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Factor and normal subgroup theorems for lattices in products of groups*

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1. Introduction

A central result in the theory of semisimple groups and their lattices is Margulis' normal subgroup theorem: any normal subgroup of an irreducible lattice in a center free, higher rank semisimple group, has finite index [Mar79,Mar91]. In the present paper we establish a Margulis-type theorem for a large family of lattices, including all Kac-Moody groups over (sufficiently large) finite fields. As in Margulis' strategy, we establish along the way a "factor theorem" for measurable quotients of boundaries, which is of independent interest. Its proof introduces new ideas relying heavily on Furstenberg's boundary theory (pertaining to harmonic functions, random walks and stationary measures for group actions). An adelic extension of the latter, and factor theorems in which the boundary is *not* a homogeneous space, follow as well.

Recall that a group is called **just infinite**, if every non-trivial normal subgroup of it has finite index. The elementary observation that every finitely generated *infinite* group admits an *infinite just infinite quotient*, is one motivation for studying this property (see [Wil00] for more on the general structure of such groups). Extending this notion, we shall call a topological group G **just non-compact**, if every non-trivial closed normal subgroup $N \triangleleft G$ is co-compact, and **topologically just infinite**, if every such N has finite index. Of course, for an abstract group G, all the three notions agree when it is viewed as a topological group with discrete topology.

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Theorem 1.1 (Normal subgroup theorem). *Let G_1, G_2 be locally compact, non-discrete, compactly generated groups, and let $\Gamma < G = G_1 \times G_2$ be a discrete co-compact subgroup which is irreducible in G, i.e., its natural projections to both G_i's have dense images. Assume that not both G_i's are isomorphic to $\mathbb{R}$ – the additive group of real numbers. If both G_i's are just non-compact, then Γ is just infinite.*

Note that the exception $N = \mathbb{Z} \lhd \mathbb{Z}^2 = \Gamma < G = \mathbb{R}^2$ (with $\mathbb{Z}^2$ "irrationally embedded"), is indeed a counterexample (the unique one) to the general statement of the theorem. It turns out that many locally compact groups arising as "sufficiently transitive" automorphism groups are just non-compact, thereby giving rise to a rich source of discrete groups to which the theorem applies. Distinguished examples are simple algebraic groups, which over local fields of positive characteristic are not always topologically just infinite. A result of Burger-Mozes [BM00a, Corollary 1.5.2] provides others, and locally compact groups with a (Tits-) (B, N)-pair for a compact B, essentially satisfy this property as well. Actually, our normal subgroup theorem holds also when Γ is only assumed to have finite co-volume in G (i.e. it is a lattice), provided an additional technical condition is satisfied. This will be crucial for one of the main new applications, elaborated upon in Theorem 1.5 below.

The scheme of proof of Theorem 1.1 relies on the same remarkable general strategy introduced by Margulis, implemented via new ideas. Namely, to show that a quotient Γ/N is finite, one shows that it is both amenable and has Kazhdan's property (T). These two properties are established in the following two completely independent results, which shall be referred to as the two "halves" of the proof of Theorem 1.1. Throughout the rest of the paper all locally compact groups are assumed second countable, and the irreducibility of a subgroup $\Gamma < G = G_1 \times G_2$ has the same meaning as in Theorem 1.1 above.

The first, "property (T) half", was established by the second named author in [Sha00a]:

Theorem 1.2 (Property (T) half [Sha00a, Theorem 0.1]**).** *Let G_1, G_2 be locally compact groups, and let $\Gamma < G = G_1 \times G_2$ be an irreducible lattice. Assume further that both G_i's are compactly generated and that Γ is co-compact. Let N be a normal subgroup of Γ. Then Γ/N has property (T) of Kazhdan if (and only if) the following two conditions are satisfied:*

- **(i)** *For every i the quotient $G_i/\overline{pr_i(N)}$ has property T (where $\overline{pr_i(N)}$ denotes the closure of the projection pr_i of N to G_i).*
- **(ii)** *If $\varphi : G \to \mathbb{R}$ is a continuous homomorphism which vanishes on N, then $\varphi = 0$.*

Notice that by irreducibility of Γ in G, $\overline{pr_i(N)}$ is indeed normal in G_i. The same unique counterexample to Theorem 1.1 accounts for the appearance of condition **(ii)**. The proof of this theorem is based on a study of the first *reduced* cohomology of unitary representations, and involves a rigidity theorem for irreducible lattices in that setting. We next state the "second half" of proof of Theorem 1.1, which was the original motivation for our present work:

Theorem 1.3 (Amenability half). *Let $\Gamma < G = G_1 \times G_2$ be as in the first sentence of Theorem 1.2. Let $N \lhd \Gamma$ be a normal subgroup. Then Γ/N is amenable if (and only if) for every i the quotient $G_i/\overline{pr_i(N)}$ is amenable.*

In Sect. 4 we derive the following concrete application:

Corollary 1.4 (Buildings' lattices). *Let Δ_1, Δ_2 be infinite, locally finite, thick irreducible buildings, and let $\Gamma < Aut(\Delta_1) \times Aut(\Delta_2)$ be a lattice. Assume that the closures of the two projections of Γ act strongly transitive on the corresponding buildings (i.e., they act transitively on pairs of chamber $\subseteq$ apartment). Then:*

1. *Every proper quotient of Γ is amenable.*
2. *If Γ is co-compact then it is just infinite.*
3. *If the minimal thickness of each Δ_i is sufficiently large (e.g. $> 2^{11 dim \Delta_i}$), and the entries of the Coxeter matrices associated to both Δ_i are finite, then again Γ is just infinite.*

In the case where Δ_1, Δ_2 are trees, part **2** strengthens Theorem 4.1 of [BM00b], by dispensing with the assumption that the closures of the Γ-projections are topologically just infinite. This, together with the many new examples of irreducible tree lattices built in [Rat03], may provide an additional rich source of groups to which Burger-Mozes' method for constructing simple groups may be applied.

At this point the reason for the co-compactness assumption on Γ in Theorem 1.1 is transparent, going back to the "property T half". However, in [Sha00a] it is shown that Theorem 1.2 does hold also in the non-cocompact case, provided a certain "square-integrability" (hereafter abbreviated **S-I**) condition holds for *some* (cocycle defined on a) fundamental domain of Γ. This condition makes sense for an arbitrary *finitely generated* lattice Γ in a locally compact group G. In [Sha00a] it is shown to be satisfied for all higher rank (S-arithmetic) irreducible lattices, and in [Sha00b] for all lattices in rank one simple Lie groups, excluding $SL_2(\mathbb{R})$ and $SL_2(\mathbb{C})$ (where it does not hold). The condition ensures that for any unitary Γ-representation π, a naturally defined injection $H^1(\Gamma, \pi) \hookrightarrow H^1(G, Ind_\Gamma^G \pi)$ exists (in this context the condition is indeed necessary, as examples with $SL_2(\mathbb{C})$ show). Simultaneously with the completion of the present paper and motivated by its results, Bertrand Rémy showed that in the situation relevant to the result described next, the **S-I** condition holds. Together with Theorem 1.3 above, this implies the following, which is new in all the non-affine cases (those not covered by Margulis' theorem):

Theorem 1.5 (Kac-Moody normal subgroup theorem – joint with Rémy). *Let F_q be the finite field of q elements, and $\mathbf{\Lambda_q}$ be the F_q-points of the Kac-Moody group (or "group valued functor") $\mathbf{\Lambda}$, having irreducible Weyl group W generated by the Coxeter generators S. Then for $q \geq |S|$, every normal subgroup of the (finitely generated) group $\mathbf{\Lambda_q}$ is either contained in its finite center, or has finite index.*

Indeed, Rémy showed in [Rém99], that for $q \geq |S|$ the group $\Gamma = \mathbf{\Lambda_q}$ fits, modulo its finite center, in the setting of part **1** in Corollary 1.4 above, where Δ_1, Δ_2 are isomorphic to the building associated with its (B, N)-pair structure. In order to pass from "amenable" to "finite" quotient Γ/N in the conclusion of part **1**, one appeals, as mentioned, to the version of Theorem 1.2 for non-uniform lattices, made available through the verification of the **S-I** condition by Rémy [Rém04]. We further remark that the ambient factors G_i appearing in the proof of Theorem 1.5, have typically a very different structure from that of algebraic groups (or automorphism groups of trees). They may not admit an amenable co-compact subgroup or a Gelfand pair structure, nor does Howe-Moore's (or Kazhdan's) property hold for them in general (see Sect. 5). Among his many results, Rémy produces examples of $\mathbf{\Lambda_q}$ having no infinite linear representations over any field (see [Rém03] and the references therein for this and further recent developments in this direction; see also the related work of Carbone-Garland [CG99]). In fact, these Kac-Moody lattices may be viewed as members of the yet more general family of groups with so-called "twin root datum", to which a similar finiteness theorem will apply (see [RR02] and the references therein for more on these groups).

Changing gear, the following result may be viewed as a generalization of Theorem 1.1, when the ambient groups G_i have in addition Kazhdan's property (T):

Theorem 1.6 (Essentially free actions). *Let G_1, G_2 be locally compact, just non-compact Kazhdan groups, and let $G = G_1 \times G_2$. Then every probability measure preserving action of G on a Lebesgue space, for which each G_i acts ergodically, is either essentially transitive, or else essentially free (i.e., the stabilizer of almost every point is trivial).*

Such result is not true in general for one simple (Kazhdan) group G. It is easy to see that for **any** non-discrete groups G_1, G_2, if $G = G_1 \times G_2$ satisfies the conclusion of the theorem, then G has the property that all its irreducible lattices are just infinite. Thus, the result may also be viewed as a generalized "virtual" (a la Mackey) normal subgroup theorem. We shall indeed see that the just infinity property of some of the Kac-Moody groups discussed in this paper follow from this result as well (this will be the case for the $\mathbf{\Lambda_q}$ which themselves have property (T)). There are, in addition, uncountably many other (discrete) groups G to which Theorem 1.6 applies. In the next subsection we describe an "intermediate factor theorem" established for the proof of the normal subgroup theorem, from which Theorem 1.6 follows as in the work of Stuck and Zimmer [SZ94], who originally established an analogous result for semisimple Lie groups.

On the approach and further main results. We next discuss our approach toward the above results, which is based on Furstenberg's boundary theory. In the next section we shall overview elements of this theory, and establish the basic ingredients required for our analysis, assuming no prior familiarity. Suffices it to say at this point that for any locally compact (second countable) group G, and any *admissible* probability measure μ on it, there is a uniquely defined measure space with a quasi-invariant, μ-stationary probability measure, (B, ν_B) (always non-trivial when G is non-amenable), called the **Poisson** boundary of (G, μ), which "minimally represents" all the *bounded* μ-harmonic functions on G in a suitable way. Here and throughout the following, as usual **admissible** stands for *being continuous with respect to the Haar measure and not being supported on a proper closed sub-semigroup*. The measure ν_B is said to be **μ-stationary** if the convolution equation $\mu * \nu_B = \nu_B$ is satisfied. We remark that throughout the paper it is in fact possible to replace the admissibility property by the weaker "spread-out" assumption on the measure μ, but this does not yield any generalization of our results.

Theorem 1.7 (Factor Theorem). *Let μ_1, μ_2 be admissible measures on the locally compact groups G_1, G_2, set $G = G_1 \times G_2$, $\mu = \mu_1 \times \mu_2$, and denote by (B, ν_B) the Poisson boundary of (G, μ). Let $\Gamma < G$ be an irreducible lattice, (Z, ζ) a measure Γ-space, and $\psi : (B, \nu_B) \to (Z, \zeta)$ a measurable Γ-map (pushing ν_B to ζ).*

Then there exists a measure G-space (C, ν_C), which as a Γ-space is isomorphic (via a Γ-map τ) to (Z, ζ), and a G-map $\pi : (B, \nu_B) \to (C, \nu_C)$, such that after identifying (a.e.) C and Z through the map τ, we have $\psi = \pi$ ν_B-a.e.

Furthermore, as a G-measure space (C, ν_C) is isomorphic to a product $(C_1, \nu_1) \times (C_2, \nu_2)$, where for each i (C_i, ν_i) is a G_i-space with a μ_i-stationary measure ν_i.

This result is the central ingredient in the proof of Theorem 1.3. In the proof of Margulis' normal subgroup theorem, his factor theorem implies the amenability of a quotient group Γ/N via a rather straightforward argument. The proof of the analogous implication (Theorem 1.7 $\Rightarrow$ Theorem 1.3) we have, involves an additional worth mentioning ingredient. This is Furstenberg's conjecture, established independently by Rosenblatt [Ros81] and Kaimanovich-Vershik [KV83], guaranteeing the existence, on any locally compact amenable group H, of some admissible measure μ whose Poisson boundary is trivial. Indeed, the measures μ_i taken on each G_i in the proof that Theorem 1.7 $\Rightarrow$ Theorem 1.3, are lifts of such chosen measures on $H_i = G_i/\overline{pr_i(N)}$, to G_i. An alternative argument not involving the proof of Furstenberg's conjecture was suggested to us by Gregory Margulis after the completion of this work, and will be presented in a forthcoming paper.

Factor-type theorems are of interest in their own right, as one of the outstanding manifestations of rigidity. In Sect. 5 we give various new and "non-standard" situations in which the above factor theorem can be made concrete. This includes, notably, an *adelic* factor theorem; results where the boundary is homogeneous but *non-compact*, as well as *non-homogeneous* factor theorems for which the measurable quotients of the ambient group G are themselves not identified. This latter phenomenon occurs in the general geometric setting covered by the following result:

Corollary 1.8 (CAT(-1) Factor Theorem). *Let X_1, X_2 be proper $CAT(-1)$ spaces, $\Gamma < \mathrm{Iso}(X_1) \times \mathrm{Iso}(X_2)$ be a discrete subgroup, and $G_i = \overline{pr_i(\Gamma)}$ ($i = 1, 2$). Assume that each G_i is non-elementary, and that Γ is a lattice in $G = G_1 \times G_2$. Then:*

1. *For any admissible measures μ_i on G_i ($i = 1, 2$), there is a unique $\mu = \mu_1 \times \mu_2$-stationary measure ν on the product of the two boundaries $\partial X_1 \times \partial X_2$, and we have: Every Γ-equivariant measurable factor of $(\partial X_1 \times \partial X_2, \nu)$ can be made a G-space, in such a way that the Γ-factor map is (ν-a.e.) equivariant under all of G.*
2. *If each G_i is convex co-compact (e.g., if it acts co-compactly on X_i) then the last rigidity statement of* **1** *above holds for the measure $\nu = \nu_1 \times \nu_2$ on $\partial X_1 \times \partial X_2$, with ν_i being the Patterson-Sullivan measure (class) on ∂X_i associated with G_i ($i = 1, 2$).*

Part **2**, which has a natural geometric setting, follows from part **1** using a recent result of Connell and Muchnik [CM03]. Corollary 1.8 also generalizes the factor theorem of Burger-Mozes [BM00b], in which the X_i's are trees, and the G_i's are assumed acting "locally ∞-transitive" on the trees and are topologically just infinite. Interesting situations to which the theorem applies arise from Kac-Moody lattices Γ, where the X_i's are (the $CAT(-1)$ realization of) a hyperbolic building as studied by Bourdon [Bou97]. Here $G_i = \mathrm{Iso}(X_i)$ are (virtually) simple topological groups which *do not* act transitively on ∂X_i, and, as remarked after Theorem 1.5, have a very different structure from that of semisimple algebraic groups over local fields (see Sect. 5 below).

