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Nonsingular star flows satisfy Axiom A and the no-cycle condition⋆

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Dedicated to Shaotao Liao and Ricardo Mañé

Abstract. We give an affirmative answer to a problem of Liao and Mañé which asks whether, for a nonsingular flow to loose the Ω-stability, it must go through a critical-element-bifurcation. More precisely, a vector field S on a compact boundaryless manifold is called a star system if S has a C^1 neighborhood $\mathcal{U}$ in the set of C^1 vector fields such that every singularity and every periodic orbit of every $X \in \mathcal{U}$ is hyperbolic. We prove that any nonsingular star flow satisfies Axiom A and the no cycle condition.

1 Introduction

Let M be a compact d-dimensional C^∞ Riemannian manifold without boundary. Denote by $\mathcal{X}(M)$ the set of C^1 vector fields on M, endowed with the C^1 topology. Denote $\phi_t = \phi_{Xt} : M \to M$ the flow generated by $X \in \mathcal{X}(M)$. Singularities and periodic orbits, sometimes called *critical elements*, are the simplest orbits of a flow. They are special kind of the so called nonwandering orbits. Recall a point $x \in M$ is called *nonwandering* of $X \in \mathcal{X}(M)$ if for any neighborhood U of x in M, there is $t \geq 1$ such that $\phi_t(U) \cap U \neq \emptyset$. Denote the nonwandering set of X by $\Omega(X)$, which then contains the recurrence and the long run behavior of all orbits of X. A vector field X is called Ω-*stable* if, briefly, small perturbations of X can

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Mathematics Subject Classification (2000): 37D30 not change the topological structure of $\Omega(X)$. There had been the following somewhat-informal problem in the literature, though exact references are seemingly hard to specify: For a flow to loose the Ω-stability, must it go through a critical-element-bifurcation? In other words, having robustly no critical-element-bifurcation, must a flow be Ω-stable? Let us be more precise.

A vector field $S \in \mathcal{X}(M)$ is called a *star vector field* or a *star flow*, denoted by $S \in \mathcal{X}^*(M)$, if S has a C^1 neighborhood $\mathcal{U}$ in $\mathcal{X}(M)$ such that every singularity and every periodic orbit of every $X \in \mathcal{U}$ is hyperbolic. Thus a star flow is exactly one that has robustly no critical-element-bifurcation. Since the definition concerns critical elements only, and since the hyperbolicity put on critical elements is merely orbit-wise but not uniform, the star condition looks, a priori, quite weak. It is not surprising that the Axiom A plus no-cycle condition, which is necessary and sufficient for a flow to be Ω-stable, looks much stronger.

Let us quickly recall the definition of Axiom A plus no-cycle condition, put on the nonwandering set. We say X satisfies *Axiom A* if $\Omega(X)$ is hyperbolic, and if $\Omega(X) = \overline{\mathrm{Sing}(X) \cup \mathrm{P}(X)}$, where $\mathrm{Sing}(X)$ and $\mathrm{P}(X)$ denote the sets of singularities and periodic points of X, respectively. Here a compact invariant set $\Lambda \subset M$ is called *hyperbolic* for X if $T_\Lambda M$ has a continuous $d\phi_t$-invariant splitting $E^s \oplus < X > \oplus E^u$, where $< X >$ denotes the 1-dimensional subspace spanned by the vector field X, such that for two uniform constants $\lambda > 0,\ T > 0$,

$$\left\| d\phi_t \middle| E^s(x) \right\| \le e^{-\lambda t} \qquad \text{and} \qquad \left\| d\phi_{-t} \middle| E^u(x) \right\| \le e^{-\lambda t}$$

for all $x \in \Lambda$ and $t \ge T$. If X satisfies Axiom A, then $\Omega(X)$ decomposes into a finite disjoint union of transitive sets $\Omega(X) = \Lambda_1 \cup \cdots \cup \Lambda_l$, called the *basic sets* of X [31]. A collection of basic sets $\Lambda_{i_1}, \ldots, \Lambda_{i_k}$ of X is called a *cycle*, if there exist points $a_j \notin \Omega(X),\ 1 \le j \le k$, such that $\alpha(a_j) \subset \Lambda_{i_j}$ and $\omega(a_j) \subset \Lambda_{i_{j+1}}$ $(k+1 \equiv 1)$. An Axiom A vector field X is said to satisfy the *no-cycle condition* if there are no cycles among the basic sets of X.

In terms of these terminologies, the above somewhat informal problem can be formally stated as follows.

Problem 1 Does every star flow satisfy Axiom A and the no-cycle condition?

The problem is striking. An affirmative answer to it would amount to an extension of the famous stability conjecture and Ω-stability conjecture of Palis and Smale [26] because, as assumptions, structural stability implies Ω-stability, which in turn implies the star condition.

For diffeomorphisms, the answer to Problem 1 is proved to be affirmative indeed (here, likewise, a diffeomorphism f is called a *star diffeomorphism*, denoted by $f \in \mathcal{F}^*(M)$, if f has a C^1 neighborhood $\mathcal{U}$ in $\mathrm{Diff}(M)$ such that every periodic orbit of every $g \in \mathcal{U}$ is hyperbolic). That is, for diffeomorphisms, the star condition is proved to imply, hence to be equivalent to, Axiom A plus the no-cycle condition, see Liao [17] and Mañé [20] for dimension 2, and Aoki [1] and Hayashi [10] for general dimensions.

However, for flows, the answer to Problem 1 is negative. A star flow may fail to have hyperbolic nonwandering set as the famous Lorenz attractor shows [9], or fail to have the critical elements dense in the nonwandering set [4] or, even with Axiom A satisfied, still fail to satisfy the no-cycle condition [14]. Thus, for flows, to have robustly no critical-element-bifurcation is far from being Ω-stable.

Nevertheless, all these counterexamples for star flows exhibit singularities. Liao and Mañé hence raised the following problem for nonsingular star flows:

Problem 2 (Liao [17], Mañé [20]) Does every nonsingular star flow S satisfy Axiom A and the no-cycle condition?

Liao emphasized this long-standing problem for several times, see for instance [19], p. 319. For dimension 3, Liao [17] solved Problem 2 affirmatively (for dimensions ≤ 2 an affirmative answer is contained in the classical work of Peixoto [27]). For general dimensions, Problem 2 remained open. The proof of the problem for star diffeomorphisms does not carry over to nonsingular star flows. One thing that causes the difference is a simple fact that, for flows, periods of periodic orbits are not necessarily integers. When a continuous arc of hyperbolic periodic orbits P_λ of a continuous arc of flows X_λ vary as the parameter λ varies, the periods of P_λ do not have to be kept the same as in the case of diffeomorphisms. In fact the periods may sweep to infinity with an arbitrarily small change of parameter λ. In other words, while for diffeomorphisms, a hyperbolic periodic orbit can not disappear through such an arc of hyperbolic periodic orbits, generally it can for flows, see the remarks of Liao in [17, p. 35] and Mañé in [20, p. 508]. On the other hand, with some C^1 generic assumptions added, the proof for diffeomorphisms does carry over to flows, see [7, 13].

In this paper we push forward the methods of Liao and Mañé to obtain an affirmative answer to Problem 2 for general dimensions:

Theorem A *Every nonsingular star flow satisfies Axiom A and the no-cycle condition.*

Let us say some words about the proof of Theorem A. The proof will take the so called minimally rambling sets approach of Liao. In this approach, to prove that a set Γ (for instance the nonwandering set) is hyperbolic, one does not have to handle the whole set Γ globally, but only has to disprove the existence of minimally rambling sets in Γ, which are of a relatively less global nature, see Sect. 2 for more details. Throughout the proof of Theorem A, a great deal of the fundamental work of Liao and Mañé will be needed. Indeed, we will see the mark and influence of Liao and Mañé everywhere in this paper.

There is a refined version of Theorem A, which is what we will prove precisely in Sect. 7 below. It does not totally ignore singularities, but requires that singularities do not appear in the so called preperiodic set.

Theorem A′ *If $S \in \mathcal{X}^*(M)$, and if the C^1 preperiodic set $\mathrm{P}_*(S)$ of S is free of singularities, then S satisfies Axiom A and the no-cycle condition.*

Here a point $x \in M$ is called *preperiodic* of S, if there is a sequence of points $x_n \in P_n$ such that $x_n \to x$, where P_n is a sequence of periodic orbits of Y_n with $Y_n \to S$ in the C^1 topology. Denote by $\mathrm{P}_*(S)$ the set of preperiodic points of S. Clearly $\mathrm{P}_*(S)$ is compact and S-invariant. Note that

$$\Omega(S) - \mathrm{Sing}(S) \subset \mathrm{P}_*(S)$$

by the C^1 closing lemma.

A key step in proving Theorem A is the following Theorem B. We single it out and state it in a flexible way for some possible general use.

Theorem B *Let $S \in \mathcal{X}^*(M)$. Let $\Gamma \subset \overline{\mathrm{P}}(S)$ be a compact invariant set of S such that $\Gamma \cap \mathrm{Sing}(S) = \emptyset$. Then Γ is hyperbolic.*

This paper is organized as follows. In Sect. 2 we present a principle of Liao for proving hyperbolicity and give an outline of the proof of Theorem A. In Sect. 3 we improve a result of Liao on non-simple minimally rambling sets for star flows. In Sect. 4 we show that star flows exhibit no heterodimensional cycles. In Sect. 5 we prove a result about determination of the index of preperiodic points. In Sect. 6 we prove Theorem B by producing a heterodimensional cycle using the connecting lemma. In Sect. 7 we prove Theorem A by a generalized shadowing lemma.

2 An outline for the proof of Theorem A

In this section we give an outline for the proof of Theorem A. We first introduce in a simplified version the beautiful idea of Liao on minimally rambling sets.

A compact invariant set Λ of $S \in \mathcal{X}(M)$ is called *minimally non-hyperbolic* if Λ is non-hyperbolic, but every nonempty compact invariant proper subset of Λ is hyperbolic. This notion resembles the notion of minimally rambling set of Liao [17], and plays an important role in the remarkable work of Pujals and Sambarino [29]. There is an introduction for the notion of minimally non-hyperbolic set in Wen [37]. The following lemma is an easy consequence of the robustness of hyperbolic sets, see [17,29] for a proof.

Lemma 2.1 *Every non-empty non-hyperbolic set contains at least one minimally non-hyperbolic set.*

We follow Liao [17] to divide minimally non-hyperbolic sets without singularities into two types, in a slightly different way by using the following Proposition 2.2. Since all minimally non-hyperbolic sets considered in this paper will be free of singularities, it will be convenient to use the linear Poincaré flow defined as follows. A vector field $X \in \mathcal{X}(M)$ generates a C^1 flow $\phi_t = \phi_{Xt} : M \to M,\ t \in \mathbf{R}$, together with the tangent flow $\mathrm{d}\phi_t : TM \to TM$. Denote

$$D = D_X = \{v \in T_xM : \langle v, X(x)\rangle = 0 \quad \text{and} \quad x \in M - \mathrm{Sing}(X)\}.$$

Projected to D, the tangent flow $\mathrm{d}\phi_t$ naturally induces the *linear Poincaré flow* $\psi_t = \psi_{Xt} : D \to D$ of X, defined to be

$$\psi_t(v) = \pi(\mathrm{d}\phi_t(v)) \text{ for any } v \in D,$$

where π is the orthogonal projection to D. For any linear subspace $A \subset D(x),\ t > 0$, denote

$$\eta_-(A, t) = \eta_-(X, A, t) = \sup_{u \in A, \|u\|=1} \{\log \|\psi_{Xt}(u)\|\},$$

$$\eta_+(A, t) = \eta_+(X, A, t) = \inf_{u \in A, \|u\|=1} \{\log \|\psi_{Xt}(u)\|\}.$$

It is well known ([16]) that a compact invariant set $\Lambda \subset M - \mathrm{Sing}(X)$ is hyperbolic if and only if D_Λ has a continuous invariant splitting $D_\Lambda = E \oplus F$ such that for some two uniform constants $\eta > 0, T > 0$, the rates for the linear Poincaré flow satisfy

$$\eta_-(X, E(x), t) \le -\eta t \quad \text{and} \quad \eta_+(X, F(x), t) \ge \eta t$$

for all $x \in \Lambda$ and $t \ge T$. We will say Λ is *of index* i, $0 \le i \le d-1$, denoted as $\mathrm{Ind}(\Lambda) = i$, if $\dim E(x) = i$ for all $x \in \Lambda$. It is easy to see if $\mathrm{Ind}(\Lambda) = 0$, then Λ is the union of finitely many expanding periodic orbits. Likewise for the case of $\mathrm{Ind}(\Lambda) = d - 1$. For $x \in M - \mathrm{Sing}(X)$, denote

$$D^s(x) = D^s(x, X) = \left\{v \in D(x, X) : \lim_{t \to +\infty} \|\psi_{Xt}(v)\| = 0\right\},$$

and

$$D^u(x) = D^u(x, X) = \left\{v \in D(x, X) : \lim_{t \to -\infty} \|\psi_{Xt}(v)\| = 0\right\}.$$

These two linear subspaces are defined at every non-singular point $x \in M$. In particular, if $\Lambda \subset M - \mathrm{Sing}(X)$ is hyperbolic, then $D^s(x) = E(x)$ for all $x \in \Lambda$. The following proposition is an equivalent characterization for hyperbolic sets without singularities, due to Selgrade [30], Sacker and Sell [33], Mañé [22] and Liao [16].

Proposition 2.2 *A compact invariant set* $\Lambda \subset M - \mathrm{Sing}(X)$ *of* X *is hyperbolic if and only if*

$$D(x) = D^s(x) \oplus D^u(x), \tag{2.1}$$

for any $x \in \Lambda$.

A point $x \in M - \mathrm{Sing}(X)$ will be called *resisting* of X if x does not satisfy the condition (2.1). Thus by Proposition 2.2, every nonsingular non-hyperbolic set contains at least one resisting point. Now we divide minimally non-hyperbolic sets without singularities into two classes. A minimally non-hyperbolic set Λ without singularities will be called of *simple type* if there is a resisting point $a \in \Lambda$ such that both $\alpha(a)$ and $\omega(a)$ are proper subsets of Λ. Otherwise, Λ will be called *non-simple type*.

A simple type minimally non-hyperbolic set Λ without singularities has a clear feature. Being a proper subset of Λ, both $\alpha(a)$ and $\omega(a)$, where $a \in \Lambda$ is a resisting point, are hyperbolic. It is then easy to see $\Lambda = \alpha(a) \cup \mathrm{Orb}(a) \cup \omega(a)$. See [37] for more details. Thus Λ is like a heteroclinic connection. The structure for a non-simple type minimally non-hyperbolic set without singularities has not been well understood in general, besides by definition for every resisting point $a \in \Lambda$, either $\Lambda = \omega(a)$, or $\Lambda = \alpha(a)$. For star flows however, Liao [17] obtains enough information for non-simple type minimally non-hyperbolic sets without singularities, which makes the following principle of Liao very powerful.

Principle Let $\Gamma \subset M - \mathrm{Sing}(X)$ be a compact invariant set of X. To prove that Γ is hyperbolic, it suffices to rule out the existence of the two kinds of nonsingular minimally non-hyperbolic subsets contained in Γ.

