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Minima of Epstein's Zeta function and heights of flat tori

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To Audrey Terras on the occasion of her 60th birthday

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1. Introduction

Let L be a lattice in $\mathbb{R}^n$ and $\mathbb{R}^n/L$ the corresponding flat torus whose volume we assume equal to one. If $v_1, \dots, v_n$ is a $\mathbb{Z}$-basis for L we let $Y = (y_{ij})_{i,j=1,\dots,n}$ with $y_{ij} = \langle v_i, v_j\rangle$, be the corresponding Gram matrix. Then $Y \in \mathcal{P}_n^\circ$, the space of positive definite $n \times n$ matrices of determinant 1. Changing basis for L by a $\gamma \in \mathrm{GL}(n,\mathbb{Z})$ yields $Y' = \gamma^t Y \gamma$ for the new Y. Also, if $k \in K = O(n)$ then $\langle kv_i, kv_j\rangle = \langle v_i, v_j\rangle$ so that $kv_1, \dots, kv_n$ produces the same point Y in $\mathcal{P}_n^\circ$, and two lattices L and L' correspond to the same $\mathrm{GL}(n,\mathbb{Z})$ orbit iff $L = kL'$ with $k \in O(n)$, or what is the same – the flat tori $\mathbb{R}^n/L$ and $\mathbb{R}^n/L'$ are isometric. In what follows we will not distinguish between L, L', Y, Y' as above. Let $\mathcal{L}_n^\circ = \mathcal{P}_n^\circ/\mathrm{GL}(n,\mathbb{Z})$ be the corresponding space of such lattices. It is a locally symmetric space of finite volume. Denote by $d\mu_n$ the corresponding volume element on $\mathcal{L}_n^\circ$ normalized to be a probability measure.

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For $s > \frac{n}{2}$ define the Epstein Zeta function for L by

$$E(L,s) = \sum_{\ell\in L}{}' \langle \ell,\ell\rangle^{-s} = \sum_{m\in\mathbb{Z}^n}{}' (m^t Ym)^{-s} := E(Y,s), \tag{1}$$

where $'$ denotes that the zero vector should be omitted. This series converges and using Poisson summation one proves ([Ep]) that $E(Y,s)$ has an analytic continuation to $\mathbb{C}$ except for a simple pole at $s = \frac{n}{2}$ with residue $\pi^{n/2}\Gamma(\frac{n}{2})^{-1}$ (note that the volume of the unit ball is $\pi^{n/2}\Gamma(\frac{n}{2}+1)^{-1}$). Moreover $E(Y,s)$ satisfies the functional equation

$$F(L,s) := \pi^{-s}\Gamma(s)E(L,s) = F(L^*, \tfrac{n}{2} - s). \tag{2}$$

Note that if L corresponds to Y then the dual lattice L^* corresponds to Y^{-1}.

It follows that $E(Y,s)$ is analytic at $s = 0$ and $E(Y,0) = -1$. Also from the notation or from (1) it is clear that $E(\gamma^t Y\gamma, s) = E(Y,s)$ for $\gamma \in \mathrm{GL}(n,\mathbb{Z})$ and $Y \in \mathcal{P}_n^\circ$. That is $E(Y,s)$ is a modular function. It is in fact a maximal parabolic Eisenstein series and as such it is an eigenfunction of the full ring $\mathcal{D}(\mathcal{P}_n^\circ)$ of invariant differential operators on $\mathcal{P}_n^\circ$ (see [Ter2]). We will make crucial use of this fact.

In this note we will examine the minimum value of $E(L,s)$ for $s > 0$ ($s \neq \frac{n}{2}$) fixed and L varying in $\mathcal{L}_n^\circ$ as well as the function $\frac{\partial E}{\partial s}(L,s)_{|s=0}$. One can show (see [Chiu], [Chua]) that if s is fixed then $E(L,s) \to \infty$ as $L \to \partial\mathcal{L}_n^\circ$ and hence its minimum value, $m_n(s)$, is finite and is attained. The problem as to which L minimizes $E(L,s)$ is of interest from a number of points of view. As $s \to \infty$, the minimizer of $E(L,s)$ corresponds to the densest lattice packing of $\mathbb{R}^n$ [Ry], or equivalently, to the flat torus $\mathbb{R}^n/L$ which has the longest systole. This is a well known and much studied problem [C-S]. At the other end $\frac{\partial E}{\partial s}(L,s)_{|s=0}$ is related to the height of the dual torus $\mathbb{R}^n/L^*$. If X is a compact Riemannian manifold its height (minus "log det of the Laplacian") is defined via a regularization to be $h(X) = Z_X'(0)$, where $Z_X(s) = \sum_{j=1}^\infty \lambda_j^{-s}$ and $0 = \lambda_0 < \lambda_1 \leqq \lambda_2 \leqq \ldots$ are the eigenvalues of the Laplacian on X. $Z_X(s)$ is known to be meromorphic and regular at $s = 0$ [M-P]. For $X = \mathbb{R}^n/L$ as above, the eigenvalues are $4\pi^2|\ell|^2$ with $\ell \in L^*$. Hence

$$h(\mathbb{R}^n/L) = h(L) = 2\log 2\pi + \frac{\partial}{\partial s}E(L^*,s)_{|s=0}. \tag{3}$$

In dimension two the extremal metrics for $h(X)$ have been studied in detail (see [O-P-S], [S]) and the height above is closely related to other notions of height. In dimensions two and three the minima of the Epstein Zeta function for other values of s, for example $s = (n+1)/2$, come up in problems of minimum energy configurations in physics and chemistry [To-S].

Let $L_n \in \mathcal{L}_n^\circ$ be a lattice yielding the densest lattice packing of space, that is

$$m(L_n) \geqq m(L) \tag{4}$$

for all $L \in \mathcal{L}_n^\circ$, where $m(L) = \min\{|v| \mid v \in L,\ v \neq 0\}$. For $n = 2, 3, \ldots, 8$, L_n is known and is the unique lattice satisfying (4). Recently it was proven in [C-K1] that the Leech lattice Λ_{24} is the unique extremal for (4) in dimension 24; thus $L_{24} = \Lambda_{24}$. An explicit description of these L_n's for $n = 2, 3, 4, 8, 24$ is given in Sect. 3. The minimum problem for $E(L, s)$ was solved completely in dimension 2 by Rankin [Ra], Cassels [Ca], Diananda [Di] and Ennola [En1]. They show that for $n = 2$,

$$E(L, s) \geqq E(L_n, s) \qquad \text{for all } s > 0,\ L \in \mathcal{L}_n^\circ, \tag{5}$$

with equality only if $L = L_n$.

For any n, we will say that L_n is *universal* (i.e. universally extremal for the Epstein Zeta function) if (5) holds. In dimension 3, Ennola [En2] has shown that for $s > 0$, $E(L, s)$ has a local minimum at L_3, the face centered cubic lattice (see Sect. 3), and he conjectures that L_3 is universal. Our first observation about the higher dimensional case is that this conjecture is false. In fact it is not possible for L_n to be universal unless the vector lengths for L_n are exactly the same as those for L_n^* (that is, $\sharp\{v \in L_n \mid |v| = y\} = \sharp\{v \in L_n^* \mid |v| = y\}$ for all $y \geqq 0$). For if $G(s) = F(L_n, s) - F(L_n^*, s)$, then according to (2), $G(s) = -G(\frac{n}{2} - s)$. Hence G is either identically zero or changes sign and so (5) fails. In particular for $n = 3$, L_3^* is the body centered cubic lattice [C-S, §4.6.3] which has $m(L_3^*) < m(L_3)$ and so L_3 is *not* universal.

For dimensions n for which L_n is unique and $L_n^* = L_n$, the lattice L_n may be universal. Our first result is that (5) holds locally in dimensions 4, 8 and 24.

Theorem 1. *For $n = 4$, 8 and 24 and $s > 0$, $E(L, s)$ has a strict local minimum at $L = L_n$, as does $h(L)$.*

We conjecture that in these dimensions L_n is universal. Evidence for this is given not only by the local result above but also by a result by Chua ([Chua]) according to which (5) holds for all $s > 0$ and any unimodular integral lattice L in dimension 24 (there are 297 such lattices, including $L = L_{24}$). Note that if L_n is in fact universal then by Theorem 1 one could in principle prove this conjecture numerically. However a direct numerical approach is not feasible in the space $\mathcal{L}_n^\circ$ (whose dimension is $(n^2+n-2)/2$) unless n is small. For $n = 3$ and the height function h, we carry this out in Sect. 4. That the following should be true was made very plausible in [Chiu].

Theorem 2. *For $L \in \mathcal{L}_3^\circ$, $h(L) \geqq h(L_3)$ with equality only if $L = L_3$.*

Remark 1. By (3), $h(L)$ is connected with $\frac{\partial}{\partial s} E(L^*, s)_{|s=0}$ and so there is no contradiction here with the fact that L_3 is not universal.

In Sect. 5 we make some remarks on minima of the *theta function* for lattices L,

$$\Theta(L, iy) = \sum_{\ell \in L} e^{-\pi y \langle \ell, \ell \rangle}, \qquad y > 0.$$

In particular we prove an analog of Theorem 1 for the theta function, and using the fact that the Epstein Zeta function is the Mellin transform of the theta function we find that the study of the theta function yields a mild simplification of one part of the proof of Theorem 1 itself. We also discuss briefly some conjectures about global minima of $\Theta(L, iy)$ which would imply that L_n is universal for $n = 4, 8, 24$.

Returning to the Epstein Zeta function, for n large it is not clear to us whether to expect L_n to be universal. For $0 < s < \frac{n}{2}$ it follows from Siegel's integration formula (see [Si] and [Ter2]) or from the theory of Eisenstein series that

$$\int_{\mathcal{L}_n^\circ} E(L, s)\, d\mu_n(L) = 0. \tag{6}$$

In particular $m_n(s) < 0$ and if L_n is universal then

$$E(L_n, s) < 0 \qquad \text{for } 0 < s < \tfrac{n}{2}. \tag{7}$$

On the other hand $E(L_n, s) > 0$ for $s > \frac{n}{2}$ and $E(L_n, s)$ has a simple pole at $s = \frac{n}{2}$. Hence if L_n is universal then $E(L_n, s)$ has no zeroes in $(0, \infty)$! It is an interesting question as to whether there is any $L \in \mathcal{L}_n^\circ$ for which $E(L, s)$ has no zeroes in $(0, \infty)$. Any attempt to construct such an L explicitly is problematic since as was shown in Terras [Ter1], if $\alpha < 1$ and n is sufficiently large (depending on α), then $E(L, s)$ has a zero in $(0, \frac{n}{2})$ for every lattice L with $m(L) \leqq \alpha\sqrt{\frac{n}{2\pi e}}$ (all explicitly known lattices in large dimension satisfy this upper bound). As for the height in large dimensions, we have

$$\int_{\mathcal{L}_n^\circ} h(L)\, d\mu_n(L) = \infty. \tag{8}$$

We show in Sect. 6 that for $n \to \infty$ and $L \in \mathcal{L}_n^\circ$,

$$h(L) \geqq 4\sqrt{\frac{\pi}{n}} \left(\frac{\sqrt{n/2\pi e}}{m(L)} \right)^n (1 + o(1)). \tag{9}$$

Thus again any explicitly known lattice will have large height. However the following shows that as $n \to \infty$, $h(L)$ concentrates at a single value. The random lattice has its height tending to $\log 4\pi - \gamma + 1$, where γ is Euler's constant. The precise statement is

Theorem 3. *Fix* $\varepsilon > 0$. *Then*

$$Prob_{\mu_n}\left\{L \in \mathcal{L}_n^\circ \;\middle|\; \left|h(L) - (\log 4\pi - \gamma + 1)\right| < \varepsilon\right\}$$

tends to 1 *as* $n \to \infty$.

Corollary 1. *If* $m_n = \min\{h(L) \mid L \in \mathcal{L}_n^\circ\}$ *then*

$$\log 4\pi - \gamma - \frac{2}{n} < m_n \leqq \log 4\pi - \gamma + 1 + o(1).$$

We end the introduction with an outline of the proofs of the Theorems. Let $Y_n \in \mathcal{P}_n^\circ$ be the Gram matrix corresponding to L_n. Theorem 1 is based on $E(Y, s)$ being an eigenfunction of $\mathcal{D}(\mathcal{P}_n^\circ)$ and the fact that for $n = 4, 8, 24$ the automorphism group $\mathrm{Aut}(L_n)$ of L_n, that is $\mathrm{Aut}(L_n) = \{B \in O(n) \mid B(L_0) = L_0\}$, is in a suitable sense a large subgroup of the orthogonal group $O(n)$. These groups act on the tangent space $\mathfrak{p}$ to $Y_n \in \mathcal{P}_n^\circ$ and hence on the corresponding symmetric algebra $\mathrm{Sym}(\mathfrak{p})$. This action preserves homogeneous polynomials of a fixed degree f and the dimensions of the $O(n)$ invariants for each f are well known (see Sect. 3). Using the conjugacy classes in $\mathrm{Aut}(L_n)$ we determine the dimensions of the $\mathrm{Aut}(L_n)$ invariants for each f in terms of the generating (Molien) series. In particular we find that the dimensions of the invariants for $\mathrm{Aut}(L_n)$ and $O(n)$ agree for $f \leqq 2$ when $n = 4$, for $f \leqq 3$ when $n = 8$ and for $f \leqq 5$ when $n = 24$. From this it follows that the Taylor expansion of $E(Y, s)$ about $Y = Y_n$ agrees with that of the spherical (i.e. $O(n)$) symmetrization of $E(s, Y)$ about Y_n, up to the above orders. On the other hand according to the Harish-Chandra/Selberg theory of spherical functions on $\mathcal{P}_n^\circ$, these spherical functions and in particular their Taylor expansions about their poles are determined by the $\mathcal{D}(\mathcal{P}_n^\circ)$ eigenvalues of $E(Y, s)$ and the value of $E(Y_n, s)$. For $0 < s < \frac{n}{2}$ in order that Y_n be a local minimum it is crucial that $E(Y_n, s) < 0$ (a condition which we encountered in (7) above) which we check is indeed the case for $n = 4, 8$ and 24.

In Sect. 4 we explain the numerical computations which lead to the rigorous verification of Theorem 2. In this case already for $f = 2$ and $Y = Y_3$, the $\mathrm{Aut}(L_3)$ invariants do not coincide with the $O(3)$ invariants. Still we find it useful to exploit the symmetry which limits the possible Taylor coefficients of $h(L)$ when developed at L_3. This analysis together with the fact that L_3 is a local minimum for h allows us to excise a reasonable neighbourhood of L_3 in $\mathcal{L}_3^\circ$. Since $h \to \infty$ as $L \to \partial\mathcal{L}_3^\circ$ we can also excise an explicit neighbourhood of the boundary. This leaves us the task of verifying that h is bigger than $h(L_3)$ by a (small) fixed amount on the remaining (compact) part of $\mathcal{L}_3^\circ$.