Building on Margulis' ideas, Zimmer established in [Zim82] a highly non-trivial generalization of Margulis' factor theorem, known as the intermediate factor theorem (abbreviated **IFT**). Over the years it has found to have various applications, and was recently extended further in the work of Nevo-Zimmer (cf. [NZ02a] [NZ02b], and the references therein). An aspect of our work which is worth mentioning, is that we establish a suitable IFT *first*, and only then deduce *from* it the factor theorem.

Theorem 1.9 (IFT). *Let $G = G_1 \times G_2$, $\mu = \mu_1 \times \mu_2$, and (B, ν_B) be as in the first sentence of the above factor theorem. Let (X, ξ) be a probability measure preserving G-space which is ergodic for the action of each G_i.*

Assume that (Y, η) *is a measure* G*-space (the "intermediate factor"), for which there exist* G*-maps:*

$$(B \times X, \nu_B \times \xi) \overset{\Psi}{\to} (Y, \eta) \overset{\rho}{\to} (X, \xi)$$

whose composition is the natural projection to X*.*

Then there exists a measure G*-space* (C, ν_C)*, a* G*-map* $\pi : (B, \nu_B) \to (C, \nu_C)$*, and a* G*-isomorphism* $\Phi : (Y, \eta) \overset{\sim}{\to} (C \times X, \nu_C \times \xi)$*, making the following diagram of factors of* G*-spaces commutative (measures are omitted for simplicity):*

$$\begin{array}{ccccc} B \times X & \overset{\Psi}{\to} & Y & \overset{\rho}{\to} & X \\ \| & & \downarrow \Phi & & \| \\ B \times X & \overset{\pi \times id}{\to} & C \times X & \overset{pr_X}{\to} & X. \end{array} \tag{1}$$

Furthermore, (C, ν_C) *decomposes as a direct product as in Theorem 1.7 above.*

By an elementary general argument one shows (see Sect. 3 below), that given *any* locally compact group G and *any* closed subgroup $\Gamma < G$, if the *conclusion* of the IFT holds for the G-space $X = G/\Gamma$ (whatever G-space B is), then the *conclusion* of the factor theorem holds in that situation. The special case of $X = G/\Gamma$ in the IFT, needed for the factor theorem, does not turn out to be any simpler than the general one. It is the IFT at which most of our effort will be directed in this paper.

Finally, we point out one key ingredient in the proof of the IFT, which may also be of independent interest. Let $G = G_1 \times G_2$ and $\mu = \mu_1 \times \mu_2$ be as in the IFT. Given a G-space (Y, η), we denote by $Y /\!\!/ G_i$ and $\pi_i : Y \to Y /\!\!/ G_i$, the "space of G_i-ergodic components" (i.e., the space corresponding via Mackey's point realization theorem to the σ-algebra of G_i-invariant subsets of Y), with π_i being the canonical projection. Notice that $Y /\!\!/ G_i$ is a G-space, with a trivial action of G_i. To simplify notation we set $Y_i = Y /\!\!/ G_i$, and $\eta_i = \pi_{i*}\eta$.

Proposition 1.10. *Let* (Y, η) *be an ergodic* G*-space. If* η *is* μ*-stationary then the map* $\pi = \pi_1 \times \pi_2 : (Y, \eta) \to (Y_1 \times Y_2, \eta_1 \times \eta_2)$ *is a* G*-map (pushing* η *to* $\eta_1 \times \eta_2$*), and this extension is relatively measure preserving.*

See Sect. 2.I below for the notion of a measure preserving extension, which is the natural relativization of the notion of a measure preserving action. Notice that in cases where it is a priori known that (Y, η) has no non-trivial relatively measure preserving factors, it follows that it itself must decompose as a direct product of G_i-spaces. Since this is always the case for Poisson boundaries, and moreover for any factor of them, this accounts for the last "furthermore" statements in both the factor and intermediate factor theorems. The space $C = Y_1 \times Y_2$ obtained by applying the proposition to the “intermediate factor” Y in Theorem 1.9, is the one making diagram (1) there commutative.

The structure of the paper is as follows. In the next section, assuming no prior familiarity, we offer a self-contained account of the relevant ingredients of Furstenberg’s boundary theory. Although none of the results there should be new to experts, some of them are not in the literature, and the approach, highlighting the uniqueness perspective, seems to have some novel aspects. In Sect. 3 we establish Theorems 1.3, 1.7, 1.9 and Proposition 1.10, including “infinite restricted product” extensions of these results. In Sect. 4 we prove Theorems 1.1, 1.5, 1.6 and Corollary 1.4. In Sect. 5 we describe several situations in which the factor theorem can be made concrete, some of them giving rise to new phenomena. In order to relate our factor theorem with Margulis’ one, for *any* semisimple algebraic group over *any* local field, and deduce the adelic generalization, we show that the “relevant G/P” is indeed the Poisson boundary, by working in the general framework of strongly transitive actions on affine buildings. This result does not seem to appear in the literature in the generality we need, even if it is ultimately of no real surprise.

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It is perhaps symbolic that our collaboration began at the conference honoring the retirement of Hillel Furstenberg, to whom this work owes a considerable intellectual debt. This paper is dedicated to Hillel, with our great admiration and affection.

2. Preliminaries

I: G-spaces. Throughout this paper all groups will be assumed locally compact and second countable. For such a group G, we call a Lebesgue space (Y, η) a ***G*-space**, if G acts measurably on Y, and the probability measure η is quasi-invariant with respect to the G-action. We say that (Y, η) is an ergodic G-space when every G-invariant subset is either null or co-null. Here and in the sequel, “G-invariance”, “G-equivariance”, and other equalities of sets or of measurable maps, always mean that the properties hold with the possible exception of a zero measure subset. A measurable map between G-spaces $\pi : (Y, \eta) \to (X, \xi)$ is called a ***G*-map**, if it is G-equivariant, i.e., $\pi \circ g = g \circ \pi$ for all $g \in G$, and $\xi = \pi_*\eta$, where by definition $(\pi_*\eta)(A) = \eta(\pi^{-1}(A))$. It is called furthermore a ***G*-isomorphism**, if π is an isomorphism measure theoretically. The following well known fact is useful:

Theorem 2.1 (Compact and Borel models). *Any G-space (Y, η) admits a* **compact model**, *namely, some compact metric space K on which G acts* *continuously, with a probability measure η_K, such that (Y, η) and (K, η_K) are measurably isomorphic G-spaces (cf.* [Zim84, Theorem 2.1.19]*). Moreover, if $\pi : (Y, \eta) \to (Z, \zeta)$ is a (measurable) G-map between two compact G-spaces, then there exists a* **Borel** *map $\sigma : Y \to Z$ such that $\sigma = \pi$ (cf.* [Zim84, Cor. B.6]*, as always, equality is η a.e.).*

The (convex, weak*-compact) space of probability measures on a compact metric space K will be denoted by $\boldsymbol{P(K)}$. Given a G-map between G-spaces $\pi : (Y, \eta) \to (X, \xi)$, we shall also refer to (Y, η) as a **(G-)extension** of (X, ξ), to (X, ξ) as a **(G-)factor** of (Y, η) (the measures will often be omitted), and to π as a (G-)factor, or (G-)extension map, depending on the context. For a G-factor map π, as above, and $x \in X$, we define the set $\pi^{-1}\{x\}$ as the **fiber over** x. Given a compact G-space (Y, η), a G-map $\pi : (Y, \eta) \to (X, \xi)$ gives rise to a **unique disintegration map** $D_\pi : X \to P(Y)$, with the property that for almost every $x \in X$ the measure $D_\pi(x)$ is supported on the fiber of x, and the barycenter of the measure $(D_\pi)_*\xi$ on $P(Y)$ is η, i.e., $\int D_\pi(x) d\xi(x) = \eta$. We define Y to be **relatively measure preserving over** X (or a **measure preserving extension** of X), if D_π is a G-equivariant map. This property, as well as many others which are defined a priori only on compact models, is well known to be a measure theoretic one. Of course, the G-action on (Y, η) is measure preserving if and only if Y is relatively measure preserving over the trivial one point space. The following basic fact is obvious using the uniqueness of the disintegration map:

Lemma 2.2. *Let $(Z, \zeta) \xrightarrow{\pi} (Y, \eta) \xrightarrow{\rho} (X, \xi)$ be G-factor maps. Then the composition $\rho \circ \pi$ is relatively measure preserving if and only if both ρ and π are.*

The same holds if we replace "relatively measure preserving" by "isomorphism".

Proof. The only non-trivial part is the "only if" of the first statement. Assume that $\rho \circ \pi$ is relatively measure preserving. Then by the uniqueness of the disintegration map, $D_\pi = D_{\rho\circ\pi} \circ \rho$ and $D_\rho = \bar{\pi} \circ D_{\rho\circ\pi}$ (where $\bar{\pi}$ denote linear extension of π to $P(X)$), proving they are both equivariant. □

Given a G-map $\pi : (Y, \eta) \to (X, \xi)$, consider the measure algebra of subsets of Y which, modulo zero measure subsets, are of the form $\pi^{-1}(A)$ where $A \subseteq X$. This is a sub-σ-algebra which is G-invariant and contains all zero measure subsets. Conversely, by Mackey's point realization theorem, every such sub-σ-algebra on Y arises in this way. A measurable function f on Y will be called **measurable with respect to the factor** X, if it is measurable with respect to the sub-$\sigma-$algebra corresponding to X, alternatively, if $f = h \circ \pi$ for some measurable map h on X.

Two special, yet important cases of this discussion are as follows. For a G-space (Y, η), denote by $B(Y)^G$ the σ-algebra of G-invariant subsets of Y

(modulo 0). By the above it corresponds to a factor (space and map) $\tilde{\eta} : Y \to Y/\!\!/G$. Here, $Y/\!\!/G$ denotes the corresponding space via Mackey's theorem, often called the space of (G-)ergodic components. The disintegration of $\tilde{\eta}$ then gives rise to the well known **ergodic decomposition** of η. The second special case we shall be needing, is that of the **Radon-Nikodym factor** of a G-space (Y, η), denoted $\boldsymbol{RN}(\boldsymbol{Y})$, first introduced in [KV83]. This is the factor of Y corresponding to the smallest sub-σ-algebra with respect to which all the Radon-Nikodym derivatives $dg\eta/d\eta : Y \to \mathbb{R}$ are measurable for all $g \in G$. The following well known basic fact will be essential:

Proposition 2.3 (cf. [NZ00, Prop. 1.8]). *Let $\pi : (Y, \eta) \to (X, \xi)$ be a G-factor map. Assume that for all $g \in G$, $dg\eta/d\eta$ is measurable with respect to X. Then π is relatively measure preserving. In particular, $Y \to RN(Y)$ is relatively measure preserving.*

In fact, $RN(Y)$ is the smallest factor of Y over which Y is relatively measure preserving (in that it is a quotient of any other such factor).

We end this subsection with two elementary observations for later reference.

Lemma 2.4. *Let (Y, η), (C, ν_C), (X, ξ) be a G-spaces, and $\varphi : (Y, \eta) \to (C, \nu_C)$, $\rho : (Y, \eta) \to (X, \xi)$ be Borel G-factor maps. Let $D_\varphi : C \to P(Y)$ be the disintegration map. If for a.e. $c \in C$ the map $\rho : (Y, D_\varphi(c)) \to (X, \xi)$ is a G-isomorphism, then the map $\varphi \times \rho : (Y, \eta) \to (C, \nu_C) \times (X, \xi)$ is a G-isomorphism.* □

Corollary 2.5. *Let (Z, ζ) be an ergodic G-space and X be a measure space with a trivial G-action on it. Let η be a measure on $Y = Z \times X$ such that the projection $pr_Z : (Z \times X, \eta) \to (Z, \zeta)$ is a measure preserving G-extension. Then $\eta = \zeta \times (pr_X)_* \eta$.*

Proof. The disintegration of the measure preserving extension pr_Z is of the form $D(z) = \delta_z \times d(z)$, where $d : Z \to P(X)$ is G-equivariant. As Z is ergodic and the action on X is trivial, the image of d is a.e. constant, which must then equal the measure $(pr_X)_* \eta = \xi$. The result now follows from the previous lemma. □

II: Stationarity and (G, μ)-spaces. Throughout the rest of the paper *all* measures μ chosen on the groups involved are assumed admissible. We say that the G-space (Y, η) is a **(G, μ)-space**, if the measure η is μ-stationary: $\mu * \eta = \eta$. It is easy to see that a μ-stationary measure for an admissible μ is necessarily G-quasi-invariant.

Proposition 2.6. *Let Y be a compact G-space, and let η_1, η_2 be two μ-stationary measures on Y. Then:*

(1) *The supremum and infimum measures $\eta_1 \vee \eta_2$ and $\eta_1 \wedge \eta_2$, are μ-stationary.*
(2) *If η_2 is continuous with respect to η_1, then the Radon-Nikodym derivative $d\eta_2/d\eta_1$ is a (η_1-a.e) G-invariant function on Y. In particular, if η_1 is ergodic and the measures have the same total mass then $\eta_1 = \eta_2$.*

Proof. **(1)** By the positivity of the operator of convolution by μ,

$$\eta_1 \leq \eta_1 \vee \eta_2 \quad \text{and} \quad \eta_2 \leq \eta_1 \vee \eta_2$$

imply

$$\eta_1 = \mu * \eta_1 \leq \mu * (\eta_1 \vee \eta_2) \quad \text{and} \quad \eta_2 = \mu * \eta_2 \leq \mu * (\eta_1 \vee \eta_2).$$

Thus, by definition of $\vee$, we have: $\eta_1 \vee \eta_2 \leq \mu * (\eta_1 \vee \eta_2)$. Equality of these measures must then hold, since convolving by a probability measure preserves total mass. This proves the stationarity of $\eta_1 \vee \eta_2$. The stationarity of $\eta_1 \wedge \eta_2$ follows similarly.