Now we give an outline for the proof of Theorem A. In Sect. 3 we improve a result of Liao on non-simple type minimally rambling sets for star flows. It is based on several deep results of Liao and Mañé on the stability conjectures, which will be used also in later sections. In Sect. 4 we prove a basic property about star flows, that is, any star flow exhibits no heterodimensional cycles. In particular, this will provide a basis for us to apply the principle of Liao by creating a heterodimensional cycle. While the result seems to be very natural, the calculations are quite involved. In Sect. 5 we prove a result about index-determination for preperiodic points, hence index-determination for dominated splittings. It will be done via creation of homoclinic orbits, by C^1 perturbations. Since we have to confirm that the created homoclinic orbit passes near some given point, rather than just to create a homoclinic orbit, more work will be involved. In Sect. 6 we prove Theorem B by using the principle of Liao. It suffices to rule out the existence of minimally non-hyperbolic sets contained in $\overline{\mathrm{P}}(S) - \mathrm{Sing}(S)$. For a simple type minimally non-hyperbolic set Λ, we try to create out of it a heterodimensional cycle, which would contradict the result of Sect. 4. Though the details are tedious, the idea is very natural: We know by the minimality of non-hyperbolicity, $\Lambda = \alpha(a) \cup \mathrm{Orb}(a) \cup \omega(a)$ is a heteroclinic connection already (going from $\alpha(a)$ to $\omega(a)$), what we do is hence to, with the help of $\overline{\mathrm{P}}(S)$, create by perturbations a second heteroclinic connection (going from $\omega(a)$ to $\alpha(a)$) without breaking the first one. Of course we have to reduce the two hyperbolic sets $\omega(a)$ and $\alpha(a)$ to two hyperbolic periodic orbits P and Q, and to confirm that P and Q have different indices. For a non-simple type minimally non-hyperbolic set Λ, we make an intensive use of the results prepared in Sects. 3 and 5. We will see that the minimality of non-hyperbolicity again plays a crucial role. In Sect. 7 we prove Theorem A. Now $\overline{\mathrm{P}}(S)$ is hyperbolic already by Theorem B. We prove if S fails to satisfy Axiom A and the no-cycle condition, by using a general shadowing lemma, there would be some periodic orbit of S outside $\overline{\mathrm{P}}(S)$, an obvious contradiction.

3 Fundamental sequences and fundamental limits

In this section we improve a classical result of Liao [17] on non-simple type minimally rambling sets. Let $X \in \mathcal{X}(M)$. Following Liao [17], we will call (P_n, Y_n) a *fundamental i-sequence* of X, where P_n is a periodic orbit of $Y_n \in \mathcal{X}(M)$ of index $0 \le i \le d-1$, if $Y_n \to X$ in the C^1 topology, and if P_n converge in the Hausdorff metric. The Hausdorff limit Λ of P_n will be called a *fundamental i-limit* of X. It is easy to see that any fundamental limit Λ of X is (compact and) X-invariant. Generally a fundamental i-limit may intersect a fundamental j-limit for $i \ne j$. Thus the "index" i for a fundamental limit may not be unique. Fundamental sequences and limits appear naturally in the process of creation of periodic orbits by C^1 perturbations. For instance, by the C^1 closing lemma, any nonwandering point $x \in M - \mathrm{Sing}(X)$ is contained in a fundamental limit of X. If $x \in M - \mathrm{Sing}(X)$ is recurrent, say $x \in \omega(x)$, then $\omega(x)$ even equals a fundamental limit of X [19, p. 257]. Corresponding to the notion of fundamental i-limits, which is at the level of sets, is the notion of i-preperiodic points at the level of points. Recall a point $x \in M$ is called *i-preperiodic* of X, where $0 \le i \le d-1$, if there is a fundamental i-sequence (Y_n, P_n) of X and a sequence of points $x_n \in P_n$ such that $x_n \to x$. A point could be i-preperiodic as well as j-preperiodic, for $i \ne j$. Denote by $\mathrm{P}_*^i(X)$ the set of i-preperiodic points of X. Clearly $\mathrm{P}_*^i(X)$ is compact and X-invariant, and equals the union of the set of fundamental i-limits of X. Denote by $\mathrm{P}_*(X) = \cup_{i=0}^{d-1} \mathrm{P}_*^i(X)$ the set of all preperiodic points of X. Thus $\Omega(X) - \mathrm{Sing}(X) \subset \mathrm{P}_*(X) \subset \mathrm{R}(X)$, where $\mathrm{R}(X)$ denotes the chain recurrent set of X.

A fundamental limit may contain singularities, a phenomenon that causes tremendous complexity and difficulty in question. However, the following Lemma 3.11 asserts that, for star flows, a fundamental limit, if free of singularities, contains quite some information. It improves a result of Liao on non-simple type minimally rambling sets ([19, Theorem 6.5.7]). The proof is based on several deep results of Liao and Mañé we now collect, some of which will be used below in later sections too. The first one is taken from Liao [15], which concerns the rates on the stable and unstable subspaces of periodic orbits for star flows. Analogous results for diffeomorphisms can be found in Mañé [20].

Theorem 3.1 [15] *Let $S \in \mathcal{X}^*(M)$. Then S has a neighborhood $\widetilde{\mathcal{U}}$ in $\mathcal{X}^*(M)$, together with two uniform constants $\widetilde{\eta} > 0$ and $\widetilde{T} > 1$ such that if $X \in \widetilde{\mathcal{U}}$ then*

(i) *Whenever x is a point on a periodic orbit of X and $\widetilde{T} \leq t < \infty$, one has*
$$\frac{1}{t}\big[\eta_+(X, D^u(X,x), t) - \eta_-(X, D^s(X,x), t)\big] \geq 2\widetilde{\eta};$$

(ii) *When P is a periodic orbit of X with period T, $x \in P$, and for an integer $m \geq 1$, $0 = t_0 < t_1 < \cdots < t_l = mT$ is a partition of $[0, mT]$ satisfying*
$$t_k - t_{k-1} \geq \widetilde{T}, \quad k = 1, 2, \cdots, l,$$
one has
$$\frac{1}{mT}\sum_{k=0}^{l-1} \eta_-\big(X, D^s(X, \phi_{Xt_k}(x)), t_{k+1} - t_k\big) \leq -\widetilde{\eta} \tag{3.2}$$
and
$$\frac{1}{mT}\sum_{k=0}^{l-1} \eta_+\big(X, D^u(X, \phi_{Xt_k}(x)), t_{k+1} - t_k\big) \geq \widetilde{\eta}. \tag{3.3}$$

The following result of Liao will be used in the proof of Lemma 3.4 below.

Lemma 3.2 [19] *Let $X \in \mathcal{X}(M)$ and Λ be a closed invariant set of X with $\Lambda \subset M - \mathrm{Sing}(X)$. Assume that for some $T > 0$ and some ϕ_T-invariant probability measure μ on Λ,*
$$\int_\Lambda \eta_-(D(x), T)d\mu < 0. \tag{3.4}$$
Then Λ contains a contracting periodic orbit of X. If the inequality (3.4) *is replaced by*
$$\int_\Lambda \eta_+(D(x), T)d\mu > 0, \tag{3.5}$$
then Λ contains an expanding periodic orbit of X.

We state without proof a simple fact that will be used several times below.

Lemma 3.3 *If a fundamental limit Λ of X intersects the basin of a contracting periodic orbit P of X, then $\Lambda = P$. Likewise for an expanding periodic orbit.*

The following result of Liao concerns fundamental 0-limits and $(d-1)$-limits.

Lemma 3.4 [15] *Let $S \in \mathcal{X}^*(M)$. Let Λ be a fundamental 0-limit of S with $\Lambda \cap \mathrm{Sing}(S) = \emptyset$. Then Λ is an expanding periodic orbit of S. Likewise, if Λ is a fundamental $(d-1)$-limit of S with $\Lambda \cap \mathrm{Sing}(S) = \emptyset$, then Λ is a contracting periodic orbit of S.*

Proof We take the case of $d-1$. Let Λ be a fundamental $(d-1)$-limit of S, which is the Hausdorff limit of a fundamental $(d-1)$-sequence (P_n, X_n). Denote $f_n = \phi_{X_n \widetilde{T}}$, $f = \phi_{\widetilde{T}}$. Given $x_n \in P_n$, let μ_n be a limit point of

$$\frac{1}{l}\sum_{i=0}^{l-1} \delta_{f_n^i(x_n)},$$

where δ_x is the atomic measure at x. μ_n is an invariant measure of f_n and $\mathrm{supp}(\mu_n) \subset P_n$. According to Theorem 3.1, it is easy to see that

$$\frac{1}{\widetilde{T}}\int \eta_-(X_n, D(X_n, x), \widetilde{T})d\mu_n(x) \le -\widetilde{\eta}.$$

We may assume that $\mu_n \to \mu$. Then μ is an invariant probability measure of f and $\mathrm{supp}(\mu) \subset \Lambda$. Since $\eta_-(X_n, D(X_n, x), \widetilde{T}) \to \eta_-(D(x), \widetilde{T})$, we have

$$\frac{1}{\widetilde{T}}\int \eta_-(D(x), \widetilde{T})d\mu(x) \le -\widetilde{\eta}.$$

Then by Lemma 3.2, Λ contains a contracting periodic orbit P of S. By Lemma 3.3, $\Lambda = P$. □

Recall a compact invariant set Λ of S that contains no singularities of S is said to have (η, T, i)-*dominated splitting* (or simply i-dominated splitting), where $1 \le i \le d-2$, if there exists a continuous invariant bundle splitting $D_\Lambda = E \oplus F$ with $\dim E = i$, together with two uniform constants $\eta > 0$ and $T > 1$ such that

$$\eta_+(F(x), t) - \eta_-(E(x), t) \ge 2\eta t$$

for any $x \in \Lambda$ and $t \ge T$.

Note that unlike the case of fundamental limits or preperiodic points, where the index i runs from 0 to $d-1$, here for dominated splittings the index runs from 1 to $d-2$, that is, neither E nor F is 0-dimensional. Also note that a compact invariant set may admit more than one dominated splittings. Nevertheless for a given index i the i-dominated splitting is unique, because dominated splittings are "nested":

Lemma 3.5 *Let $E \oplus F$ and $E' \oplus F'$ be two dominated splittings at x with $\dim E = i$ and $\dim E' = i'$.*

1) *If* $i \le i'$, *then* $E \subset E'$ *and* $F \supset F'$. *In particular, if* $i = i'$, *then* $E = E'$ *and* $F = F'$.
2) *If* $i \le \dim D^s(x)$, *then* $E \subset D^s(x)$ *and if* $i \ge \dim D^s(x)$, *then* $E \supset D^s(x)$. *In particular, if* $i = \dim D^s(x)$, *then* $E = D^s(x)$. *There is a similar relation for* F *and* $D^u(x)$.

Proof 1) is Proposition 6.4.1 in [17]. Our formulation is taken from [35, Lemma 3.7]. 2) is Lemma 3.7 in [7]. □

Since we study star flows throughout the paper, for a given $S \in \mathcal{X}^*(M)$, we will fix the neighborhood $\widetilde{\mathcal{U}}$ of S and the two uniform constants $\widetilde{T}$ and $\widetilde{\eta}$ guaranteed by Theorem 3.1. The following corollary of Theorem 3.1 will be frequently used below in this paper.

Lemma 3.6 *Let* $S \in \mathcal{X}^*(M)$ *and* $1 \le i \le d-2$. *Assume* $X \in \widetilde{\mathcal{U}}$ *and* $\Lambda \subset \mathrm{P}_*^i(X)$ *is a closed invariant set of* X *with* $\Lambda \cap \mathrm{Sing}(X) = \emptyset$. *Then there exists a* $(\widetilde{\eta}, \widetilde{T}, i)$*-dominated splitting on* Λ.

Another result we will need is the ergodic closing lemma of Mañé (see [35] for the flow version). Recall a point $x \in M - \mathrm{Sing}(S)$ is *strongly closable* for S if for any C^1 neighborhood $\mathcal{U}$ of S in $\mathcal{X}$, and any $\delta > 0$, there are $X \in \mathcal{U}$, $z \in M$, $\tau > 0$, $L > 0$ such that the following three conditions hold:

(a) $\phi_{X\tau}(z) = z$.
(b) $d(\phi_t(x), \phi_{Xt}(z)) < \delta$, for any $0 \le t \le \tau$.
(c) $X = S$ on $M - B(\phi_{[-L,0]}(x), \delta)$, where $B(\phi_{[-L,0]}(x), \delta)$ denotes the closed δ-ball of the orbital segment $\phi_{[-L,0]}(x)$.

The set of strongly closable points of S will be denoted by $\Sigma(S)$. Note that a strongly closable point must be recurrent.

Theorem 3.7 [35] *For any* $S \in \mathcal{X}(M)$, $\mu(\mathrm{Sing}(S) \cup \Sigma(S)) = 1$ *for every* $T > 0$ *and every* ϕ_T*-invariant Borel probability measure* μ.

We also need the following result, which is a reformulation of Theorem II.1 in Mañé [21].

Theorem 3.8 [21] *Let* $S \in \mathcal{X}^*(M)$. *Let* Λ *be a compact invariant set of* S *with* $\Lambda \cap \mathrm{Sing}(S) = \emptyset$ *such that* $\Omega(S|_\Lambda) = \Lambda$. *Let* $D_\Lambda = E \oplus F$ *be an* $(\widetilde{\eta}, \widetilde{T}, p)$*-dominated splitting on* Λ, $1 \le p \le d-2$. *Assume* E *is contracting and assume there exists* $\eta > 0$ *such that for points* x *in a dense set of* Λ *the following condition is satisfied:*

$$\limsup_{n\to+\infty} \frac{1}{n\widetilde{T}} \sum_{j=0}^{n-1} \eta_+\big(F(\phi_{j\widetilde{T}}(x)), \widetilde{T}\big) \ge \eta. \tag{3.6}$$

Then F *is expanding, i.e.,* Λ *is hyperbolic of index* p.

Roughly, for a dominated splitting $E \oplus F$ on such a set Λ (with good recurrence), if E is contracting and if F is, even non-uniformly, expanding on a dense subset of Λ, then F is actually (uniformly) expanding. We remark that a point in the proof (Lemma II.6 in [21]) was clarified in [24] and [39] by different methods.

The following lemma is just Lemma I.5 of Mañé [21].

Lemma 3.9 [21] *Let Λ be a compact invariant set of $f \in \mathrm{Diff}^1(M)$ and $E \subset TM|_\Lambda$ be a continuous invariant subbundle. If there exists $m > 0$ such that*

$$\int \log \|(Df^m)|_E\| d\mu < 0$$

for every ergodic $\mu \in \mathcal{M}(f^m|_\Lambda)$, then E is contracting.

The following result can be singled out from Mañé [20].

Lemma 3.10 [20] *Let $S \in \mathcal{X}^*(M)$. Let $\Lambda \subset M - \mathrm{Sing}(S)$ be a compact invariant set of S that admits a p-dominated splitting $D_\Lambda = E \oplus F$, where $1 \le p \le d-2$. If E is not contracting, then there is a fundamental r-limit contained in Λ with $r < p$.*

Proof Since E is not contracting, according to Lemma 3.9, there exists an ergodic $\phi_{\widetilde{T}}$-invariant measure μ such that

$$\int \eta_-(E(x), \widetilde{T}) d\mu(x) \ge 0$$

and $\mathrm{supp}(\mu) \subset \Lambda$. Then, according to Theorem 3.7,

$$\int_{\Lambda \cap \Sigma(S)} \eta_-(E(x), \widetilde{T}) d\mu(x) \ge 0.$$

So we can find a point $a \in \Lambda \cap \Sigma(S)$ such that

$$\lim_{m \to +\infty} \frac{1}{m\widetilde{T}} \sum_{j=0}^{m-1} \eta_-\big(E(\phi_{j\widetilde{T}}(a)), \widetilde{T}\big) \ge 0. \tag{3.7}$$

Since $a \in \Sigma(S)$, for any $n > 0$, there exist $Y_n \in \mathcal{X}^*(M)$, $b_n \in M$, $\tau_n > 0$ such that

a) $\phi_{Y_n \tau_n}(b_n) = b_n$, and τ_n is the minimum period of b_n.
b) $d(\phi_t(a), \phi_{Y_n t}(b_n)) \le 1/n$, for any $0 \le t \le \tau_n$, and
c) $\|Y_n - S\|_{C^1} \le 1/n$.

