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Hyperbolic trapped rays and global existence of quasilinear wave equations*

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1. Introduction

The purpose of this paper is to give a simple proof of global existence for quadratic quasilinear Dirichlet-wave equations outside of a wide class of compact obstacles in the critical case where the spatial dimension is three. Our results improve on earlier ones in Keel, Smith and Sogge [9] in several ways. First, and most important, we can drop the star-shaped hypothesis and handle non-trapping obstacles as well as any obstacle that has exponential local decay rate of energy for H^2 data for the linear equation (see (1.4) below). This hypothesis is fulfilled in the non-trapping case where there is actually exponential local decay of energy [19] with no loss of derivatives. This hypothesis (1.4) is also known to hold in several examples involving hyperbolic trapped rays. For instance, our results apply to situations where the obstacle is a finite union of convex bodies with smooth boundary (see [7], [8]). In addition to improving the hypotheses on the obstacles, we can also improve considerably on the decay assumptions on the initial data at infinity compared to the results in [9] which were obtained by the conformal method. Lastly, we are able to handle non-diagonal systems involving multiple wave speeds.

We shall use a refinement of techniques developed in earlier work of Keel, Smith and Sogge [10], [11]. In particular, we shall use a modification of Klainerman's commuting vector fields method [13] that only uses the collection of vector fields that seems "admissible" for boundary value problems.

The main innovation in this approach versus the classical one for the boundaryless case is the use of weighted space-time L^2 estimates to handle

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the various lower order terms that necessarily arise in obstacle problems. The weights involved are just negative powers of $\langle x\rangle$. These couple well with the pointwise estimates that we use, which involve $O(\langle x\rangle^{-1})$ decay of solutions of linear inhomogeneous Dirichlet-wave equations, as opposed to the more standard $O(t^{-1})$ decay for the boundaryless case, which are much more difficult to obtain for obstacle problems. Because of the fact that we are dealing with such problems, it does not seem that we can use vector fields such as the generators of hyperbolic rotations, $x_i\partial_t + t\partial_i$, $i = 1, 2, 3$. Additionally, it seems that these cannot be used for multiple wave speed problems since they have an associated speed (one in the above case). So, unlike in Klainerman's argument [13] for the Minkowski space case, we are only able to use the generators of spatial rotations and space-time translations

$$Z = \{\,\partial_i, x_j\partial_k - x_k\partial_j,\ 0 \le i \le 3, 1 \le j < k \le 3\}, \tag{1.1}$$

as well as the scaling vector field

$$L = t\partial_t + r\partial_r. \tag{1.2}$$

Here, and in what follows, we are using the notation that (x_1, x_2, x_3) denote the spacial coordinates, while either x_0 or t will denote the time coordinate, depending on the context. Also, $r = |x|$, and $\langle x\rangle = \langle r\rangle = \sqrt{1+r^2}$. We shall also let $u' = \partial u = \partial_{t,x}u$ denote the space-time gradient.

Another difficulty that we encounter in the obstacle case is related to the simple fact that while the vector fields

$$\Omega_{ij} = x_i\partial_j - x_j\partial_i, \quad 1 \le i < j \le 3 \tag{1.3}$$

and L preserve the equation $(\partial_t^2 - \Delta)u = 0$ in the Minkowski space case if u is replaced by either Lu or $\Omega_{ij}u$, this is not true in the obstacle case due to the fact that the Dirichlet boundary conditions are not preserved by these operators. Since the generators of spatial rotations, Ω_{ij}, have coefficients that are small near our compact obstacle, this fact is somewhat easy to get around when dealing with them; however, it is a bit harder to deal with the scaling vector field, L, since its coefficients become large on the obstacle as t goes to infinity. As a result, we are forced to consider in our estimates combinations of the Z operators and the L operators that involve relatively few of the scaling vector fields. This, together with the fact that there is necessarily a loss of smoothness in the local energy estimates for obstacles with trapped rays, makes the combinatorics that arise more complicated than in the Minkowski space case first studied by Klainerman [13].

In earlier works [9], [11] the obstacle was assumed to be star-shaped. This was a convenient assumption in proving energy estimates involving the scaling operator L. For instance, in proving energy estimates for Lu for solutions of $(\partial_t^2 - \Delta)u = 0$ one finds that if $\mathcal{K}$ is star-shaped then, although energy is not conserved, the contribution from the boundary to energy identities has a favorable sign. This is in the spirit of Morawetz's original argument [18]. If one drops the star-shaped assumption this argument of course breaks down. However, in this paper we exploit the fact that we still can prove favorable estimates for solutions of *nonlinear equations*. The additional terms arising from the boundary can be estimated using Lemma 2.9 and the $L_t^2 L_x^2(\langle x\rangle^{-1/2}dxdt)$ estimates since the forcing terms are nonlinear functions of (du, du^2) that vanish to second order.

Let us now describe more precisely our assumptions on our obstacles $\mathcal{K} \subset \mathbb{R}^3$. We shall assume that $\mathcal{K}$ that is smooth and compact. We do not assume that $\mathcal{K}$ is connected. Without loss of generality, we may assume throughout that

$$\mathcal{K} \subset \{x \in \mathbb{R}^3 : |x| < 1\}.$$

Our only additional assumption is that there is exponential local decay of energy with a possible loss of derivatives. To be specific, we require that there be a $c > 0$, a constant C so that

$$\Big(\int_{\{x\in\mathbb{R}^3\backslash\mathcal{K}:\,|x|<10\}} |u'(t,x)|^2\,dx\Big)^{1/2} \le Ce^{-ct}\sum_{|\alpha|\le 1}\big\|\partial_x^\alpha u'(0,\,\cdot\,)\big\|_2, \tag{1.4}$$
$$\text{if } \Box u = 0, \text{ and } u(0,x) = \partial_t u(0,x) = 0, \text{ for } |x| > 10.$$

We remark that our results do not actually require exponential decay of local energy. A decay rate of $O(\langle t\rangle^{-3-\delta})$, $\delta > 0$ would suffice since our main L^2-estimates involve 3 or fewer powers of the scaling operator L. By tightening the arguments one might even be able to show that $O(\langle t\rangle^{-1-\delta})$ is sufficient. On the other hand, we shall assume (1.4) throughout since the proofs under this weaker decay rate would be more technical. Moreover, in the 3-dimensional case, all of the examples that we know that involve polynomial decay actually have exponential decay of local energy. For related problems in general relativity, though, it might be much easier to establish local polynomial decay of energy.

Recall that if the obstacle is star-shaped or non-trapping a stronger version of (1.4) is always valid where in the right one just takes the $H^1 \times L^2$-norm of $(u(0,\,\cdot\,), \partial_t u(0,\,\cdot\,))$ (see Lax, Morawetz and Phillips [15] for the star-shaped case, and Morawetz, Ralston and Strauss [19] for the non-trapping case, and Melrose [17] for further results of this type). On the other hand, if $\mathbb{R}^3\backslash\mathcal{K}$ contains any trapped rays, then Ralston [20] showed that this stronger inequality cannot hold. So there must be some *"loss"* $\ell > 0$ of regularity if there is energy decay when there are trapped rays. In (1.4) we are assuming that $\ell = 1$. By interpolation, there is no loss of generality in making this assumption since if the analog of (1.4) held where the sum was taken over a given $\ell > 1$ then (1.4) would still be valid (with a different constant in the exponential). (The same argument shows that a variant of (1.4) holds where one has $\|u(0,\,\cdot\,)\|_{H_D^{1+\delta}} + \|\partial_t u(0,\,\cdot\,)\|_{H_D^\delta}$ in the right for $\delta > 0$.)

In other direction, Ikawa [7], [8] was able to show that if $\mathcal{K}$ is a finite union of convex obstacles with smooth boundary then one has exponential decay of local energy with a loss of $\ell = 7$ derivatives, which as we just pointed out leads to (1.4) here. Ikawa's theorem requires additional technical assumptions that we shall not describe (see [8]); however, they are always satisfied for instance in the case where $\mathcal{K}$ is the union of two disjoint convex obstacles or any number of balls that are sufficiently separated. Thus, even for the case where $\mathcal{K}$ is the union of 3 sufficiently separated balls one can always have infinitely many trapped rays and still have (1.4) (and the nonlinear results to follows). We also mention the work of Burq [1] who showed that for *any* compact obstacle $\mathcal{K}$ with smooth boundary, one has a local decay that is $O((\log(2+t))^{-k})$ for any k if one takes the loss of regularity to be $\ell = k$. Such a decay rate is not fast enough for us to be able to prove global existence for this class of obstacles, and it seems doubtful that such results could hold in this context since Burq's results include the case where $\mathbb{R}^3\backslash\mathcal{K}$ has trapped elliptic rays. On the other hand, an interesting question would be whether our hypothesis (1.4) might hold under the assumption that $\mathbb{R}^3\backslash\mathcal{K}$ only contains hyperbolic trapped rays.

For obstacles $\mathcal{K} \subset \mathbb{R}^3$, as above satisfying (1.4) we shall consider smooth, quadratic, quasilinear systems of the form

$$
\begin{cases} \Box u = Q(du, d^2u), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3\backslash\mathcal{K} \\ u(t,\cdot)|_{\partial\mathcal{K}} = 0 \\ u(0,\cdot) = f, \ \ \partial_t u(0,\cdot) = g. \end{cases} \tag{1.5}
$$

Here,

$$
\Box = (\Box_{c_1}, \Box_{c_2}, \dots, \Box_{c_D}), \tag{1.6}
$$

is a vector-valued multiple speed D'Alembertian with

$$
\Box_{c_I} = \partial_t^2 - c_I^2\Delta,
$$

where we assume that the wave speeds c_I are all positive but not necessarily distinct. Also, $\Delta = \partial_1^2+\partial_2^2+\partial_3^2$ is the standard Laplacian. By a simple scaling argument, in showing that (1.5) admits global small amplitude solutions, as mentioned before, we shall assume without loss of generality that $\mathcal{K} \subset \{x \in \mathbb{R}^3 : |x| < 1\}$.

By quasilinear we mean that the nonlinear term $Q(du, d^2u)$ is linear in the second derivatives of u. We shall also assume that the highest order nonlinear terms are symmetric, by which we mean that, if we let $\partial_0 = \partial_t$, then

$$
Q^I(du, d^2u) = B^I(du) + \sum_{\substack{0\le j,k,l\le 3 \\ 1\le J,K\le D}} B^{IJ,jk}_{K,l}\partial_l u^K\, \partial_j\partial_k u^J, \quad 1 \le I \le D, \tag{1.7}
$$

with $B^I(du)$ a quadratic form in the gradient of u, and $B^{IJ,jk}_{K,l}$ real constants satisfying the symmetry conditions

$$B^{IJ,jk}_{K,l} = B^{JI,jk}_{K,l} = B^{IJ,kj}_{K,l}. \tag{1.8}$$

To obtain global existence, we shall also require that the equations satisfy a form of the null condition of Christodoulou and Klainerman. Let us first assume, for simplicity that the wave speeds c_I, $I = 1, \dots, D$ are distinct. In this case, the null condition for the quasilinear terms only involves self-interactions of each wave family. Specifically, we require that self-interactions among the quasilinear terms satisfy the standard null condition for the various wave speeds:

$$\begin{aligned} &\sum_{0\le j,k,l\le 3} B^{IJ,jk}_{J,l}\xi_j\xi_k\xi_l = 0 \\ &\text{whenever} \quad \frac{\xi_0^2}{c_J^2} - \xi_1^2 - \xi_2^2 - \xi_3^2 = 0, \qquad I, J = 1, \dots, D. \end{aligned} \tag{1.9}$$

For the quasilinear terms, if one allows repeated wave speeds, it will be required that the interactions of families with the same speed satisfy a null condition. Specifically if we let $\mathcal{I}_p = \{I \,:\, c_I = c_{I_p},\ 1 \le I \le D\}$ then the above null condition is extended to

$$\begin{aligned} &\sum_{j,k,l\le 3} B^{IJ,jk}_{K,l}\xi_j\xi_k\xi_l = 0 \\ &\text{whenever } \frac{\xi_0^2}{c_{I_p}^2} - \xi_1^2 - \xi_2^2 - \xi_3^2 = 0,\ (J, K) \in \mathcal{I}_p \times \mathcal{I}_p,\ 1 \le I \le D. \end{aligned}$$

To describe the null condition for the lower order terms, we note that we can expand

$$B^I(du) = \sum_{\substack{1\le J,K\le D\\ 0\le j,k\le 3}} A^{I,jk}_{JK}\partial_j u^J \partial_k u^K.$$

We then require that each component satisfy the standard null condition for multiple wave speeds

$$\sum_{0\le j,k\le 3} A^{I,jk}_{JK}\xi_j\xi_k = 0 \quad \text{for all} \quad \xi \in \mathbb{R}\times\mathbb{R}^3, \quad 1 \le J, K \le D. \tag{1.10}$$

This means that $B^I(du)$ is an asymmetric quadratic form in du. That is, it must be a linear combination of the gauge-type null forms

$$Q^I_{JK,jk}(du) = \left(\partial_j u^J \partial_k u^K - \partial_j u^K \partial_k u^J\right),\ 0 \le j < k \le 3,\ 1 \le J \le K \le D.$$

It seems likely that one could also allow diagonal terms involving the relativistic null forms $Q^I_0(du) = (\partial_0 u^I)^2 - c_I^2|\nabla_x u^I|^2$, by using a gauge transformation to reduce to the above types of equations; however, this case will not be explored here. One should also be able to allow cubic quasilinear nonlinearities of the form $R(u, du, d^2u)$ that vanish to second order in the last two variables. Doing this, though, would require handling more powers of L, which would complicate the combinatorics in the continuity argument used to prove global existence.