(2) Set $f = d\eta_2/d\eta_1$. In order to prove the η_1-a.e. G-invariance of f, it is enough to show that the sub-level sets

$$L_a = \{y \in Y \mid f(y) < a\} \qquad L^a = \{y \in Y \mid f(y) > a\}$$

are η_1-a.e G-invariant, for a dense set of values a. We will establish this property for every *regular* a, by which we mean one satisfying the property $\eta_1(\{y \in Y | f(y) = a\}) = 0$.

Indeed, the measures

$$\eta_a = (a \vee f - f)\eta_1 = (a\eta_1) \vee \eta_2 - \eta_2$$
$$\eta^a = (f - a \wedge f)\eta_1 = \eta_2 - (a\eta_1) \wedge \eta_2$$

are μ-stationary by **(1)** above. The measure η_a (resp. η^a) is supported on the set L_a (resp. L^a), hence by the quasi-invariance of μ-stationary measures, L_a (resp. L^a) must be G-invariant modulo η_a (resp. η^a) null sets. By regularity of a, $L_a = Y - L^a$ modulo η_1 null sets, hence also modulo η_a and η^a null sets separately. It follows immediately that L_a, being a null set for η^a, must be G-invariant also w.r.t. η^a (and similarly for L^a w.r.t η_a). Therefore each one of L_a and L^a is G-invariant modulo $\eta_a + \eta^a$ null sets. But $\eta_a + \eta^a = (a \vee f - a \wedge f)\eta_1 = |f - a|\eta_1$, which, by regularity of a, is equivalent to η_1. This completes the proof of the proposition. □

The following corollary should be compared with the discussion at the end of [NZ02b, Sect. 1].

Corollary 2.7. *Let Y be a compact G-space. A μ-stationary probability measure on Y is ergodic, if and only if it is extremal in the (convex, compact) set $P(Y)^\mu$ of μ-stationary measures on Y. In particular, if for any $\nu \in P(Y)^\mu$ the G-action on (Y, ν) is ergodic, then $P(Y)^\mu$ is a singelton.*

III: Furstenberg's boundary theory (1): Harmonicity and (G, μ)-spaces. Let μ be an admissible measure on the group G. Denote by $Har(G, \mu)$ the space of *bounded* (right) μ-harmonic functions on G, i.e., the bounded functions $\phi : G \to \mathbb{R}$ satisfying $\int \phi(gg')d\mu(g') = \phi(g)$. Such functions are always continuous and the left G-action on itself gives rise to a G-action on $Har(G, \mu)$: $(g_1\phi)(g) = \phi(g_1^{-1}g)$. We now make the following general notation to be used throughout this paper. Consider the space of increments of the μ-random walk: $(\Omega, P) = (G^{\mathbb{N}}, \mu^{\mathbb{N}})$, whose elements will be denoted $\omega = (\omega_1, \omega_2, \omega_3, \dots)$. For $\omega \in \Omega$ and $n \in \mathbb{N}$, we define the corresponding n-th "random product": $\omega^{\mathbf{n}} = \omega_1 \cdot \omega_2 \cdots \omega_n$. The following classical fact is an essential tool for the development of boundary theory:

Theorem 2.8. *Retain the above notation and let $\phi \in Har(G, \mu)$. Then for P a.e. $\omega \in \Omega$ the limit: $\lim_{n\to\infty} \phi(\omega^{\mathbf{n}}) = \bar{\phi}(\omega)$ exists (and belongs to $L^\infty(\Omega, P)$). Furthermore, $\phi(e) = \int \bar{\phi}(\omega)dP(\omega)$ (where e is the identity element of G).*

Proof. The proof follows from the martingale convergence theorem (see e.g. [Fur02, Lemma 2.7], and the reference therein). □

Consider now a (G, μ)-space (Y, η). The μ-stationarity of η immediately implies that for every $f \in L^\infty(Y)$ the function $g \to \int f(gx)d\mu(x) = \int fd(g\mu)$ is bounded and μ-harmonic. We denote this map ("generalized Poisson transform") from $L^\infty(Y)$ to $Har(G, \mu)$ by $f \to \hat{f}$. It is obvious that it commutes with the (left) G-action, and that $\hat{f}$ is constant for all f if and only if the measure η is G-invariant. This accounts for the last statement in the following result, due independently to Rosenblatt [Ros81] and Kaimanovich-Vershik [KV83] (see [Kai02] for a generalization), which was originally conjectured by Furstenberg:

Theorem 2.9. *Let G be an amenable group. Then there exists an admissible measure μ on G for which $Har(G, \mu)$ consists of the constant functions only. Equivalently, for every (G, μ)-space (Y, η), the measure η is G-invariant.*

In fact it is easy to see that the existence of such a measure μ implies amenability, hence Theorem 2.9 actually gives a characterization of amenability.

We now make the following observation, which may be viewed as the starting point of Furstenberg's boundary theory. Retaining the above setup, assume furthermore that Y is a compact metric space on which G acts continuously. This induces a G-action on the space $C(Y)$ of continuous functions on Y by $(gf)(x) = f(g^{-1}x)$, and a dual action on $P(Y)$ via $(g\eta)(f) = \eta(g^{-1}f) = \int(g^{-1}f)d\eta = \int f(gx)d\eta(x) = \int fd(g\eta)$. By choosing functions f from a (normic) dense *countable* subspace of $C_0 \subseteq C(Y)$, and applying Theorem 2.8 to the harmonic functions $\hat{f}$ defined before Theorem 2.9, one deduces:

Theorem 2.10. *In the setting above, for P a.e. $\omega \in \Omega$ the limit* $\lim_{n\to\infty} \omega^{\mathbf{n}}\eta = \eta_\omega$ *exists in the weak* topology of $P(Y)$. Furthermore, $\eta = \int \eta_\omega dP(\omega)$. The measures η_ω are called the* **conditional measures** *defined by η.*

Indeed, letting $\Omega_0 \subseteq \Omega$ be the co-null subset of those ω for which the limit in Theorem 2.8 exists for all $\hat{f}$ with $f \in C_0$, we get that $f \mapsto \lim \int f d\omega^{\mathbf{n}}\eta$ is a linear functional on C_0, which is clearly bounded and therefore extends to $C(Y)$.

IV: Furstenberg's boundary theory (2): The Poisson boundary. The construction of the Poisson boundary brought here differs from the original one due to Furstenberg [Fu63] (see also Glasner's [Gla76]). It appeared in this form in [Zim78] and [KV83], although the basic idea can be traced earlier. Since our whole approach to boundary theory is aimed at some very specific applications, this subsection (and in fact all of this section) should not be viewed as any attempt to present a comprehensive treatment of the subject. Accordingly, some substantial works, such as those of Guivarc'h, Avez, Azencott, Ledrappier, Jaworski, Raugi, and particularly that of Kaimanovich (as well as some recent deep results by Erschler), will hardly be mentioned in the sequel. For more details, and a fairly complete list of references, the reader is referred to the excellent recent account of Furman [Fur02].

Let G, μ and (Ω, P) be as in the previous subsection. Consider the G-action on Ω defined by $g(\omega_1, \omega_2, \omega_3, \dots) = (g\omega_1, \omega_2, \omega_3, \dots)$, and the transformation $T : \Omega \to \Omega$ defined by $(\omega_1, \omega_2, \omega_3, \dots) \mapsto (\omega_1\omega_2, \omega_3, \omega_4, \dots)$. Because the G-action commutes with T, it preserves the sub-algebra of T-invariant subsets. The space of T-ergodic components of Ω, endowed with the image of the measure P, is called the **Poisson boundary** of G, and is denoted $(B(G), \nu_{B(G)})$, or simply (B, ν_B). The image of an element $\omega \in \Omega$ in B under the canonical projection will be denoted $\bar{\omega}$. Note that by commutativity of the G-action on Ω with T, it descends to a well defined action on B. Because $T_*P = \mu^{*2} \times \mu \times \mu \times \dots = \mu * (\mu \times \mu \times \mu \times \dots)$, the measure ν_B is μ-stationary. Since μ is admissible, ν_B is quasi-invariant under G. We remark that for all the purposes of this paper it will suffice to work with measures μ which are in the class of the Haar measure of G. For others *(admissible only)*, this construction should be done with a little more care (cf. [Fur02, §2.4]), the key point being that by the equality $T_*P = \mu * P$ and admissibility, the G-action on the subalgebra of T-invariant subsets preserves the ideal of P-null sets.

Recall from Theorem 2.8 above that for any $\phi \in Har(G, \mu)$, the limit $\lim \phi(\omega^{\mathbf{n}})$ exists a.e. on (Ω, P). It is obvious that this limit is T-invariant, hence it defines a function $\bar{\phi} \in L^\infty(B)$. We have thus defined two G-equivariant transforms: $\phi \mapsto \bar{\phi}$ from $Har(G, \mu)$ to $L^\infty(B)$, and a transform $f \mapsto \hat{f}$ going in the other direction, as defined in the previous subsection for any (G, μ)-space. We shall need the following fundamental fact:

Theorem 2.11. *For every* $f \in L^\infty(B)$ *one has* $\bar{\hat{\bar{f}}} = f$, *and for every* $\phi \in Har(G,\mu)$ *one has* $\hat{\bar{\phi}} = \phi$.

The second identity shows that every bounded μ-harmonic function on G is represented as some Poisson transform on B, which is conceptually related to the space B being a "large" G-space (for our purposes this is reflected in the amenability of the G-action on it – see Theorem 2.13 below). The first identity, which also implies the uniqueness of representation, is shown below to imply the so-called dynamical "proximality" property of (any compact model of) B, making this space "small" from a different point of view. It is the tension between these two properties which makes B (unique and) useful for rigidity theory.

Proof. To show the first equation, let $\tilde{f} \in L^\infty(\Omega)$ be the lift of $f \in L^\infty(B)$ to Ω. Of course, $\tilde{f}$ is T-invariant. Because P projects to ν_B and the projection is G-equivariant, we have by definition $\hat{f}(g) = \int \tilde{f} d(gP)$, and then for P-a.e $\omega \in \Omega$:

$$\bar{\hat{\bar{f}}}(\bar{\omega}) = \lim_{n\to\infty} \int \tilde{f} d(\omega^{\mathbf{n}} P) = \lim_{n\to\infty} \int \tilde{f}(\omega_1\omega_2 \dots \omega_n\omega')dP(\omega')$$
$$= \lim_{n\to\infty} \int \tilde{f}(T^n(\omega_1, \omega_2, \dots, \omega_n, \omega')dP(\omega')$$
$$= \lim_{n\to\infty} \int \tilde{f}(\omega_1, \omega_2, \dots, \omega_n, \omega')dP(\omega') = \tilde{f}(\omega) = f(\bar{\omega}).$$

Here, when writing $(\omega_1, \omega_2, \dots, \omega_n, \omega')$ we refer to the element of Ω whose first n coordinates are exactly those of ω, its $n+1$ coordinate is the first of ω', etc. The first equality in the last line holds because on one hand, it is clear that the sequence of functions of ω inside the limit, being the L^2-projections to increasing sequence of subspaces, converges to $\tilde{f}$ in L^2, and on the other, we already know *a priori* that this sequence has *some* (T-invariant) pointwise limit on Ω, which must therefore equal $\tilde{f}$.

For the second equality we again lift the integration of functions from $L^\infty(B)$ to $L^\infty(\Omega)$, and use the Lebesgue dominant convergence theorem to deduce:

$$\hat{\bar{\phi}}(g) = \int (\lim_{n\to\infty} \phi(\omega^{\mathbf{n}}))d(gP)(\omega) = \lim_{n\to\infty} \int \phi(\omega^{\mathbf{n}})d(gP)(\omega)$$
$$= \lim_{n\to\infty} \int \phi(g\omega_1\omega_2 \dots \omega_n)dP(\omega)$$
$$= \lim_{n\to\infty} \int d\mu(\omega_1) \int d\mu(\omega_2) \dots \int d\mu(\omega_n)\phi(g\omega_1\omega_2 \dots \omega_n)$$
$$= \lim_{n\to\infty} \phi(g) = \phi(g).$$

□

Obviously, if $\phi \in Har(G, \mu)$ is G-invariant then it is constant, and so is $\bar{\phi}$. Hence if $f \in L^{\infty}(B)$ is G-invariant then $f = \bar{\hat{f}}$ is constant, and we deduce:

Corollary 2.12. *The G-action on (B, ν_B) is ergodic.* □

A very useful property of the Poisson boundary is provided by the following result of Zimmer ([Zim78], see also [Zim84, Sect. 4] for further details).

Theorem 2.13 (Zimmer). *Let (B, ν_B) be the Poisson boundary of (G, μ). Then the G-action on B, and hence also that of any closed subgroup $H < G$, is amenable. In particular, for any continuous H-action on a compact metric space K, there is an H-equivariant measurable map $\psi : B \to P(K)$.*

Remark. The first appearance of "boundary maps" in rigidity, due to Furstenberg, arose as an application of his work on the Poisson boundary [Fu73].

V: Furstenberg's boundary theory (3): Boundaries and uniqueness. A $(\boldsymbol{G}, \boldsymbol{\mu})$-**boundary** is defined to be a G-factor of the Poisson boundary of (G, μ). Of course, any factor map $\pi : (B, \nu_B) \to (Y, \eta)$ corresponds canonically to a "lifted" map $\tilde{\pi} : \Omega \to Y$ which satisfies $\tilde{\pi} = \tilde{\pi} \circ T$, where T is as in the previous subsection, and we have $\eta = \tilde{\pi}_* P = \pi_* \nu_B$. Our next purpose is to show that for a (G, μ)-boundary, the factor map is (essentially) **unique**, and to give a dynamical characterization of (G, μ)-boundaries.

Theorem 2.14. *For a (G, μ)-space (Y, η), the following conditions are equivalent:*

- **(i)** *(Y, η) is a (G, μ)-boundary.*
- **(ii)** *For* **every** *compact model of (Y, η) (Theorem 2.1), the conditional measures η_ω defined in Theorem 2.10 above, are point measures for a.e. $\omega \in \Omega$.*
- **(iii)** *There is* **some** *compact model of (Y, η) satisfying the condition in* **(ii)**.

Furthermore, if these conditions are satisfied, then a G-factor map $\pi : (B, \nu_B) \to (Y, \eta)$ is (essentially) **unique**. *We shall denote it in the sequel by $\boldsymbol{\beta_Y}$.*

Proof. **(i) ⇒ (ii).** Let $\pi : B \to Y$ be a factor G-map. Abusing notation for simplicity, keep the notation Y for the measurably isomorphic compact metric G-model. Pick a countable dense subspace $C_0 \subseteq C(Y)$ and denote by $\Omega_0 \subseteq \Omega$ the set of those $\omega \in \Omega$ for which the first equality in Theorem 2.11 holds for all $f \in C_0$ and all $\bar{\omega}$ (where as usual $\omega \to \bar{\omega}$ is the canonical projection from Ω to B). The subset $\Omega_0 \subseteq \Omega$ is co-null by Theorem 2.11. Then for $\omega \in \Omega_0$ and $f \in C_0$, we have by that equality: $\lim \int (f \circ \pi) d(\omega^{\mathbf{n}} \nu_B) = f \circ \pi(\bar{\omega})$. By the G-equivariance of π, and the fact that $\pi_* \nu_B = \eta$, we have for every $g \in G$: $\int (f \circ \pi) d(g \nu_B) = \int f d(g\eta)$. Hence the former equality can be written as: $\lim \int f d(\omega^{\mathbf{n}} \eta) = (f \circ \pi)(\bar{\omega})$.