The proof of Theorem 3 is based on estimating the mean and variance over $\mathcal{L}_n^\circ$ of a suitable truncation $h_{\mathcal{R}_n}$ of h, where $h - h_{\mathcal{R}_n}$ is zero on a set of measure tending to 1 when $n \to \infty$. The mean is estimated using Siegel's integration formula [Si] (which is a familiar constant term computation in the theory of Eisenstein series). For the variance we use a formula of Rogers [Ro] which is a close relative of the 'Maass–Selberg' formula in the theory of Eisenstein series.

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2. Proof that L_4, L_8 and L_{24} are local minima of $E(Y, s)$

The analysis of the local minimum applies much more generally so we start with some general considerations taking place on an arbitrary symmetric space $X = K\backslash G$. [More precisely, we assume that $X = K\backslash G$ is a symmetric space of the noncompact type, meaning that G is a noncompact semisimple Lie group with finite center and K is a maximal compact subgroup, and $KG^\circ = G$. Then $K \setminus G = (K \cap G^\circ) \setminus G^\circ$ in the natural way, but it will be convenient later to not necessarily assume G connected.] The group G acts on X from the right by isometries. Denote by KAN an Iwasawa decomposition of G and by $\mathfrak{k}, \mathfrak{a}, \mathfrak{n}$ the corresponding Lie algebras. For $g \in G$, let $H(g) \in \mathfrak{a}$ be the unique element so that $g \in Ke^{H(g)}N$. Let $\mathcal{D}(X)$ be the ring of invariant differential operators on X, and for any $x_0 \in X$ let K_{x_0} denote the stabilizer of x_0 in G. According to a general lemma of Selberg [Se] any function $\phi(x)$ on X which is an eigenfunction of $\mathcal{D}(X)$ and which is K_{x_0}-invariant, i.e. $\phi(x) = \phi(xk)$ for all $k \in K_{x_0}$, is uniquely determined by $\phi(x_0)$ and its eigenvalues. Let $\rho \in \mathfrak{a}^*_{\mathbb{C}}$ be half the sum of positive roots and define, for any given $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$,

$$\phi_\lambda(g) = \int_K e^{(\lambda-\rho)(H(gk))}\,dk \qquad (x = Kg),$$

where dk denotes the (left- and right-) Haar measure on K, which we take to be normalized so that $\int_K dk = 1$. Then $\phi_\lambda(g)$ is a K-bi-invariant function on G, i.e. a K-invariant function on X; it is an eigenfunction of $\mathcal{D}(X)$ with eigenvalues corresponding to λ (cf., e.g., [D-K-V]) and $\phi_\lambda(e) = 1$. Furthermore, $\phi_\lambda = \phi_\mu$ if and only if $\lambda = w\mu$ for w an element of the Weyl group. The function ϕ_λ is called the spherical function associated to λ. It follows from the K-bi-invariance that

$$\omega_\lambda(x, x_0) = \phi_\lambda\big(gg_0^{-1}\big) \qquad (\text{for } x = Kg,\ x_0 = Kg_0) \tag{10}$$

gives a well-defined function on $X \times X$, and clearly $\omega_\lambda(xk, x_0) = \omega_\lambda(x, x_0)$ for all $k \in K_{x_0}$. Thus if $\phi(x)$ on X is spherically symmetric about x_0 as in Selberg's Lemma, then $\phi(x) = \phi(x_0)\omega_\lambda(x, x_0)$.

Now fix some $g_0 \in G$ and consider the point $x_0 = Kg_0$ in $X = K \setminus G$. Let $\mathfrak{p} = \mathfrak{a} + \mathfrak{n}$ so that $\mathfrak{p} + \mathfrak{k} = \mathfrak{g}$ is a Cartan decomposition of the Lie algebra of G. There is a diffeomorphism from $\mathfrak{p}$ onto X given by $Y \mapsto Ke^{\frac{1}{2}Y}g_0$

which maps 0 to x_0 (cf. [H, Ch. 5, Thm. 1.1]; the factor $\frac{1}{2}$ will be convenient in our special case below). We will identify $\mathfrak{p}$ and X under this map; taking the differential we also get an identification of $\mathfrak{p}$ with the tangent space $T_{x_0}(X)$. Note that this identification does not only depend on the point x_0 but also on its representative g_0. Note also $K_{x_0} = g_0^{-1} K g_0$ and that under our identification the action of $k \in K_{x_0}$ on X (or on $T_{x_0}(X)$) corresponds to $k : Y \mapsto \mathrm{Ad}(g_0 k^{-1} g_0^{-1})(Y)$ on $\mathfrak{p}$. This defines a linear action of K_{x_0} on $\mathfrak{p}$, and there is a corresponding ring of invariant polynomials on $\mathfrak{p}$.

Proposition 1. *Let H be a subgroup of K_{x_0} (which will be finite) for which the space of invariant polynomials on $\mathfrak{p}$ of degree less than or equal to f is the same as that for K_{x_0}. If $\varphi_\lambda(x)$ is an eigenfunction of $\mathcal{D}(X)$ with eigenvalues $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and if $\varphi_\lambda(x)$ is H-invariant then the Taylor expansion of $\varphi_\lambda(x)$ about x_0 (i.e. in the variables in $\mathfrak{p}$ under our identification) agrees with that of the function $\varphi_\lambda(x_0) \cdot \omega_\lambda(x, x_0)$, up to order f.*

Proof. Let $\psi(x) = \int_{K_{x_0}} \varphi_\lambda(xk)\, dk$. Then $\psi(x_0) = \varphi_\lambda(x_0)$, $\psi(x)$ is an eigenfunction of $\mathcal{D}(X)$ and $\psi(xk) = \psi(x)$ for $k \in K_{x_0}$. Hence by Selberg's Lemma, $\psi(x) = \varphi_\lambda(x_0) \cdot \omega_\lambda(x, x_0)$. The assumptions of the proposition ensure that $\psi(x)$ and $\varphi_\lambda(x)$ have the same Taylor expansion about x_0 to order f. $\square$

Since the Taylor expansion of $\omega_\lambda(x, x_0)$ at x_0 can be calculated explicitly, the proposition allows us to calculate the expansion of $\varphi_\lambda(x)$ to order f in terms of $\phi_\lambda(x_0)$ and λ.

We now turn to our setting of $\mathcal{P}_n^\circ$. It will be convenient to use

$$G = \{g \in \mathrm{GL}(n, \mathbb{R}) \mid \det g = \pm 1\}, \qquad K = O(n),$$

instead of the connected groups $\mathrm{SL}(n, \mathbb{R})$ and $SO(n)$. As before, we identify $\mathcal{P}_n^\circ = K \backslash G$ by $Kg \mapsto g^t g$, and the right action of G on $K\backslash G$ by isometries corresponds to the action $g : Y \mapsto Y[g] := g^t Y g$ on $\mathcal{P}_n^\circ$.

We have the Iwasawa decomposition $G = KAN$ with

$$N = \left\{ \begin{pmatrix} 1 & * & \dots & * \\ & 1 & * & * \\ & 0 & \ddots & \\ & & & 1 \end{pmatrix} \right\}, \quad A = \left\{ \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix} \;\middle|\; \begin{matrix} a_1, a_2, \dots, a_n > 0, \\ a_1 a_2 \cdots a_n = 1 \end{matrix} \right\},$$

and the Lie algebra of A is $\mathfrak{a} = \left\{ \begin{pmatrix} H_1 & & & \\ & H_2 & & \\ & & \ddots & \\ & & & H_n \end{pmatrix} \;\middle|\; \sum H_j = 0 \right\}.$

The Cartan decomposition of $\mathfrak{g} = \{X \in M_n(\mathbb{R}) \mid \mathrm{trace}(X) = 0\}$ is $\mathfrak{g} = \mathfrak{p} + \mathfrak{k}$ with $\mathfrak{p} = \{X \in M_n(\mathbb{R}) \mid X^t = X, \mathrm{trace}(X) = 0\}$ and $\mathfrak{k} = \{X \in M_n(\mathbb{R}) \mid X^t = -X, \mathrm{trace}(X) = 0\}$.

Denote by $\langle\cdot,\cdot\rangle$ the bilinear form $\langle X_1, X_2\rangle = \mathrm{trace}(X_1X_2)$ on $\mathfrak{g}$; this is the Killing form divided by $2n$, and in particular it restricts to a positive definite form on $\mathfrak{p}$. Given $g_0 \in G$ the identification used in Proposition 1 now takes the shape $\mathfrak{p} \ni X \mapsto g_0^t e^X g_0 \in \mathcal{P}_n^\circ$, since $Ke^{\frac{1}{2}X} g_0 \in K\backslash G$ corresponds to $g_0^t e^X g_0 \in \mathcal{P}_n^\circ$. For definiteness, let us fix the metric on $\mathcal{P}_n^\circ = K\backslash G$ so that for each $g_0 \in G$ the Riemannian tensor on the tangent space $T_{g_0^t g_0}(\mathcal{P}_n^\circ)$ agrees with $\langle\cdot,\cdot\rangle$ on $\mathfrak{p}$. (One may check that this is exactly the same metric as in [Ter2, Ch. 4 (1.11)].)

Now $E(Y,s)$ is an Eisenstein series on $\mathcal{P}_n^\circ$ and is an eigenfunction of $\mathcal{D}(G/K)$. Its eigenvalue is $\lambda_s = \rho + \frac{2s}{n}(1-n, 1, \dots, 1) \in \mathfrak{a}_{\mathbb{C}}^*$ and $\rho = (\frac{n-1}{2}, \frac{n-3}{2}, \dots, \frac{1-n}{2})$.

For $Y_0 \in \mathcal{P}_n^\circ$ let R_{Y_0} be the intersection of $\mathrm{GL}(n,\mathbb{Z})$ with K_{Y_0} (the orthogonal group fixing Y_0). Note that if we fix a representative $g_0 \in \mathrm{SL}(n,\mathbb{R})$ for Y_0, so that $Y_0 = g_0^t g_0$, then $L_0 = g_0\mathbb{Z}^n \subset \mathbb{R}^n$ is a lattice corresponding to Y_0 in the usual sense, and since $g_0 K_{Y_0} g_0^{-1} = K = O(n)$ we then have $g_0 R_{Y_0} g_0^{-1} = \mathrm{Aut}(L_0)$, the automorphism group of L_0, viz. the group of those $B \in O(n)$ such that $B(L_0) = L_0$.

Let $f(Y_0)$ be the largest integer for which the R_{Y_0}-invariant polynomials on $T_{Y_0}(\mathcal{P}_n^\circ)$ agree with the K_{Y_0}-invariant polynomials up to degree f. We will see in the next section, as a reflection of the fact that the automorphism groups $\mathrm{Aut}(L_4)$, $\mathrm{Aut}(L_8)$ and $\mathrm{Aut}(L_{24})$ are "large", that $f(L_4) = 2$, $f(L_8) = 3$ and $f(L_{24}) = 5$. With this we can determine the Taylor expansions of $E(Y,s)$ to order 2 (at least) at these L_n's as follows.

We make this second order Taylor expansion at $Y = Y_n$, identifying $\mathcal{P}_n^\circ = K \backslash G$ with $\mathfrak{p}$ as in Proposition 1. Note that the K_{Y_n}-invariant polynomials on $\mathfrak{p}$ by definition are those that are invariant under $X \mapsto kXk^t$ for all $k \in O(n)$; it is well-known that these are generated over $\mathbb{C}$ by $s_j = \mathrm{trace}(X^j)$ for $j = 2, \dots, n$ (recall that all $X \in \mathfrak{p}$ are symmetric with $\mathrm{trace}(X) = 0$; in particular $s_1 \equiv 0$ on $\mathfrak{p}$). Note that $s_2 = \sum_{j=1}^n \sum_{i=1}^n x_{ij}^2$. Hence, using Proposition 1 and $f(Y_n) \geqq 2$, if $\psi : \mathcal{P}_n^\circ \to \mathbb{R}$ is any function satisfying $\psi(Y[\gamma]) = \psi(Y)$ for $\gamma \in \mathrm{GL}(n,\mathbb{Z})$ then the Taylor expansion of $F(X) := \psi(g_0^t e^X g_0)$ for $X \in \mathfrak{p}$ near 0 has the form

$$F(X) = F(0) + a_2 s_2(X) + [\text{higher order terms}] \tag{11}$$

with $a_2 \in \mathbb{R}$.

Thus $a_2 > 0$ ensures that F has a strict local minimum at $X = 0$.

To compute a_2 for $E(\exp X, s)$ it suffices to use the Laplacian Δ on $\mathcal{P}_n^\circ$. Let $\lambda(s)$ be the negative eigenvalue, i.e.

$$\Delta E(Y,s) + \lambda(s) E(Y,s) = 0. \tag{12}$$

Then if Δ corresponds to the precise Riemannian structure specified above we have $\lambda(s) = \frac{n-1}{n} s(\frac{n}{2} - s)$, so that

$$\begin{aligned}(13)\qquad &\lambda(s) < 0 \ \text{ for } s > \frac{n}{2}, \qquad \lambda(s) > 0 \ \text{ for } 0 < s < \frac{n}{2},\\ &\lambda(0) = \lambda(\tfrac{n}{2}) = 0, \text{ and } \left.\frac{d\lambda(s)}{ds}\right|_{s=0} = \frac{n-1}{2}.\end{aligned}$$

Let us fix an orthonormal basis $\mathbf{b}_1, \ldots, \mathbf{b}_N$ in $\mathfrak{p}$ with respect to $\langle \cdot, \cdot \rangle$. Under our identification of $\mathfrak{p}$ with $T_{Y_n}(\mathcal{P}_n^\circ)$ these vectors form an orthonormal basis for the metric at Y_n; hence if we define local coordinates $x_1, \ldots, x_N$ in a neighbourhood of Y_n by $Y = \exp(x_1\mathbf{b}_1 + \ldots + x_N\mathbf{b}_N)[g_0]$ we have

$$\Delta E(x_1, \ldots, x_N; s)_{|x_1=\ldots=x_N=0} = \sum_{j=1}^{N} \left.\frac{\partial^2 E}{\partial x_j^2}\right|_{x_1=\ldots=x_N=0}.$$

On the other hand, by (11) and the definition of s_2,

$$(14)\qquad E(x_1, \ldots, x_N; s) = E(0, \ldots, 0; s) + a_2 \sum_{j=1}^{N} x_j^2 + [\text{higher order}].$$

Thus

$$\Delta E(x_1, \ldots, x_N; s)_{|x_1=\ldots=x_N=0} = 2Na_2.$$

Hence $2Na_2 + \lambda(s)E(0, \ldots, 0; s) = 0$, so

$$(15)\qquad a_2 = -\frac{\lambda(s)E(Y_n, s)}{2N}.$$

Since $E(Y_n, s)$ is a converging series of positive terms for $s > \frac{n}{2}$ it follows from (13) and (15) that

$$(16)\qquad a_2(s) > 0 \qquad \text{for } s > \tfrac{n}{2}.$$

For Y_n, $n = 4$, 8 and 24 we will show that for $0 < s < \frac{n}{2}$,

$$(17)\qquad E(Y_n, s) < 0.$$

Given this it follows from (13) and (15) that

$$(18)\qquad a_2(s) > 0 \qquad \text{for } 0 < s < \tfrac{n}{2}.$$

Differentiating (12) with respect to s, it follows from (13) and $E(Y, 0) = -1$ that

$$(19)\qquad \Delta\frac{\partial E}{\partial s}(Y, s)_{|s=0} - \frac{n-1}{2} = 0.$$

From this we see as above that $a_2 > 0$ in (11) for $F(Y) = \frac{\partial E}{\partial s}(Y, s)_{|s=0}$.

