Invent. math. 161, 629–656 (2005)
DOI: 10.1007/s00222-004-0433-9

Inventiones mathematicae

The motivic fundamental group of $\mathbf{P}^1 \setminus \{0, 1, \infty\}$ and the theorem of Siegel

Minhyong Kim

Department of Mathematics, University of Arizona, Tucson, AZ 85721, USA
(e-mail: kim@math.arizona.edu)

Oblatum 24-VI-2004 & 30-XI-2004
Published online: 7 April 2005 –

One of the strong motivations for studying the arithmetic fundamental groups of algebraic varieties comes from the hope that they will provide group-theoretic access to Diophantine geometry. This is most clearly expressed by the so-called 'section conjecture' of Grothendieck: Let F be a number field and let $X \to \mathrm{Spec}(F)$ be a smooth and compact hyperbolic curve. After some choice of base point, we get an exact sequence

$$0 \to \hat{\pi}_1(\bar{X}) \to \hat{\pi}_1(X) \xrightarrow{p} \Gamma \to 0$$

where Γ is the Galois group of F and $\hat{\pi}_1$ refers to the profinite fundamental group. The section conjecture states that there is a one-to-one correspondence between conjugacy classes of splittings of p and the geometric sections of the morphism $X \to \mathrm{Spec}(F)$. There is apparently a hope that the section conjecture will provide a direct link between the study of fundamental groups and the theorems of Siegel and Faltings.

The purpose of this paper is to illustrate a somewhat different methodology for deriving Diophantine consequences from a study of the fundamental group. To this end, we give a π_1 proof of the theorem of Siegel on the finiteness of integral points for the thrice-punctured projective line. The notion of a 'π_1 proof' may not be entirely well-defined, but it is hoped that the techniques employed will make the meaning clear. Another way of expressing the main idea is to say that we are using functions on the fundamental group to prove Diophantine finiteness. Here, the fundamental group refers to the unipotent motivic fundamental group in the sense of Deligne [7]. Although a rigorous construction of such an object (for rational varieties) has now been given by Deligne and Goncharov using Voevodsky's theory, our proof does not require more than Deligne's original construction using systems of realizations. More precisely, we will be using the local and global étale fundamental groups $\pi_{1,\text{ét}}$, the local De Rham fundamental group $\pi_{1,DR}$, the crystalline fundamental group $\pi_{1,cr}$ and comparisons between them. Here and henceforward, all π_1's will denote ($\mathbb{Q}_p$-)unipotent completions, as they are all we will be considering in this paper. In this connection, it is rather striking that the motivic theory is capable of yielding Diophantine finiteness, even though the motivic fundamental group (at least the portion we use) could be viewed as a cruder invariant of a variety than the pro-finite one. That is to say, connections between the theory of motives and Diophantine geometry via L-functions is expected in great generality. However, when viewed as invariants of varieties, L-functions, because they 'factor through' the linear category of motives, seem to provide information in general only about linearized invariants, e.g., Chow groups. It is then natural that results about non-linear sets should evoke non-linear tools like the fundamental group. What is somewhat surprising is that even a mild degree of non-linearity (coming from the subcategory of unipotent group objects in the category of motives) can still provide substantial information.

Here is a brief outline of the proof: In the discussion above, set $F = \mathbb{Q}$, and $X = \mathbf{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$. Let S be a finite set of primes. Fix any $p \notin S$ and an S-integral point x of $\mathcal{X} := \mathbf{P}^1_{\mathbb{Z}} \setminus \{0, 1, \infty\}$. Let $T = S \cup \{p\}$. Denote by Y the reduction of $\mathcal{X}$ mod p. Let $y \in Y$ be the reduction mod p of the point x. We will use a p-adic *unipotent Albanese map*

$$UAlb_x : X(\mathbb{Q}_p) \cap]Y[\rightarrow \pi_{1,DR}(X, x)(\mathbb{Q}_p)$$

associated to the basepoint x and defined on the tube of Y inside $X(\mathbb{Q}_p)$, that is, the points that reduce mod p to points of Y. This map is constructed by considering the class of the compatible pair of torsors of paths $\pi_{1,DR}(X \otimes \mathbb{Q}_p, x, x')$ and $\pi_{1,cr}(Y, y, y')$ associated to a point x' with reduction $y' \in Y$. These are torsors for the crystalline and De Rham fundamental groups $\pi_{1,DR}(X \otimes \mathbb{Q}_p, x)$ and $\pi_{1,cr}(Y, y)$, respectively. A simple classification of such compatible pairs of torsors allows us to canonically associate to the pair a point in $\pi_{1,DR}(X \otimes \mathbb{Q}_p, x)$ which we define to be $UAlb_x(x')$. Now $\pi_{1,DR}(X \otimes \mathbb{Q}_p, x)$ is a pro-unipotent algebraic group over $\mathbb{Q}_p$ with a coordinate ring A_{DR} that is generated as a $\mathbb{Q}_p$-vector space by functions α_w indexed by the words w in two letters A, B. An important point is that the functions $g_w := \alpha_w \circ UAlb$ are restrictions to $X(\mathbb{Q}_p) \cap]Y[$ of *Coleman functions* on $X(\mathbb{C}_p)$ and hence, have nice analytic properties. We will show that they are $\mathbb{C}_p$-linearly independent, so no non-zero function from $\mathcal{A}_{DR}$ pulls back to the zero function on $X(\mathbb{Q}_p)$. On the other hand, when we examine the image of the integral points $\mathcal{X}(\mathbb{Z}_S)$ under the Albanese map, we find that it is essentially contained inside the image of another 'map'

$$C : H^1_f(\Gamma_T, \pi_{1,\acute{e}t}(X, x)) \rightarrow \pi_{1,DR}(X \otimes \mathbb{Q}_p, x)$$

from a suitable continuous global cohomology set to the De Rham fundamental group. This 'map' is algebraic and is obtained from global-to-local restriction and p-adic Hodge theory. We explain the quotation marks: In fact, $H^1_f(\Gamma_T, \pi_{1,\acute{e}t}(X, x))$ has the natural structure of a pro-algebraic variety.

If we look at various quotients $[\pi_{1,DR}]_n$ and $[\pi_{1,\text{ét}}]_n$ with respect to the descending central series of these fundamental groups, what we actually have are finite-dimensional varieties and algebraic maps

$$C_n : H^1_f(\Gamma_T, [\pi_{1,\text{ét}}(X, x)]_n) \to [\pi_{1,DR}(X \otimes \mathbb{Q}_p, x)]_n,$$

whenever p is sufficiently large with respect to n (we will explain this in detail in Sect. 3). We can also consider the level n unipotent Albanese maps

$$UAlb_n : X(\mathbb{Q}_p)\cap]Y[\to [\pi_{1,DR}(X, x)]_n(\mathbb{Q}_p)$$

obtained from $UAlb$ via composition with the natural projections, and we find that

$$UAlb_n(\mathcal{X}(\mathbb{Z}_S)) \subset \text{Im}(C_n).$$

However, the descending central series filtration on $H^1_f(\Gamma_T, [\pi_{1,\text{ét}}(X, x)]_n)$ together with a Galois cohomology computation of Soulé allows us to get explicit bounds on the dimensions of these Galois cohomology varieties. The upshot then is that for large n and p, the image of $H^1_f(\Gamma_T, [\pi_{1,\text{ét}}(X, x)]_n)$ under the map to $[\pi_{1,DR}(X \otimes \mathbb{Q}_p, x)]_n$ lies inside a proper subvariety. Therefore, some non-zero element of A_{DR} vanishes on this image, and hence, on the image of $\mathcal{X}(\mathbb{Z}_S)$. This fact, together with the identity principle for Coleman functions and compactness yields the finiteness of Siegel's theorem.

The reader familiar with the method of Chabauty ([4], [6]) will immediately recognize our proof to be a 'non-abelian lift' of his. In particular, whereas Chabauty is using p-adic logarithms in his proof, what we show is that there is a linear combination of p-adic multiple polylogarithms that vanishes on the global points. In this spirit, we believe that many generalizations and refinements should be possible and hope to discuss them in the near future. The present paper, however, was motivated by the wish to work out in full detail one non-trivial example, thereby testing the strength of the techniques involved in carrying out this lift.

1. Torsor spaces

We need some elementary preliminaries on topological vector spaces. Let B be a complete Hausdorff topological field of characteristic zero. Given a finite-dimensional vector space V over B, there is a unique topology on V compatible with the vector space structure. This can be described by choosing any isomorphism $V \simeq B^n$ and using the product topology. We will be considering B-vector spaces R, possibly infinite-dimensional. It will be convenient to topologize such R by giving them the inductive limit topology coming from the family of all finite-dimensional subspaces. Thus, a map $f : R \to A$ is continuous if and only if its restriction to any finite-dimensional subspace $V \subset R$ is continuous. The inclusion $V \hookrightarrow R$ of a finite-dimensional subspace is then automatically continuous. In fact, it is a direct consequence of the definitions that any B-linear map is continuous: Say $f : R_1 \to R_2$ is B-linear. Take $V_1 \subset R_1$ finite-dimensional. Then $V_2 := f(V_1) \subset R_2$ is also finite-dimensional. Since $f|V_1$ can be factored as $V_1 \to V_2 \hookrightarrow R_2$ and both arrows are continuous, we are done. This argument is typical of those involving the inductive limit topology. Also obvious is that any vector subspace is closed. Note that the topology is Hausdorff: Let $v, w \in R$, $v \neq w$ and let $V \subset R$ be the subspace generated by v, w. By choosing a basis of R, we can construct a projection $p : R \to V$ which must be continuous. Now find $O_v, O_w \subset V$, disjoint open subsets of V containing v and w respectively. Then $p^{-1}(O_v)$ and $p^{-1}(O_w)$ separate v and w.

The inductive limit topology can be applied, in particular, to the situation where R is a B-algebra, or to any affine n-space R^n over R. Suppose $V \subset R^n$ is finite-dimensional over B. Then each of the projections $V_i := p_i(V)$ to the components is finite-dimensional and $V \subset \prod_i V_i$. So finite-dimensional subspaces of this product form are co-final. Hence, it suffices to check continuity of maps on such subspaces. In fact, it is clearly sufficient to consider subspaces of the form $W \times W \times \cdots \times W$ with W finite-dimensional in R.

Lemma 1 *Any polynomial map $f : R^n \to R^k$ is continuous.*

Proof. Consider a subspace $\prod_i W \subset R^n$ as above. Let d be the maximal degree of the monomials occurring in any component of f and let $\{a_j\}$ be the set of coefficients of f. Let $\{w_1, \ldots, w_m\}$ be a basis for W and consider the B-subspace V of R generated by all the $a_j w_1^{\alpha_1} w_2^{\alpha_2} \cdots w_m^{\alpha_m}$ as $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ runs over all multi-indices of weight $\leq d$. Obviously, f takes $\prod W$ to $\prod V$ and is continuous on $\prod W$. Since the inclusion $\prod V \hookrightarrow R^n$ is continuous, we are done. □

Given any affine R-scheme X of finite-type, we can give the R-points $X(R)$ of X the topology induced by any embedding $X \hookrightarrow A^n_R$. The lemma above shows that this topology is independent of the embedding. Also, we get

Lemma 2 *If $X \to Y$ is a map of affine R schemes of finite-type, then the induced map on R points in continuous.*

Equally obvious from the definitions is the

Lemma 3 *If $R \to T$ is a map of B-algebras, then $X(R) \to X(T)$ is continuous for any affine R-scheme X of finite-type.*

For the cohomological considerations below, it will be useful to note the following

Lemma 4 *Let R be a B-vector space with the inductive limit topology and let $C \subset R$ be compact. Then C is contained in a finite-dimensional subspace $V \subset R$.*

Proof. We cannot have $C = R$, since then, $C \cap Bv = Bv \simeq B$ for some one-dimensional $Bv \subset R$ would be compact, and there is no infinite compact-field (recall that B has characteristic zero). By choosing $v \notin C$ and translating to $C - v$, we can assume that $0 \notin C$. Suppose we had an infinite collection $\{v_n\}_{n=1}^{\infty} \dots$ of linearly independent vectors in C. By passing to a subsequence, can assume that the v_n converge to $v \in C$. Write $R = W_1 \oplus W_2$ where W_1 is the subspace generated by the v_i's and W_2 is a complement. We can write $v = \sum_i c_i v_i + w$, where $w \in W_2$. Only finitely many v_i occur in the sum, say $1 \leq i \leq N$. Let p_2 be the projection to W_2. Then $0 = p_2(v_n)$ converges to $p_2(v) = w$, so $w = 0$. Now write $W_1 = W_1' \oplus W_1''$ where W_1' is generated by $v_1, \dots, v_N$ and W_1'' is generated by $v_{N+1}, \dots$. Consider the projection $p : W_1 \to W_1'$ determined by this decomposition. Then $p(v_n)$ converges to $p(v) = \sum_i c_i v_i$. But $p(v_i) = 0$ for $i > N$. So $\sum_i c_i v_i = 0$. Therefore, we conclude that $0 = v \in C$, a contradiction. □

We wish to consider certain continuous cohomology spaces for a compact topological group G. The setting will be slightly more general than appears necessary for this paper since we wish also to look ahead to future work. Assume that G acts continuously on B by field automorphisms and denote by F the fixed field for this action. In particular, the action could be trivial and $B = F$. We also denote by K a subfield of F such that $[F : K]$ is finite. Thus, F and K are both complete with the induced topology. For any K-algebra R, we get a B-algebra $B \otimes_K R$ which we equip with a G-action via $g(b \otimes r) = (gb) \otimes r$. This action is clearly continuous.