Taking subsequences if necessary, we may assume $Q_n = \mathrm{Orb}(b_n, Y_n)$ have the same index, say r, for all n, and (Y_n, Q_n) is a fundamental r-sequence that converge to $\Gamma \subset \Lambda$. That is, Γ is a r-fundamental limit. We prove $r < p$.

Suppose $r \geq p$. First note that a is not periodic. In fact, if a is periodic, according to Theorem 3.1, we would have

$$\limsup_{m\to+\infty} \frac{1}{m\widetilde{T}} \sum_{j=0}^{m-1} \eta_-\big(D^s(\phi_{j\widetilde{T}}(a)), \widetilde{T}\big) \leq -\widetilde{\eta}.$$

By Lemma 3.5, $r \geq p$ would imply $E(\phi_t(a)) \subset D^s(\phi_t(a))$. Hence

$$\limsup_{m\to+\infty} \frac{1}{m\widetilde{T}} \sum_{j=0}^{m-1} \eta_-\big(E(\phi_{j\widetilde{T}}(a)), \widetilde{T}\big) \leq -\widetilde{\eta}.$$

This contradicts (3.7), proving a is not periodic. This implies $\lim_{n\to\infty} \tau_n = \infty$.

By (3.7), we can take m_0 large enough so that for any $m > m_0$,

$$\sum_{j=0}^{m-1} \eta_-\big(E(\phi_{j\widetilde{T}}(a)), \widetilde{T}\big) \geq -m\widetilde{T}\widetilde{\eta}/3. \tag{3.8}$$

Take a small closed neighborhood $U \subset M - \mathrm{Sing}(S)$ of Λ and a small neighborhood $\mathcal{U}$ of S such that for any $X \in \mathcal{U}$ and any closed invariant set $\Gamma \subset U$ of X, there is a dominated splitting $D_X(x) = E(X, x) \oplus F(X, x)$ for each $x \in \Gamma$ with index p. Since $\eta_-(X, E(X, \phi_{Xt}(x)), \widetilde{T})$ is continuous with respect to X and x, there exists n_0 large enough so that for any $n > n_0$, and $d(x, y) \leq 1/n$, once $E(x)$ is well-defined for Y_n and $E(y)$ is well-defined for S, we have for $t \in [0, 2\widetilde{T}]$,

$$|\eta_-(Y_n, E(Y_n, x), t) - \eta_-(E(y), t)| < \widetilde{T}\widetilde{\eta}/3. \tag{3.9}$$

Denote by

$$K = \sup_{n>n_0, t\in[0,2\widetilde{T}], x\in U} |\log \|\psi_{Y_n t}(x)\||.$$

Let $\tau_n = m_n\widetilde{T} + s_n$, $m_n \in \mathbf{Z}$, $s_n \in [0, \widetilde{T})$. Considering the partition $0 = t_0 < t_1 = \widetilde{T} < \cdots < t_{m_n-1} = (m_n - 1)\widetilde{T} < t_{m_n} = \tau_n$, according Theorem 3.1,

$$\sum_{j=0}^{m_n-2} \eta_-\big(Y_n, D^s(Y_n, \phi_{Y_n(j\widetilde{T})}(b_n)), \widetilde{T}\big) \leq -\tau_n\widetilde{\eta} + K.$$

Since $r \geq p$, by Lemma 3.5, $D^s(Y_n, \phi_{Y_n t}(b_n)) \supset E(Y_n, \phi_{Y_n t}(b_n))$, we have

$$\sum_{j=0}^{m_n-2} \eta_-\big(Y_n, E(Y_n, \phi_{Y_n(j\widetilde{T})}(b_n)), \widetilde{T}\big) \leq -\tau_n\widetilde{\eta} + K \leq -(m_n - 1)\widetilde{T}\widetilde{\eta} + K. \tag{3.10}$$

And since $d(\phi_{Y_n t}(b_n), \phi_t(a)) < 1/n$ for $0 \le t \le \tau_n$, from (3.9) and (3.10), we get

$$\sum_{j=0}^{m_n-2} \eta_-\big(E(\phi_{j\widetilde{T}}(a)), \widetilde{T}\big) \le -2(m_n-1)\widetilde{T}\widetilde{\eta}/3 + K.$$

Combining with (3.8), for n large enough so that $m_n - 1 > m_0$, we have

$$-(m_n-1)\widetilde{T}\widetilde{\eta}/3 \le -2(m_n-1)\widetilde{T}\widetilde{\eta}/3 + K.$$

But for n large enough the above inequality is impossible. This proves Lemma 3.10. □

Now we state and prove the main result of this section.

Lemma 3.11 *Let $S \in \mathcal{X}^*(M)$. Let Λ be a fundamental p-limit of S with $\Lambda \cap \mathrm{Sing}(S) = \emptyset$, where $1 \le p \le d-2$. Let $D_\Lambda = E \oplus F$ be the associated p-dominated splitting on Λ. Then,*

1) *if E is contracting, then Λ contains a hyperbolic subset of index p; if E is not contracting, then Λ contains a hyperbolic subset of index $< p$.*
2) *if F is expanding, then Λ contains a hyperbolic subset of index p; if F is not expanding, then Λ contains a hyperbolic subset of index $> p$.*
3) *if Λ is not hyperbolic, then Λ contains two hyperbolic subsets of different indices.*

We remark that Lemma 3.4 could be regarded as a complement of Lemma 3.11.

Proof We only prove 1), because 2) can be proved by reversing the time, and 3) is a direct consequence of 1) and 2).

First assume E is contracting. Let (P_n, X_n) be a fundamental p-sequence of S that converge to Λ in the Hausdorff metric. Denote $f_n = \phi_{X_n\widetilde{T}}$, $f = \phi_{\widetilde{T}}$. Given $x_n \in P_n$, as we did in the proof of Lemma 3.4, let μ_n be a limit point of

$$\frac{1}{l}\sum_{i=0}^{l-1} \delta_{f_n^i(x_n)},$$

where δ_x is the atomic measure at x. μ_n is an invariant measure of f_n and $\mathrm{supp}(\mu_n) \subset P_n$. According to Theorem 3.1, it is easily seen that

$$\frac{1}{\widetilde{T}}\int \eta_+\big(X_n, D^u(X_n, x), \widetilde{T}\big)d\mu_n(x) \ge \widetilde{\eta}.$$

We may assume that $\mu_n \to \mu$. Then μ is an invariant probability measure of f and $\mathrm{supp}(\mu) \subset \Lambda$. Since $\eta_+(X_n, D^u(X_n, x), \widetilde{T}) \to \eta_+(F(x), \widetilde{T})$, we have

$$\frac{1}{\widetilde{T}}\int \eta_+(F(x), \widetilde{T})d\mu(x) \ge \widetilde{\eta}.$$

According to Birkhoff Ergodic Theorem, there exists a positively recurrent point $b \in \omega(b) \subset \Lambda$ such that

$$\lim_{n\to+\infty} \frac{1}{n\widetilde{T}} \sum_{j=0}^{n-1} \eta_+\big(F(\phi_{j\widetilde{T}}(b)), \widetilde{T}\big) \geq \widetilde{\eta}. \tag{3.11}$$

Thus by Theorem 3.8, $\omega(b)$ is hyperbolic of index p.

Now assume that E is not contracting. Let

$$q = \min\{j : \text{there is a fundamental } j\text{−limit } \Gamma \subset \Lambda\}.$$

By Lemma 3.10, $q < p$. Take a fundamental q-limit Γ contained in Λ. Note that $q \neq 0$ because, otherwise, by Lemma 3.4, Γ must be an expanding periodic orbit of S, hence by Lemma 3.3, $\Lambda = \Gamma$, contradicting that $\Lambda = \text{limit}(P_n, X_n)$, $\text{Ind}(P_n) = p$, $1 \leq p \leq d - 2$. Let $D_\Gamma = \widetilde{E} \oplus \widetilde{F}$ be the associated q-dominated splitting on Γ. If $\widetilde{E}$ is not contracting, by Lemma 3.10, there is a fundamental r-limit Γ' contained in Γ with $r < q$, contradicting the minimality of q. Thus $\widetilde{E}$ must be contracting.

Then by the conclusion obtained above (for the case that E is contracting), $\Gamma \subset \Lambda$ contains a hyperbolic set with index q. Since $q < p$, this finishes the proof of item 1), hence the proof of Lemma 3.11. □

Remark In the proof of Lemma 3.11 we are benefited from the beautiful index-minimizing idea of Toyoshiba [34].

4 Non-existence of heterodimensional cycles for star flows

In this section we prove a basic property for star flows, that is, any star flow exhibits no heterodimensional cycles. Let P and Q be two different hyperbolic periodic orbits of $X \in \mathcal{X}(M)$. We say that P and Q form a *heterodimensional cycle* of X if both $W^s(P) \cap W^u(Q)$ and $W^u(P) \cap W^s(Q)$ are nonempty and the indices of P and Q are different. Note that for a system X to exhibit a heterodimensional cycle, there is a restriction to the dimension of M, i.e., $\dim M \geq 4$. The dynamics of systems with heterodimensional cycle are extensively studied by Diaz and other authors, see [3] and references listed there for more details.

Theorem 4.1 *Assume $X \in \mathcal{X}^*(M)$. Then X exhibits no heterodimensional cycles.*

Proof Suppose for the contrary $X \in \mathcal{X}^*(M)$ has a heterodimensional cycle $\Lambda = \text{Orb}(p) \cup \text{Orb}(q) \cup \text{Orb}(x) \cup \text{Orb}(y)$, where p and q are hyperbolic periodic points with indices i and $i + g$ $(g > 0)$ respectively, and $x, y \in M$ satisfy $\omega(x) = \text{Orb}(q)$, $\alpha(x) = \text{Orb}(p)$, $\omega(y) = \text{Orb}(p)$, $\alpha(y) = \text{Orb}(q)$. Note that we have $1 \leq i < i + g \leq d - 2$.

We claim $\Lambda \subset \text{P}^i_*(X) \cap \text{P}^{i+g}_*(X)$. This can be proved by creating homoclinic points. Indeed, since $x \in W^s(\text{Orb}(q))$ and $y \in W^u(\text{Orb}(q))$, by the λ-lemma, there exists an orbital arc A going from a point $\widetilde{x}$ near x to a point $\widetilde{y}$ near y, passing near $\mathrm{Orb}(q)$. With a small perturbation near x and y, we can create a vector filed Y with a homoclinic orbit associated with $\mathrm{Orb}(p, Y)$ passing near x, $\mathrm{Orb}(q)$, and y. (Here the perturbation can be simply some one-step-pushes, because the orbital arc A can be chosen not to wrap around, but to pass near x and y only once.) With a further perturbation if necessary, we may assume the homoclinic orbit of Y is transversal. By Birkhoff-Smale Theorem, the homoclinic orbit is approximated by periodic orbits of Y of index $\mathrm{Ind}(\mathrm{Orb}(p)) = i$. This means $x, y, q \in \mathrm{P}^i_*(X)$, proving $\Lambda \subset \mathrm{P}^i_*(X)$. Likewise for $\mathrm{P}^{i+g}_*(X)$, proving the claim.

So we have i-dominated splitting $E^i \oplus F^i$ as well as $(i+g)$-dominated splitting $E^{i+g} \oplus F^{i+g}$ over Λ. From this and Lemma 3.5 we obtain the splitting $D_\Lambda = E^i \oplus Z \oplus F^{i+g}$, where $Z = F^i \cap E^{i+g}$. It is easy to see that $\psi = \psi_X$ is contracting on E^i and expanding on F^{i+g}. In fact, any ergodic measure supported on Λ can only be supported on $\mathrm{Orb}(p)$ or $\mathrm{Orb}(q)$. But on $\mathrm{Orb}(p)$ or $\mathrm{Orb}(q)$, E^i is contracting. So by Lemma 3.9, E^i is contracting. Likewise for F^{i+g}.

For convenience, for the system X, we will drop "X" from the notations below. Take a small closed neighborhood U of Λ and a small neighborhood $\mathcal{U}_0 \subset \mathcal{X}^*(M)$ of X such that for any $Y \in \mathcal{U}_0$, $U \cap \mathrm{Sing}(Y) = \emptyset$, and any closed invariant set $\Gamma \subset U$ of Y, ψ_Y has a similar splitting, i.e., $D_{Yz} = E^i(Y, z) \oplus Z(Y, z) \oplus F^{i+g}(Y, z)$ for $z \in \Gamma$, $E^i(Y, z)$ dominates $Z(Y, z)$ and $Z(Y, z)$ dominates $F^{i+g}(Y, z)$. Moreover, ψ_Y contracts $E^i(Y, z)$ and expands $F^{i+g}(Y, z)$.

Let $r = \widetilde{\eta}/4 > 0$. Denote

$$K = \sup_{z \in U, Y \in \mathcal{U}_0, t \in [-6\widetilde{T}, 6\widetilde{T}]} \{|\log \|\psi_{Yt}|_{D_{Yz}}\||\}/\widetilde{T}.$$

It is easy to see that for any $|t| \geq \widetilde{T}$, we have $|\log \|\psi_{Yt}|_{D_{Yz}}\|| \leq K|t|$ if $\phi(Y, [0, t], z) \subset U$. Indeed, if $0 \leq t \leq 6\widetilde{T}$, this is obvious. If $t > 6\widetilde{T}$, we break t into $t = 2n\widetilde{T} + s$ with $s \in [0, 2\widetilde{T})$. Then

$$\begin{aligned} |\log \|\psi_{Yt}|_{D_{Yz}}\|| &= |\log \|(\psi_{Ys}|_{D_{Yz_n}})(\psi_{Y(2\widetilde{T})}|_{D_{Yz_{n-1}}}) \cdots (\psi_{Y(2\widetilde{T})}|_{D_{Yz_0}})\|| \\ &\leq (n+1)K\widetilde{T} \leq Kt \end{aligned}$$

where $z_i = \phi_{Y(2i\widetilde{T})}(z)$, $i = 0, 1, 2, \cdots, n$. The case $t < 0$ can be treated similarly.

According to the continuity of the splitting with respect to z, Y, t, there exist $\delta > 0$ and a neighborhood $\mathcal{U}_1 \subset \mathcal{U}_0$ of X such that if $z, z' \in U$, $d(z, z') \leq 4\delta$, and if $Z(z)$ is well-defined for X and $Z(Y, z')$ is well-defined for any $Y \in \mathcal{U}_1$, then for any $t \in [0, 6\widetilde{T}]$,

$$\begin{aligned} -r\widetilde{T} \leq |\eta_+(Z(z), t) - \eta_+(Y, Z(Y, z'), t)| \leq r\widetilde{T}, \\ -r\widetilde{T} \leq |\eta_-(Z(z), t) - \eta_-(Y, Z(Y, z'), t)| \leq r\widetilde{T}. \end{aligned} \tag{4.12}$$

We may assume that δ is small enough so that $B_{4\delta}(p)$ and $B_{4\delta}(q)$ are contained in U.

Denote by T, S the periods of p, q respectively. Take a section Σ at p and a section Γ at q, transverse to X. We may assume that δ is small enough so that $B_{4\delta}(p)-\Sigma$ and $B_{4\delta}(q)-\Gamma$ are not connected. Then for any $0<\zeta\le 4\delta$, denote $\Sigma_\zeta = B_\zeta(p)\cap\Sigma$ and $\Gamma_\zeta = B_\zeta(q)\cap\Gamma$. We may also assume that δ is so small that the Poincaré map f is defined for every $z\in\Sigma_{4\delta}$ (resp. $z\in\Gamma_{4\delta}$). For simplicity, we assume that

$$\begin{aligned}&\|D(f|_{W^s(p)\cap\Sigma_{4\delta}})\|, \quad \left\|D\left(f^{-1}|_{W^u(p)\cap\Sigma_{4\delta}}\right)\right\|,\\ &\|D(f|_{W^s(q)\cap\Gamma_{4\delta}})\|, \quad \left\|D\left(f^{-1}|_{W^u(q)\cap\Gamma_{4\delta}}\right)\right\| < 1.\end{aligned} \tag{4.13}$$

(Note that we use the same symbol f to denote two distinct Poincaré maps at p and q.)

