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Instanton counting on blowup. I. 4-dimensional pure gauge theory

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Abstract. We give a mathematically rigorous proof of Nekrasov's conjecture: the integration in the equivariant cohomology over the moduli spaces of instantons on $\mathbb{R}^4$ gives a deformation of the Seiberg-Witten prepotential for $N = 2$ SUSY Yang-Mills theory. Through a study of moduli spaces on the blowup of $\mathbb{R}^4$, we derive a differential equation for the Nekrasov's partition function. It is a deformation of the equation for the Seiberg-Witten prepotential, found by Losev et al., and further studied by Gorsky et al.

Introduction

Let $M(r, n)$ be the framed moduli space of torsion free sheaves E on $\mathbb{P}^2$ with rank r, $c_2 = n$, where the framing is a trivialization of the restricition of E at the line at infinity ℓ_∞. There is a natural action of an $(r+2)$-dimensional torus $\widetilde{T}$, coming from the symmetry of the base space $\mathbb{C}^2 = \mathbb{P}^2 \setminus \ell_\infty$ and the change of the framing.

Nekrasov's partition function [50] is the generating function of the integral of the equivariant cohomology class $1 \in H^*_{\widetilde{T}}(M(r, n))$:

$$Z(\varepsilon_1, \varepsilon_2, \vec{a}; \mathfrak{q}) = \sum_{n=0}^{\infty} \mathfrak{q}^n \int_{M(r,n)} 1,$$

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where $\varepsilon_1, \varepsilon_2, \vec{a} = (a_1, \dots, a_r)$ are generators of $H^*_{\widetilde{T}}(\mathrm{pt}) = S^*(\mathrm{Lie}\,\widetilde{T})$. When $n > 0$, $M(r,n)$ is *noncompact* and the integration is given by *formally* applying the localization formula in the equivariant cohomology. Then the integration $\int_{M(r,n)} 1$ is a rational function in $\mathbb{C}(\varepsilon_1, \varepsilon_2, a_1, \dots, a_r)$. (A precise definition will be given in the main body of the paper.)

Nekrasov conjectures that $F^{\mathrm{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}; \mathfrak{q}) = \varepsilon_1\varepsilon_2 \log Z(\varepsilon_1, \varepsilon_2, \vec{a}; \mathfrak{q})$ is regular at $\varepsilon_1, \varepsilon_2 = 0$, and $F^{\mathrm{inst}}(0, 0, \vec{a}; \mathfrak{q})$ is the instanton part of the Seiberg-Witten prepotential for $\mathcal{N} = 2$ supersymmetric 4-dimensional gauge theory [52] with gauge group $\mathrm{SU}(r)$. Nekrasov's definition is mathematically rigorous. The Seiberg-Witten prepotential is also rigorously defined by certain period integrals of hyperelliptic curves (the so-called Seiberg-Witten curves). The relation between the two, which is rather natural from a physical point of view, can be considered as a mathematically well formulated conjecture. It is very similar to the mirror symmetry. The Nekrasov partition function is a counterpart of the Gromov-Witten invariants and is on the 'symplectic' side. Seiberg-Witten prepotential is on the 'complex' side.

Let us briefly recall the history on Donaldson invariants and Seiberg-Witten prepotential. A reader can read the main body of the paper without knowing the history, but then he/she loses the motivation why we study Nekrasov's partition function. In [56] Witten described Donaldson invariants as the correlation functions of certain operators in a twisted $\mathcal{N} = 2$ supersymmetric Yang-Mills theory. Several years later Seiberg-Witten found that the prepotential, which controls the physics of the theory, can be computed via the periods of hyperelliptic curves [52]. Then Moore-Witten studied Donaldson invariants using the Seiberg-Witten prepotential [43]. In particular, they derived the blowup formula for Donaldson invariants originally given by Fintushel-Stern [19]. These arguments were physical and have no mathematically rigorous justification so far. It was very misterious why Donaldson invariants are related to periods of Seiberg-Witten curves. Nekrasov's conjecture can be considered as a first step towards the understanding of the misterious relation.

The main result in this paper can be summarized as follows. We consider a similar partition function defined via the framed moduli space $\widehat{M}(r,k,n)$ on the blowup $\widehat{\mathbb{C}}^2$. We also introduce an 'operator' $\mu(C)$ associated with the exceptional set C. We then show that the *correlation functions* $\sum_{n=0}^{\infty} \mathfrak{q}^n \int_{\widehat{M}(r,0,n)} \mu(C)^d$ vanish for $d = 1, \dots, 2r-1$. This simplest case of the blowup formula gives a differential equation (6.14) satisfied by $Z(\varepsilon_1, \varepsilon_2, \vec{a}; \mathfrak{q})$. We call it the *blowup equation*. The blowup equation is a deformation of the differential equation (7.8) for the Seiberg-Witten prepotential originally found in the study of the contact term in the twisted $\mathcal{N} = 2$ supersymmetric gauge theory by Losev et al. [34,35]. This equation was derived also from the Seiberg-Witten curve in the frame work of Whitham hierarchies by Gorsky et al. [23]. (A self-contained proof will be given in [49].) By Edelstein et al. [14] the equation determines the instanton corrections recursively (see also [40] and the references therein). An immediate application is an affirmative solution of Nekrasov's conjecture: $F^{\text{inst}}(0, 0, \vec{a}; \mathfrak{q})$ is the instanton part of the Seiberg-Witten prepotential.

Our strategy goes in the inverse direction of the above mentioned history. We define the operator $\mu(C)$, mimicking the definition of the similar operator for Donaldson invariants. Our vanishing is well-known for Donaldson invariants (see e.g., [20]) and our proof is exactly the same. But this rather trivially looking observation leads to the powerful blowup equation as we just mentioned. (We eventually recover the whole Fintushel-Stern's formula for arbitrary d and its higher rank analog given in [41] in Sect. 8.) Let us remark that a relation between Fintushel-Stern's blowup formula and the Whitham hierarchy was pointed out in [35, §3]. We also remark that there was an approach to Fintushel-Stern's blowup formula based on Uhlenbeck (partial) compactifications of framed moduli spaces [8]. The use of the simplest (or lowest) case of the blowup formula to derive constraint is *not* a new idea in the context of Donaldson invariants. The proof in [19] was done essentially by this idea. Göttsche determined the wall-crossing formula also by this idea [22].

The paper is organized as follows. In Sect. 1 we recall the Seiberg-Witten prepotential. In Sects. 2, 3, we define framed moduli spaces of coherent torsion free sheaves on the plane and its blowup. We define an action of an $(r+2)$-dimensional torus $\widetilde{T}$ on framed moduli spaces, classify the fixed point set and determine the weights of tangent spaces at fixed points. In Sect. 4 we consider a natural K-theory analog of Nekrasov's partition function and identify it with a Hilbert series of the coordinate ring of the framed moduli spaces. This result partly explains why Nekrasov's partition function is natural. But this reformulation is also used to prove the simplest blowup formula. Section 5 is a small detour. We study the rank 1 case, i.e., when the moduli spaces are Hilbert schemes of points. Nekrasov's partition function and its blowup formula is easy to derive, but some feature of the general cases can be seen in this simplest case. Sections 6, 7 are main part of this paper. We introduce the operator $\mu(C)$ and derive the blowup equation. We then prove Nekrasov's conjecture. In Sect. 8 we derive the full blowup formula for our correlation function of $\mu(C)$. In Sect. 9 we consider the case when the gauge group is not necessarily $SU(r)$. Moduli spaces of torsion-free sheaves do not have generalization to other gauge groups, so we are forced to use Uhlenbeck (partial) compactifications. Our formulation in Sect. 4 has a modification by using Uhlenbeck compactifications. We then prove the blowup equation under some technical assumptions on geometric properties of Uhlenbeck compactifications.

In this paper, we treat only the pure gauge theory. Theories with matters, as well as the inclusion of higher Casimir operators (i.e., we integrate more general cohomology classes other than 1), will be studied in the later series.

Our project started in 1997 together with I. Grojnowski. The first goal was a new proof of the blowup formula for Betti numbers of moduli spaces originally given by the second author [57]. This part was finished soon afterward, and was reported by the first author at Workshop on Complex

Differential Geometry, 14–25 July 1997, Warwick and at Verallgemeinerte Kac-Moody-Algebren, 19–25 July 1998, Oberwolfach. (There are closely related results by W-P. Li and Z. Qin [30–32]. We explain this result in [49].) We then tried to give a new proof of Fintushel-Stern's blowup formula for Donaldson invariants. The technique was to use the localization theorem in the equivariant cohomology of the framed moduli space on the blowup, which is basically the same technique taken in this paper. But we did not understand how to take the 'nonequivariant limit' since a naive limit diverges. Thus we did not succeed at that time, and a failure report was given by the first author at a workshop at RIMS Kyoto, June 2000 [48]. The correct choice of limit is provided via the use of the Nekrasov's partition function, and we finally succeed this time. And we get Nekrasov's conjecture as a bonus.

While we were writing this paper, we were informed that Nekrasov and Okounkov also proved Nekrasov's conjecture [51]. Their method is totally different from ours.

After writing the first version of this paper, the authors gave series of lectures on the subject at "Workshop on algebraic structures and moduli spaces", July 14–20, 2003, Universite de Montreal. The reader can find physical backgrounds and various related topics in the lecture notes [49].

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1. Seiberg-Witten prepotential

In this section, we briefly recall the definition of the Seiberg-Witten prepotential for the sake of the reader. See [49, §2] for detail and proofs.

We consider a family of hyperelliptic curves parametrized by $\vec{u} = (u_2, \dots, u_r)$:

$$C_{\vec{u}} : \Lambda^r \left(w + \frac{1}{w} \right) = P(z) = z^r + u_2 z^{r-2} + u_3 z^{r-3} + \cdots + u_r.$$

We call them *Seiberg-Witten curves*. The projection $C_{\vec{u}} \ni (w, z) \mapsto z \in \mathbb{P}^1$ gives a structure of hyperelliptic curves. The parameter space $\{\vec{u} \in \mathbb{C}^{r-1}\}$ is called the *u-plane*.

Let $z_1, \dots, z_r$ be the solutions of $P(z) = 0$. We will work on a region of the u-plane where $|z_\alpha - z_\beta|$, $|z_\alpha|$ are much larger than $|\Lambda|$. We can find $z_\alpha^\pm$ near z_α such that $P(z_\alpha^\pm) = \pm 2\Lambda^r$ as $|u| \gg |\Lambda|$. These are the $2r$-branched points of the projection $C_{\vec{u}} \to \mathbb{P}^1$.

The hyperelliptic curve $C_{\vec{u}}$ is made of two copies of the Riemmann sphere, glued along the r-cuts between z_α^- and z_α^+ ($\alpha = 1, \dots, r$), as usual. Let A_α be the cycle encircling the cut between z_α^- and z_α^+. We have $\sum_\alpha A_\alpha = 0$. We take cycles B_α so that $\{A_\alpha, B_\alpha \mid \alpha = 2, \dots, r\}$ form a symplectic basis of $H_1(C_{\vec{u}}, \mathbb{Z})$, i.e., $A_\alpha \cdot A_\beta = 0 = B_\alpha \cdot B_\beta$, $A_\alpha \cdot B_\beta = \delta_{\alpha\beta}$ for $\alpha, \beta = 2, \dots, r$. (The cycle A_1 is omitted.) See [49] for the precise choice.

Let us define the *Seiberg-Witten differential* by

$$dS = -\frac{1}{2\pi} z \frac{dw}{w} = -\frac{1}{2\pi} \frac{zP'(z)dz}{\sqrt{P(z)^2 - 4\Lambda^{2r}}}.$$

It is a meromorphic differential having poles at $\infty_\pm$. We define functions a_α, a_β^D on the u-plane by

$$a_\alpha = \int_{A_\alpha} dS, \quad a_\beta^D = 2\pi\sqrt{-1} \int_{B_\beta} dS, \qquad \alpha = 1, \dots, r, \ \beta = 2, \dots, r.$$

We have $\sum_\alpha a_\alpha = 0$. In the gauge theory side, $\vec{a} = (a_1, \dots, a_r)$ will be the coordinate system on Lie T.

It can be shown that there exists a locally defined function $\mathcal{F}(\vec{a}; \Lambda)$ on the $\vec{u}$-plane such that

$$a_\alpha^D = -\frac{\partial \mathcal{F}}{\partial a_\alpha}.$$

It is called the *Seiberg-Witten prepotential*. Note that

$$\tau_{\alpha\beta} = -\frac{1}{2\pi\sqrt{-1}} \frac{\partial^2 \mathcal{F}}{\partial a_\alpha \partial a_\beta}$$

is the period matrix of $C_{\vec{u}}$.

One can show that $\mathcal{F}$ has the following behaviour at $\Lambda \to 0$:

$$\mathcal{F} = \sum_{\alpha<\beta} \left[(a_\alpha - a_\beta)^2 \log\left(\frac{\sqrt{-1}(a_\alpha - a_\beta)}{\Lambda} \right) - \frac{3}{2}(a_\alpha - a_\beta)^2 \right] + \Lambda^{2r} \times O(\Lambda^{2r}). \tag{1.1}$$

The first part (resp. second part $\Lambda^{2r} \times O(\Lambda^{2r})$) is called the *perturbative part* (resp. *instanton part*) of the prepotential. For the choice of the branch of log, see [49, §2].

We rewrite (1.1) using terminology for root systems of Lie algebras. The change is useful for considering generalization to other gauge groups (see Sect. 9).

We consider $\vec{a}$ as an element of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sl}_r$. Let $\Delta \subset \mathfrak{h}^*$ be the set of roots. We take standard simple roots $\alpha_i \in \mathfrak{h}^*$ and simple coroots $\alpha_i^\vee \in \mathfrak{h}$ ($i = 1, \dots, r-1$), i.e., $\alpha_i = (0, \dots, 0, \overset{i}{1}, \overset{i+1}{-1}, 0, \dots, 0)$. Let Δ_+ denote the set of positive roots, i.e., $\Delta_+ = \{e_{\alpha,\beta} = (0, \dots, 0, \overset{\alpha}{1}, 0, \dots, 0, \overset{\beta}{-1}, 0, \dots, 0) \mid \alpha < \beta\}$. If $\vec{a} = (a_1, \dots, a_r)$, then $\langle \vec{a}, e_{\alpha,\beta} \rangle = a_\alpha - a_\beta$.

We write $\vec{a} = \sum_i a^i \alpha_i^\vee$. Let Q be the coroot lattice of $\mathfrak{h}$, i.e., $Q = \{\vec{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r \mid \sum_\alpha k_\alpha = 0\}$. We write $\vec{k} = \sum k^i \alpha_i^\vee$ as above.

The perturbative part of the prepotential is rewritten as

$$\sum_{\alpha \in \Delta_+} \left[\langle \vec{a}, \alpha \rangle^2 \log \left(\frac{\sqrt{-1}\langle \vec{a}, \alpha \rangle}{\Lambda} \right) - \frac{3}{2} \langle \vec{a}, \alpha \rangle^2 \right].$$

The period matrix is

$$\begin{aligned} \tau_{ij} &= -\frac{1}{2\pi\sqrt{-1}} \frac{\partial^2 \mathcal{F}}{\partial a^i \partial a^j} \\ &= \frac{\sqrt{-1}}{\pi} \sum_{\alpha \in \Delta_+} \langle \alpha_i^\vee, \alpha \rangle \langle \alpha_j^\vee, \alpha \rangle \log \left(\frac{\sqrt{-1}\langle \vec{a}, \alpha \rangle}{\Lambda} \right) + \Lambda^{2r} \times O(\Lambda^{2r}). \end{aligned} \tag{1.2}$$

In our proof of Nekrasov's conjecture, we use the following two equations:

$$\frac{\partial \mathcal{F}}{\partial \log \Lambda} = -2ru_2, \tag{1.3}$$

$$\frac{\partial u_2}{\partial \log \Lambda} = -\frac{2r}{\pi\sqrt{-1}} \frac{\partial u_2}{\partial a^i} \frac{\partial u_2}{\partial a^j} \frac{\partial}{\partial \tau_{ij}} \log \Theta_E(\vec{0}|\tau), \tag{1.4}$$

where

$$\Theta_E(\vec{\xi}|\tau) = \sum_{\vec{k} \in Q} \exp \left(\pi\sqrt{-1} \sum_{i,j} \tau_{ij} k^i k^j + 2\pi\sqrt{-1} \sum_i k^i \left(\xi^i + \frac{1}{2} \right) \right). \tag{1.5}$$

The first equation (1.3) is called the *renormalization group equation*, and was obtained by [53]. (See also [42, 16, 10].)

The second equation (1.4) is called the *contact term equation*. It was originally found in the context of the $\mathcal{N} = 2$ supersymmetric gauge theory [34, 35], and derived also from the above Seiberg-Witten curve in a mathematically rigorous way [23].

Remark 1.6. In order to get the exact match with the physics literature, we need to note $\vec{a} = -\sqrt{-1}\vec{a}^{\mathrm{Phys}}$, $u_p = -u_p^{\mathrm{Phys}}$.

2. Framed moduli spaces on the projective plane

In this section, we define framed moduli spaces on $\mathbb{P}^2$ and study their basic properties. All of results are straightforward generalizations of the corresponding results for Hilbert schemes on $\mathbb{C}^2$, which were explained in [47]. In fact, the results were obtained long time ago and mentioned in [47, Exercise 5.15].

Let $M(r,n)$ be the framed moduli space of torsion free sheaves on $\mathbb{P}^2$ with rank r and $c_2 = n$, which parametrizes isomorphism classes of (E, Φ) such that

(1) E is a torsion free sheaf of rank $E = r$, $\langle c_2(E), [\mathbb{P}^2]\rangle = n$ which is locally free in a neighborhood of ℓ_∞,
(2) $\Phi\colon E|_{\ell_\infty} \xrightarrow{\sim} \mathcal{O}_{\ell_\infty}^{\oplus r}$ is an isomorphism called 'framing at infinity'.

Here $\ell_\infty = \{[0 : z_1 : z_2] \in \mathbb{P}^2\} \subset \mathbb{P}^2$ is the line at infinity. Notice that the existence of a framing Φ implies $c_1(E) = 0$.

The framed moduli spaces were constructed by Huybrechts-Lehn [25] (in more general framework). The tangent space is $\operatorname{Ext}^1(E, E(-\ell_\infty))$ and the obstruction space is $\operatorname{Ext}^2(E, E(-\ell_\infty))$. In our situation, we have the following vanishing theorem:

Proposition 2.1. $\operatorname{Hom}(E, E(-\ell_\infty)) = \operatorname{Ext}^2(E, E(-\ell_\infty)) = 0.$

Proof. By the Grothendieck-Serre duality theorem, $\operatorname{Ext}^2(E, E(-\ell_\infty))$ is the dual of $\operatorname{Hom}(E, E(-2\ell_\infty))$. We shall show that $\operatorname{Hom}(E, E(-k\ell_\infty)) = 0$ for any $k \in \mathbb{Z}_{>0}$.