In order to solve (1.5) we must also assume that the data satisfies the relevant compatibility conditions. Since these are well known (see e.g., [9]), we shall describe them briefly. To do so we first let $J_k u = \{\partial_x^\alpha u : 0 \le |\alpha| \le k\}$ denote the collection of all spatial derivatives of u of order up to k. Then if m is fixed and if u is a formal H^m solution of (1.5) we can write $\partial_t^k u(0, \cdot) = \psi_k(J_k f, J_{k-1} g)$, $0 \le k \le m$, for certain compatibility functions ψ_k which depend on the nonlinear term Q as well as $J_k f$ and $J_{k-1} g$. Having done this, the compatibility condition for (1.5) with $(f, g) \in H^m \times H^{m-1}$ is just the requirement that the ψ_k vanish on $\partial\mathcal{K}$ when $0 \le k \le m-1$. Additionally, we shall say that $(f, g) \in C^\infty$ satisfy the compatibility conditions to infinite order if this condition holds for all m.

We can now state our main result:

Theorem 1.1. *Let $\mathcal{K}$ be a fixed compact obstacle with smooth boundary that satisfies* (1.4). *Assume also that $Q(du, d^2u)$ and $\Box$ are as above. Suppose that $(f, g) \in C^\infty(\mathbb{R}^3\backslash\mathcal{K})$ satisfies the compatibility conditions to infinite order. Then there is a constant $\varepsilon_0 > 0$, and an integer $N > 0$ so that for all $\varepsilon \le \varepsilon_0$, if*

$$\sum_{|\alpha|\le N} \big\| \langle x\rangle^{|\alpha|} \partial_x^\alpha f \big\|_2 + \sum_{|\alpha|\le N-1} \big\| \langle x\rangle^{1+|\alpha|} \partial_x^\alpha g \big\|_2 \le \varepsilon, \tag{1.11}$$

then (1.5) *has a unique solution $u \in C^\infty([0,\infty) \times \mathbb{R}^3\backslash\mathcal{K})$.*

This paper is organized as follows. In the next section we shall collect the L^2 estimates that will be needed for the proof of this existence theorem. In Sect. 3 we shall prove the necessary pointwise decay estimates that will be needed. The results in these two sections involve variants of those in [11]. Section 4 will include weighted estimates that are related to the null condition, which are obstacle variants of ones for the Minkowski space setting (cf. Hidano [4], Sideris and Tu [24], Sogge [27], and Yokoyama [28]). Finally, in Sect. 5, we shall use all of these estimates to prove the global existence theorem.

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2. L^2 estimates

We shall be concerned with solutions $u \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^3 \backslash \mathcal{K})$ of the Dirichlet-wave equation

$$\begin{cases} \Box_\gamma u = F \\ u|_{\partial\mathcal{K}} = 0 \\ u|_{t=0} = f, \quad \partial_t u|_{t=0} = g, \end{cases} \tag{2.1}$$

where

$$(\Box_\gamma u)^I = \big(\partial_t^2 - c_I^2 \Delta\big) u^I + \sum_{J=1}^{D} \sum_{j,k=0}^{3} \gamma^{IJ,jk}(t,x) \partial_j \partial_k u^J, \;\; 1 \le I \le D.$$

We shall assume that the $\gamma^{IJ,jk}$ satisfy the symmetry conditions

$$\gamma^{IJ,jk} = \gamma^{JI,jk} = \gamma^{IJ,kj}, \tag{2.2}$$

as well as the size condition

$$\sum_{I,J=1}^{D} \sum_{j,k=0}^{3} \|\gamma^{IJ,jk}(t,x)\|_\infty \le \delta/(1+t) \tag{2.3}$$

for $\delta > 0$ sufficiently small (depending only on the wave speeds). The energy estimate will involve bounds for the gradient of the perturbation terms

$$\|\gamma'(t,\,\cdot\,)\|_\infty = \sum_{I,J=1}^{D} \sum_{j,k,l=0}^{3} \big\| \partial_l \gamma^{IJ,jk}(t,x) \big\|_\infty ,$$

and it will of course involve the energy form associated with $\Box_\gamma$, $e_0(u) = \sum_{I=1}^{D} e_0^I(u)$, where

$$\begin{aligned} e_0^I(u) = \big(\partial_0 u^I\big)^2 + \sum_{k=1}^{3} c_I^2 \big(\partial_k u^I\big)^2 \\ + 2 \sum_{J=1}^{D} \sum_{k=0}^{3} \gamma^{IJ,0k} \partial_0 u^I \partial_k u^J - \sum_{J=1}^{D} \sum_{j,k=0}^{3} \gamma^{IJ,jk} \partial_j u^I \partial_k u^J . \end{aligned} \tag{2.4}$$

The most basic estimate will involve

$$E_M(t) = E_M(u)(t) = \int \sum_{j=0}^{M} e_0\big(\partial_t^j u\big)(t,x)\, dx.$$

Lemma 2.1. *Fix $M = 0, 1, 2, \ldots$ and assume that the perturbation terms $\gamma^{IJ,jk}$ are as above. Suppose also that $u \in C^\infty$ solves* (2.1)*, and for every t, $u(t, x) = 0$ for large x. Then there is an absolute constant C so that*

$$\partial_t E_M^{1/2}(t) \le C \sum_{j=0}^{M} \left\| \Box_\gamma \partial_t^j u(t, \cdot\,) \right\|_2 + C \|\gamma'(t, \cdot\,)\|_\infty E_M^{1/2}(t). \tag{2.5}$$

Although the result is standard, we shall present its proof since it serves as a model for the more difficult variations that are to follow. We first notice that it suffices to prove the result for $M = 0$ in view of our assumption that the $\partial_t^j u$ satisfy the Dirichlet boundary conditions for $1 \le j \le M$.

To proceed, we need to define the other components of the energy-momentum vector. For $I = 1, 2, \ldots, D$, and $k = 1, 2, 3$, we let

$$e_k^I = e_k^I(u) = -2\, c_I^2\, \partial_0 u^I \partial_k u^I + 2 \sum_{J=1}^{D} \sum_{j=0}^{3} \gamma^{IJ,jk} \partial_0 u^I \partial_j u^J. \tag{2.6}$$

Then if e_0 is the component defined before in (2.4), we have

(2.7)

$$\begin{aligned}
\partial_0 e_0^I = {} & 2\, \partial_0 u^I \partial_0^2 u^I + 2 \sum_{k=1}^{3} c_I^2 \partial_k u^I \partial_0 \partial_k u^I + 2\, \partial_0 u^I \sum_{J=1}^{D} \sum_{k=0}^{3} \gamma^{IJ,0k} \partial_0 \partial_k u^J \\
& + 2 \sum_{J=1}^{D} \sum_{k=0}^{3} \gamma^{IJ,0k} \partial_0^2 u^I \partial_k u^J \\
& - \sum_{J=1}^{D} \sum_{j,k=0}^{3} \gamma^{IJ,jk} \big[\partial_0 \partial_j u^I \partial_k u^J + \partial_j u^I \partial_0 \partial_k u^J \big] + R_0^I,
\end{aligned}$$

where

$$R_0^I = 2 \sum_{J=1}^{D} \sum_{k=0}^{3} \big(\partial_0 \gamma^{IJ,0k} \big) \partial_0 u^I \partial_k u^J - \sum_{J=1}^{D} \sum_{j,k=0}^{3} \big(\partial_0 \gamma^{IJ,jk} \big) \partial_j u^I \partial_k u^J.$$

Also,

$$\begin{aligned}
\sum_{k=1}^{3} \partial_k e_k^I = {} & -2\, \partial_0 u^I c_I^2 \Delta u^I - 2 \sum_{k=1}^{3} c_I^2 \partial_k u^I \partial_0 \partial_k u^I \\
& + 2\, \partial_0 u^I \sum_{J=1}^{D} \sum_{j=0}^{3} \sum_{k=1}^{3} \gamma^{IJ,jk} \partial_j \partial_k u^J \\
& + 2 \sum_{J=1}^{D} \sum_{j=0}^{3} \sum_{k=1}^{3} \gamma^{IJ,jk} \partial_0 \partial_k u^I \partial_j u^J + \sum_{k=1}^{3} R_k^I,
\end{aligned} \tag{2.8}$$

where

$$R_k^I = 2\sum_{J=1}^{D}\sum_{j=0}^{3}(\partial_k \gamma^{IJ,jk})\partial_0 u^I \partial_j u^J.$$

Note that by the symmetry conditions (2.2) if we sum the second to last term and the third to last terms in (2.7) over I, we get

$$-2\sum_{I,J=1}^{D}\sum_{j=0}^{3}\sum_{k=1}^{3}\gamma^{IJ,jk}\partial_0\partial_k u^I \partial_j u^J,$$

which is -1 times the sum over I of the second to last term of (2.8). From this, we conclude that if we set

$$e_j = e_j(u) = \sum_{I=1}^{D} e_j^I, \quad j = 0, 1, 2, 3,$$

and

$$R = R(u', u') = \sum_{I=1}^{D}\sum_{k=0}^{3} R_k^I,$$

then

$$\partial_t e_0 + \sum_{k=1}^{3}\partial_k e_k = 2\langle \partial_t u, \Box_\gamma u\rangle + R(u', u'),$$

with $\langle\,\cdot\,,\,\cdot\,\rangle$ denoting the standard inner product in $\mathbb{R}^D$.

If we integrate this identity over $\mathbb{R}^3\backslash\mathcal{K}$ and apply the divergence theorem, we obtain

$$\text{(2.9)}\quad \partial_t \int_{\mathbb{R}^3\backslash\mathcal{K}} e_0(t,x)\,dx - \int_{\partial\mathcal{K}}\sum_{j=1}^{3} e_j n_j\,d\sigma \\ = 2\int_{\mathbb{R}^3\backslash\mathcal{K}}\langle \partial_t u, \Box_\gamma u\rangle\,dx + \int_{\mathbb{R}^3\backslash\mathcal{K}} R(u', u')\,dx\,.$$

Here, $n = (n_1, n_2, n_3)$ is the outward normal to $\mathcal{K}$, and $d\sigma$ is surface measure on $\partial\mathcal{K}$.

Since we are assuming that u solves (2.1), and hence $\partial_t u$ vanishes on $\partial\mathcal{K}$, the integrand in the last term in the left side of (2.9) vanishes identically. Therefore, we have

$$\partial_t \int_{\mathbb{R}^3\backslash\mathcal{K}} e_0(t,x)\,dx = 2\int_{\mathbb{R}^3\backslash\mathcal{K}}\langle \partial_t u, \Box_\gamma u\rangle\,dx + \int_{\mathbb{R}^3\backslash\mathcal{K}} R(u', u')\,dx.$$

Note that if δ in (2.3) is small, then

$$(2.10)\quad \big(2\max_I\{c_I^2, c_I^{-2}\}\big)^{-1}|u'(t,x)|^2 \le e_0(t,x) \le 2\max_I\{c_I^2, c_I^{-2}\}|u'(t,x)|^2.$$

This yields

$$\partial_t\Big(\int_{\mathbb{R}^3\backslash\mathcal{K}} e_0(t,x)\,dx\Big)^{1/2} \le C\|\Box_\gamma u(t,\,\cdot\,),\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})} + C\sum_{I,J=1}^{D}\sum_{i,j,k=0}^{3}\big\|\partial_i\gamma^{IJ,jk}(t,\,\cdot\,)\big\|_\infty\Big(\int_{\mathbb{R}^3\backslash\mathcal{K}} e_0(t,x)\,dx\Big)^{1/2},$$

as desired. □

We require a minor modification of this energy estimate that involves a slight variant of the scaling vector field $L = r\partial_r + t\partial_t$.

Before stating the next result, let us introduce some notation. If $P = P(t,x,D_t,D_x)$ is a differential operator, we shall let

$$\big[P,\gamma^{kl}\partial_k\partial_l\big]u = \sum_{1\le I,J\le D}\sum_{0\le k,l\le 3}\big|\big[P,\gamma^{IJ,kl}\partial_k\partial_l\big]u^J\big|.$$

We can now state the simple variant of Lemma 2.1 that we require.

Lemma 2.2. *Fix a bump function $\eta \in C^\infty(\mathbb{R}^3)$ satisfying $\eta(x) = 0$, for $x \in \mathcal{K}$ and $\eta(x) = 1$, $|x| > 1$. Let $\tilde{L} = \eta(x)r\partial_r + t\partial_t$, and set*

$$X_{\nu,j} = \int e_0\big(\tilde{L}^\nu\partial_t^j u\big)(t,x)\,dx.$$

Then if $u \in C^\infty(\mathbb{R}_+\times\mathbb{R}^3\backslash\mathcal{K})$ solves (2.1) *and vanishes for large x for every t*

$$\begin{aligned}(2.11)\quad \partial_t X_{\nu,j} \le\; & CX_{\nu,j}^{1/2}\big\|\tilde{L}^\nu\partial_t^j\Box_\gamma u(t,\,\cdot\,)\big\|_2 + C\|\gamma'(t,\,\cdot\,)\|_\infty X_{\nu,j}\\ &+ CX_{\nu,j}^{1/2}\big\|\big[\tilde{L}^\nu\partial_t^j,\gamma^{kl}\partial_k\partial_l\big]u(t,\,\cdot\,)\big\|_2\\ &+ CX_{\nu,j}^{1/2}\sum_{\mu\le\nu-1}\big\|L^\mu\partial_t^j\Box u(t,\,\cdot\,)\big\|_2\\ &+ CX_{\nu,j}^{1/2}\sum_{\substack{\mu+|\alpha|\le j+\nu\\ \mu\le\nu-1}}\|L^\mu\partial^\alpha u'(t,\,\cdot\,)\|_{L^2(\{x\in\mathbb{R}^3\backslash\mathcal{K}:\,|x|<1\})}.\end{aligned}$$

Proof. Note that like u, $\tilde{L}^\nu\partial_t^j u(t,x)$ vanishes when $x \in \partial\mathcal{K}$. Therefore by the special case where $M = 0$ in Lemma 2.1 we have

$$(2.12)\qquad \partial_t X_{\nu,j} \le CX_{\nu,j}^{1/2}\big\|\Box_\gamma\tilde{L}^\nu\partial_t^j u(t,\,\cdot\,)\big\|_2 + C\|\gamma'(t,\,\cdot\,)\|_\infty X_{\nu,j}.$$

To proceed we need to estimate the first term in the right by noting that

$$\begin{aligned}
\left|\Box_\gamma \tilde{L}^\nu \partial_t^j u\right| &\le \left|\tilde{L}^\nu \partial_t^j \Box_\gamma u\right| + \left|\left[\tilde{L}^\nu \partial_t^j, \gamma^{kl}\partial_k\partial_l\right]u\right| + \left|[\tilde{L}^\nu, \Box]\partial_t^j u\right| \\
&\le \left|\tilde{L}^\nu \partial_t^j \Box_\gamma u\right| + \left|\left[\tilde{L}^\nu \partial_t^j, \gamma^{kl}\partial_k\partial_l\right]u\right| + \left|[L^\nu, \Box]\partial_t^j u\right| \\
&\qquad + \left|[\tilde{L}^\nu - L^\nu, \Box]\partial_t^j u\right| \\
&\le \left|\tilde{L}^\nu \partial_t^j \Box_\gamma u\right| + \left|\left[\tilde{L}^\nu \partial_t^j, \gamma^{kl}\partial_k\partial_l\right]u\right| + 2\sum_{\mu\le\nu-1} \left|L^\mu \partial_t^j \Box u\right| \\
&\qquad + C\chi_{|x|<1}(x) \sum_{\substack{\mu+|\alpha|\le j+\nu \\ \mu\le\nu-1}} |L^\mu \partial^\alpha u'(t,x)|.
\end{aligned}$$

In the last step we used the fact that $[\Box, L] = 2\Box$, and $\nabla_x \eta(x) = 0$, $|x| > 1$. If we combine the last inequality and (2.12) we get (2.11). $\square$

The last lemma involved estimates for powers of L and ∂_t. Let us now prove a simple result which shows how these lead to estimates for powers of L and $\partial = \partial_{t,x}$.