These establish that $E(Y, s)$ has a strict local minimum for Y near Y_n and for all $s > 0$ and from (3) and the above that so does $h(Y)$. Note that (12) and the sign of $E(Y, s)$ and (19) show that for $s > \frac{n}{2}$, $E(Y, s)$ and $h(Y)$ are subharmonic on $\mathcal{L}_n^\circ$.

Finally, we check (17), starting with the case $Y_n = Y_{24}$. We have by [C-S, p. 105],

$$E(L_{24}, s) = \frac{65520}{691} \cdot 2^{-s} \cdot (\zeta(s)\zeta(s-11) - L(s, \Delta)),$$

where $L(s, \Delta) := \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s}$ with $\tau(n)$ denoting the Ramanujan function. By the functional equation (2), we need only check (17) for $6 \leqq s < 12$. It is known (and easy to verify using a computer) that $L(s, \Delta) > 0$ for $0 < s < 12$; furthermore, $\zeta(s)\zeta(s-11) < 0$ for $6 \leqq s < 7$ and for $9 < s < 12$, and one checks by a computation that in the interval $7 \leqq s \leqq 9$ we have $0 \leqq \zeta(s)\zeta(s-11) < 0.02$ and $L(s, \Delta) > 0.5$ (cf. [S-St, item 3]). Hence (17) holds for $Y_n = Y_{24}$. (A more direct proof is given below at the end of Sect. 5.)

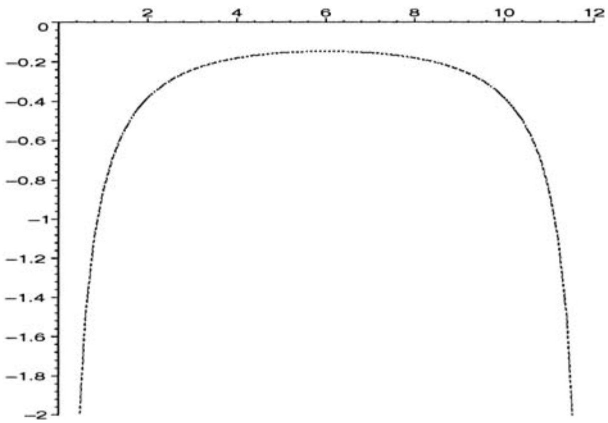

Fig. 1. The function $\pi^{-s}\Gamma(s)E(Y_{24}, s)$ for $0 < s < 12$

The cases of Y_4 and Y_8 are significantly easier: We have

$$E(Y_4, s) = 24 \cdot 2^{-\frac{s}{2}}(1 - 2^{1-s}) \cdot \zeta(s)\zeta(s-1)$$

and

$$E(Y_8, s) = 240 \cdot 2^{-s} \cdot \zeta(s)\zeta(s-3)$$

(cf. [C-S, pp. 108 (49), 122]). Hence in both cases (17) follows from the well-known behaviour of $\zeta(s)$ along the real axis.

3. Spaces of $\mathbf{Aut}(L_n)$-invariant polynomials

We first give the precise definitions of the lattices in question. The hexagonal lattice L_2 has Gram matrix $Y_2 = \frac{2}{\sqrt{3}}\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$. The face centered cubic lattice L_3 which is usually denoted by D_3 has Gram matrix $Y_3 = 2^{1/3}\begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 0 \\ 1/2 & 0 & 1 \end{pmatrix}$. The lattice L_4 is more commonly denoted by D_4, and L_8 by E_8; these two lattices may be defined eg. as follows (cf. [C-S]):

$$\begin{aligned} L_4 &= D_4 \\ &= \left\{2^{-\frac{1}{4}}(x_1, x_2, x_3, x_4) \;\middle|\; \text{all } x_i \in \mathbb{Z},\ x_1 + x_2 + x_3 + x_4 \equiv 0 \pmod 2\right\}; \\ L_8 &= E_8 \\ &= \left\{(x_1, \ldots, x_8) \;\middle|\; \text{all } x_i \in \mathbb{Z} \text{ or all } x_i \in \mathbb{Z} + \tfrac{1}{2},\ \textstyle\sum x_i \equiv 0 \pmod 2\right\}. \end{aligned}$$

Regarding the Leech lattice L_{24}, it has a large number of beautiful definitions, cf. [C-S]; for example, it can be defined as the lattice spanned by all vectors of the form

$$\frac{1}{\sqrt{8}}\big((-1)^{\varepsilon_1}x_1, \ldots, (-1)^{\varepsilon_{24}}x_{24}\big),$$

where one $x_j = -3$ and $x_k = 1$ for all $k \neq j$, and $(\varepsilon_1, \ldots, \varepsilon_{24})$ is a code word in the Golay code $\mathfrak{C}_{24}$, i.e. $(\varepsilon_1, \ldots, \varepsilon_{24}) \in \mathbf{F}_2^{24}$ belongs to the $\mathbf{F}_2$-linear span of the row vectors in the table

1												1	0	1	0	0	0	1	1	1	0	1	1
	1											1	1	0	1	0	0	0	1	1	1	0	1
		1										0	1	1	0	1	0	0	0	1	1	1	1
			1									1	0	1	1	0	1	0	0	0	1	1	1
				1								1	1	0	1	1	0	1	0	0	0	1	1
					1							1	1	1	0	1	1	0	1	0	0	0	1
						1						0	1	1	1	0	1	1	0	1	0	0	1
							1					0	0	1	1	1	0	1	1	0	1	0	1
								1				0	0	0	1	1	1	0	1	1	0	1	1
									1			1	0	0	0	1	1	1	0	1	1	0	1
										1		0	1	0	0	0	1	1	1	0	1	1	1
											1	1	1	1	1	1	1	1	1	1	1	1	0

(where empty spaces denote 0).

For $Y_n \in \mathcal{P}_n^\circ$ corresponding to one of these lattices we wish to study the space of R_{Y_n}-invariant polynomials on the tangent space $T_{Y_n}(\mathcal{P}_n^\circ)$, and compare it with its subspace of K_{Y_n}-invariant polynomials. (Recall that $R_{Y_n} = \mathrm{GL}(n, \mathbb{Z}) \cap K_{Y_n}$.) Let us fix a representative $g_0 \in \mathrm{SL}(n, \mathbb{R})$ for Y_n; we have then noticed on p. 122 that $g_0 R_{Y_n} g_0^{-1} = \mathrm{Aut}(L_n)$ for $L_n = g_0\mathbb{Z}^n$, and the action of R_{Y_n} on $T_{Y_n}(\mathcal{P}_n^\circ)$ corresponds to the action of $\mathrm{Aut}(L_n)$ on $\mathfrak{p}$ under our usual identification.

The space of homogeneous polynomials of degree d on $\mathfrak{p}$ can be identified in a natural way with the dual of the d'th symmetric power of $\mathfrak{p}$, $\mathrm{Sym}^d(\mathfrak{p})^*$. Note that $\mathfrak{p} \oplus \mathbb{R}$ can be viewed as the space of all symmetric bilinear forms on $\mathbb{R}^n$, viz. $\mathrm{Sym}^2(\mathbb{R}^n)^*$. We now have the following canonical isomorphisms of $O(n)$-modules, for any $f \geqq 0$:

$$\begin{aligned} \mathrm{Sym}^f \mathrm{Sym}^2(\mathbb{R}^n) &\cong (\mathrm{Sym}^f(\mathfrak{p} \oplus \mathbb{R}))^* \\ &\cong \bigoplus_{d=0}^{f} \left(\mathrm{Sym}^d(\mathfrak{p})^* \otimes \mathrm{Sym}^{f-d}(\mathbb{R})^*\right) \\ &\cong \bigoplus_{d=0}^{f} \mathrm{Sym}^d(\mathfrak{p})^*, \end{aligned} \tag{20}$$

where $O(n)$ acts by the standard representation on $\mathbb{R}^n$ and trivially on $\mathbb{R}$. Hence our problem is essentially equivalent to the problem of determining the subspace S_f of $\mathrm{Aut}(L_n)$-invariant vectors in $\mathrm{Sym}^f \mathrm{Sym}^2(\mathbb{R}^n)$, for each $f = 0, 1, 2, \ldots$.

The subspace of $O(n)$-invariant vectors in S_f is well understood, cf. [J]. In particular one knows that its dimension is equal to the number of partitions of f into not more than n parts. Note that this subspace in S_f by our identifications above is equal to the direct sum of the spaces of $O(n)$-invariant polynomials on $\mathfrak{p}$ – or K_{Y_n}-invariant polynomials on $T_{Y_n}(\mathcal{P}_n^\circ)$ – of degrees $d = 0, 1, \ldots, f$.

It is possible to compute the dimensions $\dim(S_f)$ for all f once representatives for all the conjugacy classes in $\mathrm{Aut}(L_n)$ are known. Let $\Phi(\lambda)$ be the Molien series for the representation $\mathrm{Sym}^2(\mathbb{R}^n)$ of $\mathrm{Aut}(L_n)$, i.e. the generating function

$$\Phi(\lambda) = \sum_{f=0}^{\infty} (\dim S_f) \lambda^f.$$

We then have the well-known identity

$$\Phi(\lambda) = \frac{1}{|\mathrm{Aut}(L_n)|} \sum_{g \in \mathrm{Aut}(L_n)} \frac{1}{\det(I - \lambda\, \pi(g))},$$

where $\pi(g) \in \mathrm{GL}(\mathrm{Sym}^2(\mathbb{R}^n))$ gives the action of $g \in \mathrm{Aut}(L_n)$. Of course, the characteristic polynomial $\det(I - \lambda\, \pi(g))$ depends only on the conjugacy class of g in $\mathrm{Aut}(L_n)$.

Before turning to the specific cases we remark that our task of determining $f(L_n)$ is partially related to the problem of determining the largest integer $t(L_n)$ such that all $\mathrm{Aut}(L_n)$-invariant vectors in $\mathrm{Sym}^t(\mathbb{R}^n)$ are $O(n)$-invariant for $0 \leqq t \leqq t(L_n)$ – this familiar problem has applications to the construction of spherical t-designs, cf. [C-S, p. 93]. It is known that $t(D_3) = 3$, $t(D_4) = 5$, $t(E_8) = 7$, $t(L_{24}) = 11$; the Molien series for the representation of $\mathrm{Aut}(L_{24})$ on $\mathbb{R}^{24}$ was computed in [Hu-S]. To see the connection to our question regarding $\mathrm{Sym}^f\mathrm{Sym}^2(\mathbb{R}^n)$, we note that $t = t(L_n)+1$ is even (since $-I \in \mathrm{Aut}(L_n)$), and for this t the space $\mathrm{Sym}^t(\mathbb{R}^n)$ contains an $\mathrm{Aut}(L_n)$-invariant vector which is not $O(n)$-invariant. On the other hand, $\mathrm{Sym}^{t/2}\mathrm{Sym}^2(\mathbb{R}^n)$ decomposes into irreducible $\mathrm{GL}(n)$-representations one of which is isomorphic to $\mathrm{Sym}^t(\mathbb{R}^n)$ (cf. [L]). Hence $f(L_n)$, the largest integer such that all $\mathrm{Aut}(L_n)$-invariant vectors in $\mathrm{Sym}^f\mathrm{Sym}^2(\mathbb{R}^n)$ are $O(n)$-invariant for all $f \leqq f(L_n)$, certainly satisfies

$$f(L_n) \leqq \frac{t(L_n) - 1}{2}.$$

In particular $f(D_3) \leqq 1$, $f(D_4) \leqq 2$, $f(E_8) \leqq 3$, $f(L_{24}) \leqq 5$. We will see below that we actually have *equality* in these four relations; however, this is not a general phenomenon, for if f is large then $\mathrm{Sym}^f\mathrm{Sym}^2(\mathbb{R}^n)$ as a representation of $O(n)$ contains many irreducible subrepresentations which do not occur in $\mathrm{Sym}^{2f}(\mathbb{R}^n)$. (For example, $\mathrm{Sym}^6\mathrm{Sym}^2(\mathbb{R}^{24})$ contains an irreducible subrepresentation which does not occur in $\mathrm{Sym}^{12}(\mathbb{R}^{24})$ but still allows $\mathrm{Aut}(L_{24})$-fixed vectors; cf. [S-St, sfunctions.pari] for more details.)

We now turn to a description of our specific cases.

The conjugacy classes of $\mathrm{Aut}(D_4)$ (which has order $1152 = 2^7 \cdot 3^2$) are of course easy to enumerate, and we obtain in this case

$$\begin{aligned}\Phi(\lambda) &= \frac{\lambda^{18} - 2\lambda^{17} + 4\lambda^{16} - \ldots + 2\lambda^6 + 2\lambda^5 - \lambda^4 + \lambda^3 + \lambda^2 - 2\lambda + 1}{\phi_1(\lambda)^{10}\phi_2(\lambda)^6\phi_3(\lambda)^3\phi_6(\lambda)^2\phi_4(\lambda)^3} \\ &= 1 + \lambda + 2\lambda^2 + 4\lambda^3 + 9\lambda^4 + 14\lambda^5 + 30\lambda^6 + 50\lambda^7 + 95\lambda^8 + \ldots\end{aligned}$$

where $\phi_r(\lambda)$ is the r-th cyclotomic polynomial. In particular, we see from (20) that the space of homogeneous $\mathrm{Aut}(D_4)$-invariant polynomials of degree d on $\mathfrak{p}$ has dimension 0, 1, 2, 5 respectively, for $d = 1, 2, 3, 4$. Since the corresponding dimensions for $O(4)$-invariant polynomials are 0, 1, 1, 2 respectively we see that $f(D_4) = 2$.

