Invent. math. 162, 551–647 (2005)
DOI: 10.1007/s00222-005-0454-z

Inventiones mathematicae

Pullbacks of Saito-Kurokawa lifts

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Oblatum 28-IX-2004 & 11-IV-2005
Published online: 18 July 2005 –

Introduction

Pullbacks of Siegel Eisenstein series have been studied by Garrett [17], [18], Böcherer [6], [7], Heim [26], and play a key role in the proof of the algebraicity of critical values of certain automorphic L-functions. More generally, one might consider pullbacks of Siegel cusp forms. For example, Ikeda [30] gave a conjectural formula for pullbacks of Ikeda lifts [29] in terms of critical values of L-functions for $\mathrm{Sp}_n \times \mathrm{GL}_2$. Also, the Gross-Prasad conjecture [22], [23], [8], [27, §8] would relate pullbacks of Siegel cusp forms of degree 2 to central critical values of L-functions for $\mathrm{GSp}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2$. Indeed, Böcherer, Furusawa, and Schulze-Pillot [8] gave an explicit formula for pullbacks of Yoshida lifts [57]. In this paper, we give an explicit formula for pullbacks of Saito-Kurokawa lifts and prove the algebraicity of central critical values of certain L-functions for $\mathrm{Sp}_1 \times \mathrm{GL}_2$.

To be precise, let κ be an odd positive integer. Let $f \in S_{2\kappa}(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized Hecke eigenform and $h \in S^+_{\kappa+1/2}(\Gamma_0(4))$ a Hecke eigenform associated to f by the Shimura correspondence. Let $F \in S_{\kappa+1}(\mathrm{Sp}_2(\mathbb{Z}))$ be the Saito-Kurokawa lift of h. For each normalized Hecke eigenform $g \in S_{\kappa+1}(\mathrm{SL}_2(\mathbb{Z}))$, we consider the period integral $\langle F|_{\mathfrak{H}\times\mathfrak{H}}, g \times g\rangle$ given by

$$\langle F|_{\mathfrak{H}\times\mathfrak{H}}, g \times g\rangle = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}} \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}} F\left(\begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}\right) \overline{g(\tau_1)g(\tau_2)} y_1^{\kappa-1} y_2^{\kappa-1}\, d\tau_1\, d\tau_2.$$

Let $\Lambda(s, \mathrm{Sym}^2(g) \otimes f)$ be the completed L-function given by

$$\Lambda(s, \mathrm{Sym}^2(g) \otimes f) = \Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s-\kappa)\Gamma_{\mathbb{C}}(s-2\kappa+1)L(s, \mathrm{Sym}^2(g) \otimes f),$$

where $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. It satisfies the functional equation

$$\Lambda(4\kappa - s, \mathrm{Sym}^2(g) \otimes f) = \Lambda(s, \mathrm{Sym}^2(g) \otimes f).$$

Our main result is as follows.

Theorem 2.1.

$$\Lambda(2\kappa, \mathrm{Sym}^2(g) \otimes f) = 2^{\kappa+1} \frac{\langle f, f\rangle}{\langle h, h\rangle} \frac{|\langle F|_{\mathfrak{H}\times\mathfrak{H}}, g \times g\rangle|^2}{\langle g, g\rangle^2}.$$

Theorem 2.1 has an application to Deligne's conjecture [13].

Corollary 2.6. *For* $\sigma \in \mathrm{Aut}(\mathbb{C})$,

$$\left(\frac{\Lambda(2\kappa, \mathrm{Sym}^2(g) \otimes f)}{\langle g, g\rangle^2 c^+(f)}\right)^\sigma = \frac{\Lambda(2\kappa, \mathrm{Sym}^2(g^\sigma) \otimes f^\sigma)}{\langle g^\sigma, g^\sigma\rangle^2 c^+(f^\sigma)}.$$

Here $c^+(f)$ *is the period of* f *as in* [52].

This paper is organized as follows. In Sect. 1, we review the theory of Saito-Kurokawa lifts. In Sect. 2, we state our main result. In Sects. 3 and 4, we recall the basic facts about automorphic forms and theta lifts, respectively. In Sect. 5, we state three seesaw identities, and show that these identities and the Kohnen-Zagier formula imply the main theorem. The rest of this paper is devoted to the proof of these identities. First, we study the Jacquet-Langlands-Shimizu correspondence in Sect. 6 and the Saito-Kurokawa lifting in Sect. 7. In Sect. 8, we prove an identity for the seesaw

$$\begin{array}{ccc} \mathrm{O}(3,2) & & \mathrm{SL}_2 \times \widetilde{\mathrm{SL}}_2 \\ \Big| & \times & \Big| \\ \mathrm{O}(2,2) \times \mathrm{O}(1) & & \widetilde{\mathrm{SL}}_2 \end{array} .$$

Next, we study the Shimura-Waldspurger correspondence in Sect. 9 and the base change for GL_2 from $\mathbb{Q}$ to an imaginary quadratic field $\mathcal{K}$ in Sect. 10. In Sect. 11, we prove an identity for the seesaw

$$\begin{array}{ccc} \widetilde{\mathrm{SL}}_2 \times \widetilde{\mathrm{SL}}_2 & & \mathrm{O}(3,1) \\ \Big| & \times & \Big| \\ \mathrm{SL}_2 & & \mathrm{O}(2,1) \times \mathrm{O}(1) \end{array} .$$

Finally, we study the local zeta integrals of Garrett, Piatetski-Shapiro and Rallis in Sect. 12. In Sect. 13, we prove an identity for the seesaw

$$\begin{array}{ccc} \mathrm{Sp}_3 & & \mathrm{R}_{\mathcal{K}/\mathbb{Q}}\, \mathrm{O}(2,2) \times \mathrm{O}(2,2) \\ \Big| & \times & \Big| \\ \mathrm{R}_{\mathcal{K}/\mathbb{Q}}\, \mathrm{SL}_2 \times \mathrm{SL}_2 & & \mathrm{O}(2,2) \end{array} .$$

Notation. Let F be a local field of characteristic not 2 and ψ a non-trivial additive character of F. Let $(\ ,\ )_F$ denote the quadratic Hilbert symbol of F and $\gamma_F(\psi)$ the Weil index [47]. For $a \in F^\times$, define a non-trivial additive character $a\psi$ of F by $(a\psi)(x) = \psi(ax)$, and put $\gamma_F(a, \psi) = \gamma_F(a\psi)/\gamma_F(\psi)$. See Sect. A.1 for more details.

Let $\psi_0 = \otimes_v \psi_v$ be the non-trivial additive character of $\mathbb{A}_\mathbb{Q}/\mathbb{Q}$ defined as follows:

- If $v = p$, then $\psi_p(x) = e^{-2\pi\sqrt{-1}x}$ for $x \in \mathbb{Z}[p^{-1}]$.
- If $v = \infty$, then $\psi_\infty(x) = e^{2\pi\sqrt{-1}x}$ for $x \in \mathbb{R}$.

We call ψ_0 (resp. ψ_v) the standard additive character of $\mathbb{A}_\mathbb{Q}$ (resp. $\mathbb{Q}_v$).

For $n \in \mathbb{N}$, let

$$\mathrm{GSp}_n = \left\{ g \in \mathrm{GL}_{2n} \,\middle|\, g \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix} {}^t g = \nu(g) \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix},\ \nu(g) \in \mathbb{G}_m \right\}$$

be the symplectic similitude group and $\nu : \mathrm{GSp}_n \to \mathbb{G}_m$ the scale map. Let $\mathrm{Sp}_n = \ker(\nu)$ denote the symplectic group. When $n = 1$, $\mathrm{GSp}_1 = \mathrm{GL}_2$ and $\mathrm{Sp}_1 = \mathrm{SL}_2$. Note that $\nu(g) = \det(g)$ for $g \in \mathrm{GL}_2$.

Let

$$\mathbf{B} = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \mathrm{GL}_2 \right\} \quad \text{and} \quad B = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \mathrm{SL}_2 \right\}$$

be the standard Borel subgroups of GL_2 and SL_2, respectively. Let

$$U = \left\{ u(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \,\middle|\, x \in \mathbb{G}_a \right\}$$

be the unipotent radical of $\mathbf{B}$ (and of B). We write

$$a(\nu) = \begin{pmatrix} \nu & 0 \\ 0 & 1 \end{pmatrix}, \quad d(\nu) = \begin{pmatrix} 1 & 0 \\ 0 & \nu \end{pmatrix}, \quad t(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}.$$

Put

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad k_1 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

Let

$$\mathrm{SO}(2) = \left\{ k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \,\middle|\, \theta \in \mathbb{R}/2\pi\mathbb{Z} \right\},$$

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \,\middle|\, \alpha, \beta \in \mathbb{C},\ |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

For $n \in \mathbb{Z}_{\geq 0}$, let ρ_n denote the irreducible representation of $\mathrm{SU}(2)$ of dimension $n + 1$. Put

$$k_\infty = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -\sqrt{-1} \\ -\sqrt{-1} & 1 \end{pmatrix} \in \mathrm{SU}(2).$$

Let $\mathfrak{su}(2)$ be the Lie algebra of SU(2) and $\mathfrak{su}(2)_{\mathbb{C}} = \mathfrak{su}(2) \otimes_{\mathbb{R}} \mathbb{C}$ its complexification. We define $H, X, Y \in \mathfrak{su}(2)_{\mathbb{C}}$ by

$$\begin{aligned} H &= -X_1 \otimes \sqrt{-1}, \\ X &= \frac{1}{2}(X_2 \otimes \sqrt{-1} - X_3 \otimes 1), \\ Y &= \frac{1}{2}(X_2 \otimes \sqrt{-1} + X_3 \otimes 1), \end{aligned}$$

where

$$X_1 = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H.$$

The Siegel upper half space $\mathfrak{H}_n$ is defined by

$$\mathfrak{H}_n = \left\{ Z \in \mathrm{M}_n(\mathbb{C}) \mid {}^t Z = Z, \mathrm{Im}(Z) > 0 \right\}.$$

When $n = 1$, $\mathfrak{H} = \mathfrak{H}_1$ is the upper half plane. For $\tau = x + \sqrt{-1}y \in \mathfrak{H}$, put $d\tau = dx\, dy$ and $q = e^{2\pi\sqrt{-1}\tau}$. Here dx, dy are the Lebesgue measures. Note that

$$\mathrm{vol}\left(\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}, y^{-2}\, d\tau\right) = \frac{\pi}{3}.$$

For $z \in \mathbb{C}$, let $|z| = \sqrt{z\bar{z}}$, and define $z^{1/2}$ so that $-\pi/2 < \arg(z^{1/2}) \leq \pi/2$. Let $\left(\frac{c}{d}\right)$ be the quadratic residue symbol as in [50]. We put $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. Let $K_\nu(z)$ denote the modified Bessel function and ${}_2F_1(\alpha, \beta; \gamma; z)$ the hypergeometric function. Recall that

$$K_\nu(z) = \frac{1}{2}\int_0^\infty e^{-z(t+t^{-1})/2} t^{\nu-1}\, dt$$

if $\mathrm{Re}(z) > 0$, and

$${}_2F_1(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{n!\Gamma(\gamma+n)} z^n$$

if $|z| < 1$.

Measures. Let F be a number field. For a connected linear algebraic group G over F, we take the Tamagawa measure on $G(\mathbb{A}_F)$. If V is a quadratic space over F, the Haar measure on $\mathrm{O}(V)(\mathbb{A}_F)$ is normalized so that

$$\mathrm{vol}(\mathrm{O}(V)(F)\backslash \mathrm{O}(V)(\mathbb{A}_F)) = 1.$$

Let $F = \mathbb{Q}$. For each prime p, let dx_p (resp. $d^\times a_p$) be the Haar measure on $\mathbb{Q}_p$ (resp. $\mathbb{Q}_p^\times$) with

$$\mathrm{vol}(\mathbb{Z}_p, dx_p) = \mathrm{vol}\left(\mathbb{Z}_p^\times, d^\times a_p\right) = 1.$$

Let dx_∞, da_∞ be the Lebesgue measures on $\mathbb{R}$ and $d^\times a_\infty = |a_\infty|_{\mathbb{R}}^{-1}\, da_\infty$ the Haar measure on $\mathbb{R}^\times$. We take the Haar measure $dk = \prod_v dk_v$ on $\mathrm{SO}(2)\,\mathrm{SL}_2(\hat{\mathbb{Z}})$ with

$$\mathrm{vol}(\mathrm{SO}(2)\,\mathrm{SL}_2(\hat{\mathbb{Z}}), dk) = \xi_{\mathbb{Q}}(2)^{-1}.$$

Here $\xi_{\mathbb{Q}}(s) = \Gamma_{\mathbb{R}}(s)\zeta(s)$. Define a Haar measure dg_v on $\mathrm{SL}_2(\mathbb{Q}_v)$ by

$$dg_v = |a_v|_{\mathbb{Q}_v}^{-2}\, dx_v\, d^\times a_v\, dk_v$$

for $g_v = u(x_v)t(a_v)k_v$ with $x_v \in \mathbb{Q}_v$, $a_v \in \mathbb{Q}_v^\times$,

$$k_v \in \begin{cases} \mathrm{SL}_2(\mathbb{Z}_p) & \text{if } v = p, \\ \mathrm{SO}(2) & \text{if } v = \infty. \end{cases}$$

Then the product measure $dg = \prod_v dg_v$ is the Tamagawa measure on $\mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}})$.

Let $F = \mathcal{K}$ be an imaginary quadratic field with discriminant $-D < 0$ and $\mathcal{O}$ the ring of integers of $\mathcal{K}$. For each prime p, let dx_p (resp. $d^\times a_p$) be the Haar measure on $\mathcal{K}_p$ (resp. $\mathcal{K}_p^\times$) with

$$\mathrm{vol}(\mathcal{O}_p, dx_p) = \mathrm{vol}\left(\mathcal{O}_p^\times, d^\times a_p\right) = 1.$$

Let dx_∞, da_∞ be the Lebesgue measures on $\mathbb{C}$ and $d^\times a_\infty = |a_\infty|_{\mathbb{C}}^{-1}\, da_\infty$ the Haar measure on $\mathbb{C}^\times$. We take the Haar measure $dk = \prod_v dk_v$ on $\mathrm{SU}(2)\,\mathrm{SL}_2(\hat{\mathcal{O}})$ with

$$\mathrm{vol}(\mathrm{SU}(2)\,\mathrm{SL}_2(\hat{\mathcal{O}}), dk) = \pi^{-1}\xi_{\mathcal{K}}(2)^{-1}.$$

Here $\xi_{\mathcal{K}}(s) = D^{s/2}\Gamma_{\mathbb{C}}(s)\zeta_{\mathcal{K}}(s)$. Define a Haar measure dg_v on $\mathrm{SL}_2(\mathcal{K}_v)$ by

$$dg_v = \begin{cases} |a_p|_{\mathcal{K}_p}^{-2}\, dx_p\, d^\times a_p\, dk_p & \text{if } v = p, \\ |a_\infty|_{\mathbb{C}}^{-2}\big(2D^{-1/2}\, dx_\infty\big)(2\, d^\times a_\infty)\, dk_\infty & \text{if } v = \infty, \end{cases}$$

for $g_v = u(x_v)t(a_v)k_v$ with $x_v \in \mathcal{K}_v$, $a_v \in \mathcal{K}_v^\times$,

$$k_v \in \begin{cases} \mathrm{SL}_2(\mathcal{O}_p) & \text{if } v = p, \\ \mathrm{SU}(2) & \text{if } v = \infty. \end{cases}$$

Then the product measure $dg = \prod_v dg_v$ is the Tamagawa measure on $\mathrm{SL}_2(\mathbb{A}_{\mathcal{K}})$.

1. Saito-Kurokawa lifts

In this section, we review the theory of Saito-Kurokawa lifts [38], [40], [1], [59].

Let κ be a positive integer. We put

$$\Gamma_0(4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, c \equiv 0 \bmod 4 \right\}$$

and define a factor of automorphy $j(\gamma, \tau)$ by

$$j(\gamma, \tau) = \left(\frac{c}{d}\right)\epsilon_d^{-1}(c\tau + d)^{1/2} \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4), \ \tau \in \mathfrak{H}.$$

Here

$$\epsilon_d = \begin{cases} 1 & \text{if } d \equiv 1 \bmod 4, \\ \sqrt{-1} & \text{if } d \equiv 3 \bmod 4. \end{cases}$$

Let $M_{\kappa+1/2}(\Gamma_0(4))$ (resp. $S_{\kappa+1/2}(\Gamma_0(4))$) denote the space of all holomorphic functions h on $\mathfrak{H}$ which satisfy

$$h(\gamma(\tau)) = j(\gamma, \tau)^{2\kappa+1} h(\tau)$$

for every $\gamma \in \Gamma_0(4)$ and which are holomorphic (resp. which vanish) at every cusp. Kohnen [32] introduced the space $M^+_{\kappa+1/2}(\Gamma_0(4))$ of all modular forms

$$h(\tau) = \sum_{n=0}^{\infty} c_h(n) q^n \in M_{\kappa+1/2}(\Gamma_0(4))$$

such that

$$c_h(n) = 0 \quad \text{unless } (-1)^\kappa n \equiv 0, 1 \bmod 4.$$

We put

$$S^+_{\kappa+1/2}(\Gamma_0(4)) = S_{\kappa+1/2}(\Gamma_0(4)) \cap M^+_{\kappa+1/2}(\Gamma_0(4)).$$

Assume that κ is odd. There is an injective linear map

$$\begin{aligned} S^+_{\kappa+1/2}(\Gamma_0(4)) &\longrightarrow S_{\kappa+1}(\mathrm{Sp}_2(\mathbb{Z})). \\ h &\longmapsto F \end{aligned}$$

Here the Fourier expansion of F is given by

$$F(Z) = \sum_B A(B) e^{2\pi\sqrt{-1}\,\mathrm{tr}(BZ)}$$

for $Z \in \mathfrak{H}_2$, where B runs over all positive definite half-integral symmetric matrices of size 2, and

$$A\left(\begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}\right) = \sum_{d|(n,r,m)} d^{\kappa} c_h\left(\frac{4nm-r^2}{d^2}\right)$$

for $n, r, m \in \mathbb{Z}$ such that $n, m > 0$ and $4nm > r^2$. We call F the Saito-Kurokawa lift of h.

Lemma 1.1. *Let $h \in S^+_{\kappa+1/2}(\Gamma_0(4))$. Let $F \in S_{\kappa+1}(\mathrm{Sp}_2(\mathbb{Z}))$ be the Saito-Kurokawa lift of h and $F|_{\mathfrak{H}\times\mathfrak{H}} \in S_{\kappa+1}(\mathrm{SL}_2(\mathbb{Z})) \otimes S_{\kappa+1}(\mathrm{SL}_2(\mathbb{Z}))$ the pullback of F via the embedding*

$$\mathfrak{H} \times \mathfrak{H} \longrightarrow \mathfrak{H}_2.$$
$$(\tau_1, \tau_2) \longmapsto \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}$$

Then

$$(T(p) \otimes \mathrm{id})(F|_{\mathfrak{H}\times\mathfrak{H}}) = (\mathrm{id} \otimes T(p))(F|_{\mathfrak{H}\times\mathfrak{H}})$$

for all primes p. Here $T(p)$ is the Hecke operator on $S_{\kappa+1}(\mathrm{SL}_2(\mathbb{Z}))$.

Proof. Since

$$F\left(\begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}\right) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \sum_{\substack{r\in\mathbb{Z} \\ r^2<4nm}} \sum_{d|(n,r,m)} d^{\kappa} c_h\left(\frac{4nm-r^2}{d^2}\right) q_1^n q_2^m,$$

$(T(p) \otimes \mathrm{id})(F|_{\mathfrak{H}\times\mathfrak{H}})(\tau_1, \tau_2)$ is equal to

$$\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \sum_{\substack{r\in\mathbb{Z} \\ r^2<4pnm}} \sum_{d|(pn,r,m)} d^{\kappa} c_h\left(\frac{4pnm-r^2}{d^2}\right) q_1^n q_2^m \\
&+ p^{\kappa} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \sum_{\substack{r\in\mathbb{Z} \\ r^2<4nm}} \sum_{d|(n,r,m)} d^{\kappa} c_h\left(\frac{4nm-r^2}{d^2}\right) q_1^{pn} q_2^m \\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \sum_{\substack{r\in\mathbb{Z} \\ r^2<4pnm}} \sum_{\substack{d|(n,r,m) \\ p\nmid d}} d^{\kappa} c_h\left(\frac{4pnm-r^2}{d^2}\right) q_1^n q_2^m \\
&+ p^{\kappa} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \sum_{\substack{r\in\mathbb{Z} \\ r^2<4nm}} \sum_{d|(n,r,m)} d^{\kappa} c_h\left(\frac{4nm-r^2}{d^2}\right) q_1^{n} q_2^{pm} \\
&+ p^{\kappa} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \sum_{\substack{r\in\mathbb{Z} \\ r^2<4nm}} \sum_{d|(n,r,m)} d^{\kappa} c_h\left(\frac{4nm-r^2}{d^2}\right) q_1^{pn} q_2^m.
\end{aligned}$$

This completes the proof. □

2. Statement of the main theorem

Let κ be an odd positive integer. Let

$$f(\tau) = \sum_{n=1}^{\infty} a_f(n) q^n \in S_{2\kappa}(\mathrm{SL}_2(\mathbb{Z}))$$

be a normalized Hecke eigenform and

$$h(\tau) = \sum_{n=1}^{\infty} c_h(n) q^n \in S^+_{\kappa+1/2}(\Gamma_0(4))$$

a Hecke eigenform associated to f by the Shimura correspondence [50], [32]. Note that h is unique up to scalars. Let $F \in S_{\kappa+1}(\mathrm{Sp}_2(\mathbb{Z}))$ be the Saito-Kurokawa lift of h. For each normalized Hecke eigenform

$$g(\tau) = \sum_{n=1}^{\infty} a_g(n) q^n \in S_{\kappa+1}(\mathrm{SL}_2(\mathbb{Z})),$$

we consider the period integral $\langle F|_{\mathfrak{H}\times\mathfrak{H}}, g \times g\rangle$ given by

$$\begin{aligned}&\langle F|_{\mathfrak{H}\times\mathfrak{H}}, g \times g\rangle \\ &= \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}} \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}} F\left(\begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}\right) \overline{g(\tau_1) g(\tau_2)} y_1^{\kappa-1} y_2^{\kappa-1}\, d\tau_1\, d\tau_2.\end{aligned}$$

Define the Petersson norms of f, g, h by

$$\begin{aligned}\langle f, f\rangle &= \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}} |f(\tau)|^2 y^{2\kappa-2}\, d\tau,\\ \langle g, g\rangle &= \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}} |g(\tau)|^2 y^{\kappa-1}\, d\tau,\\ \langle h, h\rangle &= \frac{1}{6}\int_{\Gamma_0(4)\backslash\mathfrak{H}} |h(\tau)|^2 y^{\kappa-3/2}\, d\tau,\end{aligned}$$

respectively.

For each prime p, let $\{\alpha_p, \alpha_p^{-1}\}$ and $\{\beta_p, \beta_p^{-1}\}$ denote the Satake parameters of g and f at p, respectively. Then

$$\begin{aligned}1 - a_g(p) X + p^{\kappa} X^2 &= \left(1 - p^{\kappa/2}\alpha_p X\right)\left(1 - p^{\kappa/2}\alpha_p^{-1} X\right),\\ 1 - a_f(p) X + p^{2\kappa-1} X^2 &= \left(1 - p^{\kappa-1/2}\beta_p X\right)\left(1 - p^{\kappa-1/2}\beta_p^{-1} X\right).\end{aligned}$$

We put

$$A_p = p^{\kappa}\begin{pmatrix} \alpha_p^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha_p^{-2} \end{pmatrix}, \quad B_p = p^{\kappa-1/2}\begin{pmatrix} \beta_p & 0 \\ 0 & \beta_p^{-1} \end{pmatrix}.$$

Define the L-function $L(s, \mathrm{Sym}^2(g) \otimes f)$ by an Euler product

$$L(s, \mathrm{Sym}^2(g) \otimes f) = \prod_p \det(\mathbf{1}_6 - A_p \otimes B_p \cdot p^{-s})^{-1}$$

for $\mathrm{Re}(s) \gg 0$. Let $\Lambda(s, \mathrm{Sym}^2(g) \otimes f)$ be the completed L-function given by

$$\Lambda(s, \mathrm{Sym}^2(g) \otimes f) = \Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s-\kappa)\Gamma_{\mathbb{C}}(s-2\kappa+1)L(s, \mathrm{Sym}^2(g) \otimes f),$$

where $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. It has an analytic continuation to the whole s-plane and satisfies the functional equation

$$\Lambda(4\kappa - s, \mathrm{Sym}^2(g) \otimes f) = \Lambda(s, \mathrm{Sym}^2(g) \otimes f).$$

Our main result is as follows.

Theorem 2.1.

$$\Lambda(2\kappa, \mathrm{Sym}^2(g) \otimes f) = 2^{\kappa+1} \frac{\langle f, f\rangle}{\langle h, h\rangle} \frac{|\langle F|_{\mathfrak{H}\times\mathfrak{H}}, g \times g\rangle|^2}{\langle g, g\rangle^2}.$$

Remark 2.2. Theorem 2.1 is compatible with Ikeda's conjecture [30] and the Gross-Prasad conjecture [22], [23], [8], [27, §8].

Remark 2.3. Let $g_1, g_2 \in S_{\kappa+1}(\mathrm{SL}_2(\mathbb{Z}))$ be normalized Hecke eigenforms. If $g_1 \neq g_2$, then

$$\langle F|_{\mathfrak{H}\times\mathfrak{H}}, g_1 \times g_2\rangle = 0$$

by Lemma 1.1 and the multiplicity one theorem. The assertion also follows from Theorem 1.1 of [30].

Remark 2.4. Let $E \in M_{\kappa+1}(\mathrm{Sp}_2(\mathbb{Z}))$ be either a Siegel Eisenstein series or a Klingen Eisenstein series. An explicit formula for $\langle E|_{\mathfrak{H}\times\mathfrak{H}}, g_1 \times g_2\rangle$ was proved by Garrett [17], [18], Böcherer [6], [7].

Example 2.5. We discuss the case $\kappa = 11$. Let $f \in S_{22}(\mathrm{SL}_2(\mathbb{Z}))$ be the normalized Hecke eigenform and $h \in S^+_{23/2}(\Gamma_0(4))$ the Hecke eigenform given by

$$h(\tau) = q^3 + 10q^4 - 88q^7 - 132q^8 + \cdots .$$

Then h corresponds to f by the Shimura correspondence. Let $F \in S_{12}(\mathrm{Sp}_2(\mathbb{Z}))$ be the Saito-Kurokawa lift of h and $g \in S_{12}(\mathrm{SL}_2(\mathbb{Z}))$ the normalized Hecke eigenform. Then

$$\frac{\langle F|_{\mathfrak{H}\times\mathfrak{H}}, g \times g\rangle}{\langle g, g\rangle^2} = \sum_{\substack{r\in\mathbb{Z} \\ r^2<4}} A\left(\begin{pmatrix} 1 & r/2 \\ r/2 & 1 \end{pmatrix}\right) = 2c_h(3) + c_h(4) = 12.$$

By computer calculation,

$$\langle g, g\rangle = 0.0000010353620568043209223478168122251645\ldots,$$
$$\langle f, f\rangle\langle h, h\rangle^{-1} = 1197338.2132758251275951506817254810499696\ldots,$$
$$\Lambda(22, \mathrm{Sym}^2(g)\otimes f) = 0.7570486229780282956208657580257776451825\ldots.$$

Here we have used Dokchitser's program [14]. Therefore the numerical value of

$$\langle g, g\rangle^{-2}\langle f, f\rangle^{-1}\langle h, h\rangle\Lambda(22, \mathrm{Sym}^2(g)\otimes f)$$

coincides with

$$2^{12}\cdot\frac{\langle F|_{\mathfrak{H}\times\mathfrak{H}}, g\times g\rangle^2}{\langle g, g\rangle^4} = 2^{16}\cdot 3^2.$$

Theorem 2.1 has an application to Deligne's conjecture [13].

Corollary 2.6. *For $\sigma \in \mathrm{Aut}(\mathbb{C})$,*

$$\left(\frac{\Lambda(2\kappa, \mathrm{Sym}^2(g)\otimes f)}{\langle g, g\rangle^2 c^+(f)}\right)^\sigma = \frac{\Lambda(2\kappa, \mathrm{Sym}^2(g^\sigma)\otimes f^\sigma)}{\langle g^\sigma, g^\sigma\rangle^2 c^+(f^\sigma)}.$$

Here $c^+(f)$ is the period of f as in [52].

Proof. We may assume that the Fourier coefficients of F (and hence that of h) are in $\mathbb{Q}(f)$, where $\mathbb{Q}(f)$ is the field generated over $\mathbb{Q}$ by the Fourier coefficients of f. In particular, $\langle F|_{\mathfrak{H}\times\mathfrak{H}}, g\times g\rangle \in \mathbb{R}$.

For a fundamental discriminant $-D < 0$, let χ_{-D} denote the Dirichlet character associated to $\mathbb{Q}(\sqrt{-D})/\mathbb{Q}$. Let $\Lambda(s, f, \chi_{-D})$ be the completed L-function given by

$$\Lambda(s, f, \chi_{-D}) = D^s\Gamma_{\mathbb{C}}(s)L(s, f, \chi_{-D}).$$

Then the Kohnen-Zagier formula [33] says that

$$\Lambda(\kappa, f, \chi_{-D}) = 2^{-\kappa+1}D^{1/2}c_h(D)^2\frac{\langle f, f\rangle}{\langle h, h\rangle}.$$

Note that there exists a fundamental discriminant $-D < 0$ such that $c_h(D) \neq 0$.

Let $\sigma \in \mathrm{Aut}(\mathbb{C})$. Then F^σ is the Saito-Kurokawa lift of h^σ, and

$$c_h(D)^\sigma = c_{h^\sigma}(D).$$

By the property of the period $c^+(f)$,

$$\left(\frac{D^{-1/2}\Lambda(\kappa, f, \chi_{-D})}{c^+(f)}\right)^\sigma = \frac{D^{-1/2}\Lambda(\kappa, f^\sigma, \chi_{-D})}{c^+(f^\sigma)}.$$

Obviously,

$$\left(\frac{\langle F|_{\mathfrak{H}\times\mathfrak{H}}, g\times g\rangle}{\langle g, g\rangle^2}\right)^\sigma = \frac{\langle F^\sigma|_{\mathfrak{H}\times\mathfrak{H}}, g^\sigma\times g^\sigma\rangle}{\langle g^\sigma, g^\sigma\rangle^2}.$$

This completes the proof. □

Remark 2.7. It seems that Corollary 2.6 does not follow from the algebraicity of central critical values of triple product L-functions [18], [42], [48], [19], [24], [4]. Notice that

$$\Lambda(2\kappa, g \otimes g \otimes f) = \Lambda(2\kappa, \mathrm{Sym}^2(g) \otimes f)\Lambda(\kappa, f) = 0.$$

Remark 2.8. Let κ, κ' be odd positive integers such that $\kappa \leq \kappa'$. Using differential operators as in [6], [8], one might prove the analogue of Corollary 2.6 for normalized Hecke eigenforms $f \in S_{2\kappa}(\mathrm{SL}_2(\mathbb{Z}))$ and $g \in S_{\kappa'+1}(\mathrm{SL}_2(\mathbb{Z}))$.

3. Automorphic forms

3.1. Automorphic forms on GL_2**.** Let F be a number field and $\mathbb{A}_F$ (resp. $\mathbb{A}_F^f$) the ring of adeles (resp. finite adeles) of F. Fix a non-trivial additive character ψ of $\mathbb{A}_F/F$. Let f be an automorphic form on $\mathrm{GL}_2(\mathbb{A}_F)$. For $\xi \in F$, define the ξ-th Fourier coefficient $W_{f,\xi}$ of f by

$$W_{f,\xi}(h) = \int_{F\backslash\mathbb{A}_F} f(u(x)h)\overline{\psi(\xi x)}\, dx.$$

Note that

$$W_{f,\xi_1}(d(\xi_2)h) = W_{f,\xi_1\xi_2^{-1}}(h)$$

for $\xi_1, \xi_2 \in F^\times$.

Fix an even positive integer l and a normalized Hecke eigenform

$$g(\tau) = \sum_{n=1}^{\infty} a_g(n)q^n \in S_l(\mathrm{SL}_2(\mathbb{Z})).$$

For each prime p, let $\{\alpha_p, \alpha_p^{-1}\}$ denote the Satake parameter of g at p. Then

$$1 - a_g(p)X + p^{l-1}X^2 = \left(1 - p^{(l-1)/2}\alpha_p X\right)\left(1 - p^{(l-1)/2}\alpha_p^{-1} X\right).$$

We fix $s_p \in \mathbb{C}$ such that $\alpha_p = p^{-s_p}$. Note that $\mathrm{Re}(s_p) = 0$ by the Ramanujan conjecture.

Let $F = \mathbb{Q}$ and $\psi = \psi_0$. Here ψ_0 is the standard additive character of $\mathbb{A}_\mathbb{Q}$. Then g determines a cusp form $\mathbf{g}$ on $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ by the formula

$$\mathbf{g}(h) = \det(h_\infty)^{l/2}(c\sqrt{-1} + d)^{-l} g(h_\infty(\sqrt{-1}))$$

for $h = \gamma h_\infty k \in \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ with $\gamma \in \mathrm{GL}_2(\mathbb{Q})$,

$$h_\infty = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R}),$$

and $k \in \mathrm{GL}_2(\hat{\mathbb{Z}})$. By definition, $\mathbf{g}$ satisfies

$$\mathbf{g}(hk) = \mathbf{g}(h), \tag{3.1}$$

$$\mathbf{g}(hk_\theta) = e^{\sqrt{-1}l\theta}\mathbf{g}(h), \tag{3.2}$$

for $h \in \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$, $k \in \mathrm{GL}_2(\hat{\mathbb{Z}})$, $k_\theta \in \mathrm{SO}(2)$. Moreover,

$$W_{\mathbf{g},n}(1) = a_g(n)e^{-2\pi n}$$

for $n \in \mathbb{N}$. Let $\pi = \otimes_v \pi_v$ be the irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ generated by $\mathbf{g}$. Then π_p is the principal series representation

$$\mathrm{Ind}_{\mathbf{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \left(|\ |_{\mathbb{Q}_p}^{s_p} \boxtimes |\ |_{\mathbb{Q}_p}^{-s_p}\right)$$

for each prime p, and π_∞ is the discrete series representation of weight l. In the space of π, the conditions (3.1), (3.2) characterize the cusp form $\mathbf{g}$ up to scalars. We define a cusp form $\mathbf{g}^\sharp \in \pi$ by

$$\mathbf{g}^\sharp(h) = \mathbf{g}\big(ht(2^{-1})_2\big) \tag{3.3}$$

for $h \in \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$, where

$$t(2^{-1})_2 = \begin{pmatrix} 2^{-1} & 0 \\ 0 & 2 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}_2).$$

Let $F = \mathcal{K}$ be an imaginary quadratic field with discriminant $-D < 0$ and $\mathcal{O}$ the ring of integers of $\mathcal{K}$. Let $\psi = \frac{1}{2}(\psi_0 \circ \mathrm{tr}_{\mathcal{K}/\mathbb{Q}})$. We put $\delta = \sqrt{-D}$, $\hat{\mathcal{O}} = \mathcal{O} \otimes_\mathbb{Z} \hat{\mathbb{Z}}$, and $\mathcal{K}_v = \mathcal{K} \otimes_\mathbb{Q} \mathbb{Q}_v$ for each place v of $\mathbb{Q}$. Let $\pi_\mathcal{K} = \otimes_v \pi_{\mathcal{K},v}$ be the base change of π to $\mathcal{K}$, which is an irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathcal{K})$. Then $\pi_{\mathcal{K},p}$ is the principal series representation

$$\mathrm{Ind}_{\mathbf{B}(\mathcal{K}_p)}^{\mathrm{GL}_2(\mathcal{K}_p)} \left(|\ |_{\mathcal{K}_p}^{s_p} \boxtimes |\ |_{\mathcal{K}_p}^{-s_p}\right)$$

for each prime p, and $\pi_{\mathcal{K},\infty}$ is the principal series representation

$$\mathrm{Ind}_{\mathbf{B}(\mathbb{C})}^{\mathrm{GL}_2(\mathbb{C})}(\mu^{l-1} \boxtimes \mu^{-l+1})$$

with minimal SU(2)-type ρ_{2l-2}. Here $\mu(z) = (z/\bar{z})^{1/2}$ for $z \in \mathbb{C}^\times$. There is a unique cusp form $\mathbf{g}_\mathcal{K} \in \pi_\mathcal{K}$ such that

$$\mathbf{g}_\mathcal{K}(hk) = \mathbf{g}_\mathcal{K}(h), \tag{3.4}$$

$$H \cdot \mathbf{g}_\mathcal{K} = (2l-2)\mathbf{g}_\mathcal{K}, \tag{3.5}$$

$$X \cdot \mathbf{g}_\mathcal{K} = 0, \tag{3.6}$$

for $h \in \mathrm{GL}_2(\mathbb{A}_\mathcal{K})$, $k \in \mathrm{GL}_2(\hat{\mathcal{O}})$, and such that

$$W_{\mathbf{g}_\mathcal{K},2\delta^{-1}}(1) = K_{l-1}(4\pi D^{-1/2}).$$

We remark that the conditions (3.5), (3.6) mean that $\mathbf{g}_{\mathcal{K}}$ is a highest weight vector of the minimal SU(2)-type. In the space of $\pi_{\mathcal{K}}$, the conditions (3.4)–(3.6) characterize the cusp form $\mathbf{g}_{\mathcal{K}}$ up to scalars. We define cusp forms $\mathbf{g}^{\sharp}_{\mathcal{K}}, \mathbf{g}^{\flat}_{\mathcal{K}} \in \pi_{\mathcal{K}}$ by

$$\mathbf{g}^{\sharp}_{\mathcal{K}}(h) = \mathbf{g}_{\mathcal{K}}(hk_\infty), \tag{3.7}$$

$$\mathbf{g}^{\flat}_{\mathcal{K}}(h) = \mathbf{g}_{\mathcal{K}}(hd(2^{-1}\delta)_f), \tag{3.8}$$

for $h \in \mathrm{GL}_2(\mathbb{A}_{\mathcal{K}})$. Here

$$k_\infty = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -\sqrt{-1} \\ -\sqrt{-1} & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$$

and

$$d(2^{-1}\delta)_f = \begin{pmatrix} 1 & 0 \\ 0 & 2^{-1}\delta \end{pmatrix} \in \mathrm{GL}_2\left(\mathbb{A}^f_{\mathcal{K}}\right).$$

3.2. Automorphic forms on $\widetilde{\mathrm{SL}}_2$. If F is a local field of characteristic not 2, let $\widetilde{\mathrm{SL}_2(F)}$ be the 2-fold metaplectic cover of $\mathrm{SL}_2(F)$. Let $c(g_1, g_2)$ denote Kubota's 2-cocycle defined by

$$c(g_1, g_2) = \left(x(g_1g_2)x(g_1)^{-1}, x(g_1g_2)x(g_2)^{-1}\right)_F$$

for $g_1, g_2 \in \mathrm{SL}_2(F)$. Here

$$x(g) = \begin{cases} c & \text{if } c \neq 0 \\ d & \text{if } c = 0 \end{cases} \quad \text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(F).$$

Then

$$\widetilde{\mathrm{SL}_2(F)} \simeq \mathrm{SL}_2(F) \times \{\pm 1\},$$

where the multiplication on the right-hand side is given by

$$(g_1, \epsilon_1) \cdot (g_2, \epsilon_2) = (g_1g_2, \epsilon_1\epsilon_2 \cdot c(g_1, g_2)).$$

Note that the map

$$\begin{aligned} U(F) &\longrightarrow \widetilde{\mathrm{SL}_2(F)} \\ u &\longmapsto (u, 1) \end{aligned}$$

is a homomorphism. By abuse of notation, we write g for the element $(g, 1) \in \widetilde{\mathrm{SL}_2(F)}$. For any subgroup H of $\mathrm{SL}_2(F)$, let $\tilde{H}$ denote the inverse image of H in $\widetilde{\mathrm{SL}_2(F)}$.