From a short exact sequence

$$0 \to E(-(k+1)\ell_\infty) \xrightarrow{\text{mult. by } z_0} E(-k\ell_\infty) \to E(-k\ell_\infty) \otimes \mathcal{O}_{\ell_\infty} \to 0,$$

we obtain an exact sequence

$$\begin{aligned} 0 &\to \operatorname{Hom}(E, E(-(k+1)\ell_\infty)) \to \operatorname{Hom}(E, E(-k\ell_\infty)) \\ &\to \operatorname{Hom}(E, E(-k\ell_\infty) \otimes \mathcal{O}_{\ell_\infty}). \end{aligned}$$

Since the restriction of E to ℓ_∞ is trivial, we have

$$\operatorname{Hom}(E, E(-k\ell_\infty) \otimes \mathcal{O}_{\ell_\infty}) = 0.$$

Hence we get

$$\begin{aligned} \operatorname{Hom}(E, E(-\ell_\infty)) &\cong \operatorname{Hom}(E, E(-2\ell_\infty)) \cong \cdots \\ &\cong \operatorname{Hom}(E, E(-k\ell_\infty)) \cong \cdots . \end{aligned}$$

But $\operatorname{Hom}(E, E(-k\ell_\infty)) \cong \operatorname{Ext}^2(E, E((k-3)\ell_\infty))^*$ vanishes for sufficient large k by the Serre vanishing theorem. Thus we get the assertion. □

Corollary 2.2. *$M(r,n)$ is a nonsingular variety of dimension $2nr$.*

Proof. This follows from the above vanishing theorem together with the Riemann-Roch formula. □

In fact, we have another way to define the framed moduli space and prove this corollary in our setting. By a result of Barth [5] (see [47, Theorem 2.1] for the proof), we have an isomorphism between $M(r,n)$ and the quotient space of $B_1, B_2 \in \mathrm{End}(\mathbb{C}^n)$, $i \in \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n)$ and $j \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^r)$ satisfying

(1) $[B_1, B_2] + ij = 0$,
(2) there exists no proper subspace $S \subsetneq \mathbb{C}^n$ such that $B_\alpha(S) \subset S$ $(\alpha = 1, 2)$ and $\operatorname{im} i \subset S$

modulo the action of $\mathrm{GL}_n(\mathbb{C})$ given by

$$g \cdot (B_1, B_2, i, j) = \left(gB_1g^{-1}, gB_2g^{-1}, gi, jg^{-1}\right).$$

We say (B_1, B_2, i, j) is *stable* when it satisfies the condition (2). It can be shown that the differential of the defining equation (1) is surjective and the action is free on stable points. This shows the smoothness of $M(r, n)$. (See [47, §3].)

Let $M_0(r, n)$ be the framed moduli space of ideal instantons on $S^4 = \mathbb{C}^2 \cup \{\infty\}$, that is

$$M_0(r, n) = \bigsqcup_{n'=0}^{n} M_0^{\mathrm{reg}}(r, n') \times S^{n-n'}\mathbb{C}^2,$$

where $M_0^{\mathrm{reg}}(r, n')$ is the framed moduli space of genuine instantons on S^4 and $S^k\mathbb{C}^2$ is the kth symmetric product of $\mathbb{C}^2$. We endow a topology to $M_0(r, n)$ as in [12, 4.4]. By a result of Donaldson [11] (which is based on the ADHM description [1]), $M_0^{\mathrm{reg}}(r, n)$ can be identified with the framed moduli space of *locally free sheaves* on $\mathbb{P}^2$, and also with the open subset of the space of linear data (B_1, B_2, i, j) with an extra condition that the transposes ${}^tB_1$, ${}^tB_2$, ${}^tj$ satisfy the above condition (2). Then by [12, 3.4.10] together with [47, Chap. 3], $M_0(r, n)$ can be identified (as a topological space) with

$$\{(B_1, B_2, i, j) \mid [B_1, B_2] + ij = 0\} /\!\!/ \mathrm{GL}_n(\mathbb{C}), \tag{2.3}$$

where $/\!\!/$ denotes the affine algebro-geometric quotient. The open locus $M_0^{\mathrm{reg}}(r, n)$ consists of closed orbits $\mathrm{GL}_n(\mathbb{C}) \cdot (B_1, B_2, i, j)$ such that the stabilizer is trivial.

As in [47, Chap. 3], $M(r, n)$ has a structure of hyper-Kähler manifold of dimension $4nr$ if we put the standard inner products on $\mathbb{C}^n$ and $\mathbb{C}^r$. In fact, $M(r, n)$ is isomorphic to the hyper-Kähler quotient

(2.4)
$$\left\{(B_1, B_2, i, j) \;\middle|\; \begin{array}{l} \text{(i)}\ [B_1, B_2] + ij = 0 \\ \text{(ii)}\ \left[B_1, B_1^\dagger\right] + \left[B_2, B_2^\dagger\right] + ii^\dagger - j^\dagger j = \zeta\,\mathrm{id} \end{array}\right\} \Big/ \mathrm{U}(n),$$

where $(\ )^\dagger$ is the Hermitian adjoint and ζ is a fixed positive real number. This hyper-Kähler structure plays no role later.

By these descriptions via linear data, we have a projective morphism

$$\pi \colon M(r, n) \to M_0(r, n),$$

where we endow $M_0(r,n)$ with a scheme structure by the description (2.3). (See [47, 3.51].) In terms of the original definition as framed moduli spaces, the corresponding map between closed points can be identified with

$$(2.5)\quad (E,\Phi) \longmapsto ((E^{\vee\vee},\Phi), \operatorname{Supp}(E^{\vee\vee}/E)) \in M_0^{\mathrm{reg}}(r,n') \times S^{n-n'}\mathbb{C}^2.$$

where $E^{\vee\vee}$ is the double dual of E and $\operatorname{Supp}(E^{\vee\vee}/E)$ is the support of $E^{\vee\vee}/E$ counted with multiplicities. Note that $E^{\vee\vee}$ is a locally free sheaf. (This identification can be proved easily from results in [47, Chaps. 2, 3] and details were given in [55].)

Remark 2.6. Morphisms from moduli spaces of semistable torsion-free sheaves to moduli spaces of ideal instantons on general projective surfaces were constructed by J. Li [29] and Morgan [44] in this way. (See also [26, §8.2].) But it is not clear that the scheme structure is the same as one given above.

Let T be the maximal torus of $\mathrm{GL}_r(\mathbb{C})$ consisting of diagonal matrices and let $\widetilde{T} = \mathbb{C}^* \times \mathbb{C}^* \times T$. We define an action of $\widetilde{T}$ on $M(r,n)$ as follows: For $(t_1,t_2) \in \mathbb{C}^* \times \mathbb{C}^*$, let F_{t_1,t_2} be an automorphism of $\mathbb{P}^2$ defined by

$$F_{t_1,t_2}([z_0 : z_1 : z_2]) = [z_0 : t_1 z_1 : t_2 z_2].$$

For $\operatorname{diag}(e_1,\dots,e_r) \in T$ let $G_{e_1,\dots,e_r}$ denote the isomorphism of $\mathcal{O}_{\ell_\infty}^{\oplus r}$ given by

$$\mathcal{O}_{\ell_\infty}^{\oplus r} \ni (s_1,\dots,s_r) \longmapsto (e_1 s_1,\dots,e_r s_r).$$

Then for $(E,\Phi) \in M(r,n)$, we define

$$(2.7)\qquad (t_1,t_2,e_1,\dots,e_r)\cdot(E,\Phi) = \left(\left(F_{t_1,t_2}^{-1}\right)^* E, \Phi'\right),$$

where Φ' is the composite of homomorphisms

$$\left(F_{t_1,t_2}^{-1}\right)^* E|_{\ell_\infty} \xrightarrow{\left(F_{t_1,t_2}^{-1}\right)^*\Phi} \left(F_{t_1,t_2}^{-1}\right)^* \mathcal{O}_{\ell_\infty}^{\oplus r} \longrightarrow \mathcal{O}_{\ell_\infty}^{\oplus r} \xrightarrow{G_{e_1,\dots,e_r}} \mathcal{O}_{\ell_\infty}^{\oplus r}.$$

Here the middle arrow is the homomorphism given by the action.

In a similar way, we have a $\widetilde{T}$-action on $M_0(r,n)$. The map $\pi\colon M(r,n) \to M_0(r,n)$ is equivariant.

Lemma 2.8. *These actions can be identified with the actions on the linear data defined by*

$$(B_1,B_2,i,j) \longmapsto \left(t_1 B_1, t_2 B_2, i e^{-1}, t_1 t_2 e j\right),$$
$$\text{for } t_1,t_2 \in \mathbb{C}^*,\ e = \operatorname{diag}(e_1,\dots,e_r) \in (\mathbb{C}^*)^r.$$

Note that this action preserves the equation $[B_1,B_2] + ij = 0$ *and the stability condition, and commutes with the action of* $\mathrm{GL}_n(\mathbb{C})$. *Hence it induces an action on* $M(r,n)$ *and* $M_0(r,n)$.

Proof. The sheaf E is given as the middle cohomology group of the complex

$$V\otimes\mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow[a=\begin{pmatrix} z_0B_1-z_1 \\ z_0B_2-z_2 \\ z_0 j\end{pmatrix}]{} \begin{matrix} V\otimes\mathcal{O}_{\mathbb{P}^2} \\ \oplus \\ V\otimes\mathcal{O}_{\mathbb{P}^2} \\ \oplus \\ W\otimes\mathcal{O}_{\mathbb{P}^2}\end{matrix} \xrightarrow[b=\begin{pmatrix} -(z_0B_2-z_2) & z_0B_1-z_1 & z_0 i\end{pmatrix}]{} V\otimes\mathcal{O}_{\mathbb{P}^2}(1).$$

(See [47, Chap. 2].) Let us pull back this complex by F_{t_1,t_2}^{-1}:

$$V\otimes\mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow[a=\begin{pmatrix} z_0B_1-t_1^{-1}z_1 \\ z_0B_2-t_2^{-1}z_2 \\ z_0 j\end{pmatrix}]{} \begin{matrix} V\otimes\mathcal{O}_{\mathbb{P}^2} \\ \oplus \\ V\otimes\mathcal{O}_{\mathbb{P}^2} \\ \oplus \\ W\otimes\mathcal{O}_{\mathbb{P}^2}\end{matrix}$$
$$\xrightarrow[b=\left(-\left(z_0B_2-t_2^{-1}z_2\right) \quad z_0B_1-t_1^{-1}z_1 \quad z_0 i\right)]{} V\otimes\mathcal{O}_{\mathbb{P}^2}(1).$$

Under the isomorphism

$$\begin{matrix} V\otimes\mathcal{O}_{\mathbb{P}^2} \\ \oplus \\ V\otimes\mathcal{O}_{\mathbb{P}^2} \\ \oplus \\ W\otimes\mathcal{O}_{\mathbb{P}^2}\end{matrix} \ni \begin{pmatrix} v_1 \\ v_2 \\ w\end{pmatrix} \longmapsto \begin{pmatrix} t_2^{-1}v_1 \\ t_1^{-1}v_2 \\ w\end{pmatrix},$$

the kernel of b is mapped to the kernel of

$$\left(-(z_0t_2B_2-z_2) \quad z_0t_1B_1-z_1 \quad z_0 i\right).$$

Also under the above isomorphism, the image of a is mapped to the image of

$$\frac{1}{t_1t_2}\begin{pmatrix} z_0t_1B_1-z_1 \\ z_0t_2B_2-z_2 \\ z_0t_1t_2 j\end{pmatrix}.$$

Thus the pull-back sheaf $(F_{t_1,t_2}^{-1})^*E$ corresponds to the data $(t_1B_1, t_2B_2, i, t_1t_2 j)$. Composing the change of the framing by $G_{e_1,\dots,e_r}$, we get the assertion. □

Proposition 2.9. (1) $(E,\Phi)\in M(r,n)$ *is fixed by the* $\widetilde{T}$*-action if and only if* E *has a decomposition* $E=I_1\oplus\cdots\oplus I_r$ *satisfying the following conditions for* $\alpha=1,\dots,r$:

a) I_α *is an ideal sheaf of* 0*-dimensional subscheme* Z_α *contained in* $\mathbb{C}^2=\mathbb{P}^2\setminus\ell_\infty$.
b) *Under* Φ, $I_\alpha|_{\ell_\infty}$ *is mapped to the* α*-th factor* $\mathcal{O}_{\ell_\infty}$ *of* $\mathcal{O}_{\ell_\infty}^{\oplus r}$.
c) I_α *is fixed by the action of* $\mathbb{C}^*\times\mathbb{C}^*$*, coming from that on* $\mathbb{P}^2$.

(2) *The fixed point set consists of finitely many points parametrized by r-tuple $(Y_1, \dots, Y_r)$ of Young diagrams such that $\sum_\alpha |Y_\alpha| = n$ (by a way explained in the proof).*

(3) *The fixed point set in $M_0(r,n)$ (more strongly, the fixed point set with respect to the first two factors T^2 of T^{r+2}) consists of a single point $n[0] \in S^n\mathbb{C}^2 \subset M_0(r,n)$.*

Proof. (1) $E \in M(r,n)$ is fixed by the latter T-action if and only if it decomposes as $E = I_1 \oplus \dots \oplus I_r$ ($I_\alpha \in M(1, n_\alpha)$) such that $I_\alpha|_{\ell_\infty}$ is mapped to the α-th factor $\mathcal{O}_{\ell_\infty}$ of $\mathcal{O}_{\ell_\infty}^{\oplus r}$ under Φ. Since the double dual $I_\alpha^{\vee\vee}$ of I_α is a line bundle with $c_1(I_\alpha^{\vee\vee}) = 0$, it is the structure sheaf $\mathcal{O}_{\mathbb{P}^2}$. Via the natural inclusion $I_\alpha \subset I_\alpha^{\vee\vee} = \mathcal{O}_{\mathbb{P}^2}$, I_α is an ideal sheaf of 0-dimensional subscheme Z_α contained in $\mathbb{C}^2$. Thus conditions a),b) are met for I_α. If E is fixed also by the first T^2-action, then the condition c) must be satisfied. The converse is clear.

(2) By a result of Ellingsrud and Strømme [18] (see [47, §5.2]) that I_α is fixed if and only if it is generated by monomials $x^i y^j$, where we consider I_α as an ideal of $\mathbb{C}[x,y]$, the coordinate ring of $\mathbb{C}^2$. Thus I_α corresponds to a Young diagram Y_α by the rule indicated by the Fig. 1. (A monomial $x^{i-1}y^{j-1}$ is placed at (i,j). The ideal I_α is linearly spanned by monomials outside the Young diagram Y_α. Note that our Young diagrams are rotated 90° from ones used in [39].)

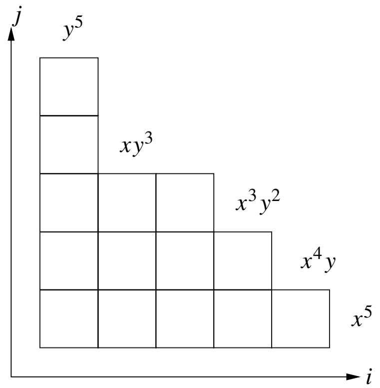

Fig. 1. Young diagram and ideal

(3) Let us use the description (2.3). Suppose that the equivalence class of (B_1, B_2, i, j) is fixed by the T^2-action. We may assume that (B_1, B_2, i, j) has a closed $GL_n(\mathbb{C})$-orbit. Then the equivalence class is fixed if and only if $(t_1B_1, t_2B_2, i, t_1t_2j)$ lies in the same $GL_n(\mathbb{C})$-orbit. Since $(t_1B_1, t_2B_2, i, t_1t_2j)$ converges to $(0,0,i,0)$ when $t_1, t_2 \to 0$, $(0,0,i,0)$ lies in the closure of the orbit. But the orbit is closed, so $(0,0,i,0)$ must be in the orbit. But $GL_n(\mathbb{C}) \cdot (0,0,i,0)$ is closed if and only if $i = 0$. Hence we have $(B_1, B_2, i, j) = (0,0,0,0)$. □

We denote by $\vec{Y}$ an r-tuple of Young diagrams $(Y_1, \ldots, Y_r)$. We write the number of boxes of Y_α by $|Y_\alpha|$ and we set $|\vec{Y}| = \sum_\alpha |Y_\alpha|$.

Let $T_{(E,\Phi)}M(r,n)$ be the tangent space of $M(r,n)$ at a point (E,Φ). If (E,Φ) is fixed by the torus action, then $T_{(E,\Phi)}M(r,n)$ is a module of the torus. In order to express the module structure in terms of Young diagrams Y_α, we introduce the following notation. Let $Y = (\lambda_1 \geq \lambda_2 \geq \ldots)$ be a Young diagram, where λ_i is the length of the ith column. Let $Y' = (\lambda'_1 \geq \lambda'_2 \geq \ldots)$ be the transpose of Y. Thus λ'_j is the length of the jth row of Y. Let $l(Y)$ denote the number of columns of Y, i.e., $l(Y) = \lambda'_1$. Let

$$a_Y(i,j) = \lambda_i - j, \qquad a'(i,j) = j - 1$$
$$l_Y(i,j) = \lambda'_j - i, \qquad l'(i,j) = i - 1.$$

Here we set $\lambda_i = 0$ when $i > l(Y)$. Similarly $\lambda'_j = 0$ when $j > l(Y')$. When the square $s = (i,j)$ lies in Y, these are called *arm-length*, *arm-colength*, *leg-length*, *leg-colength* respectively, and we usually consider in this case. But our formula below involves these also for squares outside Y. So these take negative values in general. Note that a' and l' does not depend on the diagram, and we do not write the subscript Y.

If two Young diagrams Y_α and Y_β are given, we separate Y_α into two regions ${}^\heartsuit Y_\alpha$ and ${}^\spadesuit Y_\alpha$ as

$${}^\heartsuit Y_\alpha = \left\{(i,j) \in Y_\alpha \mid j \leq l(Y'_\beta)\right\}, \qquad {}^\spadesuit Y_\alpha = \left\{(i,j) \in Y_\alpha \mid j > l(Y'_\beta)\right\}.$$

If $l(Y'_\alpha) \leq l(Y'_\beta)$, then ${}^\spadesuit Y_\alpha = \emptyset$. Exchanging the role of α and β, we divide Y_β into ${}^\heartsuit Y_\beta$ and ${}^\spadesuit Y_\beta$. Note that either ${}^\spadesuit Y_\alpha$ or ${}^\spadesuit Y_\beta$ is the empty set.

Notation 2.10. We denote by e_α $(\alpha = 1, \ldots, r)$ the one dimensional $\widetilde{T}$-module given by

$$\widetilde{T} \ni (t_1, t_2, e_1, \ldots, e_r) \mapsto e_\alpha.$$

Similarly, t_1, t_2 denote one-dimensional $\widetilde{T}$-modules. Thus the representation ring $R(\widetilde{T})$ is isomorphic to $\mathbb{Z}[t_1^\pm, t_2^\pm, e_1^\pm, \ldots, e_r^\pm]$, where e_α^{-1} is the dual of e_α.

Theorem 2.11. *Let (E,Φ) be a fixed point of $\widetilde{T}$-action corresponding to $\vec{Y} = (Y_1, \ldots, Y_r)$. Then the $\widetilde{T}$-module structure of $T_{(E,\Phi)}M(r,n)$ is given by*

$$T_{(E,\Phi)}M(r,n) = \sum_{\alpha,\beta=1}^{r} N_{\alpha,\beta}(t_1,t_2),$$

where

$$N_{\alpha,\beta}(t_1,t_2) = e_\beta\, e_\alpha^{-1} \times \left\{ \sum_{s \in Y_\alpha} \left(t_1^{-l_{Y_\beta}(s)} t_2^{a_{Y_\alpha}(s)+1} \right) + \sum_{t \in Y_\beta} \left(t_1^{l_{Y_\alpha}(t)+1} t_2^{-a_{Y_\beta}(t)} \right) \right\}.$$

Remark 2.12. (1) After the first version of this paper was written, the authors noticed that this formula already appeared in the context of the wall-crossing formula for the Donaldson invariants [17, Lemma 6.2]. Their proof does not use the ADHM description, so different from ours. We will discuss the relation between Nekrasov's prepotential and the wall-crossing formula in a future publication with L. Göttsche.