Two conclusions follow. First, viewing measures as functionals, we have as $n \to \infty$: $\omega^{\mathbf{n}}\eta(f) \to \delta_{\pi(\bar{\omega})}(f)$ for $\omega \in \Omega_0$, and for functions f in the dense subspace C_0. By continuity this limit then holds for *all* $f \in C(Y)$, namely, $\omega^{\mathbf{n}}\eta \to \delta_{\pi(\bar{\omega})}$ for $\omega \in \Omega_0$, thereby proving **(i)** $\Rightarrow$ **(ii)**. Secondly, note that in the equality: $\lim \int f d(\omega^{\mathbf{n}}\eta) = (f \circ \pi)(\bar{\omega}) = f(\pi(\bar{\omega}))$, the left hand side has no mention of π, and hence it defines π uniquely a.e.

(iii) $\Rightarrow$ **(i).** As above, we may assume that Y is already a compact G-space. By assumption the map $\omega \to \eta_\omega$ defines a measurable G-map $\tilde{\pi} : \Omega \to Y$ satisfying $\eta_\omega = \delta_{\tilde{\pi}(\omega)}$. It is clear by definition that the map $\omega \to \eta_\omega$ is T-invariant, hence $\tilde{\pi}$ descends to a measurable map $\pi : B \to Y$.

Finally, the very last statement in Theorem 2.8 now translates to the fact that $\tilde{\pi}_* P (= \pi_* \nu_B) = \eta$, thus showing that (Y, η) is a (G, μ)-boundary. □

Remark. It follows immediately that for the Poisson boundary (B, ν_B), the conditional measure $(\nu_B)_\omega$ defined via Theorem 2.10 is a.e. equal to $\delta_{\bar{\omega}}$. More generally, if (C, ν_c) is a (G, μ)-boundary, realized by the (unique) G-factor map $\beta_C : B \to C$, then $(\nu_C)_\omega = \delta_{\beta_C(\bar{\omega})}$. Although one can define the notion of the Poisson boundary of (G, μ) differently, this fact makes it particularly convenient to work with the concrete construction, $\Omega /\!/ \langle T \rangle$, of the Poisson boundary described in the previous subsection.

The verification of the following simple (yet not entirely trivial) fact is left to the reader:

Lemma 2.15. *Let N be a normal subgroup of G, and let $\bar{\mu}$ the projection of μ on G/N. The $(G/N, \bar{\mu})$-boundaries are exactly the (G, μ)-boundaries on which N acts trivially. The Poisson boundary of $(G/N, \bar{\mu})$ is $(B /\!/ N, \bar{\nu}_B)$, i.e, the space of ergodic components of the N action on B, endowed with the projection of ν_B.* □

VI: Furstenberg's boundary theory (4): The affine boundary map. Hereafter, by an **affine G-space** Q, we shall mean a G-invariant weak* compact convex subset of a dual to a separable isometric Banach G-module, endowed with the dual (weak* continuous) affine G-action. The Borel structure on Q will always be the one coming from this topology. The outstanding example here is the space of probability measures $P(Y)$, where Y is a compact metric G-space. Notice that the G-action on an affine G-space naturally extends to a convolution action of $P(G)$. Retaining the notation for (G, μ) as before, we define an **affine pointed (G, μ)-space** to be a pair (Q, q) where Q is an affine G-space and $q \in Q$ is fixed by μ. We call an affine Borel G-map A between such spaces $A : (Q, q) \to (Q', q')$ **pointed**, if $Aq = q'$.

Theorem 2.16 (The boundary map). *Let (Q, q) be a pointed affine (G, μ)-space. There exists an (essentially) unique measurable G-map, called the* **boundary map** *and denoted $\boldsymbol{\beta_q} : B \to Q$, satisfying* ***bar*** $\circ\, ((\boldsymbol{\beta}_q)_* \nu_B) = q$, *where* ***bar*** $: P(Q) \to Q$ *is the barycenter map. Consequently, for every Borel affine pointed G-map between pointed affine G-spaces, $A : (Q, q) \to (Q', q')$, one has: $\boldsymbol{\beta}_{q'} = A \circ \boldsymbol{\beta}_q$.*

Notice that given a pointed affine (G, μ)-space (Q, q), the theorem associates a canonical measure $\nu_q = (\boldsymbol{\beta_q})_* \nu_B$ on Q, which is μ-stationary and whose barycenter is q. The measure space $(Q, \nu_q) = (Q, (\boldsymbol{\beta_q})_* \nu_B)$ is called an **affine** (G, μ)**-space**, and the map $\boldsymbol{\beta}_q$ defined here is the same as the map $\boldsymbol{\beta}_Q$ corresponding to the (G, μ)-boundary (Q, ν_q) in the measurable setting of Theorem 2.14. We thus have a natural bijection between pointed affine, and affine (G, μ)-spaces, with one direction being given by the theorem and the other by taking barycenter. We call an affine Borel G-map between affine (G, μ)-spaces $A : (Q, \nu_q) \to (Q', \nu'_q)$, such that $A_* \nu_q = \nu_{q'}$, an **affine factor** G**-map**. The equality $\boldsymbol{\beta}_{q'} = A \circ \boldsymbol{\beta}_q$ shows that an affine factor G-map between (G, μ)-spaces, if exists, is essentially unique. It follows that as in the case of spaces, affine and pointed affine *G-maps* also come with a natural bijection. We now turn to the proof of Theorem 2.16.

Proof. The existence of the boundary map $\boldsymbol{\beta_q}$ follows exactly as in the proof of Theorem 2.10, from Theorem 2.8, using the fact that for every v in the predual, the map $g \mapsto (gq)(v)$ is μ-harmonic. Assume now that $\pi : (B, \nu_B) \to Q$ is a measurable G-map with the property $bar \circ (\pi_* \nu_B) = q$ (where (B, ν_B) denotes, as usual, the Poisson boundary of (G, μ)). Denote $\nu = \pi_* \nu_B$. Then $\pi : (B, \nu_B) \to (Q, \nu)$ realizes (Q, ν) as a (G, μ)-boundary. The proof of Theorem 2.14 then gives an explicit description of π: If $\tilde{\pi} : \Omega \to Q$ lifts π to Ω, then for P-a.e $\omega \in \Omega$ one has:

$$\pi(\bar{\omega}) = bar(\delta_{\tilde{\pi}(\omega)}) = bar(\lim_{n \to \infty} \omega^{\mathbf{n}} \nu) = \lim_{n \to \infty} bar(\omega^{\mathbf{n}} \nu) = \lim_{n \to \infty} \omega^{\mathbf{n}} q.$$

Thus, the right hand side determines π uniquely a.e.

The last assertion follows from the uniqueness property of $\boldsymbol{\beta_{q'}}$. □

We now specialize the above discussion to the case of a compact (G, μ)-space (Y, η). To (Y, η) we associate the pointed affine G-space $(Q, q) = (P(Y), \eta)$, and then to it the affine (G, μ)-space $(P(Y), \nu_\eta)$. In this case the construction shows that $\boldsymbol{\beta_q} : B \to Q = P(Y)$ is no other than the map: $\boldsymbol{\beta_\eta}(\bar{\omega}) = \eta_\omega$, where the latter is the conditional measure defined by η via Theorem 2.10. If, moreover, $\pi : (Y, \eta) \to (X, \xi)$ is a Borel G-map between the compact (G, μ)-spaces, then clearly the push forward of measures $A = \pi_* : P(Y) \to P(X)$ is a pointed affine Borel G-map between the two corresponding pointed affine spaces. Theorem 2.16 then yields the following important property of the conditional measures:

Corollary 2.17 (Naturality of conditional measures). *Let* $\pi : (Y, \eta) \to (X, \xi)$ *be a Borel G-factor map of compact* (G, μ)*-spaces. Then for P a.e.* ω*:* $\pi_* \eta_\omega = \xi_\omega$.

Remark. The corollary shows (the known fact) that the conditional measures associated with a (G, μ)-space are indeed a measurable object, not depending on the choice of a compact model: Given any two compact models (C_1, η_1), (C_2, η_2) for the (G, μ)-space (Y, η), any measurable isomorphism π between them, and any Borel representative σ for π, induces a measurable isomorphism $\sigma : (C_1, (\eta_1)_\omega) \to (C_2, (\eta_2)_\omega)$ for P a.e. $\omega \in \Omega$. In fact, in any situation where a measurable map between (compact) spaces "gives rise" to a well defined map on the corresponding spaces of probability measures, it behaves "naturally" with respect to the conditional measures (in a way reminiscent of ergodic decompositions). For a discussion of such phenomena in the general context of quasi-factors see also [Gla03].

VII: Some consequences.

Corollary 2.18. *If (X, ξ) is measure preserving and G-ergodic, then the product G-action on $(X \times B, \xi \times \nu_B)$ is ergodic.*

Proof. Let $E \subseteq X \times B$ be a G-invariant subset of positive measure. Define the probability measure η on $X \times B$ to be the normalized restriction of $\xi \times \nu_B$ to E. Since $\xi \times \nu_B$ is μ-stationary, clearly so is η, and the G-ergodicity on X and B readily implies that the natural projections pr_X, pr_B are G-factor maps. To show that $\eta = \xi \times \nu_B$, which clearly completes the proof, we use Corollary 2.17 and Theorem 2.10 to deduce that $(pr_X)_*\eta_\omega = \xi$, $(pr_B)_*\eta_\omega = \delta_{\bar{\omega}}$, hence $\eta = \int \eta_\omega dP(\omega) = \int (\xi \times \delta_{\bar{\omega}}) dP(\omega) = \xi \times \nu_B$. □

Remarks. In general, a product of (G, μ)-spaces (with the product measure) is *not* a (G, μ)-space. Kaimanovich recently established in [Kai03] a far reaching generalization of the corollary, showing the so-called "double ergodicity with coefficients" property of the Poisson boundary.

Much of the boundary theory discussed so far was aimed at the following auxiliary result, which will be essential for the proof of the IFT. We retain the notations introduced in the previous subsection.

Proposition 2.19 (Disintegration measures = Conditional measures). *Assume that $\varphi : (Y, \eta) \to (C, \nu_C)$ is a Borel measure preserving extension of compact (G, μ)-spaces, and let $D_\varphi : C \to P(Y)$ denote the corresponding disintegration map. If (C, ν_C) is a (G, μ)-boundary, realized by the factor G-map $\boldsymbol{\beta}_C : (B, \nu_B) \to (C, \nu_C)$, then $\boldsymbol{\beta}_\eta = D_\varphi \circ \boldsymbol{\beta}_C$, where $\boldsymbol{\beta}_\eta : B \to P(Y)$ is the map defined in Theorem 2.16 for the pointed affine (G, μ)-space $(P(Y), \eta)$. Thus, loosely speaking, the conditional measures η_ω defined by η (via Theorem 2.10) coincide almost surely with the fiber measures appearing in the disintegration of η over ν_C.*

Proof. Follows readily from the uniqueness in Theorem 2.16, and the fact that $A = D_\varphi$ is G-equivariant by the measure preserving assumption on φ, once observing that the barycenter of the measure $(D_\varphi \circ \boldsymbol{\beta}_C)_*\nu_B$, which is the same as that of $D_\varphi((\boldsymbol{\beta}_C)_*\nu_B) = D_\varphi(\nu_C)$, is indeed η. □

Corollary 2.20. *Let $\varphi : (D, \nu_D) \to (C, \nu_C)$ be a measure preserving extension of two (G, μ)-boundaries. Then $(D, \nu_D) = (C, \nu_C)$. In particular, any G-quotient of the Poisson boundary, on which the G-action is measure preserving, is trivial.*

Proof. By the previous lemma the image of the disintegration map $D_\varphi : C \to P(D)$ equals a.e. the image of the boundary map $\boldsymbol{\beta}_{\nu_D}$, which is a.e a point measure. This property is equivalent to φ being injective, hence an isomorphism. The last statement follows of course also from (ii) or (iii) in Theorem 2.14. □

3. Proofs of Theorems 1.3, 1.7, 1.9 and Proposition 1.10

The results of this section are presented in a modular fashion. In Subsect. **I**, which is concerned with boundary theory of products of groups, we prove Proposition 1.10 in the introduction. In **II** we show how this proposition, together with the basics of boundary theory developed in Sect. 2, implies the IFT. In **III** we recall (mainly for completeness), how to deduce the factor theorem from the IFT, and in **IV** show how to deduce the "amenability half" of the normal subgroup theorem (Theorem 1.3) from the factor theorem. Finally, in **V** we indicate how the results can be easily extended to arbitrary finite product of groups, and moreover, to a restricted direct product of countably many. Throughout Subsects. **I–IV** we continue to denote by $G = G_1 \times G_2$ a product of locally compact second countable groups, endowed with a product of admissible measures $\mu = \mu_1 \times \mu_2$. A concrete choice of these measures will be relevant and specified only in Subsect. **IV**.

I. Proof of Proposition 1.10. Retain the notation in and preceding the formulation of the proposition, including the definition of a (G, μ)-space (see Sect. 2.II). We shall first need the following basic fact:

Lemma 3.1. *Let (Y, η) be an ergodic G-space. If η is μ-stationary, then it is also μ_i-stationary for both $i = 1, 2$.*

Proof. The convolution of $\mu = \mu_1 \times \mu_2$ and both μ_i commute. It follows that $\mu * \mu_i * \eta = \mu_i * \mu * \eta = \mu_i * \eta$. By Proposition 2.6, $\mu_i * \eta = \eta$. □

Returning to the proof of the proposition, consider first (Y, η) as a (G_1, μ_1)-space (Lemma 3.1). We show that the G_1-factor map π_2 is relatively measure preserving. For each $g \in G_1$, apply Proposition 2.6 to the (G_2, μ_2)-stationary measures η and $g_*\eta$ (stationarity is granted, again, by Lemma 3.1). It follows that for each $g_1 \in G_1$ the Radon-Nikodym derivative $dg_1\eta/d\eta$ is G_2-invariant. Hence it is measurable with respect to the the factor (Y_2, η_2), and consequently, π_2 is a G_1-measure preserving factor map (see Proposition 2.3 above). From Lemma 2.2 we deduce that the factor map $\pi : (Y, \eta) \to (Y_1 \times Y_2, \pi_*\eta)$ is G_1-measure preserving. A similar argument shows that π is G_2-measure preserving, and we deduce that this π is a relatively measure preserving G-factor map.

