right) & \text{if } c = 0, \\ i^{-1/2}(-1)^{\frac{1}{2}(c+ad+1)} e\left(-\frac{a+d}{24c} - \frac{a}{4} + \frac{3dc}{8}\right) \cdot \omega_{-d,c}^{-1} & \text{if } c > 0. \end{cases}$$

Remark. The character χ is defined to coincide with the automorphy factor for the real analytic form $F_0(z) - G_0(z)$ when restricted to $\Gamma_0(2)$.

Throughout, let $z = x + iy$, and for $s \in \mathbb{C}$, $k \in \frac{1}{2} + \mathbb{Z}$, and $y \in \mathbb{R} \setminus \{0\}$, let

$$\mathcal{M}_s(y) := |y|^{-\frac{k}{2}} M_{\frac{k}{2}\,\mathrm{sgn}(y),\, s-\frac{1}{2}}(|y|), \tag{3.2}$$

where $M_{\nu,\mu}(z)$ is the standard M-Whittaker function which is a solution to the differential equation

$$\frac{\partial^2 u}{\partial z^2} + \left(-\frac{1}{4} + \frac{\nu}{z} + \frac{\frac{1}{4} - \mu^2}{z^2}\right) u = 0.$$

Furthermore, let

$$\varphi_{s,k}(z) := \mathcal{M}_s\left(-\frac{\pi y}{6}\right) e\left(-\frac{x}{24}\right).$$

It is straightforward to confirm that $\varphi_{s,k}(z)$ is an eigenfunction of the Casimir operator

$$\Omega_k = -4y^2 \frac{\partial^2}{\partial z \partial \bar{z}} + 2iky \frac{\partial}{\partial \bar{z}} + \frac{2k - k^2}{4}$$

with eigenvalue $s(1-s)$. Using this notation, we now define the Poincaré series $P_k(s; z)$ by

$$P_k(s; z) := \frac{2}{\sqrt{\pi}} \sum_{M \in \Gamma_\infty \backslash \Gamma_0(2)} \chi(M)^{-1} (cz + d)^{-k} \varphi_{s,k}(Mz). \tag{3.3}$$

Here Γ_∞ is the subgroup of translations in $\mathrm{SL}_2(\mathbb{Z})$

$$\Gamma_\infty := \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \ : \ n \in \mathbb{Z} \right\}.$$

Remark. Strictly speaking, the definition of $P_k(s; z)$ is not well defined because χ was only defined for those matrices with $c \geq 0$. In the definition, simply choose representatives for $\Gamma_\infty \backslash \Gamma_0(2)$ with non-negative c.

If $k \leq 1/2$ (resp. $k \geq 3/2$), then the specialization of $P_k(s; z)$ at $s = 1 - k/2$ (resp. $s = k/2$) is a weak Maass form of weight k. The defining series is absolutely convergent for $P_k\left(1 - \frac{k}{2}; z\right)$ (resp. $P_k\left(\frac{k}{2}; z\right)$) for $k < 1/2$ (resp. $k > 3/2$). We obtain the Maass forms when $k = 1/2$ (resp. $k = 3/2$) by a process of continuation using the convergence of the Fourier expansion (which is shown in Sect. 4 in the case where $k = 1/2$). Here we prove the case which is of interest in the present work.

Theorem 3.1. *If $k \in \frac{1}{2} + \mathbb{Z}$ with $k \leq \frac{1}{2}$, then the series $P_k\left(1 - \frac{k}{2}; z\right)$ is a real analytic function and satisfies, for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)$, the transformation*

$$P_k\left(1 - \frac{k}{2}; Mz\right) = \chi(M)(cz + d)^k P_k\left(1 - \frac{k}{2}; z\right).$$

In particular, the function $P_k\left(1 - \frac{k}{2}; 24z\right)$ is a weak Maass form of weight k for $\Gamma_0(144)$ with Nebentypus χ_{12}.

Proof. We first assume that $k < 1/2$. Since $\varphi_{s,k}(z) = O(y^{\mathrm{Re}(s) - \frac{k}{2}})$ as $y \to 0$, the series $P_k(1 - \frac{k}{2}; z)$ is absolutely convergent for $k < 1/2$. Furthermore, since $\varphi_{s,k}(z)$ is an eigenfunction of Ω_k with eigenvalue $(2k - k^2)/4$, it follows directly that $P_k(1 - \frac{k}{2}; z)$ is a real analytic function which is annihilated by Δ_k. That $P_k(1 - \frac{k}{2}; 24z)$ is a weak Maass form for $\Gamma_0(144)$ with Nebentypus χ_{12} follows as in the proof of Corollary 2.3.