(2) The following proof was mentioned also in [7].

(3) For the proof of the blowup equation, we only need the relation between $N_{\alpha,\beta}(t_1, t_2)$ and similar weights on the blowup (Theorem 3.4). A reader in hurry can safely skip the proof.

Proof of Theorem 2.11. We use the description by linear data for the calculation, which is very similar to that in [47, 5.8].

Let (B_1, B_2, i, j) be a datum as above. We consider a complex

$$\text{(2.13)}\quad \operatorname{Hom}(V,V) \xrightarrow{\sigma} \begin{matrix} \operatorname{Hom}(V,V)\oplus\operatorname{Hom}(V,V) \\ \oplus \\ \operatorname{Hom}(W,V)\oplus\operatorname{Hom}(V,W) \end{matrix} \xrightarrow{\tau} \operatorname{Hom}(V,V),$$

where σ and τ are defined by

$$\sigma(\xi) = \begin{pmatrix} \xi B_1 - B_1\xi \\ \xi B_2 - B_2\xi \\ \xi i \\ -j\xi \end{pmatrix}, \quad \tau\begin{pmatrix} C_1 \\ C_2 \\ I \\ J \end{pmatrix} = [B_1, C_2] + [C_1, B_2] + iJ + Ij.$$

This σ is the differential of $\operatorname{GL}(V)$-action and τ is the differential of the map $(B_1, B_2, i, j) \mapsto [B_1, B_2] + ij$. One can show that σ is injective and τ is surjective, and the tangent space of $M(r,n)$ at $\operatorname{GL}(V)\cdot(B_1, B_2, i, j)$ is isomorphic to the middle cohomology group of the above complex (cf. [47, 1.9] or [46, 3.10]).

Now suppose $\operatorname{GL}(V)\cdot(B_1, B_2, i, j)$ is fixed by the $\widetilde{T}$-action. This means that for any $(t_1, t_2, e) \in \widetilde{T}$ there exists an element $g(t_1, t_2, e) \in \operatorname{GL}(V)$ such that

$$\left(t_1 B_1, t_2 B_2, ie^{-1}, t_1 t_2 e j\right) = g(t_1, t_2, e)^{-1}\cdot(B_1, B_2, i, j).$$

Moreover, such $g(t_1, t_2, e)$ is unique since the $\operatorname{GL}(V)$-action on the set of stable points is free. In particular, it implies that the map $(t_1, t_2, e) \mapsto g(t_1, t_2, e)$ is a group homomorphism. We consider V as a $\widetilde{T}$-module via it. Also W is a $\widetilde{T}$-module via $(t_1, t_2, e) \mapsto e \in \operatorname{GL}(W)$.

We can make the complex (2.13) $\widetilde{T}$-*equivariant* by modifying it as

$$\text{(2.14)}\quad \operatorname{Hom}(V,V) \xrightarrow{\sigma} \begin{matrix} t_1\operatorname{Hom}(V,V)\oplus t_2\operatorname{Hom}(V,V) \\ \oplus \\ \operatorname{Hom}(W,V)\oplus t_1t_2\operatorname{Hom}(V,W) \end{matrix} \xrightarrow{\tau} t_1t_2\operatorname{Hom}(V,V),$$

where t_1, t_2 denote the one dimensional $\widetilde{T}$-modules as in Notation 2.10.

We have a decomposition $W = \bigoplus_{\alpha=1}^r e_\alpha$ as $\widetilde{T}$-modules. From the stability condition, it is easy to see that V decomposes as $V = \bigoplus V_\alpha e_\alpha$, where V_α is a T^2-module (i.e., T^r acts trivially on V_α). Thus $\operatorname{Ker}\tau/\operatorname{Im}\sigma$ decomposes as $\bigoplus_{\alpha,\beta}\left(\operatorname{Ker}\tau_{\beta\alpha}/\operatorname{Im}\sigma_{\beta\alpha}\right) e_\beta e_\alpha^{-1}$ where

(2.15)

$$\operatorname{Hom}(V_\alpha, V_\beta) \xrightarrow{\sigma_{\beta\alpha}} \begin{matrix} t_1 \operatorname{Hom}(V_\alpha, V_\beta) \oplus t_2 \operatorname{Hom}(V_\alpha, V_\beta) \\ \oplus \\ \operatorname{Hom}(W_\alpha, V_\beta) \oplus t_1 t_2 \operatorname{Hom}(V_\alpha, W_\beta) \end{matrix} \xrightarrow{\tau_{\beta\alpha}} t_1 t_2 \operatorname{Hom}(V_\alpha, V_\beta).$$

It is clear that each summand has the weight $e_\beta e_\alpha^{-1}$ as a latter torus T, so we suppress this factor and only consider the $\mathbb{C}^* \times \mathbb{C}^*$-module structure hereafter.

Let us write $Y_\alpha = (\lambda_{\alpha,1} \geq \lambda_{\alpha,2} \geq \dots)$, $Y'_\alpha = (\lambda'_{\alpha,1} \geq \lambda'_{\alpha,2} \geq \dots)$. Since V_α has a basis $\{x^{i-1} y^{j-1}\}$ $((i, j) \in Y_\alpha)$, we have

$$V_\alpha = \sum_{j=1}^{\lambda_{\alpha,1}} \sum_{i=1}^{\lambda'_{\alpha,j}} t_1^{-i+1} t_2^{-j+1} = \sum_{i=1}^{\lambda'_{\alpha,1}} \sum_{j=1}^{\lambda_{\alpha,i}} t_1^{-i+1} t_2^{-j+1}.$$

Hence we get

$$\begin{aligned}
&(t_1 + t_2 - 1 - t_1 t_2) V_\alpha^* \otimes V_\beta \\
&= \sum_{i=1}^{\lambda'_{\alpha,1}} \sum_{j'=1}^{\lambda_{\alpha,i}} t_1^{i-1} t_2^{j'-1} (t_2 - 1) \times \sum_{j=1}^{\lambda_{\beta,1}} \sum_{i'=1}^{\lambda'_{\beta,j}} t_1^{-i'+1} (1 - t_1)\, t_2^{-j+1} \\
&= \sum_{i=1}^{\lambda'_{\alpha,1}} \sum_{j=1}^{\lambda_{\beta,1}} \left(t_1^{i-\lambda'_{\beta,j}} - t_1^i\right)\left(t_2^{-j+\lambda_{\alpha,i}+1} - t_2^{-j+1}\right) \\
&= \sum_{i=1}^{\lambda'_{\alpha,1}} \sum_{j=1}^{\lambda_{\beta,1}} \Big[t_1^{i-\lambda'_{\beta,j}} t_2^{-j+\lambda_{\alpha,i}+1} - t_1^i t_2^{-j+1} - \left(t_1^{i-\lambda'_{\beta,j}} - t_1^i\right) t_2^{-j+1} \\
&\qquad\qquad - t_1^i \left(t_2^{-j+\lambda_{\alpha,i}+1} - t_2^{-j+1}\right)\Big].
\end{aligned}$$

Note that

$$\begin{aligned}
\sum_{i=1}^{\lambda'_{\alpha,1}} \sum_{j=1}^{\lambda_{\beta,1}} \left(t_1^{i-\lambda'_{\beta,j}} - t_1^i\right) t_2^{-j+1} &= \sum_{j=1}^{\lambda_{\beta,1}} \sum_{i=1}^{\lambda'_{\beta,j}} \left(t_1^{1-i} - t_1^{\lambda'_{\alpha,1}-i+1}\right) t_2^{-j+1} \\
&= V_\beta - \sum_{j=1}^{\lambda_{\beta,1}} \sum_{i=1}^{\lambda'_{\beta,j}} t_1^{\lambda'_{\alpha,1}-i+1} t_2^{-j+1}.
\end{aligned}$$

Similarly note that

$$\sum_{i=1}^{\lambda'_{\alpha,1}}\sum_{j=1}^{\lambda_{\beta,1}} t_1^i\left(t_2^{-j+\lambda_{\alpha,i}+1} - t_2^{-j+1}\right) = t_1t_2V_\alpha^* - \sum_{i=1}^{\lambda'_{\alpha,1}}\sum_{j=1}^{\lambda_{\alpha,i}} t_1^i t_2^{-\lambda_{\beta,1}+j}.$$

Thus we have

$$\begin{aligned}\operatorname{Ker}\tau_{\beta\alpha}/\operatorname{Im}\sigma_{\beta\alpha} &= (t_1+t_2-1-t_1t_2)V_\alpha^*\otimes V_\beta + V_\beta + t_1t_2V_\alpha^* \\ &= \sum_{i=1}^{\lambda'_{\alpha,1}}\sum_{j=1}^{\lambda_{\beta,1}}\left(t_1^{i-\lambda'_{\beta,j}}t_2^{-j+\lambda_{\alpha,i}+1} - t_1^i t_2^{-j+1}\right) + \sum_{j=1}^{\lambda_{\beta,1}}\sum_{i=1}^{\lambda'_{\beta,j}} t_1^{\lambda'_{\alpha,1}-i+1} t_2^{-j+1} \\ &\quad + \sum_{i=1}^{\lambda'_{\alpha,1}}\sum_{j=1}^{\lambda_{\alpha,i}} t_1^i t_2^{-\lambda_{\beta,1}+j}.\end{aligned} \tag{2.16}$$

This is equal to $N_{\alpha,\beta}(t_1,t_2)$, which we want to compute. We decompose it as $N_{\alpha,\beta}(t_1,t_2) = N_{\alpha,\beta}^{>0}(t_1,t_2) + N_{\alpha,\beta}^{\le 0}(t_1,t_2)$, according to the power of t_2. Then

$$\begin{aligned}N_{\alpha,\beta}^{>0}(t_1,t_2) &= \sum_{i=1}^{\lambda'_{\alpha,1}}\sum_{j=1}^{\min(\lambda_{\beta,1},\lambda_{\alpha,i})} t_1^{i-\lambda'_{\beta,j}} t_2^{-j+\lambda_{\alpha,i}+1} + \sum_{i=1}^{\lambda'_{\alpha,1}}\sum_{j=\lambda_{\beta,1}+1}^{\lambda_{\alpha,i}} t_1^i t_2^{-\lambda_{\beta,1}+j} \\ &= \sum_{s\in{}^{\heartsuit}Y_\alpha} t_1^{-l_{Y_\beta}(s)} t_2^{a_{Y_\alpha}(s)+1} + \sum_{s\in{}^{\spadesuit}Y_\alpha} t_1^{l'(s)+1} t_2^{a'(s)-l(Y'_\beta)+1},\end{aligned} \tag{2.17}$$

where the sum $\sum_{j=\lambda_{\beta,1}+1}^{\lambda_{\alpha,i}}$ is understood as 0 unless $\lambda_{\beta,1} < \lambda_{\alpha,i}$.

Noticing the symmetry $N_{\alpha,\beta}(t_1,t_2) = N_{\beta,\alpha}(t_1^{-1},t_2^{-1})t_1t_2$, we get the following from (2.16):

$$\begin{aligned}N_{\alpha,\beta}(t_1,t_2) &= \sum_{i=1}^{\lambda'_{\beta,1}}\sum_{j=1}^{\lambda_{\alpha,1}}\left(t_1^{-i+\lambda'_{\alpha,j}+1} t_2^{j-\lambda_{\beta,i}} - t_1^{-i+1}t_2^j\right) + \sum_{j=1}^{\lambda_{\alpha,1}}\sum_{i=1}^{\lambda'_{\alpha,j}} t_1^{-\lambda'_{\alpha,1}+i} t_2^j \\ &\quad + \sum_{i=1}^{\lambda'_{\beta,1}}\sum_{j=1}^{\lambda_{\beta,i}} t_1^{-i+1} t_2^{\lambda_{\alpha,1}-j+1}.\end{aligned}$$

This implies that

$$\begin{aligned}N_{\alpha,\beta}^{\le 0}(t_1,t_2) &= \sum_{i=1}^{\lambda'_{\beta,1}}\sum_{j=1}^{\min(\lambda_{\alpha,1},\lambda_{\beta,i})} t_1^{-i+\lambda'_{\alpha,j}+1} t_2^{j-\lambda_{\beta,i}} + \sum_{i=1}^{\lambda'_{\beta,1}}\sum_{j=\lambda_{\alpha,1}+1}^{\lambda_{\beta,i}} t_1^{-i+1} t_2^{\lambda_{\alpha,1}-j+1} \\ &= \sum_{t\in{}^{\heartsuit}Y_\beta} t_1^{l_{Y_\alpha}(t)+1} t_2^{-a_{Y_\beta}(t)} + \sum_{t\in{}^{\spadesuit}Y_\beta} t_1^{-l'(t)} t_2^{-a'(t)+l(Y'_\alpha)},\end{aligned} \tag{2.18}$$

where the sum $\sum_{j=\lambda_{\alpha,1}+1}^{\lambda_{\beta,i}}$ is understood as 0 unless $\lambda_{\alpha,1} < \lambda_{\beta,i}$. Combining (2.17) with (2.18), we get the assertion. We use $-l_{Y_\beta}(s) = l'(s) + 1$ for $s \in {}^{\spadesuit}Y_\alpha$ and re-order the product in $s \in {}^{\spadesuit}Y_\alpha$. □

3. Moduli spaces on the blowup

Let $\widehat{\mathbb{P}}^2$ be the blowup of $\mathbb{P}^2$ at $[1:0:0]$. Let $p\colon \widehat{\mathbb{P}}^2 \to \mathbb{P}^2$ denote the projection. The manifold $\widehat{\mathbb{P}}^2$ is the closed subvariety of $\mathbb{P}^2 \times \mathbb{P}^1$ defined by

$$\left\{\left([z_0 : z_1 : z_2], [z : w]\right) \in \mathbb{P}^2 \times \mathbb{P}^1 \middle| z_1 w = z_2 z\right\},$$

where the map $p\colon \widehat{\mathbb{P}}^2 \to \mathbb{P}^2$ is the projection to the first factor. Let us denote the inverse image of ℓ_∞ under $\widehat{\mathbb{P}}^2 \to \mathbb{P}^2$ also by ℓ_∞ for brevity. It is given by the equation $z_0 = 0$. The complement $\widehat{\mathbb{P}}^2 \setminus \ell_\infty$ is the blowup $\widehat{\mathbb{C}}^2$ of $\mathbb{C}^2$ at the origin. Let C denote the exceptional set. It is given by $z_1 = z_2 = 0$.

In this section, $\mathcal{O}$ denotes the structure sheaf of $\widehat{\mathbb{P}}^2$, $\mathcal{O}(C)$ the line bundle associated with the divisor C, $\mathcal{O}(mC)$ its mth tensor power.

Let $\widehat{M}(r,k,n)$ be the framed moduli space of torsion free sheaves (E,Φ) on $\widehat{\mathbb{P}}^2$ with rank r, $\langle c_1(E), [C]\rangle = -k$ and $\langle c_2(E) - \frac{r-1}{2r} c_1(E)^2, [\widehat{\mathbb{P}}^2]\rangle = n$.

By the same argument as in Proposition 2.1 we have $\operatorname{Hom}(E, E(-\ell_\infty)) = \operatorname{Ext}^2(E, E(-\ell_\infty)) = 0$ and $\widehat{M}(r,k,n)$ is a nonsingular variety of dimension $2nr$.

Theorem 3.1. *There is a projective morphism $\widehat{\pi}\colon \widehat{M}(r,k,n) \to M_0(r, n - \frac{1}{2r}k(r-k))$ $(0 \le k < r)$ defined by*

$$(E,\Phi) \mapsto \left(\left((p_*E)^{\vee\vee}, \Phi\right), \operatorname{Supp}\left(p_*E^{\vee\vee}/p_*E\right) + \operatorname{Supp}\left(R^1 p_* E\right)\right).$$

The proof of this result will be given in [49]. In fact, we prove also the corresponding result for arbitrary projective surfaces. For the above case with $k = 0$, we can use King's result [28] instead. Namely there is a morphism from the Uhlenbeck (partial) compactification $\widehat{M}_0(r,0,n) \to M_0(r,n)$ defined via the ADHM descriptions of both spaces. Then we compose the morphism $\widehat{M}(r,0,n) \to \widehat{M}_0(r,0,n)$. This morphism can be defined via a modification of King's description as in the case of $\mathbb{C}^2$.

We use this result to prove the vanishing result (Proposition 6.11), which is about the case $k = 0$, and we can avoid its usage for the definition of the partition function $\widehat{Z}$ on the blowup. In this sense, this paper does not rely on [49].

Let us define an action of the $(r+2)$-dimensional torus $\widetilde{T} = \mathbb{C}^* \times \mathbb{C}^* \times T$ on $\widehat{M}(r,k,n)$ by modifying the action on $M(r,n)$ as follows. For $(t_1,t_2) \in \mathbb{C}^* \times \mathbb{C}^*$, let F'_{t_1,t_2} be an automorphism of $\widehat{\mathbb{P}}^2$ defined by

$$F'_{t_1,t_2}([z_0 : z_1 : z_2], [z : w]) = ([z_0 : t_1 z_1 : t_2 z_2], [t_1 z : t_2 w]).$$

Then we define the action by replacing F_{t_1,t_2} by F'_{t_1,t_2} in (2.7). The action of the latter T is exactly the same as before. The morphism $\widehat{\pi}$ is equivariant.

Note that the fixed point set of $\mathbb{C}^* \times \mathbb{C}^*$ in $\widehat{\mathbb{C}}^2 = \widehat{\mathbb{P}}^2 \setminus \ell_\infty$ consists of two points $([1:0:0],[1:0])$, $([1:0:0],[0:1])$. Let us denote them p_1 and p_2.

Since C is invariant under the $\mathbb{C}^* \times \mathbb{C}^*$-action, the corresponding line bundle $\mathcal{O}(C)$ is an equivariant line bundle. The section $z_1/z = z_2/w$ is equivariant.

Proposition 3.2. (1) *$(E, \Phi) \in \widehat{M}(r,k,n)$ is fixed by the $\widetilde{T}$-action if and only if E has a decomposition $E = I_1(k_1C) \oplus \cdots \oplus I_r(k_rC)$ satisfying the following conditions for $\alpha = 1, \dots, r$:*

a) *$I_\alpha(k_\alpha C)$ is the tensor product $I_\alpha \otimes \mathcal{O}(k_\alpha C)$, where $k_\alpha \in \mathbb{Z}$ and I_α is an ideal sheaf of 0-dimensional subscheme Z_α contained in $\widehat{\mathbb{C}}^2 = \widehat{\mathbb{P}}^2 \setminus \ell_\infty$.*
b) *Under Φ, $I_\alpha(k_\alpha C)|_{\ell_\infty}$ is mapped to the α-th factor $\mathcal{O}_{\ell_\infty}$ of $\mathcal{O}_{\ell_\infty}^{\oplus r}$.*
c) *I_α is fixed by the action of $\mathbb{C}^* \times \mathbb{C}^*$, coming from that on $\widehat{\mathbb{P}}^2$.*

(2) *The fixed point set consists of finitely many points parametrized by r-tuples $((k_1, Y_1^1, Y_1^2), \dots, (k_r, Y_r^1, Y_r^2))$, where $k_\alpha \in \mathbb{Z}$ and Y_α^1, Y_α^2 are Young diagrams such that*

$$\sum_\alpha k_\alpha = k, \qquad \sum_\alpha \left(\left|Y_\alpha^1\right| + \left|Y_\alpha^2\right|\right) + \frac{1}{2r}\sum_{\alpha<\beta} |k_\alpha - k_\beta|^2 = n \tag{3.3}$$

(*by a way explained in the proof*).