The automorphism group of E_8 has order $|\mathrm{Aut}(E_8)| = 696729600 = 2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$, and we used the Maple package by Stembridge (cf. [Ste]) to obtain representatives for its conjugacy classes. In this case the Molien series is found to be

$$\begin{aligned}&\Phi(\lambda) \\ &= \frac{\lambda^{238} - 4\lambda^{237} + \ldots - 12\lambda^7 - \lambda^6 + 9\lambda^5 - 11\lambda^4 + 4\lambda^3 + 4\lambda^2 - 4\lambda + 1}{\phi_1^{36}\phi_2^{20}\phi_3^{12}\phi_4^{10}\phi_5^{7}\phi_6^{6}\phi_7^{5}\phi_8^{4}\phi_9^{4}\phi_{10}^{4}\phi_{12}^{3}\phi_{14}^{1}\phi_{15}^{2}\phi_{18}^{1}\phi_{20}^{1}\phi_{24}^{1}} \\ &= 1 + \lambda + 2\lambda^2 + 3\lambda^3 + 6\lambda^4 + 9\lambda^5 + 18\lambda^6 + 31\lambda^7 + 65\lambda^8 + 121\lambda^9 + \ldots\end{aligned}$$

In particular the dimensions for the $\mathrm{Aut}(E_8)$-invariant polynomials on $\mathfrak{p}$ are 0, 1, 1, 3 in degrees $d = 1, 2, 3, 4$, and since the corresponding dimensions for $O(8)$-invariant polynomials are 0, 1, 1, 2 we see that $f(E_8) = 3$.

The Automorphism group of the Leech lattice, $\mathrm{Aut}(L_{24})$, is the Conway group Co_0 of order $2^{22} \cdot 3^9 \cdot 5^4 \cdot 6 \cdot 11 \cdot 13 \cdot 23 = 8315553613086720000$. It is a central extension of degree 2 of the simple group Co_1. We read off the necessary facts about the conjugacy classes from the Atlas of finite groups, [C-C-N-P-W]. The Molien series for $\mathrm{Sym}^2(\mathbb{R}^{24})$ is found to be

$$\Phi(\lambda) = p(\lambda)/q(\lambda),$$

where

$$p(\lambda) = \lambda^{4624} + \ldots + 150679\lambda^4 - 14625\lambda^3 + 1023\lambda^2 - 46\lambda + 1$$

and

$$\begin{aligned} q(\lambda) = {} & \phi_1^{300}\phi_2^{156}\phi_3^{100}\phi_4^{78}\phi_5^{60}\phi_6^{52}\phi_7^{42}\phi_8^{32}\phi_9^{33}\phi_{10}^{30}\phi_{11}^{27}\phi_{12}^{26}\phi_{13}^{23}\phi_{14}^{22}\phi_{15}^{20}\phi_{16}^{16}\phi_{18}^{14} \\ & \cdot \phi_{20}^{14}\phi_{21}^{14}\phi_{22}^{13}\phi_{23}^{13}\phi_{24}^{11}\phi_{26}^{12}\phi_{28}^{11}\phi_{30}^{10}\phi_{33}^{7}\phi_{35}^{8}\phi_{36}^{7}\phi_{39}^{6}\phi_{40}^{6}\phi_{42}^{6}\phi_{60}^{3}. \end{aligned}$$

The first terms are

$$\begin{aligned} \Phi(\lambda) = {} & 1 + \lambda + 2\lambda^2 + 3\lambda^3 + 5\lambda^4 + 7\lambda^5 + 13\lambda^6 + 19\lambda^7 + 36\lambda^8 + 62\lambda^9 \\ & + 135\lambda^{10} + 312\lambda^{11} + 1387\lambda^{12} + 11551\lambda^{13} + 197343\lambda^{14} + \ldots \end{aligned}$$

In particular the dimensions for the $\mathrm{Aut}(L_{24})$-invariant polynomials on $\mathfrak{p}$ are 0, 1, 1, 2, 2, 6 in degrees $d = 1, 2, 3, 4, 5, 6$, and since the corresponding dimensions for $O(24)$-invariant polynomials are 0, 1, 1, 2, 2, 4 we see that $f(L_{24}) = 5$.

For the face-centered cubic lattice (fcc, alias D_3) we find Molien series:

$$\begin{aligned} \Phi(\lambda) &= \frac{\lambda^8 - \lambda^7 + \lambda^6 + \lambda^4 + \lambda^2 - \lambda + 1}{\phi_1^6\phi_2^2\phi_3^2\phi_4} \\ &= 1 + \lambda + 3\lambda^2 + 6\lambda^3 + 11\lambda^4 + 18\lambda^5 + 32\lambda^6 + 48\lambda^7 \\ &\quad + 75\lambda^8 + 111\lambda^9 + \ldots . \end{aligned}$$

Note that this shows $f(D_3) = 1$, so that the argument to prove local minimum given in last section would *not* apply to this case.

In order to get a good handle on $E(L, s)$ in a neighbourhood (of decent size!) of a special point like $L = D_3$, it is very useful to know explicit bases for the spaces of invariant polynomials. As part of our computer proof that D_3 is the unique global minimum of the height function (cf. next section), we calculated, for each $f = 2, 3, 4, \ldots, 10$, an explicit basis $\{p_{f,1}, \ldots, p_{f,n_f}\}$ in the space of $\mathrm{Aut}(D_3)$-invariant polynomials on $\mathfrak{p}$ of degree f. (The dimensions are $n_2 = 2, n_3 = 3, \ldots, n_{10} = 49$.) We present here the results for $f = 2, 3, 4$:

$$p_{2,1} = X_1^2 + X_2^2, \qquad p_{2,2} = X_3^2 + X_4^2 + X_5^2;$$

$$\text{(21)} \qquad \begin{aligned} &p_{3,1} = -3X_1^2 X_2 + X_2^3, \qquad p_{3,2} = X_3 X_4 X_5, \\ &p_{3,3} = \sqrt{3} X_1 \big(X_4^2 - X_5^2\big) + X_2 \big(2X_3^2 - X_4^2 - X_5^2\big); \end{aligned}$$

$$p_{4,1} = p_{2,1}^2, \quad p_{4,2} = p_{2,1} p_{2,2}, \quad p_{4,3} = p_{2,2}^2, \quad p_{4,4} = X_3^4 + X_4^4 + X_5^4,$$
$$p_{4,5} = 3X_1^2 X_3^2 + 2\sqrt{3} X_1 X_2 \big(X_4^2 - X_5^2\big) + X_2^2 \big(-X_3^2 + 2X_4^2 + 2X_5^2\big).$$

Here $X_1, X_2, X_3, X_4, X_5 \in \mathfrak{p}^*$ is the dual basis corresponding to the orthonormal basis $\{\mathbf{b}_1, \ldots, \mathbf{b}_5\}$ in $\mathfrak{p}$, where $\mathbf{b}_1, \ldots, \mathbf{b}_5$ are (in order):

$$2^{-\frac{1}{2}} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad 6^{-\frac{1}{2}} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad 2^{-\frac{1}{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$2^{-\frac{1}{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad 2^{-\frac{1}{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Of course the spaces of invariant polynomials depend on the choice of representative g_0; the above results are valid for any g_0 such that $g_0\mathbb{Z}^3$ is the standard form of the lattice D_3 (cf. [C-S, p. 112]), i.e.

$$\text{(22)} \quad g_0\mathbb{Z}^3 = \Big\{2^{-\frac{1}{3}}(x_1, x_2, x_3) \;\Big|\; \text{all } x_i \in \mathbb{Z},\ x_1 + x_2 + x_3 \equiv 0 \ (\mathrm{mod}\ 2)\Big\}.$$

For more details about the computations and results mentioned in this section, cf. [S-St, item 1].

4. Global study for $n = 3$

The expression for the Epstein Zeta function that one derives using Poisson summation is (see (2) and [Ter1])

$$\text{(23)} \quad F(L, s) = \frac{1}{s - \frac{n}{2}} - \frac{1}{s} + \sum_{\substack{m \in L \\ m \neq 0}} G(s, \pi |m|^2) + \sum_{\substack{m \in L^* \\ m \neq 0}} G\big(\tfrac{n}{2} - s, \pi |m|^2\big),$$

where

$$\text{(24)} \qquad G(s, x) := \int_1^\infty e^{-xt} t^{s-1}\, dt = x^{-s} \Gamma(s, x).$$

Here $\Gamma(s, x) = \int_x^\infty e^{-t} t^{s-1} dt$, the usual (complementary) incomplete Gamma function. For $Y \in \mathcal{P}_n^\circ$ corresponding to the lattice L we define

$$\text{(25)} \quad F_s(Y) = F_s(L) := \sum_{a \in \mathbb{Z}^n - \{0\}} \Big(G(s, \pi \cdot Y^{-1}[a]) + G\big(\tfrac{n}{2} - s, \pi \cdot Y[a]\big)\Big),$$

where $Y[a] := a^t Ya$. Then $F(Y,s) = \pi^{-s}\Gamma(s)E(Y,s) = (s-n/2)^{-1} - s^{-1} + F_s(Y^{-1})$. In particular for the height function, we obtain from (3), (2) together with $E(L,0)$ being -1 and a simple calculation:

$$(26)\quad h(Y) = 2\log 2\pi + \frac{\partial E}{\partial s}(Y^{-1},s)_{|s=0} = \log 4\pi - \gamma - \frac{2}{n} + F_0(Y).$$

We now let $n=3$. Our aim in this section is to describe our computer proof that D_3, the face-centered cubic lattice, which is given by $Y_3 = 2^{1/3}\begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 0 \\ 1/2 & 0 & 1 \end{pmatrix}$, is the unique global minimum of $h(Y)$ on $\mathcal{L}_3^\circ = \mathcal{P}_3^\circ/\mathrm{GL}(3,\mathbb{Z})$. For more details we refer to [S-St, item 2].

We parametrize $\mathcal{P}_3^\circ$ by Iwasawa coordinates $(y_1, y_2, t_{12}, t_{13}, t_{23})$:

$$(27)\quad Y = \begin{pmatrix} 1 & t_{12} & t_{13} \\ 0 & 1 & t_{23} \\ 0 & 0 & 1 \end{pmatrix}^t \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix} \begin{pmatrix} 1 & t_{12} & t_{13} \\ 0 & 1 & t_{23} \\ 0 & 0 & 1 \end{pmatrix}; \quad \begin{cases} d_1 = y_1^{\frac{2}{3}} y_2^{\frac{1}{3}} \\ d_2 = y_1^{-\frac{1}{3}} y_2^{\frac{1}{3}} \\ d_3 = y_1^{-\frac{1}{3}} y_2^{-\frac{2}{3}} \end{cases}.$$

This gives a diffeomorphism $(y_1, y_2, t_{12}, t_{13}, t_{23}) \mapsto Y$ from $(\mathbb{R}^+)^2 \times \mathbb{R}^3$ onto $\mathcal{P}_3^\circ$.

The best known fundamental domain for the action of $\mathrm{GL}_3(\mathbb{Z})$ on $\mathcal{P}_3^\circ$ is the Minkowski domain, but for the present problem it is perhaps slightly more convenient to use Grenier's fundamental domain, cf. [Ter2, §4.4.3]. In order to have only one copy of the fcc-point in the (closed) domain, we first shift one half of the domain by the translation $Y \mapsto Y[T_2]$, $T_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. In our coordinates, this modified Grenier's fundamental domain is given by the following inequalities (cf. [Ter2, (4.34)]):

$$\begin{aligned}
&\text{(i)} && 1 \leqq (1+t_{12}-t_{13})^2 + y_1^{-1}\big((1-t_{23})^2 + y_2^{-1}\big) \\
&\text{(ii)} && 1 \leqq (t_{12}-t_{13})^2 + y_1^{-1}\big((1-t_{23})^2 + y_2^{-1}\big) \\
&\text{(iii)} && 1 \leqq t_{12}^2 + y_1^{-1} \\
&\text{(iv)} && 1 \leqq t_{13}^2 + y_1^{-1}\big(t_{23}^2 + y_2^{-1}\big) \\
&\text{(iv}'\text{)} && 1 \leqq (t_{13}-1)^2 + y_1^{-1}\big(t_{23}^2 + y_2^{-1}\big) \\
&\text{(v)} && 1 \leqq t_{23}^2 + y_2^{-1} \\
&\text{(vi)–(viii)} && 0 \leqq t_{12} \leqq \tfrac{1}{2}, \quad 0 \leqq t_{13} \leqq 1, \quad 0 \leqq t_{23} \leqq \tfrac{1}{2}.
\end{aligned}$$

In this domain the fcc lattice has the unique representative $(y_1, y_2, t_{12}, t_{13}, t_{23}) = (\frac{4}{3}, \frac{9}{8}, \frac{1}{2}, \frac{1}{2}, \frac{1}{3})$, which corresponds to the point

$$Y_3 = 2^{1/3}\begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{pmatrix} \in \mathcal{P}_3^\circ.$$

From now on we will use Y_3 to denote *this* point (it is the image under $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \in \mathrm{GL}(n, \mathbb{Z})$ of the fcc-point written down earlier). The dual lattice, viz. the body-centered cubic lattice, also has a unique representative in our fundamental domain, namely $(\frac{9}{8}, \frac{4}{3}, \frac{1}{3}, \frac{2}{3}, \frac{1}{2})$.

According to (26), our task is to prove that $F_0(Y)$ for Y in our domain takes its unique global minimum at $Y = Y_3$. Let us write $F(Y) := F_0(Y)$ for short. One computes

$$F(Y_3) = 0.113359752603\ldots . \tag{28}$$

Our first step is to reduce to a compact domain: One computes that

$$2\,G\big(\tfrac{3}{2}, \pi \cdot 0.71\big) = 0.114813\ldots .$$

Hence if $d_1 < 0.71$ in the Iwasawa coordinates for Y, then since $G(s, x)$ is positive and decreasing with respect to x, we have

$$F(Y) > 2\,G\big(\tfrac{3}{2}, \pi \cdot 0.71\big) > 0.114,$$

because $Y[a] = d_1 < 0.71$ for the two vectors $a = \pm[1\ 0\ 0]^t \in \mathbb{Z}^3$. Hence we may from now on add the condition $y_1^2 y_2 = d_1^3 \geqq 0.71^3$ to (i)–(viii). The resulting domain in $(\mathbb{R}^+)^2 \times \mathbb{R}^3$ is compact. In fact, using (iii) and (v) we find that $0.51 \leqq y_1 \leqq \frac{4}{3}$ and $0.20 \leqq y_2 \leqq \frac{4}{3}$ must hold for each point $(y_1, y_2, t_{12}, t_{13}, t_{23})$ in the domain.