The case where $k = 1/2$ requires a little more care. In Theorem 3.2, we shall compute the Fourier coefficients of $P_{\frac{1}{2}}(\frac{3}{4}; z)$, and it will turn out that these expressions are convergent. This convergence will follow from Corollary 4.2. Combined with these additional facts, the proof of Theorem 3.1, in the case where $k = 1/2$, follows as above. □

3.2. The Fourier expansion. Here we compute the Fourier expansion of $P_{\frac{1}{2}}(\frac{3}{4};z)$, and confirm that its "holomorphic part" agrees with the conjectured expansions for the coefficients of the mock theta function $q^{-\frac{1}{24}}f(q)$. We first require some more notation. For $s \in \mathbb{C}$ and $y \in \mathbb{R}\setminus\{0\}$, let

$$\mathcal{W}_s(y) := |y|^{-\frac{1}{4}} W_{\frac{1}{4}\operatorname{sgn}(y),\, s-\frac{1}{2}}(|y|), \tag{3.4}$$

where $W_{\nu,\mu}$ denotes the usual W-Whittaker function. For $y > 0$, we shall require the following relations, which are easily deduced from standard properties of Whittaker functions (for example, see [6] or [1]):

$$\mathcal{W}_{\frac{3}{4}}(y) = e^{-\frac{y}{2}}, \tag{3.5}$$

$$\mathcal{W}_{\frac{3}{4}}(-y) = e^{\frac{y}{2}} \cdot \Gamma\left(\frac{1}{2}, y\right), \tag{3.6}$$

$$\mathcal{M}_{\frac{3}{4}}(-y) = \frac{1}{2}\left(\sqrt{\pi} - \Gamma\left(\frac{1}{2}, y\right)\right) \cdot e^{\frac{y}{2}}, \tag{3.7}$$

where

$$\Gamma(a,x) := \int_x^\infty e^{-t} t^a \frac{dt}{t}$$

is the incomplete Gamma function. Furthermore, let $J_{\frac{1}{2}}(x)$ be the usual J-Bessel function of order $1/2$.

Using this notation, we obtain the following Fourier expansion for $P_{\frac{1}{2}}(\frac{3}{4};z)$.

Theorem 3.2. *Assuming the notation above, we have that*

$$P_{\frac{1}{2}}\left(\frac{3}{4};z\right) = \left(1 - \pi^{-\frac{1}{2}} \cdot \Gamma\left(\frac{1}{2}, \frac{\pi y}{6}\right)\right) \cdot q^{-\frac{1}{24}} + \sum_{n=-\infty}^{0} \gamma_y(n) q^{n-\frac{1}{24}} + \sum_{n=1}^{\infty} \beta(n) q^{n-\frac{1}{24}},$$

where for positive integers n we have

$$\beta(n) = \pi(24n-1)^{-\frac{1}{4}} \times \sum_{k=1}^{\infty} \frac{(-1)^{\lfloor\frac{k+1}{2}\rfloor} A_{2k}\left(n - \frac{k(1+(-1)^k)}{4}\right)}{k} \cdot I_{\frac{1}{2}}\left(\frac{\pi\sqrt{24n-1}}{12k}\right),$$

and for non-positive integers n we have

$$\gamma_y(n) = \pi^{\frac{1}{2}} |24n-1|^{-\frac{1}{4}} \cdot \Gamma\left(\frac{1}{2}, \frac{\pi|24n-1| \cdot y}{6}\right) \times \sum_{k=1}^{\infty} \frac{(-1)^{\lfloor\frac{k+1}{2}\rfloor} A_{2k}\left(n - \frac{k(1+(-1)^k)}{4}\right)}{k} \cdot J_{\frac{1}{2}}\left(\frac{\pi\sqrt{|24n-1|}}{12k}\right).$$

Remark. The convergence of the coefficients $\beta(n)$ and $\gamma_y(n)$ will be established by Corollary 4.2. From this one obtains the convergence of the Fourier expansion of $P_{\frac{1}{2}}(\frac{3}{4}; z)$.

Proof of Theorem 3.2. We first compute the Fourier expansion of $P_{\frac{1}{2}}(s; z)$, and then set $s = 3/4$. First we describe a set of representatives for $\Gamma_\infty \backslash \Gamma_0(2)$. We select a single matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ for each pair (c, d), where (c, d) runs through all coprime elements in $\mathbb{N} \times \mathbb{Z}$, with $2|c$, together with the pair $(c, d) = (0, 1)$. For each such pair, we choose (a, b) arbitrarily so that $ad - bc = 1$.

We now compute the Fourier expansion by explicitly computing the contribution from each such matrix representative. The contribution in $P_{\frac{1}{2}}(s; z)$ coming from $c = 0$ equals

$$\frac{2}{\sqrt{\pi}} e\left(-\frac{x}{24}\right) \mathcal{M}_s\left(-\frac{\pi y}{6}\right).$$

Using (3.7), when $s = 3/4$ we obtain

$$\left(1 - \pi^{-\frac{1}{2}} \cdot \Gamma\left(\frac{1}{2}, \frac{\pi y}{6}\right)\right) \cdot q^{-\frac{1}{24}}.$$