We are left to show that $\pi_*\eta = \eta_1 \times \eta_2$. This follows immediately from Corollary 2.5, once we observe that G_1 acts trivially on Y_1, and ergodically on Y_2 (because $Y_2 /\!\!/ G_1 = (Y /\!\!/ G_2) /\!\!/ G_1 = Y /\!\!/ G = \star$).

This finishes the proof of Proposition 1.10. Combining it together with Corollary 2.20, we obtain a full description of (G, μ)-boundaries by means of the (G_i, μ_i)-boundaries:

Corollary 3.2. *Every (G, μ)-boundary is of the form $(C_1 \times C_2, \nu_{C_1} \times \nu_{C_2})$, where (C_i, ν_{C_i}) is a (G_i, μ_i)-boundary. In particular, $(B, \nu_B) = (B_1, \nu_{B_1}) \times (B_2, \nu_{B_2})$.*

II. Proof of the IFT (Theorem 1.9). The proof relies on two main ingredients. One is Proposition 1.10, and the other is a comparison between the boundary and the disintegration map, formulated precisely in Proposition 2.19, making essential use of the uniqueness property of boundary maps. We begin by looking at Proposition 1.10 in a way which may seem somewhat formal at first, but will lead to a conceptually clear view at the proof of the IFT, as well as to an easy "adelic" extension of it.

In the category of all G-measure spaces (not necessarily having a stationary measure), one can define the *invariants product functor* $\boldsymbol{F^G}$, which assigns to any G-space another such space, by $F^G(Y) = Y /\!/ G_1 \times Y /\!/ G_2$, where each factor is endowed with the measure projected from Y. Accordingly, F^G "operates" also on morphisms by assigning to a G-factor map $\pi : Y \to X$, the map $F^G(\pi) : Y /\!/ G_1 \times Y /\!/ G_2 \to X /\!/ G_1 \times X /\!/ G_2$, where for each i the map $Y /\!/ G_i \to X /\!/ G_i$ is Mackey's point realization of the inclusion of G_i-invariants at the level the σ-algebras.

In the category of ergodic (G, μ)-spaces, F^G has an additional important property. This is the existence of a natural transformation $T_Y : Y \to F^G(Y)$, which is a G-factor map by virtue of Proposition 1.10. It is easy to see that if $\pi : Y \to X$ is a G-map of (G, μ)-spaces, then we have a commutative diagram (omitting the measures for simplicity):

$$\begin{array}{ccc} Y & \overset{\pi}{\to} & X \\ T_Y\downarrow & & T_X\downarrow \\ F^G(Y) & \overset{F^G(\pi)}{\to} & F^G(X). \end{array} \tag{2}$$

(Again, this commutativity is immediately seen at the level of inclusion of subalgebras, which then passes to the point realization). Thus, the functor F^G enjoys the following three properties:

- **(i)** The existence of a natural transformation $T_Y : Y \to F^G(Y)$ (i.e., one making the above diagram commutative).
- **(ii)** $T_Y : Y \to F^G(Y)$ is a measure preserving extension.
- **(iii)** If $Y = B \times X$ where B is the Poisson boundary of (G, μ), and the G-action on X is measure preserving and *irreducible* (i.e. ergodic under each G_i), then $F^G(Y) = B$ and $T_Y : B \times X \to B$ is the natural projection.

Indeed, we have established the first property, and the second is a part of Proposition 1.10. Let us prove the third. By Corollary 3.2, $B = B_1 \times B_2$

where each B_i is a (G_i, μ_i)-boundary, hence $Y = B_1 \times B_2 \times X$. By Corollary 2.18, each G_i acts ergodically on $B_i \times X$ (and trivially on B_{3-i}), hence $Y /\!\!/ G_i = B_{3-i}$, and $F^G(Y) = B_1 \times B_2 = B$. We shall now forget everything about the structure of G and use only the existence of a functor F^G with these three formal properties in the proof of the IFT, to which we now turn.

Apply F^G to the sequence of factors $B \times X \overset{\Psi}{\to} Y \overset{\rho}{\to} X$, assumed in the statement of the IFT. For simplicity of notation denote $C = F^G(Y) = Y_1 \times Y_2$, and denote also $\varphi = T_Y : Y \to C$. We get the following commutative diagram of (G, μ)-spaces (where $\star$ stands for the trivial one point space):

$$\begin{array}{ccccc} B \times X & \overset{\Psi}{\to} & Y & \overset{\rho}{\to} & X \\ \downarrow & & \downarrow\varphi & & \downarrow \\ B & \overset{F^G(\Psi)}{\to} & C & \to & \star \end{array} \tag{3}$$

Here we have: (i) the vertical arrows are the natural transformations, (ii) φ is a measure preserving extension, and (iii) $F^G(B \times X) = B$, which accounts for the vertical left arrow. Choose now once and for all compact models for all the spaces appearing in the diagram, and take Borel representatives for the measurable maps (Theorem 2.1). We will be done by showing that the space C is the one required in the IFT, namely, that $\Phi = \varphi \times \rho : Y \to C \times X$ is an isomorphism (with the additional required properties). Note that the commutativity of the two paths from $B \times X$ to C, and the assumption that $\Psi \circ \rho$ is the projection to X, show together that the identification of Y and $C \times X$ through Φ would then be such that the map Ψ is indeed a product map as required in the IFT. We isolate the proof that $\Phi = \varphi \times \rho$ is an isomorphism in the following independent lemma:

Lemma 3.3. *Let G be a locally compact group and $\mu \in P(G)$ be an admissible measure. Let (X, ξ), (C, ν_C), (Y, η) be compact (G, μ)-spaces such that (X, ξ) is measure preserving, and (C, ν_C) is a (G, μ)-boundary. Let $\varphi : (Y, \eta) \to (C, \nu_C)$, $\rho : (Y, \eta) \to (X, \xi)$ be Borel factor G-maps with the following properties:*

(a) φ is a measure preserving extension.
(b) For P a.e. $\omega \in \Omega$ the conditional measure η_ω defined in Theorem 2.10 satisfies that $\rho : (Y, \eta_\omega) \to (X, \xi)$ is a G-isomorphism.

Then $\Phi = \varphi \times \rho : (Y, \eta) \to (C, \nu_C) \times (X, \xi)$ is a G-isomorphism.

Let us first see why the assumptions of the lemma are satisfied in our case. First notice that the map $F^G(\Psi)$ shows that the space C is indeed a (G, μ)-boundary. Condition (a) holds since by property (ii) φ is a measure preserving extension (property (ii) of F^G). There remains only to verify condition (b) in the lemma. For this, we trace the behavior of a “generic” conditional measure: $(\nu_B \times \xi)_\omega$ from the upper left side of the diagram through the horizontal arrows. By using the naturality of the conditional measures (Corollary 2.17), we get, using the invariance of ξ (and the remark following Theorem 2.14), for P a.e. ω the following sequence of G-factor maps:

$$(B \times X, \delta_{\bar{\omega}} \times \xi) \xrightarrow{\Psi} (Y, \eta_\omega) \xrightarrow{\rho} (X, \xi). \tag{4}$$

Because $\delta_{\bar{\omega}}$ is supported on one point, the composition $\rho \circ \Psi : B \times X \to X$ is clearly an isomorphism with respect to the conditional measures for a.e. ω, hence the same holds for ρ alone (Lemma 2.2 above). Thus, it only remains to prove Lemma 3.3:

Proof of Lemma 3.3. Let $\boldsymbol{\beta}_C : B \to C$ be the boundary map and $D_\varphi : C \to P(Y)$ be the disintegration map, which is G-equivariant by assumption. By Proposition 2.19 for a.e. $c \in C$ the fiber measure satisfies $\eta_c = D_\varphi(c) = \eta_\omega$ for the appropriate $\omega \in \Omega$, hence by condition (ii) $\rho : (Y, \eta_c) \to (X, \xi)$ is an isomorphism. An application of Lemma 2.4 completes now the proof of the lemma, and of the IFT with it. □

III. Proof of the Factor Theorem (Theorem 1.7). As mentioned at the Introduction, the deduction of Theorem 1.7 from Theorem 1.9 is a general argument which can be made for any locally compact second countable group G, a closed subgroup $\Gamma < G$, and a G-space B for which the *conclusion* of the IFT is known to hold with $X = G/\Gamma$. For brevity, throughout the rest of this subsection all maps and equalities between them (including various invariance and equivariance properties relative to group actions), are assumed to be defined and to hold almost everywhere. We preface the proof by recalling the notion of induction, which is fundamental for the argument. First, equip G and G/Γ with the unique measure class preserved by the left G-action. Denote the coset $g\Gamma \in G/\Gamma$ by $\bar{g}$. Consider G as a right Γ-space. For a (left) Γ-space (Z, ζ), consider the diagonal Γ-action on $G \times Z$ by $\gamma(g, z) = (g\gamma^{-1}, \gamma z)$. As the Γ-action on G is proper, we can form the space $G \times_\Gamma Z$ of Γ-orbits (which can measurably, but not G-equivariantly, be taken as $G/\Gamma \times Z$), with the induced measure class on it. This is the *induced* G-space. We denote its elements (Γ-invariant classes) by $[g, z]$, and the natural (measure class preserving) G-action on it by $g'[g, z] = [g'g, z]$. It is a basic fact that if Z is a G-space, then the G-space induced to G from the restriction of the G-action to Γ, is isomorphic to the product G-action on $G/\Gamma \times Z$ via $[g, z] \to (\bar{g}, gz)$ (cf. [Zim84, Proposition 4.2.22]).

Lemma 3.4. *Let $\Gamma < G$ be a closed subgroup, C be a G-space, and Z be a Γ-space (by convention both are probability measure spaces). If $\Phi : G \times_\Gamma Z \to G/\Gamma \times C$ (where the latter has the diagonal G-action) is a G-isomorphism whose composition with the projection $G/\Gamma \times C \to G/\Gamma$ is the canonical projection $[g, z] \to \bar{g}$, then there is some measure class preserving Γ-isomorphism $\tau : Z \to C$, satisfying $\Phi[g, z] = (\bar{g}, g\tau(z))$.*

Proof. Consider Φ as a map on $G \times Z$ which is Γ-invariant. Working in coordinates, the condition on the projection to G/Γ implies that there is some map $\psi : G \times Z \to C$ such that $\Phi(g, z) = (\bar{g}, \psi(g, z))$. The G-equivariance of Φ implies that $\psi(g'g, z) = g'\psi(g, z)$, hence if we define $\tau(g, z) = g^{-1}\psi(g, z)$ then τ satisfies $\tau(g'g, z) = g^{-1}g'^{-1}\psi(g'g, z) = g^{-1}g'^{-1}g'\psi(g, z) = \tau(g, z)$. Choose some compact model for C and consider the space $F(Z, C)$ of measurable maps, endowed with the topology of convergence in measure. It is a general standard fact that this is a separable metrizably complete space (where two maps are identified if they agree a.e.). It follows from the foregoing discussion that the map $T : G \to F(Z, C)$ defined by $T(g) = \tau_g$, where $\tau_g(z) = \tau(g, z)$, is G-invariant, and hence constant a.e. (here Fubini is used). In other words, there is some $\tau : Z \to C$ such that $\Phi(g, z) = (\bar{g}, g\tau(z))$. The fact that Φ is Γ-invariant immediately translates to the Γ-equivariance of τ, and because Φ is an isomorphism, so is τ. □

We can now turn to the proof of Theorem 1.7. Retain the notation of the theorem. Viewing B as a Γ-space, induce the Γ-factor map ψ to a G-map Ψ between the induced spaces, defined by: $\Psi[g, b] = [g, \psi(b)]$. This map is well defined as ψ is a Γ-map, and it commutes with the projections to G/Γ. The G-space $G \times_\Gamma B$ is isomorphic to the product G-space $G/\Gamma \times B$ via the map $\Delta([g, b]) = (\bar{g}, gb)$, and therefore this Ψ, together with $Y = G \times_\Gamma Z$ and its natural projection to $X = G/\Gamma$, fit exactly into the setting of the IFT (note that the irreducibility of Γ in G is equivalent to the ergodicity of each factor G_i on X). It then follows from the IFT that there is some G-space C and a G-isomorphism $\Phi : Y = G \times_\Gamma Z \to G/\Gamma \times C$, which commutes with the canonical projections to G/Γ. From Lemma 3.4 it follows that there is a Γ-isomorphism $\tau : Z \to C$, and that $\Phi[g, z] = (\bar{g}, g\tau(z))$. The IFT tells us moreover that after identifying $Y = G \times_\Gamma Z$ and $G/\Gamma \times C$ through Φ, the map $\Phi \circ \Psi \circ \Delta^{-1} : G/\Gamma \times B \to G \times_\Gamma B \to Y \to G/\Gamma \times C$ is a product of the identity map and a G-map: $B \to C$, showing that $\tau \circ \psi : B \to C$ is a G-map. Thus the conclusion of the Factor Theorem is satisfied for the Γ-map τ, and the G-map $\pi = \tau \circ \psi : B \to C$.

IV. Proof of Theorem 1.3. First observe that the "only if" implication is trivial, since Γ/N is projected to a dense subgroup of $G_i/\overline{pr_i(N)}$, hence its amenablity implies that of $G_i/\overline{pr_i(N)}$. Assuming now that the groups $G_i/\overline{pr_i(N)}$ are amenable, we want to show that so is Γ/N. Our strategy is to show that any continuous action of Γ/N on a compact metric space has an invariant measure (the fact that looking at compact *metric* spaces is enough in this characterization of amenability requires some argument, cf. [Mar91, Ch. IV, Lemma 4.2]). We first choose admissible measures μ_i's using Theorem 2.9, as lifts to the G_i's of admissible measures $\bar{\mu}_i$ on the quotients $G_i/\overline{pr_i(N)}$, with respect to which every bounded harmonic function is constant. As usual we shall work with the measure $\mu = \mu_1 \times \mu_2$ on $G = G_1 \times G_2$.