Proof. The proof is almost the same as that of Proposition 2.9.

$E \in \widehat{M}(r,k,n)$ is fixed by the latter T^r-action if and only if it decomposes as $E = E_1 \oplus \cdots \oplus E_r$ ($E_\alpha \in \widehat{M}(1, k_\alpha, n_\alpha)$) such that $E_\alpha|_{\ell_\infty}$ is mapped to the α-th factor $\mathcal{O}_{\ell_\infty}$ of $\mathcal{O}_{\ell_\infty}^{\oplus r}$ under Φ. Since the double dual $E_\alpha^{\vee\vee}$ is a line bundle which is trivial at ℓ_∞, it is equal to $\mathcal{O}(k_\alpha C)$ for some $k_\alpha \in \mathbb{Z}$. Thus E_α is equal to $I_\alpha(k_\alpha C) = I_\alpha \otimes \mathcal{O}(k_\alpha C)$ for some ideal sheaf I_α of 0-dimensional subscheme Z_α in $\widehat{\mathbb{C}}^2$.

If E is fixed also by the first T^2-ation, then I_α (and Z_α) is fixed. The support of Z_α must be contained in the fixed point set in $\widehat{\mathbb{C}}^2$, i.e., $\{p_1, p_2\}$. Thus Z_α is a union of Z_α^1 and Z_α^2, subschemes supported at p_1 and p_2 respectively. If we take a coordinate system $(x, y) = (z_1/z_0, w/z)$ (resp. $= (z/w, z_2/z_0)$) around p_1 (resp. p_2), then Z_α^1 (resp. Z_α^2) is generated by monomials $x^i y^j$. Then Z_α^1 (resp. Z_α^2) corresponds to a Young diagram Y_α^1 (resp. Y_α^2) as before. □

We denote by $\vec{k}$ (resp. $\vec{Y}^i$ $(i = 1, 2)$) for the r-tuple $(k_1, \dots, k_r)$ (resp. $(Y_1^i, \dots, Y_r^i)$) as before. Thus the fixed point corresponds to $(\vec{k}, \vec{Y}^1, \vec{Y}^2)$.

As in Theorem 2.11, the tangent space of $\widehat{M}(r,k,n)$ at a fixed point (E, Φ) is a $\widetilde{T}$-module.

Theorem 3.4. *Let* (E, Φ) *be a fixed point of* $\widetilde{T}$*-action corresponding to* $(\vec{k}, \vec{Y}^1, \vec{Y}^2)$*. Then the* $\widetilde{T}$*-module structure of* $T_{(E,\Phi)}\widehat{M}(r,k,n)$ *is given by*

$$\begin{aligned} &T_{(E,\Phi)}\widehat{M}(r,k,n) \\ &= \sum_{\alpha,\beta=1}^{r} \left(L_{\alpha,\beta}(t_1,t_2) + t_1^{k_\beta-k_\alpha} M^1_{\alpha,\beta}(t_1,t_2) + t_2^{k_\beta-k_\alpha} M^2_{\alpha,\beta}(t_1,t_2) \right), \end{aligned}$$

where

$$L_{\alpha,\beta}(t_1,t_2) = e_\beta\, e_\alpha^{-1} \times \begin{cases} \displaystyle\sum_{\substack{i,j\ge 0 \\ i+j\le k_\alpha-k_\beta-1}} t_1^{-i} t_2^{-j} & \text{if } k_\alpha > k_\beta, \\ \displaystyle\sum_{\substack{i,j\ge 0 \\ i+j\le k_\beta-k_\alpha-2}} t_1^{i+1} t_2^{j+1} & \text{if } k_\alpha + 1 < k_\beta, \\ 0 & \text{otherwise}, \end{cases}$$

and $M^1_{\alpha,\beta}(t_1,t_2)$ *(resp.* $M^2_{\alpha,\beta}(t_1,t_2)$*) is equal to* $N_{\alpha,\beta}(t_1, t_2/t_1)$ *(resp.* $N_{\alpha,\beta}(t_1/t_2, t_2)$*), with* (Y_α, Y_β) *is replaced by* (Y^1_α, Y^1_β) *(resp.* (Y^2_α, Y^2_β)*).*

Proof. According to the decomposition $E = I_1(k_1C) \oplus \cdots \oplus I_r(k_rC)$, the tangent space $T_{(E,\Phi)}\widehat{M}(r,k,n) = \operatorname{Ext}^1(E, E(-\ell_\infty))$ is decomposed as

$$\operatorname{Ext}^1(E, E(-\ell_\infty)) = \bigoplus_{\alpha,\beta} \operatorname{Ext}^1(I_\alpha(k_\alpha C), I_\beta(k_\beta C - \ell_\infty)).$$

The factor $\operatorname{Ext}^1(I_\alpha(k_\alpha C), I_\beta(k_\beta C - \ell_\infty))$ has weight $e_\beta e_\alpha^{-1}$ as a T-module. Thus our remaining task is to describe each factor as a T^2-module. We suppress $e_\beta e_\alpha^{-1}$ hereafter.

Let Ext^* denotes the alternating sum $\sum_i (-1)^i \operatorname{Ext}^i$ considered as an element of the representation ring. Then Ext^* defines a homomorphism from the equivariant K-group to the representation ring. By Proposition 2.1 we have $\operatorname{Ext}^*(I_\alpha(k_\alpha C), I_\beta(k_\beta C - \ell_\infty)) = -\operatorname{Ext}^1(I_\alpha(k_\alpha C), I_\beta(k_\beta C - \ell_\infty))$. Using the exact sequence $0 \to I_\alpha \to \mathcal{O} \to \mathcal{O}_{Z_\alpha} \to 0$, we have

$$\begin{aligned} &\operatorname{Ext}^*(I_\alpha(k_\alpha C), I_\beta(k_\beta C - \ell_\infty)) \\ &= \operatorname{Ext}^*(\mathcal{O}(k_\alpha C), \mathcal{O}(k_\beta C - \ell_\infty)) - \operatorname{Ext}^*(\mathcal{O}(k_\alpha C), \mathcal{O}_{Z_\beta}(k_\beta C - \ell_\infty)) \\ &\quad - \operatorname{Ext}^*(\mathcal{O}_{Z_\alpha}(k_\alpha C), \mathcal{O}(k_\beta C - \ell_\infty)) \\ &\quad + \operatorname{Ext}^*(\mathcal{O}_{Z_\alpha}(k_\alpha C), \mathcal{O}_{Z_\beta}(k_\beta C - \ell_\infty)). \end{aligned} \tag{3.5}$$

Let us first consider the term $\operatorname{Ext}^*(\mathcal{O}(k_\alpha C), \mathcal{O}(k_\beta C - \ell_\infty)) = -\operatorname{Ext}^1(\mathcal{O}(k_\alpha C), \mathcal{O}(k_\beta C - \ell_\infty)) = -H^1(\mathcal{O}((k_\beta - k_\alpha)C - \ell_\infty))$. We show that this is equal to $-L_{\alpha,\beta}$.

Set $n = k_\alpha - k_\beta$. If $n = 0$, we have $H^1(\widehat{\mathbb{P}}^2, \mathcal{O}(-\ell_\infty)) = 0$ by Proposition 2.1. Thus we have the expression $L_{\alpha,\beta}$ in this case.

Next suppose $n > 0$. Consider the cohomology long exact sequence associated with an exact sequence $0 \to \mathcal{O}(-nC) \to \mathcal{O}((-n+1)C) \to \mathcal{O}_C((-n+1)C) \to 0$. Note that this is equivariant under the $\mathbb{C}^* \times \mathbb{C}^*$-action. Since C is a projective line $\mathbb{P}^1$ with self-intersection (-1), we have $H^1(C, \mathcal{O}_C((-n+1)C)) = H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n-1)) = 0$. Thus we have

$$0 \to H^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n-1)\big) \to H^1\big(\widehat{\mathbb{P}}^2, \mathcal{O}(-nC - \ell_\infty)\big) \\ \to H^1\big(\widehat{\mathbb{P}}^2, \mathcal{O}((-n+1)C - \ell_\infty)\big) \to 0.$$

This is an exact sequence in $\mathbb{C}^* \times \mathbb{C}^*$-modules. Starting with $H^1(\widehat{\mathbb{P}}^2, \mathcal{O}(-\ell_\infty)) = 0$, we get

$$H^1\big(\widehat{\mathbb{P}}^2, \mathcal{O}(-nC - \ell_\infty)\big) = \bigoplus_{d=0}^{n-1} H^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)\big)$$

by induction. Since $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))$ is the space of homogeneous polynomials in z, w of degree d, it is equal to $\sum_{i=0}^{d} t_1^{-i} t_2^{-d+i}$ in the representation ring of T^2. Thus we have the expression $L_{\alpha,\beta}$ in this case.

Finally consider the case $n < 0$. The proof is almost the same as that for the case $n > 0$. We use $0 \to \mathcal{O}((-n-1)C) \to \mathcal{O}(-nC) \to \mathcal{O}_C(-nC) \to 0$ to get

$$0 \to H^1\big(\widehat{\mathbb{P}}^2, \mathcal{O}((-n-1)C - \ell_\infty)\big) \to H^1\big(\widehat{\mathbb{P}}^2, \mathcal{O}(-nC - \ell_\infty)\big) \\ \to H^1\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n)\big) \to 0,$$

where we have used $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n)) = 0$. Starting with $H^1(\widehat{\mathbb{P}}^2, \mathcal{O}((-n-1)C - \ell_\infty)) = 0$ for $n = -1$, we get $H^1(\widehat{\mathbb{P}}^2, \mathcal{O}(-nC - \ell_\infty)) = \bigoplus_{d=1}^{-n} H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-d))$ by induction. The canonical bundle $K_{\mathbb{P}^1}$ of $\mathbb{P}^1$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-2)$. But this isomorphism is not equivariant, and the actual formula is $K_{\mathbb{P}^1} \cong t_1^{-1} t_2^{-1} \mathcal{O}_{\mathbb{P}^1}(-2)$. Therefore the Serre duality says $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-d))$ is the dual of $t_1^{-1} t_2^{-1} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d-2))$. Thus we get the assertion also in this case.

Now we turn to the remaining three terms in (3.5). We have

(3.6)
$$\begin{aligned} &- \operatorname{Ext}^*(\mathcal{O}(k_\alpha C), \mathcal{O}_{Z_\beta}(k_\beta C - \ell_\infty)) - \operatorname{Ext}^*(\mathcal{O}_{Z_\alpha}(k_\alpha C), \mathcal{O}(k_\beta C - \ell_\infty)) \\ &\qquad + \operatorname{Ext}^*(\mathcal{O}_{Z_\alpha}(k_\alpha C), \mathcal{O}_{Z_\beta}(k_\beta C - \ell_\infty)) \\ &= - \operatorname{Ext}^*(\mathcal{O}, \mathcal{O}_{Z_\beta}((k_\beta - k_\alpha)C - \ell_\infty)) - \operatorname{Ext}^*(\mathcal{O}_{Z_\alpha}((k_\alpha - k_\beta)C), \mathcal{O}(-\ell_\infty)) \\ &\qquad + \operatorname{Ext}^*(\mathcal{O}_{Z_\alpha}, \mathcal{O}_{Z_\beta}((k_\beta - k_\alpha)C - \ell_\infty)). \end{aligned}$$

As in the proof of Proposition 3.2, we have decomposition $Z_\alpha = Z_\alpha^1 \cup Z_\alpha^2$ according to the support p_1, p_2. Hence each of the remaining terms in (3.6) is the direct sum of the corresponding terms for Z_α^1, Z_β^1 and Z_α^2, Z_β^2. (A mixed term $\operatorname{Ext}^*(\mathcal{O}_{Z_\alpha^1}(k_\alpha C), \mathcal{O}_{Z_\beta^2}(k_\beta C - \ell_\infty))$ is obviously zero.) We study each summand separately.

First consider terms for Z^1_α, Z^1_β. We take a coordinate system $(x, y) = (z_1/z_0, w/z)$ as in the proof of Proposition 3.2. Since the divisor C is given by $x = 0$, the multiplication by x^m induces an isomorphism $\mathcal{O}_{Z^1_\alpha}(mC) \cong \mathcal{O}_{Z^1_\alpha}$ of sheaves for $m \in \mathbb{Z}$. It becomes an isomorphism of *equivariant* sheaves if we twist it as $\mathcal{O}_{Z^1_\alpha}(mC) \cong t_1^m \mathcal{O}_{Z^1_\alpha}$. Hence the summand of (3.6) for p_1 is equal to

$$
\begin{aligned}
t_1^{k_\beta - k_\alpha}\big(- \operatorname{Ext}^*(\mathcal{O}, \mathcal{O}_{Z^1_\beta}(-\ell_\infty)) - \operatorname{Ext}^*(\mathcal{O}_{Z^1_\alpha}, \mathcal{O}(-\ell_\infty)) \\
+ \operatorname{Ext}^*(\mathcal{O}_{Z^1_\alpha}, \mathcal{O}_{Z^1_\beta}(-\ell_\infty))\big).
\end{aligned}
\tag{3.7}
$$

Since Z^1_α is supported at the single point p_1, we can consider it as a subscheme of $\mathbb{P}^2$ supported at the origin $[1 : 0 : 0]$, where the T^2-action on $\mathbb{P}^2$ is given by $[z_0 : z_1 : z_2] \mapsto [z_0 : t_1 z_1 : t_2/t_1 z_2]$. Let I^1_α be the corresponding ideal sheaves of $\mathcal{O}_{\mathbb{P}^2}$. Using the $0 \to I^1_\alpha \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{Z^1_\alpha} \to 0$, we find that (3.7) is equal to

$$
t_1^{k_\beta - k_\alpha}\big(\operatorname{Ext}^*\left(I^1_\alpha, I^1_\beta\right) - \operatorname{Ext}^*(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(-\ell_\infty))\big).
$$

The second term $\operatorname{Ext}^*(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(-\ell_\infty))$ is zero. Thus we can use the formula in Theorem 2.11 after replacing (t_1, t_2) by $(t_1, t_2/t_1)$, and get the expression $M^1_{\alpha,\beta}(t_1, t_2)$ in the assertion.

The terms for Z^2_α, Z^2_β can be calculated in a similar way. We get $M^2_{\alpha,\beta}(t_1, t_2)$. □

For a future reference, we record the formula of the character:

Proposition 3.8.

$$
\operatorname{ch} H^1(\widehat{\mathbb{P}}^2, \mathcal{O}(-kC - \ell_\infty)) = \begin{cases} \displaystyle\sum_{\substack{i,j \geq 0 \\ i+j \leq k-1}} t_1^{-i} t_2^{-j} & \text{if } k > 0, \\ \displaystyle\sum_{\substack{i,j \geq 0 \\ i+j \leq -k-2}} t_1^{i+1} t_2^{j+1} & \text{if } k < -1, \\ 0 & \text{if } k = 0 \text{ or } -1. \end{cases}
$$

4. Sums over Young tableaux and Hilbert series

Although our main concern is about equivariant homology groups of moduli spaces, equivariant K-groups are more natural for an explanation of meaning of Nekrasov's partition function.

Let $K^{\widetilde{T}}(M(r, n))$ be the Grothendieck group of $\widetilde{T} = T^{r+2}$-equivariant coherent sheaves on $M(r, n)$ and similarly for $K^{\widetilde{T}}(\widehat{M}(r, k, n))$, $K^{\widetilde{T}}(M_0(r, n))$. These are modules over the representation ring $R(\widetilde{T})$ of the torus $\widetilde{T}$. As in 2.10, we identify it with the Laurent polynomial ring

$\mathbb{Z}[t_1^{\pm}, t_2^{\pm}, e_1^{\pm} \dots, e_r^{\pm}]$. Since $M(r,n)$ and $\widehat{M}(r,k,n)$ are nonsingular, $K^{\widetilde{T}}(M(r,n))$, $K^{\widetilde{T}}(\widehat{M}(r,k,n))$ are isomorphic to the Grothendieck groups of $\widetilde{T}$-equivariant locally free sheaves. In particular, they have the ring structures given by tensor products. For an equivariant proper morphism f between $\widetilde{T}$-varieties, we have induced homomorphism f_* between the Grothendieck groups given by the alternating sum of higher direct image sheaves $\sum_i (-1)^i R^i f_*$. In particular, we have

$$\pi_* \colon K^{\widetilde{T}}(M(r,n)) \to K^{\widetilde{T}}(M_0(r,n)),$$
$$\widehat{\pi}_* \colon K^{\widetilde{T}}(\widehat{M}(r,k,n)) \to K^{\widetilde{T}}(M_0(r,n)).$$

Let $\mathcal{R} = \mathbb{Q}(t_1, t_2, e_1, \dots, e_m)$ be the quotient field of $R(\widetilde{T})$. Let $\iota\colon M(r,n)^{\widetilde{T}} \to M(r,n)$ be the inclusion of the $\widetilde{T}$-fixed point set. By the localization theorem for the K-theory due to Thomason [54] (a prototype of the localization theorem was in [3]), it is known that the homomorphism ι_* is an isomorphism after the localization:

$$\iota_* \colon K^{\widetilde{T}}(M(r,n)^{\widetilde{T}}) \otimes_{R(\widetilde{T})} \mathcal{R} \xrightarrow{\cong} K^{\widetilde{T}}(M(r,n)) \otimes_{R(\widetilde{T})} \mathcal{R}.$$

Since $M(r,n)^{\widetilde{T}}$ consists of finitely many points $\{\vec{Y}\}$, and $K^{\widetilde{T}}$ of the point is isomorphic to the representation ring, the left hand side is the direct sum $\#\{\vec{Y}\}$-copies of $\mathcal{R}$. Similarly, $K^{\widetilde{T}}(\widehat{M}(r,k,n)) \otimes_{R(\widetilde{T})} \mathcal{R}$ is isomorphic to $\#\{(\vec{k}, \vec{Y}^1, \vec{Y}^2)\}$-copies of $\mathcal{R}$. On the other hand, $M_0(r,n)^{\widetilde{T}}$ consists of a single point $\{0\}$, hence $K^{\widetilde{T}}(M_0(r,n)) \otimes_{R(\widetilde{T})} \mathcal{R} \cong \mathcal{R}$.

The inverse of ι_* can be explicitly given by the following formula:

$$\iota_*^{-1}(\bullet) = \bigoplus_{\vec{Y}} \frac{\iota_{\vec{Y}}^*(\bullet)}{\bigwedge_{-1} T_{\vec{Y}}^* M(r,n)},$$

where $T_{\vec{Y}}^* M(r,n)$ is the cotangent bundle of $M(r,n)$ at a fixed point of $\vec{Y}$ considered as a $\widetilde{T}$-module, $\bigwedge_{-1}$ is the alternating sum of exterior powers, and $\iota_{\vec{Y}}^*$ is the pull-back homomorphism with respect to the inclusion $\iota_{\vec{Y}}\colon \{\vec{Y}\} \to M(r,n)$. Here the pull-back homomorphism is defined via the isomorphism of $K^{\widetilde{T}}(M(r,n))$ and the Grothendieck group of $\widetilde{T}$-equivariant *locally free sheaves*.