Lemma 2.3. *Fix N_0 and ν and suppose that $u \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^3 \backslash \mathcal{K})$ solves* (2.1) *and vanishes for large x for each t. Then*

$$\begin{aligned}
(2.13)\qquad \sum_{|\alpha|\le N_0} \|L^\nu \partial^\alpha u'(t,\,\cdot\,)\|_2 &\le C \sum_{\substack{j+\mu\le\nu+N_0 \\ \mu\le\nu}} \left\|L^\mu \partial_t^j u'(t,\,\cdot\,)\right\|_2 \\
&\quad + C \sum_{\substack{|\alpha|+\mu\le N_0+\nu-1 \\ \mu\le\nu}} \|L^\mu \partial^\alpha \Box u(t,\,\cdot\,)\|_2.
\end{aligned}$$

Proof. We shall prove this inequality by induction on ν. Since, by elliptic regularity estimates, the inequality holds when $\nu = 0$, let us therefore assume that it is valid when ν is replaced by $\nu - 1$ and use this to prove it for a given $\nu = 1, 2, 3, \ldots$.

Since $\mathcal{K} \subset \{|x| < 1\}$ it is straightforward to see that

$$\sum_{|\alpha|\le N_0} \|L^\nu \partial^\alpha u'(t,\,\cdot\,)\|_{L^2(|x|>1)}$$

is dominated by the right side of (2.13). Therefore, it suffices to show that we can prove the analog of (2.13) where the norm is taken over $|x| < 2$.

For the latter, we shall use the fact that

$$\sum_{|\alpha|\le N_0} \|L^\nu \partial^\alpha u'(t,\,\cdot\,)\|_{L^2(|x|<2)} \le C \sum_{\substack{|\alpha|+\mu\le N_0+\nu \\ \mu\le\nu}} t^\mu \left\|\partial_t^\mu \partial^\alpha u'(t,\,\cdot\,)\right\|_{L^2(|x|<2)}.$$

By elliptic regularity, if we fix $R \geq 2$ then

$$\sum_{\substack{|\alpha|+\mu\leq N_0+\nu \\ \mu\leq\nu}} \left\|\partial^\alpha \partial_t^\mu u'(t,\,\cdot\,)\right\|_{L^2(|x|<R)} \leq C \sum_{\substack{j+\mu\leq N_0+\nu \\ \mu\leq\nu}} \left\|\partial_t^{j+\mu} u'(t,\,\cdot\,)\right\|_{L^2(|x|<R+2)} + C \sum_{\substack{|\alpha|+\mu\leq N_0+\nu-1 \\ \mu\leq\nu}} \left\|\partial^\alpha \partial_t^\mu \Box u(t,\,\cdot\,)\right\|_{L^2(|x|<R+2)}.$$

Therefore,

$$\begin{aligned}(2.14)\qquad \sum_{|\alpha|\leq N_0} \|L^\nu \partial^\alpha u'(t,\,\cdot\,)\|_{L^2(|x|<R)} &\leq C \sum_{\substack{j+\mu\leq N_0+\nu \\ \mu\leq\nu}} \left\|t^\mu \partial_t^{\mu+j} u'(t,\,\cdot\,)\right\|_{L^2(|x|<R+2)} \\ &\quad + C \sum_{\substack{|\alpha|+\mu\leq N_0+\nu-1 \\ \mu\leq\nu}} \left\|t^\mu \partial_t^\mu \partial^\alpha \Box u(t,\,\cdot\,)\right\|_{L^2(|x|<R+2)} \\ &\leq C \sum_{j\leq N_0} \left\|L^\nu \partial_t^j u'(t,\,\cdot\,)\right\|_2 + C \sum_{\substack{|\alpha|+\mu\leq N_0+\nu \\ \mu\leq\nu-1}} \|L^\mu \partial^\alpha u'(t,\,\cdot\,)\|_2 \\ &\quad + C \sum_{\substack{|\alpha|+\mu\leq N_0+\nu-1 \\ \mu\leq\nu}} \|L^\mu \partial^\alpha \Box u(t,\,\cdot\,)\|_2.\end{aligned}$$

As a result, we get (2.13) by the inductive step and the fact that, we can control the norms over the set where $|x| > 1$. $\square$

Using (2.13) we can prove the following estimate.

Proposition 2.4. *Suppose that the constant δ in (2.3) is small. Suppose further that*

$$(2.15)\qquad \|\gamma'(t,\,\cdot\,)\|_\infty \leq \delta/(1+t),$$

and

$$\begin{aligned}(2.16)\qquad \sum_{\substack{j+\mu\leq N_0+\nu_0 \\ \mu\leq\nu_0}} \Big(\left\|\tilde{L}^\mu \partial_t^j \Box_\gamma u(t,\,\cdot\,)\right\|_2 + \left\|\left[\tilde{L}^\mu \partial_t^j, \gamma^{kl}\partial_k\partial_l\right]u(t,\,\cdot\,)\right\|_2\Big) \\ \leq \frac{\delta}{1+t} \sum_{\substack{j+\mu\leq N_0+\nu_0 \\ \mu\leq\nu_0}} \left\|\tilde{L}^\mu \partial_t^j u'(t,\,\cdot\,)\right\|_2 + H_{\nu_0,N_0}(t),\end{aligned}$$

where N_0 and ν_0 are fixed. Then

$$\sum_{\substack{|\alpha|+\mu\le N_0+\nu_0\\ \mu\le\nu_0}} \|L^\mu\partial^\alpha u'(t,\,\cdot\,)\|_2 \tag{2.17}$$
$$\le C\sum_{\substack{|\alpha|+\mu\le N_0+\nu_0-1\\ \mu\le\nu_0}} \|L^\mu\partial^\alpha\Box u(t,\,\cdot\,)\|_2 + C(1+t)^{A\delta}\sum_{\substack{\mu+j\le N_0+\nu_0\\ \mu\le\nu_0}} X^{1/2}_{\mu,j}(0)$$
$$+ C(1+t)^{A\delta}\Big(\int_0^t \sum_{\substack{|\alpha|+\mu\le N_0+\nu_0-1\\ \mu\le\nu_0-1}} \|L^\mu\partial^\alpha\Box u(s,\,\cdot\,)\|_2\,ds + \int_0^t H_{\nu_0,N_0}(s)ds\Big)$$
$$+ C(1+t)^{A\delta}\int_0^t \sum_{\substack{|\alpha|+\mu\le N_0+\nu_0\\ \mu\le\nu_0-1}} \|L^\mu\partial^\alpha u'(s,\,\cdot\,)\|_{L^2(|x|<1)}\,ds,$$

where the constants C and A depend only on the constants in (2.11).

Note that because of (2.16) the term in (2.17) involving H_{ν_0,N_0} must involve bounds for derivatives of $\Box_\gamma u$. For our application to equations with quadratic nonlinearities (1.5), it will mainly involve involve L^2_x norms of $|L^\mu\partial^\alpha u'|^2$ with $\mu+|\alpha|$ much smaller than $N_0+\nu_0$, and so the integral involving H_{ν_0,N_0} can be dealt with using an inductive argument and weighted $L^2_tL^2_x$ estimates that will be presented at the end of this section.

Proof. We first note that by (2.3) and the definition (2.4) of the energy form

$$\sum_{\substack{j+\mu\le N_0+\nu_0\\ \mu\le\nu_0}} \big\|\tilde L^\mu\partial_t^j u'(t,\,\cdot\,)\big\|_2 \le 2\sum_{\substack{j+\mu\le N_0+\nu_0\\ \mu\le\nu_0}} X^{1/2}_{\mu,j}(t), \tag{2.18}$$

if δ is sufficiently small. Therefore, by (2.11) and (2.15)–(2.16) we have

$$\partial_t\sum_{\substack{j+\mu\le N_0+\nu_0\\ \mu\le\nu_0}} X^{1/2}_{\mu,j}(t) \le \frac{A\delta}{1+t}\sum_{\substack{j+\mu\le N_0+\nu_0\\ \mu\le\nu_0}} X^{1/2}_{\mu,j}(t) + AH_{\nu_0,N_0}(t)$$
$$+ A\sum_{\substack{\mu+j\le N_0+\nu_0-1\\ \mu\le\nu_0-1}} \big\|L^\mu\partial_t^j\Box u(t,\,\cdot\,)\big\|_2$$
$$+ A\sum_{\substack{|\alpha|+\mu\le N_0+\nu_0\\ \mu\le\nu_0-1}} \|L^\mu\partial^\alpha u'(t,\,\cdot\,)\|_{L^2(|x|<1)},$$

where A depends on the constants in (2.11). By Gronwall's inequality

$$\sum_{\substack{j+\mu\le N_0+\nu_0\\ \mu\le\nu_0}} X^{1/2}_{\mu,j}$$

is dominated by the right side of (2.17). By applying (2.13) and (2.18), we conclude that (2.17) must be valid. □

In proving our existence results for (1.5) the key step will be to obtain a priori L^2-estimates involving $L^\mu Z^\alpha u'$. The next result indicates how these can be obtained from ones involving $L^\mu \partial^\alpha u'$.

Proposition 2.5. *Fix N_0 and ν_0, and set*

$$Y_{N_0,\nu_0}(t) = \sum_{\substack{|\alpha|+\mu\le N_0+\nu_0\\ \mu\le\nu_0}} \int e_0(L^\mu Z^\alpha u)(t,x)\,dx. \tag{2.19}$$

Suppose that the constant δ in (2.3) *is small and that* (2.15) *holds. Then*

$$\begin{aligned}\partial_t Y_{N_0,\nu_0} \le C Y_{N_0,\nu_0}^{1/2} &\sum_{\substack{|\alpha|+\mu\le N_0+\nu_0\\ \mu\le\nu_0}} \big\|\Box_\gamma L^\mu Z^\alpha u(t,\,\cdot\,)\big\|_2\\ &+ C\|\gamma'(t,\,\cdot\,)\|_\infty Y_{N_0,\nu_0} + C \sum_{\substack{|\alpha|+\mu\le N_0+\nu_0+1\\ \mu\le\nu_0}} \|L^\mu\partial^\alpha u'(s,\,\cdot\,)\|^2_{L^2(|x|<1)}.\end{aligned} \tag{2.20}$$

Proof. If we argue as in the proof of Lemma 2.1 we find that

$$\begin{aligned}\partial_t Y_{N_0,\nu_0} \le C Y_{N_0,\nu_0}^{1/2} &\sum_{\substack{|\alpha|+\mu\le N_0+\nu_0\\ \mu\le\nu_0}} \big\|\Box_\gamma L^\mu Z^\alpha u(t,\,\cdot\,)\big\|_2\\ &+ C\|\gamma'(t,\,\cdot\,)\|_\infty Y_{N_0,\nu_0} + C\int_{\partial\mathcal{K}} \sum_{a=1}^3 |e_a n_a|\,d\sigma,\end{aligned} \tag{2.21}$$

where $n = (n_1, n_2, n_3)$ is the outward normal at a given point in $\partial\mathcal{K}$, and

$$e_a = \sum_{\substack{|\alpha|+\mu\le N_0+\nu_0\\ \mu\le\nu_0}} e_a(L^\mu Z^\alpha u)(t,x), \quad a = 1,2,3,$$

are the components of the energy-momentum tensor defined in (2.6). Since $\mathcal{K} \subset \{|x| < 1\}$ and since

$$\sum_{\substack{|\alpha|+\mu\le N_0+\nu_0\\ \mu\le\nu_0}} |L^\mu Z^\alpha u(t,x)| \le C \sum_{\substack{|\alpha|+\mu\le N_0+\nu_0\\ \mu\le\nu_0}} |L^\mu \partial^\alpha u(t,x)|, \quad x\in\partial\mathcal{K},$$

we have

$$\int_{\partial\mathcal{K}} \sum_{a=1}^3 |e_a n_a|\,d\sigma \le C\int_{\{x\in\mathbb{R}^3\setminus\mathcal{K}:\,|x|<1\}} \sum_{\substack{|\alpha|+\mu\le N_0+\nu_0+1\\ \mu\le\nu_0}} |L^\mu\partial^\alpha u'(t,x)|^2\,dx.$$

$\square$

As in [10] and [11] we shall control the local L^2 norms, such as the last term in (2.20) by using weighted $L^2_t L^2_x$ estimates. They will also be used in obtaining decay estimate for solutions of the nonlinear equation. To avoid cumbersome notation, for the rest of the section we shall abuse notation a bit by letting $\Box = \partial_t^2 - \Delta$ denote the unit speed D'Alembertian. The passing from the ensuing estimates involving this case to ones involving (1.6) is straightforward. Also, in what follows, we shall let

$$S_T = \{[0, T] \times \mathbb{R}^3 \backslash \mathcal{K}\}$$

denote the time strip of height T in $\mathbb{R}_+ \times \mathbb{R}^3 \backslash \mathcal{K}$.