Let K be a compact metric Γ/N-space. We view K as a Γ-space on which N acts trivially. By amenability of the Γ-action on (B, ν_B) (see Theorem 2.13), there is a measurable Γ-map $\psi : B \to P(K) = Z$. Taking the measure $\zeta = \psi_* \nu_B$ on Z, we get a Γ-factor map $\psi : (B, \nu_B) \to (Z, \zeta)$. By Theorem 1.7, there exists a (G, μ)-boundary (C, ν_C) with a boundary map $\pi : (B, \nu_B) \to (C, \nu_C)$, and a ν_C-a.e defined Γ-isomorphism $\tau : (Z, \zeta) \to (C, \nu_C)$, such that $\pi = \tau \circ \psi$ (ν_B-a.e). By the very last part of the theorem we have furthermore $(C, \nu_C) = (C_1 \times C_2, \nu_{C_1} \times \nu_{C_2})$, where (C_i, ν_{C_i}) are (G_i, μ_i)-boundaries. Since τ is Γ-equivariant, N acts trivially on Z, and G acts on each C_i via the projection to G_i, we deduce that $\overline{pr_i(N)}$ acts trivially on C_i for $i = 1, 2$. Hence each G_i acts on C_i through the quotient $G_i / \overline{pr_i(N)}$, and the measures μ_i project to the measures $\bar{\mu}_i$ chosen above under this quotient map. Thus each ν_{C_i} is $\bar{\mu}_i$-stationary, and from Theorem 2.9 it follows that ν_C is *invariant* under G. But the Poisson boundary has no non-trivial factors on which the action is measure preserving (Corollary 2.20). Hence the measure ν_C on C, and consequently also the measure ζ on Z, is supported on one point, which must then be fixed by Γ, thereby completing the proof of the theorem.

V. An "adelic" extension. Let I be a countable set of indices and $(G_i)_{i \in I}$ be a collection of locally compact second countable groups. Assume that $I = I_1 \cup I_2$, with $|I_1| < \infty$, and that we are given for every $i \in I_2$ a compact open subgroup $K_i < G_i$. In this setting we define, as usual, the restricted product $G = \prod' G_i$ to be the subgroup of $\prod G_i$ consisting of those elements (g_i) such that for all but finitely many i's, $g_i \in K_i$. We endow G with the restricted product topology, namely the weakest topology which makes the coordinate projections continuous, and the groups $G_S = \prod_{i \in S} G_i \times \prod_{i \notin S} K_i$ open for every finite S, $I_1 \subseteq S \subseteq I$. This topology makes G a locally compact second countable group. The outstanding example is of course that of algebraic groups over the ring of adeles, but we remark that any infinite direct *sum* of discrete countable groups G_i, is a discrete group covered by this setting as well, by taking K_i to be the identity element for each i.

Observe that the group G is a Borel subset of $\prod G_i$. The Borel σ-algebra of G coming from its locally compact topology coincides with the restriction of the product Borel σ-algebra on $\prod G_i$. For every $i \in I$, let μ_i be an admissible probability measure on G_i, and denote $\mu = \Pi \mu_i$. Observe that by the Borel-Cantelli lemma:

$$\mu(G) = \begin{Bmatrix} 0 & \prod \mu_i(K_i) = 0 \\ 1 & \prod \mu_i(K_i) > 0 \end{Bmatrix}.$$

Hereafter we assume that the following condition is satisfied:

$$\prod \mu_i(K_i) > 0. \tag{5}$$

Then μ gives full measure to G, and we retain the notation μ for its restriction to G, which is an admissible probability measure.

For every $j \in I$, we denote:

$$\check{G}_j = \prod_{j \neq i \in I}{}' G_i \qquad \check{\mu}_j = \prod_{j \neq i \in I} \mu_i.$$

Evidently, $(G, \mu) = (G_j, \mu_j) \times (\check{G}_j, \check{\mu}_j)$. Given a (G, μ)-space (Y, η) we denote by $\check{\pi}_i : Y \to Y /\!\!/ \check{G}_i$ the projection on the space of ergodic components of Y with respect to the $\check{G}_i$-action. To simplify notation we use $Y_i = Y /\!\!/ \check{G}_i$, and $\eta_i = (\check{\pi}_i)_* \eta$. Thus, (Y_i, η_i) becomes a (G_i, μ_i)-space (notice that the notation for (Y_i, η_i) here *does not* coincide with the previous one in the case of two factors).

Proposition 3.5. *Let (Y, η) be an ergodic (G, μ)-space. The map*

$$\check{\pi} = \prod \check{\pi}_i : (Y, \eta) \to \prod (Y_i, \eta_i)$$

is a relatively measure preserving factor G-map.

Proof. The proof is an adaption of the proof of Proposition 1.10. Viewing (G, μ) as $(G_i, \mu_i) \times (\check{G}_i, \check{\mu}_i)$, we deduce that $\check{\pi}_i$ is G_i-measure preserving. By Lemma 2.2 it follows that $(\prod Y_i, \check{\pi}_* \eta)$ is a G-measure preserving factor of (Y, η). By an iterative use of Lemma 2.5, for every finite subset $S \subseteq I$, the map

$$\check{\pi}_S = \prod_{i \in S} \check{\pi}_i : (Y, \eta) \to \prod_{i \in S} (Y_i, \eta_i)$$

satisfies $(\check{\pi}_S)_* \eta = \prod_S \eta_i$. It follows that $\check{\pi}_* \eta = \prod \eta_i$, as these two measures coincides on the generating algebra of the subsets measurable with respect to finitely many coordinates. □

Combining together Proposition 3.5 with Corollary 2.20 and Lemma 2.15, we obtain a complete description of (G, μ)-boundaries by means of the (G_i, μ_i)-boundaries:

Theorem 3.6. *Every (G, μ)-boundary (C, ν_C) is of the form $(\prod C_i, \prod \nu_{C_i})$, where (C_i, ν_{C_i}) is a (G_i, μ_i)-boundary. In particular $B(G, \mu) = \prod B(G_i, \mu_i)$.*

We now have the following result, extending the case of two factors:

Theorem 3.7. **(i)** *The IFT (Theorem 1.9) stated in the Introduction, holds for the restricted direct product G, with $\mu = \prod \mu_i$, $B = \prod B_i$, $\nu_B = \prod \nu_{B_i}$ as above, under the assumption that each (restricted)* **subproduct** *$\check{G}_j$ acts ergodically on (X, ξ).*

(ii) *The factor theorem (Theorem 1.7) stated in the Introduction, holds under this change of notation, for any lattice $\Gamma < G$ whose projection to every (individual) factor G_i is dense.*

(iii) *In both results the last "furthermore" statement holds for a decomposition of C as in Theorem 3.6 .*

(iv) *The statement of Theorem 1.3 remains true if in the first sentence we take G as a restricted direct product as above, and let $\Gamma < G$ be a lattice which projects densely to every factor G_i.*

Indeed, as in the case of two factors, one proves the IFT by using the functor $F^G(Y) = \prod Y /\!/ \check{G}_j$. As was emphasized in the proof of Theorem 1.9, for the argument one only needs to have a functor F^G with the three properties (i)–(iii) discussed there, and the same proof, using Proposition 3.5 and Theorem 3.6 above, shows that this holds in our case as well. As for the factor theorem, the irreducibility assumption on Γ is equivalent to the condition that each $\check{G}_j$ acts ergodically on $X = G/\Gamma$, and the proof that the IFT for $X = G/\Gamma$ implies the factor theorem for Γ is completely general. The proof of (iv) is then similar to Theorem 1.3, with the one additional ingredient that here one needs the following slight improvement on Furstenberg's conjecture (which follows from the proof given in [KV83] with minor changes): Given any locally compact amenable group H, a compact open subgroup $K < H$, and $\alpha < 1$, there is an admissible measure μ on H with trivial Poisson boundary, and satisfying $\mu(K) > \alpha$.

4. Proofs of Theorems 1.1, 1.5, 1.6 and Corollary 1.4

I. Proof of the normal subgroup theorem (Theorem 1.1). Retain the notations in the statement of the theorem. Consider the two possibilities for a non-trivial normal subgroup $N \lhd \Gamma$: (i) The projection $pr_i(N)$ is non-trivial for both $i = 1, 2$; (ii) This projection is trivial for one i (but then not for the other). We claim that in case (ii) the group G_i to which the projection is trivial must be discrete, contradicting the assumption. Indeed, assume without loss of generality that $i = 1$, so $N \lhd G_2$. Then it is obvious that Γ/N is irreducible in the product $G_1 \times (G_2/N)$. Since G_2/N is compact, the projection of Γ/N to G_1 must be discrete. As the projection is also dense, G_1 is discrete, contradicting the assumption.

We are left with possibility (i). To show that Γ/N is finite we show that it is both amenable and Kazhdan. The first is immediate by Theorem 1.3 and the just non-compactness assumption. To show the second we only need to verify that if the G_i's are not both isomorphic to $\mathbb{R}$ then condition (ii) in Theorem 1.2 is satisfied. In other words, if $\varphi : G \to \mathbb{R}$ is a non-zero continuous homomorphism which vanishes on N, we show that both G_i's are isomorphic to $\mathbb{R}$. If the restriction of φ to, say, G_1 vanishes, then it factors as a homomorphism of the quotient $G_2/\overline{pr_2(N)}$, which is compact by the assumption on G_2 and the non-triviality of $pr_2(N)$, so it must vanish. Consider the (non-zero) restriction of φ to each G_i, denoted $\varphi_i : G_i \to \mathbb{R}$. It has trivial kernel by the assumption that G_i is just non-compact. Let $G_i^0 < G_i$ be the connected component. If G_i^0 is trivial then G_i is totally disconnected, and a compact open subgroup of it, K_i, must be in $\mathrm{Ker}\varphi_i$. Thus K_i is trivial, which is impossible by non-discreteness of G_i. Thus G_i^0 is non-trivial and connected, and since the restriction of φ_i to G_i^0 remains injective, it must be *onto* $\mathbb{R}$, and therefore an isomorphism. Finally, we see that by injectivity of φ_i on G_i, this implies $G_i^0 = G_i$, thereby completing the proof of the theorem.

Proof of Theorem 1.6. The proof goes along the argument given by Stuck and Zimmer [SZ94] for a higher rank simple Lie group G almost verbatim. We will briefly indicate the modifications needed while recalling the structure of the proof. We first observe two simple claims:

Lemma 4.1. *Let H be a just non-compact group, and $\lambda \in P(H)$ be an admissible measure. Let (C, ν_C) be a non-trivial (H, λ)-boundary (that is, C is not a point). Then H acts faithfully on C.*

Proof. Follows from the fact that the Poisson boundary of a compact group is trivial, combined with Lemma 2.15. □

Given a group H acing on a space X, we denote $H_x = \mathrm{Stab}_H(x)$.

Lemma 4.2 (cf. [SZ94, Lemma 1.8]). *Let G_1, G_2 be just non-compact groups and (X, ξ) be a non-essentially transitive irreducible $G = G_1 \times G_2$ space. Then for ξ-a.e $x \in X$ the subgroups $(G_1)_x$ and $(G_2)_x$ are trivial.*

Proof. Assume for example that $(G_1)_x$ is non-trivial for a positive measure set of x. By ergodicity of G_2, this subgroup is constant a.e., hence by non-triviality of it and the just non-compactness of G_1, the G_1-action on X factors through the compact quotient $G_1/(G_1)_x$. As G_1 acts ergodically, it is essentially transitive. □

We are now ready to prove Theorem 1.6. We assume the action of G on (X, ξ) is not essentially transitive, nor essentially free, and derive a contradiction. By [SZ94, Lemma 1.5], this and property T imply that the G-action on X is not weakly amenable. Therefore, by the definition of weak amenability ([SZ94, Definition 1.2]) one has an affine orbital G-space A with $p : A \to X$ (see [SZ94, Definition 1.1]), having no invariant sections. We choose once and for all admissible measures μ_i on G_i, and denote $\mu = \mu_1 \times \mu_2$. As in the beginning of the proof of [SZ94, Theorem 2.1], by the amenability of the action on $B(G, \mu)$ there is a G-map $f : B(G, \mu) \times X \to A$, such that $f \circ p = pr_X$. This places us in the setting of Theorem 1.9 for $Y = A$. We deduce that A is G-isomorphic to $X \times C_1 \times C_2$, where each C_i is a (G_i, μ_i)-boundary. As in [SZ94, Theorem 2.1], we observe that for ξ-almost every $x \in X$, G_x acts trivially on $C_1 \times C_2$. By Lemma 4.2, for ξ-a.e $x \in X$, $pr_1 : G_x \to G_1$ and $pr_2 : G_x \to G_2$ are injective. Assume that for ξ-positive measure set of $x \in X$, the group G_x is non-trivial. Then for both i the G_i-actions on C_i are not faithful, thus by virtue of Lemma 4.1, both

C_i's are trivial. It follows that p has a (measurable) inverse, that is a section $X \to A$. This leads to a contradiction, by construction of A, showing that the G-action is essentially free. □

We now elaborate, for completeness, on the two remarks mentioned after the statement of Theorem 1.6.

Proposition 4.3. *Let G_1, G_2 be locally compact, non-discrete groups, such that every irreducible, non-essentially transitive, measure preserving action of $G = G_1 \times G_2$ is essentially free. Then every irreducible lattice $\Gamma < G$ is just infinite.*

Proof. Notice first that each G_i must be just non-compact, for otherwise any non-trivial closed normal non-cocompact subgroup $L < G_1$ (say), can be put in the kernel of a measure preserving irreducible G-action (e.g., by using the Gaussian measure construction as explained in [Zim84, 5.2.13] for the representation $\pi = L^2((G_1/L) \times G_2))$. Consider a non-trivial normal subgroup $N < \Gamma$ of infinite index, and take some mixing probability measure preserving Γ/N-action on a space Y (e.g., take some Bernoulli Γ/N-shift). Let X be the G-space (and action) induced to G from the Γ-action on Y. If N is non-trivial then the G-action on X is not essentially free (all stabilizers are conjugates of N). Since the Γ-action is non-transitive, so is the G-action. It is left to show that the G-action is irreducible. Indeed, assume for example that G_1 is not ergodic. Then it is not difficult to verify the non-triviality of the σ-algebra of subsets of Y, on which the Γ-action extends continuously to a G_2-action, factoring through the projection to G_2. This σ-algebra thus defines a non-trivial Γ-factor Z of Y, on which the action extends (measurably) to G_2, through $pr_2 : \Gamma \to G_2$. Since it has $\overline{pr_2(N)}$ in its kernel, the G_2-action on Z factors through a compact group K. Note that the Γ-action on Z has N in its kernel, and as a Γ/N-action it is still mixing. This implies that the image of Γ/N in K is closed (i.e., discrete) and infinite, contradicting the compactness of K. □

The proof shows that if $\Gamma < G$ is a lattice in a locally compact group, and Γ contains an infinite and infinite index normal subgroup, then one can construct ergodic measure preserving G-actions which are not essentially free nor transitive. Taking G to be the simple Lie group $Sp(n, 1)$ $(n > 1)$ and $\Gamma < G$ to be any uniform (hence hyperbolic) lattice, we see that the theorem fails for one simple (Kazhdan) group G. We can also make use of this setting, recalling the well known construction of Gromov, which starting with one hyperbolic (Kazhdan) group as before, gives a continuum of simple (Kazhdan) groups as its quotients. These exotic groups provide examples of groups G_i to which Theorem 1.6 applies as well. Below we shall also see that besides the (already known by [SZ94]) case of semisimple groups, there are other examples, arising from Kac-Moody theory, of non-discrete G containing irreducible lattices, to which Theorem 1.6 applies (see the end of proof of Corollary 1.4 below).