If $M(r,n)$ would be compact, we have a pushforward homomorphism $p_*\colon K^{\widetilde{T}}(M(r,n)) \to R(\widetilde{T})$ given by $p\colon M(r,n) \to \{pt\}$ and it can be computed by the Bott's formula:

$$p_*(\bullet) = \sum_{\vec{Y}} \frac{\iota_{\vec{Y}}^*(\bullet)}{\bigwedge_{-1} T_{\vec{Y}}^* M(r,n)}.$$

However $M(r,n)$ is noncompact, and p_* is not defined. In fact, cohomology groups $H^i(M(r,n),\bullet)$ may be infinite dimensional. But the right hand side makes sense as an element in $\mathcal{R}$. In fact, it computes the alternating sum of *Hilbert series* of cohomology groups:

Proposition 4.1. *Let E be a $\widetilde{T}$-equivariant coherent sheaf on $M(r,n)$. Then we have*

$$\sum_{i=0}^{2nr}(-1)^i \operatorname{ch} H^i(M(r,n),E) = \sum_{\vec{Y}} \frac{\iota_{\vec{Y}}^* E}{\bigwedge_{-1} T_{\vec{Y}}^* M(r,n)},$$

where ch *denotes the Hilbert series.*

Let us recall the definition of the Hilbert series. (See [9, §6.6] and the reference therein for more detailed account.) Let V be a representation of the torus $\widetilde{T}$. Let $V=\bigoplus V_\mu$ ($\mu\in X^*(\widetilde{T})$) be its weight space decomposition, i.e.,

$$V_\mu = \{v\in V \mid t\cdot v = \mu(t)v \text{ for } t\in\widetilde{T}\}.$$

Here $X^*(\widetilde{T})$ denotes the group of characters of $\widetilde{T}$. When the dimensions of weight spaces are finite dimensional, we define the character of V by

$$\operatorname{ch} V = \sum (\dim V_\mu) e^\mu.$$

We take coordinates $(t_1,t_2,e_1,\dots,e_r)\in\widetilde{T}$ as before, and we consider the right hand side as an element in the Laurent power series in t_1, t_2, $e_1,\dots,e_r$.

We want to apply this definition to the cohomology groups of a $\widetilde{T}$-equivariant coherent sheaf on $M(r,n)$. Since $M(r,n)$ is *not* projective and cohomology groups are *not* finite-dimensional in general, we first need to show that weight spaces are finite-dimensional and the above definition makes sense. For this purpose, we consider a $\widetilde{T}$-equivariant coherent sheaf E on the *affine algebraic variety* $M_0(r,n)$. Then the space $M=H^0(M_0(r,n),E)$ of global sections of E is identified with a finitely generated module over the coordinate ring of $M_0(r,n)$. (And the higher cohomology groups vanishes.) Let $M=\bigoplus M_\mu$ ($\mu\in X^*(\widetilde{T})$) be the weight space decomposition as above.

Lemma 4.2. *A weight space M_μ is finite-dimensional as a vector space over $\mathbb{C}$.*

Proof. By [37], the coordinate ring is generated by the following two types of elements

(1) $\operatorname{tr}(B_{\alpha_N}B_{\alpha_{N-1}}\cdots B_{\alpha_1}\colon V\to V)$,
(2) $\langle\chi, jB_{\alpha_N}B_{\alpha_{N-1}}\cdots B_{\alpha_1} i\rangle$,

where $\alpha_1, \dots, \alpha_N$ is 1 or 2 and χ is a linear form on $\operatorname{End}(W)$. Any of these elements is contained in a weight space with a *nonzero* weight. From this we get our assertion. □

Now the Hilbert series of E (or M) is defined by

$$\operatorname{ch} E \equiv \operatorname{ch} M = \sum_{\mu} (\dim M_\mu) e^{\mu}.$$

By a well-known argument on Hilbert series, one can show that $\operatorname{ch} E$ is a rational function, i.e., an element in $\mathcal{R}$.

Now we can return to the situation in Proposition 4.1. Let E be a $\widetilde{T}$-equivariant coherent sheaf on $M(r,n)$. Since $\pi\colon M(r,n) \to M_0(r,n)$ is a projective morphism, the higher direct image sheaves $R^i\pi_* E$ is an equivariant coherent sheaf on $M_0(r,n)$. The space of its global sections is the higher cohomology group $H^i(M(r,n), E)$. Thus we can consider the associated Hilbert series

$$\operatorname{ch} R^i\pi_* E \equiv \operatorname{ch} H^i(M(r,n), E).$$

Now we can finish the proof of Proposition 4.1 thanks to a general result of Thomason [54]. The argument appears in [24] for $r = 1$, and his argument can be applied to our situation, once the above property of the coordinate ring of $M_0(r,n)$ is established.

The proof follows from the commutativity of the following square

$$\begin{array}{ccc}
K^{\widetilde{T}}(M(r,n)) \otimes_{R(\widetilde{T})} \mathcal{R} & \xrightarrow[(\iota_*)^{-1}]{\cong} & \bigoplus_{\vec{Y}} \mathcal{R} \\
\Big\downarrow{\scriptstyle \pi_*} & & \Big\downarrow{\scriptstyle \sum_{\vec{Y}}} \\
K^{\widetilde{T}}(M_0(r,n)) \otimes_{R(\widetilde{T})} \mathcal{R} & \xrightarrow[(\iota_{0*})^{-1}]{\cong} & \mathcal{R}
\end{array}$$

and the observation $(\iota_{0*})^{-1} = \operatorname{ch}$, which is a consequence of a trivial identity $\operatorname{ch} \circ \iota_{0*} = \operatorname{id}$. Here ι_0 is the inclusion of the unique fixed point of $M_0(r,n)$.

Let us give two examples. Let $\mathcal{O}$ be the structure sheaf of $M(2,1)$. We directly check that Proposition 4.1 holds for $E = \mathcal{O}$. We have two fixed points $\vec{Y} = ([1],[\emptyset]), ([\emptyset],[1])$ in $M(2,1)$. The localization gives us

(4.3)
$$\frac{1}{(1-t_1)(1-t_2)(1-\frac{e_1}{e_2})(1-t_1t_2\frac{e_2}{e_1})} + \frac{1}{(1-t_1)(1-t_2)(1-\frac{e_2}{e_1})(1-t_1t_2\frac{e_1}{e_2})}$$
$$= \frac{1+t_1t_2}{(1-t_1)(1-t_2)(1-t_1t_2\frac{e_1}{e_2})(1-t_1t_2\frac{e_2}{e_1})}.$$

On the other hand, we have $M(2,1) \cong \mathbb{C}^2 \times T^*\mathbb{P}^1$. The $\mathbb{C}^2$-component is given by (B_1, B_2) and $T^*\mathbb{P}^1$-component is given by $(\operatorname{Ker} i, ji)$, where $\operatorname{Ker} i$

is a one-dimensional subspace in the two-dimensional space W, and $\xi = ji$ is an endomorphism of W satisfying $\xi(\operatorname{Ker} i) = 0$, $\operatorname{Im} \xi \subset \operatorname{Ker} i$. The higher cohomology groups $H^i(M(2,1), \mathcal{O}) = 0$ $(i > 0)$ vanish, and the global sections $H^0(M(2,1), \mathcal{O})$ is identified with

$$\mathbb{C}[x, y] \otimes (\mathbb{C}[s, t, u]/st = u^2),$$

where $x = B_1$, $y = B_2$, $s = j_1 i_2$, $t = j_2 i_1$, $u = j_1 i_1 = -j_2 i_2$ with $i = [\, i_1 \ i_2 \,]$, $j = \left[\begin{smallmatrix} j_1 \\ j_2 \end{smallmatrix}\right]$. Since weights of x, y, s, t, u are t_1, t_2, $t_1 t_2 e_1/e_2$, $t_1 t_2 e_2/e_1$, $t_1 t_2$ respectively, the character of $H^0(M(2,1), \mathcal{O})$ is also given by (4.3).

Remark 4.4. We have used the following convention on the action on the coordinate ring. Let $F_g \colon M(r,n) \to M(r,n)$ be the isomorphism given an element $g \in \widetilde{T}$. It induces a map F_g^* given by $f \mapsto f \circ F_g$ on the coordinate ring. The same applies to $H^i(M(r,n), E)$ for a $\widetilde{T}$-equivariant sheaf E. Accordingly when we apply Proposition 4.1, we make $\widetilde{T}$ acts on the cotangent space $T^*_{\vec{Y}} M(r,n)$ by $d(F_g)^*_{\vec{Y}}$.

Next consider the rank 1 case. The moduli space $M(1,n)$ is nothing but the Hilbert scheme $(\mathbb{C}^2)^{[n]}$ of n points in $\mathbb{C}^2$. We apply Proposition 4.1 to the structure sheaf $\mathcal{O}$ of $M(1,n)$. The fixed points are parametrized by Young diagrams Y of size n as Proposition 2.9. The weights of tangent spaces at fixed points is given by the formula 2.11. In particular, the localization gives us

$$\sum_{|Y|=n} \frac{1}{\prod_{s \in Y} \left(1 - t_1^{-l(s)} t_2^{1+a(s)}\right)\left(1 - t_1^{1+l(s)} t_2^{-a(s)}\right)}.$$

On the other hand, we have $H^0((\mathbb{C}^2)^{[n]}, \mathcal{O}) = H^0(S^n(\mathbb{C}^2), \mathcal{O}) = \mathbb{C}[x_1, y_1, \dots, x_n, y_n]^{S_n}$, where S_n acts by permuting $(x_1, y_1), \dots, (x_n, y_n)$. Higher cohomology groups $H^i((\mathbb{C}^2)^{[n]}, \mathcal{O})$ $(i > 0)$ vanish since $S^n\mathbb{C}^2$ is a rational singularity. Now $\mathbb{C}[x_1, y_1, \dots, x_n, y_n]^{S_n}$ is isomorphic to $S^n(\mathbb{C}[x, y])$, and the generating function of the Hilbert series is given by

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathfrak{q}^n \operatorname{ch} H^0(S^n(\mathbb{C}^2), \mathcal{O}) &= \prod_{p_1, p_2 \geq 0} \frac{1}{1 - t_1^{p_1} t_2^{p_2} \mathfrak{q}} \\
&= \exp\left(- \sum_{p_1, p_2 \geq 0} \log\left(1 - t_1^{p_1} t_2^{p_2} \mathfrak{q}\right) \right) \\
&= \exp\left(\sum_{p_1, p_2 \geq 0} \sum_{r=1}^{\infty} \frac{t_1^{r p_1} t_2^{r p_2} \mathfrak{q}^r}{r} \right) = \exp\left(\sum_{r=1}^{\infty} \frac{\mathfrak{q}^r}{(1 - t_1^r)(1 - t_2^r) r} \right).
\end{aligned}$$

Thus we get

$$(4.5)\qquad \sum_Y \frac{\mathfrak{q}^{|Y|}}{\prod_{s\in Y}\left(1-t_1^{-l(s)}t_2^{1+a(s)}\right)\left(1-t_1^{1+l(s)}t_2^{-a(s)}\right)} = \exp\left(\sum_{r=1}^{\infty}\frac{\mathfrak{q}^r}{\left(1-t_1^r\right)\left(1-t_2^r\right)r}\right).$$

A purely combinatorial proof of this identity can be found in [39, VI]. A different geometric proof can be found in [24, Lemma 3.2]. It also uses geometry of Hilbert schemes.

Let $H_*^{\widetilde{T}}(M(r,n))$ be the $\widetilde{T}$-equivariant Borel-Moore homology group of $M(r,n)$ with rational coefficients. We define it as in [36, §2.8], but we assign the degree as in [15] so that the fundamental class $[M(r,n)]$ has degree $2\dim M(r,n) = 4rn$.

Let us recall the definition briefly. We have a finite dimensional approximation of the classifying space $E\widetilde{T} \to B\widetilde{T}$, i.e., for any n, there exists a smooth irreducible variety U with $\widetilde{T}$-action such that

a) The quotient $U \to U/\widetilde{T}$ exists and is a principal $\widetilde{T}$-bundle.
b) $H^i(U) = 0$ for $i = 1, \dots, n$.

We then define

$$H_n^{\widetilde{T}}(X) = H_{n-2\dim\widetilde{T}+2\dim U}(X \times_{\widetilde{T}} U),$$

where $H_*(\ )$ in the right hand side is the Borel-Moore homology group (see e.g., [21, §B.2]). Note that U is smooth, and $\dim U$ makes sense. One can show that this is independent of the choice of U, using the double fibration argument.

The equivariant homology group is a module over the usual equivariant cohomology of a point $H^*_{\widetilde{T}}(pt)$. The latter is the symmetric algebra of the dual of the Lie algebra of $\widetilde{T}$, which we denote by $S(\widetilde{T})$. We choose its generators $\varepsilon_1, \varepsilon_2, a_1, \dots, a_r$ corresponding to $t_1, t_2, e_1, \dots, e_r$ respectively. We use the vector notation $\vec{a}$ for $(a_1, \dots, a_r)$. We have $H_k^{\widetilde{T}}(X) = 0$ if $k > 2\dim X$, but $H_k^{\widetilde{T}}(X)$ may be nonzero for $k < 0$.

The results given in this section have counterparts for equivariant homology groups. For example, we have a commutative diagram

$$\begin{array}{ccc} H_*^{\widetilde{T}}(M(r,n)) \otimes_{S(\widetilde{T})} \mathcal{S} & \xrightarrow[(\iota_*)^{-1}]{\cong} & \bigoplus_{\vec{Y}} \mathcal{S} \\ {\scriptstyle \pi_*}\downarrow & & \downarrow{\scriptstyle \Sigma_{\vec{Y}\bullet}} \\ H_*^{\widetilde{T}}(M_0(r,n)) \otimes_{S(\widetilde{T})} \mathcal{S} & \xrightarrow[(\iota_{0*})^{-1}]{\cong} & \mathcal{S}, \end{array}$$

where $\mathcal{S}$ is the quotient field of $S(\widetilde{T})$. The proof of the localization theorem for equivariant Borel-Moore homology can be found, for example, in [38, 4.4]. We have

$$\sum_{\vec{Y}} \frac{\iota_{\vec{Y}}^* \alpha}{e(T_{\vec{Y}})} = (\iota_{0*})^{-1} \pi_*(\alpha).$$

Further more, the right hand side has an interpretation as the *equivariant Hilbert polynomial* of E if α is the Chern character of a vector bundle E. For example, we have the following for $E = \mathcal{O}$:

$$(4.6)\quad \begin{aligned} (\iota_{0*})^{-1}\pi_*[M(r,n)] &= \sum_{\vec{Y}} \frac{1}{e(T_{\vec{Y}})} = \lim_{t\to 0} \sum_{\vec{Y}} \frac{t^{2nr}}{\bigwedge_{-1} T^*_{\vec{Y}}} \Bigg|_{\substack{t_1 = e^{-t\varepsilon_1}, t_2 = e^{-t\varepsilon_2} \\ e_\alpha = e^{-ta_\alpha}}} \\ &= \lim_{t\to 0} t^{2nr} \sum_{i=0}^{2nr} (-1)^i \operatorname{ch} H^i(M(r,n), \mathcal{O}) \Bigg|_{\substack{t_1 = e^{-t\varepsilon_1}, t_2 = e^{-t\varepsilon_2} \\ e_\alpha = e^{-ta_\alpha}}}. \end{aligned}$$

This is our interpretation of Nekrasov's partition function mentioned in the introduction. For example, for $r = 1$, we can derive the following from (4.5):

$$(4.7)\quad \begin{aligned} &\sum_Y \frac{\mathfrak{q}^{|Y|}}{\prod_{s\in Y} \{-l_Y(s)\varepsilon_1 + (1 + a_Y(s))\varepsilon_2\}\{(1 + l_Y(s))\varepsilon_1 - a_Y(s)\varepsilon_2\}} \\ &\qquad = \exp\left(\frac{\mathfrak{q}}{\varepsilon_1\varepsilon_2}\right). \end{aligned}$$

As in the proof of (4.5), we can directly obtain the right hand side as follows. We use localization on $M_0(1,n) = S^n(\mathbb{C}^2)$, instead of $M(1,n) = (\mathbb{C}^2)^{[n]}$. The point is that $S^n(\mathbb{C}^2)$ is an orbifold, and hence has an explicit formula of $(\iota_{0*})^{-1}$. This formula justifies the following definition of 'generating spaces':

$$\exp(\mathfrak{q}\mathbb{C}^2) = \sum_{n=0}^{\infty} \mathfrak{q}^n S^n(\mathbb{C}^2), \qquad \text{or } \mathfrak{q}\mathbb{C}^2 = \log\left(\sum_{n=0}^{\infty} \mathfrak{q}^n S^n(\mathbb{C}^2)\right).$$

5. Rank 1 case

This section is a detour. We study Nekrasov's partition function and its analog for blowup in the rank 1 case.

The partition function for rank 1 is

$$Z(\varepsilon_1, \varepsilon_2; \mathfrak{q}) = \sum_{n=0}^{\infty} \mathfrak{q}^n Z_n(\varepsilon_1, \varepsilon_2) = \sum_{n=0}^{\infty} \mathfrak{q}^n (\iota_{0*})^{-1} \pi_*[\operatorname{Hilb}^n \mathbb{C}^2],$$

where $\pi\colon \operatorname{Hilb}^n \mathbb{C}^2 \to S^n\mathbb{C}^2$ is the Hilbert-Chow morphism and ι_0 is the inclusion of the unique fixed point $n[0]$ in $S^n\mathbb{C}^2$. By Theorem 2.11 this is equal to (4.7).

Next we consider the Hilbert scheme $\operatorname{Hilb}^n \widehat{\mathbb{C}}^2$ of n points on the blowup $\widehat{\mathbb{C}}^2$. The fixed points with respect to the $\mathbb{C}^* \times \mathbb{C}^*$-action are parametrized by pairs of Young diagrams (Y^1, Y^2) by Proposition 3.2.

Let $\mu(C) \in H^2_{\mathbb{C}^*\times\mathbb{C}^*}(\operatorname{Hilb}^n \widehat{\mathbb{C}}^2)$ be the class attached to the exceptional divisor C. (See the next section for the definition.) We then define the partition function on the blowup by

$$\begin{aligned}\widehat{Z}(\varepsilon_1, \varepsilon_2; t; \mathfrak{q}) &= \sum_{n=0}^{\infty} \mathfrak{q}^n \sum_{d=0}^{\infty} \frac{t^d}{d!} \widehat{Z}_{n,d}(\varepsilon_1, \varepsilon_2) \\ &= \sum_{n=0}^{\infty} \mathfrak{q}^n \sum_{d=0}^{\infty} \frac{t^d}{d!} (\iota_{0*})^{-1} \widehat{\pi}_* \left(\mu(C)^d \cap [\operatorname{Hilb}^n \widehat{\mathbb{C}}^2]\right),\end{aligned}$$

where $\widehat{\pi}$ is the composite of the Hilbert-Chow morphism $\operatorname{Hilb}^n \widehat{\mathbb{C}}^2 \to S^n\widehat{\mathbb{C}}^2$ and the morphism $S^n\widehat{\mathbb{C}}^2 \to S^n\mathbb{C}^2$.