Let $F = \mathbb{Q}_p$. Let ψ_p denote the standard additive character of $\mathbb{Q}_p$ and $\chi_{-1,p}$ the quadratic character of $\mathbb{Q}_p^\times$ associated to $\mathbb{Q}_p(\sqrt{-1})/\mathbb{Q}_p$ by class field theory. Let

$$K_0(4; \mathbb{Z}_p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_p) \,\middle|\, c \equiv 0 \bmod 4\mathbb{Z}_p \right\},$$
$$K_1(4; \mathbb{Z}_p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_p) \,\middle|\, c \equiv 0, d \equiv 1 \bmod 4\mathbb{Z}_p \right\}.$$

Note that $\mathrm{SL}_2(\mathbb{Z}_p) = K_0(4; \mathbb{Z}_p) = K_1(4; \mathbb{Z}_p)$ if $p \neq 2$. We put

$$s_p(k) = \begin{cases} (c, d)_{\mathbb{Q}_p} & \text{if } cd \neq 0, \operatorname{ord}_{\mathbb{Q}_p}(c) \text{ is odd}, \\ 1 & \text{otherwise}, \end{cases}$$

for

$$k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_1(4; \mathbb{Z}_p).$$

Then the map

$$\begin{aligned} K_1(4; \mathbb{Z}_p) &\longrightarrow \widetilde{\mathrm{SL}_2(\mathbb{Q}_p)} \\ k &\longmapsto (k, s_p(k)) \end{aligned}$$

defines a splitting homomorphism. If $p = 2$, put

$$\epsilon_2(k) = \begin{cases} \gamma_{\mathbb{Q}_2}(d, \psi_2)(c, d)_{\mathbb{Q}_2} & \text{if } c \neq 0, \\ \gamma_{\mathbb{Q}_2}(d, \psi_2)^{-1} & \text{if } c = 0, \end{cases}$$

for

$$k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(4; \mathbb{Z}_2).$$

Note that $\epsilon_2(k) = s_2(k)$ for $k \in K_1(4; \mathbb{Z}_2)$ and

$$\epsilon_2(k)^2 = \chi_{-1,2}(d) \quad \text{for } k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(4; \mathbb{Z}_2).$$

Then the map

$$\begin{aligned} \widetilde{K_0(4; \mathbb{Z}_2)} &\longrightarrow \mathbb{C}^\times \\ (k, \epsilon) &\longmapsto \epsilon \cdot \epsilon_2(k) \end{aligned}$$

defines a character of $\widetilde{K_0(4; \mathbb{Z}_2)}$.

Let $F = \mathbb{R}$. There is a splitting homomorphism

$$\Gamma_1(4) \longrightarrow \widetilde{\mathrm{SL}_2(\mathbb{R})},$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \left(\frac{c}{d}\right)\right)$$

where

$$\Gamma_1(4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, c \equiv 0, d \equiv 1 \bmod 4 \right\}.$$

Note that

$$\left(\frac{c}{d}\right) = \prod_p s_p\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \quad \text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(4).$$

Let

$$\mathbb{R}/4\pi\mathbb{Z} \longrightarrow \widetilde{\mathrm{SO}(2)}$$
$$\theta \longmapsto \tilde{k}_\theta$$

be the isomorphism determined by

$$\tilde{k}_\theta = \begin{cases} (k_\theta, 1) & \text{if } -\pi < \theta \le \pi, \\ (k_\theta, -1) & \text{if } \pi < \theta \le 3\pi. \end{cases}$$

The metaplectic cover $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ acts on $\mathfrak{H}$ through $\mathrm{SL}_2(\mathbb{R})$. We define a factor of automorphy

$$\tilde{j} : \widetilde{\mathrm{SL}_2(\mathbb{R})} \times \mathfrak{H} \longrightarrow \mathbb{C}$$

by

$$\tilde{j}\left(\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \epsilon\right), \tau\right) = \begin{cases} \epsilon\sqrt{d} & \text{if } c = 0, d > 0, \\ -\epsilon\sqrt{d} & \text{if } c = 0, d < 0, \\ \epsilon(c\tau + d)^{1/2} & \text{if } c \neq 0. \end{cases}$$

Now we consider automorphic forms on the 2-fold metaplectic cover $\widetilde{\mathrm{SL}_2(\mathbb{A}_\mathbb{Q})}$ of $\mathrm{SL}_2(\mathbb{A}_\mathbb{Q})$. We identify $\mathrm{SL}_2(\mathbb{Q})$ with its image under the canonical splitting homomorphism $\mathrm{SL}_2(\mathbb{Q}) \to \widetilde{\mathrm{SL}_2(\mathbb{A}_\mathbb{Q})}$. The metaplectic cover $\widetilde{\mathrm{SL}_2(\mathbb{A}_\mathbb{Q})}$ also splits over $U(\mathbb{A}_\mathbb{Q})$ and $K_1(4; \hat{\mathbb{Z}})$.

Let l be a positive integer. Let

$$h(\tau) = \sum_{n=0}^{\infty} c_h(n) q^n \in M_{l+1/2}(\Gamma_0(4)).$$

Then h determines an automorphic form $\mathbf{h}$ on $\widetilde{\mathrm{SL}_2(\mathbb{A}_\mathbb{Q})}$ by the formula

$$\mathbf{h}(g) = \tilde{j}(g_\infty, \sqrt{-1})^{-2l-1} h(g_\infty(\sqrt{-1}))$$

for $g = \gamma g_\infty k \in \widetilde{\mathrm{SL}_2(\mathbb{A}_\mathbb{Q})}$ with $\gamma \in \mathrm{SL}_2(\mathbb{Q})$, $g_\infty \in \widetilde{\mathrm{SL}_2(\mathbb{R})}$, and $k \in K_1(4; \hat{\mathbb{Z}})$. By Proposition 3 of [55], $\mathbf{h}$ satisfies

$$\mathbf{h}(g(k, s_p(k))) = \mathbf{h}(g) \tag{3.9}$$

for $g \in \widetilde{\mathrm{SL}_2(\mathbb{A}_\mathbb{Q})}$, $k \in \mathrm{SL}_2(\mathbb{Z}_p)$ if $p \neq 2$, and

$$\mathbf{h}(gk) = \epsilon_2(k)^{(-1)^l} \mathbf{h}(g), \tag{3.10}$$

$$\mathbf{h}(g\tilde{k}_\theta) = e^{\sqrt{-1}(l+1/2)\theta} \mathbf{h}(g), \tag{3.11}$$

for $g \in \widetilde{\mathrm{SL}_2(\mathbb{A}_\mathbb{Q})}$, $k \in K_0(4; \mathbb{Z}_2)$, $\tilde{k}_\theta \in \widetilde{\mathrm{SO}(2)}$. For $\xi \in \mathbb{Q}$, define the ξ-th Fourier coefficient $W_{\mathbf{h},\xi}$ of $\mathbf{h}$ by

$$W_{\mathbf{h},\xi}(g) = \int_{\mathbb{Q}\backslash\mathbb{A}_\mathbb{Q}} \mathbf{h}(u(x)g)\overline{\psi(\xi x)}\, dx,$$

where ψ is the standard additive character of $\mathbb{A}_\mathbb{Q}$. Note that

$$W_{\mathbf{h},\xi_1}(t(\xi_2)g) = W_{\mathbf{h},\xi_1\xi_2^2}(g)$$

for $\xi_1, \xi_2 \in \mathbb{Q}^\times$. Moreover,

$$W_{\mathbf{h},n}(1) = c_h(n)e^{-2\pi n}$$

for $n \in \mathbb{Z}_{\geq 0}$.

We will give an adelic interpretation of Kohnen's plus space. We define $h|U_4, h|W_4 \in M_{l+1/2}(\Gamma_0(4))$ by

$$(h|U_4)(\tau) = \frac{1}{4} \sum_{x \in \mathbb{Z}/4\mathbb{Z}} h\left(\frac{\tau + x}{4}\right),$$

$$(h|W_4)(\tau) = (-2\sqrt{-1}\tau)^{-l-1/2} h\left(-\frac{1}{4\tau}\right),$$

for $\tau \in \mathfrak{H}$. We also define automorphic forms $\mathsf{U}(\mathbf{h})$, $\mathsf{W}(\mathbf{h})$ by

$$\mathsf{U}(\mathbf{h})(g) = \int_{\mathbb{Z}_2} \mathbf{h}(gu(x)t(2)_2)\, dx,$$

$$\mathsf{W}(\mathbf{h})(g) = \mathbf{h}\big(gw_2^{-1}t(2)_2\big),$$

for $g \in \widetilde{\mathrm{SL}_2(\mathbb{A}_\mathbb{Q})}$.

Lemma 3.1. *The automorphic form on* $\widetilde{\mathrm{SL}_2(\mathbb{A}_\mathbb{Q})}$ *associated to* $h|U_4$ *is*

$$2^{l+1/2}\mathsf{U}(\mathbf{h}).$$

Proof. It suffices to show that

$$(h|U_4)(\tau) = 2^{l+1/2}\tilde{\jmath}(g_\infty, \sqrt{-1})^{2l+1}\mathsf{U}(\mathbf{h})(g_\infty)$$

for $g_\infty \in \widetilde{\mathrm{SL}_2(\mathbb{R})}$ and $\tau = g_\infty(\sqrt{-1}) \in \mathfrak{H}$. Let $x \in \mathbb{Z}$. Since

$$t(2)_\infty^{-1}u(x)_\infty = \begin{pmatrix} 2^{-1} & 2^{-1}x \\ 0 & 2 \end{pmatrix}$$

in $\widetilde{\mathrm{SL}_2(\mathbb{R})}$, we have

$$t(2)_\infty^{-1}u(x)_\infty(\tau) = \frac{\tau + x}{4}$$

and

$$\tilde{\jmath}\big(t(2)_\infty^{-1}u(x)_\infty, \tau\big) = \sqrt{2}.$$

Hence $(h|U_4)(\tau)$ is equal to

$$\begin{aligned}
&4^{-1}\sum_{x\in\mathbb{Z}/4\mathbb{Z}} h\big(t(2)_\infty^{-1}u(x)_\infty(\tau)\big) \\
&= 4^{-1}\sum_{x\in\mathbb{Z}/4\mathbb{Z}} \tilde{\jmath}\big(t(2)_\infty^{-1}u(x)_\infty g_\infty, \sqrt{-1}\big)^{2l+1}\mathbf{h}\big(t(2)_\infty^{-1}u(x)_\infty g_\infty\big) \\
&= 2^{l-3/2}\tilde{\jmath}(g_\infty, \sqrt{-1})^{2l+1}\sum_{x\in\mathbb{Z}/4\mathbb{Z}} \mathbf{h}\big(t(2)_\infty^{-1}u(x)_\infty g_\infty\big) \\
&= 2^{l-3/2}\tilde{\jmath}(g_\infty, \sqrt{-1})^{2l+1}\sum_{x\in\mathbb{Z}/4\mathbb{Z}} \mathbf{h}\big(g_\infty u(x)_2^{-1}t(2)_2\big) \\
&= 2^{l+1/2}\tilde{\jmath}(g_\infty, \sqrt{-1})^{2l+1}\mathsf{U}(\mathbf{h})(g_\infty).
\end{aligned}$$

□

Lemma 3.2. *The automorphic form on* $\widetilde{\mathrm{SL}_2(\mathbb{A}_\mathbb{Q})}$ *associated to* $h|W_4$ *is*

$$\zeta_8^{-2l-1}\mathsf{W}(\mathbf{h}).$$

Proof. It suffices to show that

$$(h|W_4)(\tau) = \zeta_8^{-2l-1}\tilde{\jmath}(g_\infty, \sqrt{-1})^{2l+1}\mathsf{W}(\mathbf{h})(g_\infty)$$

for $g_\infty \in \widetilde{\mathrm{SL}_2(\mathbb{R})}$ and $\tau = g_\infty(\sqrt{-1}) \in \mathfrak{H}$. Since

$$t(2)_\infty^{-1}w_\infty = \begin{pmatrix} 0 & 2^{-1} \\ -2 & 0 \end{pmatrix}$$

in $\widetilde{\mathrm{SL}_2(\mathbb{R})}$, we have

$$t(2)_\infty^{-1} w_\infty(\tau) = -\frac{1}{4\tau}$$

and

$$\tilde{j}\big(t(2)_\infty^{-1} w_\infty, \tau\big) = (-2\tau)^{1/2} = \zeta_8^{-1}(-2\sqrt{-1}\tau)^{1/2}.$$

Hence $(h|W_4)(\tau)$ is equal to

$$\begin{aligned}
&(-2\sqrt{-1}\tau)^{-l-1/2} h\big(t(2)_\infty^{-1} w_\infty(\tau)\big) \\
&= (-2\sqrt{-1}\tau)^{-l-1/2} \tilde{j}\big(t(2)_\infty^{-1} w_\infty g_\infty, \sqrt{-1}\big)^{2l+1} \mathbf{h}\big(t(2)_\infty^{-1} w_\infty g_\infty\big) \\
&= \zeta_8^{-2l-1} \tilde{j}(g_\infty, \sqrt{-1})^{2l+1} \mathbf{h}\big(t(2)_\infty^{-1} w_\infty g_\infty\big) \\
&= \zeta_8^{-2l-1} \tilde{j}(g_\infty, \sqrt{-1})^{2l+1} \mathbf{h}\big(g_\infty w_2^{-1} t(2)_2\big) \\
&= \zeta_8^{-2l-1} \tilde{j}(g_\infty, \sqrt{-1})^{2l+1} \mathsf{W}(\mathbf{h})(g_\infty).
\end{aligned}$$

□

Lemma 3.3. *Let* $h \in M_{l+1/2}(\Gamma_0(4))$. *Let* $\mathbf{h}$ *denote the automorphic form on* $\widetilde{\mathrm{SL}_2(\mathbb{A}_\mathbb{Q})}$ *associated to* h. *Then* h *belongs to* $M^+_{l+1/2}(\Gamma_0(4))$ *if and only if*

$$\mathsf{W}(\mathsf{U}(\mathbf{h})) = 2^{-1/2}\zeta_8^{(-1)^l}\mathbf{h}. \tag{3.12}$$

Proof. By Proposition 2 of [32], h belongs to $M^+_{l+1/2}(\Gamma_0(4))$ if and only if

$$h|U_4W_4 = 2^l(\sqrt{-1})^{l^2+l}h.$$

Hence the assertion follows from Lemmas 3.1 and 3.2. □

Baruch and Mao [3, §9] also gave an adelic interpretation of Kohnen's plus space. We check that Lemma 3.3 is consistent with their result. Let ρ be an admissible representation of $\widetilde{\mathrm{SL}_2(\mathbb{Q}_2)}$ on $\mathcal{V}$. We fix a positive integer l and put

$$\mathcal{V}_0 = \big\{ f \in \mathcal{V} \mid \rho(k)f = \epsilon_2(k)^{(-1)^l} f \text{ for } k \in K_0(4;\mathbb{Z}_2)\big\}.$$

For $f \in \mathcal{V}$, define $\mathsf{U}(f), \mathsf{W}(f) \in \mathcal{V}$ by

$$\begin{aligned}
\mathsf{U}(f) &= \int_{\mathbb{Z}_2} \rho(u(x)t(2)) f \, dx, \\
\mathsf{W}(f) &= \rho(w^{-1}t(2)) f.
\end{aligned}$$

Lemma 3.4. *If* $f \in \mathcal{V}_0$, *then*

$$\mathsf{U}(f) \in \mathcal{V}_0.$$

Proof. Let

$$k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(4; \mathbb{Z}_2).$$

Since

$$k = \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}$$

in $\widetilde{\mathrm{SL}_2(\mathbb{Q}_2)}$, we may assume that $b = 0$ and $d = a^{-1}$. Let $x \in \mathbb{Z}_2$. We put

$$k'(x) = t(2)^{-1}u(x)^{-1}ku(x)t(2) \in \widetilde{\mathrm{SL}_2(\mathbb{Q}_2)}.$$

Then

$$k'(x) = \begin{pmatrix} a - cx & 4^{-1}x(a - d - cx) \\ 4c & d + cx \end{pmatrix}$$

in $\widetilde{\mathrm{SL}_2(\mathbb{Q}_2)}$. Since $k'(x) \in K_0(4; \mathbb{Z}_2)$, we have

$$\rho(ku(x)t(2))f = \epsilon_2(k'(x))^{(-1)^l}\rho(u(x)t(2))f.$$

It remains to show that

$$\epsilon_2(k'(x)) = \epsilon_2(k).$$

If $c = 0$, then

$$\epsilon_2(k'(x)) = \gamma_{\mathbb{Q}_2}(d, \psi_2)^{-1} = \epsilon_2(k).$$

If $c \neq 0$, then

$$\epsilon_2(k'(x)) = \gamma_{\mathbb{Q}_2}(d + cx, \psi_2)(4c, d + cx)_{\mathbb{Q}_2} = \gamma_{\mathbb{Q}_2}(d, \psi_2)(c, d)_{\mathbb{Q}_2} = \epsilon_2(k).$$

This completes the proof. □

Lemma 3.5. *If $f \in \mathcal{V}_0$, then*

$$\mathsf{W}(f) \in \mathcal{V}_0.$$

Proof. Let

$$k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(4; \mathbb{Z}_2).$$

We put

$$(k', \epsilon') = t(2)^{-1}wkw^{-1}t(2) \in \widetilde{\mathrm{SL}_2(\mathbb{Q}_2)}.$$

Since

$$k' = \begin{pmatrix} d & -4^{-1}c \\ -4b & a \end{pmatrix} \in K_0(4; \mathbb{Z}_2),$$

we have

$$\rho(k)\mathsf{W}(f) = \epsilon' \epsilon_2(k')^{(-1)^l} \mathsf{W}(f).$$

It remains to show that

$$\epsilon_2(k') = \epsilon' \epsilon_2(k).$$

If $b = c = 0$, then $\epsilon' = 1$,

$$\begin{aligned} \epsilon_2(k) &= \gamma_{\mathbb{Q}_2}(d, \psi_2)^{-1} = \gamma_{\mathbb{Q}_2}(a, \psi_2)^{-1}, \\ \epsilon_2(k') &= \gamma_{\mathbb{Q}_2}(a, \psi_2)^{-1}. \end{aligned}$$

If $b = 0$ and $c \neq 0$, then $\epsilon' = (a, -c)_{\mathbb{Q}_2}$,

$$\begin{aligned} \epsilon_2(k) &= \gamma_{\mathbb{Q}_2}(d, \psi_2)(c, d)_{\mathbb{Q}_2} = \gamma_{\mathbb{Q}_2}(a, \psi_2)(a, c)_{\mathbb{Q}_2}, \\ \epsilon_2(k') &= \gamma_{\mathbb{Q}_2}(a, \psi_2)^{-1} = \gamma_{\mathbb{Q}_2}(a, \psi_2)(a, -1)_{\mathbb{Q}_2}. \end{aligned}$$

If $b \neq 0$ and $c = 0$, then $\epsilon' = (a, b)_{\mathbb{Q}_2}$,

$$\begin{aligned} \epsilon_2(k) &= \gamma_{\mathbb{Q}_2}(d, \psi_2)^{-1} = \gamma_{\mathbb{Q}_2}(a, \psi_2)(a, -1)_{\mathbb{Q}_2}, \\ \epsilon_2(k') &= \gamma_{\mathbb{Q}_2}(a, \psi_2)(-4b, a)_{\mathbb{Q}_2} = \gamma_{\mathbb{Q}_2}(a, \psi_2)(a, -b)_{\mathbb{Q}_2}. \end{aligned}$$

If $b \neq 0$ and $c \neq 0$, then $\epsilon' = (a, -bc)_{\mathbb{Q}_2}$,

$$\begin{aligned} \epsilon_2(k) &= \gamma_{\mathbb{Q}_2}(d, \psi_2)(c, d)_{\mathbb{Q}_2} = \gamma_{\mathbb{Q}_2}(a, \psi_2)(c, d)_{\mathbb{Q}_2}, \\ \epsilon_2(k') &= \gamma_{\mathbb{Q}_2}(a, \psi_2)(-4b, a)_{\mathbb{Q}_2} = \gamma_{\mathbb{Q}_2}(a, \psi_2)(a, -b)_{\mathbb{Q}_2}, \end{aligned}$$

hence

$$(\epsilon' \epsilon_2(k))^{-1} \epsilon_2(k') = (ad, c)_{\mathbb{Q}_2} = (1 + bc, c)_{\mathbb{Q}_2} = 1.$$

This completes the proof. □

Assume that ρ is a principal series representation

$$\mathrm{Ind}_{\widetilde{B(\mathbb{Q}_2)}}^{\widetilde{\mathrm{SL}_2(\mathbb{Q}_2)}} \left((\chi_{-1,2}^l)^{\psi_2} | \ |_{\mathbb{Q}_2}^s \right)$$

(see Sect. A.3 for the notation). Then

$$\dim_{\mathbb{C}} \mathcal{V}_0 = 2$$

by Proposition 12 of [55]. There are elements $f_1, f_w \in \mathcal{V}_0$ determined by

$$f_1(1) = 1, \qquad f_1(w) = 0, \qquad f_1(k_1) = 0,$$
$$f_w(1) = 0, \qquad f_w(w) = 1, \qquad f_w(k_1) = 0.$$

With the notation of [55, p. 427],

$$f_1 = F[2, 2^2], \quad f_w = (\sqrt{-1})^{(-1)^l} F[2, 1].$$

Put $\alpha = 2^{-s}$ and

$$f^+ = \alpha^{-2} f_1 + 2^{-3/2} \zeta_8^{(-1)^{l+1}} f_w.$$

Note that $f^+ = \alpha^{-2}\tilde{\varphi}_2$, where $\tilde{\varphi}_2$ is as in (9.4) of [3].

Lemma 3.6. *Let $\mathcal{V}_0^+$ be the subspace of $\mathcal{V}_0$ given by*

$$\mathcal{V}_0^+ = \left\{ f \in \mathcal{V}_0 \mid \mathsf{W}(\mathsf{U}(f)) = 2^{-1/2} \zeta_8^{(-1)^l} f \right\}.$$

Then

$$\dim_{\mathbb{C}} \mathcal{V}_0^+ = 1$$

and

$$f^+ \in \mathcal{V}_0^+.$$

Proof. Let $f = c_1 f_1 + c_w f_w \in \mathcal{V}_0$ with $c_1, c_w \in \mathbb{C}$. Then

$$\mathsf{W}(\mathsf{U}(f))(w) = f(t(4)) = 4^{-1}\alpha^2 c_1.$$

If $f \in \mathcal{V}_0^+$, then $2^{-1/2}\zeta_8^{(-1)^l} c_w = 4^{-1}\alpha^2 c_1$. Hence $\dim_{\mathbb{C}} \mathcal{V}_0^+ \le 1$.

To prove $f^+ \in \mathcal{V}_0^+$, it suffices to show that

$$\mathsf{W}(\mathsf{U}(f^+))(g) = 2^{-1/2}\zeta_8^{(-1)^l} f^+(g)$$

for $g = 1, k_1$. For $x \in \mathbb{Q}_2$, we have

$$f_1(w^{-1}u(x)) = \begin{cases} 0 & \text{if } x \in 2^{-1}\mathbb{Z}_2, \\ \gamma_{\mathbb{Q}_2}(x, \psi_2)^{-1}((-1)^{l+1}, x)_{\mathbb{Q}_2} |x|_{\mathbb{Q}_2}^{-s-1} & \text{otherwise,} \end{cases}$$

$$f_w(w^{-1}u(x)) = \begin{cases} (-1)^l\sqrt{-1} & \text{if } x \in \mathbb{Z}_2, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\tilde{\mathfrak{G}}_\xi(a)$ be the Gauss sum as in Sect. A.2. Since $w^{-1}t(2)u(x)t(2) = t(4^{-1})w^{-1}u(4^{-1}x)$ in $\widetilde{\mathrm{SL}_2(\mathbb{Q}_2)}$, we have

$$\begin{aligned}
&\mathsf{W}(\mathsf{U}(f^+))(1)\\
&= 4\alpha^{-2}\int_{\mathbb{Z}_2} f^+(w^{-1}u(4^{-1}x))\,dx\\
&= 4\alpha^{-2}\cdot 2^{-3/2}\zeta_8^{(-1)^{l+1}}\cdot 4^{-1}(-1)^l\sqrt{-1}\\
&+ 4\alpha^{-2}\cdot\alpha^{-2}\cdot\int_{\mathbb{Z}_2^\times}\gamma_{\mathbb{Q}_2}\big(4^{-1}x,\psi_2\big)^{-1}((-1)^{l+1},4^{-1}x)_{\mathbb{Q}_2}|4^{-1}x|_{\mathbb{Q}_2}^{-s-1}\,dx\\
&= 2^{-3/2}\zeta_8^{(-1)^l}\alpha^{-2} + \alpha^{-2}\tilde{\mathfrak{G}}_0((-1)^{l+1})\\
&= 2^{-3/2}\zeta_8^{(-1)^l}\alpha^{-2} + 2^{-3/2}\zeta_8^{(-1)^l}\alpha^{-2}\\
&= 2^{-1/2}\zeta_8^{(-1)^l}\alpha^{-2}.
\end{aligned}$$

Since $k_1w^{-1}t(2)u(x)t(2) = t(4^{-1})w^{-1}u(-8^{-1}+4^{-1}x)$ in $\widetilde{\mathrm{SL}_2(\mathbb{Q}_2)}$, we have

$$\begin{aligned}
&\mathsf{W}(\mathsf{U}(f^+))(k_1)\\
&= 4\alpha^{-2}\int_{\mathbb{Z}_2} f^+(w^{-1}u(-8^{-1}+4^{-1}x))\,dx\\
&= 4\alpha^{-2}\cdot\alpha^{-2}\\
&\times\int_{\mathbb{Z}_2}\gamma_{\mathbb{Q}_2}\big(-8^{-1}+4^{-1}x,\psi_2\big)^{-1}((-1)^{l+1},-8^{-1}+4^{-1}x)_{\mathbb{Q}_2}\\
&\times|-8^{-1}+4^{-1}x|_{\mathbb{Q}_2}^{-s-1}\,dx\\
&= \alpha^{-1}\int_{\mathbb{Z}_2^\times}\gamma_{\mathbb{Q}_2}\big(8^{-1}x,\psi_2\big)^{-1}((-1)^{l+1},8^{-1}x)_{\mathbb{Q}_2}\,dx\\
&= \alpha^{-1}\tilde{\mathfrak{G}}_0((-1)^{l+1}8^{-1})\\
&= 0.
\end{aligned}$$

This completes the proof. □

Let $h \in S^+_{l+1/2}(\Gamma_0(4))$ be a Hecke eigenform and $f \in S_{2l}(\mathrm{SL}_2(\mathbb{Z}))$ the normalized Hecke eigenform associated to h. For each prime p, let $\{\alpha_p, \alpha_p^{-1}\}$ denote the Satake parameter of f at p. We fix $s_p \in \mathbb{C}$ such that $\alpha_p = p^{-s_p}$. Let $\mathbf{h}$ denote the cusp form on $\widetilde{\mathrm{SL}_2(\mathbb{A}_\mathbb{Q})}$ associated to h and $\tilde{\pi} = \otimes_v\tilde{\pi}_v$ the irreducible genuine cuspidal automorphic representation of $\widetilde{\mathrm{SL}_2(\mathbb{A}_\mathbb{Q})}$ generated by $\mathbf{h}$. By [54], $\tilde{\pi}_p$ is the principal series representation

$$\mathrm{Ind}_{\widetilde{B(\mathbb{Q}_p)}}^{\widetilde{\mathrm{SL}_2(\mathbb{Q}_p)}}\left((\chi_{-1,p}^l)^{\psi_p}|\ |_{\mathbb{Q}_p}^{s_p}\right)$$

for each prime p, and $\tilde{\pi}_\infty$ is the holomorphic discrete series representation of weight $l + 1/2$. Moreover, the multiplicity of $\tilde{\pi}$ in the space of cusp forms on $\widetilde{\mathrm{SL}_2(\mathbb{A}_\mathbb{Q})}$ is one. In the space of $\tilde{\pi}$, the conditions (3.9)–(3.12) characterize the cusp form $\mathbf{h}$ up to scalars by Lemma 3.6.

4. Theta lifts

4.1. Quadratic spaces. Let F be a field of characteristic not 2 and V a quadratic space over F. Namely, V is a vector space over F of dimension m equipped with a non-degenerate symmetric bilinear form $(\ ,\ )$. Let Q denote the associated quadratic form on V. Then

$$Q[x] = \frac{1}{2}(x, x)$$

for $x \in V$. We fix a basis $\{v_1, \dots, v_m\}$ of V and identify V with the space of column vectors F^m. Define $Q \in \mathrm{GL}_m(F)$ by

$$Q = ((v_i, v_j)),$$

and let $\det(V)$ denote the image of $\det(Q)$ in $F^\times / F^{\times,2}$. Then

$$(x, y) = {}^t x Q y$$

for $x, y \in V = F^m$. For $n \in \mathbb{N}$ and $x, y \in V^n = \mathrm{M}_{m,n}(F)$, we also write $(x, y) = {}^t x Q y$ and $Q[x] = \frac{1}{2}(x, x)$. Let

$$\mathrm{GO}(V) = \left\{ h \in \mathrm{GL}_m \mid {}^t h Q h = \nu(h) Q,\ \nu(h) \in \mathbb{G}_m \right\}$$

be the orthogonal similitude group and $\nu : \mathrm{GO}(V) \to \mathbb{G}_m$ the scale map. We let

$$\mathrm{GSO}(V) = \{ h \in \mathrm{GO}(V) \mid \det(h) = \nu(h)^{m/2} \}$$

when m is even. Let $\mathrm{O}(V) = \ker(\nu)$ denote the orthogonal group and $\mathrm{SO}(V) = \mathrm{O}(V) \cap \mathrm{SL}_m$ the special orthogonal group.

4.2. Weil representations. Let F be a local field of characteristic not 2 and V a quadratic space over F of dimension m. We fix a non-trivial additive character ψ of F. Define a quadratic character χ_V of $F^\times$ by

$$\chi_V(a) = ((-1)^{m(m-1)/2} \det(V), a)_F$$

for $a \in F^\times$, and an 8th root of unity γ_V by

$$\gamma_V = \gamma_F\big(\det(V), \tfrac{1}{2}\psi\big) \gamma_F\big(\tfrac{1}{2}\psi\big)^m h_F(V).$$

Here $h_F(V)$ is the Hasse invariant of V. Note that χ_V and γ_V depend only on the Witt class of V.

Let $\widetilde{\mathrm{Sp}_n(F)}$ be the 2-fold metaplectic cover of $\mathrm{Sp}_n(F)$. By abuse of notation, we write g for the element $(g, 1) \in \widetilde{\mathrm{Sp}_n(F)} \simeq \mathrm{Sp}_n(F) \times \{\pm 1\}$. Let $\omega = \omega_{V,n,\psi}$ denote the Weil representation of $\widetilde{\mathrm{Sp}_n(F)} \times \mathrm{O}(V)(F)$ on $\mathcal{S}(V^n) = \mathcal{S}(\mathrm{M}_{m,n}(F))$ with respect to ψ (cf. [36, §5]). Let $\varphi \in \mathcal{S}(V^n)$ and $x \in V^n$. Then

$$\omega(1, h)\varphi(x) = \varphi(h^{-1}x)$$

for $h \in \mathrm{O}(V)(F)$. The action of $\widetilde{\mathrm{Sp}_n(F)}$ is given by the following formulas:

$$\begin{aligned}
\omega\left(\begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix}, 1\right)\varphi(x) &= \chi_V(\det(a))|\det(a)|_F^{m/2}\varphi(xa) \\
&\quad \times \begin{cases} 1 & \text{if } m \text{ is even}, \\ \gamma_F\left(a, \frac{1}{2}\psi\right)^{-1} & \text{if } m \text{ is odd}, \end{cases} \\
\omega\left(\begin{pmatrix} \mathbf{1}_n & b \\ 0 & \mathbf{1}_n \end{pmatrix}, 1\right)\varphi(x) &= \varphi(x)\psi(\mathrm{tr}(bQ[x])), \\
\omega\left(\begin{pmatrix} 0 & -\mathbf{1}_n \\ \mathbf{1}_n & 0 \end{pmatrix}, 1\right)\varphi(x) &= \gamma_V^{-n}\int_{V^n}\varphi(y)\psi(-\mathrm{tr}(x, y))\,dy, \\
\omega((1, \epsilon), 1)\varphi(x) &= \epsilon\varphi(x),
\end{aligned}$$

for $a \in \mathrm{GL}_n(F)$, $b \in \mathrm{Sym}_n(F)$, and $\epsilon \in \{\pm 1\}$. Here dy is the self-dual measure on V^n with respect to the pairing $\psi(\mathrm{tr}(x, y))$, and is given by

$$dy = |\det(Q)|_F^{n/2}\prod_{i,j} dy_{ij}$$

for $y = (y_{ij}) \in V^n = \mathrm{M}_{m,n}(F)$, where dy_{ij} is the self-dual measure on F with respect to ψ. If m is even, then we may regard ω as a representation of $\mathrm{Sp}_n(F) \times \mathrm{O}(V)(F)$.

Following [25, §5.1], we extend the Weil representation ω. For simplicity, we assume that m is even. Put $R = \mathrm{G}(\mathrm{Sp}_n \times \mathrm{O}(V))$, where

$$\mathrm{G}(\mathrm{Sp}_n \times \mathrm{O}(V)) = \{(g, h) \in \mathrm{GSp}_n \times \mathrm{GO}(V) \mid \nu(g) = \nu(h)\}.$$

Then the Weil representation ω of $R(F)$ on $\mathcal{S}(V^n)$ is defined by

$$\omega(g, h)\varphi = \omega\left(g\begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & \nu(g)^{-1}\mathbf{1}_n \end{pmatrix}, 1\right)L(h)\varphi$$

for $(g, h) \in R(F)$ and $\varphi \in \mathcal{S}(V^n)$, where

$$L(h)\varphi(x) = |\nu(h)|_F^{-mn/4}\varphi(h^{-1}x)$$

for $x \in V^n$.

4.3. Theta functions and theta lifts. Let F be a number field and V a quadratic space over F of dimension m. We fix a non-trivial additive character ψ of $\mathbb{A}_F/F$. Let ω denote the Weil representation of $\widetilde{\mathrm{Sp}_n(\mathbb{A}_F)} \times \mathrm{O}(V)(\mathbb{A}_F)$ on $\mathcal{S}(V^n(\mathbb{A}_F))$ with respect to ψ. Let $S(V^n(\mathbb{A}_F))$ be the subspace of $\mathcal{S}(V^n(\mathbb{A}_F))$ consisting of functions which correspond to polynomials in the Fock model at every archimedean place. For $(g, h) \in \widetilde{\mathrm{Sp}_n(\mathbb{A}_F)} \times \mathrm{O}(V)(\mathbb{A}_F)$ and $\varphi \in S(V^n(\mathbb{A}_F))$, put

$$\theta(g, h; \varphi) = \sum_{x \in V^n(F)} \omega(g, h)\varphi(x). \tag{4.1}$$

Then $\theta(g, h; \varphi)$ is an automorphic form on $\widetilde{\mathrm{Sp}_n(\mathbb{A}_F)} \times \mathrm{O}(V)(\mathbb{A}_F)$ and is called a theta function. If m is even, then we may regard $\theta(g, h; \varphi)$ as an automorphic form on $\mathrm{Sp}_n(\mathbb{A}_F) \times \mathrm{O}(V)(\mathbb{A}_F)$.

Example 4.1. Let $F = \mathbb{Q}, m = n = 1$, and $\psi = \psi_0$. Here ψ_0 is the standard additive character of $\mathbb{A}_\mathbb{Q}$. Let $V = \mathbb{Q}$ be the quadratic space with bilinear form

$$(x, y) = 2xy.$$

We define $\varphi = \otimes_v \varphi_v^{(1)} \in S(\mathbb{A}_\mathbb{Q})$ as follows:

- If $v = p$, then $\varphi_p^{(1)}$ is the characteristic function of $\mathbb{Z}_p$.
- If $v = \infty$, then $\varphi_\infty^{(1)}(x) = e^{-2\pi x^2}$.

Note that

$$\omega((k, s_p(k)), 1)\varphi_p^{(1)} = \varphi_p^{(1)}$$

for $k \in \mathrm{SL}_2(\mathbb{Z}_p)$ if $p \neq 2$, and

$$\omega(k, 1)\varphi_2^{(1)} = \epsilon_2(k)\varphi_2^{(1)},$$
$$\omega(\tilde{k}_\theta, 1)\varphi_\infty^{(1)} = e^{\sqrt{-1}\theta/2}\varphi_\infty^{(1)},$$

for $k \in K_0(4; \mathbb{Z}_2)$, $\tilde{k}_\theta \in \widetilde{\mathrm{SO}(2)}$. For $g \in \widetilde{\mathrm{SL}_2(\mathbb{A}_\mathbb{Q})}$, put

$$\Theta(g) = \sum_{x \in \mathbb{Q}} \omega(g, 1)\varphi(x).$$

Then Θ is the automorphic form on $\widetilde{\mathrm{SL}_2(\mathbb{A}_\mathbb{Q})}$ associated to the classical theta function

$$\theta(\tau) = \sum_{n \in \mathbb{Z}} q^{n^2} \in M_{1/2}^+(\Gamma_0(4)).$$

Let f be a cusp form on $\mathrm{Sp}_n(\mathbb{A}_F)$ (resp. a genuine cusp form on $\widetilde{\mathrm{Sp}_n(\mathbb{A}_F)}$) when m is even (resp. odd). For $h \in \mathrm{O}(V)(\mathbb{A}_F)$ and $\varphi \in S(V^n(\mathbb{A}_F))$, put

$$\theta(h; f, \varphi) = \int_{\mathrm{Sp}_n(F)\backslash \mathrm{Sp}_n(\mathbb{A}_F)} \theta(g, h; \varphi) f(g)\, dg.$$

Then $\theta(f, \varphi)$ is an automorphic form on $\mathrm{O}(V)(\mathbb{A}_F)$. Let π be an irreducible cuspidal automorphic representation of $\mathrm{Sp}_n(\mathbb{A}_F)$ (resp. an irreducible genuine cuspidal automorphic representation of $\widetilde{\mathrm{Sp}_n(\mathbb{A}_F)}$) when m is even (resp. odd). We put

$$\theta(\pi) = \left\{\theta(f, \varphi) \mid f \in \pi, \varphi \in S\left(V^n(\mathbb{A}_F)\right)\right\}. \tag{4.2}$$

Then $\theta(\pi)$ is an automorphic representation of $\mathrm{O}(V)(\mathbb{A}_F)$ and is called the theta lift of π. Similarly, we define $\theta(f', \varphi)$ for a cusp form f' on $\mathrm{O}(V)(\mathbb{A}_F)$, and $\theta(\pi')$ for an irreducible cuspidal automorphic representation π' of $\mathrm{O}(V)(\mathbb{A}_F)$.

