Invent. math. 163, 455–497 (2006)
DOI: 10.1007/s00222-005-0464-x

Inventiones mathematicae

Quadratic estimates and functional calculi of perturbed Dirac operators

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Oblatum 21-V-2004 & 30-V-2005
Published online: 25 October 2005 –

Abstract. We prove quadratic estimates for complex perturbations of Dirac-type operators, and thereby show that such operators have a bounded functional calculus. As an application we show that spectral projections of the Hodge–Dirac operator on compact manifolds depend analytically on L_∞ changes in the metric. We also recover a unified proof of many results in the Calderón program, including the Kato square root problem and the boundedness of the Cauchy operator on Lipschitz curves and surfaces.

Contents

1. Introduction

We prove quadratic estimates

$$\int_0^\infty \left\| \Pi_B \left(\mathrm{I} + t^2 \Pi_B{}^2 \right)^{-1} u \right\|^2 t\, dt \;\approx\; \|u\|^2 \tag{1}$$

for all $u \in L_2(\mathbf{R}^n, \Lambda)$, where $\Pi_B = d + B^{-1} d^* B$ is the perturbation of a Dirac-type operator $\Pi = d + d^*$ by an operator B of multiplication by an L_∞ complex matrix-valued function with uniformly positive real part. Here Λ is the complex exterior algebra on $\mathbf{R}^n$ and d denotes the exterior derivative.

This estimate implies that Π_B has a bounded functional calculus. This means that

$$\|f(\Pi_B)u\| \lesssim \|f\|_\infty \|u\| \tag{2}$$

for all $u \in L_2(\mathbf{R}^n, \Lambda)$ and all bounded holomorphic functions $f : S^o_\mu \longrightarrow \mathbf{C}$, where S^o_μ is an open double sector

$$S^o_\mu := \{z \in \mathbf{C} : |\arg(\pm z)| < \mu\} \quad \text{with} \quad \mu > \omega := \sup |\arg(Bu, u)|.$$

This result in turn implies perturbation estimates of the form

$$\|f(\Pi_{B+A})u - f(\Pi_B)u\| \lesssim \|f\|_\infty \|A\|_\infty \|u\| \tag{3}$$

for all $u \in L_2(\mathbf{R}^n, \Lambda)$, provided $\|A\|_\infty$ is not too large.

The unperturbed operator Π is selfadjoint, so when $B = I$, (2) holds for all bounded Borel measurable functions f by the spectral theory of selfadjoint operators. When B is positive selfadjoint, then Π_B is selfadjoint with respect to the inner-product (Bu, v) on $L_2(\mathbf{R}^n, \Lambda)$, so (1) and (2) still hold by spectral theory. However (3) would not, were it not for the structure of the operators Π, B and A. This is because we need (2) for all small non–selfadjoint perturbations of B in order to deduce (3) for small selfadjoint perturbations.

Under our assumptions on B, the operator Π_B has spectrum in the closed double sector $S_\omega = \{z \in \mathbf{C} : |\arg(\pm z)| \leq \omega\}$ and satisfies resolvent bounds

$$\left\|(\Pi_B - \lambda\, \mathrm{I})^{-1}\right\| \lesssim \frac{1}{\operatorname{dist}(\lambda, S_\omega)}$$

for all $\lambda \in \mathbf{C} \setminus S_\omega$. This follows from operator theory, but a proof of the quadratic estimate (1) requires the full strength of the harmonic analysis. Once the estimate (1) is proven, then (2) follows if $\omega < \mu < \frac{\pi}{2}$. It can then be seen that $f(\Pi_B)$ depends holomorphically on B, from which (3) follows provided A is not too large.

Our result was inspired by the proof of the Kato square root problem by Auscher, Hofmann, Lacey, McIntosh and Tchamitchian [2], and includes not only it as a corollary, but also many results in the Calderón program such as the boundedness of the Cauchy operator on Lipschitz curves and surfaces. The proof uses many of the concepts developed over the years to prove these results in the Calderón program, and in particular the proof of the Kato problem, but is not a direct consequence, as the operator Π_B is first order, and the second order operator ${\Pi_B}^2$ is not in divergence form. Indeed, our arguments utilize only the first order structure of the operator. This enables us to exploit the algebra involved in the (non–orthogonal) Hodge decomposition of the first order system

$$L_2(\mathbf{R}^n, \Lambda) = \mathsf{N}(d) \oplus \mathsf{N}(B^{-1}d^*B)$$

where $\mathsf{N}(d)$ is the null-space of d.

Combining the Hodge decomposition with (2) in the case when $f(z) = z/\sqrt{z^2}$, we obtain the equivalence

$$\|du\| + \|d^* Bu\| \approx \|\Pi_B u\| \approx \|\sqrt{{\Pi_B}^2}u\|.$$

The square root problem of Kato follows in the special case when B splits as $B_k(x) : \Lambda^k \to \Lambda^k$ for each $0 \le k \le n$, and for almost every $x \in \mathbf{R}^n$, with $B_0 = I$ and $B_1(x) = A(x) : \mathbf{C}^n \to \mathbf{C}^n$. On making the identification

$$d : L_2(\mathbf{R}^n, \Lambda^0) \to L_2(\mathbf{R}^n, \Lambda^1) \quad \text{with} \quad \nabla : L_2(\mathbf{R}^n, \mathbf{C}) \to L_2(\mathbf{R}^n, \mathbf{C}^n) \quad \text{and}$$
$$d^* : L_2(\mathbf{R}^n, \Lambda^1) \to L_2(\mathbf{R}^n, \Lambda^0) \quad \text{with} \quad -\operatorname{div} : L_2(\mathbf{R}^n, \mathbf{C}^n) \to L_2(\mathbf{R}^n, \mathbf{C})$$

and restricting our attention to Λ^0, we obtain

$$\|\nabla u\| \approx \|\sqrt{-\operatorname{div} A \nabla} u\|$$

for all $u \in L_2(\mathbf{R}^n, \mathbf{C})$.

The choice of test–functions used in our proof of the stopping time argument in Sect. 5 has more in common with that presented in the paper on elliptic systems [4] than with [2], but the result stated above does not include the full result on systems. To remedy this, as well as to allow further consequences, our results can in fact be stated somewhat more generally than so far indicated, though without much effect on the proofs. Rather than d, we consider any first order system Γ in a space $L_2(\mathbf{R}^n, \mathbf{C}^N)$ which satisfies $\Gamma^2 = 0$, we let $\Pi = \Gamma + \Gamma^*$, and consider perturbations of the type $\Pi_B = \Gamma + B_1 \Gamma^* B_2$ where B_1 has positive real part on the range of Γ^*, B_2 has positive real part on the range of Γ, and $\Gamma^* B_2 B_1 \Gamma^* = 0$ and $\Gamma B_1 B_2 \Gamma = 0$. In this case there is a (non–orthogonal) Hodge decomposition of $\mathcal{H} = L_2(\mathbf{R}^n, \mathbf{C}^N)$ into closed subspaces:

$$\mathcal{H} = \mathsf{N}(\Pi_B) \oplus \overline{\mathsf{R}(\Gamma_B^*)} \oplus \overline{\mathsf{R}(\Gamma)}.$$

The quadratic estimates and functional calculus hold for $u \in \overline{\mathsf{R}(\Pi_B)} = \overline{\mathsf{R}(\Gamma_B^*)} \oplus \overline{\mathsf{R}(\Gamma)}$.

These results have implications for spectral projections of the Hodge–Dirac operator $d + d_g^*$ on a compact manifold M with a Riemannian metric g. The operator $d + d_g^*$ is a selfadjoint operator in the Hilbert space $\mathcal{H} = L_2(M, \wedge T^* M)$, and so there is an orthogonal decomposition

$$\mathcal{H} = \mathsf{N}(d + d_g^*) \oplus \mathcal{H}_g^+ \oplus \mathcal{H}_g^-$$

where $\mathcal{H}_g^\pm$ are the positive and negative eigenspaces of $d + d_g^*$. The projections of $\mathcal{H}$ onto $\mathcal{H}_g^\pm$ are $\mathbf{E}_g^\pm = \xi^\pm(d + d_g^*)$ where the functions $\xi^\pm : S_\mu^o \cup \{0\} \longrightarrow \mathbf{C}$ defined by

$$\xi^\pm(z) = \begin{cases} 1 & \text{if } \pm \operatorname{Re} z > 0 \\ 0 & \text{if } \pm \operatorname{Re} z \le 0 \end{cases}$$

are holomorphic on S_μ^o. The subscript g denotes dependence on the metric g.

If the metric is perturbed to $g + h$, then the adjoint of d with respect to the perturbed metric has the form $d^*_{g+h} = B^{-1} d^*_g B$ for an associated positive selfadjoint multiplication operator B. The perturbation result (3) can be transferred to this context, thus giving

$$\left\| \mathbf{E}^{\pm}_{g+h} - \mathbf{E}^{\pm}_{g} \right\| \lesssim \|h\|_\infty := \text{ess sup}_{x \in M} |h_x| \tag{4}$$

provided $\|h\|_\infty$ is not too large, where

$$|h_x| = \sup\{|h_x(v, v)| : v \in T_x M,\ g_x(v, v) = 1\}.$$

What (4) tells us is that these eigenspaces depend continuously on L_∞ changes in the metric. Indeed the eigenspaces depend analytically on L_∞ changes in the metric. This result is possibly surprising in that the local formula for d^*_{g+h} in terms of d^*_g depends on the first order derivatives of h.

Acknowledgements This research was mostly undertaken at the Centre for Mathematics and its Applications at the Australian National University, and was supported by the Australian Research Council. The second author held a visiting position at the School of Mathematics at the University of New South Wales during the final preparation of this paper, and thanks them for their hospitality.

We wish to acknowledge the contributions of Pascal Auscher and Andrea Nahmod to the development of the connections between the Kato square root problem and quadratic estimates for a corresponding perturbed Dirac operator. The framework developed in their joint paper with the third author [6] is a forerunner of that presented in the current paper.

The key results of this paper were first presented at the Conference *Analyse Harmonique et ses Applications* at Orsay in June 2003, in honour of Raphy Coifman and Yves Meyer for their profound contributions to the theory of singular integrals and to the Calderón program.

2. Statement of results

We begin by standardizing notation and terminology. All theorems and results in this paper are quantitative, in the sense that constants in estimates depend only on constants quantified in the relevant hypotheses. Such dependence will usually be clear. We use the notation $a \approx b$ and $b \lesssim c$, for $a, b, c \geq 0$, to mean that there exists $C > 0$ so that $a/C \leq b \leq Ca$ and $b \leq Cc$, respectively. The value of C varies from one usage to the next, but then is always fixed, and depends only on constants quantified in the relevant preceding hypotheses.

For an unbounded linear operator $A : \mathsf{D}(A) \longrightarrow \mathcal{H}_2$ from a domain $\mathsf{D}(A)$ in a Hilbert space $\mathcal{H}_1$ to another Hilbert spaces $\mathcal{H}_2$, we denote its null space by $\mathsf{N}(A)$ and its range by $\mathsf{R}(A)$. The operator A is said to be closed when its graph is a closed subspace of $\mathcal{H}_1 \times \mathcal{H}_2$. The space of all bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ is denoted $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$, while $\mathcal{L}(\mathcal{H}) := \mathcal{L}(\mathcal{H}, \mathcal{H})$. See for example [19] for more details.

Consider three operators $\{\Gamma, B_1, B_2\}$ in a Hilbert space $\mathcal{H}$ with the following properties.

(H1) The operator $\Gamma : \mathsf{D}(\Gamma) \longrightarrow \mathcal{H}$ is a *nilpotent* operator from $\mathsf{D}(\Gamma) \subset \mathcal{H}$ to $\mathcal{H}$, by which we mean Γ is closed, densely defined and $\mathsf{R}(\Gamma) \subset \mathsf{N}(\Gamma)$. In particular, $\Gamma^2 = 0$ on $\mathsf{D}(\Gamma)$.

(H2) The operators $B_1, B_2 : \mathcal{H} \longrightarrow \mathcal{H}$ are bounded operators satisfying the accretivity conditions for some $\kappa_1, \kappa_2 > 0$:

$$\begin{aligned} \mathrm{Re}(B_1 u, u) &\geq \kappa_1 \|u\|^2 \quad \text{for all} \quad u \in \mathsf{R}(\Gamma^*), \\ \mathrm{Re}(B_2 u, u) &\geq \kappa_2 \|u\|^2 \quad \text{for all} \quad u \in \mathsf{R}(\Gamma). \end{aligned}$$

Let the angles of accretivity be

$$\begin{aligned} \omega_1 &:= \sup_{u \in \mathsf{R}(\Gamma^*)\setminus\{0\}} |\arg(B_1 u, u)| < \tfrac{\pi}{2}, \\ \omega_2 &:= \sup_{u \in \mathsf{R}(\Gamma)\setminus\{0\}} |\arg(B_2 u, u)| < \tfrac{\pi}{2}, \end{aligned}$$

and set $\omega := \frac{1}{2}(\omega_1 + \omega_2)$.

(H3) The operators satisfy $\Gamma^* B_2 B_1 \Gamma^* = 0$ on $\mathsf{D}(\Gamma^*)$ and $\Gamma B_1 B_2 \Gamma = 0$ on $\mathsf{D}(\Gamma)$, that is, $B_2 B_1 : \mathsf{R}(\Gamma^*) \longrightarrow \mathsf{N}(\Gamma^*)$ and $B_1 B_2 : \mathsf{R}(\Gamma) \longrightarrow \mathsf{N}(\Gamma)$. This implies that $\Gamma B_1^* B_2^* \Gamma = 0$ on $\mathsf{D}(\Gamma)$ and that $\Gamma^* B_2^* B_1^* \Gamma^* = 0$ on $\mathsf{D}(\Gamma^*)$.

In some applications, B_2 satisfies the accretivity condition on all of $\mathcal{H}$ and $B_1 = B_2{}^{-1}$. In this case (H3) is automatically satisfied, and the accretivity condition for B_1 holds with $\omega_1 = \omega_2$.

Definition 2.1. Let $\Pi = \Gamma + \Gamma^*$. Also let $\Gamma_B^* = B_1 \Gamma^* B_2$ and $\Gamma_B = B_2^* \Gamma B_1^*$ and then let $\Pi_B = \Gamma + \Gamma_B^*$ and $\Pi_B^* = \Gamma^* + \Gamma_B$.

In Sect. 4, specifically in Lemma 4.1 and Corollary 4.3, we show that $\Gamma_B^* = (\Gamma_B)^*$ and $\Pi_B^* = (\Pi_B)^*$, that each of these operators is closed and densely defined, and morever that Γ_B and Γ_B^* are nilpotent. The proofs of the following two propositions are also given in Sect. 4. The first establishes a Hodge decomposition for the perturbed operators.

Proposition 2.2. *The Hilbert space $\mathcal{H}$ has the following Hodge decomposition into closed subspaces:*

$$\mathcal{H} = \mathsf{N}(\Pi_B) \oplus \overline{\mathsf{R}(\Gamma_B^*)} \oplus \overline{\mathsf{R}(\Gamma)}. \tag{5}$$

Moreover, we have $\mathsf{N}(\Pi_B) = \mathsf{N}(\Gamma_B^*) \cap \mathsf{N}(\Gamma)$ and $\overline{\mathsf{R}(\Pi_B)} = \overline{\mathsf{R}(\Gamma_B^*)} \oplus \overline{\mathsf{R}(\Gamma)}$. When $B_1 = B_2 = I$ these decompositions are orthogonal, and in general the decompositions are topological. Similarly, there is also a decomposition

$$\mathcal{H} = \mathsf{N}(\Pi_B^*) \oplus \overline{\mathsf{R}(\Gamma_B)} \oplus \overline{\mathsf{R}(\Gamma^*)}.$$

Definition 2.3. The bounded projections onto the subspaces in the Hodge decomposition (5) are denoted by $\mathbf{P}_B^0$ onto $\mathsf{N}(\Pi_B)$, $\mathbf{P}_B^1$ onto $\overline{\mathsf{R}(\Gamma_B^*)}$ and $\mathbf{P}_B^2$ onto $\overline{\mathsf{R}(\Gamma)}$. When $B_1 = B_2 = I$, these are orthogonal projections which we denote by $\mathbf{P}^0$, $\mathbf{P}^1$ and $\mathbf{P}^2$.

We now investigate the spectrum and resolvent estimates for the operator Π_B.

Definition 2.4. Given $0 \leq \omega < \mu < \frac{\pi}{2}$, define the closed and open sectors and double sectors in the complex plane by

$$\begin{aligned} S_{\omega+} &:= \{z \in \mathbf{C} : |\arg z| \leq \omega\} \cup \{0\}, \\ S^o_{\mu+} &:= \{z \in \mathbf{C} : z \neq 0,\ |\arg z| < \mu\}, \\ S_\omega &:= S_{\omega+} \cup (-S_{\omega+}), \\ S^o_\mu &:= S^o_{\mu+} \cup (-S^o_{\mu+}). \end{aligned}$$

Also let $\Psi(S^o_\mu)$ denote the collection of holomorphic functions $\psi : S^o_\mu \longrightarrow \mathbf{C}$ such that there exist $L, s > 0$ so that

$$|\psi(z)| \leq L \frac{|z|^s}{(1 + |z|^{2s})}$$

for all $z \in S^o_\mu$.

Proposition 2.5. *The spectrum $\sigma(\Pi_B)$ is contained in the double sector S_ω. Moreover the operator Π_B satisfies resolvent bounds*

$$\left\| (\mathrm{I} + \tau \Pi_B)^{-1} \right\| \lesssim \frac{|\tau|}{\operatorname{dist}(\tau, S_\omega)}$$

for all $\tau \in \mathbf{C} \setminus S_\omega$.

Such an operator is of type S_ω as defined in [1,5]. A consequence of the above proposition is that the following operators are uniformly bounded in t.

Definition 2.6. For $t \in \mathbf{R}$ ($t \neq 0$), define the bounded operators in $\mathcal{H}$:

$$\begin{aligned} R^B_t &:= (\mathrm{I} + it\Pi_B)^{-1}, \\ P^B_t &:= \left(\mathrm{I} + t^2 {\Pi_B}^2\right)^{-1} = \tfrac{1}{2}\left(R^B_t + R^B_{-t}\right) = R^B_t R^B_{-t} \quad \text{and} \\ Q^B_t &:= t\Pi_B\left(\mathrm{I} + t^2 {\Pi_B}^2\right)^{-1} = \tfrac{1}{2i}\left(-R^B_t + R^B_{-t}\right). \end{aligned}$$

In the unperturbed case $B_1 = B_2 = I$, we write R_t, P_t and Q_t for R^B_t, P^B_t and Q^B_t, respectively.