By the Lemma 6.8 below, we have

$$\iota^*_{(Y^1,Y^2)}\mu(C) = |Y^1|\varepsilon_1 + |Y^2|\varepsilon_2.$$

Together with Theorem 3.4 we have

$$\sum_{d=0}^{\infty} \frac{t^d}{d!} \sum_{(Y^1,Y^2)} \frac{\left(|Y^1|\varepsilon_1 + |Y^2|\varepsilon_2\right)^d q^{|Y^1|+|Y^2|}}{n_{Y^1}(\varepsilon_1, \varepsilon_2 - \varepsilon_1)\, n_{Y^2}(\varepsilon_1 - \varepsilon_2, \varepsilon_2)},$$

where $n_Y(\varepsilon_1, \varepsilon_2)$ is the denominator of (4.7). This is equal to

$$\begin{aligned}&\sum_{(Y^1,Y^2)} \frac{(\mathfrak{q}e^{t\varepsilon_1})^{|Y^1|}\, (\mathfrak{q}e^{t\varepsilon_2})^{|Y^2|}}{n_{Y^1}(\varepsilon_1, \varepsilon_2 - \varepsilon_1)\, n_{Y^2}(\varepsilon_1 - \varepsilon_2, \varepsilon_2)} \\ &= Z\big(\varepsilon_1, \varepsilon_2 - \varepsilon_1; \mathfrak{q}e^{t\varepsilon_1}\big) Z\big(\varepsilon_1 - \varepsilon_2, \varepsilon_2; \mathfrak{q}e^{t\varepsilon_2}\big) \\ &= \exp\left(\frac{\mathfrak{q}e^{t\varepsilon_1}}{\varepsilon_1(\varepsilon_2 - \varepsilon_1)} + \frac{\mathfrak{q}e^{t\varepsilon_2}}{(\varepsilon_1 - \varepsilon_2)\varepsilon_2}\right).\end{aligned}$$

We divide this by $Z(\varepsilon_1, \varepsilon_2; \mathfrak{q})$ and take the limit $\varepsilon_1, \varepsilon_2 \to 0$:

$$\lim_{\varepsilon_1,\varepsilon_2\to 0} \frac{\widehat{Z}(\varepsilon_1, \varepsilon_2; t; \mathfrak{q})}{Z(\varepsilon_1, \varepsilon_2; \mathfrak{q})} = \exp\left(-\frac{\mathfrak{q}t^2}{2}\right).$$

This is a prototype of the blowup formula which will be discussed in Sect. 8. It should be noticed that this is equal to the generating function of

$\int_{\mathrm{Hilb}^n \widehat{X}} \mu(C)^{2n}$ for arbitrary smooth surface X. The minus sign comes from the self-intersection number of C: $[C]^2 = -1$. This can be shown roughly as follows: first show that $\mu(C)$ is a pull-back of a class in $S^n\widehat{X}$ via the Hilbert-Chow morphism $\mathrm{Hilb}^n \widehat{X} \to S^n\widehat{X}$. Then the intersection numbers are those on $\widehat{X}^n$ divided by $n!$. The class $\mu(C)$ corresponds to $\sum_i p_i^*[C]$, where $p_i \colon \widehat{X}^n \to \widehat{X}$ is the ith projection.

6. Instanton counting

We define *the partition function* as the following generating function:

$$Z(\varepsilon_1, \varepsilon_2, \vec{a}; \mathfrak{q}) = \sum_{n=0}^{\infty} \mathfrak{q}^n Z_n(\varepsilon_1, \varepsilon_2, \vec{a}) = \sum_{n=0}^{\infty} \mathfrak{q}^n (\iota_{0*})^{-1}\pi_*[M(r,n)], \tag{6.1}$$

where $[M(r,n)]$ denote the fundamental class of $H_*^{\widetilde{T}}(M(r,n))$. As we explained, this has an expression in terms of Hilbert series (4.6). By (the equivariant homology analog of) Proposition 4.1 together with Theorem 2.11, we have

$$Z(\varepsilon_1, \varepsilon_2, \vec{a}; \mathfrak{q}) = \sum_{\vec{Y}} \frac{\mathfrak{q}^{|\vec{Y}|}}{e(T_{\vec{Y}})} = \sum_{\vec{Y}} \frac{\mathfrak{q}^{|\vec{Y}|}}{\prod_{\alpha,\beta} n^{\vec{Y}}_{\alpha,\beta}(\varepsilon_1, \varepsilon_2, \vec{a})}, \tag{6.2}$$

where

$$n^{\vec{Y}}_{\alpha,\beta}(\varepsilon_1, \varepsilon_2, \vec{a}) = \prod_{s \in Y_\alpha} \left(-l_{Y_\beta}(s)\varepsilon_1 + (a_{Y_\alpha}(s)+1)\varepsilon_2 + a_\beta - a_\alpha\right) \times \prod_{t \in Y_\beta} \left((l_{Y_\alpha}(t)+1)\varepsilon_1 - a_{Y_\beta}(t)\varepsilon_2 + a_\beta - a_\alpha\right).$$

This is nothing but Nekrasov's definition of the partition function [50, (1.6), (3.20)]. We set

$$F^{\mathrm{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}; \mathfrak{q}) = \sum_{n=1}^{\infty} \mathfrak{q}^n F^{\mathrm{inst}}_n(\varepsilon_1, \varepsilon_2, \vec{a}) = \varepsilon_1\varepsilon_2 \log Z(\varepsilon_1, \varepsilon_2, \vec{a}; \mathfrak{q}).$$

We give elementary properties of the partition function.

Lemma 6.3. (1) $Z(\varepsilon_2, \varepsilon_1, \vec{a}; \mathfrak{q}) = Z(\varepsilon_1, \varepsilon_2, \vec{a}; \mathfrak{q})$.
(2) $Z(\varepsilon_1, \varepsilon_2, w \cdot \vec{a}; \mathfrak{q}) = Z(\varepsilon_1, \varepsilon_2, \vec{a}; \mathfrak{q})$ *where w is an element of the symmetric group of r letters.*
(3) $Z(-\varepsilon_1, -\varepsilon_2, -\vec{a}; \mathfrak{q}) = Z(\varepsilon_1, \varepsilon_2, \vec{a}; \mathfrak{q})$.

Proof. (1) The exchange of ε_1 and ε_2 is compensated with the exchange of the Young diagram Y_α with its conjugate Y'_α.

(2) The exchange of a_α and a_β is compensated with the exchange of Y_α and Y_β.

(3) Clear from $n^{\vec{Y}}_{\alpha,\beta}(-\varepsilon_1, -\varepsilon_2, -\vec{a}) = (-1)^{|Y_\alpha|+|Y_\beta|} n^{\vec{Y}}_{\alpha,\beta}(\varepsilon_1, \varepsilon_2, \vec{a})$. □

We now consider the moduli spaces on the blowup. By Theorem 3.4 the Euler class of the tangent space of $\widehat{M}(r,k,n)$ at a fixed point $(\vec{k}, \vec{Y}^1, \vec{Y}^2)$ is given by

(6.4)
$$\prod_{\alpha,\beta} l^{\vec{k}}_{\alpha,\beta}(\varepsilon_1, \varepsilon_2, \vec{a})\, n^{\vec{Y}^1}_{\alpha,\beta}(\varepsilon_1, \varepsilon_2 - \varepsilon_1, \vec{a} + \varepsilon_1 \vec{k})\, n^{\vec{Y}^2}_{\alpha,\beta}(\varepsilon_1 - \varepsilon_2, \varepsilon_2, \vec{a} + \varepsilon_2 \vec{k}),$$

where

$$l^{\vec{k}}_{\alpha,\beta}(\varepsilon_1, \varepsilon_2, \vec{a}) = \begin{cases} \displaystyle\prod_{\substack{i,j\geq 0\\ i+j\leq k_\alpha - k_\beta - 1}} (-i\varepsilon_1 - j\varepsilon_2 + a_\beta - a_\alpha) & \text{if } k_\alpha > k_\beta, \\ \displaystyle\prod_{\substack{i,j\geq 0\\ i+j\leq k_\beta - k_\alpha - 2}} \big((i+1)\varepsilon_1 + (j+1)\varepsilon_2 + a_\beta - a_\alpha\big) & \text{if } k_\alpha + 1 < k_\beta, \\ 1 & \text{otherwise.} \end{cases}$$

Note that $l^{\vec{k}}_{\alpha,\beta}(\varepsilon_1, \varepsilon_2, \vec{a})$ is independent of $\vec{Y}^1, \vec{Y}^2$.

From now we use terminology for root systems of Lie algebras as in Sect. 1, i.e., $\alpha_i \in \mathfrak{h}^*$, $\alpha_i^\vee \in \mathfrak{h}$, $\vec{a} = \sum a^i \alpha_i^\vee$, etc. Recall that Q is the coroot lattice $\{(k_1, \dots, k_r) \in \mathbb{Z}^r \mid \sum_\alpha k_\alpha = 0\}$. In order to treat the case $k = \sum_\alpha k_\alpha \neq 0$ (this means that the gauge group is $\mathrm{PU}(n)$ rather than $\mathrm{SU}(n)$), we consider a normalization $\vec{l} = (k_1 - \frac{k}{r}, \dots, k_r - \frac{k}{r})$ as an element of the coweight lattice $P = \{\vec{l} = (l_1, \dots, l_r) \in \mathbb{Q}^r \mid \sum_\alpha l_\alpha = 0, \exists k \in \mathbb{Z}\ \forall \alpha\ l_\alpha \equiv -\frac{k}{r} \mod \mathbb{Z}\}$. There exists a homomorphism $P \to \mathbb{Z}/r\mathbb{Z}$ by taking the fractional part of l_α. It can be identified with the natural quotient homomorphism $P \to P/Q$. We denote it by $\vec{l} \mapsto \{\vec{l}\}$. Hereafter we identify $\vec{l}$ with $\vec{k}$ and denote both by $\vec{k}$. We write $\vec{k} = \sum_i k^i \alpha_i^\vee$ in either case $k = 0$, $\neq 0$. But k^i may be rational in the latter case. Let $(\ ,\ )$ be the standard inner product on $\mathfrak{h}$. The Killing form $B_{\mathrm{SU}(r)}$ of $\mathrm{SU}(r)$ satisfies $B_{\mathrm{SU}(r)} = 2r(\ ,\ )$. The following formulas are useful later:

(6.5)
$$\frac{1}{2r}\sum_{\alpha,\beta}(k_\alpha - k_\beta)(a_\alpha - a_\beta) = (\vec{k}, \vec{a}) = \sum_{ij} C_{ij} a^i k^j,$$
$$\frac{1}{2r}\sum_{\alpha,\beta}(k_\alpha - k_\beta)^2 = (\vec{k}, \vec{k}) = \sum_{i,j} C_{ij} k^i k^j,$$
$$\sum_{\alpha<\beta} \frac{k_\alpha - k_\beta}{2} = \langle \vec{k}, \rho\rangle = \sum_i k^i.$$

Here C_{ij} is the Cartan matrix, and ρ is the half of the sum of positive roots, as usual.

For a root $\alpha \in \Delta$, we define

$$l_\alpha^{\vec{k}}(\varepsilon_1, \varepsilon_2, \vec{a}) = \begin{cases} \prod_{\substack{i,j \geq 0 \\ i+j \leq -\langle \vec{k}, \alpha \rangle - 1}} (-i\varepsilon_1 - j\varepsilon_2 + \langle \vec{a}, \alpha \rangle) & \text{if } \langle \vec{k}, \alpha \rangle < 0, \\ \prod_{\substack{i,j \geq 0 \\ i+j \leq \langle \vec{k}, \alpha \rangle - 2}} \big((i+1)\varepsilon_1 + (j+1)\varepsilon_2 + \langle \vec{a}, \alpha \rangle\big) & \text{if } \langle \vec{k}, \alpha \rangle > 1, \\ 1 & \text{otherwise,} \end{cases} \tag{6.6}$$

where $l_{\alpha,\beta}^{\vec{k}}$ in the previous notation corresponds to $l_{e_{\beta,\alpha}}^{\vec{k}}$.

The following will be useful later:

Lemma 6.7. (1) $l_\alpha^{\vec{k}}(\varepsilon_1, \varepsilon_2, \vec{a}) = l_\alpha^{\vec{k}}(\varepsilon_2, \varepsilon_1, \vec{a})$.
(2) $l_\alpha^{\vec{k}}(\varepsilon_1, \varepsilon_2, \vec{a}) = (-1)^{\langle \vec{k}, \alpha \rangle (\langle \vec{k}, \alpha \rangle - 1)/2} \, l_{-\alpha}^{-\vec{k}}(-\varepsilon_1, -\varepsilon_2, \vec{a})$.
(3) $l_\alpha^{\vec{k}}(\varepsilon_1, \varepsilon_2, \vec{a})$ *is regular at* $(\varepsilon_1, \varepsilon_2) = 0$ *and*

$$l_\alpha^{\vec{k}}(0, 0, \vec{a}) = \langle \vec{a}, \alpha \rangle^{\langle \vec{k}, \alpha \rangle (\langle \vec{k}, \alpha \rangle - 1)/2}$$

Let $\mathcal{E}$ be a universal sheaf on $\widehat{\mathbb{P}}^2 \times \widehat{M}(r, k, n)$. We define an equivariant cohomology class $\mu(C) \in H^2_{\widetilde{T}}(\widehat{M}(r, k, n))$ by

$$\left(c_2(\mathcal{E}) - \frac{r-1}{2r} c_1(\mathcal{E})^2 \right) / [C],$$

where $/$ denotes the slant product $/ \colon H^d_{\widetilde{T}}(\widehat{\mathbb{P}}^2 \times \widehat{M}(r, k, n)) \otimes H_i^{\widetilde{T}}(\widehat{\mathbb{P}}^2) \to H^{d-i}_{\widetilde{T}}(\widehat{M}(r, k, n))$. Note that we have $c_2(\mathcal{E}) - \frac{r-1}{2r} c_1(\mathcal{E})^2 = \frac{1}{2r} c_2(\operatorname{End} \mathcal{E})$ on the open locus $\widehat{M}^{\mathrm{reg}}(r, k, n)$.

Let $\iota_{(\vec{k}, \vec{Y}^1, \vec{Y}^2)}$ be the inclusion of the fixed point $(\vec{k}, \vec{Y}^1, \vec{Y}^2)$ into $\widehat{M}(r, k, n)$.

Lemma 6.8.

$$\iota^*_{(\vec{k}, \vec{Y}^1, \vec{Y}^2)} \mu(C) = |\vec{Y}^1| \, \varepsilon_1 + |\vec{Y}^2| \, \varepsilon_2 + (\vec{k}, \vec{a}) + \frac{(\vec{k}, \vec{k})}{2} (\varepsilon_1 + \varepsilon_2).$$

Proof. Let E be a sheaf corresponding to the fixed point $(\vec{k}, \vec{Y}^1, \vec{Y}^2)$. We have

$$c_2(E) - \frac{r-1}{2r} c_1(E)^2 = |\vec{Y}^1| \, [p_1] + |\vec{Y}^2| \, [p_2] + c_2(E^{\vee\vee}) - \frac{r-1}{2r} c_1(E^{\vee\vee})^2.$$

The double dual $E^{\vee\vee}$ is a direct sum $\bigoplus_\alpha \mathcal{O}_X(k_\alpha C)e_\alpha$. Therefore

$$c_2(E^{\vee\vee}) - \frac{r-1}{2r}c_1(E^{\vee\vee})^2 = \frac{1}{2r}c_2(\operatorname{End} E^{\vee\vee})$$
$$= -\frac{1}{2r}\sum_{\alpha<\beta}\left\{(k_\alpha[C]+a_\alpha)-(k_\beta[C]+a_\beta)\right\}^2.$$

Substituting

$$\int_{\widehat{\mathbb{P}}^2}[p_1][C]=\varepsilon_1,\quad \int_{\widehat{\mathbb{P}}^2}[p_2][C]=\varepsilon_2,\quad \int_{\widehat{\mathbb{P}}^2}[C]=0,$$
$$\int_{\widehat{\mathbb{P}}^2}[C]^2=-1,\quad \int_{\widehat{\mathbb{P}}^2}[C]^3=-(\varepsilon_1+\varepsilon_2),$$

into this, we get

$$\iota^*_{(\vec{k},\vec{Y}^1,\vec{Y}^2)}\mu(C) = |\vec{Y}^1|\,\varepsilon_1 + |\vec{Y}^2|\,\varepsilon_2$$
$$+\frac{1}{2r}\sum_{\alpha<\beta}\left(2(k_\alpha-k_\beta)(a_\alpha-a_\beta)+(k_\alpha-k_\beta)^2(\varepsilon_1+\varepsilon_2)\right).$$

This is the desired formula thanks to (6.5). □

We now define the partition function on the blowup:

$$\widehat{Z}^k(\varepsilon_1,\varepsilon_2,\vec{a};t;\mathfrak{q}) = \sum_n \mathfrak{q}^n\sum_{d=0}^\infty \frac{t^d}{d!}\widehat{Z}^k_{n,d}(\varepsilon_1,\varepsilon_2,\vec{a})$$
$$=\sum_n \mathfrak{q}^n\sum_{d=0}^\infty \frac{t^d}{d!}(\iota_{0*})^{-1}\widehat{\pi}_*\left(\mu(C)^d\cap[\widehat{M}(r,k,n)]\right),$$

where n runs over $\mathbb{Z}_{\geq 0}-\frac{1}{2r}k(r-k)$. By (6.4, 6.8) this can be represented in terms of Nekrasov's partition function:

(6.9)
$$\widehat{Z}^k_{n,d}(\varepsilon_1,\varepsilon_2,\vec{a})$$
$$=\sum_{\frac{1}{2}(\vec{k},\vec{k})+l+m=n}\left(l\varepsilon_1+m\varepsilon_2+(\vec{k},\vec{a})+\frac{(\vec{k},\vec{k})}{2}(\varepsilon_1+\varepsilon_2)\right)^d\frac{1}{\prod_{\alpha\in\Delta}l^{\vec{k}}_\alpha(\varepsilon_1,\varepsilon_2,\vec{a})}$$
$$\times Z_l(\varepsilon_1,\varepsilon_2-\varepsilon_1,\vec{a}+\varepsilon_1\vec{k})Z_m(\varepsilon_1-\varepsilon_2,\varepsilon_2,\vec{a}+\varepsilon_2\vec{k}).$$

The generating function is

(6.10)
$$\widehat{Z}^k(\varepsilon_1,\varepsilon_2,\vec{a};t;\mathfrak{q})$$
$$=\sum_{\{\vec{k}\}=-\frac{k}{r}}\exp\left[t\left((\vec{k},\vec{a})+\frac{(\vec{k},\vec{k})}{2}(\varepsilon_1+\varepsilon_2)\right)\right]\frac{\mathfrak{q}^{\frac{1}{2}(\vec{k},\vec{k})}}{\prod_{\alpha\in\Delta}l^{\vec{k}}_\alpha(\varepsilon_1,\varepsilon_2,\vec{a})}$$
$$\times Z\left(\varepsilon_1,\varepsilon_2-\varepsilon_1,\vec{a}+\varepsilon_1\vec{k};\mathfrak{q}e^{t\varepsilon_1}\right)Z\left(\varepsilon_1-\varepsilon_2,\varepsilon_2,\vec{a}+\varepsilon_2\vec{k};\mathfrak{q}e^{t\varepsilon_2}\right).$$

We now only consider the case $k = 0$ for a while. We omit the superscript in this case.