For each K-algebra R, we have a category $\mathfrak{C}_R$ whose objects are modules $\mathcal{P}$ over $B \otimes_K R$ equipped with the following data:

(1) A continuous, semi-linear action of G. Here, the continuity means that the action map $G \times \mathcal{P} \to \mathcal{P}$ is continuous, while the semi-linearity refers to $g(bx) = g(b)g(x)$ for $b \in B \otimes R, x \in \mathcal{P}$. Note that $\mathcal{P}$ can simply be regarded as a B-vector space, and hence, the topology is that discussed above.

(2) An increasing filtration W indexed by $\mathbb{Z}$ such that
- $W_n\mathcal{P} = 0$ for $n < 0$ and $\cup_n W_n\mathcal{P} = \mathcal{P}$.
- each $W_n\mathcal{P}$ is stable under the G action and is a finitely generated module over $B \otimes_K R$.

The morphisms in this category consist of

$$\mathrm{Hom}(\mathcal{P}, \mathcal{Q}) = \varprojlim_n \mathrm{Hom}(W_n\mathcal{P}, W_n\mathcal{Q}),$$

where $\mathrm{Hom}(W_n\mathcal{P}, W_n\mathcal{Q})$ is the set of linear maps of $B \otimes_K R$-modules that commute with the G-action.

There is a natural tensor product in this category obtained by putting the tensor-product filtration on $\mathcal{P} \otimes_{B \otimes_K R} \mathcal{Q}$:

$$W_n(\mathcal{P} \otimes \mathcal{Q}) = \Sigma_{i+j=n} W_i\mathcal{P} \otimes W_j\mathcal{Q}.$$

Let $\mathcal{P}$ be a finitely generated B algebra in the category $\mathcal{C} := \mathcal{C}_K$. Therefore, the structure maps are all required to be $\mathcal{C}$-morphisms. If we denote $P = \mathrm{Spec}(\mathcal{P})$, we have a natural action of G on the set of points $P(B \otimes R)$. Explicitly, an element $g \in G$ takes an $B-$algebra homomorphism $\phi : \mathcal{P} \to B \otimes R$ to $g\phi g^{-1}$.

Lemma 5 *This G-action is continuous.*

Take n large enough so that $W_n\mathcal{P}$ has a set of algebra generators for $\mathcal{P}$. Then for any B-algebra S, we have

$$P(S) \hookrightarrow \mathrm{Hom}_B(W_n\mathcal{P}, S) = \mathrm{Hom}_B(W_n\mathcal{P}, B) \otimes_B S.$$

But $\mathrm{Hom}_B(W_n\mathcal{P}, B)$ is just a finite-dimensional vector space over B, so this gives us an embedding $P \hookrightarrow \mathrm{Hom}_B(W_n\mathcal{P}, B)$ into a finite-dimensional affine space over B that is compatible with the G-action on $B \otimes_K R$-points. That is,

$$P(B \otimes_K R) \subset \mathrm{Hom}_B(W_n\mathcal{P}, B) \otimes_B B \otimes_K R = \mathrm{Hom}_B(W_n\mathcal{P}, B) \otimes_K R$$

with the induced topology. Therefore, we need only check the continuity of the action on $\mathrm{Hom}_B(W_n\mathcal{P}, B) \otimes_K R$. For any finite dimensional K-subspace V of R, $\mathrm{Hom}_B(W_n\mathcal{P}, B) \otimes_K V$ is a G-invariant subspace of $\mathrm{Hom}_B(W_n\mathcal{P}, B) \otimes_K R$ and these subspaces are cofinal among finite-dimensional B-subspaces of $\mathrm{Hom}_B(W_n\mathcal{P}, B) \otimes_K R$. The map $G \times (\mathrm{Hom}_B(W_n\mathcal{P}, B) \otimes_K V) \to \mathrm{Hom}_B(W_n\mathcal{P}, B) \otimes_K V$ is continuous by the continuity of the original G-action on $W_n\mathcal{P}$ and on B. Therefore, $G \times (\mathrm{Hom}_B(W_n\mathcal{P}, B) \otimes_K R) \to \mathrm{Hom}_B(W_n\mathcal{P}, B) \otimes_K R$ is also continuous. □

We will be interested in a pro-unipotent algebraic group U in the dual category $\mathcal{C}^\circ$. Thus, in $\mathcal{C}$, the corresponding object is a B-algebra $\mathcal{A}$ with the structure of a Hopf algebra. $\mathcal{A}$ is equipped with a multiplication

$$m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A},$$

a comultiplication

$$\delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A},$$

a unit $1 \in \mathcal{A}$, a counit $e : \mathcal{A} \to B$ and an antipode $i : \mathcal{A} \to \mathcal{A}$ which are morphisms in the category $\mathcal{C}$ and are compatible in the usual sense. Let $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \ldots \subset \mathcal{A}$ denote the filtration of $\mathcal{A}$ by subalgebras corresponding to the descending central series of $U := \mathrm{Spec}(\mathcal{A})$, normalized so that $U_1 = U/[U, U] = \mathrm{Spec}(\mathcal{A}_1)$. Let $U_n = \mathrm{Spec}(\mathcal{A}_n)$. We assume that each U_n is a unipotent algebraic group over B, i.e., that each $\mathcal{A}_i$ is a finitely generated integral domain. By a U-torsor, we will mean an affine $\mathcal{C}$-scheme P (in the sense of Deligne ([7], 5.4))with an action of U that makes it into a U-torsor in the usual sense (loc. cit.). However, we recall that the structure maps of the action are required to be in the category $\mathcal{C}$. Therefore, the coordinate ring $\mathcal{P}$ of P is an object of $\mathcal{C}$ and we are given a map $a : \mathcal{P} \to \mathcal{P} \otimes \mathcal{A}$ in the category $\mathcal{C}$ that induces a free and transitive group action on points. In our definition, we also assume that the torsors are strictly compatible with the weight filtration, in the sense that $\dim W_n \mathcal{P} = \dim W_n \mathcal{A}$ for each n. Now, note that when we forget the Galois action, such a torsor is always trivial, i.e., has a B−rational point $x \in P$, since the group U is unipotent. Such a point gives rise to an isomorphism of Hopf algebras:

$$\mathcal{P} \stackrel{a}{\to} \mathcal{P} \otimes \mathcal{A} \stackrel{x\otimes 1}{\to} \mathcal{A}.$$

The isomorphism preserves the filtration W:

$$W_n(\mathcal{P}) \to \Sigma_{i+j=n} W_i(\mathcal{P}) \otimes W_j(\mathcal{A}) \to \Sigma_{j\leq n} W_j(\mathcal{A}) \subset W_n(\mathcal{A})$$

since the filtration is increasing. In fact, this map is *strict* for the filtration, i.e., induces an isomorphism $W_n\mathcal{P} \to W_n\mathcal{A}$ for each n by the equality of dimensions.

The torsor P gives rise to a collection of U_n-torsors P_n obtained by push-out:

$$P_n := (P \times U_n)/U,$$

where U acts on the product via the diagonal action, that is, $g(p, x) = (pg, g^{-1}x)$.

More generally, we can consider a $U_R := \mathrm{Spec}(\mathcal{A} \otimes_K R)$-torsor $P = \mathrm{Spec}(\mathcal{P})$ in the category $\mathcal{C}_R^o$, defined in an obvious way analogous to the above discussion. The only further requirement is that $W_n\mathcal{P}$ is a free $B\otimes_K R$-module of finite rank equal to $\dim_B W_n\mathcal{A} = \mathrm{rank}_{B\otimes_K R}\mathcal{A} \otimes_K R$. In this case as well, the choice of a point $x \in P(B \otimes R)$ determines an isomorphism

$$\mathcal{P} \simeq \mathcal{A} \otimes_K R$$

that preserves the weight filtration. By changing base to closed points y of $\mathrm{Spec}(B \otimes R)$ whereupon we get torsors P_y for $(U_R)y$ over the fields $k(y)$, we see that $W_n\mathcal{P} \otimes k(y) \simeq W_n\mathcal{A} \otimes_K R \otimes_{B\otimes R} k(y) = W_n\mathcal{A} \otimes_B k(y)$ by dimension considerations, and hence, that $W_n\mathcal{P} \simeq W_n\mathcal{A} \otimes_K R$.

The basic classification goes as follows:

Proposition 1 *The isomorphism classes of U_R torsors are in bijection with the continuous cohomology set*

$$H^1(G, U(B \otimes_K R)).$$

The continuous cohomology occurring in the statement is defined in the standard way ([11], VII Appendix) which we review briefly. Given any K-algebra R we extend scalars to the B-algebra $B \otimes_K R$ and give $U(B \otimes_K R)$ the topology and G-action discussed above. This gives rise to the set $C^i(G, U(B \otimes_K R))$ of continuous i-cochains, which are defined to be continuous maps $c : G^i \to U(B \otimes_K R)$. The boundary maps

$d : C^i(G, U(B \otimes_K R)) \to C^{i+1}(G, U(B \otimes_K R))$ are defined in a standard way at least for $i = 0$ and $i = 1$. We recall the explicit description. In degree 0, $C^0(G, U(B \otimes_K R)) = U(B \otimes_K R)$ and for $u \in U(B \otimes_K R)$, we have

$$(du)(g) = ug(u^{-1}).$$

For $c : G \to U(B \otimes_K R)$ a continuous map,

$$dc(g_1, g_2) = c(g_1g_2)(g_1c(g_2))^{-1}c(g_1)^{-1}.$$

All the C^i are pointed sets, where the point is the constant map taking values in the identity element e of $U(B \otimes_K R)$. e will also be used to denote any of the corresponding cochains. We then have the continuous 1-cocycles $Z^1(G, U(B \otimes_K R)) \subset C^1(G, U(B \otimes_K R))$ defined as $d^{-1}(e)$. Thus, it consists of continuous maps $c : G \to U(B \otimes_K R)$ such that $c(g_1g_2) = c(g_1)g_1c(g_2)$. Consider then the action of $U(B \otimes_K R)$ on $Z^1(G, U(B \otimes_K R))$ by $(uc)(g) = uc(g)g(u^{-1})$. We define

$$H^1(G, U(B \otimes_K R)) := U(B \otimes_K R))\backslash Z^1(G, U(B \otimes_K R)).$$

In the case where V is a vector group over B, we also have conventional definitions of $Z^i(G, V(B \otimes_K R))$ (the $i-$cocycles), $B^i(G, V(B \otimes_K R))$ (the $i-$coboundaries), and

$$H^i(G, V(B \otimes_K R)) = B^i(G, V(B \otimes_K R))\backslash Z^i(G, V(B \otimes_K R))$$

for all i ([11], VII.2).