Up to the multiplicative constant $2/\sqrt{\pi}$, the contribution to $P_{\frac{1}{2}}(s; z)$ for $c > 0$, can easily be seen to equal

$$i^{\frac{1}{2}} \sum_{\substack{c>0 \\ 2|c}} c^{-\frac{1}{2}} \sum_{d \pmod{c}^*} (-1)^{\frac{1}{2}(c+1+ad)} \cdot e\left(\frac{d}{24c} + \frac{a}{4} - \frac{3dc}{8}\right) \cdot \omega_{-d,c}$$
$$\sum_{n \in \mathbb{Z}} (z + d/c + n)^{-\frac{1}{2}} \mathcal{M}_s\left(-\frac{\pi y}{6c^2 |z + d/c + n|^2}\right)$$
$$e\left(\frac{1}{24c^2} \mathrm{Re}\left(\frac{1}{z + d/c + n}\right)\right) e\left(\frac{n}{24}\right).$$

To compute the Fourier expansion of this function, we let

$$f(z) := \sum_{n \in \mathbb{Z}} (z + n)^{-\frac{1}{2}} \mathcal{M}_s\left(-\frac{\pi y}{6c^2 |z + n|^2}\right) e\left(\frac{1}{24c^2} \mathrm{Re}\left(\frac{1}{z + n}\right)\right) e\left(\frac{n}{24}\right).$$

This function has a Fourier expansion

$$f(z) = \sum_{n \in \mathbb{Z}} a_y(n) e^{2\pi i \left(n - \frac{1}{24}\right) x},$$

where

$$a_y(n) = \int_{\mathbb{R}} z^{-\frac{1}{2}} \mathcal{M}_s \left(-\frac{\pi y}{6c^2|z|^2} \right) e \left(\frac{x}{24c^2|z|^2} - \left(n - \frac{1}{24} \right) x \right) dx.$$

This integral is computed on p. 357 of [18] (see also p. 33 of [10]).

An easy calculation using (2.11) shows that

$$(-1)^{\lfloor \frac{c+1}{2} \rfloor} A_{2c} \left(n - \frac{c\,(1+(-1)^c)}{4} \right) = \sum_{d \pmod{2c}^*} \omega_{-d,2c} (-1)^{\frac{2c+1+ad}{2}} e \left(\frac{a-3dc}{4} + \frac{nd}{2c} \right).$$

Using (3.5) and (3.6), it is simple to confirm the stated Fourier expansion. □

4. Some estimates for sums involving $A_k(n)$

Strictly speaking, the proof of Theorem 3.1 is incomplete when $k = 1/2$. To make it complete, it suffices to show that the formulas for the coefficients $\alpha(n)$ and $\beta(n)$ in Theorem 3.2 are convergent. Using (1.2), it turns out that it will be sufficient to obtain estimates for certain expressions involving the following sums:

$$\rho_1(n;k) := \sum_{\substack{x \pmod{48k} \\ x^2 \equiv -24n+1 \pmod{48k}}} \chi_{12}(x) \cdot e\left(\frac{x}{24k}\right), \tag{4.1}$$

$$\rho_2(n;k) := \sum_{\substack{x \pmod{12k} \\ x^2 \equiv -24n+1 \pmod{12k}}} \chi_{12}(x) \cdot i^{\frac{x^2-1}{12k}} \cdot e\left(\frac{x}{24k}\right). \tag{4.2}$$

4.1. The required estimates. The following theorem, which is obtained by generalizing an old argument of Hooley (see Sect. 6 of [20]), will give the necessary estimates for these sums.

Theorem 4.1. *If n and $k \geq 1$ are integers, then we have the estimates:*

$$\left| \sum_{k \geq 1 \text{ odd}} \frac{\rho_1(n;k)}{k} \right| = O\big(|24n-1|^{\frac{1}{2}}\big)$$

$$\left| \sum_{k \geq 2 \text{ even}} \frac{\rho_1(n;k)}{k} \right| \text{ and } \left| \sum_{k \geq 2 \text{ even}} \frac{\rho_2(n;k)}{k} \right| = O\big(|24n-1|^{\frac{1}{2}}\big).$$

Theorem 4.1 easily implies the following estimates.

Corollary 4.2. *The following estimates are true:*

(1) *For positive integers n, we have*

$$\left|\sum_{k=\lfloor\sqrt{n}\rfloor+1}^{\infty}\frac{(-1)^{\lfloor\frac{k+1}{2}\rfloor}A_{2k}\big(n-\frac{k(1+(-1)^k)}{4}\big)}{k}\cdot I_{\frac{1}{2}}\left(\frac{\pi\sqrt{24n-1}}{12k}\right)\right|$$
$$=O\big((24n-1)^{\frac{3}{4}}\big).$$

(2) *For non-positive integers n, we have*

$$\left|\sum_{k=1}^{\infty}\frac{(-1)^{\lfloor\frac{k+1}{2}\rfloor}A_{2k}\big(n-\frac{k(1+(-1)^k)}{4}\big)}{k}\cdot J_{\frac{1}{2}}\left(\frac{\pi\sqrt{|24n-1|}}{12k}\right)\right|$$
$$=O\big(|24n-1|^{\frac{3}{4}}\big).$$

Remark. We prove Theorem 4.1 by modifying an elegant argument of Hooley involving the interplay between quadratic congruences and the representation of integers by quadratic forms. Instead of modifying Hooley's argument, one could instead amplify his strategy by employing spectral methods applied to Maass forms. Although such an approach would give stronger estimates, Theorem 4.1 is sufficient for the proof of Theorem 1.1.