Fix points x_p, x_q in $\mathrm{Orb}(x)$ and y_p, y_q in $\mathrm{Orb}(y)$ so that $x_p, y_p\in\Sigma_\delta$ and $x_q, y_q\in\Gamma_\delta$. Let $y_p=\phi(s', y_q)$ and $x_q=\phi(t', x_p)$ ($t'>0, s'>0$). We may assume that $t', s'\ge 2\widetilde{T}$.

Lemma 4.2 *There exists $\delta_0>0$ small enough such that for any $\delta\in(0,\delta_0]$ and any $\varepsilon>0$, there exists an integer $L>0$ such that for each integer $n\ge L$, there exist $p_n, q_n\in U$ satisfying $p_n, fp_n, \cdots, f^n p_n\in B_{2\delta}(p)$, $q_n, fq_n, \cdots, f^n q_n\in B_{2\delta}(q)$, $d(p_n, y_p)\le\varepsilon$, $d(f^n p_n, x_p)\le\varepsilon$, $d(q_n, x_q)\le\varepsilon$, and $d(f^n q_n, y_q)\le\varepsilon$.*

Proof This is a consequence of the inclination lemma of Palis and our assumption (4.13). So the proof is omitted. We just remark that we need the successive n's in the following. □

Denote $f^n(p_n)=\phi_{T_n}(p_n)$ and $f^n(q_n)=\phi_{S_n}(q_n)$. Then for any integers $m, n\ge L$, we obtain a periodic ε-pseudoorbit

$$O(m,n)=(\phi([0,T_m],p_m), \phi([0,t'],x_p), \phi([0,S_n],q_n), \phi([0,s'],y_q)).$$

Lemma 4.3 *For any $\delta\in(0,\delta_0]$ there exist $\varepsilon>0$, $N>L$ such that if $n\ge N$ and $m\ge N$ then there exist $X_{mn}\in\mathcal{U}_1$ and a periodic point x_{mn} of X_{mn} that δ-shadows the ε-pseudoorbit $O(m,n)$.*

Proof We only have to take N large enough so that $O(m,n)$ is an ε-pseudoorbit for small enough $\varepsilon\le\delta$ (for example, with respect to $d(x_p, fx_p)$) so that we can make four small perturbations in a neighborhood of

$$\{y_p, x_p, x_q, y_q\}$$

to close the pseudoorbit $O(m,n)$ into a periodic orbit. □

Note that $\mathrm{Orb}(x_{mn}, X_{mn})$ also 3δ-shadows the periodic pseudoorbit

$$Q(m,n)=(\phi([0,mT],p), \phi([0,t'],x_p), \phi([0,nS],q), \phi([0,s'],y_q)).$$

Roughly, the periodic pseudoorbit $Q(m,n)$ of X, as well as the shadowing periodic orbit $\mathrm{Orb}(x_{mn})$ of X_{mn}, goes around $\mathrm{Orb}(p)$ m times, and around $\mathrm{Orb}(q)$ n times. We denote

$$Q(m,n,t) = \begin{cases} \phi(t,p), & \text{for } t \in [0, mT), \\ \phi(t-mT, x_p), & \text{for } t \in [mT, mT+t'), \\ \phi(t-mT-t', q), & \text{for } t \in [mT+t', mT+t'+nS), \\ \phi(t-mT-t'-nS, y_q), & \text{for } t \in [mT+t'+nS, mT+t'+nS+s']. \end{cases}$$

So for any $m, n \geq N$, there exists a strictly increasing continuous function

$$\theta_{mn} : [0, mT+nS+t'+s'] \to \mathbf{R}, \quad \theta(0) = 0$$

such that

$$d(Q(m,n,t), \phi(X_{mn}, \theta_{mn}(t), x_{mn})) \leq 3\delta$$

for $t \in [0, mT+nS+t'+s']$. We may assume that $\theta_{mn}(mT+nS+t'+s')$ is exactly the period of x_{mn} with respect to X_{mn}. We need the following lemma for further analysis, which is an extension of Lemma 5.5.3 in [19] and can be proved similarly.

Lemma 4.4 *For the above $X \in \mathcal{X}^*(M)$ and any $\bar{T} > 0, \tau > 0$, there exist a neighborhood $\mathcal{U} \subset \mathcal{X}^*(M)$ of X and $\varepsilon_1 > 0$ such that if $Y \in \mathcal{U}$, $\bar{T} \leq T' < \infty$ and $\theta(t)$ is a strictly increasing continuous function on $[0, T']$ with $\theta(0) = 0$, $\phi([0,T'], a), \phi(Y, [0,T'], b) \subset U$ and*

$$d(\phi(t,a), \phi(Y, \theta(t), b)) \leq \varepsilon_1$$

for $t \in [0, T']$, then

$$(1-\tau)T' \leq \theta(T') \leq (1+\tau)T'.$$

Fix a neighborhood $\mathcal{U} \subset \mathcal{U}_1$ in Lemma 4.4 for $\bar{T} = \widetilde{T}$ and $\tau < \min\{1/4, \widetilde{\eta}/(8K)\}$. We also assume that τ is small enough so that if $|t| \leq 6\tau\widetilde{T}$ then $|\log \|\psi_{Yt}\|| \leq r\widetilde{T}$ for each $Y \in \mathcal{U}$. Then we fix a small δ such that $4\delta < \varepsilon_1$.

Now, we follow the method in [7] to prove Theorem 4.1.

Claim. For fixed $n > N$, if m is large enough, then x_{mn} has index i. Similarly, for fixed $m > N$, if n is large enough, then x_{mn} has index $i+g$.

Roughly, this claims if $\mathrm{Orb}(x_{mn})$ goes much more times around $\mathrm{Orb}(p)$ than $\mathrm{Orb}(q)$, the index of $\mathrm{Orb}(x_{mn})$ will be the same as that of $\mathrm{Orb}(p)$, and vice versa.

According to Theorem 3.1, for any partition $0 = t_0 < t_1 < \cdots < t_l = mT$, $2\widetilde{T} \leq t_{j+1} - t_j \leq 4\widetilde{T}$ for $0 \leq j < l$,

$$\frac{1}{mT} \sum_{j=0}^{l-1} \eta_+(Z(\phi(t_j, p)), t_{j+1} - t_j) \geq \widetilde{\eta} \tag{4.14}$$

since $Z(\phi(t, p)) \subset D^u(\phi(t, p))$. Since $d(\phi(t, p), \phi(X_{mn}, \theta_{mn}(t), x_{mn})) \leq 3\delta$, by (4.12),

$$\frac{1}{mT}\sum_{j=0}^{l-1} \eta_+(X_{mn}, Z(X_{mn}, \phi(X_{mn}, \theta_{mn}(t_j), x_{mn})), t_{j+1} - t_j)$$
$$\geq \frac{1}{mT}\sum_{j=0}^{l-1} \left(\eta_+(Z(\phi(t_j, p)), t_{j+1} - t_j) - \widetilde{T}r\right) \geq \widetilde{\eta} - r. \quad (4.15)$$

Then according to Lemma 4.4, $|(\theta_{mn}(t_{j+1}) - \theta_{mn}(t_j)) - (t_{j+1} - t_j)| \leq 4\tau\widetilde{T}$ (see [7, Corollary 4.3]). According to the choice of τ, we have

$$\frac{1}{mT}\sum_{j=0}^{l-1}\eta_+(X_{mn}, Z(X_{mn}, \phi(X_{mn}, \theta_{mn}(t_j), x_{mn})), \theta_{mn}(t_{j+1}) - \theta_{mn}(t_j))$$
$$\geq \frac{1}{mT}\sum_{j=0}^{l-1} \eta_+(Z(\phi(t_j, p)), t_{j+1} - t_j) - 2r \geq \widetilde{\eta} - 2r. \quad (4.16)$$

Denote by T_{mn} $(= \theta_{mn}(mT + nS + t' + s'))$ the period of x_{mn} with respect to X_{mn}. Then

$$T_{mn} \leq (1 + \tau)(mT + nS + t' + s'), \qquad \theta_{mn}(mT) \geq (1 - \tau)(mT).$$

Therefore,

$$\begin{aligned}
&\eta_+(X_{mn}, Z(X_{mn}, x_{mn}), T_{mn}) \\
&\quad \geq \eta_+(X_{mn}, Z(X_{mn}, x_{mn}), \theta_{mn}(mT)) \\
&\quad + \eta_+(X_{mn}, Z(X_{mn}, \phi(X_{mn}, \theta_{mn}(mT), x_{mn})), T_{mn} - \theta_{mn}(mT)) \\
&\quad \geq mT(\widetilde{\eta} - 2r) - K(T_{mn} - \theta_{mn}(mT)) \\
&\quad \geq (\widetilde{\eta} - 2r - 2K\tau)(mT) - K(1 + \tau)(nS + t' + s').
\end{aligned}$$

According to the choice of r and τ, the last term is positive for large enough m. That means $Z(X_{mn}, x_{mn})$ is expanding under $\psi_{X_{mn}T_{mn}}$. This proves the first statement of the claim. The second can be proved similarly.

Take an integer $n_0 > N$ such that $n_0 > 2K/r$. Now according to the above claim, for any integer $m_0 > N$, we can take $n > n_0$ large enough so that $x_{m_0 n}$ has index $i + g$. Again, according to the above claim, there exists $m \geq m_0$ such that the index of x_{mn} is larger than i but the index of $x_{(m+1)n}$ is exactly i. We will see that such a change of index is a contradiction, because $\mathrm{Orb}(x_{(m+1)n})$ goes around $\mathrm{Orb}(p)$ only one time more than $\mathrm{Orb}(x_{mn})$ does.

First take a partition for $[0, mT + nS + t' + s']$

$$0 = t_0 < t_1 < \cdots < t_k = mT + nS + t' + s'$$

such that $2\widetilde{T} \le t_{j+1} - t_j \le 4\widetilde{T}$ for $0 \le j < k$ and $\{mT, mT + t', mT + t' + nS\} \subset \{t_0, \cdots, t_k\}$. Then

$$0 = \theta_{mn}(t_0) < \theta_{mn}(t_1) < \cdots < \theta_{mn}(t_k) = T_{mn} \tag{4.17}$$

is a partition of $[0, T_{mn}]$ such that $\theta_{mn}(t_{j+1}) - \theta_{mn}(t_j) \ge \widetilde{T}$. Since the index of x_{mn} is larger than i, $Z(X_{mn}, \phi(X_{mn}, t, x_{mn}))$ has nontrivial intersection with the contracting subspace

$$D^s(X_{mn}, \phi(X_{mn}, t, x_{mn})).$$

Therefore, according to Theorem 3.1, for x_{mn} and the partition (4.17), we have

$$\frac{1}{T_{mn}}\sum_{j=0}^{k-1}\eta_+(X_{mn}, Z(X_{mn}, \phi(X_{mn}, \theta_{mn}(t_j), x_{mn})), \theta_{mn}(t_{j+1}) - \theta_{mn}(t_j)) \le -\widetilde{\eta}. \tag{4.18}$$

By a similar argument in (4.16) and then a similar argument in (4.15), we obtain

$$\frac{1}{T_{mn}}\sum_{j=0}^{k-1}\eta_+(Q(m, n, t_j), t_{j+1} - t_j) \le -\widetilde{\eta} + 2r. \tag{4.19}$$

In the following discussion, we first assume $T \ge 2\widetilde{T}$. So now take a partition

$$\begin{aligned} 0 = t'_0 < t'_1 < \cdots < t'_l = T < t'_{l+1} = T + t_1 < \cdots < t'_{l+j} \\ = T + t_j < \cdots < t'_{l+k} = (m+1)T + ns + t' + s' \end{aligned}$$

such that $2\widetilde{T} \le t'_{j+1} - t'_j \le 4\widetilde{T}$. Then

$$0 = \theta_{(m+1)n}\big(t'_0\big) < \theta_{(m+1)n}\big(t'_1\big) < \cdots < \theta_{(m+1)n}\big(t'_{l+k}\big) = T_{(m+1)n} \tag{4.20}$$

is a partition of $[0, T_{(m+1)n}]$ such that $\theta_{(m+1)n}(t'_{j+1}) - \theta_{(m+1)n}(t'_j) \ge \widetilde{T}$. Since the index of $x_{(m+1)n}$ is i, $Z(X_{(m+1)n}, \phi(X_{(m+1)n}, t, x_{(m+1)n}))$ is a subspace of the expanding subspace

$$D^u(X_{(m+1)n}, \phi(X_{(m+1)n}, t, x_{(m+1)n})).$$

Therefore, according to Theorem 3.1, for $x_{(m+1)n}$ and the partition (4.20), we have

$$\begin{aligned} \frac{1}{T_{(m+1)n}} \sum_{j=0}^{l+k-1} \eta_+\Big(X_{(m+1)n}, Z\big(X_{(m+1)n}, \phi\big(X_{(m+1)n}, \theta_{(m+1)n}\big(t'_j\big), x_{(m+1)n}\big)\big), \\ \theta_{(m+1)n}\big(t'_{j+1}\big) - \theta_{(m+1)n}\big(t'_j\big)\Big) \ge \widetilde{\eta}. \end{aligned}$$

And again using a similar argument in (4.16) and then a similar argument in (4.15), we obtain

$$\frac{1}{T_{(m+1)n}} \sum_{j=0}^{l+k-1} \eta_+\big(Q\big(m+1, n, t'_j\big), t'_{j+1} - t'_j\big) \geq \widetilde{\eta} - 2r. \tag{4.21}$$

Since $Q(m+1, n, t'_{l+j}) = Q(m, n, t_j)$ and $t'_{l+j+1} - t'_{l+j} = t_{j+1} - t_j$ for $0 \leq j \leq k$, we have

$$\begin{aligned} 0 &< T_{(m+1)n}(\widetilde{\eta} - 2r) \\ &\leq \sum_{j=0}^{l+k-1} \eta_+\big(Q\big(m+1, n, t'_j\big), t'_{j+1} - t'_j\big) \\ &= \sum_{j=0}^{l-1} \eta_+\big(Q\big(m+1, n, t'_j\big), t'_{j+1} - t'_j\big) + \sum_{j=0}^{k-1} \eta_+(Q(m, n, t_j), t_{j+1} - t_j) \\ &\leq KT + T_{mn}(-\widetilde{\eta} + 2r) < 0. \end{aligned}$$

This is a contradiction. For the case $T < 2\widetilde{T}$, by taking two appropriate partitions for $[0, mT + nS + t' + s']$ and $[0, (m+1)T + ns + t' + s']$, it can be proved similarly. This finishes the proof of Theorem 4.1. □

5 Index-determination for preperiodic points

This section does not assume the star condition. It concerns for a general flow the question of determination of the index for preperiodic points. This will be done via creation of homoclinic orbits by C^1 perturbations. Since we have to confirm that the created homoclinic orbit passes near some given point, rather than just to create a homoclinic orbit, more work will be involved. To create by C^1 perturbations various homoclinic and heteroclinic connections in this paper, we will need the following version of the C^1 connecting lemma [38], which is more general than the original C^1 connecting lemma of Hayashi [11].

Theorem 5.1 [38] *Let $X \in \mathcal{X}(M)$, and $z \in M$ be neither singular nor periodic of X. Then for any C^1 neighborhood $\mathcal{U}$ of X in $\mathcal{X}(M)$, there exist $\rho > 1$, $T > 1$ and $\delta_0 > 0$ such that for any $0 < \delta \leq \delta_0$ and any two points x, y outside the tube $\Delta = \cup_{t \in [0,T]} B(\phi_t(z), \delta)$, if the positive X-orbit of x and the negative X-orbit of y both hit $B(z, \delta/\rho)$, then there exists $Y \in \mathcal{U}$ with $Y = X$ outside Δ such that y is on the positive Y-orbit of x. Moreover, the resulted Y-orbit segment from x to y meets $B(z, \delta)$.*

We remark that the tube Δ could be equally well taken saturated by the negative time of the flow. Also, we remark that the assertion that the resulted Y-orbit segment meets $B(z, \delta)$ is not included in the statement in [38], but obvious from the construction of the perturbations in the proof there. We include it here in the statement of Theorem 5.1 because it is important for some applications such as the following lemma.