For an operator with the spectral properties of Proposition 2.5, it is useful to know whether it satisfies quadratic estimates and whether it has a bounded holomorphic functional calculus. The hypotheses (H1–3) are not enough to imply quadratic estimates. See Remark 3.4. Thus we introduce further hypotheses which allow the use of harmonic analysis.

(H4) The Hilbert space is $\mathcal{H} = L_2(\mathbf{R}^n; \mathbf{C}^N)$, where $n, N \in \mathbf{N}$.

(H5) The operators B_1 and B_2 denote multiplication by matrix–valued functions $B_1, B_2 \in L_\infty(\mathbf{R}^n; \mathcal{L}(\mathbf{C}^N))$.

(H6) (Localisation) The nilpotent operators Γ and Γ^* are first order differential operators in the sense that if $\eta : \mathbf{R}^n \longrightarrow \mathbf{C}$ is a bounded Lipschitz function, then multiplication by η preserves $\mathsf{D}(\Gamma)$ and $\mathsf{D}(\Gamma^*)$, and the commutators

$$\Gamma_{\nabla\eta} := [\Gamma, \eta \mathrm{I}], \qquad \Gamma^*_{\nabla\eta} := [\Gamma^*, \eta \mathrm{I}]$$

are multiplication operators such that there exists $c > 0$ so that

$$|\Gamma_{\nabla\eta}(x)|,\ |\Gamma^*_{\nabla\eta}(x)| \leq c|\nabla\eta(x)|$$

for all $x \in \mathbf{R}^n$.

(H7) (Cancellation) We have $\int_{\mathbf{R}^n} \Gamma u = 0$ for all compactly supported $u \in \mathsf{D}(\Gamma)$, and we have $\int_{\mathbf{R}^n} \Gamma^* v = 0$ for all compactly supported $v \in \mathsf{D}(\Gamma^*)$.

(H8) (Coercivity) There exists $c > 0$ such that

$$\|\nabla u\| \leq c\|\Pi u\|$$

for all $u \in \mathsf{R}(\Pi) \cap \mathsf{D}(\Pi)$.

Observe that (H6–7) automatically hold if Γ is a homogeneous first order differential operator with constant coefficients. We now state the first main result of the paper.

Theorem 2.7. *Consider the operator $\Pi_B = \Gamma + B_1\Gamma^* B_2$ acting in the Hilbert space $\mathcal{H} = L_2(\mathbf{R}^n; \mathbf{C}^N)$, where $\{\Gamma, B_1, B_2\}$ satisfies the hypotheses (H1–8). Then Π_B satisfies the quadratic estimate*

$$\int_0^\infty \left\| Q_t^B u \right\|^2 \frac{dt}{t} = \int_0^\infty \left\| \Pi_B \left(\mathrm{I} + t^2 \Pi_B{}^2\right)^{-1} u \right\|^2 t\, dt \ \approx\ \|u\|^2 \tag{6}$$

for all $u \in \overline{\mathsf{R}(\Pi_B)} \subset L_2(\mathbf{R}^n; \mathbf{C}^N)$.

Let us now discuss the holomorphic functional calculus for Π_B. As a result of Proposition 2.5, one can define the operator $\psi(\Pi_B) : \mathcal{H} \longrightarrow \mathcal{H}$ whenever $\psi \in \Psi(S^o_\mu)$ for some $\mu > \omega$, in such a way that the mapping $\psi \mapsto \psi(\Pi_B)$ is an algebra homomorphism. This can be done as in the Dunford functional calculus by a contour integral

$$\psi(\Pi_B) := \frac{1}{2\pi i} \int_\gamma \psi(\lambda)(\lambda \mathrm{I} - \Pi_B)^{-1} d\lambda \tag{7}$$

where γ is the unbounded contour $\{\pm re^{\pm i\theta} : r \geq 0\}$, $\omega < \theta < \mu$, parametrised counterclockwise around S_ω. The decay estimate on ψ and the resolvent bounds of Proposition 2.5 guarantee that the integral is absolutely convergent and that $\psi(\Pi_B)$ is bounded. See for example [1,5,12,25] for a discussion of these matters.

Remark 2.8. We note in passing that each $\psi \in \Psi(S^o_\mu)$ which is nonzero on both sectors defines a quadratic seminorm on $\mathcal{H}$, and that they are all equivalent. In particular, we have $\int_0^\infty \|\psi(t\Pi_B)u\|^2 \frac{dt}{t} \approx \int_0^\infty \|Q_t^B u\|^2 \frac{dt}{t}$ for all $u \in \mathcal{H}$. Therefore, under hypotheses (H1–8), we have $\int_0^\infty \|\psi(t\Pi_B)u\|^2 \frac{dt}{t} \approx \|(\mathrm{I} - \mathbf{P}_B^0)u\|^2$ for all $u \in \mathcal{H}$.

Definition 2.9. Suppose $\omega < \mu < \frac{\pi}{2}$. We say that Π_B has a *bounded S^o_μ holomorphic functional calculus* if

$$\|\psi(\Pi_B)\| \lesssim \|\psi\|_\infty := \sup\left\{|\psi(z)| : z \in S^o_\mu\right\} \tag{8}$$

for all $\psi \in \Psi(S^o_\mu)$.

In this case one can define a bounded operator $f(\Pi_B)$ with

$$\|f(\Pi_B)\| \lesssim \|f\|_\infty := \sup\left\{|f(z)| : z \in S^o_\mu \cup \{0\}\right\} \tag{9}$$

for all bounded functions $f : S^o_\mu \cup \{0\} \longrightarrow \mathbf{C}$ which are holomorphic on S^o_μ. The operator $f(\Pi_B)$ can be defined by

$$f(\Pi_B)u = f(0)\mathbf{P}_B^0 u + \lim_{n\to\infty} \psi_n(\Pi_B)u \tag{10}$$

for all $u \in \mathcal{H}$, where the functions $\psi_n \in \Psi(S^o_\mu)$ are uniformly bounded and tend locally uniformly to f on S^o_μ; see [1,12]. The definition is independent of the choice of the approximating sequence (ψ_n). If Π_B satisfies the quadratic estimate (6) for all $u \in \overline{\mathsf{R}(\Pi_B)}$ then it has a bounded holomorphic functional calculus. Thus we have the second main result of the paper.

Theorem 2.10. *Assume the hypotheses of Theorem 2.7 and let $\omega < \mu < \frac{\pi}{2}$. Then Π_B has a bounded S^o_μ holomorphic functional calculus in $L_2(\mathbf{R}^n; \mathbf{C}^N)$.*

A consequence of this theorem is that there is a decomposition of $\mathcal{H}$ into spectral subspaces. Let $\xi^\pm$ be the holomorphic functions defined in the Introduction. Also let ξ^0 denote the characteristic function of $\{0\}$ so that $\xi^0 + \xi^+ + \xi^- = 1$ on $S^o_\mu \cup \{0\}$ and $\xi^0(\Pi_B) = \mathbf{P}_B^0$. By Theorem 2.10, the spectral projections $\mathbf{E}_B^\pm = \xi^\pm(\Pi_B)$ are bounded, and by the functional calculus, $\mathbf{P}_B^0 + \mathbf{E}_B^+ + \mathbf{E}_B^- = \mathrm{I}$. This leads to part (i) of the corollary below. Furthermore, define the function sgn by $\mathrm{sgn}(z) = z/\sqrt{z^2}$ when $z \in S^o_\mu$ and $\mathrm{sgn}(0) = 0$, so that $\mathrm{sgn}(z) = \xi_+(z) - \xi_-(z)$ and hence $\mathrm{sgn}(\Pi_B) = \mathbf{E}_B^+ - \mathbf{E}_B^-$. The boundedness of this operator together with the Hodge decomposition implies part (ii).

Corollary 2.11. *Assume the hypotheses of Theorem 2.7. Then*

(i) *there is a (non–orthogonal) spectral decomposition*

$$\mathcal{H} = \mathsf{N}(\Pi_B) \oplus \mathbf{E}_B^+\mathcal{H} \oplus \mathbf{E}_B^-\mathcal{H}$$

into spectral subspaces of Π_B corresponding to $\{0\}$, $S_{\omega+} \setminus \{0\}$ and $S_{\omega-} \setminus \{0\}$, respectively; and

(ii) *we have* $\mathsf{D}(\Gamma) \cap \mathsf{D}(\Gamma_B^*) = \mathsf{D}(\Pi_B) = \mathsf{D}(\sqrt{\Pi_B{}^2})$ *with*

$$\|\Gamma u\| + \|\Gamma_B^* u\| \approx \|\Pi_B u\| \approx \|\sqrt{\Pi_B{}^2}u\|.$$

Remark 2.12. If $u_0 \in \mathbf{E}_B^{\pm}\mathcal{H}$, then $u(x,t) = \exp(-t\sqrt{\Pi_B{}^2})u_0(x)$ is the solution of $\frac{\partial u}{\partial t} \pm \Pi_B u = 0$ for $t \geq 0$ which equals u_0 when $t = 0$ and decays as $t \to \infty$. It is a consequence of Remark 2.8 with $\psi(z) = z\exp(-\sqrt{z^2})$ that $\|u_0\|^2 \approx \int_0^\infty \|\frac{\partial}{\partial t}u(.,t)\|^2 t dt$ for $u_0 \in \mathbf{E}_B^{\pm}\mathcal{H}$.

In Sect. 3 we use Theorem 2.10 and Corollary 2.11 to give a unified proof of many results in the Calderón program, including the Kato square root problem and the boundedness of the Cauchy operator on Lipschitz curves and surfaces. We are not claiming that the approach adopted here is always better than the original proofs given by the respective authors. Nonetheless, we believe there is value in seeing that each of these results can be easily derived from Theorem 2.7. Moreover, at the end of Sect. 3 we apply Theorem 2.10 to Hodge–Dirac operators in Euclidean space, and obtain Theorem 3.11. This result is new.

Sections 6 and 7 give further consequences and developments of Theorems 2.7 and 2.10. In Sect. 6 we first demonstrate that, under the hypotheses (H1–3), the resolvents of Π_B vary holomorphically with respect to perturbations in B, as do the operators $\psi(\Pi_B)$ when $\psi \in \Psi(S_\mu^o)$. We use these results in Theorem 6.4, to show that, under all the hypotheses (H1–8), the bounded members of the functional calculus of the perturbed Dirac operator, and quadratic functions, depend holomorphically on perturbations in B. From this, we deduce Lipschitz estimates on members of the functional calculus of the perturbed Dirac operator Π_B, and also of the quadratic estimates of Π_B, in terms of small perturbations in B. In Sect. 7 we prove and then apply these results to Hodge–Dirac operators on compact Riemannian manifolds. This enables us to establish Theorem 7.1, which gives Lipschitz estimates for members of the functional calculus (including spectral projections) of the Hodge–Dirac operator on compact manifolds in terms of L_∞ changes in the metric. In Appendix A, we show that, under hypotheses (H1–3), the Hodge projections also depend holomorphically on perturbations in B, and calculate the derivatives of these projections.

We conclude this section with a brief outline of the idea behind the proofs of Theorems 2.7 and 2.10. The results in Sect. 4 just depend on hypotheses (H1–3). We prove Propositions 2.2 and 2.5, and show how to reduce Theorems 2.7 and 2.10 to a particular quadratic estimate (19). In Sect. 5 we prove this estimate under all the hypotheses (H1–8). This can be considered as a type of "$T(b)$ argument". In Sect. 5.2, we separate out the principal part γ_t of the operator appearing as the integrand in the desired quadratic estimate (19). This localization procedure relies on Propositions 2.2 and 2.5, the off–diagonal estimates established in Proposition 5.2, and the local Poincaré inequality together with the global coercivity condition (H8). We estimate the principal part γ_t of the operator in Sect. 5.3. To do this we show that

$d\mu(x,t) = |\gamma_t(x)|^2 \frac{dxdt}{t}$ is a Carleson measure, and then apply Carleson's Theorem for Carleson measures. This provides the desired result.

3. Consequences

For Consequences 3.2–3.10 we employ the following special case of Theorem 2.10.

- Let $\mathbf{C}^N = V_1 \oplus V_2$, where V_1 and V_2 are finite dimensional complex Hilbert spaces, and form the orthogonal direct sum $L_2(\mathbf{R}^n;\mathbf{C}^N) = L_2(\mathbf{R}^n;V_1) \oplus L_2(\mathbf{R}^n;V_2)$.
- Let D and D^* be adjoint homogeneous first order partial differential operators with constant coefficients

$$D : L_2\big(\mathbf{R}^n;V_1\big) \longrightarrow L_2\big(\mathbf{R}^n;V_2\big),$$
$$D^* : L_2\big(\mathbf{R}^n;V_2\big) \longrightarrow L_2\big(\mathbf{R}^n;V_1\big),$$

  such that there exists $c > 0$ so that

$$\|\nabla u\| \leq c\|Du\| \quad \text{for all} \quad u \in \mathsf{R}(D^*) \cap \mathsf{D}(D),$$
$$\|\nabla u\| \leq c\|D^*u\| \quad \text{for all} \quad u \in \mathsf{R}(D) \cap \mathsf{D}(D^*).$$

- The operators $A_i : L_2(\mathbf{R}^n;V_i) \longrightarrow L_2(\mathbf{R}^n;V_i)$, $i = 1,2$, denote multiplication by functions $A_i \in L_\infty(\mathbf{R}^n;\mathcal{L}(V_i))$ which satisfy the accretivity conditions

$$\mathrm{Re}(A_1 D^*u, D^*u) \geq \kappa_1\|D^*u\|^2 \quad \text{for all} \quad u \in \mathsf{D}(D^*),$$
$$\mathrm{Re}(A_2 Du, Du) \geq \kappa_2\|Du\|^2 \quad \text{for all} \quad u \in \mathsf{D}(D),$$

  for some $\kappa_1, \kappa_2 > 0$. Denote the angles of accretivity by

$$\omega_1 := \sup_{u \in \mathsf{D}(D^*)\setminus\mathsf{N}(D^*)} |\arg(A_1 D^*u, D^*u)| < \tfrac{\pi}{2},$$
$$\omega_2 := \sup_{u \in \mathsf{D}(D)\setminus\mathsf{N}(D)} |\arg(A_2 Du, Du)| < \tfrac{\pi}{2}.$$

In the full space $L_2(\mathbf{R}^n;\mathbf{C}^N) = L_2(\mathbf{R}^n;V_1) \oplus L_2(\mathbf{R}^n;V_2)$, consider the following operators:

$$\Gamma := \begin{bmatrix} 0 & 0 \\ D & 0 \end{bmatrix}, \quad \Gamma^* := \begin{bmatrix} 0 & D^* \\ 0 & 0 \end{bmatrix}, \quad B_1 := \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_2 := \begin{bmatrix} 0 & 0 \\ 0 & A_2 \end{bmatrix}.$$

With this choice of $\{\Gamma, B_1, B_2\}$, the operator Π_B and its square become

$$\Pi_B = \begin{bmatrix} 0 & A_1 D^* A_2 \\ D & 0 \end{bmatrix} \quad \text{and} \quad {\Pi_B}^2 = \begin{bmatrix} A_1 D^* A_2 D & 0 \\ 0 & D A_1 D^* A_2 \end{bmatrix}.$$

The operators Γ, Γ^* and Γ_B^* are clearly nilpotent, with

$$\overline{\mathsf{R}(\Gamma)} \subset L_2\big(\mathbf{R}^n;V_2\big) \subset \mathsf{N}(\Gamma) \quad \text{and}$$
$$\overline{\mathsf{R}(\Gamma^*)},\ \overline{\mathsf{R}(\Gamma_B^*)} \subset L_2\big(\mathbf{R}^n;V_1\big) \subset \mathsf{N}(\Gamma^*),\ \mathsf{N}(\Gamma_B^*).$$

Theorem 3.1. *Assume that* $\{D, A_1, A_2\}$ *have the properties listed above, and suppose* $\omega_1 + \omega_2 < 2\mu < \pi$*. Then the operator* Π_B *has a bounded* S^o_μ *holomorphic functional calculus in* $L_2(\mathbf{R}^n; \mathbf{C}^N)$*. Moreover*

(i) *the operator* $A_1 D^* A_2 D$ *has a bounded* $S^o_{2\mu+}$ *holomorphic functional calculus in* $L_2(\mathbf{R}^n; V_1)$*; and*
(ii) *we have* $\mathsf{D}((A_1 D^* A_2 D)^{1/2}) = \mathsf{D}(D)$ *with the Kato square root estimate*

$$\left\|(A_1 D^* A_2 D)^{1/2} u\right\| \approx \|Du\|$$

for all $u \in \mathsf{D}(D)$*.*
(iii) *If furthermore* $V_1 = V_2 =: V$*,* $D^* = -D$*,* $A_1 = A_2 =: A$ *and* $\omega_1 = \omega_2 = \omega < \mu < \frac{\pi}{2}$*, then* iAD *and* iDA *have bounded* S^o_μ *holomorphic functional calculi in* $L_2(\mathbf{R}^n; V)$*. In particular* $\|\mathrm{sgn}(iAD)\| < \infty$ *and* $\|\mathrm{sgn}(iDA)\| < \infty$*.*

Proof. The hypothesis of Theorem 2.7 for this Π_B is satisfied, and thus by Theorem 2.10, Π_B has a bounded S^o_μ holomorphic functional calculus.

To prove (i), let $F : S^o_{2\mu+} \cup \{0\} \longrightarrow \mathbf{C}$ be bounded and holomorphic on $S^o_{2\mu+}$, and write $f(z) := F(z^2)$, $z \in S^o_\mu \cup \{0\}$. Then

$$f(\Pi_B) = \begin{bmatrix} F(A_1 D^* A_2 D) & 0 \\ 0 & F(D A_1 D^* A_2) \end{bmatrix}$$

satisfies $\|f(\Pi_B)\| \lesssim \|f\|_\infty = \|F\|_\infty$, and thus $\|F(A_1 D^* A_2 D)\| \lesssim \|F\|_\infty$.

The Kato square root estimate in (ii) follows on applying Corollary 2.11 to $u \in \mathsf{D}(D)$.