Next, we cover this compact domain with a finite set of rectangular boxes of the form $B_j = \prod_{k=1}^{5} [b_{jk}, b'_{jk}]$, where each side $b'_{jk} - b_{jk}$ is approximately of size $\frac{1}{10}$. This is made in a way so that the fcc-point (and the bcc-point) lies close to the center in one box and not too close to any of the other boxes. Our precise construction was to first split the (t_{12}, t_{13}, t_{12})-box $[0, \frac{1}{2}] \times [0, 1] \times [0, \frac{1}{2}]$ into $5 \times 11 \times 5$ boxes, all of sizes $\approx \frac{1}{10} \times \frac{1}{11} \times \frac{1}{10}$ (but not exactly $\frac{1}{10} \times \frac{1}{11} \times \frac{1}{10}$, since we represent all coordinates by numbers in $2^{-44}\mathbb{Z}$). For each such box b_t we cover the y_1-interval $[0.51, (1 - \sup t_{12}^2)^{-1}]$ with intervals of length $\approx \frac{1}{10}$ and for each such interval I we compute from (i), (ii), (iv)–(v) a closed interval $J \subset \mathbb{R}^+$ such that every point in our domain with $y_1 \in I$, $(t_{12}, t_{13}, t_{23}) \in b_t$ must necessarily satisfy $y_2 \in J$. Finally this interval J is covered by intervals of size $\approx \frac{1}{10}$. Around the fcc-point we make a further split of the intervals for y_1, y_2, t_{12} into halves.

The construction just described leads to a set of 11427 boxes B_j which together cover our compact domain; the unique box containing the fcc-point $(\frac{4}{3}, \frac{9}{8}, \frac{1}{2}, \frac{1}{2}, \frac{1}{3})$ is

$$B_{fcc} \approx [1.29, 1.34] \times [1.1, 1.15] \times [0.45, 0.5] \times \Big[\tfrac{5}{11}, \tfrac{6}{11}\Big] \times [0.3, 0.4].$$

(Again, the precise coordinates we use are numbers in $2^{-44}\mathbb{Z}$.)

Now we verify using the computer that for each box $B_j \neq B_{fcc}$ one has

$$F(Y) > 0.11336, \qquad \forall Y \in B_j \tag{29}$$

(cf. (28)). This is done as follows.

Let $F_{(N)}(Y)$ be the *finite* sum obtained by only considering terms $a = (a_1, a_2, a_3)$ with $|a_1|, |a_2|, |a_3| \leqq N$ in (25) (for $s = 0$). What we verify on the computer is that $F_{(2)}(Y) > 0.11336$ holds for all $Y \in B_j$, $B_j \neq B_{fcc}$; this clearly implies (29), since all terms are positive in (25). To reach this aim we evaluate the value C_0 of $F_{(2)}(Y)$ at the central point $(y_1, y_2, t_{12}, t_{13}, t_{23}) = (\frac{1}{2}(b_{j1} + b'_{j1}), \dots, \frac{1}{2}(b_{j5} + b'_{j5}))$ of $B_j = \prod_{k=1}^{5}[b_{jk}, b'_{jk}]$, and then use interval arithmetic to compute rigorous bounds on the first-order partial derivatives of $F_{(2)}(Y)$, i.e. positive numbers $C_1, \dots, C_5$ such that

$$\left|\frac{\partial}{\partial y_1} F_{(2)}(Y)\right| \leqq C_1, \dots, \left|\frac{\partial}{\partial t_{23}} F_{(2)}(Y)\right| \leqq C_5, \qquad \forall Y \in B_j.$$

Clearly then, for all $Y \in B_j$, we have that $F_{(2)}(Y)$ is bounded from below by

$$C_0 - \sum_{k=1}^{5} \frac{b'_{jk} - b_{jk}}{2} C_k.$$

If this number is > 0.11336 we are done. If not, we split the box B_j into 32 smaller boxes by halving each side, and instead try to apply the method to each of these smaller boxes. This is repeated recursively if necessary.

In order to avoid uncontrollable rounding errors in floating point arithmetic, all computations in the proof are carried out using only integer arithmetic, wherein integers a are used to represent rational numbers $a/2^{44}$, and we keep track of rigorous lower and upper bounds in this format for each partial result needed (in particular, rigorous bounds on the incomplete Gamma function).

We remark that the above method of using interval arithmetic for the first derivatives turned out to be more efficient than using interval arithmetic on the function $F_{(2)}(Y)$ itself. It turns out that for most of the 11426 boxes $B_j \neq B_{fcc}$ no recursion is necessary. The total time required to prove (29) in our present implementation is approximately 19 hours, on a 1.5 GHz machine. Most time is spent on the boxes in the immediate neighbourhood of B_{fcc} or the bcc-point ($F_0(Y_3^{-1}) = 0.1139155\dots$), and for four of these boxes some parts of the search was forced to go through 6 levels of recursion.

It now remains to carry out the local analysis in the box B_{fcc}. We fix the representative

$$g_0 = 2^{-1/3} \begin{pmatrix} 1 & 0 & 1 \\ -1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

so that $I[g_0] = g_0^t g_0 = Y_3$. Note that with this choice, $g_0\mathbb{Z}^3$ is the lattice D_3 in its standard representation, as in (22).

We will use spherical coordinates about the point Y_3. Recall ([Ter2, p. 16]) that an arbitrary geodesic in $\mathcal{P}_3^\circ$ starting at $g_0^t g_0$ can be written as

$$t \mapsto Y = Y(t) := (\exp tA)[Ug_0] = \big(\exp tA[U]\big)[g_0],$$

for some $A \in \mathfrak{a}$ and $U \in SO(3)$. We can here take A to belong to the space

$$\text{(30)}\qquad \begin{aligned}\mathfrak{a}_1 &:= \left\{A = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \;\middle|\; a_1 + a_2 + a_3 = 0,\ a_1^2 + a_2^2 + a_3^2 = 1\right\} \\ &= \left\{A \in \mathfrak{a} \;\middle|\; \operatorname{tr} A^t A = 1\right\}.\end{aligned}$$

Then the geodesic $t \mapsto Y(t)$ is parametrized by arc length. It follows that the distance from $I[g_0]$ to $Y \in \mathcal{P}_3^\circ$ can be expressed as

$$\text{(31)}\qquad \rho := \sqrt{(\log\lambda_1)^2 + (\log\lambda_2)^2 + (\log\lambda_3)^2},$$

where λ_j are the three eigenvalues of $Y[g_0^{-1}]$.

Let us write

$$B(g_0, r) := \left\{(\exp tA[U])[g_0] \;\middle|\; 0 \leqq t \leqq r,\ A \in \mathfrak{a}_1,\ U \in SO(3)\right\},$$

the ball of radius $r > 0$ about the point $Y_3 = g_0^t g_0$ in $\mathcal{P}_3^\circ$. Computing the Taylor expansion of ρ^2 (cf. (31)) in terms of the Iwasawa coordinates near $(y_1, y_2, t_{12}, t_{13}, t_{23}) = (\frac{9}{8}, \frac{4}{3}, \frac{1}{3}, \frac{2}{3}, \frac{1}{2})$, with rigorous bounds, we prove that the box B_{fcc} is completely contained in the ball of radius 0.18:

$$\text{(32)}\qquad B_{fcc} \subset B(g_0, t_0), \qquad \text{where } t_0 := 0.18.$$

Cf. [S-St, sphere.pari].

Now let $\{\mathbf{b}_1, \ldots, \mathbf{b}_5\}$ be the orthonormal basis in $\mathfrak{p}$ which we fixed in last section, let

$$\begin{aligned}\mathfrak{F}_s(\mathbf{x}) &:= F_s(\exp(x_1\mathbf{b}_1 + \ldots + x_5\mathbf{b}_5)[g_0]) \qquad \text{for } \mathbf{x} = (x_1, \ldots, x_5) \in \mathbb{R}^5; \\ \mathfrak{F}(\mathbf{x}) &:= \mathfrak{F}_0(\mathbf{x}),\end{aligned}$$

and write $S^4 = \{\mathbf{x} = (x_1, \ldots, x_5) \mid \sum_{j=1}^5 x_j^2 = 1\}$ for the unit sphere in $\mathbb{R}^5$.

Note that

$$\begin{aligned}\left\{A[U] \mid A \in \mathfrak{a}_1,\ U \in SO(3)\right\} &= \left\{X \in \mathfrak{p} \;\middle|\; \operatorname{tr} X^t X = 1\right\} \\ &= \left\{x_1\mathbf{b}_1 + \ldots + x_5\mathbf{b}_5 \;\middle|\; \mathbf{x} = (x_1, \ldots, x_5) \in S^4\right\}.\end{aligned}$$

Hence by (29) and (32), our proof of global minimum will be complete if we can verify that

$$\text{(33)}\qquad \mathfrak{F}(t\mathbf{x}) > \mathfrak{F}(0) \qquad \text{for all } \mathbf{x} \in S^4,\ 0 < t \leqq t_0 = 0.18.$$

We need to get an explicit handle on the iterated derivatives of $\mathfrak{F}(t\mathbf{x})$ with respect to t. Note that $\frac{\partial^m}{\partial x^m}G(s,\pi x) = (-\pi)^m G(s+m,\pi x)$. Let $\mathfrak{R}_{m,n} \in \mathbb{Z}[X_1,\ldots,X_n]$ be the "chain rule polynomials", defined by the following identity for general one-variable functions f and g:

$$(f\circ g)^{(n)} = \sum_{m=1}^{n} \mathfrak{R}_{m,n}(g',g'',\ldots,g^{(n)})\cdot (f^{(m)}\circ g)$$

(thus $\mathfrak{R}_{1,1}(X_1)=X_1,\ \mathfrak{R}_{1,2}(X_1,X_2)=X_2,\ \mathfrak{R}_{2,2}(X_1,X_2)=X_1^2$, etc.).

Given $\mathbf{x}\in S^4$, we fix $A\in\mathfrak{a}_1$ and $U\in SO(3)$ such that $A[U]=x_1\mathbf{b}_1+\ldots+x_5\mathbf{b}_5$, and write $Y=Y(t)=(\exp tA)[Ug_0]$. It now follows from (25) that for all $t>0$ and $n\in\mathbb{Z}^+$, we have

$$\begin{aligned}\frac{d^n}{dt^n}\mathfrak{F}_s(t\mathbf{x}) = &\sum_{a\in\mathbb{Z}^3-\{0\}}\sum_{m=1}^{n}(-\pi)^m G(s+m,\pi\cdot Y^{-1}[a])\\ &\cdot \mathfrak{R}_{m,n}\left(\frac{d}{dt}Y^{-1}[a],\ldots,\frac{d^n}{dt^n}Y^{-1}[a]\right)\\ &+\sum_{a\in\mathbb{Z}^3-\{0\}}\sum_{m=1}^{n}(-\pi)^m G\big(\tfrac{3}{2}-s+m,\pi\cdot Y[a]\big)\\ &\cdot\mathfrak{R}_{m,n}\left(\frac{d}{dt}Y[a],\ldots,\frac{d^n}{dt^n}Y[a]\right).\end{aligned} \tag{34}$$

Here note that

$$\frac{d^m}{dt^m}Y[a] = (A^m\exp tA)[Ug_0a],$$
$$\frac{d^m}{dt^m}Y^{-1}[a] = ((-A)^m\exp -tA)\big[U\big(g_0^t\big)^{-1}a\big].$$

We first discuss how to use (34) to obtain an explicit upper bound on $\left|\frac{d^n}{dt^n}\mathfrak{F}(t\mathbf{x})\right|$ valid for all $\mathbf{x}\in S^4$, $t\in[0,t_0]$. To bound the $\mathfrak{R}$-factors in (34) we simply insert absolute signs on each monomial in $\mathfrak{R}_{m,n}$, and use the following bounds, valid for all $0\leqq t\leqq t_0$ and $m\geqq 0$:

$$\left|\frac{d^m}{dt^m}Y[a]\right| = \left|(A^m\exp tA)[Ug_0a]\right| \leqq \left(\frac{2}{3}\right)^{m/2} e^{\sqrt{2/3}\cdot t_0}\cdot|g_0a|^2;$$
$$\left|\frac{d^m}{dt^m}Y^{-1}[a]\right| \leqq \left(\frac{2}{3}\right)^{m/2} e^{\sqrt{2/3}\cdot t_0}\cdot\left|\big(g_0^t\big)^{-1}a\right|^2.$$

To see that the above bounds are valid, note that the three diagonal elements of any matrix $A\in\mathfrak{a}_1$ (cf. (30)) always lie in the interval $\big[-\sqrt{2/3},\sqrt{2/3}\big]$. For the G-factors we use that $G(s,x)$ is decreasing in the x-variable, and

$$Y[a]\geqq e^{-\sqrt{2/3}\cdot t_0}\cdot|g_0a|^2;\qquad Y^{-1}[a]\geqq e^{-\sqrt{2/3}\cdot t_0}\cdot\left|\big(g_0^t\big)^{-1}a\right|^2.$$

Combining these facts, we obtain an upper bound on $\left|\frac{d^n}{dt^n}\mathfrak{F}(t\mathbf{x})\right|$ in terms of an explicit finite linear combination of sums of the form

$$\sum_b |b|^{2k} \cdot G\big(s', \pi e^{-\sqrt{2/3}\cdot t_0} \cdot |b|^2\big),$$

with $s' \in \{0, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \dots\}$ and $k \in \mathbb{Z}^+$, and where b runs through all non-zero vectors in either the fcc lattice

$$g_0\mathbb{Z}^3 = D_3 = \big\{2^{-\frac{1}{3}}(x_1, x_2, x_3) \;\big|\; \text{all } x_i \in \mathbb{Z},\; x_1 + x_2 + x_3 \equiv 0 \pmod 2\big\},$$

or its dual, the bcc lattice

$$\big(g_0^t\big)^{-1}\mathbb{Z}^3 = D_3^* = \big\{2^{-\frac{2}{3}}(x_1, x_2, x_3) \;\big|\; \text{all } x_i \in \mathbb{Z},\; x_1 \equiv x_2 \equiv x_3 \pmod 2\big\}.$$

In the case of D_3, note that $|b|^2 = 2^{\frac{1}{3}}\mathbb{Z}_{\geqq 0}$ for all $b \in D_3$, and the counting function $N(m) = \sharp\{b \in D_3 \mid |b|^2 = 2^{\frac{1}{3}}m\}$ is well understood, cf. [C-S, pp. 112–113]. The above sum equals

$$2^{k/3} \sum_{m=1}^{\infty} N(m) \cdot m^k \cdot G\big(s', \pi e^{-\sqrt{2/3}\cdot t_0} \cdot 2^{\frac{2}{3}} \cdot m\big).$$

To bound this we use the explicit G-values and the known $N(m)$-values for $m \leqq 16$, while for $m > 16$ we use the (crude) bound

$$N(m) \leqq 2 \cdot \sharp\big\{(x_1, x_2) \in \mathbb{Z}^2 \;\big|\; |x_1|, |x_2| \leqq \sqrt{2m}\big\} < 30m$$

and

$$G(s, x) \leqq (s+1)e^{-x}/x \qquad \text{when } 0 \leqq s \leqq x. \tag{35}$$

(The bound (35) follows from $G(s, x) = x^{-s}\Gamma(s, x)$ and $\Gamma(s, x) = \int_x^\infty e^{-t}t^{s-1}\,dt < e^{-x}x^{s-1}$ for $s \leqq 1$, and for $s > 1$ one uses the recursion formula $\Gamma(s, x) = e^{-x}x^{s-1} + (s-1)\Gamma(s-1, x)$.) The case of D_3^* is almost identical.