Lemma 5.2 *Let P be a hyperbolic periodic orbit of X. Let $a \in \overline{W^s(P)} - P$ and $b \in \overline{W^u(P)} - P$. Assume a and b are neither periodic nor singular. Also assume there are sequences $x_n \in M$ and $s_n > 0$ with $x_n \to b$ and $\phi_{s_n}(x_n) \to a$. Then for any C^1 neighborhood $\mathcal{U}$ of X in $\mathcal{X}(M)$ and any neighborhood W of a in M, there is $Y \in \mathcal{U}$ with $Y = X$ on a neighborhood of P such that Y has in W a homoclinic point x of P.*

Proof Let $\mathcal{U}$ be a C^1 neighborhood of X. Since a is neither singular nor periodic, by Theorem 5.1, there exist $\rho_a > 0, T_a > 0, \delta_a > 0$ with those properties (to avoid too much repeat below we just say "those properties" here). Likewise, for b, there exist $\rho_b > 0, T_b > 0, \delta_b > 0$ with those properties. Let $\rho = \max\{\rho_a, \rho_b\}$, $T = \max\{T_a, T_b\}$, and $\delta_0 = \min\{\delta_a, \delta_b\}$.

We may assume a and b are not on the same orbit. Otherwise $a \in \overline{W^u(P)}$, and the proof will be easier. Thus the orbit segments $A = \phi_{[0,-T]}(a)$ and $B = \phi_{[0,T]}(b)$, as well as the periodic orbit P, are pairwise disjoint. Take $\delta \le \delta_0$ and $\eta > 0$ small such that the tubes $\cup_{t\in[0,-T]} B(\phi_t(a), \delta)$ and $\cup_{t\in[0,T]} B(\phi_t(b), \delta)$, as well as $W^s_\eta(P) - P$ and $W^u_\eta(P) - P$, are pairwise disjoint. Take $0 < \delta_1 \le \delta$ small such that $B(a, \delta_1) \subset W$ and let $\Delta_a = \cup_{t\in[0,-T]} B(\phi_t(a), \delta_1)$. Since $a \in \overline{W^s(P)}$, there is $p^s \in W^s_\eta(P)$ such that the negative X-orbit of p^s hits $B(a, \delta_1/\rho)$ at a point a^s. Note that $\mathrm{Orb}^+(p^s) \subset W^s_\eta(P)$. If the X-orbit segment C from a^s to p^s contains b, then $a^s \in \overline{W^u(P)}$, and the proof will again be easier. Thus we assume $b \notin C$. Note that C may wrap around and be close to b. Nevertheless $b \notin C \cup \mathrm{Orb}^+(p^s)$, which forms a complete positive orbit (of a^s), hence $B \cap C = \emptyset$. Hence we can take $\delta_2 \le \delta$ small such that the tube $\Delta_b = \cup_{t\in[0,T]} B(\phi_t(b), \delta_2)$ is disjoint from C. Since $b \in \overline{W^u(P)}$, there is $p^u \in W^u_\eta(P)$ such that the positive X-orbit of p^u hits $B(b, \delta_2/\rho)$ at a point b^u. Note that $\mathrm{Orb}^-(p^u) \subset W^u_\eta(P)$. Also note that the orbit segment from p^u to b^u may wrap around and even go through Δ_a (but this is all right). Finally, by assumption, there are sequences $x_n \in M$ and $s_n > 0$ with $x_n \to b$ and $\phi_{s_n}(x_n) \to a$. Fix n large such that $x_n \in B(b, \delta_2/\rho)$ and $x_n' = \phi_{s_n}(x_n) \in B(a, \delta_1/\rho)$. Note that the orbit segment from x_n to x_n' may wrap around and go through both Δ_a and Δ_b.

Now p^u and x_n' are outside the tube Δ_b, and the positive X-orbit of p^u and the negative X-orbit of x_n' both hit $B(b, \delta_2/\rho)$ (at b^u and x_n, respectively). By Theorem 5.1, there exists $Z \in \mathcal{U}$ with $Z = X$ outside Δ_b such that x_n' is on the positive Z-orbit of p^u. We emphasize that the X-orbit segment C from a^s to p^s is unchanged because C is disjoint from Δ_b. Thus p^u and p^s are outside the tube Δ_a, and the positive Z-orbit of p^u and the negative Z-orbit of p^s both hit $B(a, \delta_1/\rho)$ (at x_n'

and a^s, respectively). By Theorem 5.1, there exists $Y \in \mathcal{U}$ with $Y = Z$ outside Δ_a such that p^s is on the positive Y-orbit of p^u. Moreover, by the last assertion in Theorem 5.1, the resulted Y-orbit segment from p^u to p^s meets $B(a, \delta_1) \subset W$. Note that $\mathrm{Orb}^+(p^s, X)$ and $\mathrm{Orb}^-(p^u, X)$ are unchanged. This gives a Y-homoclinic orbit of P that meets W, proving Lemma 5.2. □

We emphasize that the two tubes Δ_a and Δ_b are disjoint. This is the reason why the same constants ρ, T, δ_0 work for both X and Z, and why the resulted perturbation Y is in $\mathcal{U}$. This is a delicate point for the use of the connecting lemma. To fully clarify this, one would have to go back to the proof of the connecting lemma. We omit the details. There is a discussion about this in [6].

We remark that the two points a and b in Lemma 5.2 are in the same situation. Hence one may create a homoclinic point near b as well. Nevertheless it is unclear if one can create homoclinic points near both a and b simultaneously.

We insert here a fact about chain transitive hyperbolic sets. Recall a compact invariant set Γ is *chain transitive* if for any $\varepsilon > 0$ and any $x, y \in \Gamma$, there are $n \geq 1$, $x_1, x_2, ..., x_n \in M$, and $t_1, ..., t_{n-1} \geq 1$ such that $x_1 = x$, $x_n = y$, and $\mathrm{d}(\phi_{t_i}(x_i), x_{i+1}) < \varepsilon$ for all $i = 1, 2, ..., n-1$. Clearly, for any $x \in M$, $\omega(x)$ and $\alpha(x)$ are chain transitive. Also recall that two hyperbolic periodic orbits P and Q are *H-related* if $W^s(P)$ and $W^u(Q)$ have a transverse intersection, and $W^s(Q)$ and $W^u(P)$ have a transverse intersection.

Lemma 5.3 *Let Γ be a chain transitive hyperbolic set of X with $\Gamma \cap \mathrm{Sing}(X) = \emptyset$ and $1 \leq \mathrm{Ind}(\Gamma) \leq d - 2$. Then there is a C^1 neighborhood $\mathcal{U}$ of X and a neighborhood U of Γ in M such that, for every $Y \in \mathcal{U}$, any two Y-periodic orbits P and Q contained in U are hyperbolic and H-related.*

Proof The proof is standard. Let $\mathcal{U}_0$ be a C^1 neighborhood of X and $\delta > 0$ be small such that for every $Y \in \mathcal{U}_0$, any compact Y-invariant set $C \subset B(\Gamma, \delta)$ is hyperbolic. There is $0 < \varepsilon < \delta$ such that for every $Y \in \mathcal{U}_0$, if $\mathrm{d}(x, y) < \varepsilon$, and if $\overline{\mathrm{Orb}(x, Y)}$ and $\overline{\mathrm{Orb}(y, Y)}$ are contained in $B(\Gamma, \delta)$, then $W^s(x, Y)$ and $W^u(y, Y)$ have a transverse intersection, and $W^s(y, Y)$ and $W^u(x, Y)$ have a transverse intersection. Since Γ is chain transitive respecting X, by the shadowing property, there is a hyperbolic periodic orbit A of X with $\Gamma \subset B(A, \varepsilon/3)$. Let $\mathcal{U} \subset \mathcal{U}_0$ be a C^1 neighborhood of X such that every $Y \in \mathcal{U}$ has a hyperbolic periodic orbit A_Y with $\Gamma \subset B(A_Y, \varepsilon/2)$. Let $U = B(\Gamma, \varepsilon/2)$. It is easy to check $\mathcal{U}$ and U satisfy Lemma 5.3. □

We prepare a technical lemma needed below for a C^1-creation of a homoclinic orbit passing near a given point $a \in M$. We know if $\omega(a)$ is hyperbolic and isolated, by the In Phase Theorem, there will be a point $y \in \omega(a)$ such that $a \in W^s(y)$. Since y is accumulated by hyperbolic periodic orbits P_n that are contained in a small neighborhood of $\omega(a)$ (since $\omega(a)$ is hyperbolic), the stable manifolds of P_n will prolongate along $W^s(y)$ hence pass near a, a situation good to our perturbations. The following technical lemma then deals with the case when $\omega(a)$ is hyperbolic but not necessarily isolated. Note that if an ω-limit set is hyperbolic, it has a homogeneous index. As usual, for any $V \subset M$, denote by $M(\overline{V})$ the maximal invariant set of $X \in \mathcal{X}(M)$ contained in $\overline{V}$. Note that if Λ is hyperbolic of index i, and V is a sufficiently small neighborhood of Λ, then $M(\overline{V})$ is hyperbolic of index i.

Lemma 5.4 *Assume $\omega(a) \subset M - \mathrm{Sing}(X)$ is hyperbolic of index $1 \le i \le d-2$ and $a \notin \omega(a)$. Then for any neighborhood U of $\omega(a)$ in M,*

1) *There exists a point $y \in U$ with $\overline{\mathrm{Orb}(y)} \subset U$ such that $a \in W^s(y)$. Moreover, there exists a sequence $\{P_n\} \subset U$ of hyperbolic periodic orbits of index i and $p_n \in P_n$ such that $p_n \to y$.*
2) *There exists a hyperbolic periodic orbit $P \subset U$ of index i such that $a \in \overline{W^s(P)}$.*

A similar result holds for $\alpha(a)$.

Note that Lemma 5.4 trivially holds when $i = 0$ or $d-1$.

Proof We may assume that U is small enough so that any two periodic orbits of X contained in U are H-related. Take a smaller neighborhood $\overline{V}$ of $\omega(a)$ with $\overline{V} \subset U$. Denote $\varepsilon_0 = d(\overline{V}, M - U)/3$. Since hyperbolic set has the shadowing property ([2,31]), there exists $0 < \delta_0 < \varepsilon_0$ such that any δ_0-pseudoorbit in V can be ε_0-shadowed. Take a point $c \in \omega(a)$ and a large enough number $\tau > 0$ such that $d(\phi_\tau a, c) < \delta_0$. For the δ_0-pseudoorbit $\phi_{(-\infty,0]}(c), \phi_{[\tau,\infty)}(a))$, let y be the shadowing point, i.e., for some orientation-preserving homeomorphism $\theta : \mathbf{R} \to \mathbf{R},\ \theta(0) = 0$,

$$d(\phi_{\theta(s)}(y), \phi_{\tau+s}(a)) < \varepsilon_0, \quad s \ge 0,$$
$$d(\phi_{\theta(s)}(y), \phi_s(c)) < \varepsilon_0, \quad s \le 0.$$

Then $a \in W^s(y)$ and $\alpha(y) = \alpha(c) \subset \omega(a) = \omega(y)$. Obviously $\overline{\mathrm{Orb}(y)} \subset U$.

Since $\alpha(y) \cap \omega(y) \neq \emptyset$, according to the shadowing property of hyperbolic sets, it is easy to see that there exists a sequence $\{P_n\} \subset U$ of periodic orbits of index i such that $p_n \in P_n$ and $p_n \to y$. Thus for any neighborhood W of a in M, there is n such that $W^s(P_n) \cap W \neq \emptyset$. Since any two periodic orbits in U are H-related, we may fix one of them, say P_1, to be P at the first place. Then $W^s(P) \cap W \neq \emptyset$. Thus $a \in \overline{W^s(P)}$. This proves Lemma 5.4. □

We also prepare the following more general result of type Lemma 5.4.

Lemma 5.5 *Let Q_n be a sequence of periodic orbits of X that converge in the Hausdorff metric to a compact invariant set K of X. Let $\Gamma \subset K - \mathrm{Sing}(X)$ be a chain transitive hyperbolic set of X of index i. Assume $K - \Gamma \neq \emptyset$. Then for any neighborhood U of Γ in M, there is a non-periodic point $a_1 \in K \cap U - \Gamma - \mathrm{Sing}(X)$ with $\mathrm{Orb}^+(a_1) \subset U$, together with a hy-*

perbolic periodic orbit $P \subset U$ of index i such that $a_1 \in \overline{W^s(P)}$. A similar result holds for W^u.

Proof Note that by Lemma 3.3, the condition $K - \Gamma \neq \emptyset$ implies $0 \neq \mathrm{Ind}(\Gamma) \neq d-1$. Take $a \in K-\Gamma$. Since $a \notin \Gamma$ and Γ is disjoint from $\mathrm{Sing}(X)$, we may assume U is small enough so that $a \notin U$ and $U \cap \mathrm{Sing}(X) = \emptyset$. We may also assume U is small enough so that $M(\overline{U})$ is hyperbolic of index i, and that any two periodic orbits contained in U are H-related. Take two small neighborhoods V, V' of Γ with $\overline{V} \subset V' \subset \overline{V}' \subset U$. Since $K = \lim Q_n$, there are $x_n \in Q_n$ with $x_n \to a$. Since $\Gamma \subset K$, there are $y_n \in Q_n$ such that $d(y_n, \Gamma) \to 0$. Since $a \notin \Gamma$, there are $t_n > 0$ such that $\phi_{[-t_n,0]}(y_n) \subset \overline{V}$ and $\phi_{-t_n}(y_n) \in \partial V$. Obviously, $t_n \to \infty$. Let b be a limit point of $\phi_{-t_n}(y_n)$. Then $b \in \partial V$ and $\mathrm{Orb}^+(b) \subset \overline{V}$. Note that $b \in K \cap U - \Gamma - \mathrm{Sing}(X)$. According to the recurrence of b, there are three cases:

Case 1: $b \notin \omega(b)$.

In this case we simply choose b to be a_1, and apply item 2) of Lemma 5.4 to get the periodic orbit P.

Case 2: $b \in \omega(b)$, but b is not periodic.

In this case we still choose b to be a_1. Note that since $\omega(a_1)$ is hyperbolic of index i, by the shadowing property, there is a sequence $\{P_n\} \subset U$ of hyperbolic periodic orbits of index i such that $p_n \in P_n$ and $p_n \to a_1$. Then we choose P_1 to be P. Note that since any two periodic orbits contained in U are H-related, $p_n \in \overline{W^s(P)}$ for any $n \geq 1$. Hence $a_1 \in \overline{W^s(P)}$.

Case 3: b is periodic.

In this case $K \cap W^s_{loc}(b) - \mathrm{Orb}(b)$ must be non-empty. We then choose any point of $K \cap W^s_{loc}(b) - \mathrm{Orb}(b)$ to be a_1, and $\mathrm{Orb}(b)$ to be P. Note that a_1 is non-periodic. Also note that if the size of $W^s_{loc}(b)$ is small enough, then $a_1 \in U - \Gamma - \mathrm{Sing}(X)$ and $\mathrm{Orb}^+(a_1) \subset U$.

Thus in all the three cases, the choice of a_1 and P satisfies the conditions of Lemma 5.5. This proves Lemma 5.5. □

Note that the assumption of Lemma 5.5 is weaker than that of Lemma 5.4 in the sense that, it is not the point a itself, but nearby points (on some periodic orbits), that approach to a chain transitive hyperbolic set. The conclusion is weaker too in the sense that we have to switch from a to another (non-periodic and nonsingular) point a_1, which is loosely connected to a (in the sense that nearby points of a approach to a_1).