Proposition 2.6. *Fix N_0 and ν_0. Suppose that $\mathcal{K}$ satisfies the local exponential energy decay bounds* (1.4)*. Suppose also that $u \in C^\infty$ solves* (2.1) *and satisfies $u(t, x) = 0$, $t < 0$. Then there is a constant $C = C_{N_0, \nu_0, \mathcal{K}}$ so that if u vanishes for large x for every fixed t*

$$(2.22) \quad \big(\log(2+T)\big)^{-1/2} \sum_{\substack{|\alpha|+\mu \le N_0+\nu_0 \\ \mu \le \nu_0}} \|\langle x\rangle^{-1/2} L^\mu \partial^\alpha u'\|_{L^2(S_T)} \le C \int_0^T \sum_{\substack{|\alpha|+\mu \le N_0+\nu_0+1 \\ \mu \le \nu_0}} \|\Box L^\mu \partial^\alpha u(s, \,\cdot\,)\|_2 \, ds + C \sum_{\substack{|\alpha|+\mu \le N_0+\nu_0 \\ \mu \le \nu_0}} \|\Box L^\mu \partial^\alpha u\|_{L^2(S_T)}.$$

Also, if N_0 and ν_0 are fixed

$$(2.23) \quad \big(\log(2+T)\big)^{-1/2} \sum_{\substack{|\alpha|+\mu \le N_0+\nu_0 \\ \mu \le \nu_0}} \|\langle x\rangle^{-1/2} L^\mu Z^\alpha u'\|_{L^2(S_T)} \le C \int_0^T \sum_{\substack{|\alpha|+\mu \le N_0+\nu_0+1 \\ \mu \le \nu_0}} \|\Box L^\mu Z^\alpha u(s, \,\cdot\,)\|_2 \, ds + C \sum_{\substack{|\alpha|+\mu \le N_0+\nu_0 \\ \mu \le \nu_0}} \|\Box L^\mu Z^\alpha u\|_{L^2(S_T)}.$$

The constants in (2.22) and (2.23) of course do not depend on the size of the set where u does not vanish. To prove these estimates we shall need a couple of lemmas. The first says that these estimates hold (with no loss of derivatives) in the boundaryless case.

Lemma 2.7. *Suppose that $v \in C^\infty(\mathbb{R}\times\mathbb{R}^3)$ vanishes for large x for every t. Then there is a uniform constant C so that if v has vanishing Cauchy data*

$$(\ln(2+T))^{-1/2}\|\langle x\rangle^{-1/2}v'\|_{L^2([0,T]\times\mathbb{R}^3)} \le C\int_0^T \|\Box v(s,\,\cdot\,)\|_{L^2(\mathbb{R}^3)}\,ds. \tag{2.24}$$

Also, given μ and α,

$$\begin{aligned}(\ln(2+T))^{-1/2}&\|\langle x\rangle^{-1/2}(L^\mu Z^\alpha v)'\|_{L^2([0,T]\times\mathbb{R}^3)}\\ &\le C\int_0^T \|\Box L^\mu Z^\alpha v(s,\,\cdot\,)\|_{L^2(\mathbb{R}^3)}\,ds.\end{aligned} \tag{2.25}$$

The first inequality, (2.24), was proved in [10]. The second follows from the first.

As was shown in [10], (2.24) follows immediately from the fact that stronger bounds hold when one restricts the norms in the left to regions where $|x|$ is bounded. In particular, just by using Huygens principle, one can show that if R is fixed then there is a uniform constant $C = C_R$ so that

$$\begin{aligned}\|v'\|_{L^2([0,T]\times\{x\in\mathbb{R}^3:|x|<R\})} &+ \|v\|_{L^2_tL^6_x([0,T]\times\{x\in\mathbb{R}^3:|x|<R\})}\\ &\le C\int_0^T \|\Box v(s,\,\cdot\,)\|_{L^2(\mathbb{R}^3)}\,ds.\end{aligned} \tag{2.26}$$

To prove Proposition 2.6 we shall need the following local estimates which follow from the local exponential energy decay (1.4).

Lemma 2.8. *Suppose that* (1.4) *holds and that $\Box u(t,x) = 0$ for $|x| > 4$ and $t > 0$. Suppose also that $u(t,x) = 0$ for $t \le 0$. Then if N_0 and ν_0 are fixed and if $c > 0$ is as in* (1.4)

$$\begin{aligned}\sum_{\substack{|\alpha|+\mu\le N_0+\nu_0\\ \mu\le\nu_0}} &\|L^\mu\partial^\alpha u'(t,\,\cdot\,)\|_{L^2(\{\mathbb{R}^3\backslash\mathcal{K}:\,|x|<4\})}\\ &\le C\sum_{\substack{|\alpha|+\mu\le N_0+\nu_0-1\\ \mu\le\nu_0}}\|L^\mu\partial^\alpha\Box u(t,\,\cdot\,)\|_2\\ &\quad + C\int_0^t e^{-(c/2)(t-s)}\sum_{\substack{|\alpha|+\mu\le N_0+\nu_0+1\\ \mu\le\nu_0}}\|L^\mu\partial^\alpha\Box u(s,\,\cdot\,)\|_2 ds.\end{aligned} \tag{2.27}$$

The proof is quite simple. By (1.4) we have that

$$\sum_{\substack{j+\mu\le N_0+\nu_0\\ \mu\le\nu_0}}\left\|\langle t\rangle^\mu\partial_t^\mu\partial_t^j u'(t,\,\cdot\,)\right\|_{L^2(\{\mathbb{R}^3\backslash\mathcal{K}:\,|x|<6\})}$$

is dominated by the last term in the right side of (2.27). By the first inequality in (2.14) with $R = 4$, this implies (2.27) since

$$\sum_{\substack{|\alpha|+\mu\le N_0+\nu_0-1\\ \mu\le\nu_0}} \left\| t^\mu \partial_t^\mu \partial^\alpha \Box u(t,\,\cdot\,)\right\|_{L^2(|x|<6)}$$

is dominated by the first term in the right side of (2.27).

Proof of Proposition 2.6. We shall only prove (2.22) since (2.23) follows from the same argument. When x is near the obstacle, our proof will rely mostly on the local energy decay (1.4). Away from the obstacle, we will refer to the related bounds for the free wave equation from Lemma 2.7.

The first step in proving (2.22) will be to show that if we take the $L_t^2 L_x^2$ norm over a region where $|x|$ is bounded then we have better estimates, i.e.,

(2.28)
$$\sum_{\substack{|\alpha|+\mu\le N_0+\nu_0\\ \mu\le\nu_0}} \|L^\mu \partial^\alpha u'\|_{L^2(S_T\cap|x|<2)} \le C\int_0^T \sum_{\substack{|\alpha|+\mu\le N_0+\nu_0+1\\ \mu\le\nu_0}} \|\Box L^\mu\partial^\alpha u(s,\,\cdot\,)\|_2 ds + C\sum_{\substack{|\alpha|+\mu\le N_0+\nu_0-1\\ \mu\le\nu_0}} \|\Box L^\mu \partial^\alpha u\|_{L^2(S_T)}.$$

To prove this, let us first assume that u is as in Lemma 2.8. Thus, if we assume that $\Box u(t,x) = 0$ when $|x| > 4$, then by (2.27) we have for $0 < \tau < T$

$$\begin{aligned}\sum_{\substack{|\alpha|+\mu\le N_0+\nu_0\\ \mu\le\nu_0}} &\|L^\mu \partial^\alpha u'(\tau,\,\cdot\,)\|^2_{L^2(|x|<2)}\\ &\le C\sum_{\substack{|\alpha|+\mu\le N_0+\nu_0-1\\ \mu\le\nu_0}} \|L^\mu\partial^\alpha\Box u(\tau,\,\cdot\,)\|_2^2\\ &\quad + C\Big(\int_0^\tau e^{-(c/2)(\tau-s)} \sum_{\substack{|\alpha|+\mu\le N_0+\nu_0+1\\ \mu\le\nu_0}} \|L^\mu\partial^\alpha\Box u(s,\,\cdot\,)\|_2 ds\Big)^2.\end{aligned}$$

After integrating τ from 0 to T we obtain (2.28) under the support assumptions of Lemma 2.8.

Note that if we had applied Young's inequality, we would have gotten

$$(2.29)\quad \sum_{\substack{|\alpha|+\mu\le N_0+\nu_0\\ \mu\le\nu_0}} \|L^\mu\partial^\alpha u'\|^2_{L^2(S_T\cap|x|<2)} \le C\sum_{\substack{|\alpha|+\mu\le N_0+\nu_0+1\\ \mu\le\nu_0}} \|L^\mu\partial^\alpha\Box u\|^2_{L^2(S_T)}, \quad \text{if } \Box u(t,x)=0\,,\ |x|>4.$$

This inequality will be useful in the last part of the proof of (2.28).

We now need to show that the inequality (2.28) holds when we assume that $\Box u(t,x)$ vanishes for $|x|<3$. To do this, we fix $\rho \in C^\infty(\mathbb{R}^3)$ satisfying $\rho(x)=1$ for $|x|<2$ and $\rho(x)=0$ for $|x|\ge 3$. We then write $u=u_0+u_r$ where u_0 solves the boundaryless wave equation $\Box u_0(t,x)=\Box u(t,x)$ if $|x|\ge 3$ and 0 otherwise with vanishing initial data . It then follows that $\tilde u = \rho u_0 + u_r$ solves the Dirichlet-wave equation $\Box \tilde u = -2\nabla_x \rho\cdot\nabla_x u_0 - (\Delta\rho)u_0$ with zero initial data. Therefore, by (2.29), we have

$$\begin{aligned}
\sum_{\substack{|\alpha|+\mu\le N_0+\nu_0\\ \mu\le\nu_0}} \|L^\mu\partial^\alpha u'\|^2_{L^2(S_T\cap|x|<2)} &= \sum_{\substack{|\alpha|+\mu\le N_0+\nu_0\\ \mu\le\nu_0}} \|L^\mu\partial^\alpha \tilde u'\|^2_{L^2(S_T\cap|x|<2)} \\
&\le \sum_{\substack{|\alpha|+\mu\le N_0+\nu_0+1\\ \mu\le\nu_0}} \|L^\mu\partial^\alpha \Box\tilde u\|^2_{L^2(S_T)} \\
&\le C\sum_{\substack{|\alpha|+\mu\le N_0+\nu_0+1\\ \mu\le\nu_0}} \Big(\big\|L^\mu\partial^\alpha u_0'\big\|^2_{L^2(S_T\cap|x|<4)} + \big\|L^\mu\partial^\alpha u_0\big\|^2_{L^2(S_T\cap|x|<4)}\Big).
\end{aligned}$$

One now gets (2.28) for this by applying (2.26) since $\Box u_0=\Box u$ in $\mathbb{R}^3\backslash\mathcal{K}$.

To finish the proof of (2.22) we must show that

$$\begin{aligned}
\big(\log(2+T)\big)^{-1/2} \sum_{\substack{|\alpha|+\mu\le N_0+\nu_0\\ \mu\le\nu_0}} \|\langle x\rangle^{-1/2}L^\mu\partial^\alpha u'\|_{L^2(S_T\cap|x|>2)} &\le C\int_0^T \sum_{\substack{|\alpha|+\mu\le N_0+\nu_0+1\\ \mu\le\nu_0}} \|\Box L^\mu\partial^\alpha u(s,\,\cdot\,)\|_2\,ds \\
&\quad + C\sum_{\substack{|\alpha|+\mu\le N_0+\nu_0-1\\ \mu\le\nu_0}} \|\Box L^\mu\partial^\alpha u\|_{L^2(S_T)}.
\end{aligned} \tag{2.30}$$

To do this, we fix $\beta\in C^\infty(\mathbb{R}^3)$ satisfying $\beta(x)=1$, $|x|\ge 2$ and $\beta(x)=0$, $|x|\le 3/2$. By assumption the obstacle is contained in the set $|x|<1$. It follows that $v=\beta u$ solves the boundaryless wave equation $\Box v = \beta\Box u - 2\nabla_x\beta\cdot\nabla_x u - (\Delta\beta)u$ with vanishing initial data. Also $u(t,x)=v(t,x)$ for $|x|>2$. We split $v=v_1+v_2$ where v_1 solves $\Box v_1=\beta\Box u$ and v_2 solves $\Box v_2 = -2\nabla_x\beta\cdot\nabla_x u-(\Delta\beta)u$ and both have zero initial data. By (2.25) if we replace u by v_1 in the left side of (2.30), then the resulting quantity is dominated by the right side of (2.30).