Proposition 6.11. (1) $\widehat{\pi}_*[\widehat{M}(r,0,n)] = [M_0(r,n)]$.
(2) $\widehat{\pi}_*\left(\mu(C)^d \cap [\widehat{M}(r,0,n)]\right) = 0$ *for* $1 \le d \le 2r-1$.

Proof. Both results are well-known in Donaldson theory (see e.g., [20, 3.8.1]). We give a proof for the completeness.

(1) By the dimension reason, the inclusion i of $M_0^{\mathrm{reg}}(r,n)$ in $M_0(r,n)$ induces an isomorphism in degree $4nr$:

$$H_{4nr}^{\widetilde{T}}(M_0(r,n)) \underset{\cong}{\xrightarrow{i^*}} H_{4nr}^{\widetilde{T}}\big(M_0^{\mathrm{reg}}(r,n)\big).$$

Therefore it is enough to show that $i^*\widehat{\pi}_*([\widehat{M}(r,0,n)]) = [M_0^{\mathrm{reg}}(r,n)]$. But this is clear since $\widehat{\pi}$ becomes an isomorphism over the set $M_0^{\mathrm{reg}}(r,n)$.

(2) First note that $\mu(C)$ is equal to $c_1(\mathcal{L})$, where $\mathcal{L}$ is the determinant line bundle over $\widehat{M}(r,0,n)$ where the fiber over (E,Φ) is

$$\left(\Lambda^{\max} H^1\big(\widehat{\mathbb{P}}^2, E(-\ell_\infty)\big)\right)^* \otimes \Lambda^{\max} H^1\big(\widehat{\mathbb{P}}^2, E(C-\ell_\infty)\big).$$

This line bundle has a natural section s whose zero set is a representative of $\mu(C) = c_1(\mathcal{L})$ and consists of bundles that restrict to C in a non-trivial way. (See [8, 4.6].)

Consider

$$\overline{\{0\} \times M_0^{\mathrm{reg}}(r,n-1)}.$$

This has complex codimension $2r$. Therefore if $i\colon U \to M_0(r,n)$ denote the inclusion of the complement, the pullback homomorphism i^* is an isomorphism in degree $\ge 4nr - 4r + 2$. Therefore we can restrict $\widehat{\pi}$ to U as in (1). Now the vanishing is clear since the section s of $\mathcal{L}$ does not vanish there as explained above. □

If we apply this to (6.9) with $d = 1, 2$, we get

$$n\varepsilon_1 Z_n(\varepsilon_1, \varepsilon_2-\varepsilon_1, \vec{a}) + n\varepsilon_2 Z_n(\varepsilon_1-\varepsilon_2, \varepsilon_2, \vec{a}) = A,$$
$$n^2\varepsilon_1^2 Z_n(\varepsilon_1, \varepsilon_2-\varepsilon_1, \vec{a}) + n^2\varepsilon_2^2 Z_n(\varepsilon_1-\varepsilon_2, \varepsilon_2, \vec{a}) = B,$$

where A and B are given by lower terms $l, m < n$ (but $\vec{a}$ may be shifted by $\vec{k}$). Therefore we can determine $Z_n(\varepsilon_1, \varepsilon_2-\varepsilon_1, \vec{a})$, $Z_n(\varepsilon_1-\varepsilon_2, \varepsilon_2, \vec{a})$ recursively. Changing ε_2 by $\varepsilon_1+\varepsilon_2$, we get $Z_n(\varepsilon_1, \varepsilon_2, \vec{a})$.

In order to express these assertions by differential equations, we introduce the following generalization of the Hirota differential:

$$\left(D_x^{(\varepsilon_1,\varepsilon_2)}\right)^m (f\cdot g) = \left(\frac{d}{dy}\right)^m f(x+\varepsilon_1 y) g(x+\varepsilon_2 y)\Big|_{y=0}$$
$$= \sum_{k=0}^m \varepsilon_1^k \varepsilon_2^{m-k} \binom{m}{k} \frac{d^k f}{dx^k}\frac{d^{m-k} g}{dx^{m-k}}.$$

$(D_x^{(1,-1)})^m$ is the ordinary Hirota differential. We have

$$
\begin{aligned}
&Z\big(\varepsilon_1, \varepsilon_2-\varepsilon_1, \vec{a}+\varepsilon_1\vec{k}; \mathfrak{q}e^{t\varepsilon_1}\big) Z\big(\varepsilon_1-\varepsilon_2, \varepsilon_2, \vec{a}+\varepsilon_2\vec{k}; \mathfrak{q}e^{t\varepsilon_2}\big)\\
&= \exp\big(tD_{\log\mathfrak{q}}^{(\varepsilon_1,\varepsilon_2)}\big)\Big(Z(\varepsilon_1, \varepsilon_2-\varepsilon_1, \vec{a}+\varepsilon_1\vec{k}; \mathfrak{q}) \cdot Z(\varepsilon_1-\varepsilon_2, \varepsilon_2, \vec{a}+\varepsilon_2\vec{k}; \mathfrak{q})\Big)
\end{aligned}
$$

Corollary 6.12. *The followings hold:*

(6.13) $Z(\varepsilon_1, \varepsilon_2, \vec{a}; \mathfrak{q})$

$$
= \sum_{\vec{k}} \frac{\mathfrak{q}^{\frac{1}{2}(\vec{k},\vec{k})}}{\prod_{\alpha\in\Delta} l_\alpha^{\vec{k}}(\varepsilon_1, \varepsilon_2, \vec{a})} Z(\varepsilon_1, \varepsilon_2-\varepsilon_1, \vec{a}+\varepsilon_1\vec{k}; \mathfrak{q}) Z(\varepsilon_1-\varepsilon_2, \varepsilon_2, \vec{a}+\varepsilon_2\vec{k}; \mathfrak{q})
$$

(6.14)

$$
\begin{aligned}
0 = \sum_{\vec{k}} \frac{\mathfrak{q}^{\frac{1}{2}(\vec{k},\vec{k})}}{\prod_{\alpha\in\Delta} l_\alpha^{\vec{k}}(\varepsilon_1, \varepsilon_2, \vec{a})} &\left(D_{\log\mathfrak{q}}^{(\varepsilon_1,\varepsilon_2)} + (\vec{k}, \vec{a}) + \frac{(\vec{k}, \vec{k})}{2}(\varepsilon_1+\varepsilon_2)\right)^d\\
&\big(Z(\varepsilon_1, \varepsilon_2-\varepsilon_1, \vec{a}+\varepsilon_1\vec{k}; \mathfrak{q}) \cdot Z(\varepsilon_1-\varepsilon_2, \varepsilon_2, \vec{a}+\varepsilon_2\vec{k}; \mathfrak{q})\big)
\end{aligned}
$$

for $1 \le d \le 2r-1$.

The second equation (6.14) will play a fundamental role in our study of the partition function Z. We call it the *blowup equation*.

Proof. (6.12) follows from Proposition 6.11(1) by setting $t = 0$ in (6.9). Proposition 6.11(2) means that $(\frac{d}{dt})^d \widehat{Z}(\varepsilon_1, \varepsilon_2, \vec{a}; t; q)|_{t=0} = 0$ with $1 \le d \le 2r-1$. We get the above if we differentiate the right hand side of (6.9). □

For a later purpose, we divide the blowup equations for $d = 1, 2$ by $Z(\varepsilon_1, \varepsilon_2-\varepsilon_1, \vec{a}; \mathfrak{q}) Z(\varepsilon_1-\varepsilon_2, \varepsilon_2, \vec{a}; \mathfrak{q})$ and write down explicitly as

(6.15) $0 =$

$$
\begin{aligned}
\sum_{\vec{k}} \frac{\mathfrak{q}^{\frac{1}{2}(\vec{k},\vec{k})}}{\prod_{\alpha\in\Delta} l_\alpha^{\vec{k}}(\varepsilon_1, \varepsilon_2, \vec{a})} &\Bigg[\frac{1}{\varepsilon_2-\varepsilon_1}\left(\mathfrak{q}\frac{\partial}{\partial\mathfrak{q}} F_a^{\text{inst}}(\vec{a}+\varepsilon_1\vec{k}) - \mathfrak{q}\frac{\partial}{\partial\mathfrak{q}} F_b^{\text{inst}}(\vec{a}+\varepsilon_2\vec{k})\right)\\
&\qquad + (\vec{k}, \vec{a}) + \frac{(\vec{k}, \vec{k})}{2}(\varepsilon_1+\varepsilon_2)\Bigg]\\
\times \exp&\left[\frac{1}{\varepsilon_2-\varepsilon_1}\left(\frac{F_a^{\text{inst}}(\vec{a}+\varepsilon_1\vec{k}) - F_a^{\text{inst}}(\vec{a})}{\varepsilon_1} - \frac{F_b^{\text{inst}}(\vec{a}+\varepsilon_2\vec{k}) - F_b^{\text{inst}}(\vec{a})}{\varepsilon_2}\right)\right],
\end{aligned}
$$

$$(6.16)\qquad 0 = \sum_{\vec{k}} \frac{\mathfrak{q}^{\frac{1}{2}(\vec{k},\vec{k})}}{\prod_{\alpha\in\Delta} l_\alpha^{\vec{k}}(\varepsilon_1,\varepsilon_2,\vec{a})} \Bigg[\Bigg\{ (\vec{k},\vec{a}) + \frac{(\vec{k},\vec{k})}{2}(\varepsilon_1+\varepsilon_2) + \frac{1}{\varepsilon_2-\varepsilon_1}\left(\mathfrak{q}\frac{\partial}{\partial\mathfrak{q}} F_a^{\text{inst}}(\vec{a}+\varepsilon_1\vec{k}) - \mathfrak{q}\frac{\partial}{\partial\mathfrak{q}} F_b^{\text{inst}}(\vec{a}+\varepsilon_2\vec{k}) \right) \Bigg\}^2 + \frac{1}{\varepsilon_2-\varepsilon_1}\left(\varepsilon_1 \left(\mathfrak{q}\frac{\partial}{\partial\mathfrak{q}}\right)^2 F_a^{\text{inst}}(\vec{a}+\varepsilon_1\vec{k}) - \varepsilon_2 \left(\mathfrak{q}\frac{\partial}{\partial\mathfrak{q}}\right)^2 F_b^{\text{inst}}(\vec{a}+\varepsilon_2\vec{k}) \right) \Bigg] \times \exp\left[\frac{1}{\varepsilon_2-\varepsilon_1} \left(\frac{F_a^{\text{inst}}(\vec{a}+\varepsilon_1\vec{k}) - F_a^{\text{inst}}(\vec{a})}{\varepsilon_1} - \frac{F_b^{\text{inst}}(\vec{a}+\varepsilon_2\vec{k}) - F_b^{\text{inst}}(\vec{a})}{\varepsilon_2} \right) \right],$$

where

$$\exp \frac{F_a^{\text{inst}}(\vec{a})}{\varepsilon_1(\varepsilon_2-\varepsilon_1)} = Z(\varepsilon_1, \varepsilon_2-\varepsilon_1, \vec{a}; \mathfrak{q}),$$
$$\exp \frac{F_b^{\text{inst}}(\vec{a})}{(\varepsilon_1-\varepsilon_2)\varepsilon_2} = Z(\varepsilon_1-\varepsilon_2, \varepsilon_2, \vec{a}; \mathfrak{q}).$$

The functions F_a^{inst}, F_b^{inst} depends also on $\varepsilon_1, \varepsilon_2$, but we omit them from the notation for brevity.

7. Behavior at $\varepsilon_1, \varepsilon_2 = 0$

We prove Nekrasov's conjecture in this section.

Lemma 7.1. $Z(\varepsilon_1, -2\varepsilon_1, \vec{a}; \mathfrak{q}) = Z(2\varepsilon_1, -\varepsilon_1, \vec{a}; \mathfrak{q})$.

Proof. We set $\varepsilon_2 = -\varepsilon_1$ in (6.9) with $d = 1$. Then we have

$$n\varepsilon_1 \left(Z_n(\varepsilon_1, -2\varepsilon_1, \vec{a}) - Z_n(2\varepsilon_1, -\varepsilon_1, \vec{a}) \right) = \sum_{\substack{\frac{1}{2}(\vec{k},\vec{k})+l+m=n \\ l\neq n, m\neq n}} \left\{ (l-m)\varepsilon_1 + (\vec{k},\vec{a}) \right\} \times \frac{Z_l(\varepsilon_1, -2\varepsilon_1, \vec{a}+\varepsilon_1\vec{k}) Z_m(2\varepsilon_1, -\varepsilon_1, \vec{a}-\varepsilon_1\vec{k})}{\prod_{\alpha\in\Delta} l_\alpha^{\vec{k}}(\varepsilon_1, -\varepsilon_1, \vec{a})}.$$

We show $Z_n(\varepsilon_1, -2\varepsilon_1, \vec{a}) = Z_n(2\varepsilon_1, -\varepsilon_1, \vec{a})$ by the induction on n. The assertion is trivial for $n = 1$. Suppose that it is true for $l, m < n$. Then the right hand side of the above equation vanishes, as terms with $(\vec{k}, l, m)$ and $(-\vec{k}, m, l)$ cancel and the term $(0, l, l)$ is 0. Here we have used $l_\alpha^{\vec{k}}(\varepsilon_1, -\varepsilon_1, \vec{a})$

$= (-1)^{\langle \vec{k},\alpha\rangle(\langle \vec{k},\alpha\rangle-1)/2} l_{-\alpha}^{-\vec{k}}(\varepsilon_1, -\varepsilon_1, \vec{a})$ which follows from Lemma 6.7, and that

$$\sum_{\alpha\in\Delta} \langle \vec{k}, \alpha\rangle(\langle \vec{k}, \alpha\rangle - 1)/2 = r(\vec{k}, \vec{k})$$

is an even number. □

The following follows from this lemma and its proof:

Corollary 7.2. *$\widehat{Z}_{n,d}(\varepsilon_1, -\varepsilon_1, \vec{a})$ vanishes for odd d.*

This is compatible with what is known for the usual blowup formula for Donaldson invariants (cf. [19].)

The following is the first part of Nekrasov's conjecture:

Proposition 7.3. *$F^{\mathrm{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}; \mathfrak{q})$ is regular at $\varepsilon_1 = \varepsilon_2 = 0$.*

Proof. The point of the proof is a recursive structure of the blowup equation (6.15, 6.16). Let us separate terms with $\vec{k} = 0$:

$$\frac{1}{\varepsilon_2 - \varepsilon_1}\left(\mathfrak{q}\frac{\partial}{\partial\mathfrak{q}} F_a^{\mathrm{inst}}(\vec{a}) - \mathfrak{q}\frac{\partial}{\partial\mathfrak{q}} F_b^{\mathrm{inst}}(\vec{a})\right) = A,$$

$$\frac{1}{(\varepsilon_2 - \varepsilon_1)^2}\left(\mathfrak{q}\frac{\partial}{\partial\mathfrak{q}} F_a^{\mathrm{inst}}(\vec{a}) - \mathfrak{q}\frac{\partial}{\partial\mathfrak{q}} F_b^{\mathrm{inst}}(\vec{a})\right)^2 + \frac{1}{\varepsilon_2 - \varepsilon_1}\left(\varepsilon_1\left(\mathfrak{q}\frac{\partial}{\partial\mathfrak{q}}\right)^2 F_a^{\mathrm{inst}}(\vec{a}) - \varepsilon_2\left(\mathfrak{q}\frac{\partial}{\partial\mathfrak{q}}\right)^2 F_b^{\mathrm{inst}}(\vec{a})\right) = B,$$

where A and B are terms with $\vec{k} \neq 0$ and hence divisible by $\mathfrak{q}$. We further replace the first term in the second equation by A^2. If we express $F_a^{\mathrm{inst}}(\vec{a})$, $F_b^{\mathrm{inst}}(\vec{a})$ by formal power series in $\mathfrak{q}$, then the above equations determine the coefficients recursively. We want to show that $F_n^{\mathrm{inst}}(\varepsilon_1, \varepsilon_2, \vec{a})$ is regular at $\varepsilon_1 = \varepsilon_2 = 0$ by the induction using this recursive system. This is equivalent to showing that A and B are regular under the assumption that $F_a^{\mathrm{inst}}(\vec{a})$, $F_b^{\mathrm{inst}}(\vec{a})$ are regular. This follows from the following lemma. □

Lemma 7.4. *Suppose that $F^{\mathrm{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}; \mathfrak{q})$ is regular at $(\varepsilon_1, \varepsilon_2) = 0$. Then the following are also regular and their values are given by*

$$\frac{1}{\varepsilon_2 - \varepsilon_1}\left(\mathfrak{q}\frac{\partial}{\partial\mathfrak{q}} F_a^{\mathrm{inst}}(\vec{a} + \varepsilon_1\vec{k}) - \mathfrak{q}\frac{\partial}{\partial\mathfrak{q}} F_b^{\mathrm{inst}}(\vec{a} + \varepsilon_2\vec{k})\right)\Bigg|_{(\varepsilon_1,\varepsilon_2)=0}$$
$$= -\sum_i k^i\, \mathfrak{q}\frac{\partial^2 F^{\mathrm{inst}}}{\partial\mathfrak{q}\partial a^i}(0, 0, \vec{a}; \mathfrak{q}),$$

$$\frac{1}{\varepsilon_2-\varepsilon_1}\left(\varepsilon_1\left(\mathfrak{q}\frac{\partial}{\partial\mathfrak{q}}\right)^2 F_a^{\mathrm{inst}}(\vec{a})-\varepsilon_2\left(\mathfrak{q}\frac{\partial}{\partial\mathfrak{q}}\right)^2 F_b^{\mathrm{inst}}(\vec{a})\right)\Bigg|_{(\varepsilon_1,\varepsilon_2)=0}$$
$$=-\left(\mathfrak{q}\frac{\partial}{\partial\mathfrak{q}}\right)^2 F^{\mathrm{inst}}(0,0,\vec{a};\mathfrak{q}),$$
$$\frac{1}{\varepsilon_2-\varepsilon_1}\left(\frac{F_a^{\mathrm{inst}}(\vec{a}+\varepsilon_1\vec{k})-F_a^{\mathrm{inst}}(\vec{a})}{\varepsilon_1}-\frac{F_b^{\mathrm{inst}}(\vec{a}+\varepsilon_2\vec{k})-F_b^{\mathrm{inst}}(\vec{a})}{\varepsilon_2}\right)\Bigg|_{(\varepsilon_1,\varepsilon_2)=0}$$
$$=-\frac{1}{2}\sum_{i,j}\frac{\partial^2 F^{\mathrm{inst}}}{\partial a^i\partial a^j}(0,0,\vec{a};\mathfrak{q})k^ik^j.$$

Proof. The regularity is a consequence of the symmetry $F_b^{\mathrm{inst}}(\vec{a}) = F_a^{\mathrm{inst}}(\vec{a})\big|_{\varepsilon_1\leftrightarrow\varepsilon_2}$. In order to show the above equalities, we just need to note

$$\frac{\partial F^{\mathrm{inst}}}{\partial\varepsilon_1}(0,0,\vec{a};\mathfrak{q})=\frac{\partial F^{\mathrm{inst}}}{\partial\varepsilon_2}(0,0,\vec{a};\mathfrak{q})=0.$$