Now make the additional assumptions stated in (iii). That iDA has a bounded S^o_μ holomorphic functional calculus in $L_2(\mathbf{R}^n; V)$ can be seen as follows. Consider a bounded function $f : S^o_\mu \cup \{0\} \longrightarrow \mathbf{C}$ holomorphic on S^o_μ. We find for $u \in L_2(\mathbf{R}^n; V)$, that

$$f(\Pi_B) \begin{bmatrix} iAu \\ u \end{bmatrix} = f\left(\begin{bmatrix} 0 & -ADA \\ D & 0 \end{bmatrix}\right) \begin{bmatrix} iAu \\ u \end{bmatrix} = \begin{bmatrix} iA(f(iDA)u) \\ f(iDA)u \end{bmatrix}.$$

Thus

$$\|f(iDA)\| \lesssim \|f(\Pi_B)\| \lesssim \|f\|_\infty .$$

Duality shows that $iAD = (iDA^*)^*$ also has a bounded S^o_μ holomorphic functional calculus in $L_2(\mathbf{R}^n; V)$, which completes the proof of (iii). □

Part (iii) can also be deduced from the quadratic estimates in Theorem 2.7, for they imply that $-ADAD$ and $-DADA$, and hence iAD and iDA, satisfy quadratic estimates.

We now consider several consequences of the above theorem.

Consequence 3.2 (The Cauchy singular integral on Lipschitz curves). Let $g : \mathbf{R} \longrightarrow \mathbf{R}$ be a Lipschitz function with Lipschitz constant

$$L := \sup_{x \neq y} \frac{|g(x) - g(y)|}{|x - y|}$$

and consider the Lipschitz graph $\gamma := \{z = x + ig(x) : x \in \mathbf{R}\}$ in $\mathbf{C}$. The operator of differentiation with respect to $z \in \gamma$ can be expressed in terms of the parameter $x \in \mathbf{R}$ as

$$D_\gamma u(x) := aDu(x) = (1 + ig'(x))^{-1} u'(x)$$

where a is the multiplication operator $a : v(x) \mapsto (1 + ig'(x))^{-1} v(x)$. Thus iD_γ is of the form considered in Theorem 3.1(iii) on making the identifications

$$\{n, V_1, V_2, D, D^*, A_1, A_2\} = \left\{1, \mathbf{C}, \mathbf{C}, \tfrac{d}{dx}, -\tfrac{d}{dx}, a, a\right\}.$$

The Cauchy singular integral operator C_γ on γ is then given as an operator on $L_2(\mathbf{R}, \mathbf{C})$ by (see [28, 1])

$$\begin{aligned} C_\gamma u(x) &:= \operatorname{sgn}(iD_\gamma) u(x) \\ &= \frac{i}{\pi} \text{p.v.} \int_{\mathbf{R}} \frac{u(y)}{(y + ig(y)) - (x + ig(x))} (1 + ig'(y)) dy. \end{aligned}$$

Using Theorem 3.1(iii) we deduce that iD_γ has a bounded S^o_μ holomorphic functional calculus in $L_2(\mathbf{R}; \mathbf{C})$ when $\arctan(L) < \mu < \frac{\pi}{2}$. In particular $\|C_\gamma\| < \infty$. The boundedness of the Cauchy integral C_γ was first proved for small L by Calderón [8], and in the general case by Coifman–McIntosh–Meyer [10]. Boundedness of other operators in the functional calculus of iD_γ have been proved by Coifman–Meyer [11], Kenig–Meyer [20] and McIntosh–Qian [28].

Consequence 3.3 (The one dimensional Kato square root problem). Let $a \in L_\infty(\mathbf{R}; \mathbf{C})$ be such that $\operatorname{Re} a(x) \geq \kappa > 0$ for almost every x, and denote the angle of accretivity by $\omega := \operatorname{ess\,sup} |\arg a(x)|$. In Theorem 3.1, let

$$\{n, V_1, V_2, D, D^*, A_1, A_2\} = \left\{1, \mathbf{C}, \mathbf{C}, \tfrac{d}{dx}, -\tfrac{d}{dx}, I, a\right\}$$

where a is the multiplication operator $a : f(x) \mapsto a(x) f(x)$, and suppose $\omega < \mu < \frac{\pi}{2}$. By Theorem 3.1(i) we deduce that $-\frac{d}{dx} a \frac{d}{dx}$ has a bounded $S^o_{2\mu+}$ holomorphic functional calculus in $L_2(\mathbf{R}; \mathbf{C})$. This can be proved by abstract methods since $-\frac{d}{dx} a \frac{d}{dx}$ is a maximal accretive operator, see [1]. However, Theorem 3.1(ii) proves the Kato square root estimate in one dimension:

$$\left\| \left(-\tfrac{d}{dx} a \tfrac{d}{dx} \right)^{1/2} u \right\| \approx \left\| \tfrac{du}{dx} \right\| \tag{11}$$

for all $u \in H^1(\mathbf{R})$. This estimate was first proved by Coifman–McIntosh–Meyer [10].

Remark 3.4. It is known that (11) may fail if D and A_2 are not differentiation and multiplication operators [23]. Working backwards, we find that hypotheses (H1–3) are not sufficient to ensure that Π_B satisfies quadratic estimates or that it has a bounded holomorphic functional calculus.

Consequence 3.5. Let $a_i \in L_\infty(\mathbf{R};\mathbf{C})$, for $i = 1, 2$, be such that there exists $\kappa > 0$ so that $\operatorname{Re} a_i(x) \geq \kappa > 0$ for almost every x, and denote the angles of accretivity by $\omega_i := \operatorname{ess\,sup}|\arg a_i(x)|$. In Theorem 3.1, let

$$\{n, V_1, V_2, D, D^*, A_1, A_2\} = \left\{1, \mathbf{C}, \mathbf{C}, \tfrac{d}{dx}, -\tfrac{d}{dx}, a_1, a_2\right\}$$

where a_i is the multiplication operator $a_i : f(x) \mapsto a_i(x) f(x)$, and suppose $\omega_1 + \omega_2 < 2\mu < \pi$. By Theorem 3.1(i) we deduce that $-a_1 \frac{d}{dx} a_2 \frac{d}{dx}$ has a bounded $S^o_{2\mu+}$ holomorphic functional calculus in $L_2(\mathbf{R};\mathbf{C})$. This result was first proved by Auscher–McIntosh–Nahmod [6] (though with $\mu > \max\{\omega_1, \omega_2\}$). Further Theorem 3.1(ii) proves the estimate

$$\left\|\left(-a_1 \tfrac{d}{dx} a_2 \tfrac{d}{dx}\right)^{1/2} u\right\| \approx \left\|\tfrac{du}{dx}\right\|$$

for all $u \in H^1(\mathbf{R})$. This estimate was first proved by Kenig–Meyer [20]. A proof is also given in [6], using a framework which can be considered a forerunner of the approach developed here.

Consequence 3.6 (The Clifford–Cauchy singular integral on a Lipschitz surface). Let $g : \mathbf{R}^n \longrightarrow \mathbf{R}$ be a Lipschitz function with Lipschitz constant L, and consider the Lipschitz graph $\Sigma := \{(x, g(x)) : x \in \mathbf{R}^n\}$ in $\mathbf{R}^{n+1}$. On identifying $\mathbf{R}^{n+1}$ with $\Lambda^0 \oplus \Lambda^1$ in the complex Clifford algebra $\mathbf{C}_{(n)} (\approx \wedge_{\mathbf{C}} \mathbf{R}^n)$ generated by $\mathbf{R}^n$, where the generating basis $\{e_i\}$ satisfies the canonical commutation relation $e_i e_j + e_j e_i = -2\delta_{ij}$, then $\Sigma = \{g(x) + x : x \in \mathbf{R}^n\}$. Furthermore, let $\mathbf{D}$ denote the Dirac operator

$$\mathbf{D}u(x) := \sum_{k=1}^{n} e_k \frac{\partial u}{\partial x_k}(x), \qquad u : \mathbf{R}^n \longrightarrow \mathbf{C}_{(n)}.$$

This first order partial differential operator $\mathbf{D}$ is elliptic and selfadjoint. In Theorem 3.1, let

$$\{n, V_1, V_2, D, D^*, A_1, A_2\} = \left\{n, \mathbf{C}_{(n)}, \mathbf{C}_{(n)}, -i\mathbf{D}, i\mathbf{D}, A, A\right\}$$

where A is the multiplication operator $A : u(x) \mapsto (1 - \mathbf{D}g(x))^{-1} u(x)$. In this case, we define the operator $\mathbf{D}_\Sigma$ on $L_2(\mathbf{R}^n, \mathbf{C}_{(n)})$ by

$$\mathbf{D}_\Sigma u(x) := A\mathbf{D}u(x) = (1 - \mathbf{D}g(x))^{-1} \mathbf{D}u(x)$$

and, parametrizing Σ with $g(x) + x$, the Cauchy singular integral operator C_Σ on Σ is given by

$$\begin{aligned} C_\Sigma u(x) &:= \operatorname{sgn}(\mathbf{D}_\Sigma) u(x) \\ &= \frac{2}{\sigma_n} \text{p.v.} \int_{\mathbf{R}^n} \frac{(g(x) - x) - (g(y) - y)}{(|y - x|^2 + (g(y) - g(x))^2)^{(n+1)/2}} (1 - \mathbf{D}g(y)) u(y)\, dy \end{aligned}$$

where σ_n is the volume of the unit n-sphere in $\mathbf{R}^{n+1}$.

Suppose $\omega := \arctan(L) < \mu < \frac{\pi}{2}$. By Theorem 3.1(iii) we deduce that $\mathbf{D}_\Sigma$ has a bounded S^o_μ holomorphic functional calculus in $L_2(\mathbf{R}^n; \mathbf{C}_{(n)})$, and in particular that $\|C_\Sigma\| < \infty$. The boundedness of the Clifford–Cauchy integral C_Σ follows from the boundedness of the Cauchy integral in Consequence 3.2 using Calderón's rotation method (c.f. [10]). A direct proof of the boundedness of C_Σ using Clifford analysis was first given by Murray [30] for surfaces with small L, and in the general case by McIntosh [26]. Boundedness of the functional calculus of $\mathbf{D}_\Sigma$ has been proved by Li–McIntosh–Semmes [22] and Li–McIntosh–Qian [21].

In the following three consequences, the differential operator D no longer has dense range.

Consequence 3.7 (The Kato square root problem). Let $A \in L_\infty(\mathbf{R}^n; \mathcal{L}(\mathbf{C}^n))$ be such that $\mathrm{Re}(A(x)v, v) \geq \kappa > 0$ for every $v \in \mathbf{C}^n$ with $|v| = 1$, and almost every x, and denote the angle of accretivity by $\omega := \operatorname{ess\,sup}_{v,x} |\arg(A(x)v, v)|$. In Theorem 3.1, let

$$\{n, V_1, V_2, D, D^*, A_1, A_2\} = \{n, \mathbf{C}, \mathbf{C}^n, \nabla, -\mathrm{div}, I, A\}$$

where A denotes the multiplication operator $A : u \mapsto Au$, and suppose $\omega < \mu < \frac{\pi}{2}$. From Theorem 3.1(i) we deduce that $-\mathrm{div} A\nabla$ has a bounded $S^o_{\mu+}$ holomorphic functional calculus in $L_2(\mathbf{R}^n; \mathbf{C})$. This can be proved by abstract methods since $-\mathrm{div} A\nabla$ is a maximal accretive operator, see [1]. More importantly, Theorem 3.1(ii) implies the full Kato square root estimate

$$\left\|(-\mathrm{div} A\nabla)^{1/2} u\right\| \approx \|\nabla u\|$$

for all $u \in H^1(\mathbf{R}^n)$. This result was proved in a series of papers by Hofmann–McIntosh [18], Auscher–Hofmann–Lewis–Tchamitchian [3], Hofmann–Lacey–McIntosh [17], and, in full generality, by Auscher–Hofmann–Lacey–McIntosh–Tchamitchian [2]. Earlier results on the Kato square root problem are due to Fabes–Jerison–Kenig [15] and Coifman–Deng–Meyer [9], where A is assumed to be close to the identity, and to McIntosh [24] when Hölder continuity of A is assumed. For many more partial results, see the book of Auscher and Tchamitchian [7]. This book provides an important bridge between the one–dimensional results and the current theory.

Consequence 3.8. Let $a \in L_\infty(\mathbf{R}^n; \mathbf{C})$ be such that $\mathrm{Re}\, a(x) \geq \kappa > 0$ for almost every x, and let $A \in L_\infty(\mathbf{R}^n; \mathcal{L}(\mathbf{C}^n))$ be such that $\mathrm{Re}(A(x)v, v) \geq \kappa > 0$ for every $v \in \mathbf{C}^n$, $|v| = 1$, and almost every x. Denote the angles of accretivity by $\omega_1 := \operatorname{ess\,sup} |\arg a(x)|$ and $\omega_2 := \operatorname{ess\,sup}_{v,x} |\arg(A(x)v, v)|$. In Theorem 3.1, let

$$\{n, V_1, V_2, D, D^*, A_1, A_2\} = \{n, \mathbf{C}, \mathbf{C}^n, \nabla, -\mathrm{div}, a, A\}$$

where a is the multiplication operator $a : u(x) \mapsto a(x)u(x)$ and A is the multiplication operator $A : v(x) \mapsto A(x)v(x)$. From Theorem 3.1(i) we deduce that $-a\, \mathrm{div} A\nabla$ has a bounded $S^o_{2\mu+}$ holomorphic functional calculus in $L_2(\mathbf{R}^n; \mathbf{C})$ when $\omega_1 + \omega_2 < 2\mu < \pi$. This was proved by McIntosh–Nahmod [27] in the case when $A = I$, and by Duong–Ouhabaz [14] under regularity assumptions on A. Theorem 3.1(ii) also proves the estimate

$$\left\|(-a\,\mathrm{div} A\nabla)^{1/2}u\right\| \approx \|\nabla u\|$$

for all $u \in H^1(\mathbf{R}^n)$.

Consequence 3.9 (The Kato square root problem for systems). Let W be a finite dimensional Hilbert space and let $A \in L_\infty(\mathbf{R}^n; \mathcal{L}(\mathbf{C}^n \otimes W))$ be such that

$$\mathrm{Re} \int_{\mathbf{R}^n} (A(x)\nabla u(x), \nabla u(x))\, dx \geq \kappa \|\nabla u\|^2$$

for all $u \in H^1(\mathbf{R}^n; W)$ and some $\kappa > 0$. In Theorem 3.1, let

$$\{n, V_1, V_2, D, D^*, A_1, A_2\} = \{n, W, \mathbf{C}^n \otimes W, \nabla, -\mathrm{div}, I, A\}$$

where A is the multiplication operator $A : f(x) \mapsto A(x) f(x)$. Theorem 3.1(ii) proves the Kato square root estimate for these elliptic systems:

$$\left\|(-\mathrm{div} A\nabla)^{1/2}u\right\| \approx \|\nabla u\|$$

for all $u \in H^1(\mathbf{R}^n; W)$. This estimate was first proved by Auscher–Hofmann–McIntosh–Tchamitchian [4].

Consequence 3.10 (Differential forms). For $n \geq 1$, let $\Lambda = \oplus_{i=0}^n \Lambda^i = \wedge_{\mathbf{C}} \mathbf{R}^n$ denote the complex exterior algebra over $\mathbf{R}^n$. Let B be a bounded multiplication operator on $L_2(\mathbf{R}^n; \Lambda)$ with bounded inverse which satisfies the following accretivity condition: there exists $\kappa > 0$ such that for almost every $x \in \mathbf{R}^n$, we have

$$\mathrm{Re}(B(x)v, v) \geq \kappa |v|^2$$

for every $v \in \Lambda$. Let d denote the exterior derivative, and consider the perturbed Hodge–Dirac operator $D_B = d + B^{-1} d^* B$. We further suppose that B splits over $L_2(\mathbf{R}^n, \Lambda^0) \oplus \cdots \oplus L_2(\mathbf{R}^n, \Lambda^n)$ as $B^0 \oplus \cdots \oplus B^n$, and so D_B can be illustrated by the following diagram.

$$\begin{array}{ccccccc}
L_2(\mathbf{R}^n, \Lambda^0) & \overset{d=\nabla}{\longrightarrow} & L_2(\mathbf{R}^n, \Lambda^1) & \overset{d}{\longrightarrow} & \cdots & \overset{d}{\longrightarrow} & L_2(\mathbf{R}^n, \Lambda^n) \\
\big\downarrow B^0 & & \big\downarrow B^1 & & & & \big\downarrow B^n \\
L_2(\mathbf{R}^n, \Lambda^0) & \overset{d^*=-\mathrm{div}}{\longleftarrow} & L_2(\mathbf{R}^n, \Lambda^1) & \overset{d^*}{\longleftarrow} & \cdots & \overset{d^*}{\longleftarrow} & L_2(\mathbf{R}^n, \Lambda^n)
\end{array}$$

Let $\omega > 0$ denote the angle of accretivity of B, and let $\omega < \mu < \frac{\pi}{2}$. We now apply Theorem 2.10 and Corollary 2.11 with $\Gamma = d$, $B_1 := B^{-1}$ and $B_2 := B$, to obtain the following new result.

Theorem 3.11. *The operator D_B has a bounded S^o_μ holomorphic functional calculus in $L_2(\mathbf{R}^n, \Lambda)$. Moreover, the operator ${D_B}^2$ has a bounded $S^o_{2\mu+}$ holomorphic functional calculus in $L_2(\mathbf{R}^n, \Lambda)$. Furthermore, $\mathsf{D}(d) \cap \mathsf{D}(d^* B) = \mathsf{D}(\sqrt{{D_B}^2})$ with*

$$\|du\| + \|d^* Bu\| \approx \left\| \sqrt{{D_B}^2} u \right\|.$$

The restriction of the second and third claims to $u \in L_2(\mathbf{R}^n, \Lambda^0)$ provides an alternative approach to the results obtained in Consequences 3.7 and 3.8, though not those of Consequence 3.9. The implications for the full exterior algebra are new and will be developed further in Remark 7.4.

4. Operator theory of Π_B

Throughout this section, we assume that the triple of operators $\{\Gamma, B_1, B_2\}$ in a Hilbert space $\mathcal{H}$ satisfies properties (H1–3). We prove Propositions 2.2 and 2.5, and then show how to reduce Theorems 2.7 and 2.10 to a quadratic estimate which will be proved in Sect. 5.