Proof of Proposition. Let $P = \mathrm{Spec}(\mathcal{P})$ be a U_R-torsor. Thus, $\mathcal{P}$ is an object of the category $\mathcal{C}_R$ and we are given a $\mathcal{C}$-morphism

$$a : \mathcal{P} \to \mathcal{P} \otimes \mathcal{A}$$

specifying the U-action. Now choose a point $x \in P(B \otimes R)$. Given an element $g \in G$, we have a unique element $u_g \in U(B \otimes R)$ such that $gx = xu_g$. We get thereby a map $G \to U(B \otimes R)$. To check continuity of this map we give an alternative description. x is an algebra homomorphism $\mathcal{P} \to B \otimes R$. Hence, as described above, we can form the composite

$$s_x : \mathcal{P} \xrightarrow{a} \mathcal{P} \otimes_{B\otimes R} (\mathcal{A} \otimes_K R) \xrightarrow{x\otimes 1} \mathcal{A} \otimes_K R$$

which is a continuous isomorphism of $G \otimes_K R$-algebras. For each $g \in G$, we then have a continuous isomorphism of algebras $c_x(g) := s_x g s_x^{-1} g^{-1} : \mathcal{A} \otimes_K R \to \mathcal{A} \otimes_K R$. As above, let n be large enough so that $W_n\mathcal{A}$ contains a generating set for $\mathcal{A}$. Then $c_x(g)$ is determined by its restriction to $W_n\mathcal{A}\otimes_K R$. Since $W_n\mathcal{A}\otimes_K R$ and $W_n\mathcal{P}$ are finite-rank over $B\otimes_K R$ and the original G-action on either side is continuous, the map $g \mapsto c_x(g)|W_n\mathcal{A}\otimes R$

is clearly continuous. On the other hand, the elements $y \in U(B \otimes_K R)$ act on $W_n\mathcal{A} \otimes R$ (and $\mathcal{A} \otimes R$) by

$$\phi_y : W_n\mathcal{A} \otimes R \xrightarrow{m} W_n(\mathcal{A} \otimes_K R \otimes_{B\otimes_K R} \mathcal{A} \otimes_K R) \xrightarrow{y^{-1}\otimes 1} W_n\mathcal{A} \otimes_K R$$

and this determines an affine embedding of U from which $U(B \otimes_K R)$ gets the induced topology.

Claim. $\phi_{u_g} = c_x(g)|W_n\mathcal{A} \otimes_K R$.

Let $f \in \mathcal{A} \otimes_K R$. We compute $c_x(g)(f)$ on points (an obvious dual argument gives the desired equality): For a point $y \in U$ and $h \in \mathcal{A} \otimes R$, the $G-$actions are related by $(gh)(y) = g(h(g^{-1}y))$. Also, given $h \in \mathcal{A} \otimes_K R$, $s_x^{-1}(h)(z) = h(y)$ where y is determined by $z = xy$. Then the proof of the claim is an exercise in careful bracketing:

$$\begin{aligned} c_x(g)(f)(y) &= \big(s_x\big(g\big(s_x^{-1}(g^{-1}f)\big)\big)\big)(y) = \big(g\big(s_x^{-1}(g^{-1}f)\big)\big)(xy) \\ &= g\big(\big(s_x^{-1}(g^{-1}f)\big)(g^{-1}(x)g^{-1}(y))\big) = g((g^{-1}f)(w)) \\ &\qquad\qquad\qquad\qquad (\text{for } xw = g^{-1}(x)g^{-1}(y)) \\ &= g(g^{-1}(f(g(w)))) = f(g(w)). \end{aligned}$$

But $xy = g(x)g(w) = xu_g g(w)$ so $g(w) = u_g^{-1}y$. That is to say, the end result is $(\phi_{u_g} f)(y)$, as desired.

Now, if $g_1, g_2 \in G$, then

$$g_1g_2x = g_1(xu_{g_2}) = g_1(x)g_1(u_{g_2}) = xu_{g_1}g_1(u_{g_2})$$

so that $g \mapsto u_g$ is a 1-cocycle. It is straightforward to check at this point that different choices of x give us equivalent cocycles, and hence, we get a well-defined class $c(P) \in H^1(G, U(B \otimes_K R))$.

In the other direction, given a 1-cocycle $c \in Z^1(G, U(B \otimes_K R))$, we can use it to twist the Galois action on $\mathcal{A} \otimes_K R$ by letting $\rho_c(g)(x) = \phi_{c(g)}gx$. The usual formula

$$\begin{gathered} \phi_{c(g_1g_2)}g_1g_2(x) = \phi_{c(g_1)g_1c(g_2)}g_1g_2(x) = \\ \phi_{c(g_1)}\phi_{g_1(c(g_2))}g_1g_2(x)) = \phi_{c(g_1)}g_1\phi_{(c(g_2))}g_1g_2(x)) = \phi_{c(g_1)}g_1(\phi_{(c(g_2))}g_2(x)) \end{gathered}$$

shows that this is a group action. The continuity of the cocyle gives the continuity of the action. Since the action by $\phi_{c(g)}$ preserves the filtration W, so does the representation ρ_c. Denote by $\mathcal{P}(c)$ the filtered algebra $\mathcal{A} \otimes_K R$ with this twisted action. We give $P(c) = \operatorname{Spec}(\mathcal{P}(c))$ the U_R-action by using the group law on U_R:

$$m : \mathcal{P}(c) \to \mathcal{P}(c) \otimes_{B\otimes_K R} \mathcal{A} \otimes_K R.$$

We need to check that this is compatible with the Galois action. Note that $\phi_y = (y^{-1} \otimes 1) \circ m$ for any $y \in U(B \otimes_K R)$. That is to say, we need to check the identity

$$[(c(g) \otimes 1 \circ m) \otimes 1](g \otimes g)(m(a)) = m((c(g) \otimes 1 \circ m)(ga)$$

for all $g \in G$ and $a \in \mathcal{A} \otimes_K R$. But $(g \otimes g) \circ m = m \circ g$ since m is G-equivariant for the original action. Thus, we have to check

$$[(c(g) \otimes 1 \circ m) \otimes 1] \circ m(a)) = m \circ (c(g) \otimes 1 \circ m)(a)$$

for all $a \in \mathcal{A}$. We can again check this by dualizing an argument on points of $U \times U$: The left-hand side evaluated on (h, k) becomes $f((gh)k)$ while the right-hand side is $f(g(hk))$.

The usual computation (cf. [11], X.2) shows that equivalent cocycles give isomorphic actions and that the two correspondences are inverses to each other. □

If $R \to S$ is a map of K-algebras, we have the induced map

$$H^1(G, U(B \otimes_K R)) \to H^1(G, U(B \otimes_K S))$$

which we view therefore as defining a functor on K-algebras. That is, we define the functor $H^1(G, U)$ by

$$H^1(G, U)(R) := H^1(G, U(B \otimes_K R))$$

for any K-algebra R. We can also define similar functors $C^1(G, U)$, $Z^1(G, U)$ and for vector groups V, $C^2(G, V)$, $Z^2(G, V)$ and $B^2(G, V)$. We denote $U^1 = U$ and $U^{i+1} = [U, U^i]$. Also, $U_i := U/U^{i+1}$.

We make the following important assumption:

$H^j(G, U^i/U^{i+1}(B))$ are finite-dimensional F-vector spaces for each i, j.

In section three we will also encounter the natural condition that the inclusion $\mathcal{A}^G \hookrightarrow \mathcal{A}$ of the G-invariants induces an isomorphism $(\mathcal{A}^G \otimes_F B) \simeq \mathcal{A}$ compatible with the G-action.

In this case, we denote by U^G, the unipotent algebraic group $\mathrm{Spec}(\mathcal{A}^G)$ and we see that for any K-algebra R, we have

$$H^0(G, U)(R) = \text{Alg-Hom}_G[\mathcal{A}, B \otimes_K R],$$

where Alg-Hom refers to B-algebra homomorphisms while the subscript G restricts to the G-invariant ones. However,

$$\begin{aligned} &\text{Alg-Hom}_G[\mathcal{A}, B \otimes_K R] = \text{Alg-Hom}_G\big[(\mathcal{A})^G \otimes_F B, B \otimes_K R\big] \\ &= \text{Alg-Hom}_G\big[(\mathcal{A})^G, B \otimes_K R\big] = \text{Alg-Hom}\big[(\mathcal{A})^G, (B \otimes_K R)^G\big] = \\ &= \text{Alg-Hom}\big[(\mathcal{A})^G, F \otimes_K R\big] = U^G(F \otimes_K R). \end{aligned}$$

That is, the functor $H^0(G, U)$ is represented by the Weil restriction $\mathrm{Res}_{F/K}(U^G)$. However, this last condition is not necesary for the following:

Proposition 2 *Suppose $H^0(G, U^i/U^{i+1})(K) = 0$ for each i. Then the functor $H^1(G, U)$ is representable by an affine pro-algebraic variety over K.*

Proof. Note that the assumption implies that $H^0(G, U^i/U^{i+1})(R) = 0$ for all i, and hence that $H^0(G, U_i)(R) = 0$, given any k-algebra R.

First we prove the elementary fact that each of the $H^1(G, U^i/U^{i+1})$ are representable. $V := U^i/U^{i+1}$ is a finite-dimensional vector group and we have $V(B \otimes_K R) = V(B) \otimes_K R$. For this, we have the

Lemma 6

$$f : C^n(G, V(B)) \otimes_K R \to C^n(G, V(B) \otimes R) \to C^n(G, V(B \otimes R))$$

is an isomorphism. Similarly,

$$Z^n(G, V(B) \otimes_K R) \simeq Z^n(G, V(B \otimes R))$$

and

$$B^n(G, V(B) \otimes_K R) \simeq B^n(G, V(B \otimes R)).$$

Proof of Lemma. We will just prove the first statement since the other two follow in exactly the same way. To check injectivity, let $\{r_i\}$ be a K-basis for R and let $c = \sum c_i \otimes r_i \in C^n(G, V(B)) \otimes_K R$. Suppose $f(c) = 0$. Then $f(c)(g) = \sum c_i(g) \otimes r_i = 0$ for all $g \in G$. Since r_i form a basis, this implies that $c_i(g) = 0$ for each i and g, so $c_i = 0$, and hence, $c = 0$. To check surjectivity, let $c : G^n \to V(B) \otimes R$ be continuous. Since G^n is compact, the image has to lie inside a finite-dimensional B-subspace. If the subspace is generated by $\sum_i b_{ij} \otimes r_i$, then at most finitely many r_i's occur in all these sums, so the image of c lies inside a subspace of the form $V(B) \otimes_K W$ where $W \subset R$ has finite K-dimension. Let $\{w_i\}$ be a basis for W. Then $c(g) = \Sigma c_i(g) \otimes w_i$ defines a finite set of $c_i : G^n \to V(B)$ and the element $\sum c_i \otimes w_i \in C^n(G, V(B)) \otimes W$ such that $f(\sum c_i \otimes w_i) = c$. □

Now since $H^1(G, V)$ is defined by the exact sequence

$$V(B \otimes R) \to Z^1(G, V(B \otimes R)) \to H^1(G, V(B \otimes R)),$$

we get

$$H^1(G, V)(R) \simeq H^1(G, V(B)) \otimes_K R$$

in a way functorial in R. So $H^1(G, V)$ is represented by the finite-dimensional vector group $H^1(G, V(B))$.

Now we will prove the theorem inductively for $H^1(G, U_n)$.

We have an exact sequence of algebraic groups

$$0 \to U^{n+1}/U^{n+2} \to U_{n+1} \to U_n \to 0$$

which realizes U_{n+1} as a U^{n+1}/U^{n+2}-torsor over U_n (and as a central extension of algebraic groups). But U^{n+1}/U^{n+2} is a vector group and U_n is affine so this torsor splits. Choose an algebraic splitting $s : U_n \to U_{n+1}$ for the projection $U_{n+1} \to U_n$. This induces a continuous map $U_n(S) \to U_{n+1}(S)$ for any B-algebra S which is in fact functorial in S. Thus, for any K algebra R, we have a split exact sequence

$$0 \to U^{n+1}/U^{n+2}(B \otimes R) \to U_{n+1}(B \otimes R) \to U_n(B \otimes R) \to 0.$$

Let $n = 1$. Then $U_1 = U/[U, U]$ is a vector group. We have shown that $H^1(G, U_1)$ is representable by a vector group.