Carrying out these computations for $n = D := 10$ we obtained the following bound:

$$\left|\frac{d^D}{dt^D}\mathfrak{F}(t\mathbf{x})\right| < 1.25 \cdot 10^9, \;\; (D = 10), \;\; \forall \mathbf{x} \in S^4, \; 0 \leqq t \leqq t_0 = 0.18. \tag{36}$$

Note that the same type of arguments and bounds as above also allows us to prove good bounds on the error when throwing away all "large a terms" in (34), and for any given $t \geqq 0$, $n \geqq 0$ and $\mathbf{x} \in \mathbb{R}^5$ we may thus compute $\frac{d^n}{dt^n}\mathfrak{F}(t\mathbf{x})$ to any desired (reasonable) precision. Let the Taylor expansion of

$\mathfrak{F}(\mathbf{x})$ at $\mathbf{x} = 0$ be

$$\mathfrak{F}(x_1, \dots, x_5) \sim \mathfrak{F}(0) + \sum_{d \geqq 1} q_d(x_1, \dots, x_5),$$

where each q_d is a homogeneous polynomial of degree d. By invariance and the results from last section, we know that the linear part of the expansion vanishes; $q_1 \equiv 0$, and for each $d = 2, 3, \dots$, the polynomial q_d is a linear combination of the invariant polynomials $p_{d,j}$ in (21). We determine the coefficients in this linear combination by computing $\frac{d^d}{dt^d}\mathfrak{F}(t\mathbf{x})_{|t=0}$ for n_d random choices of $\mathbf{x} \in \{1, 2, \dots, 10\}^5$ and equating the coefficients. In this way, and again using only integer arithmetic and rigorous bounds on each partial result, we determine each coefficient in each q_d, $d = 2, 3, \dots, 9$, up to an absolute error less than 10^{-10}. (As a test we may then repeat the computation of $\frac{d^d}{dt^d}\mathfrak{F}(t\mathbf{x})_{|t=0}$ and compare with the result from the Taylor expansion.) In particular we find

$$q_2 = c_1\big(x_1^2 + x_2^2\big) + c_2\big(x_3^2 + x_4^2 + x_5^2\big) \quad \text{with} \quad \begin{cases} c_1 = 0.0236110815\dots \\ c_2 = 0.1509259456\dots \end{cases}.$$

It follows from (36) that for all $\mathbf{x} \in S^4$ and all $0 \leqq t \leqq t_0 = 0.18$ we have, with $D = 10$,

$$\mathfrak{F}(t\mathbf{x}) = \mathfrak{F}(0) + \sum_{d=2}^{D-1} q_d(\mathbf{x})t^d + E_D t^D, \quad \text{where} \quad |E_D| \leqq \frac{1.25 \cdot 10^9}{D!} < 345.$$

Now, to prove (33), we split the unit sphere S^4 into several boxes and use interval arithmetic to compute upper and lower bounds of $q_2(\mathbf{x})$, $q_3(\mathbf{x})$ and $q_4(\mathbf{x})$ on each such box, thus proving (with $t_0 = 0.18$)

$$q_2(\mathbf{x})t_0^2 + \min(q_3(\mathbf{x}), 0)t_0^3 + \min(q_4(\mathbf{x}), 0)t_0^4 > 0.0002, \qquad \forall \mathbf{x} \in S^4. \tag{37}$$

(Note that since each q_d is $\mathrm{Aut}(D_3)$-invariant we may restrict attention to a fundamental domain of $S^4/\mathrm{Aut}(D_3)$. Our computer proof of (37) uses a covering of such a fundamental domain by 14504 boxes, and takes a few minutes to run.) For $5 \leqq d \leqq 9$ a much cruder analysis is sufficient; namely, insert absolute bounds and apply $|x_1^{m_1} \dots x_5^{m_5}| \leqq (m_1/d)|x_1|^d + \dots + (m_5/d)|x_5|^d$ individually for each monomial in q_d; adding up we obtain $|q_d(x_1, \dots x_5)| \leqq c_1|x_1|^d + \dots + c_5|x_5|^d$ for some explicit positive constants c_j, and hence $|q_d(x_1, \dots, x_5)| \leqq \max(c_1, \dots, c_5)$ for all $\mathbf{x} = (x_1, \dots, x_5) \in S^4$. In this way we obtained:

$$\sum_{d=5}^{D-1} |q_d(\mathbf{x})|t_0^d + |E_D|t_0^D < 0.00015, \qquad \forall \mathbf{x} \in S^4. \tag{38}$$

Together, (37) and (38) imply that for all $\mathbf{x} \in S^4$ and all $0 < t \leqq t_0$ we have

$$\begin{aligned} F_{\mathbf{x}}(t) &= F(0) + q_2(\mathbf{x})t^2 + \sum_{d=3}^{D-1} q_d(\mathbf{x})t^d + E_D t^D \\ &\geqq F(0) + \left(q_2 t_0^2 + \min(q_3, 0)t_0^3 + \min(q_4, 0)t_0^4\right)\frac{t^2}{t_0^2} \\ &\quad - \left(\sum_{d=5}^{D-1} |q_d| t_0^d + |E_D| t_0^D\right) \cdot \frac{t^5}{t_0^5} \\ &> F(0) + 0.0002 \cdot \frac{t^2}{t_0^2} - 0.00015 \cdot \frac{t^5}{t_0^5} \geqq F(0) + 0.00005 \cdot \frac{t^2}{t_0^2}. \end{aligned}$$

This proves (33), and concludes the proof that the face-centered lattice is the unique global minimum of the height function.

5. Some remarks on minima of theta functions

The theta function of a lattice $L \in \mathcal{L}_n^\circ$ is defined by

$$\Theta(L, z) = \sum_{\ell \in L} e^{\pi i z \langle \ell, \ell \rangle}, \qquad \text{for } z \in \mathbb{C},\ \operatorname{Im} z > 0 \tag{39}$$

(cf. [C-S]). It is related to the Epstein Zeta function by

$$\Gamma(s)\pi^{-s} E(L, s) = \int_0^\infty (\Theta(L, iy) - 1) y^{s-1}\, dy, \quad \text{for } \operatorname{Re} s > \frac{n}{2}, \tag{40}$$

which follows directly from the definitions and the formula $\Gamma(s)\pi^{-s}\langle \ell, \ell\rangle^{-s} = \int_0^\infty e^{-\pi y \langle \ell, \ell \rangle} y^{s-1}\, dy$. We will always keep $z = iy,\ y > 0$; then (39) is a positive sum. Using Poisson summation one proves

$$\Theta(L, iy) = y^{-n/2} \cdot \Theta(L^*, i/y). \tag{41}$$

Note that (40) and (41) immediately imply

$$\begin{aligned} \Gamma(s)\pi^{-s} E(L, s) &= \int_1^\infty (\Theta(L, iy) - 1)\, y^{s-1}\, dy \\ &\quad + \int_1^\infty (\Theta(L^*, iy) - 1)\, y^{\frac{n}{2}-s-1}\, dy - \frac{1}{s} + \frac{1}{s - n/2}. \end{aligned} \tag{42}$$

(Alternatively, this follows from (23) and (24) by changing order of summation and integration.) By analytic continuation this identity holds for all $s \in \mathbb{C} - \{0, \frac{n}{2}\}$.

For fixed $z = iy$, $y > 0$, each term in (39) is a decreasing function of $\langle \ell, \ell \rangle$, and it is natural to ask which $L \in \mathcal{L}_n^\circ$ yields the minimum value of $\Theta(L, iy)$.

We have the following local result, analogous to Theorem 1:

Proposition 2. *For $n = 4$, 8 and 24 and $y > 0$, $\Theta(L, iy)$ has a strict local minimum at $L = L_n$.*

We give the proof below.

One might conjecture that for $n = 4, 8$ and 24, $L = L_n$ yields the unique minimum of $\Theta(L, iy)$, for any $y > 0$. (For $n = 2$ this is a theorem, proved by Montgomery [Mo].) This would imply the corresponding conjecture for the Epstein Zeta function, (5), as we see using (40), (42) and $L_n^* = L_n$. Chua in [Chua] actually proves that $\Theta(L, iy) \geqq \Theta(L_{24}, iy)$ for all $y > 0$ and all 297 unimodular integral lattices $L \in \mathcal{L}_{24}^{\circ}$, thus giving evidence for the above global minimum conjecture for $\Theta(L, iy)$. We also mention in this vein the very interesting recent work by Cohn and Kumar, [C-K2, esp. §9].

In the case of $n = 3$, the fcc lattice $L = D_3$ cannot possibly give the global minimum of $\Theta(L, iy)$ for all $y > 0$, because of (41) and $m(D_3^*) < m(D_3)$. However, it seems to be a plausible guess that

$$\Theta(D_3, iy) \leqq \Theta(L, iy) \qquad \text{for all } \ y \geqq 1 \text{ and } L \in \mathcal{L}_3^{\circ}. \tag{43}$$

Regarding the Epstein Zeta function in $n = 3$ it appears that the following might hold:

$$E(D_3, s) \leqq E(L, s) \qquad \text{for all } \ s \geqq \tfrac{3}{4} \text{ and } L \in \mathcal{L}_3^{\circ}. \tag{44}$$

We have performed some preliminary computer tests to check the validity of (43) and (44). If true, these two inequalities may eventually turn out to be possible to prove numerically, and it may well be that the most convenient approach to (44) would be to use (42) coupled with a careful study of $\Theta(L, iy)$ in various regions, since Θ only involves the exponential function and no incomplete Gamma function.

Proof of Proposition 2. The proof is very similar to the proof of Theorem 1. Using (41) and $L_n^* = L_n$ for $n = 4, 8, 24$ we see that we may assume $y \geqq 1$. As in Sect. 2 we expand the theta functions in a Taylor series about the point $L = L_n$, and since $f(L_n) \geqq 2$ by Sect. 3, we have

$$\begin{aligned} &\Theta(\exp(x_1\mathbf{b}_1 + \ldots + x_N\mathbf{b}_N)[g_0]; iy) \\ &= \Theta(L_n, iy) + a_2(y) \cdot \sum_{j=1}^{N} x_j^2 + \begin{bmatrix}\text{higher}\\ \text{order}\end{bmatrix}, \end{aligned} \tag{45}$$

for some $a_2(y) \in \mathbb{R}$. Here, as before, $g_0^t g_0 = Y_0$, the Gram matrix of L_n, and $\mathbf{b}_1, \ldots, \mathbf{b}_N$ is any orthonormal basis in $\mathfrak{p}$. (Note, however, in contrast to Sect. 2, that $\Theta(\cdot, iy)$ is in general not an eigenfunction of the ring $\mathcal{D}(\mathcal{P}_n^{\circ})$ of invariant differential operators, and Proposition 1 does not apply.)

Next, we have the following formula for the action of the Laplace operator (in the lattice variable) on $\Theta(L, iy)$:

$$(46)\quad \Delta\Theta(L, iy) = \frac{n-1}{n} \cdot \sum_{\ell \in L - \{0\}} \big(\pi y \langle \ell, \ell\rangle - \big(\tfrac{n}{2} + 1\big)\big)\pi y \langle \ell, \ell\rangle e^{-\pi y \langle \ell, \ell \rangle}.$$

This can be proved eg. using [Ter2, p. 35 (Ex. 32)].

Recall that we are assuming $y \geqq 1$. If $n = 4$ or $n = 8$ then each term in (46) is positive, since $m(D_4) = 2^{1/4}$, $m(E_8) = \sqrt{2}$, and hence $\Delta\Theta(L_n, iy) > 0$.

The same is true for $n = 24$, but the proof is slightly more involved: Since $m(L_{24}) = 2$ we have $\pi y \cdot m(L_{24})^2 > \frac{24}{2} + 1$ whenever $y \geqq 1.04$, and hence in this case all terms in (46) are positive. It now only remains to treat $1 \leqq y < 1.04$. There are 196560 vectors of length 2 and 16773120 vectors of length $\sqrt{6}$ in L_{24}, and all other vectors have length $\geqq \sqrt{8}$ [C-S, p. 135]. Note that $\pi \cdot 6 > \frac{24}{2} + 1$. Hence if $1 \leqq y < 1.04$ we have

$$(47)\quad \begin{aligned} \Delta\Theta(L_{24}, iy) > \frac{n-1}{n} \cdot \big(&196560 \cdot (\pi \cdot 4 - 13) \cdot \pi \cdot 1.04 \cdot 4 \cdot e^{-\pi \cdot 4} \\ &+ 16773120 \cdot (\pi \cdot 6 - 13) \cdot \pi \cdot 6 \cdot e^{-\pi \cdot 1.04 \cdot 6}\big) \\ = \frac{23}{24}&(-3.884647\ldots + 5.666689\ldots) > 0. \end{aligned}$$

Hence by (45), in all three cases,

$$(48)\quad a_2(y) = \frac{1}{2N} \cdot \Delta\Theta(L_n, iy) > 0.$$

Hence $L = L_n$ is indeed a local minimum of $\Theta(L, iy)$. $\square$

Applying the Laplace operator to both sides of (40) we have (using (12) and $\lambda(s) = \frac{n-1}{n} s(\frac{n}{2} - s)$),

$$\frac{1-n}{n} s\left(\frac{n}{2} - s\right)\Gamma(s)\pi^{-s} E(L, s) = \int_0^\infty \Delta\Theta(L, iy) \cdot y^{s-1}\, dy.$$

Hence from the fact $\Delta\Theta(L_n, iy) > 0$ which we verified above we obtain a new proof of the crucial relation (17), i.e. $E(L_n, s) < 0$ for $0 < s < \frac{n}{2}$, $n = 4, 8, 24$. Note that this new proof is simpler in terms of the numerics involved.