Let us start by recording the following useful consequences of (H2):

$$\|B_1 u\| \approx \|u\| \approx \|B_1^* u\| \quad \text{for all} \quad u \in \overline{\mathsf{R}(\Gamma^*)}; \tag{12}$$

$$\|B_2 u\| \approx \|u\| \approx \|B_2^* u\| \quad \text{for all} \quad u \in \overline{\mathsf{R}(\Gamma)}. \tag{13}$$

Lemma 4.1. *The operators $\Gamma_B^* := B_1 \Gamma^* B_2$ and $\Gamma_B := B_2^* \Gamma B_1^*$ are nilpotent, and $(\Gamma_B)^* = \Gamma_B^*$.*

Proof. First note that by (H3), $\mathsf{R}(\Gamma_B^*) \subset \mathsf{N}(\Gamma_B^*)$ and $\mathsf{R}(\Gamma_B) \subset \mathsf{N}(\Gamma_B)$. To prove that the two operators are densely defined, closed and adjoint, we use the following operator theoretic fact: Let A be a closed and densely defined operator and let T be a bounded operator. Then TA is densely defined, $A^* T^*$ is closed and $(TA)^* = A^* T^*$. If furthermore $\|Tu\| \approx \|u\|$ for all $u \in \mathsf{R}(A)$, then TA and $A^* T^*$ are closed, densely defined and adjoint operators. Applying this fact first with $A = \Gamma^*$, $T = B_1$ and then with $A = \Gamma B_1^*$, $T = B_2^*$ proves the lemma. □

We next prove a lemma concerning the operators $\Pi_B := \Gamma + \Gamma_B^*$ with $\mathsf{D}(\Pi_B) = \mathsf{D}(\Gamma) \cap \mathsf{D}(\Gamma_B^*)$, and $\Pi_B^* := \Gamma^* + \Gamma_B$ with $\mathsf{D}(\Pi_B^*) = \mathsf{D}(\Gamma^*) \cap \mathsf{D}(\Gamma_B)$.

Lemma 4.2. *We have*

$$\|\Gamma u\| + \|\Gamma_B^* u\| \approx \|\Pi_B u\| \quad \textit{for all} \quad u \in \mathsf{D}(\Pi_B), \quad \textit{and}$$

$$\|\Gamma^* u\| + \|\Gamma_B u\| \approx \|\Pi_B^* u\| \quad \textit{for all} \quad u \in \mathsf{D}(\Pi_B^*).$$

Proof. The first estimate follows from the observation that (H2–3) implies

$$\|\Gamma u\|^2 \lesssim |(B_2\Gamma u, \Gamma u)| = |(B_2\Pi_B u, \Gamma u)| \lesssim \|\Pi_B u\|\,\|\Gamma u\|$$

for every $u \in \mathsf{D}(\Pi_B)$. The other claims follow by similar reasoning. □

We now prove Proposition 2.2 and then Proposition 2.5.

Proof of Proposition 2.2. It is an immediate consequence of the lemma that $\mathsf{N}(\Pi_B) = \mathsf{N}(\Gamma_B^*) \cap \mathsf{N}(\Gamma)$.

Note that once we prove

$$\mathcal{H} = \overline{\mathsf{R}(\Gamma_B^*)} \oplus \mathsf{N}(\Gamma) = \mathsf{N}(\Gamma_B^*) \oplus \overline{\mathsf{R}(\Gamma)} \tag{14}$$

then the Hodge decomposition follows since $\overline{\mathsf{R}(\Gamma_B^*)} \subset \mathsf{N}(\Gamma_B^*)$ and $\overline{\mathsf{R}(\Gamma)} \subset \mathsf{N}(\Gamma)$ by nilpotence. In the case $B_1 = B_2 = I$, (14) is orthogonal since Γ and Γ^* are adjoint operators. To prove (14) for a general B, it suffices to prove the four statements

$$\mathcal{H} \supset \overline{\mathsf{R}(\Gamma_B^*)} \oplus \mathsf{N}(\Gamma), \quad \mathcal{H} \supset \mathsf{N}(\Gamma_B^*) \oplus \overline{\mathsf{R}(\Gamma)},$$
$$\mathcal{H} \supset \overline{\mathsf{R}(\Gamma^*)} \oplus \mathsf{N}(\Gamma_B), \quad \mathcal{H} \supset \mathsf{N}(\Gamma^*) \oplus \overline{\mathsf{R}(\Gamma_B)},$$

and use duality.

Let us consider the first of these. We need to show that

$$\|\Gamma_B^* u\| + \|v\| \lesssim \|\Gamma_B^* u + v\|$$

for all $u \in \mathsf{D}(\Gamma_B^*) = \mathsf{D}(\Gamma^* B_2)$ and $v \in \mathsf{N}(\Gamma)$. This follows from

$$\begin{aligned}\|\Gamma^* B_2 u\|^2 \lesssim |\operatorname{Re}(B_1\Gamma^* B_2 u, \Gamma^* B_2 u)| &= |\operatorname{Re}(\Gamma_B^* u + v, \Gamma^* B_2 u)| \\ &\leq \|\Gamma_B^* u + v\|\|\Gamma^* B_2 u\|.\end{aligned}$$

For the second statement we need to show that

$$\|v\| + \|\Gamma u\| \lesssim \|v + \Gamma u\|$$

for all $u \in \mathsf{D}(\Gamma)$ and $v \in \mathsf{N}(\Gamma_B^*) = \mathsf{N}(\Gamma^* B_2)$. This follows from

$$\|\Gamma u\|^2 \lesssim \left|\left(\Gamma u, B_2^*\Gamma u\right)\right| = \left|\left(v + \Gamma u, B_2^*\Gamma u\right)\right| \lesssim \|v + \Gamma u\|\|\Gamma u\|.$$

The third and fourth statements have similar proofs. □

Corollary 4.3. *The operators* Π_B *and* Π_B^* *are closed, have dense domains, and satisfy* $(\Pi_B)^* = \Pi_B^*$.

This is a straightforward consequence of the preceding results. We are now in a position to prove the spectral properties stated in Sect. 2.

Proof of Proposition 2.5. Let $f = (\mathrm{I} + \tau\Pi_B)u$ where $\tau \in \mathbf{C} \setminus S_\omega$ and $u \in \mathsf{D}(\Pi_B)$. To prove the estimate $\|u\| \lesssim \|f\|$, use Proposition 2.2 to write

$$f = f_0 + f_1 + f_2,\ u = u_0 + u_1 + u_2 \in \mathsf{N}(\Pi_B) \oplus \overline{\mathsf{R}(\Gamma_B^*)} \oplus \overline{\mathsf{R}(\Gamma)}$$

and $f_1 = B_1\tilde{f}_1$, $u_1 = B_1\tilde{u}_1$, where $\tilde{f}_1, \tilde{u}_1 \in \overline{\mathsf{R}(\Gamma^*)}$. We obtain the system of equations

$$\begin{aligned} f_0 &= u_0 \\ f_1 &= u_1 + \tau\Gamma_B^* u_2, \ \text{ thus by (12), } \tilde{f}_1 = \tilde{u}_1 + \tau\Gamma^* B_2 u_2 \\ f_2 &= u_2 + \tau\Gamma u_1. \end{aligned}$$

These equations imply the identity

$$-\overline{\tau}(\tilde{u}_1, B_1\tilde{u}_1) + \tau(B_2u_2, u_2) = -\overline{\tau}(\tilde{f}_1, B_1\tilde{u}_1) + \tau(B_2u_2, f_2). \tag{15}$$

Let

$$\theta_1 = \arg(\tilde{u}_1, B_1\tilde{u}_1), \quad \text{and} \quad \theta_2 = \arg(B_2u_2, u_2)$$

so that by (H2), $|\frac{1}{2}\theta_1 - \frac{1}{2}\theta_2| \le \omega$. Suppose for a moment that $\mathrm{Im}\,\tau > 0$ and let $\mu = \arg\tau$. Then

$$\begin{aligned} &|-\overline{\tau}(\tilde{u}_1, B_1\tilde{u}_1) + \tau(B_2u_2, u_2)| \\ &\quad\ge \mathrm{Im}\, e^{-i(\theta_1+\theta_2)/2}\,(-\overline{\tau}(\tilde{u}_1, B_1\tilde{u}_1) + \tau(B_2u_2, u_2)) \\ &\quad= |\tau| \sin\big(-\tfrac{1}{2}\theta_1 + \tfrac{1}{2}\theta_2 + \mu\big)\,(|(\tilde{u}_1, B_1\tilde{u}_1)| + |(B_2u_2, u_2)|) \\ &\quad\ge \mathrm{dist}\,(\tau, S_\omega)\,(|(\tilde{u}_1, B_1\tilde{u}_1)| + |(B_2u_2, u_2)|). \end{aligned} \tag{16}$$

Therefore, by (H2), (15) and (16),

$$\begin{aligned} \|\tilde{u}_1\|^2 + \|u_2\|^2 &\lesssim |(\tilde{u}_1, B_1\tilde{u}_1)| + |(B_2u_2, u_2)| \\ &\lesssim \frac{|\tau|}{\mathrm{dist}\,(\tau, S_\omega)}(\|\tilde{f}_1\|\|\tilde{u}_1\| + \|u_2\|\|f_2\|) \end{aligned}$$

and thus

$$\|u\| \approx \|u_0\| + \|u_1\| + \|u_2\| \lesssim \frac{|\tau|}{\mathrm{dist}\,(\tau, S_\omega)}\|f\|.$$

A slight variation gives the estimate for $\mathrm{Im}\,\tau < 0$.

Finally, applying the proof above to $\mathrm{I} + \overline{\tau}\Pi_B^* = (\mathrm{I} + \tau\Pi_B)^*$ shows that $\mathrm{I} + \tau\Pi_B$ is surjective. □

Corollary 4.4. *The operator* ${\Pi_B}^2 = \Gamma B_1\Gamma^* B_2 + B_1\Gamma^* B_2\Gamma$ *is closed, has dense domain, its spectrum* $\sigma({\Pi_B}^2)$ *is contained in the sector* $S_{2\omega+}$, *and it satisfies resolvent bounds* $\|(\mathrm{I} - \tau^2{\Pi_B}^2)^{-1}\| \lesssim \frac{|\tau^2|}{\mathrm{dist}\,(\tau^2, S_{2\omega+})}$ *for all* $\tau \in \mathbf{C} \setminus S_\omega$.

Such an operator is said to be of type $S_{2\omega+}$ in [1] and of type 2ω in [5, 12].

Remark 4.5. Note that Π_B intertwines Γ and Γ_B^* in the sense that $\Pi_B\Gamma u = \Gamma_B^*\Pi_B u$ for all $u \in \mathsf{D}(\Gamma_B^*\Pi_B)$ and $\Pi_B\Gamma^* u = \Gamma\Pi_B u$ for all $u \in \mathsf{D}(\Gamma\Pi_B)$. Thus ${\Pi_B}^2$ commutes with both Γ and Γ_B^* on the appropriate domains. We find that $\Gamma P_t^B u = P_t^B\Gamma u$ for all $u \in \mathsf{D}(\Gamma)$ and $\Gamma_B^* P_t^B u = P_t^B\Gamma_B^* u$ for all $u \in \mathsf{D}(\Gamma_B^*)$.

We saw in Definition 2.6 that the operators P_t^B and Q_t^B are uniformly bounded in t. A consequence of this is the identity

$$\begin{aligned}(17)\qquad \int_0^\infty \big(Q_t^B\big)^2 u\,\frac{dt}{t} &= \lim_{\substack{\alpha\to 0\\ \beta\to\infty}} \int_\alpha^\beta \big(Q_t^B\big)^2 u\,\frac{dt}{t}\\ &= \tfrac{1}{2}\lim_{\substack{\alpha\to 0\\ \beta\to\infty}} \big(P_\alpha^B - P_\beta^B\big)u = \tfrac{1}{2}\big(\mathrm{I} - \mathbf{P}_B^0\big)u\end{aligned}$$

for all $u \in \mathcal{H}$. (Verify this on $\mathsf{N}(\Pi_B)$ and for $u \in \mathsf{D}(\Pi_B)\cap\mathsf{R}(\Pi_B)$ which is dense in $\overline{\mathsf{R}(\Pi_B)}$ and use the uniform boundedness.) For the selfadjoint operator Π this can be proved by the usual spectral theory, and has the following consequence.

Lemma 4.6. *The quadratic estimate*

$$(18)\qquad \int_0^\infty \|Q_t u\|^2\,\frac{dt}{t} \le \tfrac{1}{2}\|u\|^2$$

holds for all $u \in \mathcal{H}$.

We use the following operator in the proof of Theorem 2.7.

Definition 4.7. Define, for all $t \in \mathbf{R}$, the bounded operators

$$\Theta_t^B := t\Gamma_B^*\big(\mathrm{I} + t^2{\Pi_B}^2\big)^{-1}.$$

By Remark 4.5, $\Theta_t^B u = (\mathrm{I} + t^2{\Pi_B}^2)^{-1} t\Gamma_B^* u$ for all $u \in \mathsf{D}(\Gamma_B^*)$, and consequently $\Theta_t^B u = Q_t^B u$ for all $u \in \mathsf{N}(\Gamma)$.

Proposition 4.8. *Consider the operator $\Pi_B = \Gamma + B_1\Gamma^* B_2$ acting in a Hilbert space $\mathcal{H}$, where $\{\Gamma, B_1, B_2\}$ satisfies the hypotheses (H1–3). Also assume that the estimate*

$$(19)\qquad \int_0^\infty \big\|\Theta_t^B P_t u\big\|^2\,\frac{dt}{t} \le c\|u\|^2$$

holds for all $u \in \mathsf{R}(\Gamma)$ and some constant c, together with the three similar estimates obtained on replacing $\{\Gamma, B_1, B_2\}$ by $\{\Gamma^*, B_2, B_1\}$, $\{\Gamma^*, B_2^*, B_1^*\}$ and $\{\Gamma, B_1^*, B_2^*\}$. Then Π_B satisfies the quadratic estimate (6) for all $u \in \overline{\mathsf{R}(\Pi_B)}$, and has a bounded holomorphic S_μ^o functional calculus.

Proof. (i) We start by proving the estimate

$$\int_0^\infty \left\| \Theta_t^B (\mathrm{I} - P_t) u \right\|^2 \frac{dt}{t} \lesssim \|u\|^2 \tag{20}$$

for all $u \in \mathsf{R}(\Gamma)$. We use the orthogonal projection $\mathbf{P}^2 : \mathcal{H} \longrightarrow \overline{\mathsf{R}(\Gamma)}$ and the bounded projection $\mathbf{P}_B^1 : \mathcal{H} \longrightarrow \overline{\mathsf{R}(\Gamma_B^*)}$. Since $u \in \mathsf{R}(\Gamma)$ implies $P_t u \in \mathsf{R}(\Gamma)$, we obtain

$$\Theta_t^B (\mathrm{I} - P_t) u = \Theta_t^B \mathbf{P}^2 (\mathrm{I} - P_t) u = Q_t^B t \Gamma Q_t u = \big(\mathrm{I} - P_t^B\big) \mathbf{P}_B^1 Q_t u$$

and thus $\|\Theta_t^B (\mathrm{I} - P_t) u\| \lesssim \|Q_t u\|$ for all $u \in \mathsf{R}(\Gamma)$. This with (18) proves (20).

We remark that this use of the Hodge decompositions to handle the $(\mathrm{I} - P_t)$ term is a key step in the proof of Theorem 2.7.

(ii) A combination of (19) with (20) gives the estimate

$$\int_0^\infty \left\| Q_t^B u \right\|^2 \frac{dt}{t} = \int_0^\infty \left\| \Theta_t^B u \right\|^2 \frac{dt}{t} \lesssim \|u\|^2 \tag{21}$$

for all $u \in \overline{\mathsf{R}(\Gamma)}$.

Now the hypotheses of the theorem remain unchanged on replacing $\{\Gamma, B_1, B_2\}$ by $\{\Gamma^*, B_2, B_1\}$, in which case the estimate in (21) becomes

$$\int_0^\infty \left\| t B_2 \Gamma B_1 \big(\mathrm{I} + t^2 (\Gamma^* + B_2 \Gamma B_1)^2\big)^{-1} v \right\|^2 \frac{dt}{t} \lesssim \|v\|^2$$

for all $v \in \overline{\mathsf{R}(\Gamma^*)}$. Using the assumption $\Gamma B_1 B_2 \Gamma = 0$, we get

$$\Gamma B_1 \big(\mathrm{I} + t^2 (\Gamma^* + B_2 \Gamma B_1)^2\big)^{-1} = \Gamma \big(\mathrm{I} + t^2 \Pi_B^2\big)^{-1} B_1$$

and thus, by (12) and (13),

$$\begin{aligned} \int_0^\infty \left\| t \Gamma \big(\mathrm{I} + t^2 \Pi_B^2\big)^{-1} B_1 v \right\|^2 \frac{dt}{t} & \\ &\lesssim \int_0^\infty \left\| t B_2 \Gamma B_1 \big(\mathrm{I} + t^2 (\Gamma^* + B_2 \Gamma B_1)^2\big)^{-1} v \right\|^2 \frac{dt}{t} \\ &\lesssim \|v\|^2 \lesssim \|B_1 v\|^2 \end{aligned}$$

for all $v \in \overline{\mathsf{R}(\Gamma^*)}$. Hence

$$\int_0^\infty \left\| Q_t^B u \right\|^2 \frac{dt}{t} = \int_0^\infty \left\| t \Gamma \big(\mathrm{I} + t^2 \Pi_B{}^2\big)^{-1} u \right\|^2 \frac{dt}{t} \lesssim \|u\|^2$$

for all $u \in \overline{\mathsf{R}(\Gamma_B^*)}$.

On recalling the Hodge decompositon $\mathcal{H} = \mathsf{N}(\Pi_B) \oplus \overline{\mathsf{R}(\Gamma_B^*)} \oplus \overline{\mathsf{R}(\Gamma)}$, and noting that $Q_t^B = 0$ on $\mathsf{N}(\Pi_B)$, we conclude that

$$\int_0^\infty \left\| Q_t^B u \right\|^2 \frac{dt}{t} \lesssim \|u\|^2$$

for all $u \in \mathcal{H}$.

(iii) To prove the reverse square function estimate, consider the adjoint operator $\Pi_B^* = \Gamma^* + B_2^* \Gamma B_1^*$. From (ii) applied to Π_B^*, we get

$$\int_0^\infty \left\| \left(Q_t^B\right)^* v \right\|^2 \frac{dt}{t} \lesssim \|v\|^2$$

for all $v \in \mathcal{H}$. By (17), we have the resolution of the identity

$$\int_0^\infty \left(Q_t^B\right)^2 u \,\frac{dt}{t} = \tfrac{1}{2} u$$

for all $u \in \overline{\mathsf{R}(\Pi_B)}$, and thus

$$\begin{aligned}
\|u\| \lesssim \sup_{\|v\|=1} |(u, v)| &\approx \sup_{\|v\|=1} \left| \left(\int_0^\infty \left(Q_t^B\right)^2 u \,\frac{dt}{t}, v \right) \right| \\
&= \sup_{\|v\|=1} \left| \int_0^\infty \left(Q_t^B u, \left(Q_t^B\right)^* v \right) \frac{dt}{t} \right| \\
&\lesssim \left(\int_0^\infty \left\| Q_t^B u \right\|^2 \frac{dt}{t} \right)^{1/2}
\end{aligned}$$

for all $u \in \overline{\mathsf{R}(\Pi_B)}$. This completes the proof that (6) holds for all $u \in \overline{\mathsf{R}(\Pi_B)}$. This procedure is standard, at least when $\mathsf{N}(\Pi_B) = 0$. (See e.g. [1].)