Proof of Corollary 4.2. By (1.2), it follows that

$$\frac{A_{2k}\big(n-\frac{k(1+(-1)^k)}{4}\big)}{k}=\frac{\rho(n;k)}{\sqrt{24k}}, \tag{4.3}$$

where

$$\rho(n;k):=\sum_{\substack{x \pmod{48k}\\ x^2\equiv -24n+6k(1+(-1)^k)+1 \pmod{48k}}}\chi_{12}(x)\cdot e\left(\frac{x}{24k}\right). \tag{4.4}$$

If k is odd, then note that $\rho(n;k)=\rho_1(n;k)$.

Suppose that k is even. Using the fact that $(x+12k)^2\equiv x^2+24k \pmod{48k}$, one may group the sum defining $\rho_2(n;k)$ into two pairs of equal sums. Arguing in this way, it is not difficult to show that

$$4e\left(\frac{n}{2k}\right)\rho_2(n;k)=2\rho_1(n;k)+2i\sum_{\substack{x \pmod{48k}\\ x^2\equiv -24n+1+12k \pmod{48k}}}\chi_{12}(x)\cdot e\left(\frac{x}{24k}\right)$$
$$=2\rho_1(n;k)+2i\rho(n;k).$$

Combining (4.3) with these facts gives

$$\frac{A_{2k}\big(n-\frac{k(1+(-1)^k)}{4}\big)}{k}=\begin{cases}\frac{\rho_1(n;k)}{\sqrt{24k}} & \text{for odd } k,\\[2ex] \frac{i\cdot\rho_1(n;k)}{\sqrt{24k}}-\frac{2ie\left(\frac{n}{2k}\right)\rho_2(n;k)}{\sqrt{24k}} & \text{for even } k.\end{cases} \tag{4.5}$$

Since we have that

$$I_{\frac{1}{2}}\left(\frac{\pi\sqrt{24n-1}}{12k}\right) \sim J_{\frac{1}{2}}\left(\frac{\pi\sqrt{|24n-1|}}{12k}\right) \sim \frac{|24n-1|^{\frac{1}{4}}}{\sqrt{6k}},$$

as $k \to +\infty$, observation (4.5), the trivial bounds for those summands with $1 \leq k < \lfloor\sqrt{n}\rfloor + 1$ (*Note:* We have that $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cdot \sin(x)$), combined with the estimates in Theorem 4.1, gives the desired estimates. □

4.2. Proof of Theorem 4.1. To prove Theorem 4.1, we closely follow the proof of Theorem 1 of [20]. Given an integer N, his theorem gives estimates for sums of form

$$\sum_{k\leq X} S(h;k),$$

where

$$S(h;k) := \sum_{\substack{x \pmod{k} \\ x^2 \equiv N \pmod{k}}} e\left(\frac{hx}{k}\right).$$

Theorem 4.1 essentially involves "twisted" versions of such sums which are further modified by dividing each summand by k.

We require the following well-known estimate for incomplete Kloosterman sums (*Note:* see Lemma 3 of [20]).

Lemma 4.3. *If* $h, r \neq 0$, α, *and* β *are integers satisfying* $0 \leq \beta - \alpha \leq 2|r|$, *then we have*

$$\sum_{\substack{\alpha \leq s \leq \beta \\ \gcd(r,s)=1}} e\left(-\frac{h\bar{s}}{r}\right) = O\left(|r|^{\frac{1}{2}} \cdot \gcd(h,r)^{\frac{1}{2}} d(r) \log(2|r|)\right),$$

where $d(r)$ *denotes the number of positive divisors of* r, *and* $\bar{s}$ *denotes the inverse of* s *modulo* r.

Proof of Theorem 4.1. For brevity, we only prove the first estimate. The other cases follow in an analogous way. The argument is based on the action of subgroups of $\mathrm{SL}_2(\mathbb{Z})$ on quadratic forms. For $X \in \mathbb{R}$, we define

$$R(n;X) := \sum_{\substack{k\geq 1 \text{ odd} \\ 12k \leq X}} \frac{\rho_1(n;k)}{k}. \tag{4.6}$$

For an odd positive integer k, let

$$Q(x,y) := [12k, b, c] := 12kx^2 + bxy + cy^2 \tag{4.7}$$

be an integral binary quadratic form with discriminant $-24n+1$. Clearly, the coefficient b, of Q, solves the congruence

$$b^2 \equiv -24n+1 \pmod{48k}. \tag{4.8}$$

Conversely, every pair of integers (k, b), where k is an odd positive integer and b a solution of the congruence (4.8), corresponds to such a quadratic form (4.7). For every odd k and integer x that solves (4.8), there are integers $a, b, c, \alpha, \beta, \gamma$, and δ with $\alpha\delta - \beta\gamma = 1$, $24|\gamma$, and $a \equiv 12 \pmod{24}$ such that

$$12k = a\alpha^2 + b\alpha\gamma + c\gamma^2 =: k_{\alpha,\gamma}, \tag{4.9}$$

$$x = 2a\alpha\beta + b(\alpha\delta + \beta\gamma) + 2c\gamma\delta =: x_{\alpha,\gamma}. \tag{4.10}$$