Assume we have proved the representability for n. Consider the surjective map

$$Z^1(G, U_n(B \otimes R)) \to H^1(G, U_n)(R).$$

Taking for R the coordinate ring of $H^1(G, U_n)$, we have the element of $H^1(G, U_n)(R)$ corresponding to the identity map. Choosing a lifting to $Z^1(G, U_n(B \otimes R))$ gives us a functorial splitting

$$i : H^1(G, U_n) \to Z^1(G, U_n).$$

Composing with the section s and the boundary map $d : C^1(G, U_{n+1}) \to C^2(G, U_{n+1})$, we get a map $dsi : H^1(G, U_n) \to Z^2(G, U^{n+1}/U^{n+2})$. Define $I(G, U_n) := (dsi)^{-1}(B^2(G, U^{n+1}/U^{n+2}))$. Note that dsi composed with the natural quotient map $Z^2(G, U^{n+1}/U^{n+2}) \to H^2(G, U^{n+1}/U^{n+2})$ realizes the connecting homomorphism

$$\delta : H^1(G, U_n) \to H^2(G, U^{n+1}/U^{n+2})$$

in a functorial way (cf. [11], VII, Appendix, Prop. 2) and $I = \delta^{-1}(0)$, so it is a closed affine subvariety of $H^1(G, U_n)$. The previous lemma above shows that

$$C^1(G, U^{n+1}/U^{n+2}(B \otimes R)) \simeq C^1(G, U^{n+1}/U^{n+2}(B)) \otimes R$$

and

$$B^2(G, U^{n+1}/U^{n+2}(B \otimes R)) \simeq B^2(G, U^{n+1}/U^{n+2}(B)) \otimes R,$$

so if we choose a K-linear splitting

$$a : B^2(G, U^{n+1}/U^{n+2}(B)) \to C^1(G, U^{n+1}/U^{n+2}(B))$$

of the boundary map, then we get a functorial splitting

$$a : B^2(G, U^{n+1}/U^{n+2}) \to C^1(G, U^{n+1}/U^{n+2}).$$

Thus, we can define

$$b(x) = (si)(x)(adsi)(x)^{-1}$$

to get a map $b : I(G, U_n) \to Z^1(G, U_{n+1})$. By composing with the quotient map $Z^1(G, U_{n+1}) \to H^1(G, U_{n+1})$ we get a functorial section

$$I(G, U_n) \to H^1(G, U_{n+1})$$

of the surjection

$$H^1(G, U_{n+1}) \to I(G, U_n).$$

Claim. For each R, we have an exact sequence

$$0 \to H^1(G, U^{n+1}/U^{n+2})(R) \to H^1(G, U_{n+1})(R) \to I(G, U_n)(R) \to 0,$$

in the sense that the left hand group acts freely on the middle set, and the surjection identifies $I(G, U_n)(R)$ with the set of orbits.

Proof of Claim. The only non-evident part is the freeness of the action which follows from our assumption about the vanishing of H^0. That is, assume we have $c_{n+1} \in Z^1(G, U_{n+1})(R)$ that maps to $c_n \in Z^1(G, U_n)(R)$. Suppose $v \in Z^1(G, U^{n+1}/U^{n+2})(R)$ stabilizes the class of c_{n+1}. Then there exists a $u \in U_{n+1}$ such that $c_{n+1}(g)v(g) = u^{-1}c_{n+1}(g)g(u)$ for all $g \in G$. Projecting to U_n gives $c_n(g) = \bar{u}^{-1}c_n(g)g(\bar{u})$ for each g, where $\bar{u}$ is the projection of u to U_n. Hence, $c_n(g)g(\bar{u})c_n(g)^{-1} = \bar{u}$. By induction on n and our assumption that $H^0(G, U^i/U^{i+1}) = 0$ for all i, this implies that $\bar{u} = 0$. Therefore, $u \in U^{n+1}/U^{n+2}$ and hence, $v(g) = u^{-1}g(u)$ is a coboundary. □

The claim together with the section constructed above induces an isomorphism of functors

$$H^1(G, U^{n+1}/U^{n+2}) \times I(G, U_n) \simeq H^1(G, U_{n+1})$$

and concludes the proof that each $H^1(G, U_n)$ is represented by an affine variety. Now, the surjectivity of the map $U_{n+1}(B \otimes_K R) \to U_n(B \otimes_K R)$ can be used to show easily that $H^1(G, U) = \varprojlim H^1(G, U_n)$ as set-valued functors. □

We will be considering in Sect. 3 the important situation where we compare cohomology sets over K and B. That is to say, suppose U is defined over K and assume that the finite-dimensionality assumption preceding Prop. 2 is satisfied over both K and B. Given any K-algebra R, $U_n(R)$ acts as a subgroup on $U_n(B \otimes_K R)$ and gives rise to the 'exact sequence'

$$0 \to U_n(R) \to U_n(B \otimes_K R) \to U_n(B \otimes_K R)/U_n(R) \to 0,$$

from which we get a small part of a long exact sequence of pointed sets

$$H^0(G, U_n(B \otimes_K R)/U_n(R)) \to H^1(G, U_n(R)) \to H^1(G, U_n(B \otimes_K R)).$$

The topology on the quotient set will not be too important for us since we will be considering only H^0's for general unipotent groups. For example, it is entirely straightforward to check that one does get the portion of the long exact sequence displayed above with the H^1's being continuous cohomology. However, we will need to consider the topology somewhat in the case of a vector group V over K. The inclusion $K \hookrightarrow B$ gives K the induced topology and makes B into a topological K-vector space. Since $K \subset B$ is of course a finite-dimensional K-subspace, it has a topological complement $C \subset B$, which is a closed K-subspace such that $B = K \oplus C$ ([10], Sect. 10.7 (8)). Now we give $V(B)/V(K)$ the quotient topology and

$V(B \otimes R)/V(R) = (V(B)/V(K)) \otimes R$ the inductive limit of the topology coming from the subspaces $(V(B)/V(K)) \otimes W$, where $W \subset R$ is finite-dimensional. Then we see that the exact sequence

$$0 \to V(R) \to V(B \otimes R) \to V(B \otimes R)/V(R) \to 0$$

has a continuous K-linear splitting, giving rise to a long exact sequence of continuous cohomology groups. In particular, we see that $H^1(G,V(B)/V(K))$ is finite-dimensional and that

$$H^1(G, V(B \otimes R)/V(R)) \simeq H^1(G, V(B)/V(K)) \otimes R,$$

i.e., the functor

$$H^1(G, V^B/V)(R) := H^1(G, V(B \otimes R)/V(R))$$

is represented by the vector group $H^1(G, V(B)/V(K))$.

Proposition 3 *The functor*

$$H^0\big(G, U_n^B/U_n\big) : R \mapsto H^0(G, U_n(B \otimes_K R)/U_n(R))$$

is represented by an affine variety over K.

Proof. As before, if we examine the case of a vector group V, we get an exact sequence

$$H^0(G, V(B)) \to H^0(G, V(B)/V(K)) \to H^1(G, V(K))$$

that exhibits $H^0(G, V(B)/V(K))$ as a finite-dimensional K-vector space that represents the functor in question. Of course, in the vector group case, the functor $H^1(G, V^B/V)$ is also representable by the vector space $H^1(G, V(B)/V(K))$ as discussed above.

The general case is again proved by induction on n. That is, setting $V := U^{n+1}/U^{n+2}$, we easily verify that we have an exact sequence of G-sets

$$\begin{aligned} 0 \to V(B \otimes R)/V(R) &\to U_{n+1}(B \otimes R)/U_{n+1}(R) \\ &\to U_n(B \otimes R)/U_n(R) \to 0, \end{aligned}$$

from which we get a functorial exact sequence

$$0 \to H^0(G, V(B \otimes R)/V(R)) \to H^0(G, U_{n+1}(B \otimes R)/U_{n+1}(R))$$

$$\to H^0(G, U_n(B \otimes R)/U_n(R)) \stackrel{\delta}{\to} H^1(G, V(B \otimes R)/V(R)).$$

Taking the inverse image of the basepoint under the connecting map again defines a subfunctor

$$I_n(R) = \delta^{-1}(0) \subset H^0(G, U_n(B \otimes R)/U_n(R)).$$

When we apply the inductive hypothesis to regard $H^0(G, U_n^B/U_n)$ as an affine variety, then I_n becomes a closed subvariety. Denoting by $A(I_n)$ the coordinate ring of I_n, we then have an element in $I_n(A(I_n))$ corresponding to the identity, which can then be lifted to $H^0(G, U_{n+1}^B/U_{n+1})(A(I_n))$. This determines a splitting $I_n \to H^0(G, U_{n+1}^B/U_{n+1})$ and hence, an isomorphism of functors

$$H^0\big(G, U_{n+1}^B/U_{n+1}\big) \simeq H^0\big(G, U_n^B/U_n\big) \times I_n$$

that gives us the desired representability. □

Clearly the map

$$H^0(G, U_n(B \otimes_K R)/U_n(R)) \to H^1(G, U_n(R))$$

is functorial and hence is given by an algebraic map of varieties (when the latter is also representable as in Prop. 2). Thus, taking $R = K$, we see that the inverse image of the basepoint under the map $H^1(G, U(K)) \to H^1(G, U(B))$ is the set of K-points of a subvariety $H_f^1(G, U_n)$ of $H^1(G, U_n)$.

2. The p-adic K-Z equation

In this section, k denotes an algebraic closure of $\mathbf{F}_p$. $\mathcal{O}_F$ is a complete discrete valuation ring with residue field k, and F denotes the fraction field of $\mathcal{O}_F$. Vect_F refers to the category of finite-dimensional vector spaces over F. $\mathcal{X}$ denotes the projective line minus the three points $0, 1, \infty$: $\mathcal{X} = \mathrm{Spec}\mathcal{O}_F[z][1/z(z-1)]$. X is the generic fiber of $\mathcal{X}$ and Y the special fiber. Following Deligne, we define $\pi_{1,DR}(X, x)$, the De Rham fundamental group of X with basepoint $x \in X(F)$ to be the Tannakian fundamental group of the category Un of unipotent vector bundles with connection on X associated to the fiber functor $e_x : \mathrm{Un} \to \mathrm{Vect}_F$ that takes (V, ∇) to the vector space V_x ([7] , 10.27). This definition can be generalized to include tangential basepoints as well as to torsors of paths $\pi_{1,DR}(X, x, y)$ associated to two points $x, y \in X$ ([7], 15.28). That is, if we let $x \in \mathbf{P}^1 \setminus X$ and let v be an F-rational tangent vector to $\mathbf{P}^1$ at x we get a fiber functor e_v according to the following procedure. First we canonically extend (V, ∇) to a connection $(\bar{V}, \bar{\nabla})$ on $\mathbf{P}^1$ with log poles along $\{0, 1, \infty\}$ and nilpotent residues. Deligne then describes a procedure for associating to this data a connection $\mathrm{Res}(V, \nabla)$ on $T_x(\mathbf{P}^1) \setminus \{0\}$. $\mathrm{Res}(V, \nabla)$ is functorial in (V, ∇) and taking the fiber of $\mathrm{Res}(V, \nabla)$ at v defines the functor e_v. Now given two of these (possibly tangential) fiber functors e_x and e_y, we get the functor of isomorphisms from e_x to e_y, which is a right torsor $\pi_{1,DR}(X, x, y)$ for the group $\pi_{1,DR}(X, x)$. Define $U := \pi_{1,DR}(X, v)$ where v is the tangent vector d/dz at $0 \in \mathbf{P}^1$. Define $P(x) := \pi_{1,DR}(X, v, x)$ for $x \in X(F)$. As in the previous section, we will also consider the lower central series U_n and the associated torsors $P_n(x)$.

Let X_{an} denote the rigid analytic space associated to X. We will be using the ring Col_a of Coleman functions on X_{an} with respect to the embedding

$j : Y \hookrightarrow \mathbf{P}^1_k$ ([2], Sect. 4). For any rigid analytic space Z, we will denote by $\mathrm{An}(Z)$ the ring of rigid analytic functions on Z. Given $a \in F$, denote by $\log_a$ the branch of the p-adic log normalized by the condition $\log_a(p) = a$. Given a point $y \in \mathbf{P}^1(\bar{k})$, we have the tube $]y[\subset \mathbf{P}^1_F$ consisting of points that reduce to y. For $0 \le r < 1$, $X_{an}(r)$ denotes the rigid analytic space obtained by removing from $\mathbf{P}^1_{an}$ all closed disks of radius r around 0, 1, and ∞. Define $\mathrm{An}^a_{loc} := \Pi_{y \in \mathbf{P}^1(\bar{k})} \mathrm{An}^a_{\log}(y)$ where $\mathrm{An}^a_{\log}(y) := \mathrm{An}(]y[)$ if $y \neq 0, 1, \infty$ while $\mathrm{An}^a_{\log}(y) := \lim_{r \to 1} \mathrm{An}(]y[\cap X_{an}(r))[\log_a z_y]$ if $y = 0, 1$, or ∞ [8]. Here, z_y denotes z, $z - 1$, or $1/z$ for $y = 0, 1$, or ∞, respectively. Denote by $\mathrm{An}^\dagger$ the ring $\Gamma(\mathbf{P}^1_{an}, j^\dagger \mathcal{O}_{\mathbf{P}^1_{an}})$, where $j^\dagger$ is Berthelot's dagger functor [1]. The ring Col_a naturally contains $\mathrm{An}^\dagger$ and is equipped with a map $r_y : \mathrm{Col}_a \to \mathrm{An}_{\log}(y)$ for each y. We refer to Besser ([2], Sects. 4 and 5) for the precise definitions as well as a full discussion of Coleman integration.