(iv) It is also well-known that quadratic estimates imply the boundedness of the functional calculus. We include a proof for completeness.

Note that a direct norm estimate using (7) shows that

$$\left\| Q_t^B f(\Pi_B) Q_s^B \right\| = \| (\psi_s f \psi_t)(\Pi_B) \| \lesssim \eta(t/s) \sup_{S_\mu^o} |f|$$

for all $t, s > 0$, where $\eta(x) := \min\{x, \frac{1}{x}\}\,(1 + |\log |x||)$. A Schur estimate now gives

$$\begin{aligned}
\| f(\Pi_B) u \|^2 &\approx \int_0^\infty \left\| Q_t^B f(\Pi_B) u \right\|^2 \frac{dt}{t} \\
&\approx \int_0^\infty \left\| \int_0^\infty \left(Q_t^B f(\Pi_B) Q_s^B \right) \left(Q_s^B u \right) \frac{ds}{s} \right\|^2 \frac{dt}{t} \\
&\lesssim \sup_{S_\mu^o} |f|^2 \int_0^\infty \left(\int_0^\infty \eta(t/s) \frac{ds}{s} \right) \left(\int_0^\infty \eta(t/s) \left\| Q_s^B u \right\|^2 \frac{ds}{s} \right) \frac{dt}{t} \\
&\lesssim \sup_{S_\mu^o} |f|^2 \int_0^\infty \left\| Q_s^B u \right\|^2 \frac{ds}{s} \approx \sup_{S_\mu^o} |f|^2 \|u\|^2
\end{aligned}$$

for all $u \in \overline{\mathsf{R}(\Pi_B)}$, which proves that Π_B has a bounded S^o_μ holomorphic functional calculus in $L_2(\mathbf{R}^n; \mathbf{C}^N)$. □

What remains is for us to obtain the estimate (19) under all the hypotheses (H1–8). This is achieved in the next section.

5. Harmonic analysis of Π_B

In this section we prove the square function estimate (19) under the hypotheses (H1–8) stated in Sect. 2. By Proposition 4.8, this then suffices to prove Theorems 2.7 and 2.10. This section is an adaptation of the proof of the Kato square root problem for divergence-form elliptic operators [17, 2,4], though some estimates require new procedures. For example, we develop new methods based on hypotheses (H5–6) to prove off-diagonal estimates for resolvents of Π_B, as the arguments normally used in proving Caccioppoli-type estimates for divergence-form operators do not apply.

We use the following dyadic decomposition of $\mathbf{R}^n$. Let $\triangle = \bigcup_{j=-\infty}^{\infty} \triangle_{2^j}$ where $\triangle_t := \{2^j(k+(0,1]^n) : k \in \mathbf{Z}^n\}$ if $2^{j-1} < t \leq 2^j$. For a dyadic cube $Q \in \triangle_{2^j}$, denote by $l(Q) = 2^j$ its *sidelength*, and by $R_Q := Q \times (0, 2^j]$ the associated *Carleson box*. Let the *dyadic averaging operator* $A_t : \mathcal{H} \longrightarrow \mathcal{H}$ be given by

$$A_t u(x) := u_Q := \fint_Q u(y)\,dy = \frac{1}{|Q|}\int_Q u(y)\,dy$$

for every $x \in \mathbf{R}^n$ and $t > 0$, where $Q \in \triangle_t$ is the unique dyadic cube containing x.

Definition 5.1. By the *principal part* of the operator family Θ_t^B under consideration, we mean the multiplication operators γ_t defined by

$$\gamma_t(x)w := \big(\Theta_t^B w\big)(x)$$

for every $w \in \mathbf{C}^N$. Here we view w on the right-hand side of the above equation as the constant function defined on $\mathbf{R}^n$ by $w(x) := w$. It will be proven in Corollary 5.3 that $\gamma_t \in L_2^{\mathrm{loc}}(\mathbf{R}^n; \mathcal{L}(\mathbf{C}^N))$.

To prove the square function estimate (19), we estimate each of the following three terms separately

$$\begin{aligned} \int_0^\infty \big\|\Theta_t^B P_t u\big\|^2 \frac{dt}{t} &\lesssim \int_0^\infty \big\|\Theta_t^B P_t u - \gamma_t A_t P_t u\big\|^2 \frac{dt}{t} \\ &\quad + \int_0^\infty \|\gamma_t A_t (P_t - \mathrm{I}) u\|^2 \frac{dt}{t} + \int_0^\infty \int_{\mathbf{R}^n} |A_t u(x)|^2 |\gamma_t(x)|^2 \frac{dx dt}{t} \end{aligned} \tag{22}$$

when $u \in \mathsf{R}(\Pi)$. We estimate the first two terms in Sect. 5.2, and the last term in Sect. 5.3. In the next section we introduce crucial off–diagonal estimates for various operators involving Π_B, and also prove local L_2 estimates for γ_t.

5.1. Off–diagonal estimates. We require off–diagonal estimates for the following class of operators. Denote $\langle x\rangle := 1+|x|$, and $\operatorname{dist}(E,F) := \inf\{|x-y| : x\in E, y\in F\}$ for every $E, F\subset \mathbf{R}^n$.

Proposition 5.2. *Let U_t be given by either R_t^B for every nonzero $t\in\mathbf{R}$, or P_t^B, Q_t^B or Θ_t^B for every $t>0$ (see Remark 2.6 and Definition 4.7). Then for every $M\in\mathbf{N}$ there exists $C_M>0$ (that depends only on M and the hypotheses (H1–8)) such that*

$$\|U_t u\|_{L_2(E)} \le C_M \langle \operatorname{dist}(E,F)/t\rangle^{-M}\|u\| \tag{23}$$

whenever $E, F\subset\mathbf{R}^n$ are Borel sets, and $u\in\mathcal{H}$ satisfies $\operatorname{supp} u\subset F$.

Proof. First consider the resolvents $R_t^B = (\mathrm{I}+it\Pi_B)^{-1}$ for all nonzero $t\in\mathbf{R}$. As we have already proved uniform bounds for R_t^B in Proposition 2.5, it suffices to prove

$$\|(\mathrm{I}+it\Pi_B)^{-1}u\|_{L_2(E)} \le C_M(t/\operatorname{dist}(E,F))^M\|u\|$$

for all disjoint $E,F\subset\mathbf{R}^n$, for all $|t|\le\operatorname{dist}(E,F)$, and for all $u\in\mathcal{H}$ with $\operatorname{supp} u\subset F$.

We prove this result by induction. Proposition 2.5 proves this statement for $M=0$. Assume that the statement is true for some given $M\in\mathbf{N}$. Write

$$\widetilde{E} := \left\{x\in\mathbf{R}^n : \operatorname{dist}(x,E) < \tfrac{1}{2}\operatorname{dist}(x,F)\right\}$$

and let $\eta:\mathbf{R}^n\longrightarrow[0,1]$ be a Lipschitz function such that $\operatorname{supp}\eta\subset\widetilde{E}$, $\eta|_E = 1$ and

$$\|\nabla\eta\|_\infty \le 4/\operatorname{dist}(E,F).$$

We now use (H5–6) to calculate that

$$\left[\eta\mathrm{I}, (\mathrm{I}+it\Pi_B)^{-1}\right] = itR_t^B(\Gamma_{\nabla\eta} + B_1\Gamma^*_{\nabla\eta}B_2)R_t^B$$

and therefore

$$\begin{aligned}\left\|(\mathrm{I}+it\Pi_B)^{-1}u\right\|_{L_2(E)} &\le \left\|\eta(\mathrm{I}+it\Pi_B)^{-1}u\right\|\\ &= \left\|\left[\eta\mathrm{I}, (\mathrm{I}+it\Pi_B)^{-1}\right]u\right\|\\ &\lesssim C_0 t\|\nabla\eta\|_\infty\left\|R_t^B u\right\|_{L_2(\widetilde{E})}\\ &\lesssim C_0 t\|\nabla\eta\|_\infty C_M(t/\operatorname{dist}(\widetilde{E},F))^M\|u\|\\ &\lesssim C_0C_M(t/\operatorname{dist}(E,F))^{M+1}\|u\|.\end{aligned}$$

This completes the induction step and thus proves the proposition for the resolvents R_t^B. The result for P_t^B and Q_t^B follows, as they are linear combinations of resolvents.

Now consider $\Theta_t^B = t\Gamma_B^* P_t^B$. We have

$$\left\|\Theta_t^B u\right\|_{L_2(E)} \le \left\|\eta\Theta_t^B u\right\| \le \left\|[\eta\,\mathrm{I}, t\Gamma_B^*]P_t^B u\right\| + \left\|t\Gamma_B^*\eta P_t^B u\right\|.$$

By Lemma 4.2 the last term is bounded by

$$\left\|t\Pi_B\eta P_t^B u\right\| \le \left\|[\eta\,\mathrm{I}, t\Pi_B]P_t^B u\right\| + \left\|\eta Q_t^B u\right\|$$

and so, using (H6) and the bounds already obtained for P_t^B and Q_t^B, we conclude that for each $M \ge 0$,

$$\begin{aligned}\left\|\Theta_t^B u\right\|_{L_2(E)} &\lesssim t\|\nabla\eta\|_\infty \left\|P_t^B u\right\|_{L_2(\widetilde{E})} + \left\|Q_t^B u\right\|_{L_2(\widetilde{E})}\\ &\lesssim \langle \mathrm{dist}\,(E,F)/t\rangle^{-M}\|u\|\,.\end{aligned}$$

This completes the proof. □

A simple consequence of Proposition 5.2 is that

$$\|U_s u\|_{L_2(Q)} \le \sum_{R\in\triangle_t} \|U_s(\chi_R u)\|_{L_2(Q)} \lesssim \sum_{R\in\triangle_t} \langle \mathrm{dist}\,(R,Q)/s\rangle^{-M}\|u\|_{L_2(R)} \tag{24}$$

whenever $0 < s \le t$ and $Q \in \triangle_t$, where U_s is as specified in Proposition 5.2. We also note that the dyadic cubes satisfy

$$\sup_{Q\in\triangle_t} \sum_{R\in\triangle_t} \langle \mathrm{dist}\,(R,Q)/t\rangle^{-(n+1)} \lesssim 1 \tag{25}$$

and therefore, choosing $M \ge n+1$, we see that U_t extends to an operator $U_t : L_\infty(\mathbf{R}^n) \longrightarrow L_2^{\mathrm{loc}}(\mathbf{R}^n)$.

A consequence of the above results with $U_t = \Theta_t^B$ is:

Corollary 5.3. *The functions $\gamma_t \in L_2^{loc}(\mathbf{R}^n; \mathcal{L}(\mathbf{C}^N))$ satisfy the boundedness conditions*

$$⨍_Q |\gamma_t(y)|^2\, dy \lesssim 1$$

for all $Q \in \triangle_t$. Moreover $\|\gamma_t A_t\| \lesssim 1$ uniformly for all $t > 0$.

5.2. Principal part approximation. In this section we prove the principal part approximation $\Theta_t^B \approx \gamma_t$ in the sense that we estimate the first two terms on the right-hand side of (22). The following lemma is used in estimating the first term.

Lemma 5.4 (A weighted Poincaré inequality). *If $Q \in \triangle_t$ and $\beta < -2n$, then we have*

$$\int_{\mathbf{R}^n} |u(x) - u_Q|^2 \langle \mathrm{dist}\,(x,Q)/t\rangle^\beta\, dx \lesssim \int_{\mathbf{R}^n} |t\nabla u(x)|^2 \langle \mathrm{dist}\,(x,Q)/t\rangle^{2n+\beta}\, dx$$

for every u in the Sobolev space $H^1(\mathbf{R}^n; \mathbf{C}^N)$.

Proof. Without loss of generality we may assume that $t = 1$ and that Q is the unit cube centred at $x = 0$. By [16, p. 164] we have

$$\int_{\mathbf{R}^n} |u(y) - u_Q|^2 \chi_r(y)\,dy \lesssim \int_{\mathbf{R}^n} |\nabla u(y)|^2 r^{2n} \chi_r(y)\,dy$$

for every $r \geq 1$, where we write χ_r to denote the characteristic function of $\{y \in \mathbf{R}^n : |y| \leq r\}$. Integrating the above inequality over $(1, \infty)$ against the measure dr^β gives the desired result. □

We now estimate the first term in the right-hand side of (22).

Proposition 5.5. *For all $u \in \mathsf{R}(\Pi)$, we have*

$$\int_0^\infty \left\| \Theta_t^B P_t u - \gamma_t A_t P_t u \right\|^2 \frac{dt}{t} \lesssim \|u\|^2.$$

Proof. Using Proposition 5.2, estimate (25), Lemma 5.4 and then the coercivity assumption (H8), we get for any $v \in \mathsf{R}(\Pi)$, that

$$\begin{aligned}
\left\| \Theta_t^B v - \gamma_t A_t v \right\|^2 &= \sum_{Q \in \triangle_t} \left\| \Theta_t^B (v - v_Q) \right\|_{L_2(Q)}^2 \\
&\lesssim \sum_{Q \in \triangle_t} \Big(\sum_{R \in \triangle_t} \langle d(R, Q)/t \rangle^{-(3n+1)} \|v - v_Q\|_{L_2(R)} \Big)^2 \\
&\lesssim \sum_{Q \in \triangle_t} \int_{\mathbf{R}^n} |v(x) - v_Q|^2 \langle d(x, Q)/t \rangle^{-(3n+1)} \\
&\lesssim \sum_{Q \in \triangle_t} \int_{\mathbf{R}^n} |t \nabla v(x)|^2 \langle d(x, Q)/t \rangle^{-(n+1)} \\
&\lesssim \|t \nabla v\|^2 \lesssim \|t \Pi v\|^2
\end{aligned}$$

and therefore, taking $v = P_t u$ and using (18), that

$$\int_0^\infty \left\| \Theta_t^B P_t u - \gamma_t A_t P_t u \right\|^2 \frac{dt}{t} \lesssim \int_0^\infty \|Q_t u\|^2 \frac{dt}{t} \lesssim \|u\|^2.$$ □

We use the following lemma to estimate the second term in the right-hand side of (22), and also in the proofs of Lemmas 5.10 and 5.12. (c.f. Lemma 5.15 of [2].)

Lemma 5.6. *Let Υ be either Π, Γ or Γ^*. Then we have the estimate*

$$\left| \fint_Q \Upsilon u \right|^2 \lesssim \frac{1}{l(Q)} \Big(\fint_Q |u|^2 \Big)^{1/2} \Big(\fint_Q |\Upsilon u|^2 \Big)^{1/2} \tag{26}$$

for all $Q \in \triangle$ and $u \in \mathsf{D}(\Upsilon)$.

Proof. Let $t = (\int_Q |u|^2)^{1/2}(\int_Q |\Upsilon u|^2)^{-1/2}$. If $t \geq \frac{1}{4}l(Q)$, then (26) follows directly from the Cauchy–Schwarz inequality. If $t \leq \frac{1}{4}l(Q)$, let $\eta \in C_0^\infty(Q)$ be a real-valued bump function such that $\eta(x) = 1$ when dist $(x, \mathbf{R}^n \setminus Q) > t$, and $|\nabla\eta| \lesssim 1/t$. Using the cancellation property (H7) of Υ and the Cauchy–Schwarz inequality, we obtain

$$\left|\int_Q \Upsilon u\right| = \left|\int_Q \eta\Upsilon u + \int_Q (1-\eta)\Upsilon u\right| = \left|\int_Q [\eta, \Upsilon]u + \int_Q (1-\eta)\Upsilon u\right|$$
$$\lesssim \|\nabla\eta\|_\infty (tl(Q)^{n-1})^{1/2}\left(\int_Q |u|^2\right)^{1/2} + (tl(Q)^{n-1})^{1/2}\left(\int_Q |\Upsilon u|^2\right)^{1/2}$$

which gives (26) on substituting the chosen value of t. □

We now estimate the second term in the right-hand side of (22).

Proposition 5.7. *For all $u \in \mathcal{H}$, we have*

$$\int_0^\infty \|\gamma_t A_t(P_t - \mathrm{I})u\|^2 \frac{dt}{t} \lesssim \|u\|^2.$$

Proof. Corollary 5.3 shows that $\|\gamma_t A_t\| \lesssim 1$ and since $A_t^2 = A_t$ it suffices to prove the square function estimate with integrand $\|A_t(P_t - \mathrm{I})u\|^2$. If $u \in \mathsf{N}(\Pi)$ then this is zero. If $u \in \overline{\mathsf{R}(\Pi)}$ then write $u = 2\int_0^\infty Q_s^2 u \frac{ds}{s}$. The result will follow from another Schur estimate and (18) once we have obtained the bound

$$\|A_t(P_t - \mathrm{I})Q_s\| \lesssim \min\left\{\tfrac{s}{t}, \tfrac{t}{s}\right\}^{1/2}$$

for all $s, t > 0$.

Note that $(\mathrm{I} - P_t)Q_s = \frac{t}{s}Q_t(\mathrm{I} - P_s)$ and $P_t Q_s = \frac{s}{t}Q_t P_s$ for every $s, t > 0$. Thus, if $t \leq s$, then

$$\|A_t(P_t - \mathrm{I})Q_s\| \lesssim \|(P_t - \mathrm{I})Q_s\| \lesssim t/s,$$

while if $t > s$, then

$$\|A_t(P_t - \mathrm{I})Q_s\| \lesssim \|P_t Q_s\| + \|A_t Q_s\| \lesssim s/t + \|A_t Q_s\|.$$

To estimate $\|A_t Q_s\|$, we use Lemma 5.6 with (24) and (25) to obtain

$$\|A_t Q_s u\|^2$$
$$= \sum_{Q\in\Delta_t} |Q| \left|\fint_Q s\Pi(\mathrm{I} + s^2\Pi^2)^{-1}u\right|^2$$
$$\lesssim \frac{s}{t}\sum_{Q\in\Delta_t}\left(\int_Q |P_s u|^2\right)^{1/2}\left(\int_Q |Q_s u|^2\right)^{1/2}$$

$$\lesssim \frac{s}{t}\sum_{Q\in\triangle_t}\Big(\sum_{R\in\triangle_t}\langle d(R,Q)/t\rangle^{-(n+1)}\|u\|_{L_2(R)}\Big)^2$$
$$\lesssim \frac{s}{t}\sum_{Q\in\triangle_t}\Big(\sum_{R'\in\triangle_t}\langle d(R',Q)/t\rangle^{-(n+1)}\Big)\Big(\sum_{R\in\triangle_t}\langle d(R,Q)/t\rangle^{-(n+1)}\|u\|^2_{L_2(R)}\Big)$$
$$\lesssim \frac{s}{t}\|u\|^2$$

which completes the proof. □

We have now estimated the first two terms in the right-hand side of (22).