On Theorem 1.5 and Corollary 1.4. Parts **1** and **2** in Corollary 1.4 follow immediately from Theorem 1.3 and Theorem 1.1 respectively, once the following is shown: If Δ is a building as in the statement, and the closed subgroup $G < Aut(\Delta)$ acts strongly transitively on Δ, then G is just non-compact. Indeed, the strong transitivity assumption implies that Δ induces on G a (B, N)-pair structure (with an irreducible Weyl group by the irreducibility assumption on Δ) – cf. [Ron89, Theorem 5.2]. Here, B – the stabilizer of a chamber – is compact as Δ is locally finite. It is a general (purely algebraic) fact that in this case for every normal subgroup $H \lhd G$, either $H < B$ or $H \cdot B = G$ (cf. [Bou02, VI.2.7, Lemma 2]). The first case implies that H fixes a chamber, so by normality and transitivity of G it fixes all chambers, and hence it acts trivially on Δ. The second implies that H is co-compact, as claimed. Note, for completeness, that G cannot be discrete, by the thickness assumption on Δ (B cannot be finite, acting transitively on the chambers with a fixed w-distance from one chamber, for all $w \in W$). Part **3** of the corollary uses the work of Dymara-Januszkiewicz [DJ02]: under the conditions on the building appearing there, any closed subgroup of $Aut(\Delta)$ acting strongly transitively on it, has property (T). Thus Γ, being a lattice in the product of the closures of its projections, has property (T), and together with part **1** of the corollary, the statement follows. We remark that the various examples of Kazhdan groups which follow from [DJ02] also provide just non-compact Kazhdan groups to which Theorem 1.6 applies. Some (but not all) of the Kac-Moody lattices whose just infinity property is established in Theorem 1.5, are irreducible lattices in products of such groups.

We now elaborate briefly on the ingredients of the Proof of Theorem 1.5 already mentioned in the Introduction, referring to [Rém03] for more details and relevant references (a quick introduction to Kac-Moody groups in the functorial approach of Tits used here, can also be found in the appendix to [DJ02]). The basic result connecting our work to Kac-Moody groups over finite fields was established by Rémy [Rém99]: Let $W = < S >$ be the Weyl group associated to the Kac-Moody group $\mathbf{\Lambda}$, and let $W(t) = \Sigma_{w \in W} t^{l(w)}$ be its growth series. Then, when $W(\frac{1}{q}) < \infty$, the (discrete, with finite central kernel) embedding of the group of F_q points $\mathbf{\Lambda_q}$ in $Aut(X_-) \times Aut(X_+)$ is a non-uniform lattice, where $X_\pm$ are the isomorphic "opposite sign" buildings corresponding to the (B, N)-pair structure on $\mathbf{\Lambda_q}$ (and on which the closure of $\mathbf{\Lambda_q}$ acts strongly transitively). Dominating the growth series of W by that of a free product of $|S|$ groups of two elements (i.e. a free Coxeter group), one immediately gets the estimate $W(\frac{1}{|S|}) < \infty$. In fact, when W is not a free Coxeter group, it is not difficult to see that $W(\frac{1}{q}) < \infty$ for $q \geq |S| - 1$. Once the **S-I** condition mentioned in the Introduction was verified in this general setting by Rémy [Rém04], Theorem 1.5 follows exactly like Theorem 1.1. As in the above proof of Corollary 1.4, this uses the general fact that the closures of the projections of $\mathbf{\Lambda_q}$ in $Aut(X_\pm)$ are strongly transitive, hence are just non-compact.

Finally, a more general setting in which the framework of irreducible lattices fits naturally, arises from Ronan-Tits' theory of so-called "twin buildings" (cf. [Tit92]). One can generalize Theorem 1.5 to groups with twin root datum indexed by an irreducible Coxeter system and with finite root groups. Here we similarly have that modulo a finite normal (not necessarily central) subgroup, they are just infinite, when the minimum of the orders of the root groups (q for $\mathbf{\Lambda}_q$), is at least $|S|$ (the condition ensuring that they are of finite co-volume – see [RR02]).

5. Some factor theorems and the Poisson boundaries involved

This section is devoted to various concrete situations in which the factor theorem applies, making explicit the abstract framework of Poisson boundaries in which it is stated. In particular, we aim to deduce an adelic generalization of Margulis' factor theorem, which in its most general form is concerned with S-arithmetic groups Γ for which S can be *any* (infinite) set of places, the global field may be of positive characteristic, and the ambient algebraic group may be non simply connected. One difficulty that arises here is that not only the explicit identification of the Poisson boundary for general semisimple algebraic groups over local fields is required, but one additionally has to take into account the fact that the closure of the projection of Γ to the local factors may lie anywhere between the subgroup G^+ and G, where the former is in general co-compact, but not finite indexed in the latter. Some results on the Poisson boundary in the setting of non-Archimedean local fields can be found in [GLT98, Sect. XV] (see also the references therein), although they are not in the generality we need. We shall first recall some of the notions involved here, which will also be needed in the proofs.

The case of general algebraic groups over local fields. Recall that if k is any local field and $\mathbf{G}$ is a simple (k-isotropic) algebraic group defined over k, then $G^+ = \mathbf{G}(k)^+ < G = \mathbf{G}(k)$ is defined to be the (normal) subgroup generated by all the connected unipotent k-subgroups. It equals G in general only when $\mathbf{G}$ is simply connected, and it has finite index in general only when $char(k) = 0$. For example, when $\mathbf{G} = \mathbf{PGL_n}$ then $G^+ = PSL_n(k)$, and G/G^+ is isomorphic to $k^*/(k^*)^n$, which is infinite (and compact) when $n = char(k)$. Thus, our first main purpose is to obtain a unified adelic generalization of Margulis' factor theorem, which *a posteriori* does not involve any mention of G^+ and other "anomalies" which might be a priori encountered. Our main tool is the following general result, which may also be of some independent interest:

Theorem 5.1. *Let Δ be a locally finite, affine thick building, and assume that the locally compact group G acts properly and strongly transitively on Δ. Then there exists a closed amenable co-compact subgroup $P < G$*

such that for every admissible measure μ on G, the Poisson boundary $B(G,\mu)$ is G/P (with the unique μ-stationary measure on it).

Remark. Using his notion of a "universal irreducible affine G-space", Furstenberg showed [Fu73] that if a locally compact group G admits *some* co-compact amenable subgroup, then it admits a *maximal* subgroup with this property, in the sense that any other one is conjugate to a subgroup of it. Lemma 5.7 below shows that whenever the Poisson boundary of a group G is G/P for some amenable co-compact subgroup P, then this P must be maximal (among amenable co-compact subgroups).

The relevant consequence for the purpose of the adelic Factor Theorem is:

Corollary 5.2. *Let $\mathbf{G}$ be a connected semisimple algebraic group defined over a local field k of arbitrary characteristic, let $G = \mathbf{G}(k)$ and set $G^+ = \mathbf{G}(k)^+$. Let $H < G$ be a closed subgroup with $G^+ \le H \le G$, and $\mathbf{P} < \mathbf{G}$ be a minimal k-parabolic subgroup with $P = \mathbf{P}(k)$. Then for every admissible measure μ on H one has: (i) There is a unique μ-stationary measure ν on G/P; (ii) The measure ν belongs to the canonical (G-invariant) measure class of G/P; and (iii) The space $(G/P,\nu)$ is the Poisson boundary of (H,μ).*

We postpone the proofs of both Theorem 5.1 and Corollary 5.2 to the end of the section. The following theorem follows from the result of Borel Behr and Harder on the finite co-volume of S-arithmetic groups (cf. [Mar91, I.3.2.4]), using Theorem 3.7, and Corollary 5.2 applied to the subgroups $H_\nu = \overline{pr_\nu(\Gamma)} < G(K_\nu)$, choosing admissible measures $\mu_\nu \in P(H_\nu)$ such that condition (5) in 3.V is satisfied.

Theorem 5.3 (Adelic Factor Theorem). *Let $\mathbf{G}$ be a connected semisimple, almost K-simple algebraic group defined over a global field K of arbitrary characteristic. Let S be any set (possibly infinite) of inequivalent valuations on K, containing all Archimedean ones and at least two valuations for which $\mathbf{G}$ is isotropic (i.e., $\mathbf{G}(K_\nu)$ is non-compact). For every $\nu \in S$ denote by $\mathbf{P}_\nu$ a minimal K_ν-parabolic subgroup of $\mathbf{G}$ and set $B = \prod_{\nu\in S}\mathbf{G}(K_\nu)/\mathbf{P}_\nu(K_\nu)$ equipped with the product of the canonical measure classes. Let Γ denote (a finite index subgroup of) the S-arithmetic group $\mathbf{G}(K(S))$ acting diagonally on B. Then any Γ-factor of B is of the form $\prod_{\nu\in S}\mathbf{G}(K_\nu)/\mathbf{Q}_\nu(K_\nu)$ where each $\mathbf{Q}_\nu$ is a K_ν-parabolic subgroup of $\mathbf{G}$ containing $\mathbf{P}_\nu$, and the factor map is the natural projection.*

A remark on the comparison with Margulis' and Burger-Mozes' work. In the case where $\Delta = T$ is a tree, Theorem 5.1 immediately implies that the Poisson boundary of any group acting strongly transitively on T is ∂T, with a measure in the unique measure class preserved by $Aut(T)$. Theorem 1.7 then reduces to a generalization of the factor theorem by Burger-Mozes [BM00b, Theorem 4.6] for lattices in products of trees (which follows also from Corollary 1.8, without having to identify precisely the Poisson boundary). This factor theorem is stated in [BM00b] with the additional assumption that the closures of the projections of Γ are topologically just infinite. It is the use of Howe-Moore's theorem, as made in the original proof of Margulis, which accounts for this additional assumption. In the linear setting treated by Margulis, the use of Howe-Moore's theorem requires to work with simply connected ambient groups (or, somewhat more generally, reduce to the case of G^+ having finite index in G), as only then one knows that the action of a split torus A on G/Γ is mixing (in general an invariant subset for A will only be invariant under the co-compact subgroup G^+). A similar issue occurs in Burger-Mozes work. Our approach circumvents this problem as it reduces in the Margulis and Burger-Mozes cases to the ergodicity of P rather than A, on G/Γ, which does always hold (in fact making sense, by Mackey's virtual subgroup point of view, for *any* locally compact group G via the interpretation of G/P as the Poisson boundary of G).

Factor theorems for non-semisimple and non-compact homogeneous spaces. Theorem 1.7 can be used to give new results also in the conventional framework of Lie groups. In fact, let **G** be any algebraic group defined over $\mathbb{Q}$ and without $\mathbb{Q}$-characters. Then as long as $G = \mathbf{G}(\mathbb{R})$ is not compact by nilpotent, one can always find some admissible measure on it with non-trivial boundary B, and then get a factor rigidity theorem for irreducible lattices, such as $\Gamma = \mathbf{G}(\mathbb{Z}[\sqrt{2}])$, acting on $B \times B$ through the standard discrete embedding of Γ in $G \times G$. The boundaries B so obtained are homogeneous G-spaces, although they are *not* always compact (see [Rau77] for details on all the information brought here without proof). For example, for $G = SL_2(\mathbb{R}) \ltimes \mathbb{R}^2$ one can find admissible measures μ for which the Poisson boundary is $B = G/L$, where L is the semi-direct product of the "$ax + b$" triangular subgroup acting on the line fixed by it. Note that this L is *not* co-compact (B is a line bundle over the projective space). From Theorem 1.7 we then deduce:

Theorem 5.4. *Let $\Gamma = SL_2(\mathbb{Z}[\sqrt{2}]) \ltimes \mathbb{Z}[\sqrt{2}]^2$. Then any Γ-quotient of the product space $G/L \times G/L$, where Γ acts on the first coordinate via its natural inclusion in G and on the second via its conjugate inclusion, is one of the 9 possibilities of (homogeneous) $G \times G$ quotients.*

This generalizes Margulis' factor theorem for $SL_2(\mathbb{Z}[\sqrt{2}]) < SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$.

Non-homogeneous factor theorems – The CAT(–1) and Kac-Moody cases.

Proof of Corollary 1.8. A recent result of Connell and Muchnik [CM03], asserts that whenever X is a $CAT(-1)$ space and $G < \mathrm{Iso}(X)$ is convex co-compact, or more generally, when a Patterson-Sullivan measure ν on

∂X associated to G is supported on the radial limit set of G, there is *some* admissible measure μ on G with $\mu * \nu = \nu$ (we remark that the Patterson-Sullivan construction extends naturally to all $CAT(-1)$ spaces X, and (not necessarily discrete) subgroups $G < \mathrm{Iso}(X)$ – cf. [BM96]). Thus, part **1** of Corollary 1.8 applied to these appropriately chosen measures μ_i on G_i, yields part **2**. We now prove part **1**. By Kaimanovich's [Kai00, Theorem 2.4] we have the following.

Proposition 5.5. *Let X be a $CAT(-1)$-space, $G < \mathrm{Iso}(X)$ be a closed non-elementary subgroup, and μ be an admissible measure on G. Then there is a unique μ-stationary measure ν on ∂X, and $(\partial X, \nu)$ is* **a** *(G, μ)-boundary (see Sect. 2.V above).*

Let us only mention that the uniqueness of the μ-stationary measure follows readily from Corollary 2.7. Let now ν be any $\mu = \mu_1 \times \mu_2$-stationary measure on $\partial X_1 \times \partial X_2$. Then ν projects to two stationary measures ν_1, ν_2 on $\partial X_1, \partial X_2$ respectively, and for both $i = 1, 2$ the measure ν_i is obviously μ- as well as μ_i-stationary. By Proposition 5.5 and Lemma 2.15 above, each $(\partial X_i, \nu_i)$ forms a (G, μ)-boundary, as well as (G_i, μ_i)-boundary. From Corollary 2.17 it follows that almost every conditional measure ν_ω is projected to a point measure in both ∂X_1 and ∂X_2, hence it is itself a point measure, and $(\partial X_1 \times \partial X_2, \nu)$ is a (G, μ)-boundary. By Corollary 2.7, this implies the uniqueness of ν as a μ-stationary measure. Finally, $(\partial X_1 \times \partial X_2, \nu)$ being a (G, μ)-boundary, is a quotient of the Poisson boundary $B(G, \mu)$, so Corollary 1.8 now immediately follows from the general factor theorem (Theorem 1.7). □

An example – Bourdon's buildings. We discuss further a special case of Corollary 1.8, which brings forward some interesting phenomena that are not present in the "linear" setting. Here the X_i's in Corollary 1.8 are taken to be (the $CAT(-1)$ realization of) one (Bourdon-) hyperbolic building X, which is also associated to an appropriate countable Kac-Moody group Γ (this will be the case whenever the thicknesses at all panels is 1 + fixed prime power – cf. [Rém03]). We thank Marc Bourdon for providing useful information concerning these buildings, and refer the reader to [Bou97] and [BP00] for further details on their basic properties.