Following [25, §5.1], we also extend the theta lift. For simplicity, we assume that m is even again. For $(g, h) \in R(\mathbb{A}_F)$ and $\varphi \in S(V^n(\mathbb{A}_F))$, we can define $\theta(g, h; \varphi)$ by (4.1). Let f be a cusp form on $\mathrm{GSp}_n(\mathbb{A}_F)$. For $h \in \mathrm{GO}(V)(\mathbb{A}_F)$, choose $g' \in \mathrm{GSp}_n(\mathbb{A}_F)$ such that $\nu(g') = \nu(h)$, and put

$$\theta(h; f, \varphi) = \int_{\mathrm{Sp}_n(F)\backslash \mathrm{Sp}_n(\mathbb{A}_F)} \theta(gg', h; \varphi) f(gg')\, dg.$$

Note that this integral does not depend on the choice of g'. Then $\theta(f, \varphi)$ is an automorphic form on $\mathrm{GO}(V)(\mathbb{A}_F)$. For an irreducible cuspidal automorphic representation π of $\mathrm{GSp}_n(\mathbb{A}_F)$, the theta lift $\theta(\pi)$ of π is also defined by (4.2).

4.4. Change of polarizations. Let $n = 1$. We assume that the matrix $Q \in \mathrm{GL}_m(F)$ associated to V is of the form

$$Q = \begin{pmatrix} 0 & 0 & r \\ 0 & Q_1 & 0 \\ r & 0 & 0 \end{pmatrix}.$$

Here $Q_1 \in \mathrm{GL}_{m-2}(F)$ and $r \in F^\times$. Let $V_1 = F^{m-2}$ be the quadratic space with bilinear form

$$(v, w) = {}^t v Q_1 w.$$

The associated quadratic form on V_1 is also denoted by Q_1. For $v \in V_1$, define an element $\ell(v) \in \mathrm{O}(V)(F)$ by

$$\ell(v) = \begin{pmatrix} 1 & -r^{-1}{}^t v Q_1 & -r^{-1} Q_1[v] \\ 0 & \mathbf{1}_{m-2} & v \\ 0 & 0 & 1 \end{pmatrix}.$$

If F is a local field of characteristic not 2, let

$$\begin{aligned} \mathcal{S}(V) &\longrightarrow \mathcal{S}(V_1) \otimes \mathcal{S}(F^2) \\ \varphi &\longmapsto \hat{\varphi} \end{aligned}$$

be the partial Fourier transform given by

$$\hat{\varphi}(x_1; y) = |r|_F^{1/2} \int_F \varphi \begin{pmatrix} z \\ x_1 \\ y_1 \end{pmatrix} \psi(r y_2 z)\, dz$$

for $x_1 \in V_1$, $y = (y_1, y_2) \in F^2$. Here dz is the self-dual measure on F with respect to ψ. We define a representation $\hat{\omega}$ of $\widetilde{\mathrm{SL}_2(F)} \times \mathrm{O}(V)(F)$ on $\mathcal{S}(V_1) \otimes \mathcal{S}(F^2)$ by

$$\hat{\omega}(g, h)\hat{\varphi} = (\omega(g, h)\varphi)\hat{}.$$

If $\hat{\varphi} = \varphi_1 \otimes \varphi_2$ with $\varphi_1 \in \mathcal{S}(V_1)$ and $\varphi_2 \in \mathcal{S}(F^2)$, then

$$\hat{\omega}((g, \epsilon), 1)\hat{\varphi}(x_1; y) = \omega((g, \epsilon), 1)\varphi_1(x_1) \cdot \varphi_2(yg)$$

for $(g, \epsilon) \in \widetilde{\mathrm{SL}_2(F)}$. Also,

$$\hat{\omega}\left(1, \begin{pmatrix} a & 0 & 0 \\ 0 & h_1 & 0 \\ 0 & 0 & a^{-1} \end{pmatrix}\right) \hat{\varphi}(x_1; y) = |a|_F \hat{\varphi}\big(h_1^{-1} x_1; ay\big)$$

for $a \in F^\times$ and $h_1 \in \mathrm{O}(V_1)(F)$, and

$$\hat{\omega}(1, \ell(v))\hat{\varphi}(x_1; y) = \hat{\varphi}(x_1 - v y_1; y)\psi(-(v, x_1)y_2 + Q_1[v] y_1 y_2)$$

for $v \in V_1$.

If F is a number field, then we also obtain a representation $\hat{\omega}$ of $\widetilde{\mathrm{SL}_2(\mathbb{A}_F)} \times \mathrm{O}(V)(\mathbb{A}_F)$ on $\mathcal{S}(V_1(\mathbb{A}_F)) \otimes \mathcal{S}(\mathbb{A}_F^2)$ via the partial Fourier transform

$$\begin{aligned} \mathcal{S}(V(\mathbb{A}_F)) &\longrightarrow \mathcal{S}(V_1(\mathbb{A}_F)) \otimes \mathcal{S}\big(\mathbb{A}_F^2\big) \\ \varphi &\longmapsto \hat{\varphi} \end{aligned}$$

given by

$$\hat{\varphi}(x_1; y) = \int_{\mathbb{A}_F} \varphi \begin{pmatrix} z \\ x_1 \\ y_1 \end{pmatrix} \psi(r y_2 z)\, dz.$$

Let f be a cusp form on $\mathrm{SL}_2(\mathbb{A}_F)$ (resp. a genuine cusp form on $\widetilde{\mathrm{SL}_2(\mathbb{A}_F)}$) when m is even (resp. odd). For $\xi \in F$, the ξ-th Fourier coefficient $W_\xi = W_{f,\xi}$ of f is defined by

$$W_\xi(g) = \int_{F\backslash\mathbb{A}_F} f(u(x)g)\overline{\psi(\xi x)}\, dx.$$

Let $\varphi \in S(V(\mathbb{A}_F))$. For $\Xi \in V_1(F)$, define the Ξ-th Fourier coefficient $\mathcal{W}_\Xi = \mathcal{W}_{\theta(f,\varphi),\Xi}$ of $\theta(f,\varphi)$ by

$$\mathcal{W}_\Xi(h) = \int_{V_1(F)\backslash V_1(\mathbb{A}_F)} \theta(\ell(v)h;\, f,\varphi)\overline{\psi((\Xi,v))}\, dv.$$

Lemma 4.2. *If* $\Xi = 0$, *then*

$$\mathcal{W}_0(h) = \int_{\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A}_F)} \sum_{x_1\in V_1(F)} \hat{\omega}(g,h)\hat{\varphi}(x_1;0,0)\, f(g)\, dg.$$

If $\Xi \neq 0$, *then*

$$\mathcal{W}_\Xi(h) = \int_{U(\mathbb{A}_F)\backslash \mathrm{SL}_2(\mathbb{A}_F)} \hat{\omega}(g,h)\hat{\varphi}(-\Xi;0,1) W_{-Q_1[\Xi]}(g)\, dg.$$

Proof. The lemma follows from (5.5)–(5.7) of [43] with slight modifications. We include the proof for the sake of completeness.

By the Poisson summation formula,

$$\theta(g,h;\varphi) = \sum_{x_1\in V_1(F)} \sum_{y\in F^2} \hat{\omega}(g,h)\hat{\varphi}(x_1;y).$$

Since the map

$$\begin{aligned} U(F)\backslash \mathrm{SL}_2(F) &\longrightarrow \{y \in F^2 \mid y \neq 0\} \\ \gamma &\longmapsto (0,1)\gamma \end{aligned}$$

is bijective, we have

$$\begin{aligned}\theta(g,h;\varphi) = &\sum_{x_1\in V_1(F)} \hat{\omega}(g,h)\hat{\varphi}(x_1;0,0) \\ &+ \sum_{\gamma\in U(F)\backslash \mathrm{SL}_2(F)} \sum_{x_1\in V_1(F)} \hat{\omega}(\gamma g,h)\hat{\varphi}(x_1;0,1),\end{aligned}$$

hence

$$\begin{aligned}\theta(h;\, f,\varphi) = &\int_{\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A}_F)} \sum_{x_1\in V_1(F)} \hat{\omega}(g,h)\hat{\varphi}(x_1;0,0)\, f(g)\, dg \\ &+ \int_{U(F)\backslash \mathrm{SL}_2(\mathbb{A}_F)} \sum_{x_1\in V_1(F)} \hat{\omega}(g,h)\hat{\varphi}(x_1;0,1)\, f(g)\, dg.\end{aligned}$$

Note that

$$\begin{aligned}\hat{\omega}(g,\ell(v)h)\hat{\varphi}(x_1;0,0) &= \hat{\omega}(g,h)\hat{\varphi}(x_1;0,0), \\ \hat{\omega}(g,\ell(v)h)\hat{\varphi}(x_1;0,1) &= \hat{\omega}(g,h)\hat{\varphi}(x_1;0,1)\psi(-(x_1,v)).\end{aligned}$$

The integral

$$\int_{V_1(F)\backslash V_1(\mathbb{A}_F)} \int_{\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A}_F)} \sum_{x_1 \in V_1(F)} \hat{\omega}(g, \ell(v)h)\hat{\varphi}(x_1; 0, 0)\, f(g)\overline{\psi((\Xi, v))}\, dg\, dv$$

is equal to

$$\int_{\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A}_F)} \sum_{x_1 \in V_1(F)} \hat{\omega}(g, h)\hat{\varphi}(x_1; 0, 0)\, f(g)\, dg$$

if $\Xi = 0$, and vanishes if $\Xi \neq 0$. The integral

$$\int_{V_1(F)\backslash V_1(\mathbb{A}_F)} \int_{U(F)\backslash \mathrm{SL}_2(\mathbb{A}_F)} \sum_{x_1 \in V_1(F)} \hat{\omega}(g, \ell(v)h)\hat{\varphi}(x_1; 0, 1)\, f(g)\overline{\psi((\Xi, v))}\, dg\, dv$$

is equal to

$$\begin{aligned} &\int_{U(F)\backslash \mathrm{SL}_2(\mathbb{A}_F)} \hat{\omega}(g, h)\hat{\varphi}(-\Xi; 0, 1)\, f(g)\, dg \\ &= \int_{U(\mathbb{A}_F)\backslash \mathrm{SL}_2(\mathbb{A}_F)} \int_{F\backslash \mathbb{A}_F} \hat{\omega}(g, h)\hat{\varphi}(-\Xi; 0, 1)\psi(xQ_1[\Xi])\, f(u(x)g)\, dx\, dg \\ &= \int_{U(\mathbb{A}_F)\backslash \mathrm{SL}_2(\mathbb{A}_F)} \hat{\omega}(g, h)\hat{\varphi}(-\Xi; 0, 1)\, W_{-Q_1[\Xi]}(g)\, dg. \end{aligned}$$

This completes the proof. □

5. Seesaw identities and proof of the main theorem

In this section, we state three seesaw identities, and show that these identities and the Kohnen-Zagier formula imply the main theorem.

We retain the notation of Sect. 2. Let κ be an odd positive integer. Let $f \in S_{2\kappa}(\mathrm{SL}_2(\mathbb{Z}))$ and $g \in S_{\kappa+1}(\mathrm{SL}_2(\mathbb{Z}))$ be normalized Hecke eigenforms. Let

$$h(\tau) = \sum_{n=1}^{\infty} c_h(n)q^n \in S^+_{\kappa+1/2}(\Gamma_0(4))$$

be a Hecke eigenform associated to f and $F \in S_{\kappa+1}(\mathrm{Sp}_2(\mathbb{Z}))$ the Saito-Kurokawa lift of h. We may assume that $c_h(n) \in \mathbb{R}$ for all $n \in \mathbb{N}$. In particular, $\langle F|_{\mathfrak{H}\times\mathfrak{H}}, g \times g\rangle \in \mathbb{R}$.

Fix a fundamental discriminant $-D < 0$ with $-D \equiv 1 \bmod 8$ such that $\Lambda(\kappa, f, \chi_{-D}) \neq 0$ (and hence $c_h(D) \neq 0$). Here χ_{-D} is the Dirichlet character associated to $\mathbb{Q}(\sqrt{-D})/\mathbb{Q}$ and $\Lambda(s, f, \chi_{-D})$ is the completed L-function given by

$$\Lambda(s, f, \chi_{-D}) = D^s\Gamma_{\mathbb{C}}(s)L(s, f, \chi_{-D}).$$

Such a discriminant exists by [56], [10]. Let $\mathcal{K} = \mathbb{Q}(\sqrt{-D})$.

Let $\mathbf{f}$ and $\mathbf{g}$ denote the cusp forms on $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ associated to f and g, respectively. Let π be the irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ generated by $\mathbf{g}$ and $\pi_\mathcal{K}$ the base change of π to $\mathcal{K}$. We define cusp forms $\mathbf{g}^\sharp \in \pi$ and $\mathbf{g}^\sharp_\mathcal{K} \in \pi_\mathcal{K}$ by (3.3) and (3.7), respectively. Let $\mathbf{h}$ and Θ denote the automorphic forms on $\widetilde{\mathrm{SL}_2(\mathbb{A}_\mathbb{Q})}$ associated to h and θ, respectively. Here

$$\theta(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2} \in M^+_{1/2}(\Gamma_0(4))$$

is the classical theta function. We put

$$\langle \mathbf{h}\Theta, \mathbf{g}^\sharp\rangle = \int_{\mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{A}_\mathbb{Q})} \mathbf{h}(g)\Theta(g)\overline{\mathbf{g}^\sharp(g)}\, dg,$$

$$\mathcal{I}\left(\mathbf{g}^\sharp_\mathcal{K}, \mathbf{f}\right) = \int_{\mathbb{A}^\times_\mathbb{Q}\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})} \mathbf{g}^\sharp_\mathcal{K}(h)\mathbf{f}(h)\, dh.$$

Let $\xi_\mathbb{Q}(s) = \Gamma_\mathbb{R}(s)\zeta(s)$.

Our seesaw identities are as follows.

Proposition 5.1.

$$\langle F|_{\mathfrak{H}\times\mathfrak{H}}, g \times g\rangle = 2^{\kappa+2}\xi_\mathbb{Q}(2)\langle g, g\rangle\langle \mathbf{h}\Theta, \mathbf{g}^\sharp\rangle.$$

Proposition 5.2.

$$\langle \mathbf{h}\Theta, \mathbf{g}^\sharp\rangle = (\sqrt{-1})^\kappa D^{-1/2} c_h(D)^{-1}\langle f, f\rangle^{-1}\langle h, h\rangle \mathcal{I}\left(\mathbf{g}^\sharp_\mathcal{K}, \mathbf{f}\right).$$

Proposition 5.3.

$$\Lambda(2\kappa, \mathrm{Sym}^2(g)\otimes f)\Lambda(\kappa, f, \chi_{-D}) = -2^{2\kappa+6}D^{-1/2}\xi_\mathbb{Q}(2)^2\mathcal{I}\left(\mathbf{g}^\sharp_\mathcal{K}, \mathbf{f}\right)^2.$$

Assuming Propositions 5.1–5.3, we now prove Theorem 2.1. By Propositions 5.1 and 5.2,

$$\begin{aligned}
&2^{\kappa+1}\langle f, f\rangle\langle h, h\rangle^{-1}\langle g, g\rangle^{-2}\langle F|_{\mathfrak{H}\times\mathfrak{H}}, g\times g\rangle^2 \\
&= 2^{3\kappa+5}\xi_\mathbb{Q}(2)^2\langle f, f\rangle\langle h, h\rangle^{-1}\langle \mathbf{h}\Theta, \mathbf{g}^\sharp\rangle^2 \\
&= -2^{3\kappa+5}D^{-1}c_h(D)^{-2}\xi_\mathbb{Q}(2)^2\langle f, f\rangle^{-1}\langle h, h\rangle\mathcal{I}\left(\mathbf{g}^\sharp_\mathcal{K}, \mathbf{f}\right)^2.
\end{aligned}$$

By the Kohnen-Zagier formula [33],

$$\Lambda(\kappa, f, \chi_{-D}) = 2^{-\kappa+1}D^{1/2}c_h(D)^2\langle f, f\rangle\langle h, h\rangle^{-1},$$

hence

$$\begin{aligned}
&\Lambda(2\kappa, \mathrm{Sym}^2(g)\otimes f) \\
&= -2^{3\kappa+5}D^{-1}c_h(D)^{-2}\xi_\mathbb{Q}(2)^2\langle f, f\rangle^{-1}\langle h, h\rangle\mathcal{I}(\mathbf{g}^\sharp_\mathcal{K}, \mathbf{f})^2
\end{aligned}$$

by Proposition 5.3. This completes the proof of Theorem 2.1.

6. Theta correspondence for $(\mathrm{GL}_2, \mathrm{GO}(2,2))$

In this section, we study the Jacquet-Langlands-Shimizu correspondence in terms of theta lifts [49].

6.1. Preliminaries. Let F be a field of characteristic not 2. Let $*$ denote the involution on $\mathrm{M}_2(F)$ given by

$$x^* = \begin{pmatrix} x_4 & -x_2 \\ -x_3 & x_1 \end{pmatrix} \quad \text{for } x = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in \mathrm{M}_2(F).$$

Let $V = \mathrm{M}_2(F)$ be the quadratic space with bilinear form

$$(x, y) = \mathrm{tr}(xy^*).$$

Then the associated quadratic form is given by

$$Q[x] = \det(x).$$

Recall that there is an exact sequence

$$1 \longrightarrow \mathbb{G}_m \xrightarrow{\iota} \mathrm{GL}_2 \times \mathrm{GL}_2 \xrightarrow{\rho} \mathrm{GSO}(V) \longrightarrow 1.$$

Here

$$\iota(a) = (a\mathbf{1}_2, a\mathbf{1}_2), \quad \rho(h_1, h_2)x = h_1 x h_2^{-1},$$

for $a \in \mathbb{G}_m, h_1, h_2 \in \mathrm{GL}_2$, and $x \in V$. Note that $\nu(\rho(h_1, h_2)) = \det(h_1 h_2^{-1})$.

If F is a local field, let ω be the Weil representation of $R(F)$ on $\mathcal{S}(V)$ with respect to a fixed additive character ψ. Here $R = \mathrm{G}(\mathrm{SL}_2 \times \mathrm{O}(V))$. Let

$$\begin{aligned} \mathcal{S}(V) &\longrightarrow \mathcal{S}(V) \\ \varphi &\longmapsto \check{\varphi} \end{aligned}$$

be the partial Fourier transform given by

$$\check{\varphi}\left(\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}\right) = \int_{F^2} \varphi\left(\begin{pmatrix} x_1 & y_2 \\ x_3 & y_4 \end{pmatrix}\right) \psi(-x_4 y_2 + x_2 y_4)\, dy_2\, dy_4.$$

Here dy_2, dy_4 are the self-dual measures on F with respect to ψ. Then $\check{\check{\varphi}} = \varphi$. We define a representation $\check{\omega}$ of $R(F)$ on $\mathcal{S}(V)$ by

$$\check{\omega}(g, h)\check{\varphi} = (\omega(g, h)\varphi)\check{}.$$

Note that

$$\begin{aligned} \check{\omega}(g, 1)\check{\varphi}(x) &= \check{\varphi}(xg), \\ \check{\omega}(1, (d(\nu), d(\nu)))\check{\varphi}(x) &= |\nu|_F^{-1}\check{\varphi}(d(\nu^{-1})x), \end{aligned}$$

for $g \in \mathrm{SL}_2(F)$, $\nu \in F^\times$.

6.2. Theta lifts. Let F be a number field and fix a non-trivial additive character ψ of $\mathbb{A}_F/F$. Let π be an irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$. We may regard $\pi \boxtimes \pi^\vee$ as a representation of $\mathrm{GSO}(V)(\mathbb{A}_F)$. Let Π be a unique extension of $\pi \boxtimes \pi^\vee$ to $\mathrm{GO}(V)(\mathbb{A}_F)$ on which there is a non-zero $\mathrm{O}(V')(\mathbb{A}_F)$-invariant distribution, where $V' = \{x \in V \mid \mathrm{tr}(x) = 0\}$. By [49], [24, §7],

$$\theta(\pi) = \Pi, \quad \theta(\Pi^\vee) = \pi^\vee.$$

Let f be a cusp form on $\mathrm{GL}_2(\mathbb{A}_F)$. Let $\varphi \in S(V(\mathbb{A}_F))$. We may regard $\theta(f, \varphi)$ as an automorphic form on $\mathrm{GL}_2(\mathbb{A}_F) \times \mathrm{GL}_2(\mathbb{A}_F)$. For $\xi_1, \xi_2 \in F$, define the (ξ_1, ξ_2)-th Fourier coefficient $\mathcal{W}_{\theta(f,\varphi),\xi_1,\xi_2}$ of $\theta(f, \varphi)$ by

$$\begin{aligned}&\mathcal{W}_{\theta(f,\varphi),\xi_1,\xi_2}(h)\\ &= \int_{F\backslash\mathbb{A}_F}\int_{F\backslash\mathbb{A}_F} \theta((u(x_1), u(x_2))h;\, f, \varphi)\overline{\psi(\xi_1 x_1)\psi(\xi_2 x_2)}\, dx_1\, dx_2.\end{aligned}$$

Lemma 6.1. *Let $\xi \in F^\times$. Then*

$$\mathcal{W}_{\theta(f,\varphi),\xi,\xi}(h) = \int_{\mathrm{SL}_2(\mathbb{A}_F)} \check{\omega}(g', h)\check{\varphi}(gd(\xi))W_{f,\xi}(gd(\xi)g')\, dg$$

with $g' \in \mathrm{GL}_2(\mathbb{A}_F)$ such that $\nu(g') = \nu(h)$.

Proof. By Lemma 5.1 of [27],

$$\mathcal{W}_{\theta(f,\varphi),1,1}(h) = \int_{\mathrm{SL}_2(\mathbb{A}_F)} \check{\omega}(g', h)\check{\varphi}(g)W_{f,1}(gg')\, dg.$$

Hence $\mathcal{W}_{\theta(f,\varphi),\xi,\xi}(h)$ is equal to

$$\begin{aligned}&\mathcal{W}_{\theta(f,\varphi),1,1}((d(\xi^{-1}), d(\xi^{-1}))h)\\ &= \int_{\mathrm{SL}_2(\mathbb{A}_F)} \check{\omega}(g', (d(\xi^{-1}), d(\xi^{-1}))h)\check{\varphi}(g)W_{f,1}(gg')\, dg\\ &= \int_{\mathrm{SL}_2(\mathbb{A}_F)} \check{\omega}(g', h)\check{\varphi}(d(\xi)g)W_{f,\xi}(d(\xi)gg')\, dg\\ &= \int_{\mathrm{SL}_2(\mathbb{A}_F)} \check{\omega}(g', h)\check{\varphi}(gd(\xi))W_{f,\xi}(gd(\xi)g')\, dg.\end{aligned}$$

□

6.3. The case $F = \mathbb{Q}$. Let $\psi = \psi_0$. For a normalized Hecke eigenform $g \in S_l(\mathrm{SL}_2(\mathbb{Z}))$, let $\mathbf{g}$ denote the cusp form on $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ associated to g. We can extend the cusp form $\mathbf{g} \otimes \mathbf{g}$ on $\mathrm{GSO}(V)(\mathbb{A}_\mathbb{Q})$ to a cusp form $\mathcal{G}$ on $\mathrm{GO}(V)(\mathbb{A}_\mathbb{Q})$ such that $\mathcal{G}(hh') = \mathcal{G}(h)$ for $h \in \mathrm{GO}(V)(\mathbb{A}_\mathbb{Q})$ and $h' \in \mu_2(\mathbb{A}_\mathbb{Q})$. Here μ_2 is the subgroup of $\mathrm{O}(V)$ generated by the involution $*$ on V. We define $\varphi = \otimes_v \varphi_v^{(2)} \in S(V(\mathbb{A}_\mathbb{Q}))$ as follows:

- If $v = p$, then $\varphi_p^{(2)}$ is the characteristic function of $V(\mathbb{Z}_p)$.
- If $v = \infty$, then

$$\varphi_\infty^{(2)}(x) = (x_1 + \sqrt{-1}x_2 + \sqrt{-1}x_3 - x_4)^l e^{-\pi\,\mathrm{tr}(x^t x)}.$$

Note that

$$\omega(k, k')\varphi_p^{(2)} = \varphi_p^{(2)}, \tag{6.1}$$

$$\omega(k_\theta, (k_{\theta_1}, k_{\theta_2}))\varphi_\infty^{(2)} = e^{-\sqrt{-1}l\theta} e^{\sqrt{-1}l\theta_1} e^{\sqrt{-1}l\theta_2} \varphi_\infty^{(2)}, \tag{6.2}$$

for $(k, k') \in R(\mathbb{Z}_p)$, $k_\theta, k_{\theta_1}, k_{\theta_2} \in \mathrm{SO}(2)$.

Lemma 6.2.

$$\theta(\mathbf{g}, \varphi) = 2^l \xi_\mathbb{Q}(2)^{-1} \mathcal{G}.$$

Proof. See Proposition 5.2 of [27]. □

Let $\mathbf{g}^\sharp$ be the cusp form on $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ defined by (3.3). We define $\varphi^\sharp = \otimes_v \varphi_v^{(3)} \in S(V(\mathbb{A}_\mathbb{Q}))$ as follows:

- If $v \neq 2$, then $\varphi_v^{(3)} = \varphi_v^{(2)}$.
- If $v = 2$, then $\varphi_2^{(3)}$ is the characteristic function of $V(2\mathbb{Z}_2)$.

Lemma 6.3.

$$\theta(\bar{\mathcal{G}}, \varphi^\sharp) = 2^{l-3} \xi_\mathbb{Q}(2)^{-2} \langle g, g \rangle \bar{\mathbf{g}}^\sharp.$$

Proof. Since $\varphi_2^{(3)} = 2^{-2}\omega(t(2^{-1}), 1)\varphi_2^{(2)}$, it suffices to show that

$$\theta(\bar{\mathcal{G}}, \varphi) = 2^{l-1} \xi_\mathbb{Q}(2)^{-2} \langle g, g \rangle \bar{\mathbf{g}}.$$

By (6.1), (6.2), there exists a constant C such that

$$\theta(\bar{\mathcal{G}}, \varphi) = C\bar{\mathbf{g}}.$$

Hence

$$\langle \theta(\bar{\mathcal{G}}, \varphi), \bar{\mathbf{g}} \rangle = \langle \bar{\mathbf{g}}, \bar{\mathbf{g}} \rangle C = 2^{-1} \xi_\mathbb{Q}(2)^{-1} \langle g, g \rangle C.$$

On the other hand,

$$\langle \theta(\bar{\mathcal{G}}, \varphi), \bar{\mathbf{g}} \rangle = \langle \bar{\mathcal{G}}, \overline{\theta(\mathbf{g}, \varphi)} \rangle = 2^l \xi_\mathbb{Q}(2)^{-1} \langle \bar{\mathcal{G}}, \bar{\mathcal{G}} \rangle = 2^{l-2} \xi_\mathbb{Q}(2)^{-3} \langle g, g \rangle^2$$

by Lemma 6.2. Therefore $C = 2^{l-1} \xi_\mathbb{Q}(2)^{-2} \langle g, g \rangle$. □

6.4. The case $F = \mathcal{K}$. Let $\mathcal{K}$ be an imaginary quadratic field with discriminant $-D < 0$ and $\mathcal{O}$ the ring of integers of $\mathcal{K}$. Let $\psi = \frac{1}{2}(\psi_0 \circ \mathrm{tr}_{\mathcal{K}/\mathbb{Q}})$. We put $\delta = \sqrt{-D}$. For a normalized Hecke eigenform $g \in S_{\kappa+1}(\mathrm{SL}_2(\mathbb{Z}))$, let $\mathbf{g}$ denote the cusp form on $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ associated to g. Let $\pi = \otimes_v \pi_v$ be the irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ generated by $\mathbf{g}$ and $\pi_\mathcal{K} = \otimes_v \pi_{\mathcal{K},v}$ the base change of π to $\mathcal{K}$. Let $\mathbf{g}^\sharp_\mathcal{K}, \mathbf{g}^\flat_\mathcal{K} \in \pi_\mathcal{K}$ be the cusp forms on $\mathrm{GL}_2(\mathbb{A}_\mathcal{K})$ defined by (3.7), (3.8), respectively. We define $\varphi = \otimes_v \varphi_v^{(4)} \in S(V(\mathbb{A}_\mathcal{K}))$ as follows:

- If $v = p$, then $\varphi_p^{(4)}$ is the characteristic function of $V(\mathcal{O}_p)$.
- If $v = \infty$, then $\varphi_\infty^{(4)} = (2\kappa + 1)\omega(1, (k_\infty, k_\infty))\varphi'_\infty$. Here

$$\varphi'_\infty(x) = x_3^{2\kappa} e^{-\pi\,\mathrm{tr}(x^t\bar{x})}.$$

Note that

$$\omega(k, k')\varphi_p^{(4)} = \varphi_p^{(4)} \tag{6.3}$$

for $k \in d(2\delta^{-1})^{-1}\,\mathrm{GL}_2(\mathcal{O}_p)d(2\delta^{-1})$, $k' \in \mathrm{GO}(V)(\mathcal{O}_p)$ such that $\nu(k) = \nu(k')$, and

$$(H, 0)\cdot\varphi'_\infty = 2\kappa\varphi'_\infty, \qquad (X, 0)\cdot\varphi'_\infty = 0, \tag{6.4}$$

$$(0, H)\cdot\varphi'_\infty = 2\kappa\varphi'_\infty, \qquad (0, X)\cdot\varphi'_\infty = 0. \tag{6.5}$$

Here $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ is a subalgebra of the Lie algebra of $\mathrm{O}(V)(\mathbb{C})$. We may regard $\theta(\bar{\mathbf{g}}^\flat_\mathcal{K}, \varphi)$ as an automorphic form on $\mathrm{GL}_2(\mathbb{A}_\mathcal{K}) \times \mathrm{GL}_2(\mathbb{A}_\mathcal{K})$.

Lemma 6.4.

$$\theta\big(\bar{\mathbf{g}}^\flat_\mathcal{K}, \varphi\big) = 2^{\kappa+4}(\sqrt{-1})^\kappa D^{-\kappa/2-1}\xi_\mathcal{K}(2)^{-1}\big(\mathbf{g}^\sharp_\mathcal{K} \otimes \mathbf{g}^\sharp_\mathcal{K}\big).$$

The rest of this section is devoted to the proof of Lemma 6.4. We will compute the Fourier coefficient $\mathcal{W} = \mathcal{W}_{\theta(\bar{\mathbf{g}}^\flat_\mathcal{K}, \varphi), 2\delta^{-1}, 2\delta^{-1}}$ of $\theta(\bar{\mathbf{g}}^\flat_\mathcal{K}, \varphi)$. Let W_v be the Whittaker function of $\pi_{\mathcal{K},v}$ with respect to $2\delta^{-1}\psi_v$ which satisfies the following conditions:

- If $v = p$, then $W_p(1) = 1$, and

$$W_p(gk) = W_p(g)$$

  for $g \in \mathrm{GL}_2(\mathcal{K}_p)$, $k \in \mathrm{GL}_2(\mathcal{O}_p)$.
- If $v = \infty$, then $W_\infty(1) = K_\kappa(4\pi D^{-1/2})$, and

$$H\cdot W_\infty = 2\kappa W_\infty, \quad X\cdot W_\infty = 0.$$

Then

$$W_{\mathbf{g}^{\flat}_{\mathcal{K}}, -2\delta^{-1}} = \prod_{v} W^{\flat}_{v},$$

where

$$W^{\flat}_{v}(g) = \begin{cases} W_p(d(-1)gd(2\delta^{-1})^{-1}) & \text{if } v = p, \\ W_{\infty}(d(-1)g) & \text{if } v = \infty. \end{cases}$$

By Lemma 6.1,

$$\mathcal{W} = \pi^{-1}\xi_{\mathcal{K}}(2)^{-1} \prod_{v} \mathcal{W}_{v},$$

where

$$\mathcal{W}_v(h) = \int_{\mathrm{SL}_2(\mathcal{K}_v)} \check{\omega}(g', h)\check{\varphi}^{(4)}_v(gd(2\delta^{-1}))\overline{W^{\flat}_v(gd(2\delta^{-1})g')}\, dg$$
$$\times \begin{cases} \mathrm{vol}(\mathrm{SL}_2(\mathcal{O}_p))^{-1} & \text{if } v = p, \\ \mathrm{vol}(\mathrm{SU}(2))^{-1} & \text{if } v = \infty, \end{cases}$$

with $g' \in \mathrm{GL}_2(\mathcal{K}_v)$ such that $\nu(g') = \nu(h)$.

Lemma 6.5.

$$\mathcal{W}_p(1) = |2^{-1}\delta|_{\mathcal{K}_p}.$$

Proof. Since

$$\check{\varphi}^{(4)}_p = |2^{-1}\delta|_{\mathcal{K}_p} \times \text{the characteristic function of } \mathrm{M}_2(\mathcal{O}_p)d(2\delta^{-1}),$$

the assertion follows. □

Lemma 6.6. *Let $n \in \mathbb{Z}_{\geq 0}$. For each $0 \leq i \leq n$, let ϕ_i be the function on* $\mathrm{SU}(2)$ *defined by*

$$\phi_i\left(\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}\right) = \alpha^{n-i}\bar{\beta}^{i}.$$

Then

$$H \cdot \phi_i = n\phi_i, \quad X \cdot \phi_i = 0,$$

and

$$\mathrm{vol}(\mathrm{SU}(2))^{-1} \int_{\mathrm{SU}(2)} \phi_i(k)\overline{\phi_j(k)}\, dk = \begin{cases} (n+1)^{-1}\binom{n}{i}^{-1} & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Proof. Let V_n be the space of homogeneous polynomials of degree n in $\mathbb{C}[z_1, z_2]$. Then ρ_n is realized on V_n by the formula

$$(\rho_n(k)v)(z) = v(zk)$$

for $k \in \mathrm{SU}(2)$, $v \in V_n$, and $z = (z_1, z_2)$. Let $(\ ,\ )$ be the $\mathrm{SU}(2)$-invariant hermitian form on V_n given by

$$(v, w) = \sum_{i=0}^{n} \binom{n}{i}^{-1} a_i \bar{b}_i$$

for

$$v(z) = \sum_{i=0}^{n} a_i z_1^{n-i} z_2^i, \quad w(z) = \sum_{i=0}^{n} b_i z_1^{n-i} z_2^i.$$

For each $0 \le i \le n$, define $v_i \in V_n$ by

$$v_i(z) = (-1)^i z_1^{n-i} z_2^i.$$

Then

$$\phi_i(k) = (\rho_n(k)v_0, v_i)$$

for $k \in \mathrm{SU}(2)$. Moreover,

$$\mathrm{vol}(\mathrm{SU}(2))^{-1} \int_{\mathrm{SU}(2)} \phi_i(k)\overline{\phi_j(k)}\, dk = (n+1)^{-1}(v_0, v_0)\overline{(v_i, v_j)}$$

by the Schur orthogonality relation. This yields the lemma. □

Lemma 6.7.

$$\mathcal{W}_\infty\left(\left(k_\infty^{-1}, k_\infty^{-1}\right)\right) = 2^{\kappa+2}\pi(\sqrt{-1})^\kappa D^{-\kappa/2} K_\kappa(4\pi D^{-1/2})^2.$$

Proof. Let $x \in \mathbb{C}$, $a \in \mathbb{R}_+^\times$, and

$$k = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \in \mathrm{SU}(2).$$

Then

$$\begin{aligned}\overline{W_\infty^\flat(d(2\delta^{-1})u(x)t(a)k)} &= 2^{-\kappa-1} D^{(\kappa+1)/2} e^{\pi\sqrt{-1}\,\mathrm{tr}_{\mathbb{C}/\mathbb{R}}(x)} a^{2\kappa+2} \\ &\times \sum_{m=0}^{2\kappa} \binom{2\kappa}{m} (\sqrt{-1})^{\kappa+m} \bar{\alpha}^{2\kappa-m} \beta^m K_{\kappa-m}(2\pi a^2)\end{aligned}$$

(cf. [58, Appendix I, §2.4]). Since $\check{\varphi}'_\infty = \varphi'_\infty$, we have

$$\check{\varphi}'_\infty(d(2\delta^{-1})u(x)t(a)k) = 2^{2\kappa}(\sqrt{-1})^{2\kappa} D^{-\kappa} a^{-2\kappa} \bar{\beta}^{2\kappa} e^{-\pi(a^2+4D^{-1}a^{-2}+a^{-2}|x|^2)}.$$

Hence $\mathcal{W}_\infty((k_\infty^{-1}, k_\infty^{-1}))$ is equal to

$$\begin{aligned}
&(2\kappa+1)\,\mathrm{vol}(\mathrm{SU}(2))^{-1}\int_{\mathrm{SL}_2(\mathbb{C})}\check{\varphi}'_\infty(d(2\delta^{-1})g)\overline{W^\flat_\infty(d(2\delta^{-1})g)}\,dg\\
&=2^2D^{-1/2}(2\kappa+1)\,\mathrm{vol}(\mathrm{SU}(2))^{-1}\\
&\times\int_{\mathbb{C}\times\mathbb{C}^\times\times\mathrm{SU}(2)}\check{\varphi}'_\infty(d(2\delta^{-1})u(x)t(a)k)\overline{W^\flat_\infty(d(2\delta^{-1})u(x)t(a)k)}|a|_{\mathbb{C}}^{-2}\,dx\,d^\times a\,dk\\
&=2^{\kappa+2}\pi(\sqrt{-1})^\kappa D^{-\kappa/2}\\
&\times\int_{\mathbb{C}\times\mathbb{R}_+^\times}a^{-2}K_{-\kappa}(2\pi a^2)e^{-\pi(a^2+4D^{-1}a^{-2}+a^{-2}|x|^2)}e^{\pi\sqrt{-1}\,\mathrm{tr}_{\mathbb{C}/\mathbb{R}}(x)}\,dx\,d^\times a
\end{aligned}$$

by Lemma 6.6.

Recall that

$$\int_0^\infty e^{-2^{-1}a-2^{-1}(z^2+w^2)a^{-1}}K_\nu(zwa^{-1})\,d^\times a=2K_\nu(z)K_\nu(w)$$

if $|\arg(z)|<\pi$, $|\arg(w)|<\pi$, and $|\arg(z+w)|<\pi/4$ (cf. [20, 6.653.2]). Since

$$\int_{\mathbb{C}}e^{-\pi a^{-2}|x|^2}e^{\pi\sqrt{-1}\,\mathrm{tr}_{\mathbb{C}/\mathbb{R}}(x)}\,dx=a^2e^{-\pi a^2},$$

we have

$$\begin{aligned}
&\int_{\mathbb{C}\times\mathbb{R}_+^\times}a^{-2}K_{-\kappa}(2\pi a^2)e^{-\pi(a^2+4D^{-1}a^{-2}+a^{-2}|x|^2)}e^{\pi\sqrt{-1}\,\mathrm{tr}_{\mathbb{C}/\mathbb{R}}(x)}\,dx\,d^\times a\\
&=\int_0^\infty e^{-2\pi a^2-4\pi D^{-1}a^{-2}}K_{-\kappa}(2\pi a^2)\,d^\times a\\
&=2^{-1}\int_0^\infty e^{-2^{-1}a-16\pi^2D^{-1}a^{-1}}K_\kappa(16\pi^2D^{-1}a^{-1})\,d^\times a\\
&=K_\kappa(4\pi D^{-1/2})^2.
\end{aligned}$$

This completes the proof. □

Now we prove Lemma 6.4. By (6.3)–(6.5), there exists a constant C such that

$$\theta(\bar{\mathbf{g}}^\flat_{\mathcal{K}},\varphi)=C(\mathbf{g}^\sharp_{\mathcal{K}}\otimes\mathbf{g}^\sharp_{\mathcal{K}}).$$

Hence

$$C=K_\kappa(4\pi D^{-1/2})^{-2}\mathcal{W}\big((k_\infty^{-1},k_\infty^{-1})\big)=2^{\kappa+4}(\sqrt{-1})^\kappa D^{-\kappa/2-1}\xi_{\mathcal{K}}(2)^{-1}.$$

This completes the proof of Lemma 6.4.