Therefore, we have that

$$\frac{x}{24k} = \frac{2a\alpha\beta + b(\alpha\delta + \beta\gamma) + 2c\gamma\delta}{2(a\alpha^2 + b\alpha\gamma + c\gamma^2)}. \tag{4.11}$$

Since $\alpha \neq 0$, this equals, for $\gamma \neq 0$, the quantity

$$\begin{aligned} \frac{2\beta(a\alpha^2+b\alpha\gamma+c\gamma^2)+b\alpha+2c\gamma}{2\alpha(a\alpha^2+b\alpha\gamma+c\gamma^2)} &= -\frac{\overline{\gamma}}{\alpha} + \frac{b\alpha+2c\gamma}{2\alpha(a\alpha^2+b\alpha\gamma+c\gamma^2)} \\ &=: \vartheta_{\alpha,\gamma}. \end{aligned} \tag{4.12}$$

Here $\overline{\gamma}$ denotes the inverse of γ modulo α. In case $\gamma = 0$ (i.e., $\alpha = \pm 1$), we set

$$\vartheta_{\alpha,\gamma} := \frac{b}{2a}.$$

Using these observations, we have that

$$R(n; X) = 12 \sum_{Q=[a,b,c]} \sum_{\alpha,\gamma} \frac{\chi_{12}(x_{\alpha,\gamma})}{k_{\alpha,\gamma}} \cdot e(\vartheta_{\alpha,\gamma}), \tag{4.13}$$

where the outer sum runs over a set of representatives of quadratic forms $Q = [a, b, c]$ of discriminant $-24n+1$ (positive definite forms when $-24n+1 < 0$) with $a \equiv 12 \pmod{24}$, under the action of $\Gamma_0(24)$. Of course, the number of such quadratic forms is finite. The inner sum runs over coprime integers α, γ with $24|\gamma$, $0 < a\alpha^2 + b\alpha\gamma + c\gamma^2 \leq X$, and the summation is restricted for each quadratic form to one representation of the form (4.9) and (4.10).

Let us first consider the case that $-24n+1 < 0$ (i.e. the positive definite case). In this case, we have

$$R(n; X) = 12 \sum_{Q=[a,b,c]} \frac{1}{|\Gamma_Q|} \sum_{\alpha,\gamma} \frac{\chi_{12}(x_{\alpha,\gamma})}{k_{\alpha,\gamma}} \cdot e(\vartheta_{\alpha,\gamma}), \tag{4.14}$$

where Γ_Q denotes the isotropy subgroup of Q in $\Gamma_0(24)$, and the inner sum runs over coprime integers α, γ with $24|\gamma$, and $0 < a\alpha^2 + b\alpha\gamma + c\gamma^2 \leq X$. By the theory of reduced forms, we may assume that

$$a, b, c \ll (24n-1)^{\frac{1}{2}}, \tag{4.15}$$

where the implied constant is absolute.

Since $24|\gamma$, the case $|\alpha| = |\gamma| = 1$ cannot occur in the inner summation of (4.14). Hence, we can write the inner sum as

$$\sum_{\alpha,\gamma} \frac{\chi_{12}(x_{\alpha,\gamma})}{k_{\alpha,\gamma}} \cdot e(\vartheta_{\alpha,\gamma}) = \sum_{|\gamma|<|\alpha|} + \sum_{|\alpha|<|\gamma|} =: \sum_{10} + \sum_{11}.$$

Since both sums can be estimated in exactly the same way, we only consider Σ_{10}. In this case, we have

$$|\alpha| < A \cdot |24n-1|^{\frac{1}{4}} \cdot X^{\frac{1}{2}} =: A_{n,X}, \tag{4.16}$$

$$F_1(\alpha) \leq \gamma \leq F_2(\alpha), \tag{4.17}$$

for some positive constant A. Here $F_1(\alpha)$ and $F_2(\alpha)$ are given by

$$F_1(\alpha) := 24 \left\lfloor \frac{F_1(\alpha)'}{24} \right\rfloor,$$

$$F_2(\alpha) := 24 \left\lfloor \frac{F_2(\alpha)'}{24} \right\rfloor,$$

$$F_1(\alpha)' := \max\left(-|\alpha|, -\frac{b\alpha}{2c} - \frac{1}{c}\left(cX + \frac{(1-24n)\alpha^2}{4}\right)^{\frac{1}{2}}\right),$$

$$F_2(\alpha)' := \min\left(|\alpha|, -\frac{b\alpha}{2c} + \frac{1}{c}\left(cX + \frac{(1-24n)\alpha^2}{4}\right)^{\frac{1}{2}}\right).$$