5.3. Carleson measure estimate. In this subsection we estimate the third term in the right-hand side of (22). To do this we reduce the problem to a Carleson measure estimate, drawing upon the "$T(b)$" procedure developed by Auscher and Tchamitchian [7, Chap. 3]. Recall that a measure μ on $\mathbf{R}^n \times \mathbf{R}^+$ is said to be *Carleson* if $\|\mu\|_{\mathcal{C}} := \sup_{Q\in\triangle}|Q|^{-1}\mu(R_Q) < \infty$. Here and below $R_Q := Q \times (0, l(Q)]$ denotes the *Carleson box* of any cube Q. We recall the following theorem of Carleson.

Theorem 5.8. [31, p. 59] *If μ is a Carleson measure on $\mathbf{R}^n \times \mathbf{R}^+$ then*

$$\iint_{\mathbf{R}^n\times(0,\infty)} |A_t u(x)|^2\, d\mu(x,t) \le C\|\mu\|_{\mathcal{C}}\|u\|^2$$

for every $u \in \mathcal{H}$. Here $C > 0$ is a constant that depends only on n.

Thus, in order to prove (22) it suffices to show that

$$\iint_{R_Q} |\gamma_t(x)|^2\,\frac{dxdt}{t} \lesssim |Q| \tag{27}$$

for every dyadic cube $Q \in \triangle$. Following [2] or more precisely [4], we set $\sigma > 0$; the exact value to be chosen later. Let $\mathcal{V}$ be a finite set consisting of $\nu \in \mathcal{L}(\mathbf{C}^N)$ with $|\nu| = 1$, such that $\bigcup_{\nu\in\mathcal{V}} K_\nu = \mathcal{L}(\mathbf{C}^N)\setminus\{0\}$, where

$$K_\nu := \left\{\nu' \in \mathcal{L}(\mathbf{C}^N)\setminus\{0\} : \left|\frac{\nu'}{|\nu'|} - \nu\right| \le \sigma\right\}.$$

To prove (27) it suffices to show that

$$\iint_{\substack{(x,t)\in R_Q\\ \gamma_t(x)\in K_\nu}} |\gamma_t(x)|^2\,\frac{dxdt}{t} \lesssim |Q| \tag{28}$$

for every $\nu \in \mathcal{V}$. By the John-Nirenberg lemma for Carleson measures as applied in [2, Sect. 5], in order to prove (28) it suffices to prove the following claim.

Proposition 5.9. *There exists $\beta > 0$ such that for every dyadic cube $Q \in \triangle$ and $\nu \in \mathcal{L}(\mathbf{C}^N)$ with $|\nu| = 1$, there is a collection $\{Q_k\}_k \subset \triangle$ of disjoint subcubes of Q such that $|E_{Q,\nu}| > \beta|Q|$ where $E_{Q,\nu} = Q \setminus \bigcup_k Q_k$, and such that*

$$\iint_{\substack{(x,t)\in E^*_{Q,\nu}\\ \gamma_t(x)\in K_\nu}} |\gamma_t(x)|^2 \frac{dxdt}{t} \lesssim |Q|$$

where $E^*_{Q,\nu} = R_Q \setminus \bigcup_k R_{Q_k}$.

Fix a dyadic cube $Q \in \triangle$ and fix $\nu \in \mathcal{L}(\mathbf{C}^N)$ with $|\nu| = 1$. Choose $\hat{w}, w \in \mathbf{C}^N$ with $|\hat{w}| = |w| = 1$ and $\nu^*(\hat{w}) = w$. Let η_Q be a smooth cutoff function with range $[0, 1]$, equal to 1 on $2Q$, with support in $4Q$, and such that $\|\nabla\eta_Q\|_\infty \le \frac{1}{l}$ where $l = l(Q)$. Define $w_Q := \eta_Q w$, and for each $\epsilon > 0$, let

$$\begin{aligned} f^w_{Q,\epsilon} &:= w_Q - \epsilon li\Gamma(1 + \epsilon li\Pi_B)^{-1}w_Q \\ &= \left(1 + \epsilon li\Gamma^*_B\right)(1 + \epsilon li\Pi_B)^{-1}w_Q. \end{aligned}$$

Lemma 5.10. *We have $\|f^w_{Q,\epsilon}\| \lesssim |Q|^{1/2}$,*

$$\iint_{R_Q} \left|\Theta^B_t f^w_{Q,\epsilon}\right|^2 \frac{dxdt}{t} \lesssim \frac{1}{\epsilon^2}|Q| \quad \textit{and}$$

$$\left|\fint_Q f^w_{Q,\epsilon} - w\right| \le c\epsilon^{1/2}$$

for every $\epsilon > 0$. Here $c > 0$ is a constant that depends only on hypotheses (H1–8).

Proof. The first estimate can be deduced from Proposition 2.5 and Lemma 4.2. To obtain the second estimate, observe by the nilpotency of Γ^*_B that

$$\begin{aligned} \Theta^B_t f^w_{Q,\epsilon} &= \left(\mathrm{I} + t^2\Pi^2_B\right)^{-1} t\Gamma^*_B\left(\mathrm{I} + \epsilon li\Gamma^*_B\right)(\mathrm{I} + \epsilon li\Pi_B)^{-1}w_Q \\ &= \tfrac{t}{\epsilon l}\left(\mathrm{I} + t^2\Pi^2_B\right)^{-1}\epsilon l\Gamma^*_B(\mathrm{I} + \epsilon li\Pi_B)^{-1}w_Q \end{aligned}$$

and therefore by Proposition 2.5 and Lemma 4.2 that

$$\iint_{R_Q} \left|\Theta^B_t f^w_{Q,\epsilon}\right|^2 \frac{dxdt}{t} \lesssim |Q| \int_0^l \left(\frac{t}{\epsilon l}\right)^2 \frac{dt}{t} \lesssim \frac{1}{\epsilon^2}|Q|.$$

To obtain the last estimate, we use Lemma 5.6 with $\Upsilon = \Gamma$ and $u = (\mathrm{I} + \epsilon li\Pi_B)^{-1}w_Q$ to show that

$$\begin{aligned} &\left|\fint_Q f^w_{Q,\epsilon} - w\right| = \left|\fint_Q \epsilon l\Gamma(\mathrm{I} + \epsilon li\Pi_B)^{-1}w_Q\right| \\ &\lesssim \epsilon^{1/2}\left(\fint_Q |(\mathrm{I} + \epsilon li\Pi_B)^{-1}w_Q|^2\right)^{1/4}\left(\fint_Q |\epsilon l\Gamma(\mathrm{I} + \epsilon li\Pi_B)^{-1}w_Q|^2\right)^{1/4} \lesssim \epsilon^{1/2}. \end{aligned}$$

This completes the proof. □

For the choice $\epsilon = \frac{1}{4c^2}$, let $f_Q^w = f_{Q,\epsilon}^w$. The above lemma implies that

$$\mathrm{Re}\left(w, \fint_Q f_Q^w\right) \geq \frac{1}{2}\,.$$

Lemma 5.11. *There exists $\beta, c_1, c_2 > 0$ that depend only on (H1–8), and there exists a collection $\{Q_k\}$ of dyadic subcubes of Q such that $|E_{Q,\nu}| > \beta|Q|$ where $E_{Q,\nu} = Q \setminus \bigcup_k Q_k$, and such that*

$$\mathrm{Re}\left(w, \fint_{Q'} f_Q^w\right) \geq c_1 \quad \text{and} \quad \fint_{Q'} |f_Q^w| \leq c_2 \tag{29}$$

for all dyadic subcubes $Q' \in \triangle$ of Q which satisfy $R_{Q'} \cap E^*_{Q,\nu} \neq \emptyset$, where $E^*_{Q,\nu} = R_Q \setminus \bigcup_k R_{Q_k}$.

Proof. Fix $\alpha > 0$. Let $\mathcal{B}_1 \subset \triangle$ be the collection of maximal dyadic subcubes $S \in \triangle$ of Q such that

$$\mathrm{Re}\left(w, \fint_S f_Q^w\right) < \alpha$$

and let $\mathcal{B}_2 \subset \triangle$ be the collection of maximal dyadic subcubes $S \in \triangle$ of Q such that

$$\fint_S |f_Q^w| > \frac{1}{\alpha}.$$

Let $\{Q_k\}$ be an enumeration of the maximal cubes in $\mathcal{B}_1 \cup \mathcal{B}_2$. These are the bad cubes. By construction we have each dyadic subcube $Q' \in \triangle$ of Q with $R_{Q'} \cap E^*_{Q,\nu} \neq \emptyset$ satisfies (29) with $c_1 = \alpha$ and $c_2 = \frac{1}{\alpha}$. These are the good cubes. Thus, to prove the lemma it suffices to show that for an appropriate choice of $\alpha > 0$, that depends only on (H1–8), there exists $\beta > 0$ such that $|E_{Q,\nu}| > \beta|Q|$.

We use the rough estimate

$$|E_{Q,\nu}| \geq |Q \setminus \bigcup \mathcal{B}_1| - |\bigcup \mathcal{B}_2|.$$

By construction and by Lemma 5.10 we have

$$|\bigcup \mathcal{B}_2| = \sum_{S \in \mathcal{B}_2} |S| \leq \alpha \sum_{S \in \mathcal{B}_2} \int_S |f_Q^w| \leq \alpha \int_Q |f_Q^w| \lesssim \alpha |Q|$$

and

$$\begin{aligned}
\frac{1}{2}|Q| \leq \mathrm{Re}\left(w, \int_Q f_Q^w\right) &= \sum_{S \in \mathcal{B}_1} \mathrm{Re}\left(w, \int_S f_Q^w\right) + \mathrm{Re}\left(w, \int_{Q \setminus \bigcup \mathcal{B}_1} f_Q^w\right) \\
&\lesssim \alpha \sum_{S \in \mathcal{B}_1} |S| + \left(\int_Q |f_Q^w|^2\right)^{1/2} |Q \setminus \bigcup \mathcal{B}_1|^{1/2} \\
&\lesssim \alpha |Q| + |Q|^{1/2} |Q \setminus \bigcup \mathcal{B}_1|^{1/2}.
\end{aligned}$$

The desired estimate follows by a sufficiently small choice of $\alpha > 0$ that depends only on (H1–8). This completes the proof. □

We now choose $\sigma = \frac{c_1}{2c_2}$.

Lemma 5.12. *If* $(x,t) \in E^*_{Q,\nu}$ *and* $\gamma_t(x) \in K_\nu$ *then*

$$\left|\gamma_t(x)\left(A_t f^w_Q(x)\right)\right| \geq \tfrac{1}{2}c_1|\gamma_t(x)|.$$

Proof. To see the result apply the previous lemma to deduce that

$$\left|\nu\left(A_t f^w_Q(x)\right)\right| \geq \mathrm{Re}\left(\hat{w}, \nu\left(A_t f^w_Q(x)\right)\right) = \mathrm{Re}\left(w, A_t f^w_Q(x)\right) \geq c_1$$

and then furthermore that

$$\begin{aligned}\left|\frac{\gamma_t(x)}{|\gamma_t(x)|}\left(A_t f^w_Q(x)\right)\right| &\geq \left|\nu\left(A_t f^w_Q(x)\right)\right| - \left|\frac{\gamma_t(x)}{|\gamma_t(x)|} - \nu\right|\left|A_t f^w_Q(x)\right| \\ &\geq c_1 - \sigma c_2 = \tfrac{1}{2}c_1.\end{aligned}$$

□

Proof of Proposition 5.9. By Lemma 5.12 we have

$$\begin{aligned}\iint_{\substack{(x,t)\in E^*_{Q,\nu}\\ \gamma_t(x)\in K_\nu}} |\gamma_t(x)|^2 \frac{dxdt}{t} &\lesssim \iint_{R_Q} \left|\gamma_t(x)\left(A_t f^w_Q(x)\right)\right|^2 \frac{dxdt}{t} \\ &\lesssim \iint_{R_Q} \left|\Theta^B_t f^w_Q - \gamma_t A_t f^w_Q\right|^2 \frac{dxdt}{t} \\ &\quad + \iint_{R_Q} \left|\Theta^B_t f^w_Q\right|^2 \frac{dxdt}{t}.\end{aligned}$$

Lemma 5.10 implies that the last term in the above inequality is bounded by a constant (that depends only on (H1–8)) times $|Q|$.

It remains to show that

$$\iint_{R_Q} \left|\Theta^B_t f^w_Q - \gamma_t A_t f^w_Q\right|^2 \frac{dxdt}{t} \lesssim |Q|. \tag{30}$$

Observe that

$$\begin{aligned}\Theta^B_t f^w_Q - \gamma_t A_t f^w_Q = &-\left(\Theta^B_t - \gamma_t A_t\right) \epsilon li\Gamma(1+\epsilon li\Pi_B)^{-1} w_Q \\ &+ (\Theta^B_t - \gamma_t A_t) w_Q.\end{aligned} \tag{31}$$

Since $\epsilon li\Gamma(1+\epsilon li\Pi_B)^{-1} w_Q \in \mathsf{R}(\Gamma)$, we have by the results of Sects. 4 and 5.2 (specifically, part (i) in the proof of Proposition 4.8, and also 5.5 and 5.7) that

$$\iint_{R_Q} \left|\left(\Theta^B_t - \gamma_t A_t\right)\epsilon li\Gamma(1+\epsilon li\Pi_B)^{-1} w_Q\right|^2 \frac{dxdt}{t} \lesssim \|w_Q\|^2 \lesssim |Q|.$$

We also have

$$\left(\Theta_t^B - \gamma_t A_t\right) w_Q(x) = \Theta_t^B((\eta_Q - 1)w)(x)$$

for every $x \in Q$ and $t > 0$. Since $\left(\text{supp}\,(\eta_Q - 1)w\right) \cap 2Q = \emptyset$, then (24) implies that

$$\int_Q \left|\Theta_t^B((\eta_Q - 1)w)(x)\right|^2 dx \lesssim \frac{t|Q|}{l}$$

when $0 < t \leq l$, and therefore that

$$\iint_{R_Q} \left|\left(\Theta_t^B - \gamma_t A_t\right) w_Q(x)\right|^2 \frac{dxdt}{t} \lesssim |Q|.$$

This proves (30) and so completes the proof of Proposition 5.9. □

Proof of Theorems 2.7 and 2.10. We have demonstrated in this section that the square function estimate (19) holds for all $u \in \mathsf{R}(\Pi)$ and some constant c which depends only on the bounds in (H1–8). These hypotheses are invariant on replacing $\{\Gamma, B_1, B_2\}$ by $\{\Gamma^*, B_2, B_1\}$, $\{\Gamma^*, B_2^*, B_1^*\}$ and $\{\Gamma, B_1^*, B_2^*\}$. So, by Proposition 4.8, we conclude that Π_B satisfies the quadratic estimate (6) for all $u \in \overline{\mathsf{R}(\Pi_B)}$, and has a bounded holomorphic S_μ^o functional calculus. □

6. Holomorphic dependence

In this section we show that under the appropriate hypotheses, resolvents, projections, bounded members of the functional calculus, and quadratic estimates, all depend holomorphically on holomorphic perturbations of Π_B. Recall that if $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces and $U \subset \mathbf{C}$ is open, then an operator valued function $T : U \to \mathcal{L}(\mathcal{H}, \mathcal{K})$ is said to be *holomorphic* if it is (complex) differentiable in the uniform topology everywhere in U.

Theorem 6.1. *Let $U \subset \mathbf{C}$ be open, let $B_1, B_2 : U \to \mathcal{L}(\mathcal{H})$ be holomorphic functions such that $B_1(z)$ and $B_2(z)$ satisfy (H1–3) uniformly for each $z \in U$, and let $\tau \in \mathbf{C} \setminus S_\mu$. Then the function given by $z \mapsto (1 + \tau\Pi_{B(z)})^{-1}$ is holomorphic on U, the function given by $z \mapsto \mathbf{P}^0_{B(z)}$ is holomorphic on U, and the function given by $z \mapsto \psi(\Pi_{B(z)})$ is holomorphic on U for every $\psi \in \Psi(S_\mu^o)$.*

Remark 6.2. An interesting observation that arose from our consideration of Theorem 6.1 is that under its hypotheses, not only is the function given by $z \mapsto \mathbf{P}^0_{B(z)}$ holomorphic on U, but so too are the functions given by $z \mapsto \mathbf{P}^1_{B(z)}$ and $z \mapsto \mathbf{P}^2_{B(z)}$. This means that the Hodge decomposition (5) is holomorphic on U.

Moreover, we have

$$\frac{d}{dz}\mathbf{P}^0_B = -\mathbf{P}^0_B A_1 \tilde{\mathbf{P}}^1_B - \tilde{\mathbf{P}}^2_B A_2 \mathbf{P}^0_B, \tag{32}$$

$$\frac{d}{dz}\mathbf{P}^1_B = \mathbf{P}^0_B A_1 \tilde{\mathbf{P}}^1_B - \tilde{\mathbf{P}}^2_B A_2 \mathbf{P}^1_B, \tag{33}$$

and

$$\frac{d}{dz}\mathbf{P}^2_B = -\mathbf{P}^2_B A_1 \tilde{\mathbf{P}}^1_B + \tilde{\mathbf{P}}^2_B A_2 \mathbf{P}^0_B\,. \tag{34}$$

Here $A_1(z) = \frac{d}{dz}B_1(z)$ and $A_2(z) = \frac{d}{dz}B_2(z)$, and the operators $\tilde{\mathbf{P}}^1_B$ and $\tilde{\mathbf{P}}^2_B$ in $\mathcal{L}(\mathcal{H})$ are defined in Appendix A, and satisfy $\mathbf{P}^1_B = B_1\tilde{\mathbf{P}}^1_B$ and $\mathbf{P}^2_B = \tilde{\mathbf{P}}^2_B B_2$. The claims of this remark are verified in the Appendix A.