7. Theta correspondence for $(\widetilde{\mathrm{SL}}_2, \mathrm{PGSp}_2)$

In this section, we study the Saito-Kurokawa lifting in terms of theta lifts [43].

7.1. Preliminaries. Let F be a field of characteristic not 2. The symplectic similitude group GSp_2 acts on the space of column vectors F^4 on the left. Let

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Then $\tilde{V} = \wedge^2(F^4)$ is equipped with a non-degenerate symmetric bilinear form (,) given by

$$x \wedge y = (x, y) \cdot (e_1 \wedge e_2 \wedge e_3 \wedge e_4)$$

for $x, y \in \tilde{V}$. Let $x_0 = e_1 \wedge e_3 + e_2 \wedge e_4 \in \tilde{V}$ and

$$V = \{x \in \tilde{V} \mid (x, x_0) = 0\}.$$

Let $\tilde{\rho} : \mathrm{GSp}_2 \to \mathrm{SO}(\tilde{V})$ be the homomorphism defined by

$$\tilde{\rho}(h) = \nu(h)^{-1} \wedge^2 (h)$$

for $h \in \mathrm{GSp}_2$. Since $\tilde{\rho}(h)x_0 = x_0$, this homomorphism induces an exact sequence

$$1 \longrightarrow \mathbb{G}_m \overset{\iota}{\longrightarrow} \mathrm{GSp}_2 \overset{\rho}{\longrightarrow} \mathrm{SO}(V) \longrightarrow 1.$$

Here $\iota(a) = a\mathbf{1}_4$ for $a \in \mathbb{G}_m$.

We identify V with the space of column vectors F^5 via

$$V \longrightarrow F^5,$$
$$\sum_{i=1}^{5} x_i \mathbf{e}_i \longmapsto \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

where

$$\mathbf{e}_1 = e_2 \wedge e_1, \quad \mathbf{e}_2 = e_1 \wedge e_4, \quad \mathbf{e}_3 = e_1 \wedge e_3 - e_2 \wedge e_4,$$
$$\mathbf{e}_4 = e_2 \wedge e_3, \quad \mathbf{e}_5 = e_3 \wedge e_4.$$

Put

$$Q_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 & -1 \\ 0 & Q_1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Then $(x, y) = {}^t x Q y$ for $x, y \in V = F^5$. Let $V_1 = F^3$ be the quadratic space with bilinear form

$$(v, w) = {}^t v Q_1 w.$$

Note that

$$\rho(n(X)) = \begin{pmatrix} 1 & -x_3 & -2x_2 & x_1 & x_2^2 - x_1 x_3 \\ 0 & 1 & 0 & 0 & x_1 \\ 0 & 0 & 1 & 0 & -x_2 \\ 0 & 0 & 0 & 1 & -x_3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

for

$$n(X) = \begin{pmatrix} \mathbf{1}_2 & X \\ 0 & \mathbf{1}_2 \end{pmatrix} \quad \text{with } X = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} \in \mathrm{Sym}_2,$$

and

$$\rho(m(A, \nu)) = \det(A)^{-1} \begin{pmatrix} \nu^{-1}\det(A)^2 & 0 & 0 & 0 & 0 \\ 0 & a_1^2 & -2a_1 a_2 & -a_2^2 & 0 \\ 0 & -a_1 a_3 & a_1 a_4 + a_2 a_3 & a_2 a_4 & 0 \\ 0 & -a_3^2 & 2a_3 a_4 & a_4^2 & 0 \\ 0 & 0 & 0 & 0 & \nu \end{pmatrix}$$

for

$$m(A, \nu) = \begin{pmatrix} A & 0 \\ 0 & \nu\, {}^t A^{-1} \end{pmatrix} \quad \text{with } A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in \mathrm{GL}_2,\ \nu \in \mathbb{G}_m.$$

If F is a local field, let ω be the Weil representation of $\widetilde{\mathrm{SL}_2(F)} \times \mathrm{O}(V)(F)$ on $\mathcal{S}(V)$ with respect to a fixed additive character ψ. As in Sect. 4.4, let

$$\begin{aligned} \mathcal{S}(V) &\longrightarrow \mathcal{S}(V_1) \otimes \mathcal{S}(F^2) \\ \varphi &\longmapsto \hat{\varphi} \end{aligned}$$

be the partial Fourier transform given by

$$\hat{\varphi}(x_1; y) = \int_F \varphi \begin{pmatrix} z \\ x_1 \\ y_1 \end{pmatrix} \psi(-y_2 z)\, dz$$

for $x_1 \in V_1$, $y = (y_1, y_2) \in F^2$. Note that

$$\hat{\omega}(t(a), 1)\hat{\varphi}(x_1; y) = \gamma_F(a, \psi)^{-1} |a|_F^{3/2} \hat{\varphi}\big(a x_1; a y_1, a^{-1} y_2\big)$$

for $a \in F^\times$.

7.2. Theta lifts. Let $F = \mathbb{Q}$ and $\psi = \psi_0$. Let κ be an odd positive integer. Let $h \in S^+_{\kappa+1/2}(\Gamma_0(4))$ be a Hecke eigenform and $f \in S_{2\kappa}(\mathrm{SL}_2(\mathbb{Z}))$ the normalized Hecke eigenform associated to h. We may assume that $c_h(n) \in \mathbb{R}$ for all $n \in \mathbb{N}$. For each prime p, let $\{\alpha_p, \alpha_p^{-1}\}$ denote the Satake parameter of f at p. Let $\mathbf{h}$ denote the cusp form on $\widetilde{\mathrm{SL}_2(\mathbb{A}_\mathbb{Q})}$ associated to h and $\tilde{\pi} = \otimes_v \tilde{\pi}_v$ the irreducible cuspidal automorphic representation of $\widetilde{\mathrm{SL}_2(\mathbb{A}_\mathbb{Q})}$ generated by $\mathbf{h}$. Let $\xi \in \mathbb{Q}$. If $\xi > 0$, write $\xi = \mathfrak{d}_\xi \mathfrak{f}_\xi^2$ with $\mathfrak{d}_\xi \in \mathbb{N}$, $\mathfrak{f}_\xi \in \mathbb{Q}_+^\times$ so that $-\mathfrak{d}_\xi$ is the discriminant of $\mathbb{Q}(\sqrt{-\xi})/\mathbb{Q}$. Then

$$W_{\mathbf{h},\xi} = \begin{cases} c_h(\mathfrak{d}_\xi)\mathfrak{f}_\xi^{\kappa-1/2} \prod_v W_{\xi,v} & \text{if } \xi > 0, \\ 0 & \text{if } \xi \leq 0, \end{cases}$$

where $W_{\xi,p}$ is the Whittaker function of $\tilde{\pi}_p$ as in Sect. A.3 with $l = \kappa$ and $\alpha = \alpha_p$, and $W_{\xi,\infty}$ is the Whittaker function of $\tilde{\pi}_\infty$ defined by

$$W_{\xi,\infty}(u(x)t(a)\tilde{k}_\theta) = e^{2\pi\sqrt{-1}\xi x} a^{\kappa+1/2} e^{-2\pi\xi a^2} e^{\sqrt{-1}(\kappa+1/2)\theta} \tag{7.1}$$

for $x \in \mathbb{R}$, $a \in \mathbb{R}_+^\times$, $\theta \in \mathbb{R}/4\pi\mathbb{Z}$.

Let

$$F(Z) = \sum_B A(B) e^{2\pi\sqrt{-1}\,\mathrm{tr}(BZ)} \in S_{\kappa+1}(\mathrm{Sp}_2(\mathbb{Z}))$$

be the Saito-Kurokawa lift of h. Here B runs over all positive definite half-integral symmetric matrices of size 2. Then F determines a cusp form $\mathbf{F}$ on $\mathrm{GSp}_2(\mathbb{A}_\mathbb{Q})$ by the formula

$$\mathbf{F}(h) = \det(h_\infty)^{(\kappa+1)/2} \det(C\sqrt{-1} + D)^{-\kappa-1} F(h_\infty(\sqrt{-1}))$$

for $h = \gamma h_\infty k \in \mathrm{GSp}_2(\mathbb{A}_\mathbb{Q})$ with $\gamma \in \mathrm{GSp}_2(\mathbb{Q})$,

$$h_\infty = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GSp}_2^+(\mathbb{R}),$$

and $k \in \mathrm{GSp}_2(\hat{\mathbb{Z}})$. For $B \in \mathrm{Sym}_2(\mathbb{Q})$, define the B-th Fourier coefficient $\mathcal{W}_{\mathbf{F},B}$ of $\mathbf{F}$ by

$$\mathcal{W}_{\mathbf{F},B}(h) = \int_{\mathrm{Sym}_2(\mathbb{Q})\backslash \mathrm{Sym}_2(\mathbb{A}_\mathbb{Q})} \mathbf{F}(n(X)h)\overline{\psi(\mathrm{tr}(BX))}\, dX.$$

Let

$$h_\infty = \begin{pmatrix} z\mathbf{1}_2 & 0 \\ 0 & z\mathbf{1}_2 \end{pmatrix} \begin{pmatrix} \mathbf{1}_2 & X \\ 0 & \mathbf{1}_2 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \in \mathrm{GSp}_2^+(\mathbb{R})$$

with $z \in \mathbb{R}_+^\times$, $X \in \mathrm{Sym}_2(\mathbb{R})$, $A \in \mathrm{GL}_2^+(\mathbb{R})$, and $\mathbf{k} = \alpha + \sqrt{-1}\beta \in \mathrm{U}(2)$. Then $\mathcal{W}_{\mathbf{F},B}(h_\infty) = 0$ unless B is a positive definite half-integral symmetric matrix, in which case,

$$\mathcal{W}_{\mathbf{F},B}(h_\infty) = A(B)\det(Y)^{(\kappa+1)/2} e^{2\pi\sqrt{-1}\,\mathrm{tr}(BZ)} \det(\mathbf{k})^{\kappa+1},$$

where $Y = A^t A$ and $Z = X + \sqrt{-1}Y \in \mathfrak{H}_2$.

We define $\varphi = \otimes_v \varphi_v^{(5)} \in S(V(\mathbb{A}_\mathbb{Q}))$ as follows:

- If $v = p \neq 2$, then $\varphi_p^{(5)}$ is the characteristic function of $V(\mathbb{Z}_p)$.
- If $v = 2$, then $\varphi_2^{(5)}$ is the characteristic function of $\mathbb{Z}_2\mathbf{e}_3 + V(2\mathbb{Z}_2)$.
- If $v = \infty$, then

$$\varphi_\infty^{(5)}(x) = (x_2 + \sqrt{-1}x_1 + \sqrt{-1}x_5 - x_4)^{\kappa+1} e^{-\pi\left(x_1^2+x_2^2+2x_3^2+x_4^2+x_5^2\right)}.$$

We may regard $\theta(\mathbf{h}, \varphi)$ as an automorphic form on $\mathrm{GSp}_2(\mathbb{A}_\mathbb{Q})$.

Lemma 7.1.

$$\theta(\mathbf{h}, \varphi) = 2^{-2}\xi_\mathbb{Q}(2)^{-1}\mathbf{F}.$$

The rest of this section is devoted to the proof of Lemma 7.1. Let

$$B = \begin{pmatrix} b_1 & b_2/2 \\ b_2/2 & b_3 \end{pmatrix} \in \mathrm{Sym}_2(\mathbb{Q})$$

and $\xi = \det(B)$. We will compute the B-th Fourier coefficient $\mathcal{W}_B$ of $\theta(\mathbf{h}, \varphi)$ defined by

$$\mathcal{W}_B(h) = \int_{\mathrm{Sym}_2(\mathbb{Q})\backslash \mathrm{Sym}_2(\mathbb{A}_\mathbb{Q})} \theta(n(X)h; \mathbf{h}, \varphi)\overline{\psi(\mathrm{tr}(BX))}\,dX.$$

By Lemma 4.2 and the result of Waldspurger [54],

$$\mathcal{W}_B = \begin{cases} c_h(\mathfrak{d}_\xi)\mathfrak{f}_\xi^{\kappa-1/2}\xi_\mathbb{Q}(2)^{-1}\prod_v \mathcal{W}_{B,v} & \text{if } \xi > 0, \\ 0 & \text{if } \xi \leq 0, \end{cases}$$

where

$$\mathcal{W}_{B,v}(h) = \int_{U(\mathbb{Q}_v)\backslash \mathrm{SL}_2(\mathbb{Q}_v)} \hat{\omega}(g,h)\hat{\varphi}_v^{(5)}(\beta; 0, 1)W_{\xi,v}(g)\,dg$$
$$\times \begin{cases} \mathrm{vol}(\mathrm{SL}_2(\mathbb{Z}_p))^{-1} & \text{if } v = p, \\ \mathrm{vol}(\mathrm{SO}(2))^{-1} & \text{if } v = \infty, \end{cases}$$

with

$$\beta = \begin{pmatrix} b_3 \\ b_2/2 \\ -b_1 \end{pmatrix}.$$

7.3. The unramified case. Let $v = p \neq 2$. In this case, $\gamma_V = 1$. Let φ be the characteristic function of $V(\mathbb{Z}_p)$. Note that

$$\omega((k, s_p(k)), k')\varphi = \varphi \tag{7.2}$$

for $k \in \mathrm{SL}_2(\mathbb{Z}_p)$, $k' \in \mathrm{GSp}_2(\mathbb{Z}_p)$, and

$$\hat{\varphi} = \varphi_1 \otimes \varphi_2$$

where $\varphi_1 \in \mathcal{S}(V_1(\mathbb{Q}_p))$ and $\varphi_2 \in \mathcal{S}(\mathbb{Q}_p^2)$ are the characteristic functions of $V_1(\mathbb{Z}_p)$ and $\mathbb{Z}_p^2$, respectively.

Assume that $\xi \neq 0$. Then

$$\begin{aligned} \mathcal{W}_{B,p}(1) &= \int_{\mathbb{Q}_p^\times} \hat{\omega}(t(a), 1)\hat{\varphi}(\beta; 0, 1) W_{\xi,p}(t(a))|a|_{\mathbb{Q}_p}^{-2}\, d^\times a \\ &= \sum_{n\in\mathbb{Z}} p^{-n/2}\hat{\varphi}(p^n\beta; 0, p^{-n})\Psi_p\big(p^{2n}\xi; \alpha_p\big). \end{aligned}$$

Since $\hat{\varphi}(p^n\beta; 0, p^{-n}) = 0$ unless $-\min(\mathrm{ord}_{\mathbb{Q}_p}(b_i)) \leq n \leq 0$, we have

$$\mathcal{W}_{B,p}(1) = \begin{cases} \sum_{n=0}^{\min(\mathrm{ord}_{\mathbb{Q}_p}(b_i))} p^{n/2}\Psi_p\big(p^{-2n}\xi; \alpha_p\big) & \text{if } b_1, b_2, b_3 \in \mathbb{Z}_p, \\ 0 & \text{otherwise.} \end{cases}$$

7.4. The 2-adic case. Let $v = 2$. In this case, $\gamma_V = \zeta_8^{-1}$. Let ϕ_n denote the characteristic function of $2^n\mathbb{Z}_2$. Let φ be the characteristic function of $\mathbb{Z}_2\mathbf{e}_3 + V(2\mathbb{Z}_2)$. Note that

$$\omega(k, k')\varphi = \epsilon_2(k)\varphi \tag{7.3}$$

for $k \in K_0(4; \mathbb{Z}_2)$, $k' \in \mathrm{GSp}_2(\mathbb{Z}_2)$.

Lemma 7.2. *We define* $\varphi_1, \varphi_1', \varphi_1'' \in \mathcal{S}(V_1(\mathbb{Q}_2))$ *by*

$$\begin{aligned} \varphi_1(x_1) &= \phi_1(x_{11})\phi_0(x_{12})\phi_1(x_{13}), \\ \varphi_1'(x_1) &= \phi_{-1}(x_{11})\phi_{-1}(x_{12})\phi_{-1}(x_{13}), \\ \varphi_1''(x_1) &= \phi_0(x_{11})[\phi_{-1} - \phi_0](x_{12})\phi_0(x_{13})\psi_2\big(2^{-1}Q_1[x_1]\big), \end{aligned}$$

for $x_1 = {}^t(x_{11}, x_{12}, x_{13}) \in V_1(\mathbb{Q}_2)$*, respectively. We also define* $\varphi_2, \varphi_2', \varphi_2'' \in \mathcal{S}(\mathbb{Q}_2^2)$ *by*

$$\begin{aligned} \varphi_2(y) &= \phi_1(y_1)\phi_{-1}(y_2), \\ \varphi_2'(y) &= \phi_{-1}(y_1)\phi_1(y_2), \\ \varphi_2''(y) &= \phi_1(y_1)\phi_0(y_2) + [\phi_0 - \phi_1](y_1)[\phi_{-1} - \phi_0](y_2), \end{aligned}$$

for $y = (y_1, y_2) \in \mathbb{Q}_2^2$*, respectively. Then*

$$\begin{aligned} \hat{\varphi} &= 2^{-1}(\varphi_1 \otimes \varphi_2), \\ \hat{\omega}(w, 1)\hat{\varphi} &= 2^{-7/2}\zeta_8^{-1}(\varphi_1' \otimes \varphi_2'), \\ \hat{\omega}(k_1, 1)\hat{\varphi} &= 2^{-2}\zeta_8(\varphi_1'' \otimes \varphi_2''). \end{aligned}$$

Proof. We only check that $\omega(k_1,1)\varphi_1 = 2^{-1}\zeta_8\varphi_1''$. Note that $k_1 = t(2^{-1})u(2)w^{-1}u(2^{-1})$ in $\widetilde{\mathrm{SL}_2(\mathbb{Q}_2)}$. Since

$$\begin{aligned}\omega(u(2^{-1}),1)\varphi_1(x_1) &= \phi_1(x_{11})\phi_0(x_{12})\phi_1(x_{13})\psi_2\big(2^{-1}\big(x_{11}x_{13}+x_{12}^2\big)\big)\\ &= \phi_1(x_{11})[2\phi_1-\phi_0](x_{12})\phi_1(x_{13}),\end{aligned}$$

we have

$$\omega(w^{-1}u(2^{-1}),1)\varphi_1(x_1) = 2^{-5/2}\zeta_8\phi_{-1}(x_{11})[\phi_{-2}-\phi_{-1}](x_{12})\phi_{-1}(x_{13}),$$

hence

$$\begin{aligned}\omega(k_1,1)\varphi_1(x_1) &= 2^{3/2}\omega(w^{-1}u(2^{-1}),1)\varphi_1\big(2^{-1}x_1\big)\psi_2\big(2^{-1}Q_1[x_1]\big)\\ &= 2^{-1}\zeta_8\phi_0(x_{11})[\phi_{-1}-\phi_0](x_{12})\phi_0(x_{13})\psi_2\big(2^{-1}Q_1[x_1]\big).\end{aligned}$$

□

Lemma 7.3. *Assume that* $\xi \neq 0$. *Then*

$$\mathcal{W}_{B,2}(1) = \begin{cases} 2^{-7/2}\sum_{n=0}^{\min(\mathrm{ord}_{\mathbb{Q}_2}(b_i))} 2^{n/2}\Psi_2\big(2^{-2n+2}\xi;\alpha_2\big) & \text{if } b_1,b_2,b_3\in\mathbb{Z}_2,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. Let

$$\mathcal{J}(k) = \sum_{n\in\mathbb{Z}} 2^{n/2}\hat{\omega}(k,1)\hat{\varphi}(2^n\beta;0,2^{-n})W_{\xi,2}(t(2^n)k).$$

Then $\mathcal{W}_{B,2}(1)$ is equal to

$$\begin{aligned}&6^{-1}\int_{\mathbb{Q}_2^\times}\sum_{k\in\mathrm{SL}_2(\mathbb{Z}_2)/K_0(4;\mathbb{Z}_2)}\hat{\omega}(t(a)k,1)\hat{\varphi}(\beta;0,1)W_{\xi,2}(t(a)k)|a|_{\mathbb{Q}_2}^{-2}\,d^\times a\\ &= 6^{-1}\left[\mathcal{J}(1)+\sum_{b\in\mathbb{Z}_2/4\mathbb{Z}_2}\mathcal{J}(u(b)w)+\mathcal{J}(k_1)\right]\\ &= 6^{-1}\left[\mathcal{J}(1)+4\mathcal{J}(w)+\mathcal{J}(k_1)\right].\end{aligned}$$

Put $m_i = \mathrm{ord}_{\mathbb{Q}_2}(b_i)$, $m_0 = \min(\mathrm{ord}_{\mathbb{Q}_2}(b_i))$, and

$$m_0' = \min(\mathrm{ord}_{\mathbb{Q}_2}(b_1),\mathrm{ord}_{\mathbb{Q}_2}(b_2)-1,\mathrm{ord}_{\mathbb{Q}_2}(b_3)).$$

Since $\hat{\omega}(k,1)\hat{\varphi}(2^n\beta;0,2^{-n}) = 0$ unless

$$\begin{cases} -m_0+1\le n\le 1 & \text{if } k=1,\\ -m_0'-1\le n\le -1 & \text{if } k=w,\\ -\min(m_1,m_3)\le -m_2 = n\le 0 & \text{if } k=k_1,\end{cases}$$

we have $\mathcal{W}_{B,2}(1)=0$ unless $b_1,b_2,b_3\in\mathbb{Z}_2$.

Assume that $b_1, b_2, b_3 \in \mathbb{Z}_2$. We write $\xi = b_1 b_3 - b_2^2/4 = 2^m u$ with $u \in \mathbb{Z}_2^\times$. If $m_2 \le \min(m_1, m_3)$, then $m_0' = m_0 - 1$, $m = 2m_0 - 2$, and $u \equiv -1 \bmod 4$, hence

$$\begin{aligned}
\mathcal{J}(1) &= 2^{-1} \sum_{n=-1}^{m_0-1} 2^{n/2} \Psi_2\big(2^{-2n}\xi; \alpha_2\big), \\
\mathcal{J}(w) &= 2^{-5} \sum_{n=1}^{m_0} 2^{n/2} \Psi_2\big(2^{-2n+4}\xi; \alpha_2\big), \\
\mathcal{J}(k_1) &= 2^{(m_0-5)/2} \zeta_8 \psi_2(-2^{-2m_0-1}\xi)(2, \xi)_{\mathbb{Q}_2} = 2^{(m_0-5)/2}.
\end{aligned}$$

If $m_2 > \min(m_1, m_3)$, then $m_0' = m_0$ and $m \ge 2m_0$, hence

$$\begin{aligned}
\mathcal{J}(1) &= 2^{-1} \sum_{n=-1}^{m_0-1} 2^{n/2} \Psi_2\big(2^{-2n}\xi; \alpha_2\big), \\
\mathcal{J}(w) &= 2^{-5} \sum_{n=1}^{m_0+1} 2^{n/2} \Psi_2\big(2^{-2n+4}\xi; \alpha_2\big), \\
\mathcal{J}(k_1) &= 0.
\end{aligned}$$

This completes the proof. □

7.5. The archimedean case. Let $v = \infty$. In this case, $\gamma_V = \zeta_8$. For each $n \in \mathbb{Z}_{\ge 0}$, let

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\big(e^{-x^2}\big)$$

denote the Hermite polynomial. Note that

$$H_{n+1}(x) = 2x H_n(x) - H_n'(x).$$

Lemma 7.4. *Let $r \in \mathbb{R}_+^\times$, $c \in \mathbb{C}$, and $x \in \mathbb{R}$. Then*

$$\begin{aligned}
&\sqrt{r} \int_{-\infty}^{\infty} (c - \sqrt{-1}y)^n e^{-\pi r y^2} e^{2\pi\sqrt{-1} r x y} \, dy \\
&= (2\sqrt{\pi r})^{-n} H_n(\sqrt{\pi r}(c + x)) e^{-\pi r x^2}.
\end{aligned}$$

Proof. The lemma follows by induction on n. □

Lemma 7.5. *Let $r \in \mathbb{R}^\times$. For each $n \in \mathbb{Z}_{\ge 0}$, put*

$$J_n = \int_0^\infty a^{n-2} H_n(\sqrt{\pi}(ra + a^{-1})) e^{-\pi(ra + a^{-1})^2} \, da.$$

If $r > 0$, *then*

$$J_n = 2^{2n-1}\pi^{n/2}e^{-4\pi r}.$$

If $r < 0$, *then*

$$J_n = \begin{cases} 2^{-1} & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. It is easy to check that

$$\sum_{n=0}^{\infty} \frac{1}{n!}(-\sqrt{\pi}x)^n J_n = \int_0^{\infty} a^{-2}e^{-\pi(ra+a^{-1}+xa)^2}\, da = 2^{-1}e^{-2\pi(r+x)}e^{-2\pi|r+x|}.$$

This yields the lemma. □

We define $\varphi \in \mathcal{S}(V(\mathbb{R}))$ by

$$\varphi(x) = (x_2 + \sqrt{-1}x_1 + \sqrt{-1}x_5 - x_4)^{\kappa+1}e^{-\pi\left(x_1^2+x_2^2+2x_3^2+x_4^2+x_5^2\right)}.$$

Note that

$$\omega(\tilde{k}_\theta, k')\varphi = e^{-\sqrt{-1}(\kappa+1/2)\theta}\det(\mathbf{k})^{\kappa+1}\varphi \tag{7.4}$$

for $\tilde{k}_\theta \in \widetilde{\mathrm{SO}(2)}$,

$$k' = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \in \mathrm{Sp}_2(\mathbb{R})$$

with $\mathbf{k} = \alpha + \sqrt{-1}\beta \in \mathrm{U}(2)$. By Lemma 7.4,

$$\begin{aligned}\hat{\varphi}(x_1; y) = {} & (2\sqrt{\pi})^{-\kappa-1}H_{\kappa+1}(\sqrt{\pi}(x_{11} - x_{13} + \sqrt{-1}y_1 + y_2)) \\ & \times e^{-\pi\left(x_{11}^2+2x_{12}^2+x_{13}^2+y_1^2+y_2^2\right)}\end{aligned}$$

for $x_1 = {}^t(x_{11}, x_{12}, x_{13}) \in V_1(\mathbb{R})$, $y = (y_1, y_2) \in \mathbb{R}^2$.

Lemma 7.6. *Assume that* $\xi > 0$. *For* $A \in \mathrm{GL}_2^+(\mathbb{R})$ *and* $X \in \mathrm{Sym}_2(\mathbb{R})$, *put* $Y = A^tA$ *and* $Z = X + \sqrt{-1}Y$. *Then*

$$\mathcal{W}_{B,\infty}(n(X)m(A,1)) = \begin{cases} 2^{\kappa+1}\det(Y)^{(\kappa+1)/2}e^{2\pi\sqrt{-1}\,\mathrm{tr}(BZ)} & \text{if } B > 0, \\ 0 & \text{if } B < 0. \end{cases}$$

Proof. Put

$$h_1 = \det(A)^{-1} \begin{pmatrix} a_1^2 & -2a_1a_2 & -a_2^2 \\ -a_1a_3 & a_1a_4 + a_2a_3 & a_2a_4 \\ -a_3^2 & 2a_3a_4 & a_4^2 \end{pmatrix} \quad \text{for } A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}.$$

Then $\mathcal{W}_{B,\infty}(n(X)m(A,1))$ is equal to

$$\begin{aligned} &\int_{\mathbb{R}^\times} \hat{\omega}(t(a), n(X)m(A,1))\hat{\varphi}(\beta; 0, 1) W_{\xi,\infty}(t(a))|a|_{\mathbb{R}}^{-2}\, d^\times a \\ &= 2\det(A)e^{2\pi\sqrt{-1}\,\mathrm{tr}(BX)} \int_0^\infty a^{\kappa-1}\hat{\varphi}\big(ah_1^{-1}\beta; 0, \det(A)a^{-1}\big)e^{-2\pi\xi a^2}\, da. \end{aligned}$$

Since

$$h_1^{-1}\beta = \det(A)^{-1} \begin{pmatrix} a_2^2b_1 + a_2a_4b_2 + a_4^2b_3 \\ a_1a_2b_1 + (a_1a_4 + a_2a_3)b_2/2 + a_3a_4b_3 \\ -a_1^2b_1 - a_1a_3b_2 - a_3^2b_3 \end{pmatrix},$$

we have

$$\begin{aligned} &\hat{\varphi}\big(ah_1^{-1}\beta; 0, \det(A)a^{-1}\big) \\ &= (2\sqrt{\pi})^{-\kappa-1} H_{\kappa+1}(\sqrt{\pi}(\det(A)^{-1}\,\mathrm{tr}(BY)a + \det(A)a^{-1})) \\ &\times e^{-\pi(\det(A)^{-1}\,\mathrm{tr}(BY)a + \det(A)a^{-1})^2} e^{2\pi\xi a^2 + 2\pi\,\mathrm{tr}(BY)}. \end{aligned}$$

Hence

$$\begin{aligned} &\int_0^\infty a^{\kappa-1}\hat{\varphi}\big(ah_1^{-1}\beta; 0, \det(A)a^{-1}\big)e^{-2\pi\xi a^2}\, da \\ &= (2\sqrt{\pi})^{-\kappa-1}\det(A)^\kappa e^{2\pi\,\mathrm{tr}(BY)} \\ &\times \int_0^\infty a^{\kappa-1} H_{\kappa+1}(\sqrt{\pi}(\mathrm{tr}(BY)a + a^{-1}))e^{-\pi(\mathrm{tr}(BY)a + a^{-1})^2}\, da \\ &= \begin{cases} 2^\kappa\det(A)^\kappa e^{-2\pi\,\mathrm{tr}(BY)} & \text{if } B > 0, \\ 0 & \text{if } B < 0, \end{cases} \end{aligned}$$

by Lemma 7.5. This completes the proof. □

7.6. Proof of Lemma 7.1. By (7.2)–(7.4), it suffices to show that

$$\mathcal{W}_B(h_\infty) = 2^{-2}\xi_{\mathbb{Q}}(2)^{-1}\mathcal{W}_{\mathbf{F},B}(h_\infty)$$

for $h_\infty = n(X)m(A,1) \in \mathrm{Sp}_2(\mathbb{R})$ with $X \in \mathrm{Sym}_2(\mathbb{R})$, $A \in \mathrm{GL}_2^+(\mathbb{R})$. We may assume that $B > 0$ and $b_1, b_2, b_3 \in \mathbb{Z}$. Then $\mathcal{W}_B(h_\infty)$ is equal to

$$\begin{aligned}
&2^{\kappa-5/2}\xi_{\mathbb{Q}}(2)^{-1}\det(Y)^{(\kappa+1)/2}e^{2\pi\sqrt{-1}\,\mathrm{tr}(BZ)} \\
&\times c_h(\mathfrak{d}_\xi)\mathfrak{f}_\xi^{\kappa-1/2}\prod_p \sum_{n=0}^{\min(\mathrm{ord}_{\mathbb{Q}_p}(b_i))} p^{n/2}\Psi_p\left(\frac{4\xi}{p^{2n}};\alpha_p\right) \\
&= 2^{-2}\xi_{\mathbb{Q}}(2)^{-1}\det(Y)^{(\kappa+1)/2}e^{2\pi\sqrt{-1}\,\mathrm{tr}(BZ)} \\
&\times c_h(\mathfrak{d}_{4\xi})\mathfrak{f}_{4\xi}^{\kappa-1/2}\sum_{d|(b_1,b_2,b_3)} d^{1/2}\prod_p \Psi_p\left(\frac{4\xi}{d^2};\alpha_p\right) \\
&= 2^{-2}\xi_{\mathbb{Q}}(2)^{-1}\det(Y)^{(\kappa+1)/2}e^{2\pi\sqrt{-1}\,\mathrm{tr}(BZ)}\sum_{d|(b_1,b_2,b_3)} d^{\kappa}c_h\left(\frac{4\xi}{d^2}\right) \\
&= 2^{-2}\xi_{\mathbb{Q}}(2)^{-1}\mathcal{W}_{\mathbf{F},B}(h_\infty).
\end{aligned}$$

This completes the proof of Lemma 7.1.

8. Proof of Proposition 5.1

Let $F = \mathbb{Q}$. Let V and V' be the quadratic spaces as in Sects. 7.1 and 6.1, respectively. We may identify the quadratic space $\{x \in V \mid (x, \mathbf{e}_3) = 0\}$ with V' via

$$\begin{aligned}
\{x \in V \mid (x, \mathbf{e}_3) = 0\} &\longrightarrow V'. \\
x_1\mathbf{e}_2 + x_2\mathbf{e}_1 + x_3\mathbf{e}_5 + x_4\mathbf{e}_4 &\longmapsto \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}
\end{aligned}$$

Then the homomorphisms ρ and the embedding

$$\begin{aligned}
\mathrm{G}(\mathrm{SL}_2 \times \mathrm{SL}_2) &\longrightarrow \mathrm{GSp}_2 \\
\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right) &\longmapsto \begin{pmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ c_1 & 0 & d_1 & 0 \\ 0 & c_2 & 0 & d_2 \end{pmatrix}
\end{aligned}$$

induce a natural embedding $\mathrm{SO}(V') \subset \mathrm{SO}(V)$.

Let $l = \kappa + 1$. We define

$$\begin{aligned}
\varphi &= \otimes_v \varphi_v^{(5)} \in S(V(\mathbb{A}_{\mathbb{Q}})), \\
\varphi' &= \otimes_v \varphi_v^{(3)} \in S(V'(\mathbb{A}_{\mathbb{Q}})), \\
\varphi'' &= \otimes_v \varphi_v^{(1)} \in S(\mathbb{A}_{\mathbb{Q}}),
\end{aligned}$$

as in Sect. 7.2, Sect. 6.3, Example 4.1, respectively. Note that $\varphi = \varphi' \otimes \varphi''$. Then a seesaw identity

$$\int_{\mathrm{O}(V')(\mathbb{Q})\backslash \mathrm{O}(V')(\mathbb{A}_{\mathbb{Q}})} \theta(h; \mathbf{h}, \varphi)\overline{\mathcal{g}(h)}\, dh = \langle \mathbf{h}, \overline{\theta(\bar{\mathcal{g}}, \varphi')\Theta}\rangle$$

holds. The left-hand side is equal to

$$2^{-3}\xi_{\mathbb{Q}}(2)^{-1} \int_{\mathrm{SO}(V')(\mathbb{Q})\backslash \mathrm{SO}(V')(\mathbb{A}_{\mathbb{Q}})} \mathbf{F}(h)\overline{\mathcal{g}(h)}\, dh = 2^{-4}\xi_{\mathbb{Q}}(2)^{-3}\langle F|_{\mathfrak{H}\times\mathfrak{H}}, g \times g\rangle$$

by Lemma 7.1, and the right-hand side is equal to

$$2^{\kappa-2}\xi_{\mathbb{Q}}(2)^{-2}\langle g, g\rangle\langle \mathbf{h}\Theta, \mathbf{g}^{\sharp}\rangle$$

by Lemma 6.3. This completes the proof of Proposition 5.1.

9. Theta correspondence for $(\widetilde{\mathrm{SL}}_2, \mathrm{PGL}_2)$

In this section, we study the Shimura-Waldspurger correspondence in terms of theta lifts [53], [41], [54].

9.1. Preliminaries. Let F be a field of characteristic not 2. Fix $\Delta \in F^{\times}$. Let

$$\begin{aligned} V &= \{x \in \mathrm{M}_2(F) \mid \mathrm{tr}(x) = 0\} \\ &= \left\{ \begin{pmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{pmatrix} \middle| \, x_1, x_2, x_3 \in F \right\} \end{aligned}$$

be the quadratic space with bilinear form

$$(x, y) = -\Delta\, \mathrm{tr}(xy) = -\Delta(2x_1y_1 + x_2y_3 + x_3y_2).$$

Then the associated quadratic form is given by

$$Q[x] = \Delta \det(x) = -\Delta\big(x_1^2 + x_2x_3\big).$$

Also, let $V_1 = F$ be the quadratic space with bilinear form

$$(v, w) = -2\Delta vw.$$

Recall that there is an exact sequence

$$1 \longrightarrow \mathbb{G}_m \xrightarrow{\iota} \mathrm{GL}_2 \xrightarrow{\rho} \mathrm{SO}(V) \longrightarrow 1.$$

Here

$$\iota(a) = a\mathbf{1}_2, \quad \rho(h)x = hxh^{-1},$$

for $a \in \mathbb{G}_m$, $h \in \mathrm{GL}_2$, and $x \in V$. Note that $\rho(h)x$ is equal to

$$\det(h)^{-1}\begin{pmatrix}(ad+bc)x_1 - acx_2 + bdx_3 & -2abx_1 + a^2x_2 - b^2x_3 \\ 2cdx_1 - c^2x_2 + d^2x_3 & -(ad+bc)x_1 + acx_2 - bdx_3\end{pmatrix}$$

for

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2, \quad x = \begin{pmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{pmatrix} \in V.$$

If F is a local field, let ω be the Weil representation of $\widetilde{\mathrm{SL}_2(F)} \times \mathrm{O}(V)(F)$ on $\mathcal{S}(V)$ with respect to a fixed additive character ψ. As in Sect. 4.4, let

$$\begin{aligned}\mathcal{S}(V) &\longrightarrow \mathcal{S}(V_1) \otimes \mathcal{S}(F^2) \\ \varphi &\longmapsto \hat{\varphi}\end{aligned}$$

be the partial Fourier transform given by

$$\hat{\varphi}(x_1; y) = |\Delta|_F^{1/2} \int_F \varphi\left(\begin{pmatrix} x_1 & z \\ y_1 & -x_1 \end{pmatrix}\right) \psi(-\Delta y_2 z)\, dz$$

for $x_1 \in V_1$, $y = (y_1, y_2) \in F^2$. Note that

$$\begin{aligned}\hat{\omega}(t(a), 1)\hat{\varphi}(x_1; y) &= \gamma_F(a, \psi)(\Delta, a)_F |a|_F^{1/2} \hat{\varphi}\big(ax_1; ay_1, a^{-1}y_2\big), \\ \hat{\omega}(1, u(b))\hat{\varphi}(x_1; y) &= \hat{\varphi}(x_1 - by_1; y)\psi\big(2\Delta bx_1y_2 - \Delta b^2 y_1 y_2\big), \\ \hat{\omega}(1, w)\hat{\varphi}(x_1; y) &= |\Delta|_F \int_{F^2} \hat{\varphi}(-x_1; z_1, z_2)\psi(\Delta(y_2z_1 - y_1z_2))\, dz_1\, dz_2,\end{aligned}$$

for $a \in F^\times$, $b \in F$.