The main result of this section is the following lemma that concerns the index-determination for preperiodic points.

Lemma 5.6 *Let $X \in \mathcal{X}(M)$. Assume $a \in \overline{\mathrm{P}}(X)$ and $\omega(a) \subset M - \mathrm{Sing}(X)$ is hyperbolic of index i. Then $a \in \mathrm{P}^i_*(X)$. Similarly, if $a \in \overline{\mathrm{P}}(X)$ and $\alpha(a) \subset M - \mathrm{Sing}(X)$ is hyperbolic of index i, then $a \in \mathrm{P}^i_*(X)$.*

Note that the assumption $a \in \overline{\mathrm{P}}(X)$ merely means periodic orbits (of X) accumulate on the point a, without specifying the indices of these periodic orbits. The indices may well be different from i. In fact i is given as the index of $\omega(a)$. Observe that a is "nearly homoclinic" associate with $\omega(a)$, in the sense that the positive orbit of a accumulates on $\omega(a)$, and the negative orbits of some nearby points (on those periodic orbits) accumulate on $\omega(a)$ too. We hence try to create by C^1 perturbations, arbitrarily close to a, a homoclinic point associated with some hyperbolic periodic orbit P of index i. Then by Birkhoff-Smale Theorem a will be i-preperiodic.

Proof If $a \in \omega(a)$, then the conclusion is obvious. So we assume $a \notin \omega(a)$. Since $a \in \overline{\mathrm{P}}(X)$, by Lemma 3.3, $0 \neq \mathrm{Ind}(\omega(a)) \neq d-1$. Take a sequence of periodic orbits $Q_n \subset \mathrm{P}(X)$ and $x_n \in Q_n$ such that $x_n \to a$. Taking subsequence if necessary, we may assume Q_n converge to a compact X-invariant set K. Then $a \in K$ and hence $\omega(a) \subset K$.

Take a small neighborhood U of $\omega(a)$ with $a \notin U$ such that $M(\overline{U}) = \cap_{t \in \mathbf{R}} \phi_t(\overline{U})$ is hyperbolic of index i. We may assume that U is small enough so that any periodic orbits contained in U are H-related.

By item 2) of Lemma 5.4, there exists a hyperbolic periodic orbit $P \subset U$ of index i such that $a \in \overline{W^s(P)}$. By Lemma 5.5 (for the case of W^u), there is a non-periodic point $a_1 \in K \cap U - \omega(a) - \mathrm{Sing}(X)$ with $\mathrm{Orb}^-(a_1) \subset U$, together with a hyperbolic periodic orbit Q of index i contained in U such that $a_1 \in \overline{W^u(Q)}$. Since P and Q are H-related, $a_1 \in \overline{W^u(P)}$. Thus we may forget about Q.

Note that a is not periodic since $a \notin \omega(a)$. Also, a is not singular since otherwise $\omega(a)$ would be in $\mathrm{Sing}(X)$. Since a and a_1 are both in K, there are sequences $x_n \in M$ and $s_n > 0$ with $x_n \to a_1$ and $\phi_{s_n}(x_n) \to a$. By Lemma 5.2, for any C^1 neighborhood $\mathcal{U}$ of X in $\mathcal{X}(M)$ and any neighborhood W of a in M, there is $Y \in \mathcal{U}$ such that Y has in W a homoclinic point x of P. With a further perturbation if necessary, we may assume x is a transverse Y-homoclinic point of P. By Birkhoff-Smale Theorem, $x \in \overline{\mathrm{P}}(Y)$. Thus $a \in \mathrm{P}_*^i(X)$, proving Lemma 5.6. □

Note that in the proof of Lemma 5.6 we have not used the conditions $a_1 \in U - \omega(a)$ and $\mathrm{Orb}^-(a_1) \subset U$. These will be used in the proof of Theorem B below.

We remark a delicate point involved here. The assumption of Lemma 5.6 says that in the direction of positive time it is the point a itself that approaches to Γ (in this case $\Gamma = \omega(a)$), while in the direction of negative time it is nearby points x_n of a that approach to Γ. If the assumption is weakened to that, in both directions, it is merely nearby points of a that approach to Γ, though we still can create homoclinic points of Γ, it will be unclear if we can create a homoclinic point of Γ *near* a. In the proof of Theorem B we will have such a situation (for heteroclinic case), but it will be all right because there we will not need the resulted heteroclinic connection to pass near some given point.

6 The proof of Theorem B

In this section we prove Theorem B.

Theorem B *Let* $S \in \mathcal{X}^*(M)$. *Let* $\Gamma \subset \overline{\mathrm{P}}(S)$ *be a compact invariant set of* S *such that* $\Gamma \cap \mathrm{Sing}(S) = \emptyset$. *Then* Γ *is hyperbolic.*

To prove Theorem B, by the principle of Liao, it suffices to rule out the existence of minimally non-hyperbolic sets contained in $\overline{\mathrm{P}}(S) - \mathrm{Sing}(S)$, that is, to prove the following Lemma 6.1 and Lemma 6.2.

Lemma 6.1 *Assume* $S \in \mathcal{X}^*(M)$. *Then there are no simple type minimally non-hyperbolic sets contained in* $\overline{\mathrm{P}}(S) - \mathrm{Sing}(S)$.

Proof Assume for the contrary there is a simple type minimally non-hyperbolic set $\Lambda \subset \overline{\mathrm{P}}(S) - \mathrm{Sing}(S)$. We prove that a heterodimensional cycle can be created by perturbations. This will contradict Theorem 4.1.

The idea is briefly this: By the definition of simple type minimally non-hyperbolic set, there exists a resisting point $a \in \Lambda$ such that both $\omega(a)$ and $\alpha(a)$ are proper subsets of Λ, hence are hyperbolic. Moreover, $\Lambda = \alpha(a) \cup \mathrm{Orb}(a) \cup \omega(a)$, because $\alpha(a) \cup \mathrm{Orb}(a) \cup \omega(a)$ is itself a non-hyperbolic compact invariant set. Thus we have, loosely, already a heteroclinic connection going from the hyperbolic set $\alpha(a)$ to the hyperbolic set $\omega(a)$. Here we say "loosely", because this connection is between two hyperbolic sets, rather than two hyperbolic periodic orbits and, by definition, a heterodimensional cycle is between hyperbolic periodic orbits. In other words, later we should pass from the two sets to two periodic orbits. It is easy to see by dominated splitting that the two hyperbolic sets $\omega(a)$ and $\alpha(a)$ have different indices. Now since $a \in \overline{\mathrm{P}}(S)$, there is a sequence Q_n of periodic orbits of S that converge in the Hausdorff metric to a compact invariant set K of S such that $a \in K$. (K may contain singularities.) Clearly $\Lambda \subset K$. By Lemma 3.3, neither $\mathrm{Ind}(\omega(a))$ nor $\mathrm{Ind}(\alpha(a))$ is 0 or $d-1$. With the help of K (which, being the Hausdorff limit of a sequence of periodic orbits, is roughly "circle-like"), we will create by perturbation a second heteroclinic connection, going from $\omega(a)$ to $\alpha(a)$, without breaking the first one that goes from $\alpha(a)$ to $\omega(a)$, to form a heteroclinic cycle between $\omega(a)$ and $\alpha(a)$ (actually, in the proof below, we will take the opposite order). As just remarked, to really get a heterodimensional cycle which is defined to be between hyperbolic periodic orbits, we need to carry out the whole program for two hyperbolic periodic orbits P and Q (rather than the two hyperbolic sets $\omega(a)$ and $\alpha(a)$).

Now we start with the formal proof.

First we claim $\mathrm{Ind}\alpha(a) \neq \mathrm{Ind}\omega(a)$. This is argued by dominated splitting. Note that by Lemma 5.6, $a \in \mathrm{P}^i_*(S)$, where $i = \mathrm{Ind}\omega(a)$. Since $\Lambda = \overline{\mathrm{Orb}(a)}$ contains no singularities, $\Lambda \subset \mathrm{P}^i_*(S)$. By Lemma 3.6, Λ has i-dominated splitting $D_\Lambda = E \oplus F$. Since $\dim D^s(a) = \mathrm{Ind}\omega(a)$

and $\dim D^u(a) = d - 1 - \mathrm{Ind}\alpha(a)$, from Lemma 3.5, if $\mathrm{Ind}\alpha(a) = \mathrm{Ind}\omega(a)$ then $D^s(a) = E(a)$, $D^u(a) = F(a)$, which implies $D(a) = D^s(a) \oplus D^u(a)$, contradicting that a is a resisting point. This proves the claim.

Denote $\mathrm{Ind}\omega(a) = i$, and $\mathrm{Ind}\alpha(a) = j$. Thus $i \neq j$. In particular $\alpha(a) \cap \omega(a) = \emptyset$. Thus $\Lambda = \alpha(a) \cup \mathrm{Orb}(a) \cup \omega(a)$ is a disjoint union. Take a neighborhood U of $\omega(a)$ and a neighborhood V of $\alpha(a)$ respectively such that $\overline{U} \cap \overline{V} = \emptyset$. We may assume U and V have been taken small enough so that, together with some C^1 neighborhood $\mathcal{W}$ of S, Lemma 5.3 holds. That is, for every $X \in \mathcal{W}$, any two X-periodic orbits contained in U are H-related, and any two X-periodic orbits contained in V are H-related.

Next we specify two hyperbolic periodic orbits P and Q, between which a heterodimensional cycle will be created, and specify three points, through which three disjoint tubes will be prolongated to support our connecting perturbations. This is prepared by Lemma 5.5. Indeed, since $K - \omega(a) \neq \emptyset$ ($a \in K - \omega(a)$), by Lemma 5.5 (for the case of W^u), there is a non-periodic point $a_1^* \in K \cap U - \omega(a) - \mathrm{Sing}(S)$ with $\mathrm{Orb}^-(a_1^*) \subset U$, together with a hyperbolic periodic orbit P of index i contained in U such that $a_1^* \in \overline{W^u(P)}$. Likewise, there is a non-periodic point $a_2^* \in K \cap V - \alpha(a) - \mathrm{Sing}(S)$ with $\mathrm{Orb}^+(a_2^*) \subset V$, together with a hyperbolic periodic orbit Q of index j contained in V such that $a_2^* \in \overline{W^s(Q)}$. P and Q will be the two hyperbolic periodic orbits, between which we will create a heterodimensional cycle. Clearly $a_1^* \notin \alpha(a)$. Moreover, $a_1^* \notin \mathrm{Orb}(a)$ because $\mathrm{Orb}^-(a_1^*) \subset U$. Thus $a_1^* \in K - \Lambda$. Likewise, $a_2^* \in K - \Lambda$. The two points a_1^* and a_2^*, together with the point a we have had already, will serve as the point z in Theorem 5.1, through which three disjoint tubes will be prolongated.

Let $\mathcal{U}$ be any C^1 neighborhood of S. We may assume $\mathcal{U} \subset \mathcal{W}$. Since a_1^* is neither singular nor periodic, by Theorem 5.1, treating a_1^* as z, there exist $\rho > 0$, $T > 0$, $\delta_0 > 0$ with those properties. Likewise for a_2^* and a. We assume $\rho > 0$, $T > 0$, $\delta_0 > 0$ have been chosen to work for all the three points a_1^*, a_2^* and a.

We prolongate the tubes. We may assume a_1^* and a_2^* are not on the same orbit. Otherwise $a_2^* \in \overline{W^u(P)}$, and the proof will be easier. Thus the two orbital segments $A = \phi_{[0,T]}(a_1^*)$ and $B = \phi_{[0,-T]}(a_2^*)$ are disjoint. In fact A, B and Λ are pairwise disjoint since $a_1^*, a_2^* \notin \Lambda$. Take $\delta \leq \delta_0$ small and let

$$\Delta_1 = \cup_{t\in[0,T]} B\big(\phi_t\big(a_1^*\big), \delta\big),$$

$$\Delta_2 = \cup_{t\in[0,-T]} B\big(\phi_t\big(a_2^*\big), \delta\big),$$

$$\Delta = \cup_{t\in[0,T]} B(\phi_t(a), \delta).$$

We assume δ has been taken small enough such that $\overline{\Delta_1}$, $\overline{\Delta_2}$, and $\overline{\Delta}$ are pairwise disjoint, and such that $\overline{\Delta_1}$, $\overline{\Delta_2}$ and Λ are pairwise disjoint. Our perturbations X, Y and Z will all be given by Theorem 5.1 supported in the three disjoint tubes, hence will all be in $\mathcal{U}$. Take $\eta > 0$ small such that $W^u_\eta(P) \cup W^s_\eta(P) \subset U$ and $W^u_\eta(Q) \cup W^s_\eta(Q) \subset V$, and such that $W^u_\eta(P) \cup W^s_\eta(P)$ and $W^u_\eta(Q) \cup W^s_\eta(Q)$ are disjoint from the three tubes $\overline{\Delta_1}$, $\overline{\Delta_2}$ and $\overline{\Delta}$.

We remark that the tube Δ is for creating a connection from Q to P, and the two tubes Δ_1 and Δ_2, combined together, are for creating a connection from P to Q. At this point, two of the three tubes, Δ_1 and Δ, are preliminary choices. Later, to minimize the interference of orbits, we will shrink the radius δ for Δ_1 and Δ respectively, to get two tubes of the same length but thinner, as the final choices.

Having the tubes at hand, we will locate five orbital segments passing near the three points a_1^*, a_2^*, a, respectively, for the connecting perturbation guaranteed by Theorem 5.1. Since $\overline{\Delta_1}$ and $\overline{\Delta_2}$ are disjoint from $\omega(a)$ and $\alpha(a)$, there is a neighborhood $U_1 \subset U$ of $\omega(a)$ and a neighborhood $V_1 \subset V$ of $\alpha(a)$ such that $(\overline{U_1} \cup \overline{V_1}) \cap (\overline{\Delta_1} \cup \overline{\Delta_2}) = \emptyset$. By item 1) of Lemma 5.4, there is a point $y \in U_1$ with $\overline{\mathrm{Orb}(y)} \subset U_1$ such that $a \in W^s(y)$. Moreover, there exists a sequence $\{P_n\} \subset U_1$ of hyperbolic periodic orbits of index i and $p_n \in P_n$ such that $p_n \to y$. Likewise, there is a point $z \in V_1$ with $\overline{\mathrm{Orb}(z)} \subset V_1$ such that $a \in W^u(z)$. Moreover, there exists a sequence $\{Q_n\} \subset V_1$ of hyperbolic periodic orbits of index j and $q_n \in Q_n$ such that $q_n \to z$. Later, we will locate by these information two orbital segments passing near a, to create a connection going from Q to P. In the next two paragraphs we first create a connection going from P to Q.

To create a connection going from P to Q, we locate three orbital segments passing near a_1^* and a_2^*, respectively. Since $a_2^* \in \overline{W^s(Q)}$, there is $q^s \in W^s_\eta(Q)$ such that the negative S-orbit of q^s hits $B(a_2^*, \delta/\rho)$ at a point a^s. Note that $\mathrm{Orb}^+(q^s) \subset W^s_\eta(Q)$. If the S-orbit segment C from a^s to q^s contains a_1^*, then $a^s \in \overline{W^u(P)}$, and the proof will again be easier. Thus we assume $a_1^* \notin C$. In fact $a_1^* \notin C \cup \mathrm{Orb}^+(q^s)$, which forms a complete positive orbit (of a^s), hence $B \cap C = \emptyset$. Hence we can take $\delta^u \le \delta$ small such that the closure of the tube

$$\Delta^u = \cup_{t\in[0,T]} B\big(\phi_t\big(a_1^*\big), \delta^u\big)$$

is disjoint from C. Note that Δ^u has the same length as Δ_1, but thinner. Since $a_1^* \in \overline{W^u(P)}$, there is $p^u \in W^u_\eta(P)$ such that the positive S-orbit of p^u hits $B(a_1^*, \delta^u/\rho)$ at a point a^u. Note that $\mathrm{Orb}^-(p^u) \subset W^u_\eta(P)$. Finally, since a_1^*, $a_2^* \in K$, where K is a Hausdorff limit of a sequence of periodic orbits of S, there are sequences $x_n \in M$ and $s_n > 0$ with $x_n \to a_1^*$ and $\phi_{s_n}(x_n) \to a_2^*$. Fix n large such that $x_n \in B(a_1^*, \delta^u/\rho)$ and $x_n' = \phi_{s_n}(x_n) \in B(a_2^*, \delta/\rho)$. These three orbital segments, $[p^u, a^u]$, $[x_n, x_n']$ and $[a^s, q^s]$, are for creating a connection going from P to Q.