If we let

$$\varphi(\alpha,\gamma) := \frac{\chi_{12}(x_{\alpha,\gamma})}{k_{\alpha,\gamma}} \cdot e\left(\frac{b\alpha + 2c\gamma}{2\alpha(a\alpha^2 + b\alpha\gamma + c\gamma^2)}\right),$$

then

$$\sum_{10} = \sum_{1\leq|\alpha|<A_{n,X}} \sum_{\substack{F_1(\alpha)\leq\gamma\leq F_2(\alpha)\\ (\alpha,\gamma)=1\\ 24|\gamma\neq 0}} e\left(-\frac{\overline{\gamma}}{\alpha}\right)\varphi(\alpha,\gamma) + e\left(\frac{b}{2a}\right). \tag{4.18}$$

By partial summation, for $\alpha \neq \pm 1$, the inner sum equals

(4.19)

$$\sum_{\substack{F_1(\alpha)\leq\mu\leq F_2(\alpha)\\ (\alpha,\gamma)=1\\ 24|\mu}} g(\mu)\left(\varphi(\alpha,\mu) - \varphi(\alpha,\mu+24)\right) + g\left(F_2(\alpha)\right)\varphi\left(\alpha, F_2(\alpha)+24\right),$$

where

$$g(\mu) := \sum_{\substack{\frac{F_1(\alpha)}{24} \le \gamma \le \frac{\mu}{24} \\ (\alpha,\gamma)=1}} e\left(-\frac{\overline{24} \cdot \overline{\gamma}}{\alpha} \right).$$

When $\alpha = \pm 1$, we let the summation run over $\mu \neq 0$, and we include the extra term $e(\frac{b}{2a})$. Using the series expansion for the exponential function, one finds that

$$\varphi(\alpha, \mu) - \varphi(\alpha, \mu + 24) \ll |\alpha|^{-3} (24n - 1)^{\frac{1}{2}}, \tag{4.20}$$

and one trivially finds that

$$\varphi\,(\alpha, F_2(\alpha) + 24) \ll |\alpha|^{-2}. \tag{4.21}$$

Moreover, by Lemma 4.3, we have that

$$g(\mu) \ll |\alpha|^{\frac{1}{2}} d(\alpha) \log(2|\alpha|). \tag{4.22}$$

Inserting (4.20), (4.21), and (4.22) in (4.19), we find that (4.19) can be estimated by

$$\begin{aligned} &|\alpha|^{-\frac{5}{2}} d(\alpha) \log(2|\alpha|)(24n-1)^{\frac{1}{2}} \sum_{F_1(\alpha) \le \mu \le F_2(\alpha)} 1 + |\alpha|^{-\frac{3}{2}} d(\alpha) \log(2|\alpha|) \\ &\ll |\alpha|^{-\frac{3}{2}} d(\alpha) \log(2|\alpha|)(24n-1)^{\frac{1}{2}} + |\alpha|^{-\frac{3}{2}} d(\alpha) \log(2|\alpha|) \\ &\ll (24n-1)^{\frac{1}{2}} |\alpha|^{-\frac{3}{2}+\epsilon}. \end{aligned}$$

Inserting this into (4.18), we find that

$$\sum_{10} \ll (24n-1)^{\frac{1}{2}} \sum_{1 \le |\alpha| < A_{n,X}} |\alpha|^{-\frac{3}{2}+\epsilon} + 1 \ll (24n-1)^{\frac{1}{2}},$$

where we obtained the last estimate by comparing the sum with an integral. The sum $\sum_{11}$ is estimated in exactly the same way, and this gives the desired estimate in the case that $-24n + 1 < 0$.

We just make some short comments for the case where $-24n + 1 > 0$. For simplicity, suppose that the quadratic forms considered are primitive (the general case is treated similarly). We can moreover assume that $a > 0$ and $c < 0$. It is well known that every representation of $12k$ by $ax^2 + bxy + cy^2$ contains exactly one representation of $12k$ such that

$$x, y > 0 \tag{4.23}$$

$$y \le \frac{au}{t - bu} \cdot x, \tag{4.24}$$

where (t, u) is a solution of the Pell equation

$$t^2 + (24n - 1)u^2 = 4,$$

where t and u are positive integers. Moreover, if the inequalities (4.23) hold, then the quadratic form only attains positive values, and so the inner sum in (4.13) corresponding to a primitive form $Q = [a, b, c]$ can be written as

$$\sum_{\alpha,\gamma} \frac{\chi_{12}(x_{\alpha,\gamma})}{k_{\alpha,\gamma}} \cdot e(\vartheta_{\alpha,\gamma}),$$

where the sum runs over positive coprime integers α, γ with $24|\gamma$, $a\gamma^2 + b\alpha\gamma + c\gamma^2 \leq X$ and $\gamma \leq \frac{au\alpha}{t-bu}$. This sum can now be estimated as in the positive definite case above. □

5. Proof of Theorem 1.1

Here we combine the main results of the previous sections to prove Theorem 1.1. For convenience, we let

$$P(z) := P_{\frac{1}{2}}\left(\frac{3}{4}; 24z\right). \tag{5.1}$$

Canonically decompose $P(z)$ into a non-holomorphic and a holomorphic part

$$P(z) = P_{nh}(z) + P_h(z). \tag{5.2}$$

In particular, we have that

$$P_h(z) = q^{-1} + \sum_{n=1}^{\infty} \beta(n) q^{24n-1},$$

where the $\beta(n)$ are defined in Theorem 3.2.