Therefore, to finish the proof, we must show that

$$\begin{aligned}
(\log(2+T))^{-1/2} \sum_{\substack{|\alpha|+\mu\le N_0+\nu_0\\ \mu\le\nu_0}} \big\|\langle x\rangle^{-1/2}L^\mu\partial^\alpha v_2'\big\|_{L^2(S_T\cap|x|>2)} \\
\le C\int_0^T \sum_{\substack{|\alpha|+\mu\le N_0+\nu_0+1\\ \mu\le\nu_0}} \|\Box L^\mu\partial^\alpha u(s,\,\cdot\,)\|_2 ds + C\sum_{\substack{|\alpha|+\mu\le N_0+\nu_0-1\\ \mu\le\nu_0}} \|\Box L^\mu\partial^\alpha u\|_{L^2(S_T)}.
\end{aligned} \tag{2.31}$$

To prove this, we note that $G = -2\nabla_x\beta\cdot\nabla_x u - (\Delta\beta)u = \Box v_2$ vanishes unless $1 < |x| < 2$. To use this, fix $\chi \in C_0^\infty(\mathbb{R})$ satisfying $\chi(s) = 0$ for $s \notin [1/2, 2]$ and $\sum_j \chi(s-j) = 1$. We then split $G = \sum_j G_j$ where $G_j(s,x) = \chi(s-j)G(s,x)$, and let $v_{2,j}$ be the solution of the inhomogeneous wave equation $\Box v_{2,j} = G_j$ in Minkowski space with zero initial data. Since v_2 also has vanishing Cauchy data, by the sharp Huygens principle the functions $v_{2,j}$ have finite overlap, so that we have $|L^\mu\partial^\alpha v_2'|^2 \le C\sum_j |L^\mu\partial^\alpha v_{2,j}'|^2$ for some uniform constant C. Therefore, by (2.25), the square of the left side of (2.31) is dominated by

$$\begin{aligned}
&\sum_{\substack{|\alpha|+\mu\le N_0+\nu_0\\ \mu\le\nu_0}} \sum_j \Big(\int_0^T \big\|L^\mu\partial^\alpha G_j(s,\,\cdot\,)\big\|_{L^2(\mathbb{R}^3)}\,ds\Big)^2 \\
&\qquad\le C \sum_{\substack{|\alpha|+\mu\le N_0+\nu_0\\ \mu\le\nu_0}} \|L^\mu\partial^\alpha G\|^2_{L^2([0,T]\times\mathbb{R}^3)} \\
&\qquad\le C \sum_{\substack{|\alpha|+\mu\le N_0+\nu_0\\ \mu\le\nu_0}} \Big(\|L^\mu\partial^\alpha u'\|^2_{L^2([0,T]\times|x|<2)} + \|L^\mu\partial^\alpha u\|^2_{L^2([0,T]\times|x|<2)}\Big) \\
&\qquad\le \sum_{\substack{|\alpha|+\mu\le N_0+\nu_0\\ \mu\le\nu_0}} \|L^\mu\partial^\alpha u'\|^2_{L^2(S_T\cap|x|<2)}.
\end{aligned}$$

Consequently, the bound (2.31) follows from (2.28).

This finishes the proof of (2.22). Since the other part of the proposition follow from the same argument, this completes the proof of Proposition 2.6. $\Box$

To be able to handle the last term in the right side of (2.17) we shall need the following result which follows from a similar argument.

Lemma 2.9. *Suppose that* (1.4) *holds, and suppose that* $u \in C^\infty$ *solves* (2.1) *and satisfies* $u(t,x) = 0$, $t < 0$. *Then if* ν_0 *and* N_0 *are fixed and if* $c > 0$ *is as in* (1.4)

$$\begin{aligned}
(2.32)\qquad &\sum_{\substack{|\alpha|+\mu\le N_0+\nu_0\\ \mu\le\nu_0}} \|L^\mu\partial^\alpha u'(t,\,\cdot\,)\|_{L^2(|x|<2)} \\
&\le C \sum_{\substack{|\alpha|+\mu\le N_0+\nu_0+1\\ \mu\le\nu_0}} \Big[\int_0^t e^{-\frac{c}{2}(t-s)}\|L^\mu\partial^\alpha\Box u(s,\,\cdot\,)\|_{L^2(|x|<4)}\,ds \\
&\qquad\qquad + \|L^\mu\partial^\alpha\Box u(t,\,\cdot\,)\|_{L^2(|x|<4)}\Big]
\end{aligned}$$

$$+ C \sum_{\substack{|\alpha|+\mu\le N_0+\nu_0+1\\ \mu\le\nu_0}} \int_0^t e^{-\frac{c}{2}(t-s)}\Big(\int_0^s \|L^\mu\partial^\alpha\Box u(\tau,\,\cdot\,)\|_{L^2(|\,|x|-(s-\tau)\,|<10)}\,d\tau\Big)ds$$

$$+ C \sum_{\substack{|\alpha|+\mu\le N_0+\nu_0+1\\ \mu\le\nu_0}} \int_0^t \|L^\mu\partial^\alpha\Box u(s,\,\cdot\,)\|_{L^2(|\,|x|-(t-s)\,|<10)}\,ds.$$

Additionally, if $t>2$,

$$\sum_{\substack{|\alpha|+\mu\le N_0+\nu_0\\ \mu\le\nu_0}} \int_0^t \|L^\mu\partial^\alpha u'(s,\,\cdot\,)\|_{L^2(|x|<2)}\,ds \tag{2.33}$$
$$\le C \sum_{\substack{|\alpha|+\mu\le N_0+\nu_0+1\\ \mu\le\nu_0}} \int_0^t \left(\int_0^s \|L^\mu\partial^\alpha\Box u(\tau,\,\cdot\,)\|_{L^2(|\,|x|-(s-\tau)\,|<10)}\,d\tau\right)ds.$$

Proof. Since the first inequality obviously implies the second, we shall only prove (2.32).

If $\Box u(s,x)$ vanishes when $|x|>4$, the result follows from (2.27). In this case a stronger inequality holds where the last two terms in the right are not present.

To finish we need to show that the inequality is valid when $\Box u(s,x)$ vanishes for $|x|<3$. In this case, as in the proof of Proposition 2.6 we write $u=u_0+u_r$ where u_0 solves $\Box u_0=\Box u$ with vanishing Cauchy data. Then if as above $\rho\in C^\infty(\mathbb{R}^3)$ equals 1 for $|x|<2$ and 0 for $|x|>3$, then $\tilde u=\rho u_0+u_r$ has vanishing Cauchy data and solves $\Box\tilde u=-2\nabla_x\rho\cdot\nabla_x u_0-(\Delta\rho)u_0$. Thus, since $\Box\tilde u=0$ for $|x|>3$, by the above case

$$\sum_{\substack{|\alpha|+\mu\le N_0+\nu_0\\ \mu\le\nu_0}} \|L^\mu\partial^\alpha u'(t,\,\cdot\,)\|_{L^2(|x|<2)}$$
$$= \sum_{\substack{|\alpha|+\mu\le N_0+\nu_0\\ \mu\le\nu_0}} \|L^\mu\partial^\alpha \tilde u'(t,\,\cdot\,)\|_{L^2(|x|<2)}$$
$$\le C \sum_{\substack{|\alpha|+\mu\le N_0+\nu_0+1\\ \mu\le\nu_0}} \Big[\int_0^t e^{-\frac{c}{2}(t-s)}\|L^\mu\partial^\alpha\Box\tilde u(s,\,\cdot\,)\|_2 ds + \|L^\mu\partial^\alpha\Box\tilde u(t,\,\cdot\,)\|_2\Big]$$
$$\le C \sum_{\substack{|\alpha|+\mu\le N_0+\nu_0+1\\ \mu\le\nu_0}} \Big[\int_0^t e^{-\frac{c}{2}(t-s)}\Big(\big\|L^\mu\partial^\alpha u_0'(s,\,\cdot\,)\big\|_{L^2(|x|<4)}$$
$$+ \big\|L^\mu\partial^\alpha u_0(s,\,\cdot\,)\big\|_{L^2(|x|<4)}\Big)ds$$
$$+ \big\|L^\mu\partial^\alpha u_0'(t,\,\cdot\,)\big\|_{L^2(|x|<4)} + \big\|L^\mu\partial^\alpha u_0(t,\,\cdot\,)\big\|_{L^2(|x|<4)}\Big].$$

Since $\Box u = \Box u_0$ one can use the sharp Huygens principle to see that the last term is dominated by the last term in the right side of (2.32), which finishes the proof. $\square$

3. Pointwise estimates

We will estimate solutions of the scalar inhomogeneous wave equation

$$\begin{cases} (\partial_t^2 - \Delta)w(t,x) = F(t,x), & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3 \backslash \mathcal{K} \\ w(t,x) = 0, & x \in \partial\mathcal{K} \\ w(t,x) = 0, & t \le 0. \end{cases} \tag{3.1}$$

If we assume, as before, that $\mathcal{K} \subset \{x \in \mathbb{R}^3 : |x| < 1\}$ then we have the following

Theorem 3.1. *Suppose that the local energy decay bounds* (1.4) *hold for* $\mathcal{K}$. *Suppose also that* $|\alpha| = M$. *Then*

$$\begin{aligned} (1+t+|x|)|L^\nu Z^\alpha w(t,x)| \\ \le C \int_0^t \int_{\mathbb{R}^3\backslash\mathcal{K}} \sum_{\substack{|\beta|+\mu\le M+\nu+7 \\ \mu\le\nu+1}} |L^\mu Z^\beta F(s,y)| \frac{dyds}{|y|} \\ + C\int_0^t \sum_{\substack{|\beta|+\mu\le M+\nu+4 \\ \mu\le\nu+1}} \|L^\mu \partial^\beta F(s,\,\cdot\,)\|_{L^2(\{x\in\mathbb{R}^3\backslash\mathcal{K}:\,|x|<2\})}\,ds. \end{aligned} \tag{3.2}$$

The special case of this estimate where $\nu = 0$ was handled in [11] in the non-trapping case. Since it is technically harder to handle pointwise bounds involving powers of L, we shall give the proof of (3.2) for the sake of completeness. Handling the case where there is a loss of regularity in the energy decay as in (1.4) does not present any added difficulty. The fact that (1.4) involves a loss of one derivative accounts why when $\nu = 0$ the right side of (3.2) involves on extra derivative versus the results in [11].

The proof will resemble that of Proposition 2.6. We shall prove the estimate when x is near the obstacle primarily by using the local energy decay estimates (1.4), while away from the obstacle we shall mainly use the fact that related bounds hold in Minkowski space.

The Minkowski space estimates we shall use say that if w_0 is a solution of the inhomogeneous wave equation

$$\begin{cases} \big(\partial_t^2 - \Delta\big)w_0(t,x) = G(t,x), & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3 \\ w_0(0,x) = \partial_t w_0(0,x) = 0, \end{cases} \tag{3.3}$$

then

(3.4)

$$(1+t+|x|)\big|L^\nu Z^\alpha w_0(t,x)\big| \le C\int_0^t\int_{\mathbb{R}^3}\sum_{\substack{|\beta|+\mu\le|\alpha|+\nu+3\\ \mu\le\nu+1}} |L^\mu Z^\beta G(s,y)|\,\frac{dyds}{|y|}.$$

This follows from inequalities (2.3) and (2.9) in [11] and the fact that $[\partial_t^2-\Delta, Z]=0$, and $[\partial_t^2-\Delta, L]=2(\partial_t^2-\Delta)$. The estimate where the weight in the left is $(1+t)$ was the main pointwise estimate in [11], while the contribution of the weight $|x|$ in the left just follows from the fact that

$$\text{(3.5)}\quad |x|\,|w_0(t,x)| \le C\int_0^t\int_{|\,|x|-(t-s)\,|}^{|x|+(t-s)} \sup_{|\theta|=1}|G(s,r\theta)|\,rdrds$$
$$\le C\int_0^t\int_{\{y\in\mathbb{R}^3:\,|y|\in[\,|\,|x|-(t-s)\,|,|x|+(t-s)]\}}\sum_{|a|\le 2}|\Omega^a\,G(s,y)|\,\frac{dyds}{|y|}.$$

Recall that we are assuming that $\mathcal{K}\subset\{x\in\mathbb{R}^3:\,|x|<1\}$. With this in mind, the first step is to see that (3.4) and (3.5) yield

(3.6)

$$(1+t+|x|)|L^\nu Z^\alpha w(t,x)| \le \int_0^t\int_{\mathbb{R}^3\backslash\mathcal{K}}\sum_{\substack{|\beta|+\mu\le|\alpha|+\nu+3\\ \mu\le\nu+1}} |L^\mu Z^\beta F(s,y)|\,\frac{dyds}{|y|}$$
$$+C\sup_{|y|\le 2,0\le s\le t}(1+s)\big(|L^\nu Z^\alpha w'(s,y)|+|L^\nu Z^\alpha w(s,y)|\big).$$

The proof is exactly like that of Lemma 4.2 in [11]. One fixes $\rho\in C^\infty(\mathbb{R})$ satisfying $\rho(r)=1$, $r\ge 2$, $\rho(r)=0$, $r\le 1$, and then applies (3.4)–(3.5) to $w_0(t,x)=\rho(|x|)L^\nu Z^\alpha w(t,x)$, which solves the inhomogeneous wave equation

$$\big(\partial_t^2-\Delta\big)w_0(t,x)=\rho(|x|)\big(\partial_t^2-\Delta\big)L^\nu Z^\alpha w(t,x)$$
$$-2\rho'(|x|)\frac{x}{|x|}\cdot\nabla_x L^\nu Z^\alpha w(t,x)-(\Delta\rho(|x|))L^\nu Z^\alpha w(t,x),$$

with zero initial data. When one applies (3.4), the first term in the right side of this equation results in the first term in the right side of of (3.6), while if one applies the first inequality in (3.5) one sees that the last two terms of the equation result in the last two terms of (3.6).

It remains to prove pointwise bounds in the region where $|x|<2$. Additionally since the coefficients of Z are bounded, it suffices to show that if $|\gamma| \le |\alpha| + 1 = M + 1$, then

$$\text{(3.7)} \qquad t \sup_{|x|<2} |L^\nu \partial^\gamma w(t,x)| \le C \int_0^t \int_{\mathbb{R}^3\backslash\mathcal{K}} \sum_{\substack{|\beta|+\mu \le M+\nu+7 \\ \mu \le \nu+1}} |L^\mu Z^\beta F(s,y)| \frac{dyds}{|y|} + C \int_0^t \sum_{\substack{|\beta|+\mu \le M+\nu+4 \\ \mu \le \nu+1}} \|L^\mu \partial^\beta F(s,\cdot)\|_{L^2(\{x\in\mathbb{R}^3\backslash\mathcal{K}:\,|x|<4\})}\, ds.$$

Using cutoffs for the forcing terms, we can split things into proving (3.7) for the following two cases

- **Case 1:** $F(s,y) = 0$ if $|y| > 4$
- **Case 2:** $F(s,y) = 0$ if $|y| < 3$.