Now we apply Theorem 5.1 to get the ensured connection. Since p^u and x_n' are outside the tube Δ^u, and since the positive S-orbit of p^u and the negative S-orbit of x_n' both hit $B(a_1^*, \delta^u/\rho)$ (at a^u and x_n, respectively), by Theorem 5.1, there exists $X \in \mathcal{U}$ with $X = S$ outside Δ^u such that x_n' is on the positive X-orbit of p^u. We emphasize that the S-orbit segment C from a^s to p^s is unchanged because C is disjoint from $\overline{\Delta^u}$. Thus p^u and q^s are outside the tube Δ_2, and the positive X-orbit of p^u and the negative X-orbit of q^s both hit $B(a_2^*, \delta/\rho)$ (at x_n' and a^s, respectively). By Theorem 5.1, there exists $Y \in \mathcal{U}$ with $Y = X$ outside Δ_2 such that q^s is on the positive Y-orbit of p^u. Note that $\mathrm{Orb}^+(q^s, S)$ and $\mathrm{Orb}^-(p^u, S)$ are unchanged. Thus $\mathrm{Orb}(p^u, Y)$ gives a Y-heteroclinic connection from P to Q, that is, $p^u \in W^u(P, Y) \cap W^s(Q, Y)$. This is the connection created from P to Q.

Now we are to create the other connection, going from Q to P, without breaking the one just created. Note that Λ is unchanged. In particular $\mathrm{Orb}(a, Y) = \mathrm{Orb}(a, S)$, $\omega(a, Y) = \omega(a, S)$, and $\alpha(a, Y) = \alpha(a, S)$. Also note that by the last assertion in Theorem 5.1, $\mathrm{Orb}(p^u, Y)$ meets $B(a_2^*, \delta) \subset \Delta_2$. But $\mathrm{Orb}(a, Y) \subset \Lambda$ is disjoint from $\overline{\Delta_2}$, hence $\mathrm{Orb}(a, Y) \cap \mathrm{Orb}(p^u, Y) = \emptyset$. Thus the Y-orbit segment $D = \phi_{[0,T]}(a, Y)$ is disjoint from the compact invariant set $\Lambda^* = P \cup \mathrm{Orb}(p^u, Y) \cup Q$. Take δ_a small such that the closure of the tube

$$\Delta_a = \cup_{t\in[0,T]} B(\phi_t(a), \delta_a)$$

is disjoint from Λ^*. Note that Δ_a has the same length as Δ, but thinner.

We claim $a \in \overline{W^s(P, Y)} \cap \overline{W^u(Q, Y)}$. In fact, since $\Lambda \cap (\overline{\Delta^u} \cup \overline{\Delta_2}) = \emptyset$, the orbit of a is unchanged. That is, $\mathrm{Orb}(a, Y) = \mathrm{Orb}(a, S)$. Similarly, since $(U_1 \cup V_1) \cap (\overline{\Delta^u} \cup \overline{\Delta_2}) = \emptyset$, the orbit of the above $y \in U_1$, as well as the sequences P_n and p_n obtained by applying item 1) of Lemma 5.4 for S, are unchanged. Thus $\overline{\mathrm{Orb}(y, Y)} \subset U_1$ and $a \in W^s(y, Y)$ (note that $W^s(y)$ may change). The sequence $\{P_n\} \subset U_1$ now become Y-hyperbolic periodic orbits of index i with $p_n \in P_n$ and $p_n \to y$ (note that $W^s(P_n)$ may change). Take $L > 0$ large such that $a \in W^s_L(y, Y)$. Since compact parts of stable manifolds $W^s(x, Y)$ of Y vary continuously when x vary, for any neighborhood W of a in M, there is n such that $W^s_L(p_n, Y) \cap W \neq \emptyset$. But $Y \in \mathcal{W}$, hence any two Y-periodic orbits contained in U are H-related, hence $W^s(P, Y) \cap W \neq \emptyset$. Thus $a \in \overline{W^s(P, Y)}$. Likewise, $a \in \overline{W^u(Q, Y)}$. This proves the claim.

We locate two orbital segments passing near a, to create a connection going from Q to P. Since $a \in \overline{W^s(P, Y)}$, there is $p^s \in W^s_\eta(P, Y) = W^s_\eta(P, S)$ such that the negative Y-orbit of p^s hits $B(a, \delta_a/\rho)$ at a point b^s. Note that $\mathrm{Orb}^+(p^s, Y) \subset W^s_\eta(P, Y)$. Likewise, there is $q^u \in W^u_\eta(Q, Y) = W^u_\eta(Q, S)$ such that the positive Y-orbit of q^u hits $B(a, \delta_a/\rho)$ at a point b^u. Note that $\mathrm{Orb}^-(q^u, Y) \subset W^u_\eta(Q, Y)$. The two Y-orbital segments, $[q^u, b^u]$ and $[b^s, p^s]$, are for creating a connection going from Q to P.

We apply Theorem 5.1 to get the connection. Since q^u and p^s are outside the tube Δ_a, and since the positive Y-orbit of q^u and the negative Y-orbit of p^s both hit $B(a, \delta_a/\rho)$ (at b^u and b^s, respectively), by Theorem 5.1, there exists $Z \in \mathcal{U}$ with $Z = Y$ outside Δ_a such that p^s is on the positive Z-orbit of q^u. Note that $\mathrm{Orb}^+(p^s, X)$ and $\mathrm{Orb}^-(q^u, X)$ are unchanged. Thus $\mathrm{Orb}(q^u, Z)$ gives a Z-heteroclinic connection going from Q to P, that is, $q^u \in W^u(Q, Z) \cap W^s(P, Z)$. Since $\Delta_a \cap \Lambda^* = \emptyset$, the previously obtained Y-heteroclinic connection $\Lambda^* = P \cup \mathrm{Orb}(p^u, Y) \cup Q$ going from P to Q is unchanged. This gives a Z-heterodimensional cycle between P and Q, proving Lemma 6.1. □

Lemma 6.2 *Assume $S \in \mathcal{X}^*(M)$. Then there are no non-simple type minimally non-hyperbolic sets contained in $\overline{\mathrm{P}}(S) - \mathrm{Sing}(S)$.*

Proof Assume for the contrary there is a non-simple type minimally non-hyperbolic set $\Lambda \subset \overline{\mathrm{P}}(S) - \mathrm{Sing}(S)$. We may assume there is a resisting point $a \in \Lambda$ such that $\omega(a) = \Lambda$. Let

$$p = \min\{j : \text{There is a fundamental } j - \text{limit } K \subset \Lambda\}.$$

Note that by the C^1 closing lemma, Λ itself is a fundamental limit [19, p. 257]. Hence p is well-defined. Also note that $0 \neq p \neq d - 1$, because otherwise by Lemma 3.3, Λ would reduce to an expanding or contracting periodic orbit, contradicting that Λ is non-hyperbolic.

Claim 1 $\Lambda \subset \mathrm{P}_*^p(S)$.

Assume for the contrary that $\Gamma = \Lambda \cap \mathrm{P}_*^p(S)$ is a proper subset of Λ. Of course $\Gamma \neq \emptyset$. Since Λ is minimally non-hyperbolic, Γ is hyperbolic. Denote by Γ^p the part of Γ that has index p. Note that $\Gamma^p \neq \emptyset$. In fact, there is a fundamental p-limit $K \subset \Lambda$, hence by Lemma 3.11, there is a hyperbolic set $K' \subset K$ of index $q \leq p$, hence a fundamental q-limit contained in K'. (Any hyperbolic set of index q contains a minimal set, which by shadowing lemma is a fundamental q-limit.) By the minimality of p, $q = p$. This proves $\Gamma^p \neq \emptyset$. Clearly, Γ^p is compact and invariant, and is open in Γ. Note that $a \notin \Gamma^p$ since a is resisting. Take a small open neighborhood U of Γ^p in M such that any compact invariant set contained in $\overline{U}$ is hyperbolic of index p, and such that $a \notin \overline{U}$ and $\Gamma \cap \overline{U} = \Gamma^p$. Since $\Gamma^p \subset \omega(a)$ and $a \notin \Gamma^p$, there is a point $b \in \Lambda \cap \partial U$ such that $\mathrm{Orb}^+(b) \subset \overline{U}$. Then $\omega(b)$ is a hyperbolic set of index p. By Lemma 5.6, $b \in \mathrm{P}_*^p(S)$. Hence $b \in \Gamma$, contradicting $\Gamma \cap \overline{U} = \Gamma^p$. This proves Claim 1.

Thus by Lemma 3.6, there exists a p-dominated splitting

$$D_\Lambda = E \oplus F$$

over Λ.

Claim 2 E is contracting.

Otherwise, by Lemma 3.10, Λ would contain a fundamental q-limit with $q < p$, contradicting the minimality in the definition of p. This proves Claim 2.

Denote by G the set of points $x \in \Lambda$ which satisfy the following condition

$$\limsup_{n\to+\infty} \frac{1}{n\widetilde{T}} \sum_{j=0}^{n-1} \eta_+\big(F\big(\phi_{j\widetilde{T}}(x)\big), \widetilde{T}\big) \geq \widetilde{\eta}. \tag{6.22}$$

Obviously, $\overline{G}$ is a nonempty compact invariant subset of Λ.

Claim 3 $\overline{G} = \Lambda$.

Since $\Lambda = \omega(a)$, it suffices to prove $a \in \overline{G}$. Assume for the contrary $a \notin \overline{G}$. Then $\overline{G}$ is a proper subset of Λ. Since Λ is minimally nonhyperbolic, $\overline{G}$ is hyperbolic. According to the definition of G (6.22), $\overline{G}$ has index p. In fact, by the definition of G, the subbundle F (restricted to G) contains no contracting vectors. Since E (with $\dim E = p$) is contracting, $\overline{G}$ has index p.

Take a small neighborhood U of $\overline{G}$ in M such that $a \notin \overline{U}$ and every compact invariant set contained in U is hyperbolic of index p. Since $a \notin \overline{U}$, we can find $b \in \Lambda \cap (\partial U)$ such that $\mathrm{Orb}^+(b) \subset \overline{U}$ and hence $\omega(b)$ is hyperbolic of index p. (Note that every hyperbolic set in Λ with index p is contained in $\overline{G}$. Thus $\omega(b) \subset \overline{G}$.) In particular, $b \notin G$. That is,

$$\limsup_{n\to+\infty} \frac{1}{n\widetilde{T}} \sum_{j=0}^{n-1} \eta_+\big(F\big(\phi_{j\widetilde{T}}(b)\big), \widetilde{T}\big) < \widetilde{\eta}.$$

Hence there exists $0 < \eta < \widetilde{\eta}$ such that for n large enough,

$$\frac{1}{n\widetilde{T}} \sum_{j=0}^{n-1} \eta_+\big(F\big(\phi_{j\widetilde{T}}(b)\big), \widetilde{T}\big) \leq \eta.$$

Take a subsequence $\{n_i\}$ such that $\phi_{n_i\widetilde{T}}(b) \to c \in \omega(b)$.

Fix $\eta_1 < \eta_2$ in $(\eta, \widetilde{\eta})$. For every i, if $k - i$ is sufficiently large, then

$$\frac{1}{(n_k - n_i)\widetilde{T}} \sum_{j=n_i}^{n_k-1} \eta_+\big(F\big(\phi_{j\widetilde{T}}(b)\big), \widetilde{T}\big) \leq \eta_1.$$

For i large, the orbital arc $A = \phi_{[n_i\widetilde{T}, n_k\widetilde{T}]}(b)$ is contained in a small neighborhood of the hyperbolic set $\omega(b)$, and the two end points of A that corresponding to time $n_i\widetilde{T}$ and $n_k\widetilde{T}$ are very close (they are both near c). Thus A can be shadowed by a periodic orbit P such that for some $x \in P$ and some partition

$$0 = t_0 < t_1 < \cdots < t_{n_k - n_i} = \tau$$

of $[0, \tau]$ (τ is the period of P), $t_{j+1} - t_j$ is approximately $\widetilde{T}$ ($j = 0, 1, \cdots, n_k - n_i - 1$), and

$$\frac{1}{\tau} \sum_{j=0}^{n_k - n_i - 1} \eta_+ \big(D^u(\phi_{t_j}(x)), t_{j+1} - t_j\big) \le \eta_2.$$

This contradicts Theorem 3.1, proving Claim 3.

Thus by Theorem 3.8, Λ is hyperbolic (of index p), contradicting that Λ is non-hyperbolic. This proves Lemma 6.2. □

We remark that, since any non-simple type minimally non-hyperbolic set Λ is the ω-limit set of a (resisting) point a and hence is a fundamental limit, by item 3) of Lemma 3.11, Λ contains two hyperbolic sets Γ_1 and Γ_2 of different indices. Thus to prove Lemma 6.2, it would be natural at the first place to try to create a heterodimensional cycle. We have been unable to go this way. (Unlike the case of Lemma 6.1, we do not have in advance one heteroclinic connection.)

7 The proof of Theorem A

We first introduce a generalized shadowing lemma. It generalizes the standard shadowing lemma for hyperbolic set.

Let ϕ_t be the flow generated by $S \in \mathcal{X}^*(M)$. As usual, given $L > 0$ and $\alpha > 0$, $\{t_i, x_i\}_{i=-\infty}^{\infty}$ will be called an (L, α) *pseudo-orbit* if $d(\phi_{t_i}(x_i), x_{i+1}) \le \alpha$, $t_i \ge L$. We will say a point $y \in M$ *ε-shadows a pseudo-orbit* $\{t_i, x_i\}_{i=-\infty}^{\infty}$ if there exists an orientation-preserving homeomorphism $g : \mathbf{R} \to \mathbf{R}$, $g(0) = 0$ such that $d(\phi_{g(t)}(y), \phi_{t-T_i}(x_i)) \le \varepsilon$ for $T_i \le t \le T_{i+1}$, where T_i is defined as

$$T_i = \begin{cases} t_0 + \cdots + t_{i-1}, & \text{if } i > 0, \\ 0, & \text{if } i = 0, \\ -t_{-1} - \cdots - t_i, & \text{if } i < 0. \end{cases} \tag{7.23}$$

If for some $m \ge 1$, $x_i = x_{i+m}, t_i = t_{i+m}$ for all i, then $\{t_i, x_i\}$ will be called a *periodic pseudo-orbit*. It is well known that hyperbolic sets have the shadowing property.