By Theorem 3.1, $P(z)$ is a weak Maass form of weight $1/2$ for $\Gamma_0(144)$ with Nebentypus χ_{12}. Similarly, the function

$$M(z) := F_0(24z) - G_0(24z),$$

where F_0 and G_0 are defined by (2.2) and (2.3), by Corollary 2.3, is also a weak Maass form of weight $1/2$ on $\Gamma_0(144)$ with Nebentypus χ_{12}. Decompose $M(z)$ into a non-holomorphic and a holomorphic part

$$M(z) = M_{nh}(z) + M_h(z). \tag{5.3}$$

In particular, observe that

$$M_h(z) = q^{-1} f(q^{24}).$$

Thanks to Theorem 3.2, to prove Theorem 1.1 it suffices to show that $M_h(z) = P_h(z)$. This identity obviously follows if we establish that

$$P(z) = M(z).$$

First we establish the following lemma.

Lemma 5.1. *In the notation above, we have*

$$P_{nh}(z) = M_{nh}(z).$$

Proof. To prove Lemma 5.1, we apply an anti-linear differential operator ξ_k defined by

$$\xi_k(g)(z) := 2iy^k \overline{\tfrac{\partial}{\partial \bar{z}} g(z)}. \tag{5.4}$$

In their work on Theta lifts, Bruinier and Funke (see Proposition 3.2 of [11]) show that if g is a weak Maass form of weight k for the group $\Gamma_0(4N)$ with Nebentypus χ, then $\xi_k(g)$ is a weakly holomorphic modular form of weight $2-k$ on $\Gamma_0(4N)$ with Nebentypus χ (i.e. those whose poles (if there are any) are supported at the cusps of $\Gamma_0(4N)$). Furthermore, ξ_k has the property that its kernel consists of those weight k weak Maass forms which are weakly holomorphic modular forms.

We first apply $\xi_{\frac{1}{2}}$ to the Fourier expansion of $P(z)$ given in Theorem 3.2. Since $\xi_{\frac{1}{2}}(g) = 0$ for holomorphic g, and since $\xi_{\frac{1}{2}}$ is anti-linear, we just have to compute $\xi_{\frac{1}{2}}((\Gamma(\frac{1}{2}, 4\pi\,|24n-1|\,y)))$, where n is a non-positive integer. In this case, we have that

$$\xi_{\frac{1}{2}}\left(\Gamma\left(\frac{1}{2}, 4\pi|24n-1|y\right)\right) = -(4\pi|24n-1|)^{1/2} e^{4\pi(24n-1)y}.$$

Therefore

$$\xi_{\frac{1}{2}}(P(z)) = \sum_{n=0}^{\infty} a(n) e^{2\pi i(24n+1)z},$$

where $a(n)$, for $n \neq 0$, is given by

$$-2\pi(24n+1)^{\frac{1}{4}} \sum_{k=1}^{\infty} \frac{(-1)^{\lfloor \frac{k+1}{2} \rfloor} \cdot \overline{A_{2k}\Big(-n-\frac{k(1+(-1)^k)}{4}\Big)}}{k} \cdot \overline{J_{\frac{1}{2}}\left(\frac{\pi\sqrt{24n+1}}{12k}\right)}, \tag{5.5}$$

and, for $n = 0$, is given by

$$2 - 2\pi \sum_{k=1}^{\infty} \frac{(-1)^{\lfloor \frac{k+1}{2} \rfloor} \cdot \overline{A_{2k}\Big(-\frac{k(1+(-1)^k)}{4}\Big)}}{k} \cdot \overline{J_{\frac{1}{2}}\left(\frac{\pi}{12k}\right)}. \tag{5.6}$$

By Corollary 4.2, this implies that the weakly holomorphic modular form $\xi_{\frac{1}{2}}(P(z))$ is indeed a holomorphic modular form of weight $3/2$ on $\Gamma_0(144)$ with Nebentypus χ_{12}. Its non-zero coefficients are also easily seen to be supported on exponents in the arithmetic progression $1 \pmod{24}$.

Now we apply $\xi_{\frac{1}{2}}$ to $M(z)$. It is easily seen that

$$\xi_{\frac{1}{2}}(M(z)) = -24 \cdot \overline{g_1(-24\bar{z})} = 4\vartheta(\psi; z),$$

and so it is a cusp form of weight $3/2$ on $\Gamma_0(144)$ with Nebentypus χ_{12}. Obviously, its Fourier coefficients are supported on exponents in the arithmetic progression $1 \pmod{24}$.