For either case, we shall use the following immediate consequence of the Fundamental Theorem of Calculus:

$$|tL^\nu \partial^\gamma w(t,x)| \le \sum_{j=0,1} \int_0^t \left|(s\partial_s)^j L^\nu \partial^\gamma w(s,x)\right| ds.$$

If we apply the Sobolev lemma, using the fact that $|\gamma| \le M+1$, and that Dirichlet conditions allow us to control w locally by w', then we get

$$\begin{aligned} t \sup_{|x|<2} |L^\nu \partial^\gamma w(t,x)| &\le C \int_0^t \sum_{|\beta| \le M+2,\, \mu \le 1} \left\|(s\partial_s)^\mu L^\nu \partial^\beta w'(s,\cdot)\right\|_{L^2(\mathbb{R}^3\backslash\mathcal{K}:\,|x|<4)} ds \\ &\le C \int_0^t \sum_{\substack{|\beta|+\mu \le M+\nu+3 \\ \mu \le \nu+1}} \|L^\mu \partial^\beta w'(s,\,\cdot\,)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K}:\,|x|<4)}\, ds. \end{aligned}$$

If we are in Case 1, we apply (2.27) to get the variant of (3.7) involving only the second term in the right.

In Case 2, we need to write $w = w_0 + w_r$ where w_0 solves the boundaryless wave equation $(\partial_t^2 - \Delta)w_0 = F$ with zero initial data. Fix $\eta \in C_0^\infty(\mathbb{R}^3)$ satisfying $\eta(x) = 1$, $|x| < 2$ and $\eta(x) = 0$, $|x| \ge 3$. It then follows that if we set $\tilde{w} = \eta w_0 + w_r$, then since $\eta F = 0$, $\tilde{w}$ solves the Dirichlet-wave equation

$$\left(\partial_t^2 - \Delta\right)\tilde{w} = G = -2\nabla_x \eta \cdot \nabla_x w_0 - (\Delta\eta) w_0$$

with zero initial data. The forcing term vanishes unless $2 \le |x| \le 4$. Hence, by Case 1

$$\begin{aligned} t \sup_{|x|<2} |L^\nu \partial^\gamma w(t,x)| &= t \sup_{|x|<2} |L^\nu \partial^\gamma \tilde{w}(t,x)| \\ &\le C \int_0^t \sum_{\substack{|\beta|+\mu \le M+\nu+4 \\ \mu \le \nu+1}} \|L^\mu \partial^\beta G(s,\,\cdot\,)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}\, ds \\ &\le C \int_0^t \sum_{\substack{|\beta|+\mu \le M+\nu+5 \\ \mu \le \nu+1}} \big\|L^\mu \partial^\beta w_0(s,\,\cdot\,)\big\|_{L^\infty(2\le|x|\le 4)}\, ds. \end{aligned}$$

To finish the argument, we apply (3.5) to obtain

$$\begin{aligned} &\big\|L^\mu \partial^\beta w_0(s,\,\cdot\,)\big\|_{L^\infty(2\le|x|\le 4)} \\ &\qquad \le C \sum_{|a|\le 2} \int_0^s \int_{|s-\tau-|y||\le 4} |L^\mu \partial^\beta \Omega^a\ F(\tau,y)|\, \frac{dy d\tau}{|y|}. \end{aligned}$$

Note that the sets $\Lambda_s = \{(\tau,y) : 0 \le \tau \le s, |s-\tau-|y|| \le 4\}$ satisfy $\Lambda_s \cap \Lambda_{s'} = \emptyset$ if $|s-s'| \ge 10$. Therefore, if in the preceding inequality we sum over $|\beta|+\mu \le M+\nu+5$, $\mu \le \nu+1$, and then integrate over $s \in [0,t]$ we conclude that (3.7) must hold for Case 2, which finishes the proof. □

4. Estimates related to the null condition

Here we shall prove simple bounds for the null forms. They must involve the weight $< c_J t - r >$ due to the fact that we are not using the generators of Lorentz rotations. The estimates will involve the admissible homogeneous vector fields that we are using $\{\Gamma\} = \{Z, L\}$. Also, as before, ∂ denotes the space-time gradient $\nabla_{t,x}$.

Lemma 4.1. *Suppose that the quasilinear null condition* (1.9) *holds. Then*

$$\begin{aligned} (4.1) \quad &\Big| \sum_{0\le j,k,l\le 3} B^{IJ,jk}_{J,l} \partial_l u \partial_j \partial_k v \Big| \\ &\le C < r >^{-1} \big(|\Gamma u|\,|\partial^2 v| + |\partial u|\,|\partial \Gamma v|\big) + C \frac{< c_J t - r >}{< t+r >} |\partial u|\,|\partial^2 v|. \end{aligned}$$

Also, if the asymmetric semilinear null condition (1.10) *holds*

$$(4.2) \qquad \Big| \sum_{0\le j,k\le 3} A^{I,jk}_{JK} \partial_j u \partial_k v \Big| \le C < r >^{-1} \big(|\Gamma u|\,|\partial v| + |\partial u|\,|\Gamma v|\big).$$

Proof. The first estimate is well known. See, e.g., [24], [27]. It also follows from the proof of (4.2).

Proving (4.2) is straightforward. Since we are assuming (1.10) the quadratic form involved must be skew symmetric. If we write $\nabla_x = \frac{x}{r}\partial_r + \frac{x}{r^2}\wedge\Omega$ where $\wedge$ is the usual vector cross product and $\Omega = (\Omega_{23}, \Omega_{13}, \Omega_{12})$, then since $|(\frac{x}{r^2}\wedge\Omega)u| \le C\langle r\rangle^{-1}|\Gamma u|$, we conclude that the left side of (4.2) must be dominated by

$$\langle r\rangle^{-1}\big(|\Omega u|\,|\partial v| + |\partial u|\,|\Omega v|\big) + |\partial_t u\partial_r v - \partial_r u\partial_t v|.$$

If we write $\partial_r = r^{-1}L + \frac{t}{r}\partial_t$ then we can estimate the last term

$$|\partial_t u\partial_r v - \partial_r u\partial_t v| \le \tfrac{1}{r}\big(|Lu|\,|\partial_t v| + |\partial_t u|\,|Lv|\big).$$

Combining these two steps yields (4.2). $\square$

We also need the following result.

Lemma 4.2. *If $h \in C_0^\infty$ has Dirichlet boundary conditions then if $R < t/2$ and $t \ge 1$*

$$\begin{aligned}(4.3)\quad &\|\partial h'(t,\,\cdot\,)\|_{L^2(R/2<|x|<R)}\\ &\quad\le Ct^{-1}\Big(\sum_{|\alpha|\le 1}\|\Gamma^\alpha h'(t,\,\cdot\,)\|_{L^2(R/4<|x|<2R)}\\ &\qquad\qquad + t\big\|\big(\partial_t^2-\Delta\big)h(t,\,\cdot\,)\big\|_{L^2(R/4<|x|<2R)}\Big)\\ &\qquad + C\|\langle x\rangle^{-1}h'(t,\,\cdot\,)\|_{L^2(R/4<|x|<2R)} + C\|\langle x\rangle^{-2}h(t,\,\cdot\,)\|_{L^2(R/4<|x|<2R)}.\end{aligned}$$

Also,

$$\begin{aligned}(4.4)\quad &\|<t-r>\partial h'(t,\,\cdot\,)\|_{L^2(|x|>t/4)}\\ &\quad\le C\sum_{|\alpha|\le 1}\|\Gamma^\alpha h'(t,\,\cdot\,)\|_2 + C\big\|<t+r>\big(\partial_t^2-\Delta\big)h(t,\,\cdot\,)\big\|_2,\end{aligned}$$

and if $\delta > 0$ is fixed then

$$\begin{aligned}(4.5)\quad &\|h'(t,\,\cdot\,)\|_{L^6(|x|\notin[(1-\delta)t,(1+\delta)t],\ |x|>\delta t)}\\ &\quad\le Ct^{-1}\Big(\sum_{|\alpha|\le 1}\|\Gamma^\alpha h'(t,\,\cdot\,)\|_2 + \|<t+r>\big(\partial_t^2-\Delta\big)h(t,\,\cdot\,)\|_2\Big).\end{aligned}$$

Proof. To prove (4.3) we need to use the fact (see [14], Lemma 2.3) that

$$\begin{aligned}(4.6)\quad &\langle t-r\rangle\big(|\partial\partial_t h(t,x)| + |\Delta h(t,x)|\big)\\ &\quad\le C\sum_{|\alpha|\le 1}|\partial\Gamma^\alpha h(t,x)| + C\langle t+r\rangle\big|\big(\partial_t^2-\Delta\big)h(t,x)\big|.\end{aligned}$$

Also, elliptic regularity gives

$$\|\nabla_x h'(t,\,\cdot\,)\|_{L^2(|x|\in[R/2,R])} \le C\|\Delta h(t,\,\cdot\,)\|_{L^2(|x|\in[R/4,2R])} \\ + CR^{-1}\|h'(t,\,\cdot\,)\|_{L^2(|x|\in[R/4,2R])} + CR^{-2}\|h(t,\,\cdot\,)\|_{L^2(|x|\in[R/4,2R])}.$$

If we combine these two inequalities then we get (4.3).

To prove (4.4) we need to use another estimate from [14], namely, if $g \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^3)$,

$$\big\|\langle t-r\rangle \nabla_x^2 g(t,\,\cdot\,)\big\|_{L^2(\mathbb{R}^3)} \le C \sum_{|\alpha|\le 1} \|\Gamma^\alpha g'(t,\,\cdot\,)\|_{L^2(\mathbb{R}^3)} \\ + C\big\|\langle t+r\rangle\big(\partial_t^2 - \Delta\big)g\big\|_{L^2(\mathbb{R}^3)}.$$

If we fix $\eta \in C^\infty(\mathbb{R}^3)$ satisfying $\eta(x) = 1$, $|x| > 1/4$ and $\eta(x) = 0$, $|x| < 1/8$ and let $g(t,x) = \eta(x/\langle t\rangle)h(t,x)$ then we conclude that the analog of (4.4) must hold where $\nabla h'$ is replaced by $\nabla_x h'$. Since (4.6) yields the same bounds for $\partial_t h'$, we get (4.4).

Inequality (4.5) follows from the fact that its left side is dominated by

$$\|\nabla_x h'(t,\,\cdot\,)\|_{L^2(|x|\notin[(1-\delta/2)t,(1+\delta/2)t],\ |x|>\delta t/2)} + t^{-1}\|h'(t,\,\cdot\,)\|_2.$$

Since the proof of (4.4) implies that the first term is dominated by the right side of (4.5) if $\delta > 0$ is fixed, we are done. □

The following result will be useful for dealing with waves interacting at different speeds.

Corollary 4.3. *Fix $c_1, c_2 > 0$ satisfying $c_1 \ne c_2$. Then if $u, v \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^3\backslash\mathcal{K})$ vanish on $\mathbb{R}_+ \times \partial\mathcal{K}$*

$$(4.7)\quad \int_{\mathbb{R}^3\backslash\mathcal{K}} |\partial^2 u(t,x)|\,|v'(t,x)| < x >^{-1}\, dx \\ \le Ct^{-1}\Big(\sum_{|\alpha|\le 1}\|\Gamma^\alpha u'(t,\,\cdot\,)\|_2 + \big\| < t+r > \big(\partial_t^2 - c_1^2\Delta\big)u(t,\,\cdot\,)\big\|_2\Big) \\ \times \| < x >^{-1}\, v'(t,\,\cdot\,)\|_2 \\ + C\sum_{R=2^k\le t/2}\big(\|\langle x\rangle^{-1}u'(t,\,\cdot\,)\|_{L^2(R/2<|x|<R)} + \|\langle x\rangle^{-2}u(t,\,\cdot\,)\|_{L^2(R/2<|x|<R)}\big) \\ \times \| < x >^{-1}\, v'(t,\,\cdot\,)\|_{L^2(R/2<|x|<R)} \\ + Ct^{-4/3}\Big(\sum_{|\alpha|\le 1}\|\Gamma^\alpha u'(t,\,\cdot\,)\|_2 + \big\| < t+r > \big(\partial_t^2 - c_1^2\Delta\big)u(t,\,\cdot\,)\big\|_2\Big) \\ \times\Big(\sum_{|\alpha|\le 1}\|\Gamma^\alpha v'(t,\,\cdot\,)\|_2 + \big\| < t+r > \big(\partial_t^2 - c_2^2\Delta\big)v(t,\,\cdot\,)\big\|_2\Big).$$

Proof. Let $\delta < |c_1 - c_2|$. Then if we use Schwarz's inequality, (4.3) and (4.4) we see that we can bound

$$\int_{|x|\notin((1-\delta)c_1t,(1+\delta)c_1t)} |\partial^2 u(t,x)|\,|v'(t,x)| < x >^{-1} dx$$

by the first two terms in the right side of (4.7).

For a given $j = 0, 1, 2, \dots$ we can use Hölder's inequality, to find that

$$\begin{aligned}&\int_{<c_1t-r>\in[2^j,2^{j+1})} |\partial^2 u(t,x)|\,|v'(t,x)| < x >^{-1} dx \\ &\quad\le Ct^{-1/3}2^{j/3}\|\partial^2 u(t,\,\cdot\,)\|_{L^2(<c_1t-r>\in(2^j,2^{j+1}))}\|v'(t,\,\cdot\,)\|_{L^6(<c_1t-r>\in(2^j,2^{j+1}))},\end{aligned}$$

assuming that r is bounded below by a fixed multiple of t when $< c_1t-r >\in [2^j, 2^{j+1})$. Since $\delta < |c_1 - c_2|$, if $\{x :< c_1t - r >\in [2^j, 2^{j+1})\} \cap \{x : r \in ((1-\delta)c_1t, (1+\delta)c_1t)\} \ne \emptyset$, we can apply (4.4) and (4.5) to see that the right side is bounded by $2^{-2j/3}$ times the third term in the right side of (4.7). After summing over j, this implies that when we restrict the integration in the left side of (4.7) to the the set where $r \in ((1-\delta)c_1t, (1+\delta)c_1t)$, the resulting expression is dominated by the third term in the right of (4.7). This completes the proof. □

To handle same-speed interactions, we shall need the following similar result.