9.2. Theta lifts. Let $F = \mathbb{Q}$, $\Delta = -D$, and $\psi = \psi_0$. Here $-D < 0$ is a fundamental discriminant with $-D \equiv 1 \bmod 8$. Let κ be an odd positive integer. Let $f \in S_{2\kappa}(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized Hecke eigenform and $h \in S^+_{\kappa+1/2}(\Gamma_0(4))$ a Hecke eigenform associated to f. We may assume that $c_h(n) \in \mathbb{R}$ for all $n \in \mathbb{N}$. Let $\mathbf{f}$ (resp. $\mathbf{h}$) denote the cusp form on $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ (resp. $\widetilde{\mathrm{SL}_2(\mathbb{A}_\mathbb{Q})}$) associated to f (resp. h) and $\sigma = \otimes_v \sigma_v$ (resp. $\tilde{\pi} = \otimes_v \tilde{\pi}_v$) the irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ (resp. $\widetilde{\mathrm{SL}_2(\mathbb{A}_\mathbb{Q})}$) generated by $\mathbf{f}$ (resp. $\mathbf{h}$). Let W_p (resp. W_∞) be the Whittaker function of $\tilde{\pi}_p$ (resp. $\tilde{\pi}_\infty$) as in Sect. A.3 (resp. (7.1)) with $\xi = D$, $l = \kappa$, and $\alpha = \alpha_p$. Here $\{\alpha_p, \alpha_p^{-1}\}$ is the Satake parameter of f at p. Then

$$W_{\mathbf{h},D} = c_h(D) \prod_v W_v.$$

We assume that $\Lambda(\kappa, f, \chi_{-D}) \neq 0$. By [54],

$$\theta(\pi \otimes \chi_{-D}) = \sigma, \quad \theta(\sigma^\vee) = (\pi \otimes \chi_{-D})^\vee.$$

We define $\varphi = \otimes_v \varphi_v^{(6)} \in S(V(\mathbb{A}_\mathbb{Q}))$ as follows:

- If $v = p$ with $p \nmid 2D$, then $\varphi_p^{(6)}$ is the characteristic function of $V(\mathbb{Z}_p)$.
- If $v = q$ with $q \mid D$, then

$$\varphi_q^{(6)} = (1+q)^{-1} \sum_{k,k' \in \mathrm{SL}_2(\mathbb{Z}_q)/K_0(q;\mathbb{Z}_q)} \omega((k, s_q(k)), k')\varphi_q'.$$

Here

$$\varphi_q'(x) = \begin{cases} (\Delta, x_2)_{\mathbb{Q}_q} & \text{if } x_1 \in \mathbb{Z}_q, x_2 \in q^{-1}\mathbb{Z}_q^\times, x_3 \in \mathbb{Z}_q, \\ 0 & \text{otherwise.} \end{cases}$$

- If $v = 2$, then

$$\varphi_2^{(6)}(x) = \begin{cases} 1 & \text{if } x_1 \in \mathbb{Z}_2, x_2 \in 2\mathbb{Z}_2, x_3 \in 2\mathbb{Z}_2, \\ 0 & \text{otherwise.} \end{cases}$$

- If $v = \infty$, then

$$\varphi_\infty^{(6)}(x) = (2x_1 - \sqrt{-1}x_2 - \sqrt{-1}x_3)^\kappa e^{-\pi D \operatorname{tr}(x^t x)}.$$

Lemma 9.1.

$$\theta(\mathbf{f} \otimes \chi_{-D}, \varphi) = -2^{-1}\sqrt{-1}D^{-\kappa+1/2}c_h(D)\xi_\mathbb{Q}(2)^{-1}\langle f, f\rangle\langle h, h\rangle^{-1}\mathbf{h}.$$

The rest of this section is devoted to the proof of Lemma 9.1. We will compute the Fourier coefficient $\mathcal{W}$ of $\theta(\bar{\mathbf{h}}, \varphi)$ defined by

$$\mathcal{W}(h) = \int_{\mathbb{Q}\backslash\mathbb{A}_\mathbb{Q}} \theta(u(x)h; \bar{\mathbf{h}}, \varphi)\overline{\psi(-x)}\, dx.$$

By Lemma 4.2,

$$\mathcal{W} = c_h(D)\xi_\mathbb{Q}(2)^{-1} \prod_v \mathcal{W}_v,$$

where

$$\mathcal{W}_v(h) = \int_{U(\mathbb{Q}_v)\backslash \mathrm{SL}_2(\mathbb{Q}_v)} \hat{\omega}(g, h)\hat{\varphi}_v^{(6)}(-2^{-1}\Delta^{-1}; 0, 1)\overline{W_v(t(-2^{-1}\Delta^{-1})g)}\, dg$$
$$\times \begin{cases} \mathrm{vol}(\mathrm{SL}_2(\mathbb{Z}_p))^{-1} & \text{if } v = p, \\ \mathrm{vol}(\mathrm{SO}(2))^{-1} & \text{if } v = \infty. \end{cases}$$

9.3. The unramified case. Let $v = p$ with $p \nmid 2D$. In this case, $\Delta \in \mathbb{Z}_p^\times$ and $\gamma_V = 1$. Let φ be the characteristic function of $V(\mathbb{Z}_p)$. Note that

$$\omega((k, s_p(k)), k')\varphi = \varphi \tag{9.1}$$

for $k \in \mathrm{SL}_2(\mathbb{Z}_p)$, $k' \in \mathrm{GL}_2(\mathbb{Z}_p)$, and

$$\hat{\varphi} = \varphi_1 \otimes \varphi_2$$

where $\varphi_1 \in \mathcal{S}(V_1(\mathbb{Q}_p))$ and $\varphi_2 \in \mathcal{S}(\mathbb{Q}_p^2)$ are the characteristic functions of $V_1(\mathbb{Z}_p)$ and $\mathbb{Z}_p^2$, respectively. Then

$$\begin{aligned}\mathcal{W}_p(1) &= \int_{\mathbb{Q}_p^\times} \hat{\omega}(t(a), 1)\hat{\varphi}(-2^{-1}\Delta^{-1}; 0, 1)\overline{W_p(t(-2^{-1}\Delta^{-1})t(a))}|a|_{\mathbb{Q}_p}^{-2}\, d^\times a \\ &= \sum_{n\in\mathbb{Z}} p^{n/2}(\Delta, p^n)_{\mathbb{Q}_p}\hat{\varphi}(-2^{-1}\Delta^{-1}p^n; 0, p^{-n})\overline{\Psi_p\big(p^{2n}D; \alpha_p\big)}.\end{aligned}$$

Since $\hat{\varphi}(-2^{-1}\Delta^{-1}p^n; 0, p^{-n}) = 0$ unless $n = 0$, we have

$$\mathcal{W}_p(1) = 1.$$

9.4. The ramified case. Let $v = q$ with $q \mid D$. In this case, $\Delta \in q\mathbb{Z}_q^\times$ and $\gamma_V = \gamma_{\mathbb{Q}_q}(\Delta, \psi_q)^{-1}$. Let χ be the quadratic character of $\mathbb{Q}_q^\times$ associated to $\mathbb{Q}_q(\sqrt{\Delta})/\mathbb{Q}_q$ by class field theory. Let ϕ_n denote the characteristic function of $q^n\mathbb{Z}_q$. We define $\phi_\chi \in \mathcal{S}(\mathbb{Q}_q)$ by

$$\phi_\chi(x) = \begin{cases}\chi(x) & \text{if } x \in q^{-1}\mathbb{Z}_q^\times, \\ 0 & \text{otherwise.}\end{cases}$$

Let

$$\begin{aligned}K_0 &= \left\{ \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_q) \,\middle|\, c \equiv 0 \bmod q\mathbb{Z}_q \right\}, \\ \mathbf{K}_0 &= \left\{ \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_q) \,\middle|\, c \equiv 0 \bmod q\mathbb{Z}_q \right\}.\end{aligned}$$

We define $\varphi' \in \mathcal{S}(V(\mathbb{Q}_q))$ by

$$\varphi'(x) = \phi_0(x_1)\phi_\chi(x_2)\phi_0(x_3).$$

Note that

$$\omega((k, s_q(k)), k')\varphi' = \chi(\det(k'))\varphi'$$

for $k \in K_0$, $k' \in \mathbf{K}_0$. Let

$$\varphi = (1+q)^{-1} \sum_{k,k'\in\mathrm{SL}_2(\mathbb{Z}_q)/K_0} \omega((k, s_q(k)), k')\varphi'.$$

Since $\mathrm{SL}_2(\mathbb{Z}_q)/K_0 \simeq \mathrm{GL}_2(\mathbb{Z}_q)/\mathbf{K}_0$, we have

$$\omega((k, s_q(k)), k')\varphi = \chi(\det(k'))\varphi \tag{9.2}$$

for $k \in \mathrm{SL}_2(\mathbb{Z}_q)$, $k' \in \mathrm{GL}_2(\mathbb{Z}_q)$.

Lemma 9.2.

$$\begin{aligned}
\hat{\varphi}'(x_1; y) &= \gamma_{\mathbb{Q}_q}(\Delta, \psi_q)^{-1}\phi_0(x_1)\phi_0(y_1)\phi_\chi(y_2),\\
\hat{\omega}(1, w)\hat{\varphi}'(x_1; y) &= q^{-1/2}\chi(-1)\phi_0(x_1)\phi_\chi(y_1)\phi_{-1}(y_2),\\
\hat{\omega}(w, 1)\hat{\varphi}'(x_1; y) &= q^{-1/2}\chi(-1)\phi_{-1}(x_1)\phi_\chi(y_1)\phi_0(y_2),\\
\hat{\omega}(w, w)\hat{\varphi}'(x_1; y) &= q^{-1}\gamma_{\mathbb{Q}_q}(\Delta, \psi_q)^{-1}\phi_{-1}(x_1)\phi_{-1}(y_1)\phi_\chi(y_2).
\end{aligned}$$

Proof. By the result of Kahn [31], we have $\epsilon(1/2, \chi, \psi_q) = \gamma_{\mathbb{Q}_q}(\Delta, \psi_q)^{-1}$, hence

$$\hat{\phi}_\chi(-\Delta x) = q^{1/2}\gamma_{\mathbb{Q}_q}(\Delta, \psi_q)^{-1}\phi_\chi(x)$$

for $x \in \mathbb{Q}_q$. Here $\hat{\phi}_\chi$ is the Fourier transform of ϕ_χ. This yields the lemma. □

Lemma 9.3.

$$\mathcal{W}_q(1) = q^{3/2}(\Delta, 2)_{\mathbb{Q}_q}.$$

Proof. Let

$$\begin{aligned}
\mathcal{J}(k, k') = \sum_{n\in\mathbb{Z}} & q^{3n/2}\gamma_{\mathbb{Q}_q}\big(\Delta^n, \psi_q\big)(-2, \Delta^n)_{\mathbb{Q}_q}\\
&\times \hat{\omega}(k, k')\hat{\varphi}'(-2^{-1}\Delta^{n-1}; 0, \Delta^{-n})\overline{W_q(t(-2^{-1}\Delta^{n-1}))}.
\end{aligned}$$

Then $\mathcal{W}_q(1)$ is equal to

$$\begin{aligned}
&\int_{\mathbb{Q}_q^\times} \hat{\omega}(t(a), 1)\hat{\varphi}(-2^{-1}\Delta^{-1}; 0, 1)\overline{W_q(t(-2^{-1}\Delta^{-1})t(a))}|a|_{\mathbb{Q}_q}^{-2}\, d^\times a\\
&= (1+q)^{-1}\Bigg[\mathcal{J}(1, 1) + \sum_{b'\in\mathbb{Z}_q/q\mathbb{Z}_q}\mathcal{J}(1, u(b')w) + \sum_{b\in\mathbb{Z}_q/q\mathbb{Z}_q}\mathcal{J}(u(b)w, 1)\\
&\qquad + \sum_{b,b'\in\mathbb{Z}_q/q\mathbb{Z}_q}\mathcal{J}(u(b)w, u(b')w)\Bigg]\\
&= (1+q)^{-1}\left[\mathcal{J}(1, 1) + q\mathcal{J}(1, w) + q\mathcal{J}(w, 1) + q^2\mathcal{J}(w, w)\right].
\end{aligned}$$

Since $\hat{\omega}(k, k')\hat{\varphi}'(-2^{-1}\Delta^{n-1}; 0, \Delta^{-n}) = 0$

$$\begin{cases}
\text{unless } n = 1 & \text{if } k = 1, k' = 1,\\
\text{for all } n & \text{if } k = 1, k' = w,\\
\text{for all } n & \text{if } k = w, k' = 1,\\
\text{unless } n = 1 & \text{if } k = w, k' = w,
\end{cases}$$

we have

$$\begin{aligned}\mathcal{J}(1,1) &= q^{3/2}(\Delta,2)_{\mathbb{Q}_q},\\ \mathcal{J}(1,w) &= 0,\\ \mathcal{J}(w,1) &= 0,\\ \mathcal{J}(w,w) &= q^{1/2}(\Delta,2)_{\mathbb{Q}_q}.\end{aligned}$$

This completes the proof. □

9.5. The 2-adic case. Let $v=2$. In this case, $\Delta\in\mathbb{Z}_2^\times\cap\mathbb{Q}_2^{\times,2}$ and $\gamma_V=\zeta_8$. Let ϕ_n denote the characteristic function of $2^n\mathbb{Z}_2$. We define $\varphi\in\mathcal{S}(V(\mathbb{Q}_2))$ by

$$\varphi(x)=\phi_0(x_1)\phi_1(x_2)\phi_1(x_3).$$

Note that

$$\omega(k,k')\varphi=\epsilon_2(k)^{-1}\varphi \tag{9.3}$$

for $k\in K_0(4;\mathbb{Z}_2)$, $k'\in\mathrm{GL}_2(\mathbb{Z}_2)$.

Lemma 9.4. *Let*

$$\mathsf{W}(\mathsf{U}(\varphi))=\int_{\mathbb{Z}_2}\omega(w^{-1}t(2)u(b)t(2),1)\varphi\,db.$$

Then

$$\mathsf{W}(\mathsf{U}(\varphi))=2^{-1/2}\zeta_8^{-1}\varphi.$$

Proof. Note that $w^{-1}t(2)u(b)t(2)=t(4^{-1})w^{-1}u(4^{-1}b)$ in $\widetilde{\mathrm{SL}_2(\mathbb{Q}_2)}$. Let

$$\varphi'=\int_{\mathbb{Z}_2}\omega(u(4^{-1}b),1)\varphi\,db.$$

Since

$$\varphi'(x)=\int_{\mathbb{Z}_2}\varphi(x)\psi_2\big(-4^{-1}\Delta\big(x_1^2+x_2x_3\big)b\big)\,db=\phi_1(x_1)\phi_1(x_2)\phi_1(x_3),$$

we have

$$\omega(w^{-1},1)\varphi'(x)=2^{-7/2}\zeta_8^{-1}\phi_{-2}(x_1)\phi_{-1}(x_2)\phi_{-1}(x_3),$$

hence

$$\mathsf{W}(\mathsf{U}(\varphi))(x)=2^3\omega(w^{-1},1)\varphi'(4^{-1}x)=2^{-1/2}\zeta_8^{-1}\phi_0(x_1)\phi_1(x_2)\phi_1(x_3).$$

□

Lemma 9.5. *We define* $\varphi_2, \varphi_2', \varphi_2'' \in \mathcal{S}(\mathbb{Q}_2^2)$ *by*

$$\begin{aligned}
\varphi_2(y) &= \phi_1(y_1)\phi_{-1}(y_2),\\
\varphi_2'(y) &= \phi_{-1}(y_1)\phi_1(y_2),\\
\varphi_2''(y) &= \phi_1(y_1)\phi_0(y_2) + [\phi_0 - \phi_1](y_1)[\phi_{-1} - \phi_0](y_2),
\end{aligned}$$

for $y = (y_1, y_2) \in \mathbb{Q}_2^2$*, respectively. Then*

$$\begin{aligned}
\hat{\varphi} &= 2^{-1}(\phi_0 \otimes \varphi_2),\\
\hat{\omega}(w, 1)\hat{\varphi} &= 2^{-3/2}\zeta_8(\phi_{-1} \otimes \varphi_2'),\\
\hat{\omega}(k_1, 1)\hat{\varphi} &= 2^{-1}([\phi_{-1} - \phi_0] \otimes \varphi_2'').
\end{aligned}$$

Proof. We only check that $\omega(k_1, 1)\phi_0 = \phi_{-1} - \phi_0$. Note that $k_1 = t(2^{-1})u(2)w^{-1}u(2^{-1})$ in $\widetilde{\mathrm{SL}_2(\mathbb{Q}_2)}$. Since

$$\omega(u(2^{-1}), 1)\phi_0(x_1) = \phi_0(x_1)\psi_2\big(-2^{-1}\Delta x_1^2\big) = [2\phi_1 - \phi_0](x_1),$$

we have

$$\omega(w^{-1}u(2^{-1}), 1)\phi_0(x_1) = 2^{-1/2}\zeta_8^{-1}[\phi_{-2} - \phi_{-1}](x_1),$$

hence

$$\begin{aligned}
\omega(k_1, 1)\phi_0(x_1) &= 2^{1/2}\omega(w^{-1}u(2^{-1}), 1)\phi_0\big(2^{-1}x_1\big)\psi_2\big(-2^{-1}\Delta x_1^2\big)\\
&= [\phi_{-1} - \phi_0](x_1).
\end{aligned}$$

□

Lemma 9.6.

$$\mathcal{W}_2(1) = -2^{-3/2}\sqrt{-1}.$$

Proof. Let

$$\mathcal{J}(k) = \sum_{n\in\mathbb{Z}} 2^{3n/2}\hat{\omega}(k, 1)\hat{\varphi}(-2^{n-1}\Delta^{-1}; 0, 2^{-n})\overline{W_2(t(-2^{n-1}\Delta^{-1})k)}.$$

Then $\mathcal{W}_2(1)$ is equal to

$$\begin{aligned}
&6^{-1}\int_{\mathbb{Q}_2^\times} \sum_{k\in\mathrm{SL}_2(\mathbb{Z}_2)/K_0(4;\mathbb{Z}_2)} \hat{\omega}(t(a)k, 1)\hat{\varphi}(-2^{-1}\Delta^{-1}; 0, 1)\\
&\times \overline{W_2(t(-2^{-1}\Delta^{-1})t(a)k)}|a|_{\mathbb{Q}_2}^{-2}\, d^\times a\\
&= 6^{-1}\left[\mathcal{J}(1) + \sum_{b\in\mathbb{Z}_2/4\mathbb{Z}_2} \mathcal{J}(u(b)w) + \mathcal{J}(k_1)\right]\\
&= 6^{-1}\left[\mathcal{J}(1) + 4\mathcal{J}(w) + \mathcal{J}(k_1)\right].
\end{aligned}$$

Since $\hat{\omega}(k, 1)\hat{\varphi}(-2^{n-1}\Delta^{-1}; 0, 2^{-n}) = 0$

$$\begin{cases} \text{unless } n = 1 & \text{if } k = 1, \\ \text{for all } n & \text{if } k = w, \\ \text{unless } n = 0 & \text{if } k = k_1, \end{cases}$$

we have

$$\begin{aligned} \mathcal{J}(1) &= 2^{1/2}\overline{W_2(t(-\Delta^{-1}))} = -2^{1/2}\sqrt{-1}, \\ \mathcal{J}(w) &= 0, \\ \mathcal{J}(k_1) &= 2^{-1}\overline{W_2\big(t(-2^{-1}\Delta^{-1})k_1\big)} = -2^{-1/2}\sqrt{-1}. \end{aligned}$$

This completes the proof. □

9.6. The archimedean case. Let $v = \infty$. In this case, $\Delta < 0$ and $\gamma_V = \zeta_8$. We define $\varphi \in \mathcal{S}(V(\mathbb{R}))$ by

$$\varphi(x) = (2x_1 - \sqrt{-1}x_2 - \sqrt{-1}x_3)^\kappa e^{-\pi D\left(2x_1^2 + x_2^2 + x_3^2\right)}.$$

Note that

$$\omega(\tilde{k}_\theta, k_{\theta'})\varphi = e^{\sqrt{-1}(\kappa+1/2)\theta} e^{-2\sqrt{-1}\kappa\theta'}\varphi \tag{9.4}$$

for $\tilde{k}_\theta \in \widetilde{\mathrm{SO}(2)}$, $k_{\theta'} \in \mathrm{SO}(2)$. By Lemma 7.4,

$$\hat{\varphi}(x_1; y) = (2\sqrt{\pi D})^{-\kappa} H_\kappa(\sqrt{\pi D}(2x_1 - \sqrt{-1}y_1 + y_2))e^{-\pi D\left(2x_1^2 + y_1^2 + y_2^2\right)}$$

for $x_1 \in V_1(\mathbb{R})$, $y = (y_1, y_2) \in \mathbb{R}^2$.

Lemma 9.7.

$$\mathcal{W}_\infty(1) = 2^{-1/2} D^{-\kappa-1} e^{-2\pi}.$$

Proof. The quantity $\mathcal{W}_\infty(1)$ is equal to

$$\begin{aligned} &\int_{\mathbb{R}^\times} \hat{\omega}(t(a), 1)\hat{\varphi}(-2^{-1}\Delta^{-1}; 0, 1)\overline{W_\infty(t(-2^{-1}\Delta^{-1})t(a))}|a|_{\mathbb{R}}^{-2}\, d^\times a \\ &= 2\int_0^\infty a^{-5/2}\hat{\varphi}(2^{-1}D^{-1}a; 0, a^{-1})\overline{W_\infty(t(2^{-1}D^{-1}a))}\, da \\ &= 2^{-2\kappa+1/2}\pi^{-\kappa/2}D^{-(3\kappa+1)/2} \\ &\times \int_0^\infty a^{\kappa-2}H_\kappa(\sqrt{\pi D}(D^{-1}a + a^{-1}))e^{-\pi(D^{-1}a^2 + Da^{-2})}\, da. \end{aligned}$$

By Lemma 7.5,

$$\begin{aligned} &\int_0^\infty a^{\kappa-2}H_\kappa(\sqrt{\pi D}(D^{-1}a + a^{-1}))e^{-\pi(D^{-1}a^2 + Da^{-2})}\, da \\ &= 2^{2\kappa-1}\pi^{\kappa/2}D^{(\kappa-1)/2}e^{-2\pi}. \end{aligned}$$

This completes the proof. □

9.7. Proof of Lemma 9.1. By (9.1)–(9.4), there exists a constant C such that

$$\theta(\bar{\mathbf{h}}, \varphi) = C\overline{\mathbf{f} \otimes \chi_{-D}}.$$

Since $\prod_{q|D}(\Delta, 2)_{\mathbb{Q}_q} = 1$, we have

$$C = e^{2\pi}\mathcal{W}(1) = -2^{-2}\sqrt{-1}D^{-\kappa+1/2}c_h(D)\xi_{\mathbb{Q}}(2)^{-1}.$$

Again, by (9.1)–(9.4) and Lemma 9.4, there exists a constant C' such that

$$\theta(\mathbf{f} \otimes \chi_{-D}, \varphi) = C'\mathbf{h}.$$

Hence

$$\langle\theta(\mathbf{f} \otimes \chi_{-D}, \varphi), \mathbf{h}\rangle = \langle\mathbf{h}, \mathbf{h}\rangle C' = 2^{-1}\xi_{\mathbb{Q}}(2)^{-1}\langle h, h\rangle C'.$$

On the other hand,

$$\begin{aligned}\langle\theta(\mathbf{f} \otimes \chi_{-D}, \varphi), \mathbf{h}\rangle &= \langle\mathbf{f} \otimes \chi_{-D}, \overline{\theta(\bar{\mathbf{h}}, \varphi)}\rangle = \langle\mathbf{f} \otimes \chi_{-D}, \mathbf{f} \otimes \chi_{-D}\rangle C \\ &= \xi_{\mathbb{Q}}(2)^{-1}\langle f, f\rangle C.\end{aligned}$$

Therefore $C' = 2\langle f, f\rangle\langle h, h\rangle^{-1}C$. This completes the proof of Lemma 9.1.

10. Theta correspondence for $(\mathrm{GL}_2, \mathrm{GO}(3, 1))$

In this section, we study the base change for GL_2 from $\mathbb{Q}$ to an imaginary quadratic field $\mathcal{K}$ in terms of theta lifts [34], [2], [16], [11], [12].

10.1. Preliminaries. Let F be a field of characteristic not 2 and E an abelian semisimple algebra over F of dimension 2. Let τ denote the non-trivial automorphism of E over F. Fix $\delta \in E^\times$ such that $\delta^\tau = -\delta$ and put $\Delta = \delta^2 \in F^\times$. Let $*$ denote the involution on $\mathrm{M}_2(E)$ given by

$$x^* = \begin{pmatrix} x_4^\tau & -x_2^\tau \\ -x_3^\tau & x_1^\tau \end{pmatrix} \quad \text{for } x = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in \mathrm{M}_2(E).$$

Let

$$\begin{aligned} V &= \{x \in \mathrm{M}_2(E) \mid x^* = x\} \\ &= \left\{ \begin{pmatrix} x_1 & \delta x_2 \\ \delta x_3 & x_1^\tau \end{pmatrix} \middle| \; x_1 \in E, x_2, x_3 \in F \right\} \end{aligned}$$

be the quadratic space with bilinear form

$$(x, y) = \mathrm{tr}(xy^\tau) = \mathrm{tr}_{E/F}\left(x_1 y_1^\tau\right) - \Delta(x_2 y_3 + x_3 y_2).$$

Then the associated quadratic form is given by

$$Q[x] = \det(x) = \mathrm{N}_{E/F}(x_1) - \Delta x_2 x_3.$$

Also, let $V_1 = E$ be the quadratic space with bilinear form

$$(v, w) = \mathrm{tr}_{E/F}(vw^\tau).$$

Recall that there is an exact sequence

$$1 \longrightarrow \mathrm{R}_{E/F}\,\mathbb{G}_m \xrightarrow{\iota} \mathbb{G}_m \times \mathrm{R}_{E/F}\,\mathrm{GL}_2 \xrightarrow{\rho} \mathrm{GSO}(V) \longrightarrow 1.$$

Here

$$\iota(a) = \left(\mathrm{N}_{E/F}(a)^{-1}, a\mathbf{1}_2\right), \quad \rho(z, h)x = zhxh^*,$$

for $a \in \mathrm{R}_{E/F}\,\mathbb{G}_m, z \in \mathbb{G}_m, h \in \mathrm{R}_{E/F}\,\mathrm{GL}_2$, and $x \in V$. Note that $\nu(\rho(z, h)) = z^2\,\mathrm{N}_{E/F}(\det(h))$, and

$$\rho(1, h)x = \begin{pmatrix} \tilde{x}_1 & \delta\tilde{x}_2 \\ \delta\tilde{x}_3 & \tilde{x}_1^\tau \end{pmatrix}$$

with

$$\begin{aligned} \tilde{x}_1 &= ad^\tau x_1 - bc^\tau x_1^\tau - \delta ac^\tau x_2 + \delta bd^\tau x_3, \\ \tilde{x}_2 &= \mathrm{tr}_{E/F}\left(\delta^{-1}a^\tau b x_1^\tau\right) + \mathrm{N}_{E/F}(a)x_2 - \mathrm{N}_{E/F}(b)x_3, \\ \tilde{x}_3 &= \mathrm{tr}_{E/F}\left(\delta^{-1}cd^\tau x_1\right) - \mathrm{N}_{E/F}(c)x_2 + \mathrm{N}_{E/F}(d)x_3, \end{aligned}$$

for

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{R}_{E/F}\,\mathrm{GL}_2, \quad x = \begin{pmatrix} x_1 & \delta x_2 \\ \delta x_3 & x_1^\tau \end{pmatrix} \in V.$$

If F is a local field, let ω be the Weil representation of $R(F)$ on $\mathcal{S}(V)$ with respect to a fixed additive character ψ. Here $R = \mathrm{G}(\mathrm{SL}_2 \times \mathrm{O}(V))$. As in Sect. 4.4, let

$$\begin{aligned} \mathcal{S}(V) &\longrightarrow \mathcal{S}(V_1) \otimes \mathcal{S}(F^2) \\ \varphi &\longmapsto \hat{\varphi} \end{aligned}$$

be the partial Fourier transform given by

$$\hat{\varphi}(x_1; y) = |\Delta|_F^{1/2} \int_F \varphi\left(\begin{pmatrix} x_1 & \delta z \\ \delta y_1 & x_1^\tau \end{pmatrix}\right) \psi(-\Delta y_2 z)\, dz$$

for $x_1 \in V_1$, $y = (y_1, y_2) \in F^2$. Note that

$$\begin{aligned} \hat{\omega}(t(a), 1)\hat{\varphi}(x_1; y) &= (\Delta, a)_F |a|_F \hat{\varphi}\left(ax_1; ay_1, a^{-1}y_2\right), \\ \hat{\omega}(1, u(b))\hat{\varphi}(x_1; y) &= \hat{\varphi}(x_1 - \delta b y_1; y) \\ &\quad \times \psi\left(-\mathrm{tr}_{E/F}\left(\delta b x_1^\tau\right) y_2 - \Delta\,\mathrm{N}_{E/F}(b) y_1 y_2\right), \end{aligned}$$

$$\hat{\omega}(1, w)\hat{\varphi}(x_1; y) = |\Delta|_F \int_{F^2} \hat{\varphi}\big(x_1^\tau; z_1, z_2\big)\psi(\Delta(y_2 z_1 - y_1 z_2))\, dz_1\, dz_2,$$
$$\hat{\omega}(\nu \mathbf{1}_2, a(\nu))\hat{\varphi}(x_1; y) = |\nu|_F \hat{\varphi}(x_1; \nu y),$$

for $a \in F^\times$, $b \in E$, $\nu \in F^\times$.

10.2. Theta lifts. Let $F = \mathbb{Q}$, $E = \mathcal{K}$, and $\psi = \psi_0$. Here $\mathcal{K}$ is an imaginary quadratic field with discriminant $-D < 0$. We assume that $-D \equiv 1 \bmod 8$. Let $\mathcal{O}$ be the ring of integers of $\mathcal{K}$. We put $\delta = \sqrt{-D}$. Let $g \in S_{\kappa+1}(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized Hecke eigenform. Let $\mathbf{g}$ denote the cusp form on $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ associated to g and $\pi = \otimes_v \pi_v$ the irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ generated by $\mathbf{g}$. Let $\pi_\mathcal{K} = \otimes_v \pi_{\mathcal{K},v}$ be the base change of π to $\mathcal{K}$, which is an irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathcal{K})$. We define cusp forms $\mathbf{g}^\sharp \in \pi$ and $\mathbf{g}^\sharp_\mathcal{K} \in \pi_\mathcal{K}$ by (3.3) and (3.7), respectively. We may regard $\theta(\pi)$ as a representation of $\mathbb{A}_\mathbb{Q} \times \mathrm{GL}_2(\mathbb{A}_\mathcal{K})$. By the strong multiplicity one theorem,

$$\theta(\pi) = \chi_{-D} \boxtimes \pi_\mathcal{K}.$$

We define $\varphi = \otimes_v \varphi_v^{(7)} \in S(V(\mathbb{A}_\mathbb{Q}))$ as follows:

- If $v = p$ with $p \nmid 2D$, then $\varphi_p^{(7)}$ is the characteristic function of $V(\mathbb{Z}_p)$.
- If $v = q$ with $q \mid D$, then

$$\varphi_q^{(7)} = (1+q)^{-1} \sum_{k,k' \in \mathrm{SL}_2(\mathbb{Z}_q)/K_0(q;\mathbb{Z}_q)} \omega(k, k')\varphi'_q.$$

  Here

$$\varphi'_q(x) = \begin{cases} (\Delta, x_2)_{\mathbb{Q}_q} & \text{if } x_1 \in \mathcal{O}_q,\ x_2 \in q^{-1}\mathbb{Z}_q^\times,\ x_3 \in \mathbb{Z}_q, \\ 0 & \text{otherwise.} \end{cases}$$

- If $v = 2$, then

$$\varphi_2^{(7)}(x) = \begin{cases} 1 & \text{if } x_1 \in \mathbb{Z}_2 + 2\mathcal{O}_2,\ x_2 \in 2\mathbb{Z}_2,\ x_3 \in 2\mathbb{Z}_2, \\ 0 & \text{otherwise.} \end{cases}$$

- If $v = \infty$, then $\varphi_\infty^{(7)} = (-2\delta^{-1})^\kappa \omega(1, k_\infty)\varphi'_\infty$. Here

$$\varphi'_\infty(x) = \bar{x}_1^\kappa e^{-\pi\, \mathrm{tr}(x^t \bar{x})}.$$

We may regard $\theta(\bar{\mathbf{g}}^\sharp, \varphi)$ as an automorphic form on $\mathrm{GL}_2(\mathbb{A}_\mathcal{K})$.

Lemma 10.1.

$$\theta(\bar{\mathbf{g}}^\sharp, \varphi) = -2^{-1}(\sqrt{-1})^{\kappa+1} D^{-\kappa} \xi_\mathbb{Q}(2)^{-1} \mathbf{g}^\sharp_\mathcal{K}.$$

The rest of this section is devoted to the proof of Lemma 10.1. We will compute the Fourier coefficient $\mathcal{W}$ of $\theta(\bar{\mathbf{g}}^\sharp, \varphi)$ defined by

$$\mathcal{W}(h) = \int_{\mathcal{K}\backslash\mathbb{A}_\mathcal{K}} \theta(u(x)h; \bar{\mathbf{g}}^\sharp, \varphi)\overline{\psi\big(\operatorname{tr}_{\mathcal{K}/\mathbb{Q}}(\delta^{-1}x)\big)}\, dx.$$

Let W_v be the Whittaker function of π_v with respect to ψ_v which satisfies the following conditions:

- If $v = p$, then $W_p(1) = 1$, and

$$W_p(gk) = W_p(g)$$

  for $g \in \mathrm{GL}_2(\mathbb{Q}_p)$, $k \in \mathrm{GL}_2(\mathbb{Z}_p)$.
- If $v = \infty$, then $W_\infty(1) = e^{-2\pi}$, and

$$W_\infty(gk_\theta) = e^{\sqrt{-1}(\kappa+1)\theta} W_\infty(g)$$

  for $g \in \mathrm{GL}_2(\mathbb{R})$, $\theta \in \mathbb{R}/2\pi\mathbb{Z}$.

Then

$$W_{\mathbf{g}^\sharp,\Delta^{-2}} = \prod_v W_v^\sharp,$$

where

$$W_v^\sharp(g) = \begin{cases} W_v(a(\Delta^{-2})g) & \text{if } v \neq 2, \\ W_2(a(\Delta^{-2})gt(2^{-1})) & \text{if } v = 2. \end{cases}$$

By Lemma 4.2,

$$\mathcal{W} = \xi_\mathbb{Q}(2)^{-1} \prod_v \mathcal{W}_v,$$

where

$$\mathcal{W}_v(h) = \int_{U(\mathbb{Q}_v)\backslash \mathrm{SL}_2(\mathbb{Q}_v)} \hat{\omega}(gg', h)\hat{\varphi}_v^{(7)}(-\Delta^{-1}; 0, 1)\overline{W_v^\sharp(gg')}\, dg$$
$$\times \begin{cases} \mathrm{vol}(\mathrm{SL}_2(\mathbb{Z}_p))^{-1} & \text{if } v = p, \\ \mathrm{vol}(\mathrm{SO}(2))^{-1} & \text{if } v = \infty, \end{cases}$$

with $g' \in \mathrm{GL}_2(\mathbb{Q}_v)$ such that $\nu(g') = \nu(h)$.

10.3. The unramified case. Let $v = p$ with $p \nmid 2D$. In this case, $\Delta \in \mathbb{Z}_p^\times$ and $\gamma_V = 1$. Let φ be the characteristic function of $V(\mathbb{Z}_p)$. Note that

$$\omega(k, k')\varphi = \varphi \tag{10.1}$$

for $k \in \mathrm{GL}_2(\mathbb{Z}_p)$, $k' \in \mathrm{GL}_2(\mathcal{O}_p)$ such that $\nu(k) = \nu(k')$, and

$$\hat{\varphi} = \varphi_1 \otimes \varphi_2$$

where $\varphi_1 \in \mathcal{S}(V_1(\mathbb{Q}_p))$ and $\varphi_2 \in \mathcal{S}(\mathbb{Q}_p^2)$ are the characteristic functions of $V_1(\mathbb{Z}_p)$ and $\mathbb{Z}_p^2$, respectively. Then

$$\begin{aligned} \mathcal{W}_p(1) &= \int_{\mathbb{Q}_p^\times} \hat{\omega}(t(a), 1)\hat{\varphi}(-\Delta^{-1}; 0, 1)\overline{W_p^\sharp(t(a))}|a|_{\mathbb{Q}_p}^{-2}\, d^\times a \\ &= \sum_{n\in\mathbb{Z}} p^n (\Delta, p^n)_{\mathbb{Q}_p}\hat{\varphi}(-\Delta^{-1}p^n; 0, p^{-n})\overline{W_p^\sharp(t(p^n))}. \end{aligned}$$

Since $\hat{\varphi}(-\Delta^{-1}p^n; 0, p^{-n}) = 0$ unless $n = 0$, we have

$$\mathcal{W}_p(1) = 1.$$

10.4. The ramified case. Let $v = q$ with $q \mid D$. In this case, $\Delta \in q\mathbb{Z}_q^\times$ and $\gamma_V = \gamma_{\mathbb{Q}_q}(\Delta, \psi_q)^{-1}$. Let χ be the quadratic character of $\mathbb{Q}_q^\times$ associated to $\mathcal{K}_q/\mathbb{Q}_q$ by class field theory. Let ϕ_n (resp. Φ_n) denote the characteristic function of $q^n\mathbb{Z}_q$ (resp. $\delta^n\mathcal{O}_q$). We define $\phi_\chi \in \mathcal{S}(\mathbb{Q}_q)$ by

$$\phi_\chi(x) = \begin{cases} \chi(x) & \text{if } x \in q^{-1}\mathbb{Z}_q^\times, \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$\begin{aligned} K_0 &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_q) \,\middle|\, c \equiv 0 \bmod q\mathbb{Z}_q \right\}, \\ \mathbf{K}_0 &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_q) \,\middle|\, c \equiv 0 \bmod q\mathbb{Z}_q \right\}, \\ \mathbb{K}_0 &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_q) \,\middle|\, c \equiv 0 \bmod \delta\mathcal{O}_q \right\}. \end{aligned}$$

We define $\varphi' \in \mathcal{S}(V(\mathbb{Q}_q))$ by

$$\varphi'(x) = \Phi_0(x_1)\phi_\chi(x_2)\phi_0(x_3).$$

Note that

$$\omega(k, k')\varphi' = \varphi'$$

for $k \in \mathbf{K}_0$, $k' \in \mathbb{K}_0$ such that $\nu(k) = \nu(k')$. Let

$$\varphi = (1+q)^{-1} \sum_{k,k' \in \mathrm{SL}_2(\mathbb{Z}_q)/K_0} \omega(k,k')\varphi'.$$

Since $\mathrm{SL}_2(\mathbb{Z}_q)/K_0 \simeq \mathrm{GL}_2(\mathbb{Z}_q)/\mathbf{K}_0 \simeq \mathrm{GL}_2(\mathcal{O}_q)/\mathbb{K}_0$, we have

$$\omega(k,k')\varphi = \varphi \tag{10.2}$$

for $k \in \mathrm{GL}_2(\mathbb{Z}_q)$, $k' \in \mathrm{GL}_2(\mathcal{O}_q)$ such that $\nu(k) = \nu(k')$.