Therefore, $\xi_{\frac{1}{2}}(P(z))$ and $\xi_{\frac{1}{2}}(M(z))$ are both holomorphic modular forms of weight $3/2$ on $\Gamma_0(144)$ with Nebentypus χ_{12}. Using the dimension formulas for spaces of half-integral weight modular forms (for example, see [15]), it follows that

$$\dim_{\mathbb{C}}(S_{1/2}(\Gamma_0(144), \chi_{12})) = -24 + \dim_{\mathbb{C}}(M_{3/2}(\Gamma_0(144), \chi_{12})),$$

where $S_{1/2}(\Gamma_0(144), \chi_{12})$ (resp. $M_{3/2}(\Gamma_0(144), \chi_{12})$) denotes the space of cusp (resp. holomorphic modular) forms of weight $1/2$ (resp. $3/2$) on $\Gamma_0(144)$ with Nebentypus χ_{12}. The Serre-Stark Basis Theorem for modular forms of weight $1/2$ [24] implies that

$$\dim_{\mathbb{C}}(M_{1/2}(\Gamma_0(144), \chi_{12})) = \dim_{\mathbb{C}}(S_{1/2}(\Gamma_0(144), \chi_{12})) = 0, \tag{5.7}$$

since $576 = 4 \cdot \mathfrak{f}(\chi_{12})^2 \nmid 144$, where $\mathfrak{f}(\chi_{12}) = 12$ is the conductor of χ_{12}. Therefore, we find that

$$\dim_{\mathbb{C}}\left(M_{3/2}(\Gamma_0(144), \chi_{12})\right) = 24.$$

Since $\xi_{\frac{1}{2}}(P(z)), \xi_{\frac{1}{2}}(M(z)) \in M_{3/2}(\Gamma_0(144), \chi_{12})$ both have the property that their Fourier coefficients are supported on exponents of the form $24n + 1 \geq 1$, choose a constant c so that the coefficient of q, and hence all the coefficients up to and including q^{24}, of $\xi_{\frac{1}{2}}(P(z))$ and $c \cdot \xi_{\frac{1}{2}}(M(z))$ agree. By dimensionality, this in turn implies that $\xi_{\frac{1}{2}}(P(z)) = c \cdot \xi_{\frac{1}{2}}(M(z))$, and so we have that

$$P_{nh}(z) = c \cdot M_{nh}(z).$$

To complete the proof of Lemma 5.1, we must establish that $c = 1$. To this end, we let

$$E(z) := P(z) - c \cdot M(z).$$

This function is a weakly holomorphic modular form of weight $1/2$ on $\Gamma_0(144)$ with Nebentypus χ_{12}. By (1.7) and Corollary 4.2, we have that

$$E(z) = P_h(z) - c \cdot M_h(z) = (1 - c)q^{-1} f(q^{24}) + \sum_{n \geq 0} A(n) q^{24n-1},$$

where $|A(n)| = O((24n - 1)^{\frac{3}{4}+\epsilon})$ for positive integers n. By the proof of Theorem 2.1 (see [26] and [27]), applying the map $z \mapsto -\frac{1}{z}$ returns a non-holomorphic contribution unless $c = 1$. Since $E(z)$ does not have a non-holomorphic component, it follows that $c = 1$, which in turn proves that $P_{nh}(z) = M_{nh}(z)$. $\square$

Remark. In the proof of Lemma 5.1, it is shown that $\xi_{\frac{1}{2}}(P(z)) = \xi_{\frac{1}{2}}(M(z)) = 4\vartheta(\psi; z)$. We illustrate the rate of convergence of the formulas in (5.5) and (5.6). By truncating these infinite series expansions after 750 terms, one obtains the following numerical approximation

$$\begin{aligned}\frac{1}{4}\xi_{\frac{1}{2}}(P(z)) \sim 0.989q - 5.008q^{25} + 7.019q^{49} + 0.110q^{73}\\ - 0.043q^{97} - 10.939q^{121} + \cdots\\ = q - 5q^{25} + 7q^{49} - 11q^{121} + \cdots .\end{aligned}$$

Returning to the proof of Theorem 1.1, by Lemma 5.1, it follows that

$$\begin{aligned}P(z) - M(z) = P_h(z) - M_h(z) &= q^{-1} + \sum_{n=1}^{\infty} \beta(n)q^{24n-1} - q^{-1}f(q^{24})\\ &= \sum_{n=1}^{\infty} \nu(n)q^{24n-1}\end{aligned}$$

is a weakly holomorphic modular form of weight 1/2 on $\Gamma_0(144)$ with Nebentypus χ_{12}. By (1.7) and Corollary 4.2, it follows that

$$|\nu(n)| = O\left(n^{\frac{3}{4}+\epsilon}\right).$$

Therefore, $P(z) - M(z)$ is a holomorphic modular form. However, by (5.7), this space is trivial, and so we find that $P(z) - M(z) = 0$, which in turn implies that

$$q^{-1} + \sum_{n=1}^{\infty} \beta(n)q^{24n-1} = q^{-1}f(q^{24}).$$

Theorem 1.1 now follows from Theorem 3.2.

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