There is a crystalline interpretation of the De Rham fundamental group via overconvergent connections. That is, we consider the category Un_{an} of unipotent overconvergent (iso-)crystals on Y. In fact, there is an equivalence of categories $E : \mathrm{Un} \simeq \mathrm{Un}_{an}$ ([5], Prop. 2.4.1). If we let $y \in Y$, there is the fiber functor $s_y : \mathrm{Un}_{an} \to Vect_F$ which associates to a crystal M the horizontal sections on the tube $]y[$ ([2], p.26). Suppose $x \in X$ reduces to $y \in Y$. Then the map ev_x evaluating horizontal sections at the point x defines an isomorphism of functors $ev_x : s_y \circ E \simeq e_x$. We can define s_y also for $y = 0, 1$ or ∞. For example, if $y = 0$, then s_0 associates to the overconvergent crystal M, a full set of solutions with coefficients in $\mathrm{An}^a_{\log}(0)$. A similar discussion holds near 1 or ∞. For any points $y, y' \in \mathbf{P}^1(k)$, we have the group of isomorphisms from s_y to $s_{y'}$ giving rise to the crystalline fundamental groups $\pi_{1,cr}(Y, y)$, which are pro-unipotent algebraic groups over F, and the torsors of paths $\pi_{1,cr}(Y, y, y')$ ([2], Sect. 3). Now if $x, x' \in X(F)$ reduce to $y, y' \in Y$, then we have a natural isomorphism $\pi_{1,DR}(X, x, x') \simeq \pi_{1,cr}(Y, y, y')$ defined by the evaluation isomorphisms ev_x and $ev_{x'}$. On the other hand, the crystalline torsors have the natural action of Frobenius endomorphisms $\phi : \pi_{1,cr}(Y, y, y') \to \pi_{1,cr}(Y, y, y')$ (loc. cit.). That is to say, if $f : \mathbf{P}^1_k \to \mathbf{P}^1_k$ is the geometric Frobenius of $\mathbf{P}^1_k$, then there is a power f^a of f that fixes y and y'. Therefore, the pull-back $(f^a)^*$ defines an automorphism of the category Un_{an} that commutes with the functors s_y and $s_{y'}$. The endomorphism $\pi_{1,cr}(Y, y, y') \to \pi_{1,cr}(Y, y, y')$ induced by any such pull-back will be called a Frobenius endomorphism and denoted by the same letter ϕ when no danger of confusion is present. Besser and Furusho define fiber functors s_w on unipotent crystals also for tangential basepoints $w \in T_y(\mathbf{P}^1)$, $y \in \mathbf{P}^1 \setminus Y$ ([3], Sect. 2). This construction *uses* the equivalence $E : \mathrm{Un} \simeq \mathrm{Un}_{an}$ discussed above. That is, given a unipotent crystal M, $\mathrm{Res}_y(M) := E(\mathrm{Res}_x(E^{-1}(M)))$ where $x = 0, 1$ or ∞ following the value of y. Here, $\mathrm{Res}_x(E^{-1}(M))$ is an overconvergent connection on $T_x(\mathbf{P}^1_F)_{an} \setminus \{0\}$, the analytic space associated to $T_x(\mathbf{P}^1_F) \setminus \{0\}$. Hence, $\mathrm{Res}_y(M)$ is an overconvergent unipotent crystal on $T_y(\mathbf{P}^1_k) \setminus \{0\}$. Then s_w associates to M, the horizontal sections to $E(\mathrm{Res}_y(E^{-1}(M)))$ on the tube

$]w[$ of w in $T_x(\mathbf{P}^1_F)_{an} \setminus \{0\}$. If an F-rational v in $T_x(\mathbf{P}^1_F) \setminus \{0\}$ reduces to w, then the construction makes it clear that evaluation of horizontal sections at v defines a isomorphism of functors $ev_v : s_w \circ E \simeq e_v$.

According to Besser ([2], Cor. 3.2) and Besser-Furusho ([3], Thm. 2.8), for any two points y, y', possibly tangential, we have a unique Frobenius invariant path $\gamma_F(y, y') \in \pi_{1,cr}(Y, y, y')$. Thus, if x and x' reduce to y and y', we get a Frobenius invariant path in $\pi_{1,DR}(X, x, x')$ which we will denote by $\gamma_F(x, x')$. Note that if x and x' in $X(F)$ reduce to the same point $y \in Y$, then this path is described on connections (V, ∇) as follows: Given an element $t_x \in V_x$, let s be the unique horizontal section on the tube of y such that $s(x) = t_x$. Then $\gamma_F(x, x')(t_x) = s(x')$. Now let $v = d/dz$, a tangent vector in $\mathbf{P}^1$. Then for a point $x \in X(F)$ reducing to $y \in Y$, the description of the path $\gamma_F(v, x) \in \pi_{1,DR}(X, v, x) = P(x)$ is slightly more intricate ([3], Prop. 2.11). As above, let $(\bar{V}, \bar{\nabla})$ be the canonical extension of (V, ∇). Then we have a canonical isomorphism $\mathrm{Res}_0(V, \nabla)_v \simeq \bar{V}_0$. An element t of $s_0(E(V, \nabla))$ can be identified with an element of $\mathrm{An}^a_{\log}(0) \otimes \bar{V}_0$, and hence, can be written $t = t_0 + t_1(\log_a(z)) + t_2(\log_a(z))^2 + \cdots$ where the t_i are Laurent series with values in $\bar{V}_0$ converging in some annulus. The constant term $c_0(t) \in \bar{V}_0$ is then defined to be the constant term of the Laurent series t_0. Now if we start with a vector $t_v \in \mathrm{Res}_0(V, \nabla) = \bar{V}_0$, we can find a unique $t \in s_0(E(v, \nabla))$ such that $c_0(t) = t_v$, and then $\gamma_F(v, x)(t_v) = \gamma_F(0, y)(t)(x)$.

As described in Deligne ([7], Sect. 12), there is also a canonical element $\gamma_{DR}(x) \in P(x)$ corresponding to the 'De Rham trivialization' of unipotent connections. So for any $x \in]Y[$, we get an element $g_x \in U$ such that $\gamma_{DR}(x) g_x = \gamma_F(x)$. We call this map $U_v\mathrm{Alb} : x \in]Y[\mapsto g_x \in U$ the unipotent Albanese map. The corresponding images $U_v\mathrm{Alb}_n(x) \in U_n$ are p-adic analogues of the 'higher' Albanese maps of Hain [9], except for the fact that in the p-adic case, we can dispense of periods using the Frobenius invariant path. We note that if we choose a different base-point z, then there are obviously Albanese maps $U_z\mathrm{Alb}$ and $U_z\mathrm{Alb}_n$ defined analogously to $U_v\mathrm{Alb}$.

Let $V := F << A, B >>$ be the ring of non-commutative power series over F in the variables A and B. A and B act as endomorphisms of the fiber functor of evaluation at v via the residue of a connection. That is, if we are given an object (E, ∇), then the residue at zero defines an endomorphism

$$a = \mathrm{Res}_0 : \bar{E}_0 \to \bar{E}_0,$$

while the residue at 1 defines

$$b : \bar{E}_0 \simeq \bar{E}_1 \xrightarrow{\mathrm{Res}_1} \bar{E}_1 \simeq \bar{E}_0,$$

(where the first and last isomorphisms are the canonical De Rham paths) giving us the actions of A and B. Then U can be identified with the group-like elements of V while U_n is identified with the group-like elements of $V_n := V/I^{n+1}$ ([7], 16.1.4 and Prop. 16.4).

Now consider the trivial pro-vector bundle $\mathcal{V}$ on X with fiber V and connection

$$\nabla(1) = Adz/z + Bdz/(z-1).$$

As in Furusho ([8], Thm. 3.4), there is a unique horizontal section $G_a(z)$ with coefficients in the Coleman functions Col_a such that $G_a(z) \to z^A := \exp(\log_a(z)A)$ as $z \to 0$. This means that $G_a(z)z^{-A}$ is of the form $1 + u(z)$ where $u(z) \in \mathrm{An}(]0[) \otimes V$ and $u(0) = 0$. Write $G_a(z) = \Sigma_w g_w^a(z)w$ where w runs over the words in A, B. In fact, $G_a(z) \in U$ (i.e., it is group-like) for each z ([8], proof of Prop. 3.39).

Proposition 4 *For any $z \in]Y[$, $UAlb(z) = G_a(z) := r_y(G_a)(z)$ where y is the reduction of z.*

Proof. The representation of U associated to the bundle $\mathcal{V}$ is simply V with the canonical action of U by left multiplication. The De Rham trivialization assigns to any point z, the isomorphism $V_v \simeq V \simeq V_z$. We know that $G_a(z)$ is group-like for each z. If we expand $G_a(z)$ as a power series in $\log_a(z)$, its constant term is 1. Hence, given any vector $l \in V$, $G_a(z)l$ is a horizontal section of $\mathcal{V}$ such that its constant term near zero is l. Furthermore, it is a horizontal section of $\mathcal{V}$ with coefficients in the Coleman functions. As described above, according to Besser and Furusho, taking the constant term of a horizontal section near zero corresponds to the fiber functor associated to the tangential base-point $v = d/dz$. It is essentially tautological then that $g_F(z)l = G_a(z)l$, but let us briefly sketch the logic: To construct $G_a(z)$, one constructs the solution $t = \Sigma t^w w$ to the K-Z equation satisfying the asymptotic condition in $\mathrm{An}(]0[)[\log_a(z)] \otimes V$ and then translates it to a solution $t_y = \Sigma t_y^w w$ in $\mathrm{An}_{\log}^a(y) \otimes V$ using the Frobenius invariant isomorphism $\gamma_F(y) : (\mathrm{An}_{\log}^a(0) \otimes V)^{\nabla=0} \simeq (\mathrm{An}_{\log}^a(y) \otimes V)^{\nabla=0}$. One gets thereby a Coleman function $g_w^a = [(\mathcal{V}, f_w, \{t_y\})]$ for each word w, where f_w denote the projection to $\mathcal{O}_X^\dagger$ given by the w-component. One then gets the solution $[G_a] := \Sigma g_w^a w$ which has values in V and coefficients in Col_a. From the construction, we have $r_y(g_w^a) = t_y^w$ for each w and y. Now, if $l \in V$, by the construction above, we get that $\gamma_F(z)l = t_y(z)l = r_y(G_a)(z)l = G_a(z)l$. Since the action of U on V is faithful, we get $\gamma_F(z) = G_a(z)$. □

Theorem 1 *The functions $g_w^a(z)$ on $]Y[$ are linearly independent over F.*

Proof. Given two different choices of branches $a, b \in F$ for the p-adic logarithm, there is an isomorphism of rings $Col_a \simeq Col_b$ characterized by $f(z) \mapsto f(z)$ for $f(z) \in \mathrm{An}^\dagger$ and $\log_a(z) \mapsto \log_b(z)$. The proof will use the comparison between $G_a(z)$ and $G_b(z)$. First, note that for functions in Col_a, linear independence over F is equivalent to independence over the ring of locally constant functions. To see this, suppose f_i are Coleman functions linearly independent over the constants and assume that $\sum_i c_i f_i = 0$ where the c_i are locally constant. For any point z in $]Y[$, there exists a neighborhood

$z \in O$ such that the c_i are constant on O. Then $\sum_i c_i(z) f_i = 0$ on O. By the identity principle for Coleman functions, this implies $\sum_i c_i(z) f_i = 0$ everywhere. Then, using the linear independence over the constants, this implies that $c_i(z) = 0$ for all i.

Using the differential equation, it is easy to see that $G_a = G_b c(z)$ for a locally constant function $c(z)$. By the asymptotic condition at 0, we get that $c(z) \approx z_b^{-A} z_a^A$ as $z \to 0$, where we write the subscripts a, b to denote the dependence on the different logs. In fact, let's write this out a bit more precisely.

We have that $G_a(z) z_a^{-A} \approx 1$ and $G_b(z) z_b^{-A} \approx 1$, so from, $G_a(z) z_a^{-A} = G_b z_b^{-A} z_b^A c(z) z_a^{-A}$ we get

$$z_b^A c(z) z_a^{-A} \approx 1,$$

or more precisely,

$$z_b^A c(z) z_a^{-A} = 1 + u_A(z) A + u_B(z) B + \cdots ,$$

where $u_A(z)$ etc. are rigid analytic functions that vanish at the origin. So

$$c(z) = 1 + (\log_a(z) - \log_b(z) + u_A(z)) A + u_B(z) B + \cdots .$$

Since the difference of the locally analytic log functions $\log_a(z) - \log_b(z)$ is locally constant and $c(z)$ is locally constant, we get $u_A = 0$, $u_B = 0$. That is, $c(z) = 1 + (\log_a(z) - \log_b(z)) A + d(z)$ where $d(z)$ involves only words of length ≥ 2.