Lemma 10.2.

$$\begin{aligned}
\hat{\varphi}'(x_1; y) &= \gamma_{\mathbb{Q}_q}(\Delta, \psi_q)^{-1} \Phi_0(x_1)\phi_0(y_1)\phi_\chi(y_2),\\
\hat{\omega}(w,1)\hat{\varphi}'(x_1; y) &= q^{-1/2}\chi(-1)\Phi_{-1}(x_1)\phi_\chi(y_1)\phi_0(y_2),\\
\hat{\omega}(1,w)\hat{\varphi}'(x_1; y) &= q^{-1/2}\chi(-1)\Phi_0(x_1)\phi_\chi(y_1)\phi_{-1}(y_2),\\
\hat{\omega}(w,w)\hat{\varphi}'(x_1; y) &= q^{-1}\gamma_{\mathbb{Q}_q}(\Delta, \psi_q)^{-1}\Phi_{-1}(x_1)\phi_{-1}(y_1)\phi_\chi(y_2).
\end{aligned}$$

Proof. See the proof of Lemma 9.2. □

Lemma 10.3.

$$\mathcal{W}_q(1) = q\gamma_{\mathbb{Q}_q}(\Delta, \psi_q)^{-1}.$$

Proof. Let

$$\mathcal{J}(k,k') = \sum_{n \in \mathbb{Z}} q^n (\Delta, \Delta^n)_{\mathbb{Q}_q} \hat{\omega}(k,k')\hat{\varphi}'(-\Delta^{n-1}; 0, \Delta^{-n})\overline{W_q^\sharp(t(\Delta^n))}.$$

Then $\mathcal{W}_q(1)$ is equal to

$$\begin{aligned}
&\int_{\mathbb{Q}_q^\times} \hat{\omega}(t(a),1)\hat{\varphi}(-\Delta^{-1}; 0, 1)\overline{W_q^\sharp(t(a))}|a|_{\mathbb{Q}_q}^{-2}\, d^\times a\\
&= (1+q)^{-1}\Bigg[\mathcal{J}(1,1) + \sum_{b' \in \mathbb{Z}_q/q\mathbb{Z}_q} \mathcal{J}(1, u(b')w)\\
&\qquad + \sum_{b \in \mathbb{Z}_q/q\mathbb{Z}_q} \mathcal{J}(u(b)w, 1) + \sum_{b,b' \in \mathbb{Z}_q/q\mathbb{Z}_q} \mathcal{J}(u(b)w, u(b')w)\Bigg]\\
&= (1+q)^{-1}\left[\mathcal{J}(1,1) + q\mathcal{J}(1,w) + q\mathcal{J}(w,1) + q^2\mathcal{J}(w,w)\right].
\end{aligned}$$

Since $\hat{\omega}(k,k')\hat{\varphi}'(-\Delta^{n-1}; 0, \Delta^{-n}) = 0$

$$\begin{cases}
\text{unless } n = 1 & \text{if } k = 1, k' = 1,\\
\text{for all } n & \text{if } k = 1, k' = w,\\
\text{for all } n & \text{if } k = w, k' = 1,\\
\text{unless } n = 1 & \text{if } k = w, k' = w,
\end{cases}$$

we have

$$\begin{aligned}\mathcal{J}(1,1) &= q\gamma_{\mathbb{Q}_q}(\Delta,\psi_q)^{-1},\\ \mathcal{J}(1,w) &= 0,\\ \mathcal{J}(w,1) &= 0,\\ \mathcal{J}(w,w) &= \gamma_{\mathbb{Q}_q}(\Delta,\psi_q)^{-1}.\end{aligned}$$

This completes the proof. □

10.5. The 2-adic case. Let $v = 2$. In this case, $\Delta \in \mathbb{Z}_2^\times \cap \mathbb{Q}_2^{\times,2}$ and $\gamma_V = 1$. Fix $\delta_0 \in \mathbb{Z}_2^\times$ such that $\Delta = \delta_0^2$. We identify $\mathcal{K}_2$ with $\mathbb{Q}_2 \oplus \mathbb{Q}_2$ via

$$\begin{aligned}\mathcal{K}_2 &\longrightarrow \mathbb{Q}_2 \oplus \mathbb{Q}_2.\\ x_1 + \delta x_2 &\longmapsto (x_1 + \delta_0 x_2, x_1 - \delta_0 x_2)\end{aligned}$$

Let $\mathcal{O}_2^\natural = \mathbb{Z}_2 + 2\mathcal{O}_2$. Note that $\{(1,0),(0,1)\}$ (resp. $\{1,\delta\}$) is a basis of $\mathcal{O}_2$ (resp. $\mathcal{O}_2^\natural$) over $\mathbb{Z}_2$. Let ϕ_n (resp. $\Phi^\natural$) denote the characteristic function of $2^n\mathbb{Z}_2$ (resp. $\mathcal{O}_2^\natural$). Then

$$\Phi^\natural(x) = \phi_1(x')\phi_1(x'') + [\phi_0 - \phi_1](x')[\phi_0 - \phi_1](x'')$$

for $x = (x', x'') \in \mathcal{K}_2$. Let

$$\begin{aligned}\mathbf{K}_0(4;\mathbb{Z}_2) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_2) \,\middle|\, c \equiv 0 \bmod 4\mathbb{Z}_2 \right\},\\ \mathbf{\Gamma} &= \{h \in \mathrm{GL}_2(\mathbb{Z}_2) \mid h \equiv \mathbf{1}_2 \bmod 2\mathbb{Z}_2\}.\end{aligned}$$

We define $\varphi \in \mathcal{S}(V(\mathbb{Q}_2))$ by

$$\varphi(x) = \Phi^\natural(x_1)\phi_1(x_2)\phi_1(x_3).$$

Note that

$$\omega(k,k')\varphi = \varphi \tag{10.3}$$

for $k \in \mathbf{K}_0(4;\mathbb{Z}_2)$, $k' \in (\mathbf{\Gamma} \times \mathbf{\Gamma})\,\mathrm{GL}_2(\mathbb{Z}_2)$ such that $\nu(k) = \nu(k')$.

Lemma 10.4. *We define $\varphi_1, \varphi_1', \varphi_1'' \in \mathcal{S}(V_1(\mathbb{Q}_2))$ by*

$$\begin{aligned}\varphi_1(x_1) &= \Phi^\natural(x_1),\\ \varphi_1'(x_1) &= \Phi^\natural(2x_1),\\ \varphi_1''(x_1) &= \phi_1(x_1')[\phi_0 - \phi_1](x_1'') + [\phi_0 - \phi_1](x_1')\phi_1(x_1''),\end{aligned}$$

for $x_1 = (x_1', x_1'') \in V_1(\mathbb{Q}_2)$, *respectively. We also define* $\varphi_2, \varphi_2', \varphi_2'', \varphi_3, \varphi_3', \varphi_3'' \in \mathcal{S}(\mathbb{Q}_2^2)$ *by*

$$\begin{aligned}
\varphi_2(y) &= \phi_1(y_1)\phi_{-1}(y_2),\\
\varphi_2'(y) &= \phi_{-1}(y_1)\phi_1(y_2),\\
\varphi_2''(y) &= \phi_1(y_1)\phi_0(y_2) + [\phi_0 - \phi_1](y_1)[\phi_{-1} - \phi_0](y_2),\\
\varphi_3(y) &= \phi_2(y_1)\phi_0(y_2),\\
\varphi_3'(y) &= \phi_0(y_1)\phi_2(y_2),\\
\varphi_3''(y) &= \phi_2(y_1)\phi_1(y_2) + [\phi_1 - \phi_2](y_1)[\phi_0 - \phi_1](y_2),
\end{aligned}$$

for $y = (y_1, y_2) \in \mathbb{Q}_2^2$, *respectively. Then*

$$\begin{aligned}
\hat{\varphi} &= 2^{-1}(\varphi_1 \otimes \varphi_2),\\
\hat{\omega}(w, 1)\hat{\varphi} &= 2^{-2}(\varphi_1' \otimes \varphi_2'),\\
\hat{\omega}(k_1, 1)\hat{\varphi} &= 2^{-1}(\varphi_1'' \otimes \varphi_2''),\\
\hat{\omega}(2^{-1}, a(2^{-1}))\hat{\varphi} &= \varphi_1 \otimes \varphi_3,\\
\hat{\omega}(2^{-1}w, a(2^{-1}))\hat{\varphi} &= 2^{-1}(\varphi_1' \otimes \varphi_3'),\\
\hat{\omega}(2^{-1}k_1, a(2^{-1}))\hat{\varphi} &= \varphi_1'' \otimes \varphi_3''.
\end{aligned}$$

Proof. We only check that $\omega(k_1, 1)\varphi_1 = \varphi_1''$. Note that $k_1 = t(2^{-1})u(2)w^{-1} \times u(2^{-1})$. Since

$$\begin{aligned}
\omega(u(2^{-1}), 1)\varphi_1(x_1) &= \big(\phi_1(x_1')\phi_1(x_1'') + [\phi_0 - \phi_1](x_1')[\phi_0 - \phi_1](x_1'')\big)\\
&\quad \times \psi_2\big(2^{-1}x_1'x_1''\big)\\
&= \phi_1(x_1')\phi_0(x_1'') + \phi_0(x_1')\phi_1(x_1'') - \phi_0(x_1')\phi_0(x_1''),
\end{aligned}$$

we have

$$\begin{aligned}
\omega(w^{-1}u(2^{-1}), 1)\varphi_1(x_1) &= 2^{-1}\phi_0(x_1')\phi_{-1}(x_1'') + 2^{-1}\phi_{-1}(x_1')\phi_0(x_1'')\\
&\quad - \phi_0(x_1')\phi_0(x_1''),
\end{aligned}$$

hence

$$\begin{aligned}
\omega(k_1, 1)\varphi_1(x_1) &= 2\omega(w^{-1}u(2^{-1}), 1)\varphi_1\big(2^{-1}x_1\big)\psi_2\big(2^{-1}x_1'x_1''\big)\\
&= \phi_1(x_1')\phi_0(x_1'') + \phi_0(x_1')\phi_1(x_1'') - 2\phi_1(x_1')\phi_1(x_1'').
\end{aligned}$$

□

Lemma 10.5.

$$\mathcal{W}_2(1) = 2^{-2}.$$

Proof. Let

$$\mathcal{J}(k) = \sum_{n \in \mathbb{Z}} 2^n \hat{\omega}(k, 1) \hat{\varphi}(-2^n \Delta^{-1}; 0, 2^{-n}) \overline{W_2^\sharp(t(2^n)k)}.$$

Then $\mathcal{W}_2(1)$ is equal to

$$\begin{aligned}
&6^{-1} \int_{\mathbb{Q}_2^\times} \sum_{k \in \mathrm{SL}_2(\mathbb{Z}_2)/K_0(4;\mathbb{Z}_2)} \hat{\omega}(t(a)k, 1) \hat{\varphi}(-\Delta^{-1}; 0, 1) \overline{W_2^\sharp(t(a)k)} |a|_{\mathbb{Q}_2}^{-2} \, d^\times a \\
&= 6^{-1} \left[\mathcal{J}(1) + \sum_{b \in \mathbb{Z}_2/4\mathbb{Z}_2} \mathcal{J}(u(b)w) + \mathcal{J}(k_1) \right] \\
&= 6^{-1} \left[\mathcal{J}(1) + 4\mathcal{J}(w) + \mathcal{J}(k_1) \right].
\end{aligned}$$

Since $\hat{\omega}(k, 1) \hat{\varphi}(-2^n \Delta^{-1}; 0, 2^{-n}) = 0$

$$\begin{cases} \text{unless } n = 0, 1 & \text{if } k = 1, \\ \text{unless } n = -1 & \text{if } k = w, \\ \text{for all } n & \text{if } k = k_1, \end{cases}$$

we have

$$\begin{aligned}
\mathcal{J}(1) &= 2^{-1} \overline{W_2^\sharp(1)} + \overline{W_2^\sharp(t(2))} = 1, \\
\mathcal{J}(w) &= 2^{-3} \overline{W_2^\sharp(t(2^{-1})w)} = 2^{-3}, \\
\mathcal{J}(k_1) &= 0.
\end{aligned}$$

This completes the proof. □

Lemma 10.6.

$$\mathcal{W}_2(a(2^{-1})) = 0.$$

Proof. Let

$$\mathcal{J}'(k) = \sum_{n \in \mathbb{Z}} 2^n \hat{\omega}(2^{-1}k, a(2^{-1})) \hat{\varphi}(-2^n \Delta^{-1}; 0, 2^{-n}) \overline{W_2^\sharp(2^{-1}t(2^n)k)}.$$

Then $\mathcal{W}_2(a(2^{-1}))$ is equal to

$$\begin{aligned}
&6^{-1} \int_{\mathbb{Q}_2^\times} \sum_{k \in \mathrm{SL}_2(\mathbb{Z}_2)/K_0(4;\mathbb{Z}_2)} \hat{\omega}(2^{-1}t(a)k, a(2^{-1})) \hat{\varphi}(-\Delta^{-1}; 0, 1) \\
&\times \overline{W_2^\sharp(2^{-1}t(a)k)} |a|_{\mathbb{Q}_2}^{-2} \, d^\times a \\
&= 6^{-1} \left[\mathcal{J}'(1) + \sum_{b \in \mathbb{Z}_2/4\mathbb{Z}_2} \mathcal{J}'(u(b)w) + \mathcal{J}'(k_1) \right] \\
&= 6^{-1} \left[\mathcal{J}'(1) + 4\mathcal{J}'(w) + \mathcal{J}'(k_1) \right].
\end{aligned}$$

Since $\hat{\omega}(2^{-1}k, a(2^{-1}))\hat{\varphi}(-2^n\Delta^{-1}; 0, 2^{-n}) = 0$

$$\begin{cases} \text{unless } n = 0 & \text{if } k = 1, \\ \text{for all } n & \text{if } k = w, \\ \text{for all } n & \text{if } k = k_1, \end{cases}$$

we have

$$\begin{aligned} \mathcal{J}'(1) &= \overline{W_2^\sharp(1)} = 0, \\ \mathcal{J}'(w) &= 0, \\ \mathcal{J}'(k_1) &= 0. \end{aligned}$$

This completes the proof. □

Lemma 10.7. *The Whittaker function $\mathcal{W}_2$ of $\pi_{\mathcal{K},2}$ satisfies*

$$\mathcal{W}_2(hk') = \mathcal{W}_2(h)$$

for $h \in \mathrm{GL}_2(\mathcal{K}_2)$, $k' \in \mathrm{GL}_2(\mathcal{O}_2)$.

Proof. For convenience, we write $\pi = \pi_2$ and $\Pi = \pi_{\mathcal{K},2}$. Since π is the principal series representation

$$\mathrm{Ind}_{\mathbf{B}(\mathbb{Q}_2)}^{\mathrm{GL}_2(\mathbb{Q}_2)} \left(| \, |_{\mathbb{Q}_2}^{s_2} \boxtimes | \, |_{\mathbb{Q}_2}^{-s_2} \right),$$

we have $\dim_{\mathbb{C}} \pi^{\mathbf{\Gamma}} = 3$. We may regard $\pi^{\mathbf{\Gamma}}$ as a representation of $\mathrm{GL}_2(\mathbb{Z}_2)/\mathbf{\Gamma} \simeq \mathrm{GL}_2(\mathbb{F}_2)$. Then

$$\pi^{\mathbf{\Gamma}} \simeq \mathbf{1} \oplus r,$$

where r is the irreducible representation of $\mathrm{GL}_2(\mathbb{F}_2)$ of dimension 2. Define elements $f_0, f_1, f_2, f_3 \in \pi$ so that $f_0|_{\mathrm{GL}_2(\mathbb{Z}_2)} \equiv 1$,

$$\begin{aligned} f_1|_{\mathbf{\Gamma}} &\equiv 1, & \mathrm{supp}(f_1) &= \mathbf{B}(\mathbb{Q}_2)\mathbf{\Gamma}, \\ f_2|_{w\mathbf{\Gamma}} &\equiv 1, & \mathrm{supp}(f_2) &= \mathbf{B}(\mathbb{Q}_2)w\mathbf{\Gamma}, \\ f_3|_{wu(1)\mathbf{\Gamma}} &\equiv 1, & \mathrm{supp}(f_3) &= \mathbf{B}(\mathbb{Q}_2)wu(1)\mathbf{\Gamma}. \end{aligned}$$

Note that $\{f_1, f_2, f_3\}$ is a basis of $\pi^{\mathbf{\Gamma}}$.

We may regard Π as a representation $\pi \boxtimes \pi$ of $\mathrm{GL}_2(\mathbb{Q}_2) \times \mathrm{GL}_2(\mathbb{Q}_2)$. Computing the multiplicity of the trivial representation of $\mathrm{GL}_2(\mathbb{F}_2)$ in $\pi^{\mathbf{\Gamma}} \otimes \pi^{\mathbf{\Gamma}}$, we obtain

$$\dim_{\mathbb{C}} \Pi^{(\mathbf{\Gamma} \times \mathbf{\Gamma})\,\mathrm{GL}_2(\mathbb{Z}_2)} = 2.$$

Define elements $\mathcal{F}_0, \mathcal{F}_1 \in \Pi$ by

$$\begin{aligned} \mathcal{F}_0 &= f_0 \otimes f_0, \\ \mathcal{F}_1 &= f_1 \otimes f_1 + f_2 \otimes f_2 + f_3 \otimes f_3. \end{aligned}$$

Note that $\{\mathcal{F}_0, \mathcal{F}_1\}$ is a basis of $\Pi^{(\Gamma\times\Gamma)\,\mathrm{GL}_2(\mathbb{Z}_2)}$. Let $\mathcal{W}_{\mathcal{F}_i}$ be the Whittaker function of Π defined by

$$\mathcal{W}_{\mathcal{F}_i}(h) = \int_{\mathcal{K}_2} \mathcal{F}_i(w^{-1}u(x)h)\overline{\psi_2\big(\mathrm{tr}_{\mathcal{K}_2/\mathbb{Q}_2}(\delta^{-1}x)\big)}\, dx.$$

A routine calculation shows that

$$\mathcal{W}_{\mathcal{F}_0}(a(2^{-1})) = 0,$$
$$\mathcal{W}_{\mathcal{F}_1}(a(2^{-1})) \neq 0.$$

Hence the assertion follows from (10.3) and Lemma 10.6. □

10.6. The archimedean case. Let $v = \infty$. In this case, $\Delta < 0$ and $\gamma_V = \sqrt{-1}$. We define $\varphi' \in \mathcal{S}(V(\mathbb{R}))$ by

$$\varphi'(x) = \bar{x}_1^{\kappa} e^{-\pi\left(2|x_1|^2+Dx_2^2+Dx_3^2\right)}.$$

Note that

$$\omega(k_\theta, 1)\varphi' = e^{\sqrt{-1}(\kappa+1)\theta}\varphi'$$

for $k_\theta \in \mathrm{SO}(2)$,

$$H\cdot\varphi' = 2\kappa\varphi', \quad X\cdot\varphi' = 0, \tag{10.4}$$

and

$$\hat{\varphi}'(x_1; y) = \bar{x}_1^{\kappa} e^{-\pi\left(2|x_1|^2+Dy_1^2+Dy_2^2\right)}$$

for $x_1 \in V_1(\mathbb{R})$, $y = (y_1, y_2) \in \mathbb{R}^2$.

Lemma 10.8.

$$\mathcal{W}_\infty(k_\infty^{-1}) = 2(\sqrt{-1})^\kappa D^{-\kappa-1} K_\kappa(4\pi D^{-1/2}).$$

Proof. The quantity $\mathcal{W}_\infty(k_\infty^{-1})$ is equal to

$$(-2\delta^{-1})^\kappa \int_{\mathbb{R}^\times} \hat{\omega}(t(a), 1)\hat{\varphi}'(-\Delta^{-1}; 0, 1)\overline{W_\infty^\sharp(t(a))}|a|_{\mathbb{R}}^{-2}\, d^\times a$$
$$= 2^{\kappa+1}(\sqrt{-1})^\kappa D^{-\kappa/2} \int_0^\infty a^{-1}\hat{\varphi}'(D^{-1}a; 0, a^{-1})\overline{W_\infty^\sharp(t(a))}\, d^\times a.$$

Since

$$W_\infty^\sharp(t(a)) = D^{-\kappa-1}a^{\kappa+1}e^{-2\pi D^{-2}a^2}$$

for $a \in \mathbb{R}_+^\times$, we have

$$\int_0^\infty a^{-1}\hat{\varphi}'(D^{-1}a; 0, a^{-1})\overline{W^\sharp(t(a))}\, d^\times a$$
$$= D^{-2\kappa-1}\int_0^\infty a^{2\kappa}e^{-4\pi D^{-2}a^2-\pi D a^{-2}}\, d^\times a$$
$$= 2^{-\kappa}D^{-\kappa/2-1}K_\kappa(4\pi D^{1/2}).$$

This completes the proof. □

10.7. Proof of Lemma 10.1. By (10.1), (10.2), (10.4), and Lemma 10.7, there exists a constant C such that

$$\theta(\bar{\mathbf{g}}^\sharp, \varphi) = C\mathbf{g}^\sharp_{\mathcal{K}}.$$

Since $\prod_{q|D} \gamma_{\mathbb{Q}_q}(\Delta, \psi_q)^{-1} = \gamma_{\mathbb{R}}(\Delta, \psi_\infty) = -\sqrt{-1}$, we have

$$C = K_\kappa (4\pi D^{-1/2})^{-1} \mathcal{W}\big(k_\infty^{-1}\big) = -2^{-1}(\sqrt{-1})^{\kappa+1} D^{-\kappa} \xi_{\mathbb{Q}}(2)^{-1}.$$

This completes the proof of Lemma 10.1.

11. Proof of Proposition 5.2

Let $F = \mathbb{Q}$ and $E = \mathcal{K}$. Let V and V' be the quadratic spaces as in Sects. 10.1 and 9.1, respectively. We may identify the quadratic space $\{x \in V \mid (x, \mathbf{1}_2) = 0\}$ with V' via

$$\begin{aligned} \{x \in V \mid (x, \mathbf{1}_2) = 0\} &\longrightarrow V'. \\ \delta \begin{pmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{pmatrix} &\longmapsto \begin{pmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{pmatrix} \end{aligned}$$

Then the homomorphisms ρ and the embedding

$$\begin{aligned} \mathrm{GL}_2 &\longrightarrow \mathbb{G}_m \times \mathrm{R}_{\mathcal{K}/\mathbb{Q}} \mathrm{GL}_2 \\ h &\longmapsto (\det(h)^{-1}, h) \end{aligned}$$

induce a natural embedding $\mathrm{SO}(V') \subset \mathrm{SO}(V)$.

We define

$$\begin{aligned} \varphi &= \otimes_v \varphi_v^{(7)} \in S(V(\mathbb{A}_{\mathbb{Q}})), \\ \varphi' &= \otimes_v \varphi_v^{(6)} \in S(V'(\mathbb{A}_{\mathbb{Q}})), \\ \varphi'' &= \otimes_v \varphi_v^{(1)} \in S(\mathbb{A}_{\mathbb{Q}}), \end{aligned}$$

as in Sect. 10.2, Sect. 9.2, Example 4.1, respectively. Note that $\varphi = \varphi' \otimes \varphi''$. Then a seesaw identity

$$\langle \theta(\mathbf{f} \otimes \chi_{-D}, \varphi')\Theta, \mathbf{g}^\sharp \rangle = \int_{\mathbb{A}_{\mathbb{Q}}^\times \mathrm{GL}_2(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})} (\mathbf{f} \otimes \chi_{-D})(h) \theta(h; \bar{\mathbf{g}}^\sharp, \varphi)\, dh$$

holds. The left-hand side is equal to

$$-2^{-1}\sqrt{-1} D^{-\kappa+1/2} c_h(D) \xi_{\mathbb{Q}}(2)^{-1} \langle f, f \rangle \langle h, h \rangle^{-1} \langle \mathbf{h}\Theta, \mathbf{g}^\sharp \rangle$$

by Lemma 9.1, and the right-hand side is equal to

$$-2^{-1}(\sqrt{-1})^{\kappa+1} D^{-\kappa} \xi_{\mathbb{Q}}(2)^{-1} \mathcal{I}\left(\mathbf{g}^\sharp_{\mathcal{K}}, \mathbf{f}\right)$$

by Lemma 10.1. This completes the proof of Proposition 5.2.

12. Triple product L-functions

In this section, we study the local zeta integrals of Garrett [18], Piatetski-Shapiro and Rallis [44].

12.1. Preliminaries. Let F be a local field of characteristic not 2 and fix a non-trivial additive character ψ of F. Let E be an abelian semisimple algebra over F of dimension 2. Let τ denote the non-trivial automorphism of E over F. Fix $\delta \in E^\times$ such that $\delta^\tau = -\delta$ and put $\Delta = \delta^2 \in F^\times$. Let $\mathbf{G} = \mathrm{GSp}_3(F)$ and

$$G = \{(g, g') \in \mathrm{GL}_2(E) \times \mathrm{GL}_2(F) \mid \nu(g) = \nu(g')\}.$$

Let $Z_{\mathbf{G}}$ denote the center of $\mathbf{G}$. We identify G with its image under the embedding

$$\begin{aligned} G &\longrightarrow \mathbf{G}. \\ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\right) &\longmapsto \begin{pmatrix} a_1 & a_2 & 0 & b_1 & \Delta b_2 & 0 \\ \Delta a_2 & a_1 & 0 & \Delta b_2 & \Delta b_1 & 0 \\ 0 & 0 & a' & 0 & 0 & b' \\ c_1 & c_2 & 0 & d_1 & \Delta d_2 & 0 \\ c_2 & \Delta^{-1} c_1 & 0 & d_2 & d_1 & 0 \\ 0 & 0 & c' & 0 & 0 & d' \end{pmatrix} \end{aligned}$$

Here $a = a_1 + \delta a_2$, $b = b_1 + \delta b_2$, $c = c_1 + \delta c_2$, $d = d_1 + \delta d_2$.

Let V be the quadratic space as in Sect. 6.1. Put $R = \mathrm{G}(\mathrm{SL}_2 \times \mathrm{O}(V))$. Then $R(F)$ (resp. $R(E)$) acts on $\mathcal{S}(V(F))$ (resp. $\mathcal{S}(V(E))$) via the Weil representation ω with respect to ψ (resp. $\frac{1}{2}(\psi \circ \mathrm{tr}_{E/F})$). Also, $\mathrm{G}(\mathrm{Sp}_3 \times \mathrm{O}(V))(F)$ acts on $\mathcal{S}(V^3(F))$ via the Weil representation ω with respect to ψ. Let $\Phi \in \mathcal{S}(V(E))$ and $\varphi \in \mathcal{S}(V(F))$. We identify $\mathcal{S}(V^3(F))$ with $\mathcal{S}(V(E)) \otimes \mathcal{S}(V(F))$ via

$$\begin{aligned} V^3(F) &\longrightarrow V(E) \oplus V(F). \\ (y_1, y_2, y_3) &\longmapsto (y_1 + \delta y_2, y_3) \end{aligned}$$

Then

$$\omega((g, g'), h)(\Phi \otimes \varphi) = \omega(g, h)\Phi \otimes \omega(g', h)\varphi$$

for $(g, g') \in G$, $h \in \mathrm{GO}(V)(F)$ such that $\nu(g) = \nu(g') = \nu(h)$.

Let

$$\mathbf{P} = \left\{ \begin{pmatrix} A & * \\ 0 & \nu\, {}^t A^{-1} \end{pmatrix} \in \mathbf{G} \,\middle|\, A \in \mathrm{GL}_3(F),\ \nu \in F^\times \right\}$$

be the Siegel parabolic subgroup of $\mathbf{G}$. For $s \in \mathbb{C}$, let $I(s) = \mathrm{Ind}_{\mathbf{P}}^{\mathbf{G}}(\rho_{\mathbf{P}}^s)$ denote the degenerate principal series representation of $\mathbf{G}$ consisting of smooth functions $f^{(s)}$ on $\mathbf{G}$ which satisfy

$$f^{(s)}\left(\begin{pmatrix} A & * \\ 0 & \nu\, {}^t A^{-1} \end{pmatrix} g\right) = |\det(A)|_F^{2(s+1)} |\nu|_F^{-3(s+1)} f^{(s)}(g).$$

Let $\Psi \in \mathcal{S}(V(E)) \otimes \mathcal{S}(V(F))$. For $g \in \mathbf{G}$, choose $h \in \mathrm{GO}(V)(F)$ such that $\nu(h) = \nu(g)$, and put

$$f_{\Psi}^{(0)}(g) = \omega(g, h)\Psi(0).$$

It does not depend on the choice of h and defines an element $f_{\Psi}^{(0)}$ of $I(0)$. Put

$$\eta = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix} \in \mathrm{Sp}_3(\mathbb{Z}).$$

Lemma 12.1. *Let $\Psi = \Phi \otimes \varphi$ with $\Phi \in \mathcal{S}(V(E))$ and $\varphi \in \mathcal{S}(V(F))$. Then*

$$f_{\Psi}^{(0)}(\eta) = \int_{V(F)} \Phi(y)\varphi(y)\, dy.$$

Here $V(F) \subset V(E)$ is a natural embedding and dy is the self-dual measure on $V(F)$ with respect to the pairing $\psi((x, y))$.

Proof. We may regard Ψ as an element of $\mathcal{S}(V^3(F))$. Let $\eta = pw'p'$ be the Bruhat decomposition of η given by

$$p = \begin{pmatrix} 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix},$$

$$w' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix},$$

$$p' = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Since

$$\omega(w'p',1)\Psi(y_1,y_2,y_3) = \int_{V(F)} \omega(p',1)\Psi(y_1,y_2,y')\psi((y_3,y'))\,dy'$$
$$= \int_{V(F)} \Psi(y',y_2,y_1+y')\psi((y_3,y'))\,dy',$$

we have

$$f_\Psi^{(0)}(\eta) = f_\Psi^{(0)}(w'p') = \omega(w'p',1)\Psi(0) = \int_{V(F)} \Psi(y',0,y')\,dy'.$$

This completes the proof. □

12.2. Local zeta integrals. Let $\mathcal{K}$ be an imaginary quadratic field with discriminant $-D<0$. We assume that $-D \equiv 1 \bmod 8$. Let $\mathcal{O}$ be the ring of integers of $\mathcal{K}$. We put $\delta=\sqrt{-D}$ and fix $\delta_0 \in \mathbb{Z}_2^\times$ such that $-D=\delta_0^2$. Let κ be an odd positive integer. Let $g \in S_{\kappa+1}(\mathrm{SL}_2(\mathbb{Z}))$ and $f \in S_{2\kappa}(\mathrm{SL}_2(\mathbb{Z}))$ be normalized Hecke eigenforms. Let $\mathbf{g}$ (resp. $\mathbf{f}$) denote the cusp form on $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ associated to g (resp. f) and $\pi=\otimes_v\pi_v$ (resp. $\sigma=\otimes_v\sigma_v$) the irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ generated by $\mathbf{g}$ (resp. $\mathbf{f}$). Let $\pi_\mathcal{K}=\otimes_v\pi_{\mathcal{K},v}$ be the base change of π to $\mathcal{K}$, which is an irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathcal{K})$. We define cusp forms $\mathbf{g}_\mathcal{K}^\sharp, \mathbf{g}_\mathcal{K}^\flat \in \pi_\mathcal{K}$ by (3.7), (3.8), respectively.

For each prime p, let $\{\alpha_p,\alpha_p^{-1}\}$ and $\{\beta_p,\beta_p^{-1}\}$ denote the Satake parameters of g and f at p, respectively. Note that $|\alpha_p|=|\beta_p|=1$ by the Ramanujan conjecture. Let $L(s,\pi_{\mathcal{K},p}\otimes\sigma_p)$ be the local L-factor given by

$$L(s,\pi_{\mathcal{K},p}\otimes\sigma_p) = \big[\big(1-\alpha_p^2\beta_p p^{-s}\big)\big(1-\alpha_p^2\beta_p^{-1}p^{-s}\big)\big(1-\beta_p p^{-s}\big)^2$$
$$\times\big(1-\beta_p^{-1}p^{-s}\big)^2\big(1-\alpha_p^{-2}\beta_p p^{-s}\big)\big(1-\alpha_p^{-2}\beta_p^{-1}p^{-s}\big)\big]^{-1}$$

if p splits in $\mathcal{K}$,

$$L(s,\pi_{\mathcal{K},p}\otimes\sigma_p) = \big[\big(1-\alpha_p^2\beta_p p^{-s}\big)\big(1-\alpha_p^2\beta_p^{-1}p^{-s}\big)\big(1-\beta_p^2 p^{-2s}\big)$$
$$\times\big(1-\beta_p^{-2}p^{-2s}\big)\big(1-\alpha_p^{-2}\beta_p p^{-s}\big)\big(1-\alpha_p^{-2}\beta_p^{-1}p^{-s}\big)\big]^{-1}$$

if p is inert in $\mathcal{K}$,

$$L(s,\pi_{\mathcal{K},p}\otimes\sigma_p) = \big[\big(1-\alpha_p^2\beta_p p^{-s}\big)\big(1-\alpha_p^2\beta_p^{-1}p^{-s}\big)\big(1-\beta_p p^{-s}\big)$$
$$\times\big(1-\beta_p^{-1}p^{-s}\big)\big(1-\alpha_p^{-2}\beta_p p^{-s}\big)\big(1-\alpha_p^{-2}\beta_p^{-1}p^{-s}\big)\big]^{-1}$$

if p is ramified in $\mathcal{K}$.

Let $\psi_0=\otimes_v\psi_v$ be the standard additive character of $\mathbb{A}_\mathbb{Q}$. Let W_v be the Whittaker function of $\pi_{\mathcal{K},v}$ with respect to $\delta^{-1}(\psi_v\circ\mathrm{tr}_{\mathcal{K}_v/\mathbb{Q}_v})$ which satisfies the following conditions:

- If $v = p$, then $W_p(1) = 1$, and
$$W_p(gk) = W_p(g)$$
for $g \in \mathrm{GL}_2(\mathcal{K}_p)$, $k \in \mathrm{GL}_2(\mathcal{O}_p)$.
- If $v = \infty$, then $W_\infty(1) = K_\kappa(4\pi D^{-1/2})$, and
$$H \cdot W_\infty = 2\kappa W_\infty, \quad X \cdot W_\infty = 0.$$

Then
$$W_{\mathbf{g}^\flat_{\mathcal{K}}, -2\delta^{-1}} = \prod_v W^\flat_v,$$
where
$$W^\flat_v(g) = \begin{cases} W_p(d(-1)gd(2\delta^{-1})^{-1}) & \text{if } v = p, \\ W_\infty(d(-1)g) & \text{if } v = \infty. \end{cases}$$

Also, let W'_v be the Whittaker function of σ_v with respect to ψ_v which satisfies the following conditions:

- If $v = p$, then $W'_p(1) = 1$, and
$$W'_p(gk) = W'_p(g)$$
for $g \in \mathrm{GL}_2(\mathbb{Q}_p)$, $k \in \mathrm{GL}_2(\mathbb{Z}_p)$.
- If $v = \infty$, then $W'_\infty(1) = e^{-2\pi}$, and
$$W'_\infty(gk_\theta) = e^{2\sqrt{-1}\kappa\theta} W'_\infty(g)$$
for $g \in \mathrm{GL}_2(\mathbb{R})$, $\theta \in \mathbb{R}/2\pi\mathbb{Z}$.

Then
$$W_{\mathbf{f},1} = \prod_v W'_v.$$

Let $F = \mathbb{Q}_v$, $E = \mathcal{K}_v$, and $\psi = \psi_v$. Let $K_{\mathbf{G}} = \mathrm{GSp}_3(\mathbb{Z}_p)$ if $v = p$, and
$$K_{\mathbf{G}} = \left\{ \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \middle| \ \alpha + \sqrt{-1}\beta \in \mathrm{U}(3) \right\}$$
if $v = \infty$. Put $K'_{\mathbf{G}} = \tilde{\gamma} K_{\mathbf{G}} \tilde{\gamma}^{-1}$, where
$$\tilde{\gamma} = \begin{pmatrix} \gamma & 0 \\ 0 & {}^t\gamma^{-1} \end{pmatrix} \in \mathbf{G} \quad \text{with } \gamma = \begin{cases} \mathbf{1}_3 & \text{if } v = p \neq 2, \\ \begin{pmatrix} 1 & 1 & 0 \\ \delta_0 & -\delta_0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{if } v = 2, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{D} & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{if } v = \infty. \end{cases}$$

Let

$$K_G = \begin{cases} G \cap \left(d(2\delta^{-1})^{-1}\,\mathrm{GL}_2(\mathcal{O}_p)d(2\delta^{-1}) \times \mathrm{GL}_2(\mathbb{Z}_p)\right) & \text{if } v = p, \\ \mathrm{SU}(2) \times \mathrm{SO}(2) & \text{if } v = \infty. \end{cases}$$

Then

$$K_G \subset K'_{\mathbf{G}}.$$

Let $l = 2\kappa$. We define $\Phi = \varphi_v^{(4)} \in \mathcal{S}(V(\mathcal{K}_v))$ and $\varphi = \varphi_v^{(2)} \in \mathcal{S}(V(\mathbb{Q}_v))$ as in Sects. 6.4 and 6.3, respectively. Put $\Psi = \Phi \otimes \varphi$. We extend $f_\Psi^{(0)}$ to a holomorphic section $f_\Psi^{(s)}$ of $I(s)$ so that its restriction to $K'_{\mathbf{G}}$ is independent of s. As in [44], define the local zeta integral $Z_v(s)$ by

$$Z_v(s) = \int_{Z_{\mathbf{G}}U_0\backslash G} f_\Psi^{(s)}(\eta(g, g'))\overline{W_v^\flat(d(2\delta^{-1})g)}\,W'_v(g')\,dg\,dg'$$
$$\times \begin{cases} \mathrm{vol}(\mathrm{SL}_2(\mathcal{O}_p))^{-1}\,\mathrm{vol}(\mathrm{SL}_2(\mathbb{Z}_p))^{-1} & \text{if } v = p, \\ \mathrm{vol}(\mathrm{SU}(2))^{-1}\,\mathrm{vol}(\mathrm{SO}(2))^{-1} & \text{if } v = \infty, \end{cases}$$

where

$$U_0 = \left\{ (u(x), u(x')) \,\middle|\, x \in \mathcal{K}_v, x' \in \mathbb{Q}_v, \frac{1}{2}\,\mathrm{tr}_{\mathcal{K}_v/\mathbb{Q}_v}(x) + x' = 0 \right\}.$$

If $v = p$ with $p \nmid 2D$, then

$$Z_p(s) = \zeta_p(2s+2)^{-1}\zeta_p(4s+2)^{-1} L\left(s + \frac{1}{2}, \pi_{\mathcal{K},p} \otimes \sigma_p\right)$$

by Theorem 3.1 of [44]. Here $\zeta_p(s) = (1 - p^{-s})^{-1}$.

Lemma 12.2. (i) *Let $v = p$. Then*

$$Z_p(0) = |2^{-1}\delta|_{\mathcal{K}_p}\zeta_p(2)^{-2} L\left(\frac{1}{2}, \pi_{\mathcal{K},p} \otimes \sigma_p\right).$$

(ii) *Let $v = \infty$. Then*

$$Z_\infty(0) = 2^{-3\kappa-2}\pi^{-4\kappa+2}(\sqrt{-1})^{-\kappa} D^{(\kappa+1)/2}\Gamma(\kappa)^2\Gamma(2\kappa).$$

The rest of this section is devoted to the proof of Lemma 12.2. Following [21], [27, §6], we will compute the local zeta integral $Z_v(s)$.