The first equality is the consequence of Lemma 6.3(1), and the second equality follows from Lemma 7.1. □

We now take the limit $\varepsilon_1,\varepsilon_2\to 0$ of (6.16). (The limit of (6.15) becomes the trivial identity $0=0$.) We set $\mathcal{F}^{\mathrm{inst}}(\vec{a};q)=F^{\mathrm{inst}}(0,0,\vec{a};\mathfrak{q})$. By Lemma 6.7(3), we have

$$\prod_{\alpha\in\Delta} l_\alpha^{\vec{k}}(0,0,\vec{a})=\prod_{\alpha\in\Delta^+}(-1)^{\langle\vec{k},\alpha\rangle(\langle\vec{k},\alpha\rangle+1)/2}\langle\vec{a},\alpha\rangle^{\langle\vec{k},\alpha\rangle^2}$$
$$=(-1)^{\langle\vec{k},\rho\rangle}\prod_{\alpha\in\Delta^+}\left\{\sqrt{-1}\langle\vec{a},\alpha\rangle\right\}^{\langle\vec{k},\alpha\rangle^2}.$$

Therefore we get

(7.5)
$$0=\sum_{\vec{k}}\frac{(-1)^{\langle\vec{k},\rho\rangle}\mathfrak{q}^{\frac{1}{2}(\vec{k},\vec{k})}}{\prod_{\alpha\in\Delta^+}\left\{\sqrt{-1}\langle\vec{a},\alpha\rangle\right\}^{\langle\vec{k},\alpha\rangle^2}}$$
$$\times\left[\left\{\sum_i k^i\left(\sum_j C_{ij}a^j-\mathfrak{q}\frac{\partial^2\mathcal{F}^{\mathrm{inst}}}{\partial\mathfrak{q}\partial a^i}(\vec{a};\mathfrak{q})\right)\right\}^2-\left(\mathfrak{q}\frac{\partial}{\partial\mathfrak{q}}\right)^2\mathcal{F}^{\mathrm{inst}}(\vec{a};\mathfrak{q})\right]$$
$$\times\exp\left(-\frac{1}{2}\sum_{i,j}\frac{\partial^2\mathcal{F}^{\mathrm{inst}}}{\partial a^i\partial a^j}(\vec{a};\mathfrak{q})k^ik^j\right).$$

In order to compare this with the formula in literature, we introduce the following functions:

(7.6)

$$\tau_{ij} = \frac{\sqrt{-1}}{\pi} \sum_{\alpha\in\Delta_+} \langle\alpha_i^\vee, \alpha\rangle\langle\alpha_j^\vee, \alpha\rangle \log\left(\frac{\sqrt{-1}\langle\vec{a}, \alpha\rangle}{\mathfrak{q}^{\frac{1}{2r}}}\right) - \frac{1}{2\pi\sqrt{-1}} \frac{\partial^2 \mathcal{F}^{\text{inst}}}{\partial a^i \partial a^j}(\vec{a}; \mathfrak{q}),$$

$$u_2 = \frac{1}{2}(\vec{a}, \vec{a}) - \mathfrak{q}\frac{\partial \mathcal{F}^{\text{inst}}}{\partial \mathfrak{q}}(\vec{a}; \mathfrak{q}). \tag{7.7}$$

Now (7.5) can be written as

$$\left(\mathfrak{q}\frac{\partial}{\partial\mathfrak{q}}\right)^2 \mathcal{F}^{\text{inst}}(\vec{a}; \mathfrak{q}) = \sum_{i,j} \frac{\partial u_2}{\partial a^i}\frac{\partial u_2}{\partial a^j}\frac{1}{\pi\sqrt{-1}}\frac{\partial}{\partial\tau_{ij}} \log\Theta_E(0|\tau), \tag{7.8}$$

where we have used (6.5) several times and Θ_E is as in (1.5). This, combined with (7.7), is exactly the contact term equation (1.4) if we replace $\mathfrak{q}^{\frac{1}{2r}}$ by Λ. Note also that (7.6) coincides with (1.2). And (7.7) is nothing but (1.3).

The equation (7.8) has the same structure as the blowup equation (6.14). When we expand $\mathcal{F}^{\text{inst}}$ as a formal power series in $\mathfrak{q}$, coefficients are determined inductively. In particular, the solution to the above equation is *unique*. This observation was due to [14]. (See also [42] for an earlier result for SU(2).) Since the Seiberg-Witten prepotential satisfies (7.8), we conclude that $\mathcal{F}^{\text{inst}}$ coincides with its instanton part. This is our confirmation of Nekrasov's conjecture.

8. Blowup formula

We divide (6.10) by $Z(\varepsilon_1, \varepsilon_2-\varepsilon_1, \vec{a}; \mathfrak{q})Z(\varepsilon_1-\varepsilon_2, \varepsilon_2, \vec{a}; \mathfrak{q})$ and take the limit $\varepsilon_1, \varepsilon_2 \to 0$. We need the following generalization of the third equation in Lemma 7.4:

$$\frac{1}{\varepsilon_2-\varepsilon_1}\left(\frac{F^{\text{inst}}(\varepsilon_1, \varepsilon_2-\varepsilon_1, \vec{a}+\varepsilon_1\vec{k}; \mathfrak{q}e^{t\varepsilon_1}) - F^{\text{inst}}(\varepsilon_1, \varepsilon_2-\varepsilon_1, \vec{a}; \mathfrak{q})}{\varepsilon_1}\right.$$
$$\left.\left. - \frac{F^{\text{inst}}(\varepsilon_1-\varepsilon_2, \varepsilon_2, \vec{a}+\varepsilon_2\vec{k}; \mathfrak{q}e^{t\varepsilon_2}) - F^{\text{inst}}(\varepsilon_1-\varepsilon_2, \varepsilon_2, \vec{a}; \mathfrak{q})}{\varepsilon_2}\right)\right|_{(\varepsilon_1,\varepsilon_2)=0}$$
$$= -\frac{1}{2}\left(\mathfrak{q}\frac{\partial}{\partial\mathfrak{q}}\right)^2 \mathcal{F}^{\text{inst}}(\vec{a}; \mathfrak{q})\, t^2 - \frac{1}{2}\sum_{i,j}\frac{\partial^2\mathcal{F}^{\text{inst}}}{\partial a^i\partial a^j}(\vec{a}; \mathfrak{q})k^i k^j$$
$$-\sum_i \mathfrak{q}\frac{\partial^2\mathcal{F}^{\text{inst}}}{\partial\mathfrak{q}\partial a^i}(\vec{a}; \mathfrak{q})\, tk^i.$$

Here we have used $\frac{\partial}{\partial\log\mathfrak{q}} = \mathfrak{q}\frac{\partial}{\partial\mathfrak{q}}$. Therefore we get

Theorem 8.1. $\widehat{Z}^k(\varepsilon_1,\varepsilon_2,\vec{a};t;\mathfrak{q})/Z(\varepsilon_1,\varepsilon_2,\vec{a};q)$ *is regular at* $(\varepsilon_1,\varepsilon_2)=0$. *Its value is*

$$\exp\left(-\frac{1}{2}\left(\mathfrak{q}\frac{\partial}{\partial\mathfrak{q}}\right)^2\mathcal{F}^{\text{inst}}(\vec{a};\mathfrak{q})\,t^2\right)$$
$$\times\sum_{\vec{k}\in P:\{\vec{k}\}=-\frac{k}{r}}\Bigg\{\frac{(-1)^{\langle\vec{k},\rho\rangle}\mathfrak{q}^{\frac{1}{2}(\vec{k},\vec{k})}}{\prod_{\alpha\in\Delta^+}\left\{\sqrt{-1}\langle\vec{a},\alpha\rangle\right\}^{(\vec{k},\alpha)^2}}$$
$$\exp\Bigg(-\frac{1}{2}\sum_{i,j}\frac{\partial^2\mathcal{F}^{\text{inst}}}{\partial a^i\partial a^j}(\vec{a};\mathfrak{q})k^ik^j$$
$$+\sum_i\left(\sum_j C_{ij}a^j-\mathfrak{q}\frac{\partial^2\mathcal{F}^{\text{inst}}}{\partial\mathfrak{q}\partial a^i}(\vec{a};\mathfrak{q})\right)tk^i\Bigg)\Bigg\}$$
$$\times\left[\sum_{\vec{k}\in Q}\frac{(-1)^{\langle\vec{k},\rho\rangle}\mathfrak{q}^{\frac{1}{2}(\vec{k},\vec{k})}}{\prod_{\alpha\in\Delta^+}\left\{\sqrt{-1}\langle\vec{a},\alpha\rangle\right\}^{(\vec{k},\alpha)^2}}\exp\left(-\frac{1}{2}\sum_{i,j}\frac{\partial^2\mathcal{F}^{\text{inst}}}{\partial a^i\partial a^j}(\vec{a};\mathfrak{q})k^ik^j\right)\right]^{-1}.$$

If we use the theta function in (1.5), this can be written simply as

$$\exp\left(-\frac{1}{2}\left(\mathfrak{q}\frac{\partial}{\partial\mathfrak{q}}\right)^2\mathcal{F}^{\text{inst}}(\vec{a};\mathfrak{q})\,t^2\right)\frac{\Theta_k(\vec{\xi}|\tau)}{\Theta_E(0|\tau)},\quad\text{where }\xi^i=\frac{t}{2\pi\sqrt{-1}}\frac{\partial u_2}{\partial a^i}.$$

Here Θ_k is defined as in (1.5) where the summation is over $\vec{k}\in P$ with $\{\vec{k}\}=-\frac{k}{r}$. This form of the blowup formula for Donaldson invariants and its higher rank analog coincides with one given in [43,34,41].

9. General gauge groups

Our proof relies only on the blowup formula for degree $d=1,2$. Hence it has a natural generalization to more general gauge groups. The point is that we do not need the explicit formula (6.2) in terms of Young tableaux.

Let G be a compact semisimple Lie group. Let $M^{\text{reg}}(G,n)$ be the framed moduli space of G-instantons on $S^4=\mathbb{R}^4\cup\{\infty\}$ with instanton number n, which corresponds to $\pi_3(G)\cong\mathbb{Z}$. By [2] it is a nonsingular manifold, whose dimension can be computed by the index theorem (and a standard calculation in the Lie algebra of G). By the Hitchin-Kobayashi correspondence, the moduli space can be identified with the framed moduli space of principal G^c-bundles on $\mathbb{P}^2=\mathbb{C}^2\cup\ell_\infty$, where G^c is the complexification of G. When G is a classical group, this version of the Hitchin-Kobayashi correspondence was proved in [11] via the ADHM description. Bando's analytic argument [4] works for arbitrary G. It is not clear, as far as the authors know, whether we have a natural generalization of $M(r,n)$ for the group G. Thus we can use only the Uhlenbeck compactification $M_0(G, n) = \bigsqcup_{m \le n} M^{\text{reg}}(G, m) \times S^{n-m}\mathbb{C}^2$. We also consider the framed moduli spaces $\widehat{M}^{\text{reg}}(G, k, n)$ and its Uhlenbeck compactification $\widehat{M}_0(G, k, n)$ on the blowup. Here k is the characteristic class in H^2, which is considered as an element in $\pi_1(G)$.

Let T be a maximal torus of G. (In the other part of the paper, T was a complex torus. We hope that this does not make a confusion.) Then we have an action of $\widetilde{T} = T^2 \times T$ on the moduli spaces $M_0(G, n)$, $\widehat{M}_0(G, k, n)$. Let $H_*^T(M_0(G, n))$, $H_*^T(\widehat{M}_0(G, k, n))$ denote the equivariant homology groups. The only fixed point in $M_0(G, n)$ is the ideal instanton consisting of the trivial connection and the singularity concentrated at the origin. We denote this point by 0, and the inclusion $0 \to M_0(G, n)$ by ι_0. We *assume* that the localization theorem is applicable to $M_0(G, n)$. This is guaranteed when $M_0(G, n)$ can be equivariantly embedded in a finite dimensional representation of $\widetilde{T}$, or $M_0(G, n)$ can be endowed with a structure of $\widetilde{T}$-algebraic variety. We define the *partition function* by

$$Z(\varepsilon_1, \varepsilon_2, \vec{a}; \mathfrak{q}) = \sum_{n=0}^{\infty} \mathfrak{q}^n (\iota_{0*})^{-1}[M_0(G, n)],$$

where $[M_0(G, n)]$ is the fundamental class of $M_0(G, n)$. The fundamental class is defined since the singular locus is lower dimensional as fundamental classes of algebraic cycles are always defined.

Proposition 9.1. *The fixed points in* $\widehat{M}_0(G, k, n)$ *are parametrized by triples* $(\vec{k}, l, m)$ *where* $\vec{k} \in \pi_1(T) \cong \operatorname{Hom}(S^1, T)$ *and* l, m *are nonnegative integers. They satisfy the constraint* $\rho(\vec{k}) = k$ *and* $\frac{1}{2}(\vec{k}, \vec{k}) + l + m = n$, *where* ρ *is the homomorphism* $\pi_1(T) \to \pi_1(G)$ *induced by the inclusion* $T \subset G$, *and* $(\ ,\ )$ *is the inner product on* $\operatorname{Lie} T$ *such that the square of the length of the highest root* θ *with respect to the induced inner product on the dual space* $\operatorname{Lie} T^*$ *is equal to* 2.

If we choose simple coroots $\alpha_i^\vee$ ($1 \le i \le \dim T = \operatorname{rank} G$), $\vec{k}$ can be identified with an r-tuple of rational numbers $(k^1, \dots, k^r) \in \mathbb{Q}^r$ by $\vec{k} = \sum_i k^i \alpha_i^\vee$.

Proof. A fixed point in $\widehat{M}_0(G, k, n)$ is $(A, l[p_1] + m[p_2])$, where A is a reducible instanton (or a G^c-principal bundle which is reducible to a T^c-bundle) with instanton number $n(A)$ and l, m are integers with $n(A) + l + m = n$. A reducible instanton A on the blowup is classified by $\vec{k} \in \pi_1(T)$. We have constraint $\rho(\vec{k}) = k$, so that the induced bundle has the right characteristic k. We also have

$$n(A) = \frac{1}{2}(\vec{k}, \vec{k}),$$

where (,) is the inner product as above. This can be proved as follows. Let $\mathfrak{g}^c$ be the complexification of the vector bundle associated with the adjoint representation. We have

$$c_2(\mathfrak{g}^c) = \frac{1}{2}\sum_{\alpha\in\Delta}\langle\vec{k},\alpha\rangle^2 = \frac{1}{2}B_G(\vec{k},\vec{k}) = h^\vee(\vec{k},\vec{k}),$$

where B_G is the Killing form, and $h^\vee$ is the dual Coxeter number. For the last equality, see [27, Exercise 6.1]. On the other hand, the instanton number is given by $\frac{c_2(\mathfrak{g}^c)}{2h^\vee}$. (See [2, §8].) □

For $G = \mathrm{SU}(r)$, the inner product (,) is the standard one used in earlier sections, and we have $h^\vee = r$. Note that $c_2(\mathfrak{g}^c)$ is the complex dimension of the framed moduli space $\widehat{M}(G,k,n(A))$, so it is given by $2h^\vee n(A)$, as was shown in [2].

For a root $\alpha\in\Delta$, we define $l_\alpha^{\vec{k}}(\varepsilon_1,\varepsilon_2,\vec{a})$ by the same formula as as (6.6). The Euler class of tangent space of $\widehat{M}^{\mathrm{reg}}(G,k,\frac{1}{2}(\vec{k},\vec{k}))$ at the reducible instanton A is given by $\prod_{\alpha\in\Delta} l_\alpha^{\vec{k}}(\varepsilon_1,\varepsilon_2,\vec{a})$.

Conjecture 9.2. (1) There exists a proper continuous map $\widehat{\pi}_0\colon \widehat{M}_0(G,k,n) \to M_0(G,n')$ for some n'.

(2) A neighborhood of the fixed point $(\vec{k},l,m)$ in $\widehat{M}_0(G,k,n)$ is isomorphic to a neighborhood of $(\vec{k},0,0)\times 0\times 0$ in $\widehat{M}_0(G,k,n-l-m)\times M_0(G,l)\times M_0(G,m)$ as a $\widetilde{T}$-space, where the T^2-actions on the latter two factors are modified as $(t_1,t_2)\mapsto(t_1,t_2/t_1)$ and $(t_1,t_2)\mapsto(t_1/t_2,t_2)$ respectively.

We define an equivariant cohomology class $\mu(C)\in H^2_{\widetilde{T}}(\widehat{M}^{\mathrm{reg}}(G,k,n))$ by

$$-\frac{1}{2h^\vee}p_1(\widetilde{\mathfrak{g}})/[C],$$

where $\widetilde{\mathfrak{g}}$ is the universal *adjoint* bundle, i.e., the fiber is the Lie algebra $\mathfrak{g}$.

Conjecture 9.3. (1) The class $\mu(C)$ extends to a class in $H^2_{\widetilde{T}}(\widehat{M}_0(G,k,n))$. We denote the extended class by the same notation.

(2) If $\iota_{(\vec{k},l,m)}$ denotes the inclusion of the fixed point $(\vec{k},l,m)$ in $\widehat{M}_0(G,k,n)$, we have

$$\iota^*_{(\vec{k},l,m)}(\mu(C)) = l\varepsilon_1 + m\varepsilon_2 + (\vec{k},\vec{a}) + \frac{(\vec{k},\vec{k})}{2}(\varepsilon_1+\varepsilon_2).$$

We define the *partition function* on the blowup by

$$\widehat{Z}(\varepsilon_1,\varepsilon_2,\vec{a};t;\mathfrak{q}) = \sum_{n=0}^\infty \mathfrak{q}^n\sum_{d=0}^\infty\frac{t^d}{d!}\,(\iota_{0*})^{-1}\widehat{\pi}_{0*}\left(\mu(C)^d\cap[\widehat{M}_0(G,0,n)]\right).$$

Then (6.10) holds if we assume Conjectures 9.2, 9.3. Proposition 6.11 can be modified as

$$\widehat{\pi}_{0*}[\widehat{M}_0(G,0,n)] = [M_0(G,n)],$$
$$\widehat{\pi}_{0*}\big(\mu(C)^d \cap [\widehat{M}_0(G,0,n)]\big) = 0 \quad \text{for } 1 \le d \le 2h^\vee - 1.$$

The proof of the first equality is exactly the same. For the proof of the second equality, we need a line bundle $\mathcal{L}$ and a section which does not vanish on $\widehat{\pi}_0^{-1}(\overline{\{0\} \times M_0^{\text{reg}}(G,n-1)})$. I do not know such things exists for genuine $\mu(C)$. But probably there exists such things for $2h^\vee\mu(C)$. If this is indeed true, the rest of the argument is the same as before.

We can now proceed as in the $\mathrm{SU}(r)$ case, we use this formula $d = 1, 2$ to get (6.15, 6.16). Considering the limit $\varepsilon_1, \varepsilon_2 \to 0$, we get (7.8) exactly as before. On the other hand, the proof that the Seiberg-Witten prepotential satisfies (7.8) was generalized to classical groups [13].

Remark 9.4. The assumption that the localization theorem is applicable to $M_0(G,n)$ and $\widehat{M}(G,k,n)$ follows from the description in [28] for a classical group G, since they are algebraic varieties. For general group, one can probably use the method in [6]. Conjectures 9.2, 9.3 are true in view of King's description, except 9.2(2). We believe that 9.2(2) can be also checked, but we need a further study.

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