Recall that there are also functions G_a^1 and G_b^1 satisfying the same differential equations and the asymptotic condition

$$G_a^1 \approx (1 - z)_a^B, \; G_b^1 \approx (1 - z)_b^B$$

near as $z \to 1$. By an argument entirely similar to that above, this gives us $G_a^1(z) = G_b^1(z) c^1(z)$ for a locally constant function

$$c^1(z) = 1 + (\log_a(1 - z) - \log_b(1 - z)) B + d^1(z)$$

where $d^1(z)$ also involves only words of length ≥ 2. Meanwhile, the p-adic Drinfeld associator Φ relates the two asymptotics, so that $G_a(z) = G_a^1(z)\Phi$ and $G_b(z) = G_b^1(z)\Phi$ for the same Φ [8]. This gives us the additional relation

$$G_a(z) = G_a^1(z)\Phi = G_b^1(z) c^1(z)\Phi = G_b(z)\Phi^{-1} c^1(z)\Phi,$$

or $c(z) = \Phi^{-1} c^1(z)\Phi$ near $z = 1$. The explicit formula of Furusho (Example 3.35) shows that Φ involves no linear terms, so that $c(z) = 1 + (\log_a(1 - z) - \log_b(1 - z)) B + d'(z)$ near $z = 1$, again for d' with words of length ≥ 2. Write

$$G_a(z) = \sum g_w^a(z) w, \quad G_b(z) = \sum g_w^b(z) w.$$

Now we will prove that the g_w^a are linearly independent by induction on the length $n = l(w)$ of w. The statement is clearly true for $n = 1$. Assume that the g_w^a are linearly independent for w of length $\leq n - 1$.

Suppose $\sum a_w g_w^a = 0$ for some a_w running over w of length $\leq n$. Then $\sum_w a_w g_w^b = 0$, that is, linear relations are preserved under a change of logs. Write the relations as

$$\begin{aligned}&\textstyle\sum_{l(w)=n-1} a_{wA} g_{wA}^a + \sum_{l(w)=n-1} a_{wB} g_{wB}^a\\ &+ \textstyle\sum_{l(w)=n-2} a_{wA} g_{wA}^a(z) + \sum_{l(w)=n-2} a_{wB} g_{wB}^a\\ &+ \textstyle\sum_{l(w)\leq n-2} a_w g_w^a(z) = 0\end{aligned}$$

and

$$\begin{aligned}&\textstyle\sum_{l(w)=n-1} a_{wA} g_{wA}^b + \sum_{l(w)=n-1} a_{wB} g_{wB}^b\\ &+ \textstyle\sum_{l(w)=n-2} a_{wA} g_{wA}^b(z) + \sum_{l(w)=n-2} a_{wB} g_{wB}^b\\ &+ \textstyle\sum_{l(w)\leq n-2} a_w g_w^b(z) = 0.\end{aligned}$$

Using the relation between G_a and G_b near $z = 0$ derived above, that is,

$$G_a = G_b c(z) = G_b(z)(1 + (\log_a(z) - \log_b(z))A + d(z)),$$

we get from the first equation

$$\begin{aligned}&\textstyle\sum_{l(w)=n-1} a_{wA}\big(g_{wA}^b(z) + (\log_a(z) - \log_b(z)) g_w^b(z)\big) + \sum_{l(w)=n-1} a_{wB} g_{wB}^b\\ &+ \textstyle\sum_{l(w)=n-2} a_{wA}\big(g_{wA}^b + (\log_a(z) - \log_b(z)) g_w^b(z)\big) + \sum_{l(w)=n-2} a_{wB} g_{wB}^b\\ &+ \textstyle\sum_{l(w)\leq n-2} e_w g_w^b(z) = 0,\end{aligned}$$

and subtracting this from the second equation, we get

$$(\log_a(z) - \log_b(z)) \sum_{l(w)=n-1} a_{wA} g_w^b(z) + \sum_{l(w)\leq n-2} e'_w g_w^b(z) = 0.$$

From this we get $a_{wA} = 0$ when $l(w) = n - 1$. Now, using the expression for $c(z)$ near $z = 1$, an entirely similar argument gives us $a_{wB} = 0$ for $l(w) = n - 1$. Then by induction, we get $a_w = 0$ for all w. □

We will now suppress the choice of log from the notation. Suppose we take another basepoint x instead of v. We wish to compare $U_x\mathrm{Alb}$ and $U_v\mathrm{Alb}$. Using the uniqueness of the Frobenius invariant path we see that $U_x\mathrm{Alb}(y) = (\gamma_{DR}(x))U_v\mathrm{Alb}(y)\gamma_F^{-1}(x)$. If α_w is the function on U that picks off the coefficient of the word w, then the formula for the comultiplication on U shows that

$$\alpha_w\big(U_v\mathrm{Alb}(\cdot)\gamma_F^{-1}(x)\big) = \alpha_w(U_v\mathrm{Alb}(\cdot)) + \sum_{\{w'|w=w'w''\}} c_{w'}^w \alpha_{w'}(U_v\mathrm{Alb}(\cdot)),$$

where the $c_{w'}^w$ are constants (depending on x). Thus, by induction on the length of w, the linear independence of the g_w implies the linear independence of the $\alpha_w(U_v\mathrm{Alb}(\cdot)\gamma_F^{-1}(x))$. Similarly, by considering multiplication on the other side, $g_w(x, \cdot) := \alpha_w((\gamma_{DR}(x))U_v\mathrm{Alb}(\cdot)\gamma_F^{-1}(x)$ are linearly independent.

Although we will not need it, as discussed in Furusho [8], the function $G_a(z)$ extends to a function on $X(\mathbb{C}_p)$ and the same argument as that given above shows that the coefficients are linearly independent over $\mathbb{C}_p$.

3. The non-abelian method of Chabauty

Now let $\mathcal{X} = \mathbf{P}^1_{\mathbb{Z}[1/S]} - \{0, 1, \infty\}$. Let X be the generic fiber of $\mathcal{X}$ and Y the special fiber over a prime $p \notin S$. Let $\bar{x}$ be a $\mathbb{Z}_p$ point of $\mathcal{X}$ and $x \in X$ its restriction to $X \otimes \mathbb{Q}_p$. $y \in Y$ denotes its reduction to $\mathbf{F}_p$. Associated to $X \otimes \mathbb{Q}_p$ we have the $\mathbb{Q}_p$-unipotent étale fundamental group $\pi_{1,\text{ét}}(X \otimes \mathbb{Q}_p, x)$ and the De Rham fundamental group $\pi_{1,DR}(X \otimes \mathbb{Q}_p, x)$ which is a unipotent group over $\mathbb{Q}_p$. We denote the corresponding coordinate rings by $A_{\text{ét}}$ and A_{DR}. Let $G = G_{\mathbb{Q}_p}$ be the Galois group of $\mathbb{Q}_p$. We also denote by $[A_{\text{ét}}]_n$ and $[A_{DR}]_n$ the subalgebras corresponding to the lower central series of the fundamental groups, where we set the index so that the abelianization corresponds to $n = 1$. In the following, if a definition, statement, or proof of a statement works the same way for a group or a quotient in the descending central series, we will give it only for one of them unless a danger of confusion presents itself. In fact, the only cause for concern arises from the p-adic comparison theorem. According to Vologodsky ([13], 1.8, Thm A), $[A_{\text{ét}}]_n$ is a De Rham representation of G for $(p-1)/2 \geq n+1$ and $[A_{DR}]_n \simeq D([A_{\text{ét}}]_n) := ([A_{\text{ét}}]_n) \otimes_{\mathbb{Q}_p} B_{DR})^G$ together with the induced weight and Hodge filtration. We elaborate briefly on the relation between this statement and the statement of Vologodsky's theorem: If x is an $\mathbb{Q}_p$-point of $X \otimes \mathbb{Q}_p$, denote by $e_x(n)$ the fiber functor of the previous section restricted to the category of unipotent vector bundles on X with connection having index of unipotence $\leq n$. Similarly, denote by $e'_x(n)$ the natural fiber functor defined by x on the category of unipotent $\mathbb{Q}_p$-étale local systems on $X \otimes_{\mathbb{Q}_p} R \otimes_{\mathbb{Q}_p} \bar{\mathbb{Q}}_p$, again restricting to those of index of unipotence $\leq n$. Denote as before by e_x the fiber functor on all unipotent connections and by e'_x the corresponding fiber functor on all $\mathbb{Q}_p$-local systems. What he considers are two categories C^{et} and C^{dr} (Vologodsky uses the notation $\mathcal{P}^{et}$ and $\mathcal{P}^{dr}$) whose objects are $\mathbb{Q}_p$-points of $X \otimes \mathbb{Q}_p$ and such that $Mor_{C^{et}}(x, y) = \mathrm{Hom}(e'_x(n), e'_y(n))$ and $Mor_{C^{dr}}(x, y) = \mathrm{Hom}(e_x(n), e_y(n))$. Vologodsky defines an equivalence of categories

$$C^{et} \otimes B_{DR} \simeq C^{dr} \otimes B_{DR}.$$

But then, we have an isomorphism between $\mathrm{Hom}(e'_x(n), e'_x(n)) \otimes B_{DR}$ and $\mathrm{Hom}(e_x(n), e_y(n)) \otimes B_{DR}$, and hence, between $\mathrm{Hom}(e'_x(n), e'_x(n))^* \otimes B_{DR}$ and $\mathrm{Hom}(e_x(n), e_y(n))^* \otimes B_{DR}$. Now, the symmetric tensor product on connections and étale local systems gives rise to a co-commutative co-product on the Hom spaces $\mathrm{Hom}(e'_x, e'_x)$ and $\mathrm{Hom}(e_x, e_x)$, and hence, a commutative product on the dual spaces $\mathrm{Hom}(e'_x, e'_x)^*$ and $\mathrm{Hom}(e_x, e_x)^*$. We have the family of inclusions

$$\cdots \mathrm{Hom}(e'_x(n), e'_x(n))^* \hookrightarrow \mathrm{Hom}(e'_x(n+1), e'_x(n+1))^* \hookrightarrow \cdots \hookrightarrow \mathrm{Hom}(e'_x, e'_x)^*$$

and

$$\cdots \mathrm{Hom}(e_x(n), e_x(n))^* \hookrightarrow \mathrm{Hom}(e_x(n+1), e_x(n+1))^* \hookrightarrow \cdots \hookrightarrow \mathrm{Hom}(e_x, e_x)^*.$$

But in fact, $[A_{\text{ét}}]_n$ and $[A_{DR}]_n$ are generated as algebras by $\mathrm{Hom}(e'_x(n), e'_x(n))^*$ and $\mathrm{Hom}(e_x(n), e_y(n))^*$. That is,

$$[A_{\text{ét}}]_n \simeq T\big(\mathrm{Hom}(e'_x(n), e'_x(n))^*\big)/I'$$

and

$$[A_{DR}]_n \simeq T(\mathrm{Hom}(e_x(n), e_x(n))^*)/I$$

and we have an isomorphism

$$T(\mathrm{Hom}(e'_x(n), e'_x(n))^*) \otimes B_{DR} \simeq T(\mathrm{Hom}(e_x(n), e_x(n))^*) \otimes B_{DR}$$

which carries I' to I, since the isomorphisms are compatible with the inclusions (from smaller n) and relations come from the shuffle relations on either side ([7], Sect. 16). Therefore, we get

$$[A_{\text{ét}}]_n \otimes B_{DR} \simeq [A_{DR}]_n \otimes B_{DR}$$

compatibly with all the structures. Here of course, the Hodge filtration on $A_{\text{ét}}$ is trivial ($F^0 A_{\text{ét}} = A_{\text{ét}}, F^1 A_{\text{ét}} = 0$), the weight filtration on B_{DR} is trivial ($W_0 B_{DR} = B_{DR}, W_{-1} B_{DR} = 0$), and the Galois action on A_{DR} is trivial. For $\mathbb{Q}_p$-algebras R of finite type, we will be considering torsors P of $[\pi^R_{1,\text{ét}}]_n := [\pi_{1,\text{ét}}(X, x)]_n \otimes_{\mathbb{Q}_p} R$ of *De Rham type*. (We will also simply call these 'De Rham torsors.) We first define it for finite field extensions L of $\mathbb{Q}_p$ by the condition that the coordinate ring $\mathcal{P}$ of the $\pi^L_{1,\text{ét}}$-torsor P has the property that the natural inclusion $D(\mathcal{P}) \to \mathcal{P} \otimes B_{DR}$ induces an isomorphism $D(\mathcal{P}) \otimes B_{DR} \simeq \mathcal{P} \otimes B_{DR}$, where $D(\mathcal{P}) := (\mathcal{P} \otimes B_{DR})^G$. Here, we assume that the isomorphism respects all the structures, namely, the Galois action, the weight filtration, and the Hodge filtration. Since each level W_n of the weight filtration is finite-dimensional, the De Rham property is equivalent to the condition $\dim W_n(D(\mathcal{P})) = \dim W_n \mathcal{P}$ for each n. Now for R of finite type, say that P is De Rham if its base change to all the closed points of $\mathrm{Spec}(R)$ is De Rham. Denote by $\pi^B_{1,\text{ét}}$, the base change $\pi^B_{1,\text{ét}} := \mathrm{Spec}(A_{\text{ét}} \otimes_{\mathbb{Q}_p} B_{DR})$. The construction of Sect. 1, namely Proposition 2, allows us to interpret $H^1(G, [\pi_{1,\text{ét}}]_n)$ canonically as an affine variety (for $B = F = K = \mathbb{Q}_p$ with trivial action) whose $\mathbb{Q}_p$-rational points (which is the cohomology set of interest) we will now denote by