Corollary 4.4. *Let $u, v \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^3\backslash\mathcal{K})$ vanish on $\mathbb{R}_+ \times \partial\mathcal{K}$. Then,*

$$\begin{aligned}(4.8)\ &\int_{\mathbb{R}^3\backslash\mathcal{K}} \frac{\langle t-r\rangle}{\langle t+r\rangle}|\partial^2 u(t,x)|\,|v'(t,x)|\,\langle x\rangle^{-1} dx \\ &\le Ct^{-1}\Big(\sum_{|\alpha|\le 1}\|\Gamma^\alpha u'(t,\,\cdot\,)\|_2 + \|\langle t+r\rangle\Box u(t,\,\cdot\,)\|_2\Big)\|\langle x\rangle^{-1}v'(t,\,\cdot\,)\|_2 \\ &+ C\sum_{R=2^k<t/2}\Big(\|\langle x\rangle^{-1}u'(t,\,\cdot\,)\|_{L^2(R/4<|x|<2R)} + \|\langle x\rangle^{-2}u(t,\,\cdot\,)\|_{L^2(R/4<|x|<2R)}\Big) \\ &\qquad\qquad\times\|\langle x\rangle^{-1}v'(t,\,\cdot\,)\|_{L^2(R/4<|x|<2R)}.\end{aligned}$$

Proof of Corollary 4.4. To prove (4.8) we just use Schwarz's inequality and (4.3) and (4.4) to see that its left side is dominated by

$$\begin{aligned}&t^{-1}\|\langle t-r\rangle\partial^2 u(t,\,\cdot\,)\|_{L^2(|x|>t/4)}\|\langle x\rangle^{-1}v'(t,\,\cdot\,)\|_{L^2(|x|>t/4)} \\ &+\sum_{R=2^k<t/2} t^{-1}\|\langle t-r\rangle\partial^2 u(t,\,\cdot\,)\|_{L^2(R/2<|x|<R)}\|\langle x\rangle^{-1}v'(t,\,\cdot\,)\|_{L^2(R/2<|x|<R)}\end{aligned}$$

$$
\begin{aligned}
&\leq Ct^{-1}\Big(\sum_{|\alpha|\leq 1}\|\Gamma^\alpha u'(t,\,\cdot\,)\|_2+\|\langle t+r\rangle\Box u(t,\,\cdot\,)\|_2\Big)\|\langle x\rangle^{-1}v'(t,\,\cdot\,)\|_2\\
&+C\sum_{R=2^k<t/2}\Big(\|\langle x\rangle^{-1}u'(t,\,\cdot\,)\|_{L^2(R/4<|x|<2R)}+\|\langle x\rangle^{-2}u(t,\,\cdot\,)\|_{L^2(R/4<|x|<2R)}\Big)\\
&\qquad\qquad\qquad\qquad\times\|\langle x\rangle^{-1}v'(t,\,\cdot\,)\|_{L^2(R/4<|x|<2R)},
\end{aligned}
$$

which completes the proof. □

We also need the following consequence of the Sobolev lemma (see [13]).

Lemma 4.5. *Suppose that $h\in C^\infty(\mathbb{R}^3)$. Then for $R\geq 1$*

$$
\|h\|_{L^\infty(R/2<|x|<R)}\leq CR^{-1}\sum_{|\alpha|+|\beta|\leq 2}\big\|\Omega^\alpha\partial_x^\beta h\big\|_{L^2(R/4<|x|<2R)}.
$$

Also,

$$
\|h\|_{L^\infty(R<|x|<R+1)}\leq CR^{-1}\sum_{|\alpha|+|\beta|\leq 2}\big\|\Omega^\alpha\partial_x^\beta h\big\|_{L^2(R-1<|x|<R+2)}.
$$

5. Continuity argument

In this section we shall prove our main result, Theorem 1.1. We shall take $N=101$ in its smallness hypothesis (1.11), but this certainly is not optimal.

We start out with a number of straightforward reductions that will allow us to use the estimates from Sects. 2–4.

First, let us assume that the wave speeds c_I all are distinct since straightforward modifications of the argument give the more general case where the various components are allowed to have the same speed.

To prove our global existence theorem we shall need a standard local existence theorem:

Theorem 5.1. *Suppose that f and g are as in Theorem 1.1 with $N\geq 6$ in* (1.11). *Then there is a $T>0$ so that the initial value problem* (1.5) *with this initial data has a C^2 solution satisfying*

$$
u\in L^\infty([0,T];H^N(\mathbb{R}^3\backslash\mathcal{K}))\cap C^{0,1}([0,T];H^{N-1}(\mathbb{R}^3\backslash\mathcal{K})).
$$

The supremum of such T is equal to the supremum of all T such that the initial value problem has a C^2 solution with $\partial^\alpha u$ bounded for $|\alpha|\leq 2$. Also, one can take $T\geq 2$ if $\|f\|_{H^N}+\|g\|_{H^{N-1}}$ is sufficiently small.

This essentially follows from the local existence results Theorem 9.4 and Lemma 9.6 in [9]. The latter were only stated for diagonal single-speed systems; however, since the proof relied only on energy estimates, it extends to the multi-speed non-diagonal case if the symmetry assumptions (1.8) are satisfied.

Next, as in [11], in order to avoid dealing with compatibility conditions for the Cauchy data, it is convenient to reduce the Cauchy problem (1.5) to an equivalent equation with a nonlinear driving force but vanishing Cauchy data. We then can set up a continuity argument for the new equation using the estimates from Sects. 2–4 to prove Theorem 1.1.

Recall that our smallness condition on the data is

$$\sum_{|\alpha|\le 101} \big\| \langle x\rangle^{\alpha} \partial_x^{\alpha} f \big\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})} + \sum_{|\alpha|\le 100} \big\| \langle x\rangle^{1+|\alpha|} \partial_x g \big\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})} \le \varepsilon. \tag{5.1}$$

To make the reduction to an equation with zero initial data, we first note that if the data satisfies (5.1) with $\varepsilon > 0$ small, then we can find a solution u to the system (1.5) on a set of the form $0 < ct < |x|$ where $c = 5 \max_I c_I$, and that this solution satisfies

$$\sup_{0<t<\infty} \sum_{|\alpha|\le 101} \| \langle x\rangle^{\alpha} \partial^{\alpha} u(t,\,\cdot\,)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K}:\,|x|>ct)} \le C_0\varepsilon, \tag{5.2}$$

where C_0 is an absolute constant.

To prove this we shall repeat an argument from [11]. We note that by scaling in the t-variable we may assume that $\max_I c_I = 1/2$. The above local existence theorem yields a solution u to (1.5) on the set $0 < t < 2$ satisfying the bounds (5.2). To see that this solution extends to the larger set $0 < ct < |x|$, we let $R \ge 4$ and consider data (f_R, g_R) supported in the set $R/4 < |x| < 4R$ which agrees with the data (f, g) on the set $R/2 < |x| < 2R$. Let $u_R(t, x)$ satisfy the boundaryless equation

$$\Box u_R = Q\big(du_R,\, R^{-1} d^2 u_R\big)$$

with Cauchy data $(f_R(R\cdot), Rg_R(R\cdot))$. The solution u_R then exists for $0 < t < 1$ by standard results (see [5]) and satisfies

$$\begin{aligned} \sup_{0<t<1} \|u_R(t,\,\cdot\,)\|_{H^{101}(\mathbb{R}^3)} &\le C\big(\|f_R(R\cdot)\|_{H^{101}(\mathbb{R}^3)} + R\|g_R(R\cdot)\|_{H^{100}(\mathbb{R}^3)}\big) \\ &\le CR^{-3/2}\Big(\sum_{|\alpha|\le 101} \big\|(R\partial_x)^{\alpha} f_R\big\|_{L^2(\mathbb{R}^3)} + R\sum_{|\alpha|\le 100} \big\|(R\partial_x)^{\alpha} g_R\big\|_{L^2(\mathbb{R}^3)}\Big). \end{aligned}$$

The smallness condition on $|u_R'|$ implies that the wave speeds for the quasilinear equation are bounded above by 1. A domain of dependence argument shows that the solutions $u_R(R^{-1}t, R^{-1}x)$ restricted to $|\,|x| - R\,| < \frac{R}{2} - t$ agree on their overlaps, and also with the local solution, yielding a solution to (1.5) on the set $\{\mathbb{R}^3\backslash\mathcal{K} : 2t < |x|\}$. A partition of unity argument now yields (5.2).

We use the local solution u to set up the continuity argument. Fix a cutoff function $\chi \in C^{\infty}(\mathbb{R})$ satisfying $\chi(s) = 1$ if $s \le \frac{1}{2c}$ and $\chi(s) = 0$ if $s > \frac{1}{c}$, and set

$$u_0(t, x) = \eta(t, x)u(t, x), \qquad \eta(t, x) = \chi(|x|^{-1}t),$$

assuming as we may that $0 \in \mathcal{K}$. Note that since $|x|$ is bounded below on the complement of $\mathcal{K}$, the function $\eta(t,x)$ is smooth and homogeneous of degree 0 in (t,x). Also

$$\Box u_0 = \eta Q(du, d^2u) + [\Box, \eta]u.$$

Thus, u solves $\Box u = Q(du, d^2u)$ for $0 < t < T$ if and only if $w = u - u_0$ solves

$$\begin{cases} \Box w = (1-\eta)Q(du, d^2u) - [\Box, \eta]u \\ w|_{\partial\mathcal{K}} = 0 \\ w(t,x) = 0, \quad t \le 0 \end{cases} \tag{5.3}$$

for $0 < t < T$.

The key step in proving that (5.3) admits a global solution is to prove uniform dispersive estimates for w on intervals of existence. To do this, let us first note that since $u_0 = \eta u$ by (5.2) and Lemma 4.5 there is an absolute constant C_1 so that

$$\begin{aligned}(1+t+|x|) \sum_{\mu+|\alpha|\le 99} \left|L^\mu Z^\alpha u_0(t,x)\right| \\ + \sum_{\mu+|\alpha|+|\beta|\le 101} \left\|\langle t+r\rangle^{|\beta|} L^\mu Z^\alpha \partial^\beta u_0(t,\,\cdot\,)\right\|_2 \le C_1\varepsilon.\end{aligned} \tag{5.4}$$

Furthermore, if we let v be the solution of the linear equation

$$\begin{cases} \Box v = -[\Box, \eta]u \\ v|_{\partial\mathcal{K}} = 0 \\ v(t,x) = 0, \quad t \le 0, \end{cases} \tag{5.5}$$

then (5.2) and Theorem 3.1 implies that there is an absolute constant C_2 so that

$$(1+t+|x|) \sum_{\mu+|\alpha|\le 90} |L^\mu Z^\alpha v(t,x)| \le C_2\varepsilon. \tag{5.6}$$

Indeed, by (3.2) the left side of (5.6) is dominated by

$$\begin{aligned}\int_0^t \int_{|x|>cs} \sum_{\mu+|\alpha|\le 97} |L^\mu Z^\alpha([\Box, \eta]u)(s,x)| \,\frac{dxds}{|x|} \\ + \int_0^t \sum_{\mu+|\alpha|\le 94} \|L^\mu \partial^\beta([\Box, \eta]u)(s,\cdot)\|_{L^2(\mathbb{R}\backslash\mathcal{K}\,:\,|x|<2)}\, ds\end{aligned}$$

which by the Schwarz inequality is bounded by

$$\sum_{\mu+|\alpha|\le 97} \sum_{j=0}^{\infty} \sup_{0<cs<2^{j+1}} \|\langle x\rangle^{3/2} L^\mu Z^\alpha [\Box, \eta]u(s,\,\cdot\,)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K}:\,2^j<|x|<2^{j+1})}.$$

Since this is bounded by

$$\sup_{0<t<\infty} \sum_{\mu+|\alpha|\le 97} \|\langle x\rangle^2 L^\mu Z^\alpha [\Box, \eta] u(t,\,\cdot\,)\|_2,$$

one gets (5.6) by (5.2) and the homogeneity of η.

Using this we can set up the continuity argument. If $\varepsilon > 0$ is as above we shall assume that we have a C^2 solution of our equation (1.5) for $0 \le t \le T$ such that for $t \in [0, T]$ and small $\varepsilon > 0$ we have the pointwise dispersive estimates

$$(5.7) \qquad (1+t+r) \sum_{|\alpha|\le 40} \Big(|Z^\alpha w(t,x)| + |Z^\alpha w'(t,x)| \Big) \le A_0 \varepsilon$$

$$(5.8) \qquad (1+t+r) \sum_{\substack{|\alpha|+\nu\le 55\\ \nu\le 2}} |L^\nu Z^\alpha w(t,x)| \le B_1 \varepsilon (1+t)^{1/5} \log(2+t),$$

as well as the L^2_x and weighted $L^2_t L^2_x$ estimates

$$(5.9) \qquad \sum_{|\alpha|\le 100} \|\partial^\alpha w'(t,\,\cdot\,)\|_2 \le B_2 \varepsilon (1+t)^{1/20}$$

$$(5.10) \qquad \sum_{\substack{|\alpha|+\nu\le 70\\ \nu\le 3}} \|L^\nu Z^\alpha w'(t,\,\cdot\,)\|_2 \le B_3 \varepsilon (1+t)^{1/10}$$

$$(5.11) \qquad \sum_{\substack{|\alpha|+\nu\le 68\\ \nu\le 3}} \|\langle x\rangle^{-1/2} L^\nu Z^\alpha w'\|_{L^2(S_t)} \le B_4 \varepsilon (1+t)^{1/10} (\log(2+t))^{1/2}.$$

Here, as before the L^2_x-norms are taken over $\mathbb{R}^3\backslash\mathcal{K}$, and the weighted $L^2_t L^2_x$-norms are taken over $S_t = [0,t] \times \mathbb{R}^3\backslash\mathcal{K}$.

In our main estimate, (5.7), $A_0 = 4C_2$, where C_2 is the constant occurring for the bounds (5.6) for v. Clearly if ε is small then all of these estimates are valid if $T = 2$, by Theorem 5.1. Keeping this in mind, we shall then prove that for $\varepsilon > 0$ smaller than some number depending on the constants $B_1, \dots, B_4$ that

i) (5.7) is valid with A_0 replaced by $A_0/2$;
ii) (5.8)–(5.11) are a consequence of (5.7) for suitable constants B_i.

By the local existence theorem it will follow that a solution exists for all $t > 0$ if ε is small enough.