Lemma 12.3. *Let $v = p$. Then*

$$\int_{\mathbb{Q}_p} \max(p^{-m}, |x|_{\mathbb{Q}_p})^{-s}\psi(p^n x)\,dx$$
$$= \begin{cases} p^{m(s-1)}(1 - p^{-(m+n+1)(s-1)})(1 - p^{-s})(1 - p^{-s+1})^{-1} & \text{if } m \ge -n, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. See [44, p. 54]. □

Lemma 12.4. *Let $v = p$. Then*

$$Z_p(s) = |2^{-1}\delta|_{\mathcal{K}_p}\big(Z_p^{(0)}(s) + Z_p^{(1)}(s)\big),$$

where

$$\begin{aligned} Z_p^{(0)}(s) = &\int_{\mathbb{Q}_p\times\mathcal{K}_p^\times\times\mathbb{Q}_p^\times} f_\Psi^{(s)}(\eta(t(a), u(x)t(a'))) \\ &\times \overline{W_p^\flat(d(2\delta^{-1})t(a))}\,W_p'(u(x)t(a'))|a|_{\mathcal{K}_p}^{-2}|a'|_{\mathbb{Q}_p}^{-2}\,dx\,d^\times a\,d^\times a', \\ Z_p^{(1)}(s) = &\int_{\mathbb{Q}_p\times\mathcal{K}_p^\times\times\mathbb{Q}_p^\times} f_\Psi^{(s)}(\eta(a(p)t(a), a(p)u(x)t(a'))) \\ &\times \overline{W_p^\flat(d(2\delta^{-1})a(p)t(a))}\,W_p'(a(p)u(x)t(a'))|a|_{\mathcal{K}_p}^{-2}|a'|_{\mathbb{Q}_p}^{-2}\,p^2\,dx\,d^\times a\,d^\times a'. \end{aligned}$$

Proof. It is easy to check that the function

$$(g, g') \longmapsto f_\Psi^{(s)}(\eta(g, g'))\overline{W_p^\flat(d(2\delta^{-1})g)}\,W_p'(g')$$

on G is right K_G-invariant. Hence the assertion follows as in [21, §4], [27, §6.2]. □

12.3. The ramified case. Let $v = q$ with $q \mid D$. For convenience, we write $\alpha = \alpha_q$ and $\beta = \beta_q$. Then $Z_q^{(0)}(s)$ is equal to

$$\begin{aligned} &\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\int_{\mathbb{Q}_q} f_\Psi^{(s)}(\eta(t(\delta^{n_1}), u(x)t(q^{n_2}))) \\ &\times q^{-n_1}\frac{\alpha^{2n_1+1}-\alpha^{-2n_1-1}}{\alpha-\alpha^{-1}}\cdot\psi(x)q^{-n_2}\frac{\beta^{2n_2+1}-\beta^{-2n_2-1}}{\beta-\beta^{-1}}\cdot q^{2n_1+2n_2}\,dx, \end{aligned}$$

and $Z_q^{(1)}(s)$ is equal to

$$\begin{aligned} &\sum_{n_1=-1}^{\infty}\sum_{n_2=0}^{\infty}\int_{\mathbb{Q}_q} q^{-s-1} f_\Psi^{(s)}(\eta(t(\delta^{n_1}), u(x)t(q^{n_2}))) \\ &\times q^{-n_1-1}\frac{\alpha^{2n_1+3}-\alpha^{-2n_1-3}}{\alpha-\alpha^{-1}} \\ &\times \psi(qx)q^{-n_2-1/2}\frac{\beta^{2n_2+2}-\beta^{-2n_2-2}}{\beta-\beta^{-1}}\cdot q^{2n_1+2n_2+2}\,dx. \end{aligned}$$

Lemma 12.5.

$$\begin{aligned} &f_\Psi^{(s)}(\eta(t(\delta^{n_1}), u(x)t(q^{n_2}))) \\ &= \begin{cases} q^{-(4m+2n_2)(s+1)}\max\left(q^{-2\min(m,n_2)}, |x|_{\mathbb{Q}_q}\right)^{-2s-2} & \text{if } n_1 = 2m, \\ q^{-(4m+2n_2+2)(s+1)}\max\left(q^{-2\min(m,n_2)}, |x|_{\mathbb{Q}_q}\right)^{-2s-2} & \text{if } n_1 = 2m+1. \end{cases} \end{aligned}$$

Proof. A routine calculation shows that $f_\Psi^{(s)}(\eta(t(\delta^{n_1}), u(x)t(q^{n_2})))$ is equal to

$$f_\Psi^{(0)}(\eta(t(\delta^{n_1}), u(x)t(q^{n_2}))) \times \begin{cases} q^{-(4m+2n_2)s} \max\left(q^{-2\min(m,n_2)}, |x|_{\mathbb{Q}_q}\right)^{-2s} & \text{if } n_1 = 2m, \\ q^{-(4m+2n_2+2)s} \max\left(q^{-2\min(m,n_2)}, |x|_{\mathbb{Q}_q}\right)^{-2s} & \text{if } n_1 = 2m+1. \end{cases}$$

By Lemma 12.1,

$$f_\Psi^{(0)}(\eta(t(\delta^{n_1}), u(x)t(q^{n_2}))) = \begin{cases} q^{-(4m+2n_2)} \max\left(q^{-2\min(m,n_2)}, |x|_{\mathbb{Q}_q}\right)^{-2} & \text{if } n_1 = 2m, \\ q^{-(4m+2n_2+2)} \max\left(q^{-2\min(m,n_2)}, |x|_{\mathbb{Q}_q}\right)^{-2} & \text{if } n_1 = 2m+1. \end{cases}$$

This completes the proof. □

Let $X = q^{-s-1/2}$. For each $a, b \in \mathbb{Z}$, put

$$P_{a,b}(X) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} X^{4m+2n-4\min(m,n)}(\alpha^{4m+a} - \alpha^{-4m-a})(\beta^{2n+b} - \beta^{-2n-b}),$$

$$P'_{a,b}(X) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} X^{4m+2n}(\alpha^{4m+a} - \alpha^{-4m-a})(\beta^{2n+b} - \beta^{-2n-b}).$$

Then $P_{a,b}(X)$ is equal to

$$\begin{aligned}
&(1-\beta^2X^2)^{-1}[\alpha^a\beta^b(1-\alpha^4\beta^2X^2)^{-1} - \alpha^{-a}\beta^b(1-\alpha^{-4}\beta^2X^2)^{-1}] \\
&- (1-\beta^{-2}X^2)^{-1}[\alpha^a\beta^{-b}(1-\alpha^4\beta^{-2}X^2)^{-1} - \alpha^{-a}\beta^{-b}(1-\alpha^{-4}\beta^{-2}X^2)^{-1}] \\
&+ \alpha^4X^4(1-\alpha^4X^4)^{-1}[\alpha^a\beta^b(1-\alpha^4\beta^2X^2)^{-1} - \alpha^a\beta^{-b}(1-\alpha^4\beta^{-2}X^2)^{-1}] \\
&- \alpha^{-4}X^4(1-\alpha^{-4}X^4)^{-1} \\
&\times [\alpha^{-a}\beta^b(1-\alpha^{-4}\beta^2X^2)^{-1} - \alpha^{-a}\beta^{-b}(1-\alpha^{-4}\beta^{-2}X^2)^{-1}],
\end{aligned}$$

and $P'_{a,b}(X)$ is equal to

$$\begin{aligned}
&[\alpha^a(1-\alpha^4X^4)^{-1} - \alpha^{-a}(1-\alpha^{-4}X^4)^{-1}] \\
&\times [\beta^b(1-\beta^2X^2)^{-1} - \beta^{-b}(1-\beta^{-2}X^2)^{-1}].
\end{aligned}$$

By Lemma 12.3, $Z_q^{(0)}(s)$ is equal to

$$\begin{aligned}
&(1-q^{-1}X^2)(1-X^2)^{-1}(\alpha-\alpha^{-1})^{-1}(\beta-\beta^{-1})^{-1} \\
&\times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} X^{4m+2n-4\min(m,n)} \\
&\times (1-X^{4\min(m,n)+2})(\alpha^{4m+1} - \alpha^{-4m-1})(\beta^{2n+1} - \beta^{-2n-1}) \\
&+ (1-q^{-1}X^2)(1-X^2)^{-1}(\alpha-\alpha^{-1})^{-1}(\beta-\beta^{-1})^{-1}
\end{aligned}$$

$$\times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} X^{4m+2n-4\min(m,n)+2}$$
$$\times (1-X^{4\min(m,n)+2})(\alpha^{4m+3}-\alpha^{-4m-3})(\beta^{2n+1}-\beta^{-2n-1}),$$

and $Z_q^{(1)}(s)$ is equal to

$$(1-q^{-1}X^2)(1-X^2)^{-1}(\alpha-\alpha^{-1})^{-1}(\beta-\beta^{-1})^{-1}$$
$$\times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} X^{4m+2n-4\min(m,n)+1}$$
$$\times (1-X^{4\min(m,n)+4})(\alpha^{4m+3}-\alpha^{-4m-3})(\beta^{2n+2}-\beta^{-2n-2})$$
$$+ (1-q^{-1}X^2)(1-X^2)^{-1}(\alpha-\alpha^{-1})^{-1}(\beta-\beta^{-1})^{-1}$$
$$\times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} X^{4m+2n-4\min(m,n)+3}$$
$$\times (1-X^{4\min(m,n)+4})(\alpha^{4m+5}-\alpha^{-4m-5})(\beta^{2n+2}-\beta^{-2n-2}).$$

Hence

$$Z_q^{(0)}(s)+Z_q^{(1)}(s)=(1-q^{-1}X^2)(1-X^2)^{-1}(\alpha-\alpha^{-1})^{-1}(\beta-\beta^{-1})^{-1}P(X),$$

where

$$P(X)=P_{1,1}(X)-X^2P'_{1,1}(X)+X^2P_{3,1}(X)-X^4P'_{3,1}(X)$$
$$+XP_{3,2}(X)-X^5P'_{3,2}(X)+X^3P_{5,2}(X)-X^7P'_{5,2}(X).$$

By a direct calculation, $P(X)$ is equal to

$$(\alpha-\alpha^{-1})(\beta-\beta^{-1})(1-X^2)(1-X^4)$$
$$\times [(1-\alpha^2\beta X)(1-\alpha^2\beta^{-1}X)(1-\beta X)(1-\beta^{-1}X)$$
$$\times (1-\alpha^{-2}\beta X)(1-\alpha^{-2}\beta^{-1}X)]^{-1}.$$

Therefore

$$Z_q^{(0)}(s)+Z_q^{(1)}(s)=\zeta_q(2s+2)^{-1}\zeta_q(4s+2)^{-1}L\left(s+\frac{1}{2},\pi_{\mathcal{K},q}\otimes\sigma_q\right).$$

This completes the proof of (i) of Lemma 12.2 in this case.

12.4. The 2-adic case. Let $v=2$. We identify $\mathcal{K}_2$ with $\mathbb{Q}_2\oplus\mathbb{Q}_2$ via

$$\mathcal{K}_2 \longrightarrow \mathbb{Q}_2\oplus\mathbb{Q}_2.$$
$$x_1+\delta x_2 \longmapsto (x_1+\delta_0x_2,\, x_1-\delta_0x_2)$$

For convenience, we write $\alpha = \alpha_2$ and $\beta = \beta_2$. Then $Z_2^{(0)}(s)$ is equal to

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\int_{\mathbb{Q}_2} f_{\Psi}^{(s)}(\eta((t(2^{n_1}), t(2^{n_2})), u(x)t(2^{n_3})))$$
$$\times\, 2^{-n_1-n_2}\frac{\alpha^{2n_1+1}-\alpha^{-2n_1-1}}{\alpha-\alpha^{-1}}\frac{\alpha^{2n_2+1}-\alpha^{-2n_2-1}}{\alpha-\alpha^{-1}}$$
$$\times\, \psi(x)2^{-n_3}\frac{\beta^{2n_3+1}-\beta^{-2n_3-1}}{\beta-\beta^{-1}}\cdot 2^{2n_1+2n_2+2n_3}\,dx,$$

and $Z_2^{(1)}(s)$ is equal to

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\int_{\mathbb{Q}_2} 2^{-s-1} f_{\Psi}^{(s)}(\eta((t(2^{n_1}), t(2^{n_2})), u(x)t(2^{n_3})))$$
$$\times\, 2^{-n_1-n_2-1}\frac{\alpha^{2n_1+2}-\alpha^{-2n_1-2}}{\alpha-\alpha^{-1}}\frac{\alpha^{2n_2+2}-\alpha^{-2n_2-2}}{\alpha-\alpha^{-1}}$$
$$\times\, \psi(2x)2^{-n_3-1/2}\frac{\beta^{2n_3+2}-\beta^{-2n_3-2}}{\beta-\beta^{-1}}\cdot 2^{2n_1+2n_2+2n_3+2}\,dx.$$

Lemma 12.6.

$$f_{\Psi}^{(s)}(\eta((t(2^{n_1}), t(2^{n_2})), u(x)t(2^{n_3})))$$
$$= 2^{-(2n_1+2n_2+2n_3)(s+1)}\max\left(2^{-2\min(n_1,n_2,n_3)}, |x|_{\mathbb{Q}_2}\right)^{-2s-2}.$$

Proof. A routine calculation shows that $f_{\Psi}^{(s)}(\eta((t(2^{n_1}), t(2^{n_2})), u(x)t(2^{n_3})))$ is equal to

$$f_{\Psi}^{(0)}(\eta((t(2^{n_1}), t(2^{n_2})), u(x)t(2^{n_3})))$$
$$\times\, 2^{-(2n_1+2n_2+2n_3)s}\max\left(2^{-2\min(n_1,n_2,n_3)}, |x|_{\mathbb{Q}_2}\right)^{-2s}.$$

By Lemma 12.1,

$$f_{\Psi}^{(0)}(\eta((t(2^{n_1}), t(2^{n_2})), u(x)t(2^{n_3})))$$
$$= 2^{-(2n_1+2n_2+2n_3)}\max\left(2^{-2\min(n_1,n_2,n_3)}, |x|_{\mathbb{Q}_2}\right)^{-2}.$$

This completes the proof. □

Let $X = 2^{-s-1/2}$. By Lemma 12.3, $Z_2^{(0)}(s)$ is equal to

$$(1-2^{-1}X^2)(1-X^2)^{-1}(\alpha-\alpha^{-1})^{-2}(\beta-\beta^{-1})^{-1}$$
$$\times \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty} X^{2n_1+2n_2+2n_3-4\min(n_1,n_2,n_3)}(1-X^{4\min(n_1,n_2,n_3)+2})$$
$$\times\, (\alpha^{2n_1+1}-\alpha^{-2n_1-1})(\alpha^{2n_2+1}-\alpha^{-2n_2-1})(\beta^{2n_3+1}-\beta^{-2n_3-1}),$$

and $Z_2^{(1)}(s)$ is equal to

$$(1-2^{-1}X^2)(1-X^2)^{-1}(\alpha-\alpha^{-1})^{-2}(\beta-\beta^{-1})^{-1}$$
$$\times \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty} X^{2n_1+2n_2+2n_3-4\min(n_1,n_2,n_3)+1}(1-X^{4\min(n_1,n_2,n_3)+4})$$
$$\times (\alpha^{2n_1+2}-\alpha^{-2n_1-2})(\alpha^{2n_2+2}-\alpha^{-2n_2-2})(\beta^{2n_3+2}-\beta^{-2n_3-2}).$$

Hence

$$Z_2^{(0)}(s)+Z_2^{(1)}(s)=\zeta_2(2s+2)^{-1}\zeta_2(4s+2)^{-1}L\left(s+\frac{1}{2},\pi_{\mathcal{K},2}\otimes\sigma_2\right)$$

as in the proof of Theorem 3.1 of [44]. This completes the proof of (i) of Lemma 12.2 in this case.

12.5. The archimedean case. Let $v=\infty$.

Lemma 12.7.

$$\text{(i)}\quad \sum_{i=0}^{n}\binom{n}{i}\Gamma(z+i)\Gamma(w-i)=\frac{\Gamma(z)\Gamma(z+w)\Gamma(w-n)}{\Gamma(z+w-n)}.$$

$$\text{(ii)}\quad \sum_{i=0}^{n}(-1)^i\binom{n}{i}\frac{\Gamma(z+i)}{\Gamma(w+i)}=\frac{\Gamma(z)\Gamma(w-z+n)}{\Gamma(w-z)\Gamma(w+n)}.$$

Proof. It is easy to verify (i) by induction on n. For (ii), see Lemma 2.1 of [28]. □

Lemma 12.8. *For each* $n\in\mathbb{Z}_{\geq 0}$, *put*

$$I_n(\alpha,\beta,\varrho)=\int_0^{\infty}\int_{-\infty}^{\infty}(1+\sqrt{-1}x)^{-\alpha}(1-\sqrt{-1}x)^{-\beta}$$
$$\times t^{\alpha+\beta-1}(t-1-\sqrt{-1}x)^n(t+1-\sqrt{-1}x)^{-\varrho-n}\,dx\,d^{\times}t.$$

If $\mathrm{Re}(\varrho)>\mathrm{Re}(\alpha+\beta-1)>0$, *then*

$$I_n(\alpha,\beta,\varrho)=2^{-\varrho+1}\pi\frac{\Gamma(\alpha+\beta-1)\Gamma(\beta+n)\Gamma(\varrho-\alpha-\beta+1)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\varrho-\alpha+n+1)}.$$

Proof. Recall that

$$\int_0^{\infty}t^{\alpha}(t+z)^{-\beta}\,d^{\times}t=z^{\alpha-\beta}\frac{\Gamma(\alpha)\Gamma(\beta-\alpha)}{\Gamma(\beta)}$$

if $|\arg(z)|<\pi$ and $\mathrm{Re}(\beta)>\mathrm{Re}(\alpha)>0$, and

$$\int_{-\infty}^{\infty}(1+\sqrt{-1}x)^{-\alpha}(1-\sqrt{-1}x)^{-\beta}\,dx=2^{2-\alpha-\beta}\pi\frac{\Gamma(\alpha+\beta-1)}{\Gamma(\alpha)\Gamma(\beta)}$$

if $\mathrm{Re}(\alpha+\beta) > 1$ (cf. [18], [28, §2]). Put $\tilde{x} = 2(1+\sqrt{-1}x)^{-1}$ for $x \in \mathbb{R}$. Then

$$\begin{aligned}
&\int_0^\infty t^{\alpha+\beta-1}(t-1-\sqrt{-1}x)^n(t+1-\sqrt{-1}x)^{-\varrho-n}\,d^\times t \\
&= \sum_{i=0}^n \binom{n}{i}(-1-\sqrt{-1}x)^{n-i}\int_0^\infty t^{\alpha+\beta+i-1}(t+1-\sqrt{-1}x)^{-\varrho-n}\,d^\times t \\
&= \sum_{i=0}^n \binom{n}{i}(-1-\sqrt{-1}x)^{n-i}(1-\sqrt{-1}x)^{-\varrho+\alpha+\beta-n+i-1} \\
&\times \Gamma(\alpha+\beta+i-1)\Gamma(\varrho-\alpha-\beta+n-i+1)\Gamma(\varrho+n)^{-1} \\
&= (-1-\sqrt{-1}x)^n(1-\sqrt{-1}x)^{-\varrho+\alpha+\beta-n-1}\Gamma(\varrho+n)^{-1} \\
&\times \sum_{i=0}^n \binom{n}{i}\Gamma(\alpha+\beta+i-1)\Gamma(\varrho-\alpha-\beta+n-i+1)(1-\tilde{x})^i \\
&= (-1-\sqrt{-1}x)^n(1-\sqrt{-1}x)^{-\varrho+\alpha+\beta-n-1}\Gamma(\varrho+n)^{-1} \\
&\times \sum_{i=0}^n\sum_{j=0}^i \binom{n}{i}\binom{i}{j}\Gamma(\alpha+\beta+i-1)\Gamma(\varrho-\alpha-\beta+n-i+1)(-\tilde{x})^j \\
&= (-1-\sqrt{-1}x)^n(1-\sqrt{-1}x)^{-\varrho+\alpha+\beta-n-1}\Gamma(\varrho+n)^{-1} \\
&\times \sum_{j=0}^n\sum_{i=0}^{n-j} \binom{n}{j}\binom{n-j}{i}\Gamma(\alpha+\beta+i+j-1) \\
&\times \Gamma(\varrho-\alpha-\beta+n-i-j+1)(-\tilde{x})^j \\
&= (-1-\sqrt{-1}x)^n(1-\sqrt{-1}x)^{-\varrho+\alpha+\beta-n-1}\Gamma(\varrho-\alpha-\beta+1) \\
&\times \sum_{j=0}^n \binom{n}{j}\frac{\Gamma(\alpha+\beta+j-1)}{\Gamma(\varrho+j)}(-\tilde{x})^j.
\end{aligned}$$

Hence $I_n(\alpha,\beta,\varrho)$ is equal to

$$\begin{aligned}
&(-1)^n\Gamma(\varrho-\alpha-\beta+1)\sum_{j=0}^n(-1)^j 2^j\binom{n}{j}\frac{\Gamma(\alpha+\beta+j-1)}{\Gamma(\varrho+j)} \\
&\times \int_{-\infty}^\infty (1+\sqrt{-1}x)^{-\alpha+n-j}(1-\sqrt{-1}x)^{-\varrho+\alpha-n-1}\,dx \\
&= (-1)^n 2^{-\varrho+1}\pi\frac{\Gamma(\varrho-\alpha-\beta+1)}{\Gamma(\varrho-\alpha+n+1)}\sum_{j=0}^n(-1)^j\binom{n}{j}\frac{\Gamma(\alpha+\beta+j-1)}{\Gamma(\alpha-n+j)}
\end{aligned}$$

$$= (-1)^n 2^{-\varrho+1}\pi \frac{\Gamma(\alpha+\beta-1)\Gamma(-\beta+1)\Gamma(\varrho-\alpha-\beta+1)}{\Gamma(\alpha)\Gamma(-\beta-n+1)\Gamma(\varrho-\alpha+n+1)}$$
$$= 2^{-\varrho+1}\pi \frac{\Gamma(\alpha+\beta-1)\Gamma(\beta+n)\Gamma(\varrho-\alpha-\beta+1)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\varrho-\alpha+n+1)}.$$

□

Lemma 12.9. *Put*

$$\begin{aligned} &I(\lambda,\mu,\nu,n_1,n_2)\\ &= \int_0^\infty\int_0^\infty\int_{-\infty}^\infty (a_1+a_2+\sqrt{-1}x)^{-s-\lambda}(a_1+a_2-\sqrt{-1}x)^{-s-\mu}\\ &\times K_\nu(a_1)e^{-a_2+\sqrt{-1}x}a_1^{2s+n_1}a_2^{s+n_2}\,dx\,d^\times a_1\,d^\times a_2. \end{aligned}$$

Then

$$\begin{aligned} &I(n_2+1,\mu,\nu,n_1,n_2)\\ &= 2^{-3s-n_1-n_2+1}\pi^{3/2}\frac{\Gamma(s-\mu+\nu+n_1)\Gamma(s-\mu-\nu+n_1)\Gamma(s+n_2)}{\Gamma\left(s-\mu+n_1+\frac{1}{2}\right)\Gamma(s+n_2+1)}. \end{aligned}$$

Proof. Recall that

$$\begin{aligned} &\int_0^\infty t^\mu e^{-\alpha t}K_\nu(\beta t)\,d^\times t\\ &= \frac{\sqrt{\pi}(2\beta)^\nu}{(\alpha+\beta)^{\mu+\nu}}\frac{\Gamma(\mu+\nu)\Gamma(\mu-\nu)}{\Gamma\left(\mu+\frac{1}{2}\right)}{}_2F_1\left(\mu+\nu,\nu+\frac{1}{2};\mu+\frac{1}{2};\frac{\alpha-\beta}{\alpha+\beta}\right) \end{aligned}$$

if $\mathrm{Re}(\mu) > |\mathrm{Re}(\nu)|$ and $\mathrm{Re}(\alpha+\beta) > 0$ (cf. [20, 6.621.3]). Since

$$\begin{aligned} &\int_0^\infty K_\nu(a_1)e^{-(t-\sqrt{-1}x)a_1}a_1^{2s+n_1}\,d^\times a_1\\ &= 2^\nu\pi^{1/2}(t+1-\sqrt{-1}x)^{-2s-\nu-n_1}\frac{\Gamma(2s+\nu+n_1)\Gamma(2s-\nu+n_1)}{\Gamma\left(2s+n_1+\frac{1}{2}\right)}\\ &\times {}_2F_1\left(2s+\nu+n_1,\nu+\frac{1}{2};2s+n_1+\frac{1}{2};\frac{t-1-\sqrt{-1}x}{t+1-\sqrt{-1}x}\right) \end{aligned}$$

and

$$\int_0^\infty e^{-(t+1-\sqrt{-1}x)a_2}a_2^{s+n_2}\,d^\times a_2 = (t+1-\sqrt{-1}x)^{-s-n_2}\Gamma(s+n_2),$$

$\Gamma(2s+\lambda+\mu-1)I(\lambda,\mu,\nu,n_1,n_2)$ is equal to

$$\int_0^\infty \int_0^\infty \int_0^\infty \int_{-\infty}^\infty e^{-(a_1+a_2)t} t^{2s+\lambda+\mu-1} (1+\sqrt{-1}x)^{-s-\lambda} (1-\sqrt{-1}x)^{-s-\mu}$$
$$\times K_\nu(a_1) e^{\sqrt{-1}a_1 x} e^{-a_2(1-\sqrt{-1}x)} a_1^{2s+n_1} a_2^{s+n_2}\, dx\, d^\times a_1\, d^\times a_2\, d^\times t$$
$$= 2^\nu \pi^{1/2} \frac{\Gamma(2s+\nu+n_1)\Gamma(2s-\nu+n_1)\Gamma(s+n_2)}{\Gamma\left(2s+n_1+\frac{1}{2}\right)}$$
$$\times \int_0^\infty \int_{-\infty}^\infty (1+\sqrt{-1}x)^{-s-\lambda}(1-\sqrt{-1}x)^{-s-\mu} t^{2s+\lambda+\mu-1}$$
$$\times (t+1-\sqrt{-1}x)^{-3s-\nu-n_1-n_2}$$
$$\times {}_2F_1\left(2s+\nu+n_1, \nu+\frac{1}{2}; 2s+n_1+\frac{1}{2}; \frac{t-1-\sqrt{-1}x}{t+1-\sqrt{-1}x}\right) dx\, d^\times t.$$

For $t \in \mathbb{R}_+^\times$ and $x \in \mathbb{R}$, we have

$$\left| \frac{t-1-\sqrt{-1}x}{t+1-\sqrt{-1}x} \right| < 1,$$

hence $I(\lambda, \mu, \nu, n_1, n_2)$ is equal to

$$2^\nu \pi^{1/2} \frac{\Gamma(2s-\nu+n_1)\Gamma(s+n_2)}{\Gamma(2s+\lambda+\mu-1)\Gamma\left(\nu+\frac{1}{2}\right)}$$
$$\times \sum_{n=0}^\infty \frac{\Gamma(2s+\nu+n_1+n)\Gamma\left(\nu+\frac{1}{2}+n\right)}{n!\Gamma\left(2s+n_1+\frac{1}{2}+n\right)}$$
$$\times I_n(s+\lambda, s+\mu, 3s+\nu+n_1+n_2).$$

Therefore $I(n_2+1, \mu, \nu, n_1, n_2)$ is equal to

$$2^{-3s-n_1-n_2+1} \pi^{3/2} \frac{\Gamma(s-\mu+\nu+n_1)\Gamma(2s-\nu+n_1)\Gamma(s+n_2)}{\Gamma(s+\mu)\Gamma\left(\nu+\frac{1}{2}\right)\Gamma(s+n_2+1)}$$
$$\times \sum_{n=0}^\infty \frac{\Gamma(s+\mu+n)\Gamma\left(\nu+\frac{1}{2}+n\right)}{n!\Gamma\left(2s+n_1+\frac{1}{2}+n\right)}.$$

By the Gauss summation formula,

$$\sum_{n=0}^\infty \frac{\Gamma(s+\mu+n)\Gamma(\nu+\frac{1}{2}+n)}{n!\Gamma(2s+n_1+\frac{1}{2}+n)} = \frac{\Gamma(s-\mu-\nu+n_1)\Gamma(s+\mu)\Gamma(\nu+\frac{1}{2})}{\Gamma(s-\mu+n_1+\frac{1}{2})\Gamma(2s-\nu+n_1)}.$$

This completes the proof. □

Now we compute the local zeta integral $Z_\infty(s)$. As in [21, §6], [27, §6.3], $Z_\infty(s)$ is equal to

$$\begin{aligned}
&\mathrm{vol}(\mathrm{SU}(2))^{-1}\,\mathrm{vol}(\mathrm{SO}(2))^{-1}\\
&\times \int_{\mathbb{R}\times\mathbb{C}^\times\times\mathbb{R}^\times}\int_{\mathrm{SU}(2)\times\mathrm{SO}(2)} f_\Psi^{(s)}(\eta(t(a_1)k,\,u(x)t(a_2)k_\theta))\\
&\times \overline{W_\infty^\flat(d(2\delta^{-1})t(a_1)k)}\,W'_\infty(u(x)t(a_2)k_\theta)|a_1|_{\mathbb{C}}^{-2}|a_2|_{\mathbb{R}}^{-2}\,dk\,dk_\theta\,dx\,d^\times a_1\,d^\times a_2\\
&= 2^2\pi\,\mathrm{vol}(\mathrm{SU}(2))^{-1}\,\mathrm{vol}(\mathrm{SO}(2))^{-1}\\
&\times \int_{\mathbb{R}\times\mathbb{R}_+^\times\times\mathbb{R}_+^\times}\int_{\mathrm{SU}(2)\times\mathrm{SO}(2)} f_\Psi^{(s)}(\eta(t(a_1)k,\,u(x)t(a_2)k_\theta))\\
&\times 2^{-\kappa-1}D^{(\kappa+1)/2}a_1^{2\kappa+2}\sum_{m=0}^{2\kappa}\binom{2\kappa}{m}(\sqrt{-1})^{\kappa+m}\bar{\alpha}^{2\kappa-m}\beta^m K_{\kappa-m}(2\pi a_1^2)\\
&\times e^{2\pi\sqrt{-1}x}a_2^{2\kappa}e^{-2\pi a_2^2}e^{2\sqrt{-1}\kappa\theta}\cdot a_1^{-4}a_2^{-2}\,dk\,dk_\theta\,dx\,d^\times a_1\,d^\times a_2.
\end{aligned}$$

Lemma 12.10. *Let $x\in\mathbb{R}$, $a_1, a_2\in\mathbb{R}_+^\times$,*

$$k=\begin{pmatrix}\alpha & \beta\\ -\bar{\beta} & \bar{\alpha}\end{pmatrix}\in\mathrm{SU}(2),$$

and $k_\theta\in\mathrm{SO}(2)$. Then

$$\begin{aligned}
&f_\Psi^{(s)}(\eta(t(a_1)k,\,u(x)t(a_2)k_\theta))\\
&=\pi^{-2\kappa}(2\kappa+1)!\left(a_1^2+a_2^2+\sqrt{-1}x\right)^{-s-2\kappa-1}\left(a_1^2+a_2^2-\sqrt{-1}x\right)^{-s-1}\\
&\times a_1^{4s+2\kappa+4}a_2^{2s+2\kappa+2}\sum_{m=0}^{2\kappa}\binom{2\kappa}{m}(\sqrt{-1})^{2\kappa-m}\alpha^{2\kappa-m}\bar{\beta}^m e^{-2\sqrt{-1}\kappa\theta}.
\end{aligned}$$

Proof. A routine calculation shows that $f_\Psi^{(s)}(\eta(t(a_1)k,\,u(x)t(a_2)k_\theta))$ is equal to

$$f_\Psi^{(0)}(\eta(t(a_1)k,\,u(x)t(a_2)k_\theta))\cdot a_1^{4s}a_2^{2s}\left(\left(a_1^2+a_2^2\right)^2+x^2\right)^{-s}.$$

By Lemma 12.1, $f_\Psi^{(0)}(\eta(t(a_1)k,\,u(x)t(a_2)k_\theta))$ is equal to the product of

$$2^{-2\kappa}(2\kappa+1)a_1^{2\kappa+4}a_2^{2\kappa+2}\sum_{m=0}^{2\kappa}\binom{2\kappa}{m}(\sqrt{-1})^{2\kappa-m}\alpha^{2\kappa-m}\bar{\beta}^m e^{-2\sqrt{-1}\kappa\theta}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^4}\left((y_1+y_4)^2+(y_2+y_3)^2\right)^{2\kappa}\\
&\times e^{-\pi\left(a_1^2+a_2^2\right)\left(y_1^2+y_2^2+y_3^2+y_4^2\right)}e^{-2\pi\sqrt{-1}x(y_1y_4+y_2y_3)}\,dy_1\,dy_2\,dy_3\,dy_4.
\end{aligned}$$

By Lemma 6.9 of [27], this integral is equal to

$$2^{2\kappa}\pi^{-2\kappa}(2\kappa)!\big(a_1^2+a_2^2+\sqrt{-1}x\big)^{-2\kappa-1}\big(a_1^2+a_2^2-\sqrt{-1}x\big)^{-1}.$$

This completes the proof. □

By Lemma 6.6, $Z_\infty(s)$ is equal to

$$\begin{aligned}
&2^{-\kappa+1}\pi^{-2\kappa+1}(\sqrt{-1})^{-\kappa}D^{(\kappa+1)/2}(2\kappa)!\sum_{m=0}^{2\kappa}\binom{2\kappa}{m}\\
&\times\int_0^\infty\int_0^\infty\int_{-\infty}^\infty\big(a_1^2+a_2^2+\sqrt{-1}x\big)^{-s-2\kappa-1}\big(a_1^2+a_2^2-\sqrt{-1}x\big)^{-s-1}\\
&\times K_{\kappa-m}\big(2\pi a_1^2\big)e^{-2\pi a_2^2+2\pi\sqrt{-1}x}a_1^{4s+4\kappa+2}a_2^{2s+4\kappa}\,dx\,d^\times a_1\,d^\times a_2\\
&=2^{-s-3\kappa-1}\pi^{-s-4\kappa+1}(\sqrt{-1})^{-\kappa}D^{(\kappa+1)/2}(2\kappa)!\\
&\times\sum_{m=0}^{2\kappa}\binom{2\kappa}{m}I(2\kappa+1,1,\kappa-m,2\kappa+1,2\kappa).
\end{aligned}$$

Hence $Z_\infty(0)$ is equal to

$$\begin{aligned}
&2^{-7\kappa-1}\pi^{-4\kappa+5/2}(\sqrt{-1})^{-\kappa}D^{(\kappa+1)/2}\frac{\Gamma(2\kappa)}{\Gamma\big(2\kappa+\frac{1}{2}\big)}\\
&\times\sum_{m=0}^{2\kappa}\binom{2\kappa}{m}\Gamma(\kappa+m)\Gamma(3\kappa-m)\\
&=2^{-7\kappa-1}\pi^{-4\kappa+5/2}(\sqrt{-1})^{-\kappa}D^{(\kappa+1)/2}\frac{\Gamma(\kappa)^2\Gamma(4\kappa)}{\Gamma\big(2\kappa+\frac{1}{2}\big)}\\
&=2^{-3\kappa-2}\pi^{-4\kappa+2}(\sqrt{-1})^{-\kappa}D^{(\kappa+1)/2}\Gamma(\kappa)^2\Gamma(2\kappa)
\end{aligned}$$

by Lemmas 12.7, 12.9, and the duplication formula. This completes the proof of (ii) of Lemma 12.2.

13. Proof of Proposition 5.3

We retain the notation of Sect. 12.2. Let V be the quadratic space as in Sect. 6.1. Let $l=2\kappa$. We define

$$\begin{aligned}
\Phi&=\otimes_v\varphi_v^{(4)}\in S(V(\mathbb{A}_{\mathcal{K}})),\\
\varphi&=\otimes_v\varphi_v^{(2)}\in S(V(\mathbb{A}_{\mathbb{Q}})),
\end{aligned}$$

as in Sect. 6.4, Sect. 6.3, respectively. Put $\Psi=\Phi\otimes\varphi$. We may regard $\theta(\bar{\mathbf{g}}_{\mathcal{K}}^\flat\otimes\mathbf{f},\Psi)$ as an automorphic form on

$$\mathrm{GL}_2(\mathbb{A}_{\mathcal{K}})\times\mathrm{GL}_2(\mathbb{A}_{\mathcal{K}})\times\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})\times\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}).$$

Let S be a finite set of places of $\mathbb{Q}$ which contains ∞ and all primes dividing $2D$. As in the proof of Main Identity 9.1 of [24], the integral representation of the triple product L-function [44] and the regularized Siegel-Weil formula [37] imply a seesaw identity

$$\begin{aligned}
&\pi^{-1}\xi_{\mathbb{Q}}(2)^{-1}\xi_{\mathcal{K}}(2)^{-1}Z_S(0)\zeta^S(2)^{-2}L^S\left(\frac{1}{2},\pi_{\mathcal{K}}\otimes\sigma\right)\\
&=2\operatorname{vol}(\mathbb{A}_{\mathbb{Q}}^{\times}\operatorname{GL}_2(\mathbb{Q})\backslash\operatorname{GL}_2(\mathbb{A}_{\mathbb{Q}}))^{-2}\\
&\times\int_{(\mathbb{A}_{\mathbb{Q}}^{\times}\operatorname{GL}_2(\mathbb{Q})\backslash\operatorname{GL}_2(\mathbb{A}_{\mathbb{Q}}))^2}\theta((h_1,h_2,h_1,h_2);\bar{\mathbf{g}}_{\mathcal{K}}^{\flat}\otimes\mathbf{f},\Psi)\,dh_1\,dh_2.
\end{aligned}$$

The left-hand side is equal to

$$2^{\kappa-3}(\sqrt{-1})^{-\kappa}D^{-(\kappa+1)/2}\xi_{\mathbb{Q}}(2)^{-3}\xi_{\mathcal{K}}(2)^{-1}\Lambda(2\kappa,\operatorname{Sym}^2(g)\otimes f)\Lambda(\kappa,f,\chi_{-D})$$

by Lemma 12.2, and the right-hand side is equal to

$$2^{3\kappa+3}(\sqrt{-1})^{\kappa}D^{-\kappa/2-1}\xi_{\mathbb{Q}}(2)^{-1}\xi_{\mathcal{K}}(2)^{-1}\mathcal{I}(\mathbf{g}_{\mathcal{K}}^{\sharp},\mathbf{f})^2$$

by Lemmas 6.2 and 6.4. This completes the proof of Proposition 5.3.

Remark 13.1. By [45], [46], [39],

$$\dim_{\mathbb{C}}\operatorname{Hom}_{\operatorname{GL}_2(\mathbb{Q}_v)}(\pi_{\mathcal{K},v}\otimes\sigma_v,\mathbb{C})=1$$

for all places v of $\mathbb{Q}$.

Appendix. Whittaker functions on $\widetilde{\mathrm{SL}}_2$

In this appendix, we compute certain Whittaker functions on $\widetilde{\mathrm{SL}_2(\mathbb{Q}_p)}$.