6. Heights in large dimension

We recall from Sect. 4 that the height function in arbitrary dimension n is given by

$$(49)\quad h(L) = \log 4\pi - \gamma - \frac{2}{n} + F_0(L),$$

where

$$F_0(L) = \sum_{\substack{m\in L^* \\ m\neq 0}} G(0,\pi|m|^2) + \sum_{\substack{m\in L \\ m\neq 0}} G\big(\tfrac{n}{2},\pi|m|^2\big). \tag{50}$$

We will use (49) to study $h(L)$ for $L \in \mathcal{L}_n^\circ$. To do so we need to examine $G(s,x)$ (cf. (24)) as $s \to \infty$ with x in various ranges. Temme in [Tem2] gave a uniform asymptotic expansion of the incomplete Gamma function in complex variables. For us it will be sufficient to consider the leading order terms, and real variables; in fact the asymptotics that we need could alternatively be derived fairly easily from scratch using elementary calculus. The expansion involves the complementary error function, which is defined by

$$\operatorname{erfc}(t) = \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-w^2}\,dw.$$

We note the following properties, all of which are easily verified:

$$\operatorname{erfc}(t) = \frac{e^{-t^2}}{\sqrt{\pi}t}(1+O(t^{-2})) \qquad \text{as } t\to\infty, \tag{51}$$

$$\operatorname{erfc}(0) = 1, \qquad \text{and} \tag{52}$$

$$\operatorname{erfc}(t) = 2-\operatorname{erfc}(-t) = 2 - O(e^{-t^2}/t) \qquad \text{as } t\to -\infty. \tag{53}$$

Proposition 3. ([Tem2]) *The following asymptotic relation holds uniformly for all $x > 0$, as $s \to \infty$:*

(54)

$$G(s,x) \sim \sqrt{\frac{\pi}{2s}}\left(\frac{ex}{s}\right)^{-s}\cdot\left\{\operatorname{erfc}(\eta\sqrt{s/2}) + \sqrt{\frac{2}{\pi}}\frac{e^{-s\eta^2/2}}{\sqrt{s}}\left(\frac{1}{\lambda-1}-\frac{1}{\eta}\right)\right\}.$$

Here $\lambda = x/s$ and $\eta = \eta(\lambda) = \sqrt{2(\lambda-1-\log\lambda)}$, an analytic function of λ in some complex domain containing all $\lambda > 0$, with branch chosen so that $\eta > 0$ for $\lambda > 1$.

In particular, for any fixed constant $\alpha > \frac{1}{2}$, we have uniformly in the range $0 < x \leqq s - s^\alpha$, as $s \to \infty$,

$$G(s,x) \sim \sqrt{\frac{2\pi}{s}}\left(\frac{xe}{s}\right)^{-s}, \tag{55}$$

and in the range $x \geqq s + s^\alpha$,

$$G(s,x) \sim \frac{e^{-x}}{x-s}. \tag{56}$$

Proof. By [Tem2, (1.1), (2.5), (2.13), with $N = 1$]) we have

$$G(s,x) = x^{-s}\Gamma(s)A(s,x) + \varepsilon(s,x), \tag{57}$$

where

$$A(s,x) = \frac{1}{2}\mathrm{erfc}\big(\eta\sqrt{s/2}\big) + \frac{e^{-(1/2)s\eta^2}}{\sqrt{2\pi s}}\left(\frac{1}{\lambda-1} - \frac{1}{\eta}\right), \tag{58}$$

and by [Tem2, (2.13), (2.14)] and Stirling's formula the error $\varepsilon(s,x)$ is bounded by

$$|\varepsilon(s,x)| \ll \frac{(xe/s)^{-s}}{s(s+x)}\begin{Bmatrix} e^{-(1/2)s\eta^2} & \text{if } \eta \geqq 0 \\ 2 & \text{if } \eta \leqq 0 \end{Bmatrix} + \frac{G(s,x)}{s^2} \tag{59}$$

uniformly for $s, x > 0$.

The auxiliary function $\eta(\lambda)$ is increasing for $\lambda > 0$, and $\eta(\lambda) = \lambda - 1 + O((\lambda-1)^2)$ as $\lambda \to 1$. Hence the function $\frac{1}{\lambda-1} - \frac{1}{\eta}$ is uniformly bounded for all $\lambda > 0$. Now fix any constant $\frac{1}{2} < \alpha < \frac{2}{3}$. Using (51) and (53) we then deduce that as $s \to \infty$,

$$A(s,x) \sim \begin{cases} 1 & \text{if } 0 < x \leqq s - s^\alpha \\ \frac{1}{2}\mathrm{erfc}(\eta\sqrt{s/2}) & \text{if } s - s^\alpha \leqq x \leqq s + s^\alpha \\ \dfrac{e^{-(1/2)s\eta^2}}{\sqrt{2\pi s}\,(\lambda-1)} & \text{if } s + s^\alpha \leqq x. \end{cases} \tag{60}$$

(To prove (60) in the third range, i.e. $1 + s^{\alpha-1} \leqq \lambda$, one uses the fact that $\eta \gg \lambda - 1$ for $1 < \lambda \leqq 2$ and $\eta \gg \sqrt{\lambda}$ for all $\lambda \geqq 2$ to see that $\frac{1}{\lambda-1} + O(\frac{1}{\eta^3 s}) \sim \frac{1}{\lambda-1}$.)

We now divide through with $x^{-s}\Gamma(s)A(s,x)$ in (57). Regarding the error (cf. (59)) we first verify by a quick computation using (60) that

$$\frac{1}{x^{-s}\Gamma(s)A(s,x)} \cdot \frac{(xe/s)^{-s}}{s(s+x)}\begin{Bmatrix} e^{-(1/2)s\eta^2} & \text{if } \eta \geqq 0 \\ 2 & \text{if } \eta \leqq 0 \end{Bmatrix} \ll s^{-1},$$

uniformly for $x > 0$, $s \to \infty$. Hence by (59) and (57),

$$\left|\frac{G(s,x)}{x^{-s}\Gamma(s)A(s,x)} - 1\right| \ll s^{-1} + s^{-2} \cdot \frac{G(s,x)}{x^{-s}\Gamma(s)A(s,x)}. \tag{61}$$

This implies

$$\frac{G(s,x)}{x^{-s}\Gamma(s)A(s,x)} \to 1 \tag{62}$$

uniformly for $x > 0$, $s \to \infty$. This is equivalent to (54), by (58) and Stirling's formula. Now (55) and (56) follow using (60) and $\eta = \sqrt{2(\lambda - 1 - \log\lambda)}$. □

We remark that Temme gives explicit numerical bounds for the implied constants in error bounds related to (59) (cf. [Tem2, §4]).

As a corollary to Proposition 3 we note some bounds for later use.

Corollary 2. *The following bound holds uniformly for all $x > 0$, $s \geqq 1$,*

$$G(s, x) \ll \frac{1}{\sqrt{s}} \left(\frac{ex}{s}\right)^{-s}. \tag{63}$$

In the case $x \geqq s \geqq 1$ we also have the stronger bound

$$G(s, x) \ll \frac{e^{-x}}{\sqrt{s}}. \tag{64}$$

Proof. For s large these two bounds follow from (62) since $A(s, x) \ll 1$ for all $x > 0$ and $A(s, x) \ll e^{-(1/2)s\eta^2} = e^{s-x}(x/s)^s$ when $x \geqq s$ (cf. (60)).

It remains to treat the case $1 \leqq s \leqq B$ where B is some constant; this is easily done using the definition, $G(s, x) = \int_1^\infty e^{-xy} y^s \frac{dy}{y}$. For (63) it suffices to note that $G(s, x) = x^{-s} \int_x^\infty e^{-u} u^{s-1}\, du \ll_B x^{-s}$ for all $x > 0$, $1 \leqq s \leqq B$. For (64) we note $y^{s-1} \ll_B e^y$ for all $y \geqq 1$, $1 \leqq s \leqq B$, and hence if $x \geqq 2$, $G(s, x) \ll_B \int_1^\infty e^{-(x-1)y}\, dy \leqq e^{1-x}$. □

It is now easy to prove a first lower bound for $h(L)$ in terms of the length of the shortest vector, $m(L)$. First note that since $F_0(L) > 0$ we have by (49), for all $L \in \mathcal{L}_n^\circ$, $n \geqq 2$,

$$h(L) > \log 4\pi - \gamma - \frac{2}{n} > 1.95 - \frac{2}{n} \geqq 0.95. \tag{65}$$

Recall that since $-L = L$, there are at least two vectors $v \in L$ attaining $|v| = m(L)$, and thus, by (50), $h(L) > F_0(L) > 2G(\frac{n}{2}, \pi m(L)^2)$. Hence by (55), as $n \to \infty$ we have for all $L \in \mathcal{L}_n^\circ$ such that (say) $\pi m(L)^2 < \frac{2}{5}n$,

$$h(L) \geqq 2\sqrt{\frac{4\pi}{n}} \left(\frac{2\pi e}{n} m(L)^2\right)^{-\frac{n}{2}} (1 + o(1)). \tag{66}$$

But it follows from known upper bounds on the density of sphere packings [C-S, p. 19 (45), or (41)] that $\pi m(L)^2 < \frac{2}{5}n$ is true for *all* $L \in \mathcal{L}_n^\circ$ (for n large), and thus (66) holds unconditionally. (Alternatively, we may simply note that $\pi m(L)^2 \geqq \frac{2}{5}n$ would imply $\frac{2\pi e}{n} m(L)^2 > \frac{2e}{3} > 1$, and then (66) follows trivially from (65).) Note that (66) is equivalent to the lower bound (9) stated in the introduction.

For the proof of Theorem 3 we will need certain integration formulae of Siegel and Rogers respectively ([Si], [Ro]). Siegel's formula, which is familiar in the theory of the constant term of Eisenstein series asserts that if f defined on $\mathbb{R}^n$ is nonnegative and

$$F(L) = \sum_{\substack{m \in L \\ m \neq 0}} f(m)$$

then

$$\int_{\mathcal{L}_n^\circ} F(L)\, d\mu(L) = \int_{\mathbb{R}^n} f(x)\, dx. \tag{67}$$

(Here dx is Lebesgue measure on $\mathbb{R}^n$ giving $[0, 1] \times \ldots \times [0, 1]$ measure equal to 1).

Rogers' formula is related to the familiar formula for the inner product of two incomplete Eisenstein series and it asserts the following: Let ρ be a nonnegative function on $\mathbb{R}^n \times \mathbb{R}^n$ satisfying $\rho(\pm x_1, \pm x_2) = \rho(x_1, x_2)$. Then

$$\begin{aligned} &\int_{\mathcal{L}_n^\circ} \sum_{m_1 \in L}{}^{*} \sum_{m_2 \in L}{}^{*} \rho(m_1, m_2)\, d\mu(L) \\ &= \frac{1}{\zeta(n)^2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \rho(x_1, x_2)\, dx_1 dx_2 + \frac{2}{\zeta(n)} \int_{\mathbb{R}^n} \rho(x, x)\, dx. \end{aligned} \tag{68}$$

Here $*$ denotes that the sum is restricted to the primitive lattice vectors in L (i.e. non-zero vectors which are not positive integral multiples of another lattice vector).

Proposition 4. *Let R_n be a sequence of positive numbers satisfying*

$$\frac{R_n^n \omega_n}{n} = o(1) \qquad \text{as } n \to \infty,$$

where ω_n is the volume of the $n-1$ sphere. Then

$$\mu\left\{L \in \mathcal{L}_n^\circ \;\middle|\; m(L) \leqq R_n\right\} \to 0 \qquad \text{as } n \to \infty.$$

Proof. Apply Siegel's formula (67) with $f(x) = \chi_{R_n}(x) = \begin{cases} 1 & \text{if } |x| \leqq R_n \\ 0 & \text{if } |x| > R_n \end{cases}$.

Then

$$\int_{\mathcal{L}_n^\circ} \sum_{\substack{m \in L \\ m \neq 0}} \chi_{R_n}(m)\, d\mu(L) = \int_{\mathbb{R}^n} \chi_{R_n}(x)\, dx = \frac{\omega_n R_n^n}{n}.$$

If $m(L) \leqq R_n$ then the sum on the left hand side is at least 1. Hence

$$\mu\left\{L \in \mathcal{L}_n^\circ \;\middle|\; m(L) \leqq R_n\right\} \leqq \frac{\omega_n R_n^n}{n} \to 0.$$

$\square$

Note that

$$\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)} \tag{69}$$

and hence from Stirling's series

$$\omega_n \sim \left(\frac{2\pi e}{n}\right)^{n/2} \sqrt{\frac{n}{\pi}} \qquad \text{as } n \to \infty. \tag{70}$$

So

$$R_n = \left(\frac{n}{2\pi e}\right)^{1/2} \tag{71}$$

satisfies the assumption in Proposition 4 and we will use this choice of R_n below.

In order to study F_0 in (50) we start with H given by

$$H(L) = \sum_{\substack{m\in L\\ m\neq 0}} G\big(\tfrac{n}{2}, \pi|m|^2\big). \tag{72}$$

According to (67),

$$\int_{\mathcal{L}_n^\circ} H(L)\, d\mu(L) = \int_{\mathbb{R}^n} G\big(\tfrac{n}{2}, \pi|x|^2\big)\, dx = \omega_n \int_0^\infty G\big(\tfrac{n}{2}, \pi r^2\big)\, r^{n-1}\, dr$$
$$= \omega_n \int_0^\infty \int_1^\infty e^{-\pi r^2 y} y^{\frac{n}{2}} \frac{dy}{y}\, r^{n-1}\, dr = \omega_n \int_1^\infty y^{\frac{n}{2}} \int_0^\infty e^{-\pi r^2 y} r^n \frac{dr}{r} \frac{dy}{y}$$
$$= \omega_n \int_1^\infty \int_0^\infty e^{-\pi\xi^2} \xi^n \frac{d\xi}{\xi} \frac{dy}{y} = \infty.$$

Thus to find small values of $H(L)$ we truncate H. Set

$$H_{R_n}(L) = \sum_{\substack{m\in L\\ m\neq 0}} G\big(\tfrac{n}{2}, \pi|m|^2\big) I_{R_n}(m),$$

where

$$I_{R_n}(x) = \begin{cases} 1 & \text{if } |x| > R_n \\ 0 & \text{if } |x| \leqq R_n. \end{cases}$$

Note that if $m(L) > R_n$ then

$$H_{R_n}(L) = H(L).$$

Consider

$$E(H_{R_n}) := \int_{\mathcal{L}_n^\circ} H_{R_n}(L)\, d\mu(L).$$

Applying (67) yields

$$E(H_{R_n}) = \int_{\mathbb{R}^n} G\big(\tfrac{n}{2}, \pi|x|^2\big) I_{R_n}(x)\, dx = \omega_n \int_{R_n}^{\infty} G\big(\tfrac{n}{2}, \pi r^2\big) r^{n-1}\, dr.$$

We take R_n as in (71) and break up the integral according to the intervals in Proposition 3. We keep α fixed, $\frac{1}{2} < \alpha < 1$, and set $\delta_n = (n/2)^{\alpha-1}$. Write

$$E(H_{R_n}) = I + II + III$$

with

$$I = \omega_n \int_{\sqrt{\frac{n}{2\pi e}}}^{\sqrt{\frac{n}{2\pi}(1-\delta_n)}} G\big(\tfrac{n}{2}, \pi r^2\big) r^{n-1}\, dr$$

$$II = \omega_n \int_{\sqrt{\frac{n}{2\pi}(1-\delta_n)}}^{\sqrt{\frac{n}{2\pi}(1+\delta_n)}} G\big(\tfrac{n}{2}, \pi r^2\big) r^{n-1}\, dr$$

$$III = \omega_n \int_{\sqrt{\frac{n}{2\pi}(1+\delta_n)}}^{\infty} G\big(\tfrac{n}{2}, \pi r^2\big) r^{n-1}\, dr.$$

According to (55) in Proposition 3 we have

$$\begin{aligned} I &\sim \omega_n \int_{\sqrt{\frac{n}{2\pi e}}}^{\sqrt{\frac{n}{2\pi}(1-\delta_n)}} 2\sqrt{\frac{\pi}{n}} \left(\frac{2\pi r^2 e}{n}\right)^{-\frac{n}{2}} r^{n-1}\, dr \\ &= 2\omega_n \left(\frac{n}{2\pi e}\right)^{\frac{n}{2}} \sqrt{\frac{\pi}{n}} \int_{\sqrt{\frac{n}{2\pi e}}}^{\sqrt{\frac{n}{2\pi}(1-\delta_n)}} \frac{dr}{r}. \end{aligned}$$

Using (70) it follows that

$$I \sim 2 \log\left(\frac{\sqrt{1-\delta_n}}{\sqrt{1/e}}\right) \sim 1 \qquad \text{as } n \to \infty. \tag{73}$$

As for II, we use (63) to get

$$\begin{aligned} II &\ll \omega_n \int_{\sqrt{\frac{n}{2\pi}(1-\delta_n)}}^{\sqrt{\frac{n}{2\pi}(1+\delta_n)}} \frac{1}{\sqrt{n}} \left(\frac{2\pi r^2 e}{n}\right)^{-\frac{n}{2}} r^{n-1}\, dr \\ &\ll \left(\frac{2\pi e}{n}\right)^{\frac{n}{2}} \sqrt{n} \left(\frac{2\pi e}{n}\right)^{-\frac{n}{2}} \frac{1}{\sqrt{n}} \log\left(\frac{1+\delta_n}{1-\delta_n}\right) = o(1) \end{aligned} \tag{74}$$

as $n \to \infty$.