Let G be the topological Kac-Moody group associated with Γ above, namely, the closure of the image of Γ in $Aut(X)$ (see the proof of Theorem 1.5 above). Then G acts co-compactly on X (in fact it is virtually all of $\mathrm{Iso}(X)$), however it *does not* act transitively on ∂X. The G-orbits on ∂X are in 1-1 bijection with the orbits of the corresponding Weyl (here virtually a surface-) group on the boundary of an apartment (isometric to the hyperbolic plane). It is interesting to apply here Connell-Muchnik (methods and) results mentioned above [CM03] to *two* Patterson-Sullivan measures defined on ∂X. The usual one, associated with the $CAT(-1)$ metric on X, and a combinatorial one, arising from the combinatorial metric on the (chambers of the) building (we remark that it can be shown that the latter measure is not supported on one G-orbit). Both measures satisfy the usual properties of Patterson-Sullivan measures, however they are mutually singular. Thus, one deduces in this way from Corollary 1.8, measurable (non-homogeneous) factor theorems defined on the same topological boundary, but with mutually singular measures.

It would be interesting to understand what are the measurable G-factors of ∂X with respect to (either one of) the Patterson-Sullivan measures. It does not seem clear even if there are finitely many of them. A conjecture which arises naturally in this framework is the topological counterpart to Corollary 1.8 for minimal continuous Γ-actions on compact spaces (in general involving the limit set of the G_i's), with analogy to Dani's topological factor theorem in the linear higher rank case [Dan84].

Continuing with the same group(s) G as above, we now elaborate on the remark made following Theorem 1.5, concerning the substantial structural difference between these topological Kac-Moody (simple) groups, and simple algebraic groups over local fields, emphasizing the features relevant to the factor theorems.

We have already seen that unlike the case of semisimple algebraic groups over local fields (Theorem 5.1 above), the Poisson boundary of G may not be a homogeneous space (at least with respect to some, but probably with respect to all, admissible measures). In fact, a remark mentioned to us by Rémy with a somewhat different argument, once we know that $G < \mathrm{Iso}(X)$ acts non-transitively on its limit set $L(G)$ (the whole boundary in our case), then it cannot posses a co-compact amenable subgroup $P < G$. Indeed, such P would preserve a measure on $L(G)$, which must have support of at most two points (otherwise the center of mass in X would be fixed by P and G would be compact). Hence, by passing to a finite index subgroup we may assume that P fixes a point in $L(G)$ and by co-compactness its G-orbit in $L(G)$ will be closed, contradicting minimality and non-transitivity of G.

Another feature which is crucial for Margulis' factor theorem is the Howe-Moore property for unitary representations (i.e., all representations without a fixed vector are mixing). This property completely fails for G by the following reason: The stabilizer $H < G$ of a "tree-wall" in Bourdon's building (cf. [BP00]) is an open subgroup which is non-compact and of infinite index. Thus the G-representation on $L^2(G/H)$, in which H fixes the identity coset, has a vector which is invariant under a non-compact subgroup H, but not invariant under (a finite index, or even a co-compact subgroup of) all of G. In fact, the G-representation on $L^2(G/H)$ can be shown to be *irreducible*, since it is not difficult to see via the geometric definition of H, that it is exactly its own commensurator in G. A well known result of Mackey [Mac76, Sect. 3.5, Cor. 2] guarantees irreducibility of $L^2(G/H)$ in this situation. However, since any compact subgroup $K < G$ will have an infinite dimensional subspace of K-invariant vectors in $L^2(G/H)$ (by non-compactness of G/H), it follows that G does not admit a Gelfand pair structure with respect to *any* (compact) K, as this would force at most 1-dimensional subspace of K-invariants. Finally, we mention that because the building has also a natural structure of a "space with walls" invariant under G, it follows also that G, and hence also the Kac-Moody lattice $\Gamma < G \times G$, fails to have Kazhdan's property (T) (cf. [HP98]).

Proofs of Theorem 5.1 and Corollary 5.2.

Proof of Theorem 5.1. Our strategy is to show that there exists a compact subgroup $K < G$ which acts transitively on $B = B(G, \mu)$. This fact alone implies the result, because first, it shows that B is a homogeneous space G/P for some co-compact subgroup $P < G$, second, G/P must be an amenable G-space (Theorem 2.13 above), and third, since the G-action on G/P is amenable (if and) only if P is amenable [Zim84, Proposition 4.3.2], it follows that P is an amenable subgroup. We begin by reducing to the case where the G-action is *type preserving*. Indeed, G has a finite index subgroup $G_0 < G$ which is type preserving. However, it is a general fact that for any (admissible) measure μ on a group G, there exists *some* (admissible) measure μ_0 on the finite index subgroup $G_0 < G$, such that $B(G, \mu)$ is a (G_0, μ_0)-space, and forms the Poisson boundary of (G_0, μ_0) (μ_0 will be the hitting measure on G_0 of the μ-random walk on G – cf. [Fu71, Sect. 4.3]). But once we know that for **every** (admissible) μ_0 there exists a compact subgroup $K_0 < G_0$ that acts transitively on $B(G_0, \mu_0)$, it follows that the same subgroup $K = K_0 < G$ acts transitively on $B(G, \mu)$.

To continue the proof of Theorem 5.1 we shall need the following notion:

Definition. A real valued function ϕ on a G-space X is called **μ-harmonic**, if it satisfies $\mu * \phi = \int \phi(gx) d\mu(g) = \phi$.

Notice that for a compact subgroup $K < G$, K acts transitively on $B(G, \mu)$ if and only if every μ-harmonic function on G/K is constant. Indeed, the latter is equivalent to the fact that every K-invariant μ-harmonic function on G is constant. By Theorem 2.11 this is equivalent to the ergodicity of K on $B(G, \mu)$. However, for a compact group K, ergodicity is equivalent to (essential) transitivity.

Theorem 3.1 of [Fu63] asserts that if G is a connected semisimple Lie group and $K < G$ is a maximal compact subgroup, then for any admissible measure μ on G every μ-harmonic function on the symmetric space $X = G/K$ is constant. The proof of this result uses only one feature of the G-action on X (besides its being isometric), proved in Lemma 3.1 there: for any neighborhood Q of the identity $e \in G$, there is some $\epsilon > 0$, such that for any $p \in X$ the set Qp contains an ϵ-neighborhood of p. In the case where X is (the vertices of) a **graph**, on which some group G acts (continuously and) isometrically, an easy modification (and even simplification) of the argument shows the following: if there is **some** bounded neighborhood Q of e in G, such that for any $p \in X$ the set Qp contains all the neighbors of p, then every μ-harmonic function on X is constant, for any measure μ which dominates a multiple of the Haar measure on Q. However for any admissible measure μ on G, using the boundedness of Q one can always replace μ by a measure μ' which is a finite convex combination of convolution powers of μ, which has the latter property. Since any μ-harmonic function will be harmonic also with respect to μ', the same criterion for triviality of μ-harmonic functions applies to all admissible μ.

Suppose that we show that for G as in Theorem 5.1, there exists a G-transitive graph X with a compact (open) vertex stabilizer K, such that the above criterion holds for the G action on X. Then it follows that every μ-harmonic function on G/K is constant, which we have shown to imply Theorem 5.1. Therefore the theorem is now reduced to the following geometric result:

Proposition 5.6. *Let Δ be a locally finite, affine thick building, on which the locally compact group G acts properly and strongly transitively. Then there exists a proper isometric G-action on a locally finite graph, which is transitive on its set of vertices X, satisfying the following property: There exists a bounded neighborhood Q of e in G, such that for any vertex $p \in X$ the set Qp contains all the neighbors of p.*

Proof. Fix an apartment $A \in \mathcal{A}$. Denote the stabilizer and the fixator of A in G by N and T respectively. The affine Weyl group associated to A is $W = N/T$. Denote by V the group of translations of A and define $L = W \cap V$. By [Bou02, V.3.10], there exist a **special point** $s \in A$, that is, a point such that $W = W_s \ltimes L$, where $W_s = \mathrm{Stab}_W(s)$. Remark that $L \simeq \mathbb{Z}^n$, and the crucial fact here, which distinguishes the affine buildings, is its being abelian.

Let $X = Gs \subset \Delta$. We define a graph structure on X by letting

$$E = \left\{(x, y) \in X^2 \;\middle|\; link_\Delta(x) \cap link_\Delta(y) \neq \emptyset\right\}.$$

Equip X with the graph metric d. Denote

$$Q = \{g \in G \mid d(gs, s) \leq 1\}.$$

Q is an open and compact neighborhood of e in G, because G acts properly on Δ. We will show that Q possesses the desired property.

Let $A_1 = A \cap X = Ws$ (i.e., A_1 is the set of vertices in A of the same type as s). From the fact that s is a special point, L is easily seen to act simply transitive on A_1. Denote $N' = Q \cap N$ and let $L' = L \cap (N'/T)$, where N'/T is the image of N' in $W = N/T$. For every $a \in A_1$ we have, as L is abelian:

$$L'a = \{x \in A_1 \mid d(x, a) \leq 1\}. \tag{6}$$

Let $C \subset A$ be a chamber such that $s \in \bar{C}$. Denote by $\rho = \rho_{A;C}$ the retraction of Δ on A with center C (see [BT72, Definition 2.3.5]). It is easily seen that $\rho : X \to A_1$ is 1-Lipschitz retraction. Denote $B = \mathrm{Stab}_G(C)$. For every $x \in X$, we choose an element $g_x \in B$ such that $\rho(x) = g_x(x)$.

Given $(x, y) \in E$, choose $n \in N'$ such that $nT \in L'$ maps $\rho(x)$ to $\rho(y)$ (using 6). Then

$$y = g_y^{-1} n g_x(x)$$

and

$$g_y^{-1} n g_x \in Q.$$

This completes the proof of Theorem 5.1. □

We preface the proof of Corollary 5.2 with the following:

Lemma 5.7. *Let G be any locally compact group with an admissible measure μ, and let $M < L < G$ be co-compact amenable subgroups of G. Then the natural projection $G/M \to G/L$ is relatively measure preserving, where the two spaces are endowed with the (unique) μ-stationary measure. In particular, since the Poisson boundary has no relatively measure preserving quotients (Corollary 2.20 above), if $P < G$ is amenable and co-compact, such that G/P is the Poisson boundary of (G, μ), then P is a maximal amenable co-compact subgroup.*

Proof. Let $\pi : G/M \to G/L$ be the G-equivariant map. Denote by η and $\xi = \pi_*\eta$ the (unique) stationary measures on G/M and G/L respectively. Since L/M is compact and L is amenable, there exists a finite L-invariant measure ν_0 on L/M. Viewed as an element of $P(G/M)$, this measure naturally defines a G-equivariant map $\psi : G/L \to P(G/M)$. Obviously, $\mathbf{bar}(\psi_*\eta) \in P(G/M)$ is stationary, thus it is equal to ξ. It follows that ψ satisfies the defining property of the disintegration map $D_\pi : G/L \to P(G/M)$ (see Sect. 2.I), hence $\psi = D_\pi$, and in particular the latter is G-equivariant. □

Proof of Corollary 5.2. We refer the reader to Sects. 1, 2 of Chap. I in [Mar91] for all the relevant facts concerning algebraic groups in general, and properties of the group G^+, mentioned here without proofs. Let $\tilde{\mathbf{G}}$ be the simply connected covering of $\mathbf{G}$, $\boldsymbol{\phi} : \tilde{\mathbf{G}} \to \mathbf{G}$ be the associated k-morphism, and $\tilde{\mathbf{P}} = \boldsymbol{\phi}^{-1}(\mathbf{P})$. Then $\tilde{\mathbf{P}}$ is a minimal k-parabolic subgroup of the k-group $\tilde{\mathbf{G}}$, and we denote $\tilde{G} = \tilde{\mathbf{G}}(k)$, $\tilde{P} = \tilde{\mathbf{P}}(k)$, and $\phi : \tilde{G} \to G$. We have $\phi(\tilde{G}) = G^+$. Let $\mathbf{S}$ be a maximal k-split torus of $\mathbf{G}$ contained in $\mathbf{P}$. Then P contains the centralizer $Z_{\mathbf{G}}(\mathbf{S})(\mathbf{k})$. Because $G = G^+ \cdot Z_{\mathbf{G}}(\mathbf{S})(\mathbf{k})$, the G^+-action on G/P is transitive with a point stabilizer being $G^+ \cap P = \phi(\tilde{P})$. Thus the G^+-action on G/P is isomorphic to its action on $\phi(\tilde{G})/\phi(\tilde{P})$, and because $Ker\phi$ is central and $\tilde{P}$ contains the center of $\tilde{G}$, the $\tilde{G}$-action on $\tilde{G}/\tilde{P}$ factors through $Ker\phi$ to the action of G^+ on G/P. Because $G^+ < H$ this shows in particular that the H-action on G/P is transitive, and hence stationary measures for any admissible measure on H are unique, and are in the same unique measure class of quasi-invariant measures under all of G.

By the work of Bruhat-Tits [BT84] the affine building associated with $\tilde{G}$ and with G may be identified in such a way that the $\tilde{G}$-action factors through ϕ (to an action of G^+). Because $\tilde{G}$ acts strongly transitively, so does G^+, and hence so does H (although no longer necessarily in a type preserving way). It then follows from Theorem 5.1 that the Poisson boundary of (H, μ) is of the form H/L for some amenable co-compact subgroup $L < H$. We note at this point that this implies that the Poisson boundary of $\tilde{G}$ itself is $\tilde{G}/\tilde{P}$, by Lemma 5.7 above and the fact that any subgroup of $\tilde{G}$ containing $\tilde{P}$ is (the rational points of) a parabolic subgroup, and hence can be amenable (if and) only if it is $\tilde{P}$ itself.

Now, because H acts transitively on G/P, the latter may be identified as an H-space with H/P_H where $P_H = H \cap P$. We will be done by showing that P_H is (conjugate to) L. Otherwise by maximality of L as a co-compact amenable subgroup, we may assume that $L > P_H$, and with respect to the μ-stationary measures the natural projection from H/P_H to H/L is measure preserving (see Lemma 5.7 above). Identifying back H/P_H with G/P as an H-space, the latter projection would define also a relatively measure preserving proper factor map of G^+-spaces. But using the discussion in the first paragraph this may be viewed as a measure preserving *proper* factor of the $\tilde{G}$-action on $\tilde{G}/\tilde{P}$, which is impossible since we saw above that the latter is the Poisson boundary of $\tilde{G}$, and hence has no non-trivial relatively measure preserving quotients (Corollary 2.20). This contradiction proves the corollary.

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