$H^1(G, [\pi_{1,\acute{e}t}]_n)(\mathbb{Q}_p)$. To check the hypothesis of the proposition, it suffices to note that for all $n \geq 1$, $[\pi_{1,\acute{e}t}]_n/[\pi_{1,\acute{e}t}]_{n-1} \simeq \mathbb{Q}_p(n)^{r_n}$ for some positive integer r_n. The map $H^1(G, [\pi_{1,\acute{e}t}]_n) \to H^1(G, [\pi^B_{1,\acute{e}t}]_n)$ also defines a subset $H^1_f(G, [\pi_{1,\acute{e}t}]_n)(\mathbb{Q}_p) \subset H^1(G, [\pi_{1,\acute{e}t}]_n)(\mathbb{Q}_p)$ as the inverse image of the basepoint, which in fact is the set of $\mathbb{Q}_p$ points of a subvariety $H^1_f(G, [\pi_{1,\acute{e}t}]_n) \subset H^1(G, [\pi_{1,\acute{e}t}]_n)$ as discussed in Sect. 1.

Proposition 5 *Assume* $(p-1)/2 \geq n+1$. *Let* L *be a finite extension of* $\mathbb{Q}_p$ *and let*

$$[P] \in H^1(G, [\pi_{1,\acute{e}t}]_n)(L)$$

be the cohomology class of the torsor P. *Then* P *is De Rham if and only if* $[P] \in H^1_f(G, [\pi_{1,\acute{e}t}]_n)(L)$.

Proof. Suppose the class $[P]$ becomes trivial in $H^1(G, [\pi^B_{1,\acute{e}t}]_n)(L)$. This means $\mathcal{P} \otimes B_{DR} \simeq A_{\acute{e}t} \otimes_{\mathbb{Q}_p} L \otimes_{\mathbb{Q}_p} B_{DR}$ as Galois representations with weight filtrations. But then, $D(\mathcal{P}) \simeq D([A_{\acute{e}t}]_n \otimes L) = D([A_{\acute{e}t}]_n) \otimes L$ so that $\dim_L W_m D(\mathcal{P}) = \dim_L W_m D([A_{\acute{e}t}]_n) \otimes L = \dim_L W_m [A_{\acute{e}t}]_n \otimes L = \dim_L \dim W_m \mathcal{P}$. So P is De Rham. Conversely, assume $\mathcal{P} \otimes B_{DR} \simeq D(\mathcal{P}) \otimes B_{DR}$. $D(\mathcal{P})$ is a torsor for $D([A_{\acute{e}t}]_n \otimes L)$. By unipotence, there is a point $x \in D(P)$ inducing an isomorphism

$$D(\mathcal{P}) \simeq D([A_{\acute{e}t}]_n \otimes L).$$

Also, we have an equality of dimensions

$$\begin{aligned} \dim W_m D(\mathcal{P}) = \dim W_m \mathcal{P} &= \dim W_m([A_{\acute{e}t}]_n \otimes L) \\ &= \dim(W_m D([A_{\acute{e}t}]_n)) \otimes L. \end{aligned}$$

Thus, $D([A_{\acute{e}t}]_n \otimes L) \simeq D(\mathcal{P})$ together with the weight filtration. Since the Galois actions are trivial on $D([A_{\acute{e}t}]_n \otimes L)$ and $D(\mathcal{P})$, we get

$$\begin{aligned} \mathcal{P} \otimes B_{DR} &\simeq D(\mathcal{P}) \otimes B_{DR} \\ &\simeq D([A_{\acute{e}t}]_n \otimes L) \otimes B_{DR} \simeq [A_{\acute{e}t}]_n \otimes L \otimes B_{DR} \end{aligned}$$

as torsors with Galois action. Therefore, $[\mathcal{P} \otimes B_{DR}]$ is trivial. □

Recall from the previous section that we have a Frobenius map

$$\phi : \pi_{1,DR} = \pi_{1,DR}(X \otimes \mathbb{Q}_p, x) \to \pi_{1,DR}(X \otimes \mathbb{Q}_p, x)$$

induced from the comparison with the crystalline fundamental group.

If R is a finitely-generated $\mathbb{Q}_p$-algebra, by a torsor for $\pi_{1,DR} \otimes R$, we will mean an R-scheme $P = \mathrm{Spec}(\mathcal{P})$ equipped with a non-negatively indexed decreasing (Hodge) filtration $F^i\mathcal{P}$, an increasing weight filtration $W_n\mathcal{P}$ by finitely generated R-submodules, and an R-algebra (Frobenius) automorphism $\phi_P : \mathcal{P} \to \mathcal{P}$. P is equipped with an action of $\pi_{1,DR} \otimes R$ that gives it the structure of a torsor for the pro-unipotent group $\pi_{1,DR}$ in the usual sense. However, we require that the structure map $a : \mathcal{P} \to \mathcal{P} \otimes A_{DR}$ for the action preserves both filtrations and the Frobenius map. For any maximal ideal m of R, we get a torsor $P \otimes R/m$ for $\pi_{1,DR} \otimes R/m$. We will say the family is admissible if for each of these fibers, $P \otimes R/m$ is of the form $D(P_{et}(m))$ for a De Rham $\pi_{1,\text{ét}}^{R/m}$-torsor $P_{et}(m) = \text{Spec}(\mathcal{P}_{et}(m))$. Note that we have $\dim W_n \mathcal{P} \otimes R/m = \dim W_n \mathcal{P}_{et}(m) = \dim W_n A_{\text{ét}} \otimes R/m$ is constant over maximal ideals m, and hence, $W_n\mathcal{P}$ is locally-free of constant rank equal to this dimension. We can extend the definition above in an obvious way to torsors for $[\pi_{1,DR}]_n \otimes R$.

According to Besser ([2], Thm 3.1), the Lang map $L : \pi_{1,DR} \to \pi_{1,DR}$ that sends g to $g^{-1}\phi(g)$ is an isomorphism. Given any finitely-generated $\mathbb{Q}_p$-algebra R and a torsor P_{DR} for $\pi_{1,DR} \otimes R$ this implies ([2], proof of Cor. 3.2)that there exists a unique Frobenius invariant element $p_F(R)$ in $P_{DR}(R)$. The filtration F on the coordinate ring $\mathcal{P}_{DR}$ determines a filtration by subschemes $F^i P_{DR}$. $F^0 P_{DR}$ in particular is the subscheme defined by the ideal $F^1\mathcal{P}_{DR}$.

Lemma 7 *Let P_{DR} be a torsor for $\pi_{1,DR} \otimes R$ (or $[\pi_{1,DR}]_n \otimes R$). If P_{DR} is admissible, then there is a unique element $p_{DR}(R)$ in $F^0 P_{DR}(R)$.*

Proof. If R is a field, this follows from Deligne ([7], Prop. 7.12) and the fact that both the filtrations F on $\mathcal{P}_{DR}$ and on A_{DR} are gradable (by Wintenberger [14]). That is, this implies that P_{DR} is associated to $F^0 P_{DR}$ which is a $F^0\pi_{1,DR}$-torsor, and $F^0\pi_{1,DR}$ is the trivial group ([7], Sect. 12), so $F^0 P_{DR}$ is scheme-theoretically a point. Now, $P_{DR} \otimes R/m$ is a torsor for $\pi_{1,DR} \otimes R/m$ for any maximal ideal m of R. Consider the structure map $R \to \mathcal{P}_{DR}/F^1\mathcal{P}_{DR}$. By the result over fields, we see that $R/m \to \mathcal{P}_{DR}/F^1\mathcal{P}_{DR} \otimes R/m$ is an isomorphism for each maximal ideal m. Hence, $R \simeq \mathcal{P}_{DR}/F^1\mathcal{P}_{DR}$, the inverse of which provides the element of $F^0 P_{DR}$. Uniqueness is obvious from this construction. □

Define the variety $Z^1_f(G, [\pi_{1,\text{ét}}]_n)$ to be the inverse image of $H^1_f(G, [\pi_{1,\text{ét}}]_n)$ under the natural projection

$$Z^1(G, [\pi_{1,\text{ét}}]_n) \to H^1(G, [\pi_{1,\text{ét}}]_n).$$

If we consider the map

$$Z^1(G, [\pi_{1,\text{ét}}]_n)(R) \to Z^1\left(G, \left[\pi^B_{1,\text{ét}}\right]_n\right)(R),$$

$Z^1_f(G, [\pi_{1,\text{ét}}]_n)(R)$ consists exactly of those cocycles whose image in $Z^1(G, [\pi^B_{1,\text{ét}}]_n)(R)$ lie in the image of the map

$$\left[\pi^B_{1,\text{ét}}\right]_n(R) \to Z^1\left(G, \left[\pi^B_{1,\text{ét}}\right]_n\right)(R).$$

Taking the fiber product, we get a surjective map of functors

$$[\mathcal{Z}]_n := \left[\pi^B_{1,\text{ét}}\right]_n \times_{Z^1\left(G,\left[\pi^B_{1,\text{ét}}\right]_n\right)} Z^1_f(G, [\pi_{1,\text{ét}}]_n) \to Z^1_f(G, [\pi_{1,\text{ét}}]_n).$$

Now, since the map $[\pi_{1,\text{ét}}^B]_n(R) \to Z^1(G, [\pi_{1,\text{ét}}^B]_n)(R)$ takes u to $ug(u^{-1})$, we easily see that $[\mathcal{Z}]_n$ has the structure of a torsor over the group-valued functor $[\pi_{1,\text{ét}}^B]_n^G$ whose value on R is just the G-fixed part of $[\pi_{1,\text{ét}}(B \otimes R)]_n$. To understand this functor better, we need to consider the G-invariants in the B-algebra homomorphisms from $[A_{\text{ét}}]_n \otimes B$ to $B \otimes R$. Now, let $(p-1)/2 \geq n+1$. Then $[A_{\text{ét}}]_n \otimes B = [A_{DR}]_n \otimes B$ and $[A_{DR}]_n$ has trivial G-action. So

$$\begin{aligned}\text{Alg-Hom}_B([A_{\text{ét}}]_n \otimes B, B \otimes R) &= \text{Alg-Hom}_B([A_{DR}]_n \otimes B, B \otimes R)\\ &= \text{Alg-Hom}_{\mathbb{Q}_p}([A_{DR}]_n, B \otimes R).\end{aligned}$$

Hence, the G-invariants are just

$$\text{Alg-Hom}_{\mathbb{Q}_p}([A_{DR}]_n, (B \otimes R)^G) = \text{Alg-Hom}_F([A_{DR}]_n, R),$$

that is, $[\mathcal{Z}]_n$ is a torsor for $[\pi_{1,DR}]_n$ over $Z^1_f(G, [\pi_{1,\text{ét}}]_n)$. Now, pulling back to $H^1_f(G, [\pi_{1,\text{ét}}]_n)$ via a section $H^1_f(G, [\pi_{1,\text{ét}}]_n) \to Z^1_f(G, [\pi_{1,\text{ét}}]_n)$, we get a $[\pi_{1,DR}]_n$-torsor over $H^1_f(G, [\pi_{1,\text{ét}}]_n)$, which we will also denote by $[\mathcal{Z}]_n$. This torsor has two sections $\gamma_{DR} \in F^0[\mathcal{Z}]_n(R)$ and $\gamma_F \in [\mathcal{Z}(R)]_n^{\phi=1}$ where R is now the coordinate ring of $H^1_f(G, [\pi_{1,\text{ét}}]_n)$. The transporter in $[\pi_{1,DR}]_n(R)$ between these two points gives us an R-point of $[\pi_{1,DR}]_n$, and hence, an algebraic map $\mathcal{D} : H^1_f(G_F, [\pi_{1,\text{ét}}]_n) \to [\pi_{1,DR}]_n$. Given a class $c \in H^1_f(G_F, [\pi_{1,\text{ét}}]_n)$, there is the torsor $P_c = \text{Spec}(\mathcal{P}(c))$ corresponding to it and a corresponding $[\pi_{1,DR}]_n$-torsor $D(P_c)$. By considering the transporter between $F^0D(P_c)$ and $D(P_c)^{\phi=1}$, we get an element $c' \in [\pi_{1,DR}]_n$.