Before proving Theorem 6.1 we recall some standard results from operator theory. The function $T : U \to \mathcal{L}(\mathcal{H}, \mathcal{K})$ is holomorphic if and only if it is locally uniformly bounded (that is, uniformly bounded on each compact subset of U), and strongly differentiable (see [19, p. 365]). Cauchy's Theorem, and indeed many standard results about complex-valued holomorphic functions extend to the operator valued setting. A suitable reference is [13, III.14]. In particular, the following holds:

Lemma 6.3. *Let $U \subset \mathbf{C}$ be an open set, and let $T_n, T : U \longrightarrow \mathcal{L}(\mathcal{H}, \mathcal{K})$ be functions with T_n holomorphic for each $n \in \mathbf{N}$. Suppose that $T_n(z)u \to T(z)u$ as $n \to \infty$, for every $z \in U$ and $u \in \mathcal{H}$, and that for every compact $K \subset U$ there exists $L > 0$ such that $\|T_n(z)\| \leq L$ for every $z \in K$ and $n \in \mathbf{N}$. Then T is holomorphic, and moreover for every $u \in \mathcal{H}$, we have $(T_n u)$ and $(\frac{d}{dz}T_n u)$ converge locally uniformly to Tu and $\frac{d}{dz}Tu$ respectively. (i.e. the convergence is uniform on each compact subset of U.)*

A sequence $(T_n) \subset \mathcal{L}(\mathcal{H})$ is said to converge to $T \in \mathcal{L}(\mathcal{H})$ *strongly* if for every $u \in \mathcal{H}$ we have $\|T_n u - Tu\| \to 0$ as $n \to \infty$. We use the fact that, for any pair of sequences $(S_n), (T_n) \subset \mathcal{L}(\mathcal{H})$ with $S_n \to S$ and $T_n \to T$ strongly as $n \to \infty$, where $S, T \in \mathcal{L}(\mathcal{H})$, then $S_n T_n \to ST$ strongly.

Proof of Theorem 6.1. Fix $\tau \in \mathbf{C} \setminus S^o_\mu$. Then

$$\begin{aligned}\frac{d}{dz}(\mathrm{I}+\tau\Pi_B)^{-1} = &-(\mathrm{I}+\tau\Pi_B)^{-1}A_1\tau\Gamma^* B_2(\mathrm{I}+\tau\Pi_B)^{-1}\\ &-(\mathrm{I}+\tau\Pi_B)^{-1}B_1\tau\Gamma^* A_2(\mathrm{I}+\tau\Pi_B)^{-1}\end{aligned} \tag{35}$$

where $A_1(z) = \frac{d}{dz}B_1(z)$ and $A_2(z) = \frac{d}{dz}B_2(z)$. The fact that the above operators are all uniformly bounded can be obtained from (12), (13) and Lemma 4.2. This proves the first claim. Thus $\{z \mapsto (\mathrm{I}+in\Pi_B)^{-1}\}_n$ is a collection of uniformly bounded functions holomorphic on U. Moreover

$\mathbf{P}^0_B u = \lim_{n\to\infty}(\mathrm{I}+in\Pi_{B(z)})^{-1}u$ for all $u \in \mathcal{H}$. (This is proved in a setting similar to ours in [12, Theorem 3.8]; we also prove it as a part of Lemma A.1). The second claim now follows from Lemma 6.3. We now prove the third claim. Fix $\psi \in \Psi(S^o_\mu)$. The desired result can now be deduced from the first claim and Lemma 6.3 by using a Riemann sum to approximate the contour integral representation of $\psi(\Pi_{B(z)})$ as in (7). This completes the proof. □

We now adopt the notation from hypotheses (H1–8) and consider the Hilbert space

$$\mathcal{K} = L_2\left(\mathbf{R}^n \times (0,\infty), \frac{dxdt}{t}; \mathbf{C}^N\right)$$

and for every $\psi \in \Psi(S^o_\mu)$ and $z \in U$, define the operator $S_{B(z)}(\psi) : \mathcal{H} \longrightarrow \mathcal{K}$ by

$$(S_{B(z)}(\psi)u)(x,t) = \big(\psi(t\Pi_{B(z)})u\big)(x)$$

for every $u \in \mathcal{H}$, $t > 0$ and almost every $x \in \mathbf{R}^n$.

Theorem 6.4. *Let $U \subset \mathbf{C}$ be open, let $B_1, B_2 : U \to \mathcal{L}(\mathcal{H})$ be holomorphic functions such that $B_1(z)$ and $B_2(z)$ satisfy (H1–8) uniformly for each $z \in U$, and let $\omega < \mu < \frac{\pi}{2}$. Then the function given by $z \mapsto f(\Pi_{B(z)})$ is holomorphic on U for every bounded $f : S^o_\mu \cup \{0\} \longrightarrow \mathbf{C}$ holomorphic on S^o_μ, and the function given by $z \mapsto S_{B(z)}(\psi)$ is holomorphic on U for every $\psi \in \Psi(S^o_\mu)$.*

Proof. We prove the first claim. Let f be as above. Since by Theorem 6.1, the function $z \mapsto \mathbf{P}^0_{B(z)}$ is holomorphic on U, we can without loss of generality further assume that $f(0) = 0$. Choose a uniformly bounded sequence $(\psi_n) \subset \Psi(S^o_\mu)$ that converges locally uniformly to f on S^o_μ. By Theorem 6.1 we have each function $z \mapsto \psi_n(\Pi_{B(z)})$ is holomorphic on U. Moreover, by Theorem 2.10 and (10), we have that $\psi_n(\Pi_{B(z)})$ is uniformly bounded (with respect to $n \in \mathbf{N}$ and $z \in U$) and that $\big(\psi_n(\Pi_{B(z)})\big)$ converges strongly to $f(\Pi_{B(z)})$ for every $z \in U$. The first claim of Theorem 6.4 now follows from Lemma 6.3.

We now prove the second claim. Let $n \in \mathbf{N}$, and define $\psi^n_t : S^o_\mu \longrightarrow \mathbf{C}$ by $\psi^n_t(\zeta) = \psi(t\zeta)$ whenever $\zeta \in S^o_\mu$ and $1/n < t < n$, and $\psi^n_t = 0$ otherwise. Next let $S^n_{B(z)}(\psi) : \mathcal{H} \longrightarrow \mathcal{K}$ be given by

$$\big(S^n_{B(z)}(\psi)u\big)(x,t) = \big(\psi^n_t(\Pi_{B(z)})u\big)(x)$$

for every $z \in U$, $u \in \mathcal{H}$, $t > 0$ and almost every $x \in \mathbf{R}^n$. We deduce from Theorem 6.1 that for every $t > 0$, the function $z \mapsto \psi^n_t(\Pi_{B(z)})$ is holomorphic on U, and by Theorem 2.10 that this family of functions is uniformly bounded with respect to $t > 0$. This with the fact that ψ^n_t is only non-zero for $t \in (1/n, n)$ allows us to deduce that the function given by $z \mapsto S^n_{B(z)}(\psi)$ is holomorphic on U. However, by Remark 2.8 we have $\|S^n_{B(z)}(\psi)\|$ is uniformly bounded over every $z \in U$ and $n \in \mathbf{N}$, and that $S^n_{B(z)}(\psi)$ strongly converges to $S_{B(z)}(\psi)$ as $n \to \infty$ for every $z \in U$. The second claim now follows from Lemma 6.3. This completes the proof. □

We use the previous theorem to prove Lipschitz estimates on members of the functional calculus of the perturbed Dirac operator Π_B, and Lipschitz estimates on quadratic functions of Π_B.

Theorem 6.5. *Let $\mathcal{H}$, Γ, B_1, B_2, κ_1, κ_2 and n be as outlined in (H1–8). For $i = 1, 2$, fix $\eta_i < \kappa_i$, and then let $0 < \hat{\omega}_i < \frac{\pi}{2}$ be given by $\cos\hat{\omega}_i = \frac{\kappa_i - \eta_i}{\|B_i\| + \eta_i}$. Next let $\hat{\omega} = \frac{1}{2}(\hat{\omega}_1 + \hat{\omega}_2)$ and $\hat{\omega} < \mu < \frac{\pi}{2}$. Then we have*

$$\|f(\Pi_B) - f(\Pi_{B+A})\| \lesssim (\|A_1\|_\infty + \|A_2\|_\infty)\|f\|_\infty$$

for every bounded $f : S^o_\mu \cup \{0\} \longrightarrow \mathbf{C}$ holomorphic on S^o_μ, and every $A_i \in L_\infty(\mathbf{R}^n, \mathcal{L}(\mathbf{C}^N))$ with $\|A_i\|_\infty \le \eta_i$. Moreover, given $\psi \in \Psi(S^o_\mu)$, we have

$$\int_0^\infty \|\psi(t\Pi_B)u - \psi(t\Pi_{B+A})u\|^2 \frac{dt}{t} \lesssim \left(\|A_1\|^2_\infty + \|A_2\|^2_\infty\right)\|u\|^2$$

for all $u \in \mathcal{H}$, and every $A_i \in L_\infty(\mathbf{R}^n, \mathcal{L}(\mathbf{C}^N))$ with $\|A_i\|_\infty \le \eta_i$.

Proof. For each $i = 1, 2$, define the functions $G_i : \mathbf{C} \longrightarrow \mathcal{L}(\mathcal{H})$ by $z \mapsto B_i + zA_i$, and let

$$U = \left\{z \in \mathbf{C} : |z| \le \min\left\{\eta_1\|A_1\|^{-1}, \eta_2\|A_2\|^{-1}\right\}\right\}.$$

For all $z \in U$ and $i = 1, 2$ we have

$$\mathrm{Re}((B_i + zA_i)u, u) \ge (\kappa_i - \eta_i)\|u\|^2$$

for every $u \in \mathcal{H}$, and therefore

$$\cos \sup_{u \in \mathsf{R}(\Gamma^*)\setminus\{0\}} |\arg((B_i + zA_i)u, u)| \ge \frac{\kappa_i - \eta_i}{\|B_i\| + \eta_i} = \cos\hat{\omega}_i.$$

We conclude that $G_1(z)$ and $G_2(z)$ satisfy (H2) with ω_1 and ω_2 replaced by $\hat{\omega}_1$ and $\hat{\omega}_2$, and thence by Theorem 6.4, that the function given by $z \mapsto \Pi_{G(z)}$ is holomorphic on U. The first claim of the theorem then follows by Schwarz's Lemma. The second claim is proved by a similar argument. □

7. Applications to Riemannian manifolds

We now consider applications to compact Riemannian manifolds M with metric g. For each $x \in M$ let $\wedge T_x^* M$ denote the complex exterior algebra over the cotangent space $T_x^* M$. We then let $\wedge T^* M$ and $\mathcal{L}_M$ denote the bundles over M whose fibres at each $x \in M$ are given by $\wedge T_x^* M$ and $\mathcal{L}(\wedge T_x^* M)$, respectively. We let $\mathcal{H} = L_2(\wedge T^* M)$ denote the collection of L_2 integrable sections of $\wedge T^* M$, and let $L_\infty(\mathcal{L}_M)$ denote the bounded measurable sections of $\mathcal{L}_M$. We let d_g^* denote the dual of d in $\mathcal{H}$, and consider the Hodge–Dirac operator $D_g := d + d_g^*$.

Theorem 7.1. *Let M be a compact Riemannian manifold with metric g, let $B \in L_\infty(\mathcal{L}_M)$ be invertible and so that there exists $\kappa > 0$ such that for almost every $x \in \mathbf{R}^n$, we have*

$$\mathrm{Re}(B(x)v, v) \geq \kappa |v|^2$$

for every $v \in \wedge T_x^* M$. Let $\omega < \mu < \frac{\pi}{2}$ where

$$\omega := \operatorname{ess\,sup}_{\substack{x \in M \\ v \in \wedge T_x^* M}} |\arg(B(x)v, v)|.$$

Then the operator $D_B = d + B^{-1} d_g^* B$ has a bounded S_μ^o holomorphic functional calculus in $\mathcal{H}$. The constant in this bound depends only on (M, g), $\|B\|$ and κ.

We begin the proof of Theorem 7.1 with a localization lemma. Let $\rho : U \longrightarrow B(0, 4\delta)$ be a diffeomorphism (or bi-Lipschitz mapping) for some open $U \subset M$, $\delta > 0$. Here we let $B(x, r)$ denote the ball in $\mathbf{R}^n$ with centre $x \in \mathbf{R}^n$ and radius $r > 0$, where n is the dimension of M. Let ρ^* denote the pullback by a function ρ. Let Θ_t^B be as given in Definition 4.7 with $\Gamma := d$ and $\Pi_B := D_B$.

Lemma 7.2. *We have*

$$\int_0^1 \left\| \Theta_t^B u \right\|^2 \frac{dt}{t} \lesssim \|u\|^2$$

for every $u \in \mathcal{H}$ with $\operatorname{supp} u \subset \rho^{-1}(B(0, \delta))$. *The bound here depends on δ, the hypothesis of Theorem 7.1, and the gradient bounds of ρ and ρ^{-1}.*

Proof. By Proposition 5.2 (adapted to the setting of a compact Riemannian manifold) we have that

$$\int_{M \setminus \rho^{-1}(B(0,2\delta))} \left| \Theta_t^B u \right|^2 dx \lesssim t^2 \|u\|^2$$

and therefore that

$$\int_0^1 \int_{M \setminus \rho^{-1}(B(0,2\delta))} \left| \Theta_t^B u \right|^2 dx \frac{dt}{t} \lesssim \int_0^1 t^2 \|u\|^2 \frac{dt}{t} \lesssim \|u\|^2.$$

It remains to show that

$$(36)\qquad \int_0^1 \int_{\rho^{-1}(B(0,2\delta))} \left|\Theta_t^B u\right|^2 dx \frac{dt}{t} \lesssim \|u\|^2.$$

We do this by pushing the problem onto $\mathbf{R}^n$.

Let $\hat{B}$ be the multiplication operator on $L_2(\mathbf{R}^n; \wedge_{\mathbf{C}}\mathbf{R}^n)$ that coincides with the identity on $\mathbf{R}^n \setminus B(0, 4\delta)$, and is otherwise fixed by the condition that $(\rho^{-1})^* D_B \rho^* = D_{\hat{B}}$, where we write $D_{\hat{B}} := d + (\hat{B})^{-1} d^* \hat{B}$, and where d^* denotes the adjoint of d under the standard Euclidean metric. Here $\hat{B} = (\rho_*/J_\rho) B \rho^*$, where $\rho_*/J_\rho : L_2(\wedge T^* U) \to L_2(B(0, 4\delta); \wedge_{\mathbf{C}}\mathbf{R}^n)$ is the adjoint of $\rho^* : L_2(B(0, 4\delta); \wedge_{\mathbf{C}}\mathbf{R}^n) \to L_2(\wedge T^* U)$ and ρ_* denotes the pushforward and J_ρ the Jacobian determinant of ρ. By our hypotheses on B we then have $B_1 = (\hat{B})^{-1}$ and $B_2 = \hat{B}$ satisfy (H2,3,5) with bounds that depend only on the hypotheses and the gradient bounds on ρ and ρ^{-1}. By Theorem 2.7 with $\{\Gamma = d, (\hat{B})^{-1}, \hat{B}\}$ we then have

$$(37)\qquad \int_0^1 \left\| t D_{\hat{B}} \left(\mathrm{I} + t^2 D_{\hat{B}}{}^2\right)^{-1} v \right\|^2 \frac{dt}{t} \lesssim \|u\|^2$$

where, here and after we fix $v = (\rho^{-1})^* u$.

To complete the proof it suffices to show that

(38)

$$\left\| (\rho^{-1})^* it D_B \left(\mathrm{I} + t^2 D_B{}^2\right)^{-1} \rho^* v - it D_{\hat{B}} \left(\mathrm{I} + t^2 D_{\hat{B}}{}^2\right)^{-1} v \right\|_{L_2(B(0,2\delta))} \lesssim t\|v\|$$

for every $0 < t \leq 1$, and that these bounds depend on the hypotheses and the gradient bounds on ρ and ρ^{-1}. (Indeed, we can then apply the triangle inequality with (37) to the bound the left-hand side of (36) by a controlled constant times $\|u\|^2 + \int_0^1 t^2 \|u\|^2 \frac{dt}{t} \lesssim \|u\|^2$.) To see (38) holds, let $\eta_1, \eta_2 : \mathbf{R}^n \longrightarrow \mathbf{R}$ be smooth cut-off functions with

$$\eta_i(x) = \begin{cases} 1 & \text{if } x \in B(0, (i+1)\delta) \\ 0 & \text{if } x \in \mathbf{R}^n \setminus B(0, (i+2)\delta) \end{cases}$$

and $|\nabla \eta_i| \leq 2\delta^{-1}$ for $i = 1, 2$. Observe that

$$\begin{aligned}
&(\mathrm{I} + itD_{\hat{B}})^{-1} v(x) - (\rho^{-1})^* (\mathrm{I} + itD_B)^{-1} \rho^* v(x) \\
&\quad = (\rho^{-1})^* (\mathrm{I} + itD_B)^{-1} \rho^* \eta_2 \big((\rho^{-1})^* (\mathrm{I} + itD_B) \rho^* \eta_1 \\
&\qquad\qquad - (\mathrm{I} + itD_{\hat{B}}) \big) (\mathrm{I} + itD_{\hat{B}})^{-1} v(x) \\
&\quad = (\rho^{-1})^* (\mathrm{I} + itD_B)^{-1} \rho^* \eta_2 (\mathrm{I} + itD_{\hat{B}}) (\eta_1 - 1) (\mathrm{I} + itD_{\hat{B}})^{-1} v(x) \\
&\quad = (\rho^{-1})^* (\mathrm{I} + itD_B)^{-1} \rho^* \eta_2 [itD_{\hat{B}}, \eta_1] (\mathrm{I} + itD_{\hat{B}})^{-1} v(x)
\end{aligned}$$

for almost every $x \in B(0, 2\delta)$, and by Proposition 5.2 has norm in $L_2(B(0, 2\delta); \wedge_{\mathbf{C}}\mathbf{R}^n)$ bounded by a constant multiple of $t\|\nabla \eta_1\| \|v\| \lesssim t\|v\|$, where the constant depends only on the assumed constants of the hypotheses. Estimate (38) now follows by writing $Q_t^B = \frac{1}{2i}(-R_t^B + R_{-t}^B)$. This completes the proof. □

Proof of Theorem 7.1. By Proposition 4.8, we need to establish (19) for every $u \in \mathsf{R}(\Gamma)$, for each case where $\{\Gamma, B^{-1}, B\}$ is given by $\{d, B^{-1}, B\}$, $\{d_g^*, B, B^{-1}\}$, $\{d_g^*, B^*, (B^{-1})^*\}$ and $\{d, (B^{-1})^*, B^*\}$. Let H be the Hodge-star operator on M and let N be the operator that changes sign of forms of odd degree. Then we have the unitary equivalence

$$H^*\big(d_g^* + BdB^{-1}\big)H = Nd + \tilde{B}^{-1}(Nd)^*\tilde{B}$$

where $\tilde{B} = H^*B^{-1}H$ satisfies the same hypothesis as B. Consequently, all four cases are essentially of the form $\{d, B^{-1}, B\}$ which we now consider.