A.1. Weil indices. Let F be a local field of characteristic not 2 and fix a non-trivial additive character ψ of F. Let

$$\begin{aligned}\mathcal{S}(F)&\longrightarrow\mathcal{S}(F)\\ \phi&\longmapsto\hat{\phi}\end{aligned}$$

be the Fourier transform given by

$$\hat{\phi}(x)=\int_F\phi(y)\psi(xy)\,dy.$$

Here dy is the self-dual measure on F with respect to ψ. There is an 8th root of unity $\gamma_F(\psi)$ such that

$$\int_F\phi(x)\psi(x^2)\,dx=\gamma_F(\psi)|2|_F^{-1/2}\int_F\hat{\phi}(x)\psi(-x^2/4)\,dx.$$

Following the appendix of [47], we put $\gamma_F(a,\psi) = \gamma_F(a\psi)/\gamma_F(\psi)$ for $a \in F^\times$. It satisfies $\gamma_F(ac^2,\psi) = \gamma_F(a,\psi)$ and

$$\gamma_F(ab,\psi) = (a,b)_F \gamma_F(a,\psi)\gamma_F(b,\psi).$$

When $F = \mathbb{Q}_v$, the Weil index is given by the following formulas.

Lemma A.1. *Let ψ be the standard additive character of $\mathbb{Q}_v$.*

(i) *Let $v = p \neq 2$. Then $\gamma_{\mathbb{Q}_p}(a\psi) = \gamma_{\mathbb{Q}_p}(a,\psi) = 1$ for $a \in \mathbb{Z}_p^\times$, and*

$$\gamma_{\mathbb{Q}_p}(p\psi) = \gamma_{\mathbb{Q}_p}(p,\psi) = \begin{cases} 1 & \text{if } p \equiv 1 \bmod 4, \\ -\sqrt{-1} & \text{if } p \equiv 3 \bmod 4. \end{cases}$$

(ii) *Let $v = 2$. Then*

$$\gamma_{\mathbb{Q}_2}(a\psi) = \begin{cases} \zeta_8^{-1} & \text{if } a \equiv 1 \bmod 4, \\ \zeta_8 & \text{if } a \equiv 3 \bmod 4, \end{cases}$$

$$\gamma_{\mathbb{Q}_2}(a,\psi) = \begin{cases} 1 & \text{if } a \equiv 1 \bmod 4, \\ \sqrt{-1} & \text{if } a \equiv 3 \bmod 4, \end{cases}$$

and

$$\gamma_{\mathbb{Q}_2}(2a\psi) = \zeta_8^{-a}, \quad \gamma_{\mathbb{Q}_2}(2a,\psi) = \begin{cases} 1 & \text{if } a \equiv 1 \bmod 8, \\ -\sqrt{-1} & \text{if } a \equiv 3 \bmod 8, \\ -1 & \text{if } a \equiv 5 \bmod 8, \\ \sqrt{-1} & \text{if } a \equiv 7 \bmod 8, \end{cases}$$

for $a \in \mathbb{Z}_2^\times$.

(iii) *Let $v = \infty$. Then*

$$\gamma_{\mathbb{R}}(a\psi) = \begin{cases} \zeta_8 & \text{if } a > 0, \\ \zeta_8^{-1} & \text{if } a < 0, \end{cases} \quad \gamma_{\mathbb{R}}(a,\psi) = \begin{cases} 1 & \text{if } a > 0, \\ -\sqrt{-1} & \text{if } a < 0. \end{cases}$$

A.2. Gauss sums. Let ψ be the standard additive character of $\mathbb{Q}_p$. For $\xi \in \mathbb{Q}_p$ and $a \in \mathbb{Q}_p^\times$, put

$$\mathfrak{G}_\xi(a) = \int_{\mathbb{Z}_p^\times} (a,x)_{\mathbb{Q}_p} \overline{\psi(\xi x)}\, dx,$$

$$\tilde{\mathfrak{G}}_\xi(a) = \int_{\mathbb{Z}_p^\times} \gamma_{\mathbb{Q}_p}(x,\psi)^{-1} (a,x)_{\mathbb{Q}_p} \overline{\psi(\xi x)}\, dx.$$

Then

$$\mathfrak{G}_\xi(ab^2) = \mathfrak{G}_\xi(au) = \mathfrak{G}_\xi(a),$$

$$\tilde{\mathfrak{G}}_\xi(ab^2) = \tilde{\mathfrak{G}}_\xi(au) = \tilde{\mathfrak{G}}_\xi(a),$$

for $a, b \in \mathbb{Q}_p^\times$ and

$$u \in \begin{cases} \mathbb{Z}_p^\times & \text{if } p \neq 2, \\ 1 + 4\mathbb{Z}_2 & \text{if } p = 2. \end{cases}$$

Moreover, $\mathfrak{G}_\xi(a) = \tilde{\mathfrak{G}}_\xi(a)$ if $p \neq 2$. An easy computation proves the following formulas for the Gauss sum.

Lemma A.2. (i) *Let $p \neq 2$. Then*

$$\mathfrak{G}_\xi(1) = \begin{cases} 1 - p^{-1} & \text{if } \xi \in \mathbb{Z}_p, \\ -p^{-1} & \text{if } \xi \in p^{-1}\mathbb{Z}_p^\times, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathfrak{G}_\xi(p) = \begin{cases} p^{-1/2}(p, u)_{\mathbb{Q}_p}\epsilon_p & \text{if } \xi = p^{-1}u,\ u \in \mathbb{Z}_p^\times, \\ 0 & \text{otherwise.} \end{cases}$$

Here

$$\epsilon_p = \begin{cases} 1 & \text{if } p \equiv 1 \bmod 4, \\ \sqrt{-1} & \text{if } p \equiv 3 \bmod 4. \end{cases}$$

(ii) *Let $p = 2$. Then*

$$\mathfrak{G}_\xi(1) = \begin{cases} 2^{-1} & \text{if } \xi \in \mathbb{Z}_2, \\ -2^{-1} & \text{if } \xi \in 2^{-1}\mathbb{Z}_2^\times, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathfrak{G}_\xi(-1) = \begin{cases} 2^{-1}(-1, u)_{\mathbb{Q}_2}\sqrt{-1} & \text{if } \xi = 4^{-1}u,\ u \in \mathbb{Z}_2^\times, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathfrak{G}_\xi(\pm 2) = \begin{cases} 2^{-3/2}(\pm 2, u)_{\mathbb{Q}_2}\sqrt{\pm 1} & \text{if } \xi = 8^{-1}u,\ u \in \mathbb{Z}_2^\times, \\ 0 & \text{otherwise.} \end{cases}$$

(iii) *Let $p = 2$. Then*

$$\tilde{\mathfrak{G}}_\xi(\pm 1) = \begin{cases} 2^{-3/2}\zeta_8^{\mp 1} & \text{if } \xi \in \mathbb{Z}_2, \\ -2^{-3/2}\zeta_8^{\mp 1} & \text{if } \xi = 2^{-1}u,\ u \in \mathbb{Z}_2^\times, \\ 2^{-3/2}(\sqrt{-1})^u\zeta_8^{\pm 1} & \text{if } \xi = 4^{-1}u,\ u \in \mathbb{Z}_2^\times, \\ 0 & \text{otherwise,} \end{cases}$$

$$\tilde{\mathfrak{G}}_\xi(\pm 2) = \begin{cases} 2^{-1}\zeta_8^u & \text{if } \xi = 8^{-1}u,\ u \in \mathbb{Z}_2^\times,\ u \equiv \mp 1 \bmod 4, \\ 0 & \text{otherwise.} \end{cases}$$

A.3. Principal series representations. Let ψ be the standard additive character of $\mathbb{Q}_p$ and $\chi_{-1,p}$ the quadratic character of $\mathbb{Q}_p^\times$ associated to $\mathbb{Q}_p(\sqrt{-1})/\mathbb{Q}_p$ by class field theory. We fix a positive integer l. Let $\xi \in \mathbb{Q}_+^\times$. We write $\xi = \mathfrak{d}_\xi \mathfrak{f}_\xi^2$ with $\mathfrak{d}_\xi \in \mathbb{N}$, $\mathfrak{f}_\xi \in \mathbb{Q}_+^\times$ so that $(-1)^l \mathfrak{d}_\xi$ is the discriminant of $\mathbb{Q}(\sqrt{(-1)^l\xi})/\mathbb{Q}$. Let $\chi_{(-1)^l\xi}$ denote the primitive Dirichlet character associated to $\mathbb{Q}(\sqrt{(-1)^l\xi})/\mathbb{Q}$. As in [29, p. 647], define $\Psi_p(\xi; X) \in \mathbb{C}[X+X^{-1}]$ by

$$\Psi_p(\xi; X) = \begin{cases} \dfrac{X^{e+1} - X^{-e-1}}{X - X^{-1}} - p^{-1/2}\chi_{(-1)^l\xi}(p)\dfrac{X^e - X^{-e}}{X - X^{-1}} & \text{if } e \geq 0, \\ 0 & \text{if } e < 0, \end{cases}$$

where $e = \mathrm{ord}_{\mathbb{Q}_p}(\mathfrak{f}_\xi)$. Note that $\Psi_p(\xi; X) = 1$ if $\mathrm{ord}_{\mathbb{Q}_p}(\mathfrak{f}_\xi) = 0$.

Let

$$\rho = \mathrm{Ind}_{\widetilde{B(\mathbb{Q}_p)}}^{\widetilde{\mathrm{SL}_2(\mathbb{Q}_p)}} \left((\chi_{-1,p}^l)^\psi |\ |_{\mathbb{Q}_p}^s \right)$$

be a principal series representation of $\widetilde{\mathrm{SL}_2(\mathbb{Q}_p)}$ on the space $\mathcal{V}$ of all locally constant functions f such that

$$f((u(x)t(a), \epsilon)g) = \epsilon \gamma_{\mathbb{Q}_p}(a, \psi)^{-1} \chi_{-1,p}^l(a)|a|_{\mathbb{Q}_p}^{s+1} f(g)$$

for $x \in \mathbb{Q}_p$, $a \in \mathbb{Q}_p^\times$, $\epsilon \in \{\pm 1\}$, and $g \in \widetilde{\mathrm{SL}_2(\mathbb{Q}_p)}$. Put $\alpha = p^{-s}$. Let $W_{\xi,p}$ be the Whittaker function of ρ with respect to $\xi\psi$ which satisfies the following conditions:

- $W_{\xi,p}\big(t(\mathfrak{f}_\xi^{-1})\big) = |\mathfrak{f}_\xi^{-1}|_{\mathbb{Q}_p} \gamma_{\mathbb{Q}_p}(\mathfrak{f}_\xi^{-1}, \psi)^{-1} \chi_{-1,p}^l(\mathfrak{f}_\xi^{-1})$.
- If $p \neq 2$, then

  $$W_{\xi,p}(g(k, s_p(k))) = W_{\xi,p}(g)$$

  for $g \in \widetilde{\mathrm{SL}_2(\mathbb{Q}_p)}$, $k \in \mathrm{SL}_2(\mathbb{Z}_p)$.
- If $p = 2$, then

  $$W_{\xi,2}(gk) = \epsilon_2(k)^{(-1)^l} W_{\xi,2}(g)$$

  for $g \in \widetilde{\mathrm{SL}_2(\mathbb{Q}_2)}$, $k \in K_0(4; \mathbb{Z}_2)$, and

  $$\mathsf{W}(\mathsf{U}(W_{\xi,2})) = 2^{-1/2}\zeta_8^{(-1)^l} W_{\xi,2}.$$

Lemma A.3. (i) *Let* $p \neq 2$. *Then*

$$W_{\xi,p}(t(p^n)) = p^{-n}\gamma_{\mathbb{Q}_p}(p^n, \psi)^{-1}\chi_{-1,p}^l(p^n)\Psi_p(p^{2n}\xi; \alpha).$$

(ii) *Let* $p = 2$. *Let* $\xi = 2^m u$ *with* $u \in \mathbb{Z}_2^\times$. *Then*

$$W_{\xi,2}(t(2^n)) = 2^{-n}\Psi_2(2^{2n}\xi; \alpha).$$

If m is even and $u \equiv (-1)^l \bmod 4$, then

$$W_{\xi,2}(t(2^n)w) = \begin{cases} 2^{-n-3/2}\zeta_8^{(-1)^{l+1}}\Psi_2(2^{2n+4}\xi;\alpha) & \text{if } n \geq -m/2-1, \\ 0 & \text{otherwise,} \end{cases}$$
$$W_{\xi,2}(t(2^n)k_1) = \begin{cases} 2^{-n-1/2}(2,\xi)_{\mathbb{Q}_2} & \text{if } n = -m/2-1, \\ 0 & \text{otherwise.} \end{cases}$$

If m is even and $u \equiv (-1)^{l+1} \bmod 4$, or m is odd, then

$$W_{\xi,2}(t(2^n)w) = 2^{-n-3/2}\zeta_8^{(-1)^{l+1}}\Psi_2(2^{2n+4}\xi;\alpha),$$
$$W_{\xi,2}(t(2^n)k_1) = 0.$$

These formulas for $W_{\xi,p}$ will be proved by computing the integral

$$W_{f,\xi}(g) = \int_{\mathbb{Q}_p} f(w^{-1}u(x)g)\overline{\psi(\xi x)}\,dx$$

for $f \in \mathcal{V}$, $g \in \widetilde{\mathrm{SL}_2(\mathbb{Q}_p)}$. Note that

$$W_{f,\xi}(t(a)g) = \gamma_{\mathbb{Q}_p}(a,\psi)^{-1}\chi_{-1,p}^l(a)|a|_{\mathbb{Q}_p}^{-s+1}W_{f,a^2\xi}(g)$$

for $a \in \mathbb{Q}_p^\times$.

A.4. The unramified case. Let $p \neq 2$. We may assume that l is even since $\chi_{-1,p}$ is unramified. Define an element $f \in \mathcal{V}$ so that $f(1) = 1$ and

$$\rho((k, s_p(k)))f = f$$

for $k \in \mathrm{SL}_2(\mathbb{Z}_p)$. To prove (i) of Lemma A.3, it suffices to compute $W_{f,\xi}(1)$.

Lemma A.4. *Let $\xi = p^m u$ with $u \in \mathbb{Z}_p^\times$. If $m \geq 0$ and m is even, then*

$$W_{f,\xi}(1) = \alpha^{m/2}\big(1 + p^{-1/2}(p,\xi)_{\mathbb{Q}_p}\alpha\big)(\alpha - \alpha^{-1})^{-1}$$
$$\times \left[\alpha^{m/2+1} - \alpha^{-m/2-1} - p^{-1/2}(p,\xi)_{\mathbb{Q}_p}(\alpha^{m/2} - \alpha^{-m/2})\right].$$

If $m \geq 0$ and m is odd, then

$$W_{f,\xi}(1) = \alpha^{(m-1)/2}(1 - p^{-1}\alpha^2)(\alpha - \alpha^{-1})^{-1}(\alpha^{(m-1)/2+1} - \alpha^{-(m-1)/2-1}).$$

If $m < 0$, then $W_{f,\xi}(1) = 0$.

Proof. If $\xi \notin \mathbb{Z}_p$, then $W_{f,\xi}(1) = 0$ since

$$W_{f,\xi}(1) = W_{f,\xi}(u(x)) = \psi(\xi x) W_{f,\xi}(1)$$

for $x \in \mathbb{Z}_p$.

Assume that $\xi \in \mathbb{Z}_p$. For $x \in \mathbb{Z}_p$,

$$w^{-1}u(x) = \left(\begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix}, 1 \right)$$

with

$$\begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_p), \quad s_p\left(\begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix} \right) = 1.$$

For $x \in \mathbb{Q}_p - \mathbb{Z}_p$,

$$w^{-1}u(x) = \left(\begin{pmatrix} x^{-1} & -1 \\ 0 & x \end{pmatrix}, (-1, x)_{\mathbb{Q}_p} \right) \cdot \left(\begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix}, 1 \right)$$

with

$$\begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_p), \quad s_p\left(\begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix} \right) = 1.$$

Hence

$$\begin{aligned} W_{f,\xi}(1) &= \mathrm{vol}(\mathbb{Z}_p) + \sum_{n=1}^{\infty} \int_{p^{-n}\mathbb{Z}_p^\times} (-1, x)_{\mathbb{Q}_p} \gamma_{\mathbb{Q}_p}(x^{-1}, \psi)^{-1} |x^{-1}|_{\mathbb{Q}_p}^{s+1} \overline{\psi(\xi x)}\, dx \\ &= 1 + \sum_{n=1}^{\infty} p^{-ns} \int_{\mathbb{Z}_p^\times} \gamma_{\mathbb{Q}_p}(p^n x^{-1}, \psi) \overline{\psi(p^{-n}\xi x)}\, dx \\ &= 1 + \sum_{n=1}^{\infty} \alpha^n \gamma_{\mathbb{Q}_p}(p^n, \psi) \mathfrak{G}_{p^{-n}\xi}(p^n). \end{aligned}$$

If m is even, then

$$\begin{aligned} W_{f,\xi}(1) &= 1 + (1 - p^{-1}) \sum_{n=1}^{m/2} \alpha^{2n} + p^{-1/2}(p, \xi)_{\mathbb{Q}_p} \alpha^{m+1} \\ &= (\alpha^2 - 1)^{-1} \big(1 + p^{-1/2}(p, \xi)_{\mathbb{Q}_p} \alpha\big) \\ &\quad \times \left[\alpha^{m+2} - 1 - p^{-1/2}(p, \xi)_{\mathbb{Q}_p}(\alpha^{m+1} - \alpha)\right]. \end{aligned}$$

If m is odd, then

$$\begin{aligned} W_{f,\xi}(1) &= 1 + (1 - p^{-1}) \sum_{n=1}^{(m-1)/2} \alpha^{2n} - p^{-1}\alpha^{m+1} \\ &= (\alpha^2 - 1)^{-1}(1 - p^{-1}\alpha^2)(\alpha^{m+1} - 1). \end{aligned}$$

□

A.5. The 2-adic case. Let $p = 2$. We put

$$\mathcal{V}_0 = \{f \in \mathcal{V} \mid \rho(k)f = \epsilon_2(k)^{(-1)^l} f \text{ for } k \in K_0(4; \mathbb{Z}_2)\}.$$

Define elements $f_1, f_w \in \mathcal{V}_0$ so that

$$\begin{aligned} &f_1(1) = 1, && f_1(w) = 0, && f_1(k_1) = 0, \\ &f_w(1) = 0, && f_w(w) = 1, && f_w(k_1) = 0. \end{aligned}$$

Lemma A.5. *Let $\xi = 2^m u$ with $u \in \mathbb{Z}_2^\times$.*

(i) *If $m \geq 0$, m is even, and $u \equiv (-1)^l \bmod 4$, then*

$$\begin{aligned} W_{f_1,\xi}(1) = 2^{-3/2}\zeta_8^{(-1)^l}(\alpha - \alpha^{-1})^{-1} \\ \times \left[\alpha^{m+3} - \alpha + 2^{1/2}(2,\xi)_{\mathbb{Q}_2}(\alpha^{m+4} - \alpha^{m+2})\right]. \end{aligned}$$

If $m \geq 0$, m is even, and $u \equiv (-1)^{l+1} \bmod 4$, then

$$W_{f_1,\xi}(1) = -2^{-3/2}\zeta_8^{(-1)^l}(\alpha - \alpha^{-1})^{-1}(\alpha^{m+3} - 2\alpha^{m+1} + \alpha).$$

If $m \geq 0$ and m is odd, then

$$W_{f_1,\xi}(1) = -2^{-3/2}\zeta_8^{(-1)^l}(\alpha - \alpha^{-1})^{-1}(\alpha^{m+2} - 2\alpha^{m} + \alpha).$$

If $m < 0$, then $W_{f_1,\xi}(1) = 0$.

(ii) *If $m \geq 0$, then*

$$W_{f_w,\xi}(1) = (-1)^l\sqrt{-1}.$$

If $m < 0$, then $W_{f_w,\xi}(1) = 0$.

Proof. Let $f \in \mathcal{V}_0$. If $\xi \notin \mathbb{Z}_2$, then $W_{f,\xi}(1) = 0$ since

$$W_{f,\xi}(1) = W_{f,\xi}(u(x)) = \psi(\xi x)W_{f,\xi}(1)$$

for $x \in \mathbb{Z}_2$.

Assume that $\xi \in \mathbb{Z}_2$. For $x \in \mathbb{Z}_2$,

$$w^{-1}u(x) = \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, (-1,-1)_{\mathbb{Q}_2}\right) \cdot (w, 1) \cdot \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, 1\right)$$

with

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in K_1(4; \mathbb{Z}_2), \quad s_2\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) = 1.$$

For $x \in 2^{-1}\mathbb{Z}_2^\times$,

$$w^{-1}u(x) = \left(\begin{pmatrix} x^{-1} & -1 \\ 0 & x \end{pmatrix}, (-1,x)_{\mathbb{Q}_2}\right) \cdot (k_1, 1) \cdot \left(\begin{pmatrix} 1 & 0 \\ -2 + x^{-1} & 1 \end{pmatrix}, 1\right)$$

with

$$\begin{pmatrix} 1 & 0 \\ -2+x^{-1} & 1 \end{pmatrix} \in K_1(4;\mathbb{Z}_2), \quad s_2\left(\begin{pmatrix} 1 & 0 \\ -2+x^{-1} & 1 \end{pmatrix}\right) = 1.$$

For $x \in \mathbb{Q}_2 - 2^{-1}\mathbb{Z}_2$,

$$w^{-1}u(x) = \left(\begin{pmatrix} x^{-1} & -1 \\ 0 & x \end{pmatrix}, (-1,x)_{\mathbb{Q}_2}\right) \cdot \left(\begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix}, 1\right)$$

with

$$\begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix} \in K_1(4;\mathbb{Z}_2), \quad s_2\left(\begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix}\right) = 1.$$

Hence

$$\begin{aligned} W_{f_1,\xi}(1) &= \sum_{n=2}^{\infty} \int_{2^{-n}\mathbb{Z}_2^\times} \gamma_{\mathbb{Q}_2}(x^{-1},\psi)^{-1}((-1)^{l+1},x)_{\mathbb{Q}_2}|x^{-1}|_{\mathbb{Q}_2}^{s+1}\overline{\psi(\xi x)}\,dx \\ &= \sum_{n=2}^{\infty} 2^{-ns} \int_{\mathbb{Z}_2^\times} \gamma_{\mathbb{Q}_2}(2^n x^{-1},\psi)^{-1}((-1)^{l+1},x)_{\mathbb{Q}_2}\overline{\psi(2^{-n}\xi x)}\,dx \\ &= \sum_{n=2}^{\infty} \alpha^n \tilde{\mathfrak{G}}_{2^{-n}\xi}((-1)^{l+1}2^n), \end{aligned}$$

and

$$W_{f_w,\xi}(1) = \gamma_{\mathbb{Q}_2}(-1,\psi)^{-1}((-1)^{l+1},-1)_{\mathbb{Q}_2}\operatorname{vol}(\mathbb{Z}_2) = (-1)^l\sqrt{-1}.$$

If m is even and $u \equiv (-1)^l \bmod 4$, then

$$\begin{aligned} W_{f_1,\xi}(1) &= 2^{-3/2}\zeta_8^{(-1)^l}\sum_{n=1}^{m/2}\alpha^{2n} + 2^{-3/2}(\sqrt{-1})^u\zeta_8^{(-1)^{l+1}}\alpha^{m+2} + 2^{-1}\zeta_8^u\alpha^{m+3} \\ &= 2^{-3/2}\zeta_8^{(-1)^l}(\alpha^2-1)^{-1}\left[\alpha^{m+4}-\alpha^2+2^{1/2}\zeta_8^{u-(-1)^l}(\alpha^{m+5}-\alpha^{m+3})\right]. \end{aligned}$$

If m is even and $u \equiv (-1)^{l+1} \bmod 4$, then

$$\begin{aligned} W_{f_1,\xi}(1) &= 2^{-3/2}\zeta_8^{(-1)^l}\sum_{n=1}^{m/2}\alpha^{2n} + 2^{-3/2}(\sqrt{-1})^u\zeta_8^{(-1)^{l+1}}\alpha^{m+2} \\ &= -2^{-3/2}\zeta_8^{(-1)^l}(\alpha^2-1)^{-1}(\alpha^{m+4}-2\alpha^{m+2}+\alpha^2). \end{aligned}$$

If m is odd, then

$$\begin{aligned} W_{f_1,\xi}(1) &= 2^{-3/2}\zeta_8^{(-1)^l}\sum_{n=1}^{(m-1)/2}\alpha^{2n} - 2^{-3/2}\zeta_8^{(-1)^l}\alpha^{m+1} \\ &= -2^{-3/2}\zeta_8^{(-1)^l}(\alpha^2-1)^{-1}(\alpha^{m+3}-2\alpha^{m+1}+\alpha^2). \end{aligned}$$

□

Lemma A.6. *Let $\xi = 2^m u$ with $u \in \mathbb{Z}_2^\times$.*

(i) *If $m \geq -2$, then*

$$W_{f_1,\xi}(w) = 2^{-2}.$$

If $m < -2$, then $W_{f_1,\xi}(w) = 0$.

(ii) *If $m \geq -2$, m is even, and $u \equiv (-1)^l \bmod 4$, then*

$$W_{f_w,\xi}(w) = 2^{-3/2}\zeta_8^{(-1)^l}(\alpha - \alpha^{-1})^{-1} \times \left[\alpha^{m+3} - \alpha^{-1} + 2^{1/2}(2,\xi)_{\mathbb{Q}_2}(\alpha^{m+4} - \alpha^{m+2})\right].$$

If $m \geq -2$, m is even, and $u \equiv (-1)^{l+1} \bmod 4$, then

$$W_{f_w,\xi}(w) = -2^{-3/2}\zeta_8^{(-1)^l}(\alpha - \alpha^{-1})^{-1}(\alpha^{m+3} - 2\alpha^{m+1} + \alpha^{-1}).$$

If $m \geq -2$ and m is odd, then

$$W_{f_w,\xi}(w) = -2^{-3/2}\zeta_8^{(-1)^l}(\alpha - \alpha^{-1})^{-1}(\alpha^{m+2} - 2\alpha^{m} + \alpha^{-1}).$$

If $m < -2$, then $W_{f_w,\xi}(w) = 0$.

Proof. Let $f \in \mathcal{V}_0$. If $\xi \notin 4^{-1}\mathbb{Z}_2$, then $W_{f,\xi}(w) = 0$ since

$$W_{f,\xi}(w) = W_{f,\xi}\left(w\begin{pmatrix}1 & 0\\ -x & 1\end{pmatrix}\right) = W_{f,\xi}(u(x)w) = \psi(\xi x)W_{f,\xi}(w)$$

for $x \in 4\mathbb{Z}_2$.

Assume that $\xi \in 4^{-1}\mathbb{Z}_2$. For $x \in 4\mathbb{Z}_2$,

$$w^{-1}u(x)w = \left(\begin{pmatrix}1 & 0\\ -x & 1\end{pmatrix}, 1\right)$$

with

$$\begin{pmatrix}1 & 0\\ -x & 1\end{pmatrix} \in K_1(4;\mathbb{Z}_2), \quad s_2\left(\begin{pmatrix}1 & 0\\ -x & 1\end{pmatrix}\right) = 1.$$

For $x \in 2\mathbb{Z}_2^\times$,

$$w^{-1}u(x)w = (k_1, 1)\cdot\left(\begin{pmatrix}1 & 0\\ -2-x & 1\end{pmatrix}, 1\right)$$

with

$$\begin{pmatrix}1 & 0\\ -2-x & 1\end{pmatrix} \in K_1(4;\mathbb{Z}_2), \quad s_2\left(\begin{pmatrix}1 & 0\\ -2-x & 1\end{pmatrix}\right) = 1.$$

For $x \in \mathbb{Q}_2 - 2\mathbb{Z}_2$,

$$w^{-1}u(x)w = \left(\begin{pmatrix}x^{-1} & -1\\ 0 & x\end{pmatrix}, (-1,x)_{\mathbb{Q}_2}\right)\cdot(w,1)\cdot\left(\begin{pmatrix}1 & -x^{-1}\\ 0 & 1\end{pmatrix}, 1\right)$$

with

$$\begin{pmatrix} 1 & -x^{-1} \\ 0 & 1 \end{pmatrix} \in K_1(4; \mathbb{Z}_2), \quad s_2\left(\begin{pmatrix} 1 & -x^{-1} \\ 0 & 1 \end{pmatrix}\right) = 1.$$

Hence

$$W_{f_1,\xi}(w) = \mathrm{vol}(4\mathbb{Z}_2) = 2^{-2},$$

and

$$\begin{aligned} W_{f_w,\xi}(w) &= \sum_{n=0}^{\infty} \int_{2^{-n}\mathbb{Z}_2^\times} \gamma_{\mathbb{Q}_2}(x^{-1}, \psi)^{-1}((-1)^{l+1}, x)_{\mathbb{Q}_2} |x^{-1}|_{\mathbb{Q}_2}^{s+1} \overline{\psi(\xi x)}\, dx \\ &= \sum_{n=0}^{\infty} 2^{-ns} \int_{\mathbb{Z}_2^\times} \gamma_{\mathbb{Q}_2}(2^n x^{-1}, \psi)^{-1}((-1)^{l+1}, x)_{\mathbb{Q}_2} \overline{\psi(2^{-n}\xi x)}\, dx \\ &= \sum_{n=0}^{\infty} \alpha^n \tilde{\mathfrak{G}}_{2^{-n}\xi}((-1)^{l+1} 2^n). \end{aligned}$$

If m is even and $u \equiv (-1)^l \bmod 4$, then

$$\begin{aligned} W_{f_w,\xi}(w) &= 2^{-3/2}\zeta_8^{(-1)^l} \sum_{n=0}^{m/2} \alpha^{2n} + 2^{-3/2}(\sqrt{-1})^u \zeta_8^{(-1)^{l+1}} \alpha^{m+2} + 2^{-1}\zeta_8^u \alpha^{m+3} \\ &= 2^{-3/2}\zeta_8^{(-1)^l}(\alpha^2 - 1)^{-1}\left[\alpha^{m+4} - 1 + 2^{1/2}\zeta_8^{u-(-1)^l}(\alpha^{m+5} - \alpha^{m+3})\right]. \end{aligned}$$

If m is even and $u \equiv (-1)^{l+1} \bmod 4$, then

$$\begin{aligned} W_{f_w,\xi}(w) &= 2^{-3/2}\zeta_8^{(-1)^l} \sum_{n=0}^{m/2} \alpha^{2n} + 2^{-3/2}(\sqrt{-1})^u \zeta_8^{(-1)^{l+1}} \alpha^{m+2} \\ &= -2^{-3/2}\zeta_8^{(-1)^l}(\alpha^2 - 1)^{-1}(\alpha^{m+4} - 2\alpha^{m+2} + 1). \end{aligned}$$

If m is odd, then

$$\begin{aligned} W_{f_w,\xi}(w) &= 2^{-3/2}\zeta_8^{(-1)^l} \sum_{n=0}^{(m-1)/2} \alpha^{2n} - 2^{-3/2}\zeta_8^{(-1)^l} \alpha^{m+1} \\ &= -2^{-3/2}\zeta_8^{(-1)^l}(\alpha^2 - 1)^{-1}(\alpha^{m+3} - 2\alpha^{m+1} + 1). \end{aligned}$$

□

Lemma A.7. *Let $\xi = 2^m u$ with $u \in \mathbb{Z}_2^\times$.*

(i) *If $m = -2$ and $u \equiv (-1)^l \bmod 4$, then*

$$W_{f_1,\xi}(k_1) = 2^{-1}\zeta_8^u \alpha.$$

If $m \neq -2$ or $u \not\equiv (-1)^l \bmod 4$, then $W_{f_1,\xi}(k_1) = 0$.

(ii) *If* $m = -2$ *and* $u \equiv (-1)^l \bmod 4$*, then*

$$W_{f_w,\xi}(k_1) = (-1)^l\sqrt{-1}.$$

If $m \neq -2$ *or* $u \not\equiv (-1)^l \bmod 4$*, then* $W_{f_w,\xi}(k_1) = 0$.

Proof. Let $f \in \mathcal{V}_0$. If $\xi \notin 4^{-1}\mathbb{Z}_2$, then $W_{f,\xi}(k_1) = 0$ since

$$\begin{aligned} W_{f,\xi}(k_1) &= W_{f,\xi}\left(k_1\begin{pmatrix} 1+2x & x \\ -4x & 1-2x \end{pmatrix}\right) \\ &= W_{f,\xi}(u(x)k_1) = \psi(\xi x)W_{f,\xi}(k_1) \end{aligned}$$

for $x \in 4\mathbb{Z}_2$.

Assume that $\xi \in 4^{-1}\mathbb{Z}_2$. For $x \in \mathbb{Z}_2$,

$$\begin{aligned} &w^{-1}u(x)k_1 \\ &= \left(\begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}, (-1,-1)_{\mathbb{Q}_2}\right)\cdot(w,1)\cdot\left(\begin{pmatrix} 1+2x & x \\ -4x & 1-2x \end{pmatrix}, 1\right) \end{aligned}$$

with

$$\begin{pmatrix} 1+2x & x \\ -4x & 1-2x \end{pmatrix} \in K_0(4;\mathbb{Z}_2), \quad \epsilon_2\left(\begin{pmatrix} 1+2x & x \\ -4x & 1-2x \end{pmatrix}\right) = \psi(4^{-1}x).$$

For $x \in 2^{-1}\mathbb{Z}_2^\times$,

$$w^{-1}u(x)k_1 = \left(\begin{pmatrix} x^{-1} & 1 \\ 0 & x \end{pmatrix}, (-1,x)_{\mathbb{Q}_2}\right)\cdot\left(\begin{pmatrix} -1-4x & -2x \\ 2+x^{-1} & 1 \end{pmatrix}, 1\right)$$

with

$$\begin{pmatrix} -1-4x & -2x \\ 2+x^{-1} & 1 \end{pmatrix} \in K_1(4;\mathbb{Z}_2), \quad s_2\left(\begin{pmatrix} -1-4x & -2x \\ 2+x^{-1} & 1 \end{pmatrix}\right) = 1.$$

For $x \in \mathbb{Q}_2 - 2^{-1}\mathbb{Z}_2$,

$$w^{-1}u(x)k_1 = \left(\begin{pmatrix} x^{-1} & -1 \\ 0 & x \end{pmatrix}, (-1,x)_{\mathbb{Q}_2}\right)\cdot(k_1,1)\cdot\left(\begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix}, 1\right)$$

with

$$\begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix} \in K_1(4;\mathbb{Z}_2), \quad s_2\left(\begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix}\right) = 1.$$

Hence

$$\begin{aligned} W_{f_1,\xi}(k_1) &= \int_{2^{-1}\mathbb{Z}_2^\times} \gamma_{\mathbb{Q}_2}(x^{-1},\psi)^{-1}((-1)^{l+1},x)_{\mathbb{Q}_2}|x^{-1}|_{\mathbb{Q}_2}^{s+1}\overline{\psi(\xi x)}\,dx \\ &= 2^{-s}\int_{\mathbb{Z}_2^\times} \gamma_{\mathbb{Q}_2}(2x^{-1},\psi)^{-1}((-1)^{l+1},x)_{\mathbb{Q}_2}\overline{\psi(2^{-1}\xi x)}\,dx \\ &= \alpha\tilde{\mathfrak{G}}_{2^{-1}\xi}((-1)^{l+1}2), \end{aligned}$$

and

$$W_{f_w,\xi}(k_1) = \gamma_{\mathbb{Q}_2}(-1,\psi)^{-1}((-1)^{l+1},-1)_{\mathbb{Q}_2}\int_{\mathbb{Z}_2}\psi(4^{-1}x)^{(-1)^l}\overline{\psi(\xi x)}\,dx$$
$$= (-1)^l\sqrt{-1}\int_{\mathbb{Z}_2}\psi(((-1)^l 4^{-1}-\xi)x)\,dx.$$

This completes the proof. □

Put

$$f^+ = \alpha^{-2} f_1 + 2^{-3/2}\zeta_8^{(-1)^{l+1}} f_w.$$

Then

$$\mathsf{W}(\mathsf{U}(f^+)) = 2^{-1/2}\zeta_8^{(-1)^l} f^+$$

by Lemma 3.6. It is easy to verify the following formulas for $W_{f^+,\xi}$.

Lemma A.8. *Let* $\xi = 2^m u$ *with* $u \in \mathbb{Z}_2^\times$.

(i) *If* $m \geq 0$, *m is even, and* $u \equiv (-1)^l \bmod 4$, *then*

$$W_{f^+,\xi}(1) = 2^{-1/2}\zeta_8^{(-1)^l}\alpha^{m/2}(1+2^{-1/2}(2,\xi)_{\mathbb{Q}_2}\alpha)(\alpha-\alpha^{-1})^{-1}$$
$$\times\left[\alpha^{m/2+1}-\alpha^{-m/2-1}-2^{-1/2}(2,\xi)_{\mathbb{Q}_2}(\alpha^{m/2}-\alpha^{-m/2})\right].$$

If $m \geq 0$, *m is even, and* $u \equiv (-1)^{l+1} \bmod 4$, *then*

$$W_{f^+,\xi}(1) = 2^{-1/2}\zeta_8^{(-1)^l}\alpha^{m/2-1}(1-2^{-1}\alpha^2)(\alpha-\alpha^{-1})^{-1}(\alpha^{m/2}-\alpha^{-m/2}).$$

If $m \geq 0$ *and m is odd, then*

$$W_{f^+,\xi}(1) = 2^{-1/2}\zeta_8^{(-1)^l}\alpha^{(m-3)/2}(1-2^{-1}\alpha^2)(\alpha-\alpha^{-1})^{-1}$$
$$\times(\alpha^{(m-1)/2}-\alpha^{-(m-1)/2}).$$

If $m < 0$, *then* $W_{f^+,\xi}(1) = 0$.

(ii) *If* $m \geq -2$, *m is even, and* $u \equiv (-1)^l \bmod 4$, *then*

$$W_{f^+,\xi}(w) = 2^{-2}\alpha^{m/2}(1+2^{-1/2}(2,\xi)_{\mathbb{Q}_2}\alpha)(\alpha-\alpha^{-1})^{-1}$$
$$\times\left[\alpha^{m/2+3}-\alpha^{-m/2-3}-2^{-1/2}(2,\xi)_{\mathbb{Q}_2}(\alpha^{m/2+2}-\alpha^{-m/2-2})\right].$$

If $m \geq -2$, *m is even, and* $u \equiv (-1)^{l+1} \bmod 4$, *then*

$$W_{f^+,\xi}(w) = 2^{-2}\alpha^{m/2-1}(1-2^{-1}\alpha^2)(\alpha-\alpha^{-1})^{-1}(\alpha^{m/2+2}-\alpha^{-m/2-2}).$$

If $m \geq -2$ *and m is odd, then*

$$W_{f^+,\xi}(w) = 2^{-2}\alpha^{(m-3)/2}(1-2^{-1}\alpha^2)(\alpha-\alpha^{-1})^{-1}(\alpha^{(m+3)/2}-\alpha^{-(m+3)/2}).$$

If $m < -2$, *then* $W_{f^+,\xi}(w) = 0$.

(iii) *If* $m = -2$ *and* $u \equiv (-1)^l \bmod 4$, *then*

$$W_{f^+,\xi}(k_1) = 2^{-1}\zeta_8^u \alpha^{-1}\bigl(1 + 2^{-1/2}(2,\xi)_{\mathbb{Q}_2}\alpha\bigr).$$

If $m \neq -2$ *or* $u \not\equiv (-1)^l \bmod 4$, *then* $W_{f^+,\xi}(k_1) = 0$.

This completes the proof of (ii) of Lemma A.3.

Acknowledgements. The author expresses gratitude to Masaaki Furusawa and Tamotsu Ikeda for their advice and encouragement. Thanks are also due to the referee for helpful comments.

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