Finally for III we have, by (56) in Proposition 3,

$$III \sim \int_{\sqrt{\frac{n}{2\pi}(1+\delta_n)}}^{\infty} \frac{e^{-\pi r^2}}{\pi r^2 - \frac{n}{2}} r^{n-1}\, dr.$$

Now

$$e^{-\pi r^2} r^n \leqq \left(\frac{n}{2\pi e}\right)^{\frac{n}{2}} \qquad \text{for } r \geqq \left(\frac{n}{2\pi}\right)^{1/2},$$

and hence

$$\begin{aligned} III &\ll \omega_n \left(\frac{n}{2\pi e}\right)^{\frac{n}{2}} \int_{\sqrt{\frac{n}{2\pi}(1+\delta_n)}}^{\infty} \frac{1}{\left(\pi r^2 - \frac{n}{2}\right) r}\, dr \\ &\ll \sqrt{n} \int_{\frac{n}{2\pi}(1+\delta_n)}^{n} \frac{1}{\left(\pi u - \frac{n}{2}\right)\sqrt{n}} \cdot \frac{du}{\sqrt{n}} + \sqrt{n} \int_{n}^{\infty} \frac{1}{u \cdot \sqrt{u}} \cdot \frac{du}{\sqrt{u}} \qquad (75) \\ &\ll \frac{1}{\sqrt{n}} \log\left(\frac{1}{\delta_n}\right) = o(1) \qquad \text{as } n \to \infty. \end{aligned}$$

Combining (73), (74) and (75) we have

Proposition 5.

$$E(H_{R_n}) \sim 1 \qquad \textit{as } n \to \infty.$$

Next we estimate the variance of H_{R_n}.

Proposition 6.

$$V(H_{R_n}) = E\big((H_{R_n} - E(H_{R_n}))^2\big) \ll n^{-3/2}, \qquad \forall n \geqq 3.$$

Proof. We have

$$\begin{aligned} &\int_{\mathcal{L}_n^\circ} H_{R_n}(L)^2\, d\mu(L) \\ &= \int_{\mathcal{L}_n^\circ} \sum_{\substack{m_1 \in L \\ m_1 \neq 0}} G\left(\tfrac{n}{2}, \pi |m_1|^2\right) I_{R_n}(m_1) \sum_{\substack{m_2 \in L \\ m_2 \neq 0}} G\left(\tfrac{n}{2}, \pi |m_2|^2\right) I_{R_n}(m_2)\, d\mu(L) \\ &= \sum_{\substack{d_1 \geqq 1 \\ d_2 \geqq 1}} \int_{\mathcal{L}_n^\circ} \sum_{\nu_1, \nu_2 \in L}^{*} G\left(\tfrac{n}{2}, \pi |\nu_1|^2 |d_1|^2\right) I_{R_n}(d_1 \nu_1) \\ &\qquad \cdot G\left(\tfrac{n}{2}, \pi |\nu_2|^2 {d_2}^2\right) I_{R_n}(d_2 \nu_2)\, d\mu(L), \end{aligned}$$

where $*$ denotes that the ν_1, ν_2-sum is over pairs of primitive vectors in L. Applying (68) yields

$$\begin{aligned}
&\int_{\mathcal{L}_n^\circ} H_{R_n}(L)^2\, d\mu(L) \\
&= \sum_{d_1,d_2} \Bigg[\frac{1}{\zeta(n)^2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} G\big(\tfrac{n}{2}, \pi |x_1|^2 {d_1}^2\big)\, I_{R_n}(d_1 x_1) \\
&\qquad\qquad \cdot G\big(\tfrac{n}{2}, \pi |x_2|^2 {d_2}^2\big) I_{R_n}(d_2 x_2)\, dx_1\, dx_2 \\
&\qquad + \frac{2}{\zeta(n)} \int_{\mathbb{R}^n} G\big(\tfrac{n}{2}, \pi |x|^2 {d_1}^2\big)\, I_{R_n}(d_1 x)\, G\big(\tfrac{n}{2}, \pi |x|^2 {d_2}^2\big)\, I_{R_n}(d_2 x)\, dx \Bigg] \\
&= \sum_{d_1,d_2} \frac{1}{\zeta(n)^2 d_1^n d_2^n} \int_{\mathbb{R}^n} G\big(\tfrac{n}{2}, \pi |x_1|^2\big)\, I_{R_n}(x_1)\, dx_1 \int_{\mathbb{R}^n} G\big(\tfrac{n}{2}, \pi |x_2|^2\big)\, I_{R_n}(x_2)\, dx_2 \\
&\qquad + \frac{2}{\zeta(n)} \sum_{d_1,d_2} \int_{\mathbb{R}^n} G\big(\tfrac{n}{2}, \pi |x|^2 {d_1}^2\big)\, I_{R_n}(d_1 x)\, G\big(\tfrac{n}{2}, \pi |x|^2 {d_2}^2\big)\, I_{R_n}(d_2 x)\, dx.
\end{aligned}$$

Now the first sum is just $\left(\int_{\mathbb{R}^n} G(\frac{n}{2}, \pi |x|^2)\, I_{R_n}(x)\, dx\right)^2$ which is $E(H_{R_n})^2$. Hence we have

$$\begin{aligned}
&\int_{\mathcal{L}_n^\circ} H_{R_n}(L)^2\, d\mu(L) = E(H_{R_n})^2 \\
&\qquad + \frac{2}{\zeta(n)} \sum_{d_1,d_2} \int_{\mathbb{R}^n} G\big(\tfrac{n}{2}, \pi |x|^2 {d_1}^2\big)\, I_{R_n}(d_1 x)\, G\big(\tfrac{n}{2}, \pi |x|^2 {d_2}^2\big)\, I_{R_n}(d_2 x)\, dx,
\end{aligned}$$

or what is the same,

(76)

$$V(H_{R_n}) = \frac{2}{\zeta(n)} \sum_{d_1,d_2} \int_{\mathbb{R}^n} G\big(\tfrac{n}{2}, \pi |x|^2 {d_1}^2\big) I_{R_n}(d_1 x) G\big(\tfrac{n}{2}, \pi |x|^2 {d_2}^2\big) I_{R_n}(d_2 x) dx.$$

Apply the inequality (63) to get

$$\begin{aligned}
V(H_{R_n}) &\ll \frac{2}{\zeta(n)} \sum_{d_1,d_2} \int_{\max\left(\frac{R_n}{d_1}, \frac{R_n}{d_2}\right)}^{\infty} \frac{1}{\sqrt{n}} \left(\frac{2\pi d_1^2 r^2 e}{n}\right)^{-\frac{n}{2}} \frac{1}{\sqrt{n}} \left(\frac{2\pi d_2^2 r^2 e}{n}\right)^{-\frac{n}{2}} \\
&\qquad\qquad \cdot \omega_n r^{n-1}\, dr \\
&= \frac{2\omega_n}{\zeta(n) n} \left(\frac{2\pi e}{n}\right)^{-n} \sum_{d_1,d_2} d_1^{-n} d_2^{-n} \int_{R_n/\min(d_1,d_2)}^{\infty} r^{-n-1}\, dr \\
&= \frac{2\omega_n}{\zeta(n) n} \left(\frac{2\pi e}{n}\right)^{-n} \frac{R_n^{-n}}{n} \sum_{d_1,d_2} \max(d_1, d_2)^{-n}.
\end{aligned}$$

But we have $\sum_{d_1,d_2} \max(d_1, d_2)^{-n} = \sum_{m=1}^{\infty} (2m-1) m^{-n}$, and this sum is uniformly bounded for all $n \geqq 3$. Recalling $\omega_n \sim \left(\frac{2\pi e}{n}\right)^{n/2} \sqrt{\frac{n}{\pi}}$ and

$R_n = \left(\frac{n}{2\pi e}\right)^{1/2}$ we obtain

$$V(H_{R_n}) \ll n^{-3/2}.$$

□

We remark that for $n = 2$ we have $V(H_R) = \infty$ for all $R > 0$. This is easily seen from (76) (with R in place of R_n), using $G(1, x) = x^{-1}e^{-x}$ and only considering the terms with (e.g.) $1 \leqq d_1 \leqq d_2 \leqq 2d_1$.

Proposition 7. *For $\varepsilon > 0$,*

$$\mu\left\{L \in \mathcal{L}_n^\circ \;\middle|\; |H(L) - 1| > \varepsilon\right\} \to 0 \qquad \text{as } n \to \infty.$$

Proof. This is a matter of collecting what we have. For n large enough, according to Proposition 5, we have $|E(H_{R_n}) - 1| < \varepsilon/2$, hence

$$\left\{L \in \mathcal{L}_n^\circ \;\middle|\; |H(L) - 1| > \varepsilon\right\} \subset \left\{L \in \mathcal{L}_n^\circ \;\middle|\; H(L) \neq H_{R_n}(L)\right\} \;\cup$$
$$\left\{L \in \mathcal{L}_n^\circ \;\middle|\; |H_{R_n}(L) - E(H_{R_n})| > \frac{\varepsilon}{2}\right\}.$$

Now Proposition 4 asserts that the measure of the first set on the right hand side goes to zero while Proposition 6 does the same for the second set. □

We turn to the second series in (50). Set

$$J(L) = \sum_{\substack{m \in L^* \\ m \neq 0}} G(0, \pi |m|^2).$$

As before, let $R_n = \left(\frac{n}{2\pi e}\right)^{1/2}$ and define

$$J_{R_n}(L) = \sum_{\substack{m \in L^* \\ m \neq 0}} G(0, \pi |m|^2) I_{R_n}(m).$$

Then

$$\mu\left\{L \in \mathcal{L}_n^\circ \;\middle|\; J_{R_n}(L) \neq J(L)\right\} \to 0 \qquad \text{as } n \to \infty. \tag{77}$$

Again applying (67) and using the fact that the measure μ_n is invariant under the homeomorphism $L \mapsto L^*$ of $\mathcal{L}_n^\circ$ onto itself,

$$E(J_{R_n}) = \int_{\mathbb{R}^n} G(0, \pi |x|^2) I_{R_n}(x)\, dx = \omega_n \int_{R_n}^\infty G(0, \pi r^2) r^{n-1}\, dr.$$

Now $G(0, x) \leqq \int_1^\infty e^{-xy}\, dy = x^{-1}e^{-x}$, hence

$$E(J_{R_n}) \leqq \omega_n \int_{R_n}^\infty \frac{e^{-\pi r^2}}{\pi r^2} r^{n-1}\, dr.$$

The maximum of $r^n e^{-\pi r^2}$ for $0 < r < \infty$ occurs when $r = \left(\frac{n}{2\pi}\right)^{1/2}$, thus

$$E(J_{R_n}) \leqq \omega_n \left(\frac{n}{2\pi}\right)^{n/2} e^{-n/2} \int_{R_n}^{\infty} \frac{dr}{\pi r^3} = \frac{\omega_n}{2\pi} \left(\frac{n}{2\pi e}\right)^{n/2} R_n^{-2}$$
$$\sim \frac{1}{2\pi} \left(\frac{n}{2\pi e}\right)^{-n/2} \sqrt{\frac{n}{\pi}} \left(\frac{n}{2\pi e}\right)^{n/2} \left(\frac{n}{2\pi e}\right)^{-1} \ll n^{-1/2}.$$

It follows that for each $\varepsilon > 0$,

(78) $$\mu\left\{L \in \mathcal{L}_n^\circ \;\middle|\; J_{R_n}(L) > \varepsilon\right\} \to 0 \qquad \text{as } n \to \infty.$$

Combining (78) with (77) we have

Proposition 8. *For each $\varepsilon > 0$,*

$$\mu\left\{L \in \mathcal{L}_n^\circ \;\middle|\; J(L) > \varepsilon\right\} \to 0 \qquad \textit{as } n \to \infty.$$

Since the function $F_0(L)$ in (50) is equal to $H(L) + J(L)$, we may conclude from Propositions 7 and 8 that for each $\varepsilon > 0$,

$$\mu\left\{L \in \mathcal{L}_n^\circ \;\middle|\; |F_0(L) - 1| > \varepsilon\right\} \to 0 \qquad \text{as } n \to \infty.$$

Recalling the formula (49) for the height we have established our main result of this section:

(79) $$\mu\left\{L \in \mathcal{L}_n^\circ \;\middle|\; |h(L) - (\log 4\pi - \gamma + 1)| > \varepsilon\right\} \to 0 \qquad \text{as } n \to \infty.$$

In other words, we have now completed the proof of Theorem 3 stated in the introduction.

If $m_n = \min\{h(L) \mid L \in \mathcal{L}_n^\circ\}$ then according to (79) and (65) we have that

(80) $$\log 4\pi - \gamma - \frac{2}{n} < m_n \leqq \log 4\pi - \gamma + 1 + o(1).$$

Hence Corollary 1 is now proved.

By way of comparison, Theorem 2 asserts that

$$m_3 = h(D_3) = \log 4\pi - \gamma - \frac{2}{3} + 0.113359\ldots,$$

while we expect that

$$m_{24} = h(L_{24}) = \log 4\pi - \gamma - \frac{1}{12} + 0.270863\ldots.$$

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