Since M is compact we can use Lemma 7.2 with a standard local chart/partition of unity argument to deduce that

$$\int_0^1 \big\|\Theta_t^B u\big\|^2 \frac{dt}{t} \lesssim \|u\|^2.$$

Again because M is compact, and also because $u \in \mathsf{R}(d)$ and thus $P_t u \in \mathsf{R}(D)$, we can apply the Gaffney-Gårding inequality (see [29, Theorem 7.3.2]) to deduce that $\|P_t u\| \lesssim \|DP_t u\|$, and therefore conclude that

$$\int_1^\infty \big\|\Theta_t^B P_t u\big\|^2 \frac{dt}{t} \lesssim \int_1^\infty \|P_t u\|^2 \frac{dt}{t} \lesssim \int_1^\infty \|tDP_t u\|^2 \frac{dt}{t^3}$$
$$\lesssim \int_1^\infty \|u\|^2 \frac{dt}{t^3} \lesssim \|u\|^2.$$

This with Lemma 7.2 proves (19) and so completes the proof of Theorem 7.1. □

We now state an application of the above theorem. Given a smooth perturbation $g + h$ of g we let

$$|h_x| = \sup\{|h_x(v, v)| : v \in \wedge T_x M\,,\ g_x(v, v) = 1\}$$

for all $x \in M$, and define $\|h\|_\infty := \sup_{x\in M} |h_x|$. (This norm is equivalent to the one given in the Introduction, but more useful for our purposes.)

Theorem 7.3. *Let M be a compact Riemannian manifold with metric g, let $g + h$ be a measurable perturbation of g with $\|h\|_\infty < 1/4$, and let $0 < \mu < \frac{\pi}{2}$ be given by $\mu = \cos^{-1}(1/4)$. Then we have*

$$\|f(D_{g+h}) - f(D_g)\| \lesssim \|f\|_\infty \|h\|_\infty$$

for every bounded $f : S_\mu^o \cup \{0\} \longrightarrow \mathbf{C}$ holomorphic on S_μ^o. The constant in the above bound depends only on (M, g).

Remark 7.4. Lipschitz estimates like those in Theorem 7.3 also hold in terms of the quadratic estimates appearing in the second part of Theorem 6.5. These results are a consequence of the deeper fact that the mapping given by $z \mapsto f(D_{g+zh})$ depends holomorphically on $z \in \mathbf{C}$ when $|z| < \|h\|_\infty^{-1}$. These same results hold for any manifold bi-Lipschitz equivalent to Euclidean space, and follow by arguments similar to those used in this section. We leave the details to the reader.

Proof of Theorem 7.3. We can implicitly define $A \in L_\infty(\mathcal{L}_M)$ by the formula

$$((I + A(x))u(x), v(x))_g = (u(x), v(x))_{g+h}$$

for every $u, v \in L_2(\wedge_{\mathbf{C}} T^*M)$. Here we let $(\cdot, \cdot)_{g+h}$ and $(\cdot, \cdot)_g$ denote the metrics on M corresponding to $g+h$ and g, respectively. Our hypothesis on $g + h$ implies that $A \in L_\infty(\mathcal{L}_M)$ with $\|A\|_\infty = \|h\|_\infty \leq 1/4$ and therefore also

$$\|I - (I + A)^{-1}\|_\infty \leq \frac{\|h\|_\infty}{1 - \|h\|_\infty} \leq 1/3.$$

Moreover, we have

$$\begin{aligned}\left((I + A)d^*_{g+h}u, v\right)_g &= \left(d^*_{g+h}u, v\right)_{g+h} = (u, dv)_{g+h} \\ &= ((I + A)u, dv)_g = \left(d^*_g(I + A)u, v\right)_g\end{aligned}$$

for every $u, v \in L_2(\wedge_{\mathbf{C}} T^*M)$ with $u \in \mathsf{D}(d^*_{g+h})$ and $v \in \mathsf{D}(d)$, and therefore

$$D_{g+h} = d + d^*_{g+h} = d + (I + A)^{-1} d^*_g (I + A).$$

The desired result now follows from an application of Theorem 7.1 and results analogous to Theorem 6.5 with $A_2 = A$, $A_1 = (I + A)^{-1} - I$, $\eta_i = 1/2$, $\kappa_i = 1$, and $B_i = I$ for $i = 1, 2$. □

Appendix A. Further properties of the Hodge decomposition

In this appendix we verify the claim of Remark 6.2. As in Sect. 4, we assume that the triple of operators $\{\Gamma, B_1, B_2\}$ in a Hilbert space $\mathcal{H}$ satisfies properties (H1–3). We begin with a lemma.

Lemma A.1. *The Hodge projections can be represented as limits of resolvents in the following ways:*

$$\mathbf{P}^0_B u = \lim_{n\to\infty} (\mathrm{I} + in\Pi_B)^{-1} u = \lim_{n\to\infty} (\mathrm{I} - in\Pi_B)^{-1} u \quad \textit{for all} \quad u \in \mathcal{H};$$

$$\mathbf{P}^1_B u = \lim_{n\to\infty} in\Gamma^*_B (\mathrm{I} + in\Pi_B)^{-1} u = \lim_{n\to\infty} in\Gamma^*_B (-\mathrm{I} + in\Pi_B)^{-1} u \quad \textit{for all} \quad u \in \mathcal{H};$$

$$\mathbf{P}^1_B u = \lim_{n\to\infty} (\mathrm{I} + in\Pi_B)^{-1} in\Gamma u = \lim_{n\to\infty} (-\mathrm{I} + in\Pi_B)^{-1} in\Gamma u \quad \textit{for all} \quad u \in \mathsf{D}(\Gamma);$$

$$\mathbf{P}^2_B u = \lim_{n\to\infty} in\Gamma(\mathrm{I} + in\Pi_B)^{-1} u = \lim_{n\to\infty} in\Gamma(-\mathrm{I} + in\Pi_B)^{-1} u \quad \textit{for all} \quad u \in \mathcal{H};$$

$$\mathbf{P}^2_B u = \lim_{n\to\infty} (\mathrm{I} + in\Pi_B)^{-1} in\Gamma^*_B u = \lim_{n\to\infty} (-\mathrm{I} + in\Pi_B)^{-1} in\Gamma^*_B u \quad \textit{for all} \quad u \in \mathsf{D}(\Gamma^*_B).$$

Remark A.2. If $\mathsf{N}(\Pi_B) = \{0\}$, then $\mathbf{P}^1_B = \Gamma^*_B \Pi_B^{-1} = \Pi_B^{-1}\Gamma$ and $\mathbf{P}^2_B = \Gamma\Pi_B^{-1} = \Pi_B^{-1}\Gamma^*_B$ on the appropriate domains, in which case the proofs would be somewhat more direct.

Note that, by Proposition 2.5 and Lemma 4.2, each of the operator sequences $(\mathrm{I} + in\Pi_B)^{-1}$, $in\Gamma^*_B(\mathrm{I} + in\Pi_B)^{-1}$, etc, is uniformly bounded in n. It is not a-priori clear that their strong limits exist. This will be shown in the course of the proof.

Proof. We begin by showing that

$$Q^B_n u = n\Pi_B\big(\mathrm{I} + n^2{\Pi_B}^2\big)^{-1} u \to 0 \tag{39}$$

as $n \to \infty$ for every $u \in \mathcal{H}$. The expression on the left vanishes if $u \in \mathsf{N}(\Pi_B)$, so it suffices to consider the case when $u \in \overline{\mathsf{R}(\Pi_B)}$. If $u = \Pi_B v \in \mathsf{R}(\Pi_B)$, then

$$\|Q^B_n u\| = \|Q^B_n \Pi_B v\| = \tfrac{1}{n}\|v - P^B_n v\| \lesssim \tfrac{1}{n}\|v\| \to 0$$

as $n \to \infty$. Since, by Proposition 2.5, the sequence $\|Q^B_n\|$ is uniformly bounded, we conclude by a standard continuity argument that (39) holds for every $u \in \overline{\mathsf{R}(\Pi_B)}$.

Define operators T_0, T_1 and T_2 on $\mathcal{H}$ by

$$T_0 u = \lim_{n\to\infty} (\mathrm{I} + in\Pi_B)^{-1} u,$$

$$T_1 u = \lim_{n\to\infty} in\Gamma^*_B(\mathrm{I} + in\Pi_B)^{-1} u \quad \text{and} \quad T_2 u = \lim_{n\to\infty} in\Gamma(\mathrm{I} + in\Pi_B)^{-1} u$$

whenever $u \in \mathcal{H}$ and the corresponding limit exists. We next show that

$$T_0 u = \lim_{n\to\infty} (\mathrm{I} - in\Pi_B)^{-1} u, \tag{40}$$

$$\begin{aligned} T_1 u = \lim_{n\to\infty} in\Gamma^*_B(-\mathrm{I} + in\Pi_B)^{-1} u &= \lim_{n\to\infty} (\mathrm{I} + in\Pi_B)^{-1} in\Gamma u \\ &= \lim_{n\to\infty} (-\mathrm{I} + in\Pi_B)^{-1} in\Gamma u \end{aligned} \tag{41}$$

and

$$T_2u = \lim_{n\to\infty} in\Gamma(-\mathrm{I}+in\Pi_B)^{-1}u = \lim_{n\to\infty}(\mathrm{I}+in\Pi_B)^{-1}in\Gamma_B^*u \tag{42}$$
$$= \lim_{n\to\infty}(-\mathrm{I}+in\Pi_B)^{-1}in\Gamma_B^*u$$

whenever $u \in \mathcal{H}$ (and when required, $u \in \mathsf{D}(\Gamma)$ or $u \in \mathsf{D}(\Gamma_B^*)$) and the corresponding limit exists. Here we interpret the above as saying that if one such limit exists, then the limits that are indicated to be equal, also exist.

Equation (40) follows by (39) and the fact that

$$(\mathrm{I}+in\Pi_B)^{-1} - (\mathrm{I}-in\Pi_B)^{-1} = -2in\Pi_B\big(\mathrm{I}+n^2\Pi_B^2\big)^{-1}.$$

To see the first equality in (42), observe that by (39) and Lemma 4.2 we have

$$\big\|in\Gamma(\mathrm{I}+in\Pi_B)^{-1}u - in\Gamma(-\mathrm{I}+in\Pi_B)^{-1}u\big\| = \big\|2in\Gamma(\mathrm{I}+n^2{\Pi_B}^2)^{-1}u\big\|$$
$$\lesssim \big\|n\Pi_B(\mathrm{I}+n^2{\Pi_B}^2)^{-1}u\big\| \to 0$$

as $n \to \infty$. The second equality in (42) follows from (39) and the identity

$$\begin{aligned} &in\Gamma(-\mathrm{I}+in\Pi_B)^{-1}u - (\mathrm{I}+in\Pi_B)^{-1}in\Gamma_B^*u \\ &= (\mathrm{I}+in\Pi_B)^{-1}\big((\mathrm{I}+in\Pi_B)in\Gamma - in\Gamma_B^*(-\mathrm{I}+in\Pi_B)\big)(-\mathrm{I}+in\Pi_B)^{-1}u \\ &= (\mathrm{I}+in\Pi_B)^{-1}(in\Gamma + in\Gamma_B^*)(-\mathrm{I}+in\Pi_B)^{-1}u \\ &= -in\Pi_B(\mathrm{I}+n^2{\Pi_B}^2)^{-1}u = -iQ_n^Bu \end{aligned} \tag{43}$$

for all $u \in \mathsf{D}(\Gamma_B^*)$.

The remaining equality in (42) as well as Equation (41) can be proved by similar arguments.

We note that $T_0u = u$ when $u \in \mathsf{N}(\Pi_B)$, and, by adapting the proof of (39), that $T_0u = 0$ when $u \in \mathsf{R}(\Pi_B)$ and hence when $u \in \overline{\mathsf{R}(\Pi_B)}$. Therefore $T_0 = \mathbf{P}_B^0$.

Now investigate T_1. By (41), $T_1u = 0$ when $u \in \mathsf{N}(\Gamma)$. If $u \in \mathsf{R}(\Gamma_B^*)$, let $u = \Gamma_B^*v$, where, by Proposition 2.2, we may assume that $v \in \overline{\mathsf{R}(\Gamma)}$. Using the facts that $T_0v = 0$ and that Γ_B^* is closed, we obtain

$$T_1u = \lim_{n\to\infty} in\Gamma_B^*(\mathrm{I}+in\Pi_B)^{-1}\Pi_Bv = \lim_{n\to\infty}\Gamma_B^*\big(\mathrm{I}-(\mathrm{I}+in\Pi_B)^{-1}\big)v \tag{44}$$
$$= \Gamma_B^*v = u.$$

By a standard argument, we find that $T_1u = u$ when $u \in \overline{\mathsf{R}(\Gamma_B^*)}$. Therefore $T_1 = \mathbf{P}_B^1$.

Similarly, $T_2u = 0$ when $u \in \mathsf{N}(\Gamma_B^*)$, and $T_2u = u$ when $u \in \overline{\mathsf{R}(\Gamma)}$, so that $T_2 = \mathbf{P}_B^2$. □

Define operators $\tilde{\mathbf{P}}^1_B$ and $\tilde{\mathbf{P}}^2_B$ on $\mathcal{H}$ by

$$\tilde{\mathbf{P}}^1_B u = \lim_{n\to\infty} in\Gamma^* B_2(\mathrm{I}+in\Pi_B)^{-1}u \quad \text{for all} \quad u \in \mathcal{H}\,,$$
$$\tilde{\mathbf{P}}^2_B u = \lim_{n\to\infty} (\mathrm{I}+in\Pi_B)^{-1}inB_1\Gamma^* u \quad \text{for all} \quad u \in \mathsf{D}(\Gamma^*).$$

The fact that the limits defining $\tilde{\mathbf{P}}^1_B$ and $\tilde{\mathbf{P}}^2_B$ exist and define bounded operators, as well as the fact that $\mathbf{P}^1_B = B_1\tilde{\mathbf{P}}^1_B$ and $\mathbf{P}^2_B = \tilde{\mathbf{P}}^2_B B_2$, now follow from (12), (13). We remark that for (32), (33) and (34) to be true, the sum of the right-hand sides must equal zero, which requires

$$\mathbf{P}^2_B A_1 \tilde{\mathbf{P}}^1_B + \tilde{\mathbf{P}}^2_B A_2 \mathbf{P}^1_B = 0.$$

Indeed, this is a consequence of the assumption $\Gamma^* B_2 B_1 \Gamma^* = 0$.

Proof of Remark 6.2. Let $T^n_0 = (\mathrm{I}+in\Pi_B)^{-1}$, $T^n_1 = in\Gamma^*_B(\mathrm{I}+in\Pi_B)^{-1}$ and $T^n_2 = in\Gamma(\mathrm{I}+in\Pi_B)^{-1}$. By Proposition 2.2 and Lemma 4.2 we have that the mappings $z \mapsto T^n_0$, $z \mapsto T^n_1$ and $z \mapsto T^n_2$ are uniformly bounded. Furthermore Lemma A.1 shows that $T^n_0 \to \mathbf{P}^0_B$, $T^n_1 \to \mathbf{P}^1_B$ and $T^n_2 \to \mathbf{P}^2_B$ strongly. Thus it will follow from Lemma 6.3 that $z \mapsto \mathbf{P}^0_{B(z)}$, $z \mapsto \mathbf{P}^1_{B(z)}$ and $z \mapsto \mathbf{P}^2_{B(z)}$ are holomorphic with derivatives as stated in (32), (33) and (34) once we prove that T^n_i, $i = 1, 2, 3$ are holomorphic functions and that $\frac{d}{dz}T^n_i$ have as strong limits the right hand sides in (32), (33) and (34) respectively.

For T^n_0, we see that

$$\begin{aligned}
\frac{d}{dz}T^n_0 u &= \frac{d}{dz}(\mathrm{I}+in\Pi_B)^{-1}u \\
&= -(\mathrm{I}+in\Pi_B)^{-1}A_1 in\Gamma^* B_2(\mathrm{I}+in\Pi_B)^{-1}u \\
&\quad - (\mathrm{I}+in\Pi_B)^{-1}B_1 in\Gamma^* A_2(\mathrm{I}+in\Pi_B)^{-1}u \\
&\to \big(-\mathbf{P}^0_B A_1 \tilde{\mathbf{P}}^1_B - \tilde{\mathbf{P}}^2_B A_2 \mathbf{P}^0_B\big)u.
\end{aligned} \tag{45}$$

For T^n_1, we see that when $u \in \mathsf{D}(\Gamma)$,

$$\begin{aligned}
\frac{d}{dz}T^n_1 u &= \frac{d}{dz}(\mathrm{I}+in\Pi_B)^{-1}in\Gamma u \\
&= -(\mathrm{I}+in\Pi_B)^{-1}A_1 in\Gamma^* B_2(\mathrm{I}+in\Pi_B)^{-1}in\Gamma u \\
&\quad - (\mathrm{I}+in\Pi_B)^{-1}B_1 in\Gamma^* A_2(\mathrm{I}+in\Pi_B)^{-1}in\Gamma u.
\end{aligned} \tag{46}$$

The second term on the right-hand side converges to $-\tilde{\mathbf{P}}^2_B A_2 \mathbf{P}^1_B u$. In order to calculate the first term on the right-hand side, we note by an argument similar to (43) that

$$\begin{aligned}
&in\Gamma^*_B(\mathrm{I}+in\Pi_B)^{-1} - (-\mathrm{I}+in\Pi_B)^{-1}in\Gamma \\
&\qquad = -in\Pi_B(\mathrm{I}+in\Pi_B)^{-1}(-\mathrm{I}+in\Pi_B)^{-1}.
\end{aligned}$$

This with the fact that $\Gamma^2 = 0$ implies that

$$in\Gamma_B^*(\mathrm{I}+in\Pi_B)^{-1}in\Gamma = -in\Pi_B(\mathrm{I}+in\Pi_B)^{-1}(-\mathrm{I}+in\Pi_B)^{-1}in\Gamma$$
$$\to \big(\mathbf{P}_B^0 - I\big)\mathbf{P}_B^1 = -\mathbf{P}_B^1$$

as $n \to \infty$. Therefore the first term on the right-hand side of (46) converges to $\mathbf{P}_B^0 A_1 \tilde{\mathbf{P}}_B^1 u$ as $n \to \infty$.

A similar argument shows that $\frac{d}{dz}T_2^n u \to (-\mathbf{P}_B^2 A_1 \tilde{\mathbf{P}}_B^1 + \tilde{\mathbf{P}}_B^2 A_2 \mathbf{P}_B^0)u$. This completes the proof. □

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