Let $\Lambda \subset M - \mathrm{Sing}(S)$ be a closed invariant set of $S \in \mathcal{X}(M)$ that has a continuous invariant splitting $D_\Lambda = E \oplus F$ with $\dim E = p$, $1 \le p \le d - 2$. For two real numbers $T > 0$ and $\eta > 0$, an orbit arc $\{x, t\} = \phi_{[0,t]}(x)$ will be called an *(η, T, p)-quasi hyperbolic orbit arc* of S with respect to the splitting $E \oplus F$, if $[0, t]$ has a partition

$$0 = t_0 < t_1 < \cdots < t_l = t$$

such that $t_k - t_{k-1} \in [T, 2T]$, $k = 1, 2, \cdots, l$, with the following three conditions are satisfied:

$$\frac{1}{t_k} \sum_{j=1}^{k} \eta_-\big(E(\phi_{t_{j-1}}(x)), t_j - t_{j-1}\big) \le -\eta, \tag{7.24}$$

$$\frac{1}{t_l - t_{k-1}} \sum_{j=k}^{l} \eta_+\big(F(\phi_{t_{j-1}}(x)), t_j - t_{j-1}\big) \ge \eta, \tag{7.25}$$

$$\eta_+\big(F(\phi_{t_{k-1}}(x)), t_k - t_{k-1}\big) - \eta_-\big(E(\phi_{t_{k-1}}(x)), t_k - t_{k-1}\big) \ge 2\eta \tag{7.26}$$

for $k = 1, 2, \cdots, l$.

Quasi hyperbolic orbit arcs are conceptually weaker than hyperbolic orbit arcs. Nevertheless they also have the shadowing property as the following lemma asserts.

Lemma 7.1 [8,18,21] *Let $S \in \mathcal{X}(M)$ and Λ be a closed invariant set containing no singularity. Assume there exists a continuous invariant splitting $D_\Lambda = E \oplus F$ over Λ and* $\dim E = p$*, $1 \le p \le d-2$. Then for any $\eta > 0, T > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that if $\{x_i, t_i\}_{i=-\infty}^{\infty}$ is a (T, δ) pseudo-orbit and if, for every i, $\{x_i, t_i\}$ is an (η, T, p)-quasi hyperbolic orbit arc of S with respect to the splitting $E \oplus F$, then there exists $y \in M$ ε-shadowing $\{x_i, t_i\}$. Moreover, if $\{x_i, t_i\}$ is periodic, then y can be taken to be a periodic point.*

Important cases of Lemma 7.1 that involve the main idea were proved by Liao [18] and Mañé [21]. A proof for the general case can be found in [8].

We also need the following classical result of Liao [18].

Proposition 7.2 [18] *Let $S \in \mathcal{X}(M)$. If* $\mathrm{P}_*(S)$ *is hyperbolic, then S satisfies Axiom A and the no-cycle condition.*

Note that Proposition 7.2 does not assume the star condition and is for general flows. Now we proceed to the main result of this section.

Proposition 7.3 *Let $S \in \mathcal{X}^*(M)$. If* $\mathrm{P}_*(S) \cap \mathrm{Sing}(S) = \emptyset$*, then* $\mathrm{P}_*(S) = \overline{\mathrm{P}}(S)$.

Note that Proposition 7.3 may not hold if $\mathrm{P}_*(S) \cap \mathrm{Sing}(S) \neq \emptyset$, see examples in [4,14].

Proof Since $\overline{\mathrm{P}}(S) \subset \mathrm{P}_*(S)$, we have $\overline{\mathrm{P}}(S) \cap \mathrm{Sing}(S) = \emptyset$. By Theorem B, $\overline{\mathrm{P}}(S)$ is hyperbolic (treating $\overline{\mathrm{P}}(S)$ itself to be Λ of Theorem B). Hence S has the $\overline{\mathrm{P}}$-spectral decomposition

$$\overline{\mathrm{P}}(S) = B_1 \cup B_2 \cup \cdots \cup B_s,$$

where B_i, $1 \le i \le s$, are the $\overline{\mathrm{P}}$-basic sets of S.

Lemma 7.4 *There are no cycles among $B_1, B_2, \cdots, B_s$.*

Proof of Lemma 7.4 Suppose without loss of generality there is a cycle among $B_1, B_2, \cdots, B_k$, $1 \le k \le s$, i.e., suppose there exist points $a_i \in M - \overline{\mathrm{P}}(S)$, $1 \le i \le k$ such that $\alpha(a_i) \subset B_i$ and $\omega(a_i) \subset B_{i+1}$ $(k+1 \equiv 1)$.

We claim $\operatorname{Ind} B_i = \operatorname{Ind} B_j$ for all $1 \le i < j \le k$. Suppose on the contrary $\operatorname{Ind} B_i \ne \operatorname{Ind} B_j$ for some $1 \le i < j \le k$. For each l with $i \ne l \ne j$, by λ-lemma, there is an orbital arc A_l going from a point x_{l-1} near a_{l-1} to a point x_l near a_l. By one-step-pushes near a_l (as in the beginning of the proof of Theorem 4.1) for all $i \ne l \ne j$ simultaneously, we can get a 2-cycle between the two basic sets B_i and B_j. (Here we do not need the connecting lemma, because the orbital arcs can be chosen not to wrap around, and not to be close to each other except at the end points.) Since periodic orbits are dense in B_i and B_j, with a further perturbation we can get a heteroclinic cycle between two periodic orbits $P \subset B_i$ and $Q \subset B_j$, which is a heterodimensional one because P and Q have different indices. (Here we do not need the connecting lemma, for the same reason.) This contradicts Theorem 4.1, proving the claim.

Thus $\operatorname{Ind} B_i$ are all the same, say r, for all $1 \le i \le k$. It is easily seen that there is an r-dominated splitting $T_\Gamma M = E \oplus F$ over the cycle $\Gamma = \cup_{i=1}^{k}(B_i \cup \operatorname{Orb}(a_i))$. Since $\dim D^s(a_i) = r$, $\dim D^u(a_i) = d - r - 1$, according to Lemma 3.5, we have $D^s(a_i) = E(a_i)$ and $D^u(a_i) = F(a_i)$ for $1 \le i \le k$. So the cycle is a transversal cycle and hence $a_i \in \overline{\mathrm{P}}(S)$. This contradiction proves Lemma 7.4. □

We continue the proof of Proposition 7.3. Suppose for the contrary $\mathrm{P}_*(S) \ne \overline{\mathrm{P}}(S)$. Since $\mathrm{P}_*(S)$ is the union of the set of fundamental limits of S, there exists a fundamental limit Γ of S such that $\Gamma - \overline{\mathrm{P}}(S) \ne \emptyset$. Let

$$k = \min\{i : \text{ there is a fundamental limit } \Gamma \text{ with } \Gamma - \overline{\mathrm{P}}(S) \ne \emptyset \text{ such that } \Gamma \text{ intersects exactly } i \text{ of the } \overline{\mathrm{P}}\text{–basic sets of } S\}.$$

Fix Λ a fundamental limit of S with $\Lambda - \overline{\mathrm{P}}(S) \ne \emptyset$ such that Λ intersects exactly k of the $\overline{\mathrm{P}}$-basic sets of S. Since $\mathrm{P}_*(S) \cap \operatorname{Sing}(S) = \emptyset$, it follows that $\Lambda \cap \operatorname{Sing}(S) = \emptyset$. Note that $k \ne 0$. In fact, if Λ is hyperbolic, it obviously intersects $\overline{\mathrm{P}}(S)$ (any nonsingular hyperbolic set contains a nonsingular hyperbolic minimal set, which is contained in $\overline{\mathrm{P}}(S)$). If Λ is non-hyperbolic, by Lemma 3.11, Λ contains hyperbolic sets, hence intersects $\overline{\mathrm{P}}(S)$ anyway. We may assume Λ intersects $B_1, B_2, \cdots, B_k$. Let

$$m = \min\{\operatorname{Ind} B_1, \operatorname{Ind} B_2, \cdots, \operatorname{Ind} B_k\}.$$

We may assume $\operatorname{Ind} B_1 = m$. Note that $0 \ne m \ne d-1$ because otherwise B_1 is an expanding or contracting periodic orbit, hence by Lemma 3.3 $\Lambda = B_1$, contradicting $\Lambda - \overline{\mathrm{P}}(S) \ne \emptyset$.

We claim that, for each $1 \le i \le k$, there is a point $a_i \in W^s(B_i) \cap \Lambda - \overline{\mathrm{P}}(S)$. Indeed, since Λ is a fundamental limit set, there exist $X_n \to S$ and periodic orbits P_n of X_n such that $P_n \to \Lambda$ in the Hausdorff metric. Take $x \in \Lambda - \overline{P}(S)$. Then there exists $x_n \in P_n$ such that $x_n \to x$. Given $1 \leq i \leq k$, since $B_i \cap \Lambda \neq \emptyset$, there exists $y_n \in P_n$ such that $y_n \to y \in B_i$. Since B_i is a basic set, there exists a small neighborhood U of B_i such that for any positive orbit $\mathrm{Orb}^+(z) \subset U$, we have $z \in W^s(B_i)$. Fix such a (small closed) neighborhood U of B_i such that $x \notin U$ and $U \cap \overline{P}(S) = B_i$. Let $[t_n, s_n]$ be the largest interval such that $0 \in [t_n, s_n]$ and for every $t \in [t_n, s_n]$, $\phi_{X_n t}(y_n) \in U$. Let $z_n = \phi_{X_n t_n}(y_n) \in P_n$ and $\tau_n = s_n - t_n$. Since $x_n \in P_n$ and $x_n \notin U$, we have $z_n \in \partial U$. Then $\phi_{X_n [0,\tau_n]}(z_n) \subset U$. Taking subsequences if necessary, we may assume $z_n \to a_i \in \partial U \cap \Lambda$. It is easy to see $\tau_n \to \infty$ because, otherwise, by the continuity of $\phi_{Xt}(z)$ with respect to X, t, z, we would get $a_i \in B_i$, contradicting $a_i \in \partial U$. Again, by the continuity of $\phi_{Xt}(z)$ with respect to X, t, z, we have $\mathrm{Orb}^+(a_i) \subset U$ and hence $a_i \in W^s(B_i)$. Thus $a_i \in W^s(B_i) \cap \Lambda \cap \partial U \subset W^s(B_i) \cap \Lambda - \overline{P}(S)$, proving the claim.

If $\alpha(a_1) \subset B_{i_2}$ for some $1 \leq i_2 \leq k$, we go on to look at a_{i_2}. If $\alpha(a_{i_2}) \subset B_{i_3}$ for some $1 \leq i_3 \leq k$, we go on to look at a_{i_3}. If this process goes without end, we would trace out a cycle among B_i, $1 \leq i \leq k$, contradicting Lemma 7.4. Thus for some $l \geq 0$, $\alpha(a_{i_l})$ is not contained in any of B_i, $1 \leq i \leq k$.

Claim $\mathrm{Ind}\, B_{i_l} = m$, and $\alpha(a_{i_l}) \cap B_i \neq \emptyset$ for any $1 \leq i \leq k$.

Since $\alpha(a_{i_l})$ is not contained in any of B_i, $1 \leq i \leq k$, there is $b \in \alpha(a_{i_l}) - \overline{\mathrm{P}}(S)$. According to the C^1 closing lemma, we can find a fundamental q-sequence (Q_n, Y_n) such that $Q_n \to \Gamma$ in the sense of Hausdorff and $b \in \Gamma \subset \alpha(a_{i_l})$. Suppose for the contrary $\alpha(a_{i_l}) \cap B_j = \emptyset$ for some $1 \leq j \leq k$, then $\Gamma \cap B_j = \emptyset$. Since $\Gamma \subset \alpha(a_{i_l}) \subset \Lambda$, the number of $\overline{\mathrm{P}}$-basic sets which intersect Γ would be less than k, contradicting the minimality of k. Thus $\alpha(a_{i_l}) \cap B_i \neq \emptyset$ for any $1 \leq i \leq k$.

We prove $\mathrm{Ind}\, B_{i_l} = m$. Suppose for the contrary $\mathrm{Ind}\, B_{i_l} \neq m$. Since $\alpha(a_{i_l}) \cap B_1 \neq \emptyset$ we can take $b \in \alpha(a_{i_l}) \cap W^u(B_1) - \overline{\mathrm{P}}(S)$. By using the C^1 connecting lemma, we will obtain a cycle among $B_1 = B_{i_1}, \cdots, B_{i_l}$. Since $\mathrm{Ind}\, B_1 = m \neq \mathrm{Ind}\, B_{i_l}$, by a similar argument in the proof of Lemma 7.4, we will get a heterodimensional cycle, contradicting Theorem 4.1. This proves the claim.

Thus $a_{i_l} \in W^s(B_{i_l}) \cap \Lambda - \overline{\mathrm{P}}(S)$, and $\alpha(a_{i_l}) \cap B_{i_l} \neq \emptyset$. Since B_{i_l} is a basic set hence the In Phase Theorem holds, a standard application of the C^1 connecting lemma shows $a_{i_l} \in \mathrm{P}_*^m(S)$. Since $\mathrm{Ind}\, B_{i_l} = m$, we have $B_{i_l} \cup \overline{\mathrm{Orb}}(a_{i_l}) \subset \mathrm{P}_*^m(S)$. Then there is an m-dominated splitting $D_\Delta = E \oplus F$ over $\Delta = B_{i_l} \cup \overline{\mathrm{Orb}}(a_{i_l})$. Moreover, there is a fundamental m-sequence (Q_n, Y_n) such that $Q_n \to \Gamma \subset B_{i_l} \cup \overline{\mathrm{Orb}}(a_{i_l}) \subset B_{i_l} \cup \Lambda$. Hence if E is not contracting, by Lemma 3.11, Δ would contain a hyperbolic subset with index $j < m$, hence

$$\Lambda \cap \overline{\mathrm{P}}_j(S) = (B_{i_l} \cup \Lambda) \cap \overline{\mathrm{P}}_j(S) \supset \Gamma \cap \overline{\mathrm{P}}_j(S) \neq \emptyset$$

for some $j < m$, contradicting the minimality of m.

Thus E is contracting. Since B_{i_l} is hyperbolic, there exist $T > \widetilde{T} > 0$, $\eta \in (0, \widetilde{\eta}]$ such that for any $x \in B_{i_l}, t \geq T$, $\eta_+(F(x), t) > \eta$, and for any $x \in \Delta$, $\eta_-(E(x), T) < -\eta$. Take a small enough neighborhood U of B_{i_l} so that for any $x \in \overline{\mathrm{Orb}}(a_{i_l}) \cap U$, $\eta_+(F(x), T) \geq \eta$. We may assume $\mathrm{Orb}^+(a_{i_l}), \mathrm{Orb}^-(b) \subset U$. For any $\varepsilon > 0$, fix a large T' so that $d(\phi_{-T'}(a_{i_l}), b) < \varepsilon$. It is easy to see that for T'' large enough, $\phi_{[-T',T'']}(a_{i_l})$ is an $(m, T, \eta/2)$-quasi hyperbolic orbit arc. Apply Lemma 7.1 to the pseudo-orbit $\phi_{(-\infty,0]}(b) \cup \phi_{[-T',T'']}(a_{i_l}) \cup \phi_{[T'',+\infty)}(a_{i_l})$. Let $c = c_\varepsilon$ be a shadowing point that shadows a_{i_l}. It is easy to see that c_ε is a transverse homoclinic point of B_{i_l} with $\lim_{\varepsilon\to 0} c_\varepsilon = a_{i_l}$. That means c_ε is approximated by periodic orbits of S and hence a_{i_l} is approximated by periodic orbits of S, contradicting $a_{i_l} \notin \overline{\mathrm{P}}(S)$. This proves Proposition 7.3. □

Now we finish the proof of Theorem A. It suffices to prove the following refined version of Theorem A.

Theorem A′ *If $S \in \mathcal{X}^*(M)$, and if $\mathrm{P}_*(S) \cap \mathrm{Sing}(S) = \emptyset$, then S satisfies Axiom A and the no-cycle condition.*

Proof By Theorem B, $\overline{\mathrm{P}}(S)$ (treated as Λ of Theorem B) is hyperbolic. By Proposition 7.3, $\mathrm{P}_*(S) = \overline{\mathrm{P}}(S)$. Thus $\mathrm{P}_*(S)$ is hyperbolic. By Proposition 7.2, S satisfies Axiom A and the no cycle condition. This proves Theorem A′. □

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