Lemma 8 $c' = \mathcal{D}(c)$.

Proof. It suffices to check that $D(P_c)$ coincides with the torsor $([\mathcal{Z}]_n)_c$. We check this on points (for arbitrary R). $([\mathcal{Z}]_n)_c(R)$ consists of $u \in \pi_{1,\text{ét}}(B \otimes R)$ such that $ug(u^{-1}) = c(g)$ for all $g \in G$. This is equivalent to $u = c(g)g(u)$. Now we examine the points of $D(P_c)$. These are the G-invariant homomorphisms $x : \mathcal{P}(c) \to B \otimes R$. Since $\mathcal{P}(c)$ is just $[A_{\text{ét}}]_n \otimes R$ with the action twisted by c, this is equivalent to looking at the G-invariant homomorphisms $x : [A_{\text{ét}}]_n \otimes B \otimes R \to B \otimes R$. Such homomorphisms are in particular points in $[\pi_{1,\text{ét}}]_n(B \otimes R)$. Let us impose the invariance condition. For this, recall also the formula $c(g^{-1}) = g^{-1}c(g)^{-1}$ for a 1-cocycle c. Now the c-twisted G-action takes x to the homomorphisms whose value on a is

$$\begin{aligned}g\bigl(x\bigl(\phi_{c(g^{-1})}(g^{-1}(a))\bigr)\bigr) &= g((g^{-1}(a))(c(g^{-1})^{-1}x))\\ &= g(g^{-1}(a(g(c(g^{-1})^{-1})g(x)))) = a(c(g)g(x)),\end{aligned}$$

that is, the point $c(g)g(x)$. Thus, G-invariance is the same as $c(g)g(x) = x$, which is exactly the condition for $([\mathcal{Z}]_n)_c(R)$. □

Proof of Siegel's theorem. Let $\Gamma := \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ and consider a ring $\mathbb{Z}[1/S]$ of S-integers. Choose a $p \notin S$ and let $T = S \cup \{p\}$. Below, we will put further restrictions on p. We will consider the pro-unipotent completion of the p-adic étale fundamental group of $\bar{X} = (\mathbf{P}^1 \setminus \{0, 1, \infty\}) \otimes \bar{\mathbb{Q}}$ and denote it by $\pi_{1,\text{ét}}$. Here we choose an $S-$integral point x as a base-point, if it exists (otherwise, we are done). Thus, we get an action of Γ_T, the Galois group of the maximal extension of $\mathbb{Q}$ unramified outside T, on $\pi_{1,\text{ét}}$. Let G be a decomposition group at p. We can then restrict the representation to G, which can also be interpreted as the action on the $\mathbb{Q}_p$-unipotent fundamental group of $(\mathbf{P}^1 \setminus \{0, 1, \infty\}) \otimes \bar{\mathbb{Q}}_p$.

Any other integral point y determines a point $[[\pi_{1,\text{ét}}(\bar{X}, x, y)]_n]$ in $H^1(\Gamma_T, [\pi_{1,\text{ét}}]_n)(\mathbb{Q}_p)$, which we regard as an affine variety following the construction of Sect. 1 (again with $K = F = B = \mathbb{Q}_p$). In fact, if we define $H^1_f(\Gamma_T, [\pi_{1,\text{ét}}]_n)$ to be the inverse image of $H^1_f(G, [\pi_{1,\text{ét}}]_n)$ with respect to the restriction map

$$H^1(\Gamma_T, [\pi_{1,\text{ét}}]_n) \to H^1(G, [\pi_{1,\text{ét}}]_n)$$

the class of the point y lies in $H^1_f(\Gamma_T, \pi_{1,\text{ét}})(\mathbb{Q}_p)$ for $(p-1)/2 \geq n+1$ ([13], 1.9, Thm. A). There is a commutative diagram

$$\begin{array}{ccc} \mathcal{X}(\mathbb{Z}[1/S]) & \longrightarrow & X(\mathbb{Q}_p)\cap]Y[\\ \downarrow & & \downarrow \\ H^1_f(\Gamma_T, \pi_{1,\text{ét}})(\mathbb{Q}_p) & \longrightarrow & H^1_f(G, \pi_{1,\text{ét}})(\mathbb{Q}_p) \end{array}$$

simply from the compatibility with base change of the fundamental groups and torsors of paths while loc. cit. yields the commutative diagram

$$\begin{array}{ccc} X(\mathbb{Q}_p)\cap]Y[& \longrightarrow & [\pi_{1,DR}]_n(\mathbb{Q}_p) \\ \downarrow & & \downarrow{=} \\ H^1_f(G, [\pi_{1,\text{ét}}]_n)(\mathbb{Q}_p) & \longrightarrow & [\pi_{1,DR}]_n(\mathbb{Q}_p) \end{array}$$

for $(p-1)/2 \geq r+1$, where the upper horizontal arrow is of course the level n Albanese map $UAlb_n$. That is, the commutativity here is simply a restatement of Vologodsky's p-adic comparison isomorphism for torsors of paths, showing that

$$D([\pi_{1,\text{ét}}]_n(X \otimes \bar{\mathbb{Q}}_p; , x, y)) \simeq \pi_{1,DR}(X; x, y).$$

Of course, one establishes the relation between Vologodsky's theorem and the statement here again as in the argument at the beginning of the section. Hence, $UAlb_n(X(\mathbb{Q}_p)\cap]Y[)$ lies inside the image of $H^1_f(\Gamma_T, [\pi_{1,\text{ét}}]_n)(\mathbb{Q}_p)$.

But there is a bound on the dimension of the latter. That is, clearly,

$$\dim H^1_f(\Gamma_T, [\pi_{1,\text{ét}}]_n) \leq \dim H^1(\Gamma_T, [\pi_{1,\text{ét}}]_n)(\mathbb{Q}_p),$$

and for the latter, we have the exact sequences

$$0 \to H^1\big(\Gamma_T, \pi^n_{1,\text{ét}}/\pi^{n+1}_{1,\text{ét}}\big)(\mathbb{Q}_p) \to H^1(\Gamma_T, [\pi_{1,\text{ét}}]_n)(\mathbb{Q}_p) \to H^1(\Gamma_T, [\pi_{1,\text{ét}}]_{n-1})(\mathbb{Q}_p) \to 0,$$

as discussed in the proof of Proposition 2, in the sense that the middle term is a fibration with fibers isomorphic to the first term. We also have an exact sequence for the De Rham fundamental group:

$$0 \to \pi^n_{DR}/\pi^{n+1}_{1,DR} \to [\pi_{1,DR}]_{n+1} \to [\pi_{1,DR}]_n \to 0$$

([7], Prop. 6.3, 16.4) On the other hand, we have $\pi^n_{1,\text{ét}}/\pi^{n+1}_{1,\text{ét}} \simeq \mathbb{Q}_p(n)^{r_n}$ and $\pi^n_{DR}/\pi^{n+1}_{DR} \simeq \mathbb{Q}_p(n)^{r_n}$ for some rank r_n growing to infinity and, importantly, independent of the choice of p. So

$$\dim \pi_{DR,n} = r_1 + r_2 + \cdots r_n.$$

But by Soulé's vanishing theorem ([12]), $H^1(G_T, \mathbb{Q}_p(2n)) = 0$ for each positive n, while $H^1(G_T, \mathbb{Q}_p(2n+1))$ has dimension 1 for $n \geq 3$. Hence,

$$\dim H^1_f(\Gamma_T, (\pi_{1,\text{ét}})_n)(\mathbb{Q}_p) \leq R + r_3 + r_5 + \cdots + r_{2\lfloor n/2 \rfloor}$$

where $R = 2\text{rank}(\mathbb{Z}[1/T]^*)$. Therefore, for n sufficiently large, $\dim H^1_f(\Gamma_T, [\pi_{1,\text{ét}}]_n) < \dim[\pi_{1,DR}]_n$. Now we choose p such that $(p-1)/2 \geq n+1$ so that we have the commutative diagrams above, and we get that some element of the coordinate ring of $[\pi_{1,DR}]_n$ must vanish on the image of $H^1_f(\Gamma_T, [\pi_{1,\text{ét}}]_n)(\mathbb{Q}_p)$. But when pulled back to $X(\mathbb{Q}_p) \cap]Y[$ via the unipotent Albanese map, the elements of this coordinate rings are exactly linear combinations of the $g_w(x, \cdot)$ of Sect. 2. Thus, there is a non-zero Coleman function that vanishes on $\mathcal{X}(\mathbb{Z}[1/S])$. Since $X(\mathbb{Q}_p) \cap]Y[$ is compact, this implies the desired finiteness. □

Acknowledgements. I seem to have accumulated quite a debt of gratitude even in writing this simple paper: (1) to Dick Hain for teaching me about iterated integrals; (2) to Jean-Marc Fontaine and Luc Illusie for many conversations and lectures on p-adic Hodge theory as well as an invitation to Orsay in September 2003; (3) to Hidekazu Furusho for patiently explaining to me his results on p-adic multiple polylogs; (4) to Makoto Matsumoto, who generously supported my visit to Kyoto and Hiroshima from October to December, 2003; (5) to Akio Tamagawa, who also arranged the visit and listened patiently to a premature version of the proof presented here; (6) to Ofer Gabber who helped me clarify the proof of Prop. 2; (7) to Gerd Faltings for his interest, encouragement, and helpful suggestions; (8) to a referee who gave the first submission a very thorough reading and made many helpful suggestions; (9) and to the unbelievably hospitable staff in the international office at RIMS, especially Ms. Eriko Edo. But my deepest gratitude goes to Shinichi Mochizuki, who explained to me his profound ideas relating fundamental groups to Diophantine problems in innumerable hours of stimulating conversation, and without whose strong encouragement I would never have thought through and written down the details of this paper. More specifically, he corrected an earlier naive approach to the map 'C' by emphasizing to me the importance of non-linearity.

References

1. Berthelot, P.: Finitude et pureté cohomologique en cohomologie rigide. With an appendix in English by Aise Johan de Jong. Invent. Math. **128**, 329–377 (1997)
2. Besser, A.: Coleman integration using the Tannakian formalism. Math. Ann. **322**, 19–48 (2002)
3. Besser, A., Furusho, H.: The double shuffle relations for p-adic multiple zeta values. math.NT/0310177
4. Chabauty, C.: Sur les points rationnels des courbes algébriques de genre supérieur à l'unité. C. R. Acad. Sci., Paris **212**, 882–885 (1941)
5. Chiarellotto, B., Le Stum, B.: F-isocristaux unipotents. Compos. Math. **116**, 81–110 (1999)
6. Coleman, R.F.: Effective Chabauty. Duke Math. J. **52**, 765–770 (1985)
7. Deligne, P.: Le groupe fondamental de la droite projective moins trois points. Galois groups over Q (Berkeley, CA, 1987), 79–297, Math. Sci. Res. Inst. Publ., vol. 16. New York: Springer 1989
8. Furusho, H.: p-adic multiple zeta values I – p-adic multiple polylogarithms and the p-adic KZ equation. Invent. Math. **155**, 253–286 (2004)
9. Hain, R.M.: Higher Albanese manifolds. Hodge theory (Sant Cugat, 1985), pp. 84–91. Lect. Notes Math., vol. 1246. Berlin: Springer 1987
10. Köthe, G.: Topological Vector Spaces I. 2nd printing, revised. Berlin, Heidelberg, New York: Springer (1983)
11. Serre, J.-P.: Local Fields. Berlin, Heidelberg, New York: Springer 1979
12. Soulé, C.: K-théorie des anneaux d'entiers de corps de nombres et cohomologie étale. Invent. Math. **55**, 251–295 (1979)
13. Vologodsky, V.: Hodge structure on the fundamental group and its application to p-adic integration. Mosc. Math. J. **3**, 205–247, 260 (2003)
14. Wintenberger, J.-P.: Un scindage de la filtration de Hodge pour certaines variétés algébriques sur les corps locaux. Ann. Math. (2) **119**, 511–548 (1984)