Let us first deal with i). Since we already know that v satisfies (5.6) to achieve i), by Theorem 3.1 it suffices to show that

$$(5.12) \qquad I + II \le C\varepsilon^2,$$

where

$$I=\int_0^t\int_{\mathbb{R}^3\backslash\mathcal{K}}\sum_{\substack{|\alpha|+\mu\le 48\\ \mu\le 1}}|L^\mu Z^\alpha Q(du,d^2u)(s,y)|\frac{dsdy}{|y|}\tag{5.13}$$

$$II=\int_0^t\sum_{\substack{|\alpha|+\mu\le 45\\ \mu\le 1}}\|L^\mu\partial^\alpha Q(du,d^2u)(s,\,\cdot\,)\|_{L^2(|x|<2)}\,ds,\tag{5.14}$$

since this implies the same sort of bounds where Q is replaced by $(1-\eta)Q$ in (5.13) and (5.14).

Let us first deal with I. This term was the only one that had to be dealt with in the boundaryless case, and the argument for it is similar to the corresponding one in [27].

To handle I we shall have to employ a different argument for the quadratic terms satisfying the null condition and the quasilinear ones that do not. Therefore, let us write

$$Q=\Box u=N(u',u'')+\sum_{J\neq K}\sum_{j,k,l=0}^{3}B_{K,l}^{IJ,jk}\partial_l u^K\partial_j\partial_k u^J,\;\; u=u_0+w,\tag{5.15}$$

where the "null term" $N(u',u'')$ satisfies the bounds in Lemma 4.1, while the second term in the right of (5.15) involves interactions between waves of different speeds.

Let us first handle the contribution of $N(u',u'')$ to I. By Lemma 4.1

$$\begin{aligned}\sum_{\substack{|\alpha|+\mu\le 48\\ \mu\le 1}}|L^\mu Z^\alpha N(u',u'')|\le\frac{C}{|y|}\sum_{\substack{|\alpha|+\mu\le 50\\ \mu\le 2}}|L^\mu Z^\alpha u|\sum_{\substack{|\alpha|+\mu\le 50\\ \mu\le 2}}|L^\mu Z^\alpha u'|\\ +C\sum_J\frac{\langle c_Jt-r\rangle}{\langle t+r\rangle}\sum_{\substack{|\alpha|+\mu\le 48\\ \mu\le 1}}|L^\mu Z^\alpha\partial u|\sum_{\substack{|\alpha|+\mu\le 48\\ \mu\le 1}}|L^\mu Z^\alpha\partial^2 u|.\end{aligned}\tag{5.16}$$

To handle the contribution of the first term in the right side of (5.16) to I, we apply (5.8) to get that

$$\sum_{\substack{|\alpha|+\mu\le 50\\ \mu\le 2}}|L^\mu Z^\alpha u(y,s)|\le C\varepsilon(|y|+s)^{-4/5}\log(2+s),$$

which means that the first term in the right side of (5.16) has a contribution to I which is dominated by

$$\varepsilon \int_0^t \int_{\mathbb{R}^3\backslash\mathcal{K}} \sum_{\substack{|\alpha|+\mu\le 50\\ \mu\le 2}} |L^\mu Z^\alpha u'(s,y)| \frac{\log(2+s)\,dyds}{|y|^2(|y|+s)^{4/5}}$$
$$\le C_\delta \varepsilon \int_0^t \sum_{\substack{|\alpha|+\mu\le 50\\ \mu\le 2}} \|\langle y\rangle^{-1/2} L^\mu Z^\alpha u'(s,\,\cdot\,)\|_2 \langle s\rangle^{-4/5+\delta}\,ds,$$

if $\delta > 0$. But if δ is chosen small enough so that $4/5 - \delta > 1/2 + 1/10$ then we can use Schwarz's inequality along with (5.11) to see that the last expression is $O(\varepsilon^2)$. We are using here the fact that $u = u_0 + w$, as well as the fact that u_0 satisfies better bounds than those in (5.11) because of (5.4).

Let us see that the contribution of the second term in the right side of (5.16) enjoys the same bound. For a given J we can use (4.8) to see that the contribution is dominated by

(5.17)
$$\int_0^t \langle s\rangle^{-1}\Big(\sum_{\substack{|\alpha|+\mu\le 51\\ \mu\le 2}} \|L^\mu Z^\alpha u'(s,\,\cdot\,)\|_2 + \sum_{\substack{|\alpha|+\mu\le 50\\ \mu\le 1}} \|\langle t+r\rangle L^\mu Z^\alpha \Box u(s,\,\cdot\,)\|_2\Big)$$
$$\times \sum_{\substack{|\alpha|+\mu\le 50\\ \mu\le 1}} \|\langle y\rangle^{-1} L^\mu Z^\alpha u'(s,\,\cdot\,)\|_2\,ds$$
$$+ \int_0^t \sum_{R=2^k<\frac{c_0 s}{2}} \sum_{\substack{|\alpha|+\mu\le 50\\ \mu\le 1}} \Big(\|\langle y\rangle^{-1} L^\mu Z^\alpha u'(s,\,\cdot\,)\|_{L^2(|y|\approx R)}$$
$$+ \|\langle y\rangle^{-1} L^\mu Z^\alpha u(s,\,\cdot\,)\|_{L^2(|y|\approx R)}\Big)$$
$$\times \sum_{\substack{|\alpha|+\mu\le 50\\ \mu\le 1}} \|\langle y\rangle^{-1} L^\mu Z^\alpha u'(s,\,\cdot\,)\|_{L^2(|y|\approx R)}\,ds,$$

with $c_0 = \min_I c_I$, and $L^2(|y| \approx R)$ indicating L^2-norms over $\{y \in \mathbb{R}^3\backslash\mathcal{K} : |y| \in [R/4, 2R]\}$. If one uses (5.4) and (5.8) to estimate the first factor in the last term, one concludes that this term is dominated by

$$\varepsilon \int_0^t (\log(2+s))^2 \langle s\rangle^{-4/5} \sum_{\substack{|\alpha|+\mu\le 50\\ \mu\le 1}} \|\langle y\rangle^{-1/2} L^\mu Z^\alpha u'(s,\,\cdot\,)\|_2\,ds = O(\varepsilon^2),$$

using (5.11) and (5.4) in the last step. For the first term of (5.17), we note that by (5.8)

$$\langle s+r\rangle \sum_{\substack{|\alpha|+\mu\le 50\\ \mu\le 1}} |L^\mu Z^\alpha \Box u| \le C\langle s+r\rangle \sum_{\substack{|\alpha|+\mu\le 51\\ \mu\le 1}} |L^\mu Z^\alpha u'|^2 \\ \le C\varepsilon \log(2+s)(1+s)^{1/5} \sum_{\substack{|\alpha|+\mu\le 51\\ \mu\le 1}} |L^\mu Z^\alpha u'|, \tag{5.18}$$

assuming, as we may, that $\varepsilon \le 1$. Thus, by (5.10) and (5.4) the contribution of the first term in the right side of (5.17) must be dominated by

$$\int_0^t \langle s\rangle^{-1}\Big(\varepsilon\langle s\rangle^{1/10} + \varepsilon\log(2+s)\langle s\rangle^{1/10+1/5}\Big) \\ \times \sum_{\substack{|\alpha|+\mu\le 51\\ \mu\le 2}} \|\langle y\rangle^{-1} L^\mu Z^\alpha u'(s,\,\cdot\,)\|_2\, ds,$$

which is also $O(\varepsilon^2)$ by (5.11) and (5.4).

This concludes the proof that the null form terms have $O(\varepsilon^2)$ contributions to I. If we use (4.7) it is clear that the multi-speed quadratic terms

$$\sum_{j,k,l=0}^{3} B^{IJ,jk}_{K,l}\partial_l u^K \partial_j\partial_k u^J, \quad J\ne K$$

will have the same contribution. This completes the proof that I satisfies the bounds in (5.12).

It is also easy to see now that II is $O(\varepsilon^2)$. If we use (5.18), we see that II is dominated by

$$\varepsilon\int_0^t \langle s\rangle^{-4/5}\log(2+s) \sum_{\substack{|\alpha|+\mu\le 51\\ \mu\le 1}} \|L^\mu Z^\alpha u'(s,\,\cdot\,)\|_{L^2(|y|<4)}\, ds,$$

which is $O(\varepsilon^2)$ by (5.11) and (5.4).

This completes step i) of the proof, which was to show that (5.8)–(5.11) imply (5.7).

To finish the proof of Theorem 1.1 we need to show how (5.7) implies (5.8)–(5.12). In proving the L^2 estimates we shall use the fact that, in the notation of Sect. 1, $\Box_\gamma u = B(du)$, where the quadratic form $B(du)$ is the semilinear part of the nonlinearity Q, and

$$\gamma^{IJ,jk} = \gamma^{IJ,jk}(u') = -\sum_{\substack{0\le l\le 3\\ 1\le K\le D}} B^{IJ,jk}_{K,l}\partial_l u^K.$$

Depending on the linear estimates we shall employ, at times we shall prove certain L^2 bounds for u while at other times, we shall prove them for w. Since $u = w + u_0$ and u_0 satisfies the bounds in (5.4) it will always be the case that bounds for w will imply those for u and vice versa. Also note that by (5.7)

$$\|\gamma'(s,\,\cdot\,)\|_\infty \le \frac{C\varepsilon}{(1+s)}. \tag{5.19}$$

Using these facts we can prove (5.9). Let us first notice that if we use (2.5) and (5.7) then we can estimate the energy of $\partial_t^j u$ for $j \le M \le 100$. We shall use induction on M.

We first notice that by (2.5) and (5.19) we have

$$\partial_t E_M^{1/2}(u)(t) \le C \sum_{j\le M} \left\|\Box_\gamma \partial_t^j u(t,\,\cdot\,)\right\|_2 + \frac{C\varepsilon}{1+t} E_M^{1/2}(u)(t). \tag{5.20}$$

Note that for $M = 1, 2, \ldots$

$$\begin{aligned}\sum_{j\le M} \left|\Box_\gamma \partial_t^j u\right| &\le C\Big(\sum_{j\le M} \left|\partial_t^j u'\right| + \sum_{j\le M-1} \left|\partial_t^j \partial^2 u\right|\Big) \sum_{|\alpha|\le 40} |\partial^\alpha u'| \\ &\qquad + C \sum_{|\alpha|\le M-41} |\partial^\alpha u'| \sum_{40<|\alpha|\le M/2} |\partial^\alpha u'| \\ &\le \frac{C\varepsilon}{1+t}\Big(\sum_{j\le M} \left|\partial_t^j u'\right| + \sum_{j\le M-1} \left|\partial_t^j \partial^2 u\right|\Big) \\ &\qquad + C \sum_{|\alpha|\le M-41} |\partial^\alpha u'| \sum_{|\alpha|\le M/2} |\partial^\alpha u'|,\end{aligned}$$

since (5.7) and (5.4) imply $|\partial^\alpha u'| \le C\varepsilon/(1+t)$ if $|\alpha| \le 40$. Also, if we use elliptic regularity and repeat this argument we get

$$\begin{aligned}\sum_{j\le M-1} \left\|\partial_t^j \partial^2 u(t,\,\cdot\,)\right\|_2 &\le C \sum_{j\le M} \left\|\partial_t^j u'(t,\,\cdot\,)\right\|_2 + C \sum_{j\le M-1} \left\|\partial_t^j \Box u(t,\,\cdot\,)\right\|_2 \\ &\le C \sum_{j\le M} \left\|\partial_t^j u'(t,\,\cdot\,)\right\|_2 + \frac{C\varepsilon}{1+t} \sum_{j\le M-1} \left\|\partial_t^j \partial^2 u(t,\,\cdot\,)\right\|_2 \\ &\qquad + C \sum_{|\alpha|\le M-41, |\beta|\le M/2} \|\partial^\alpha u'(t,\,\cdot\,)\partial^\beta u'(t,\,\cdot\,)\|_2.\end{aligned}$$

If ε is small we can absorb the second to last term into the left side of the preceding inequality. Therefore, if we combine the last two inequalities we conclude that

$$\begin{aligned}\sum_{j\le M} \left\|\Box_\gamma \partial_t^j u(t,\,\cdot\,)\right\|_2 &\le \frac{C\varepsilon}{1+t} \sum_{j\le M} \left\|\partial_t^j u'(t,\,\cdot\,)\right\|_2 \\ &\qquad + C \sum_{|\alpha|\le M-41, |\beta|\le M/2} \|\partial^\alpha u'(t,\,\cdot\,)\partial^\beta u'(t,\,\cdot\,)\|_2.\end{aligned}$$

If we combine this with (5.20) we get that for small $\varepsilon > 0$

$$\begin{aligned}(5.21)\quad \partial_t E_M^{1/2}(u)(t) \le \frac{C\varepsilon}{1+t} E_M^{1/2}(u)(t) \\ + C \sum_{|\alpha|\le M-41, |\beta|\le M/2} \|\partial^\alpha u'(t,\,\cdot\,)\partial^\beta u'(t,\,\cdot\,)\|_2,\end{aligned}$$

since when ε is small $\frac{1}{2}E_M^{1/2}(u)(t) \le \sum_{j\le M} \|\partial_t^j u'(t,\,\cdot\,)\|_2 \le 2E_M^{1/2}(u)(t)$.

If $M = 40$, the last term in (5.21) drops out and so

$$\partial_t E_{40}^{1/2}(u)(t) \le \frac{C\varepsilon}{1+t} E_{40}^{1/2}(u)(t).$$

Since $E_{100}^{1/2}(u)(0) \le C\varepsilon$, an application of Gronwall's inequality yields

$$(5.22)\qquad \sum_{j\le 40} \left\|\partial_t^j u'(t,\,\cdot\,)\right\|_2 \le 2E_{40}^{1/2}(u)(t) \le C\varepsilon(1+t)^{C\varepsilon}.$$

By elliptic regularity and (5.7) this leads to the bounds

$$\sum_{|\alpha|\le 40} \|\partial^\alpha u'(t,\,\cdot\,)\|_2 \le C\varepsilon(1+t)^{C\varepsilon}.$$

If $M > 40$ we have to deal with the last term in (5.21). To do this we first note that by Lemma 4.5 we have

$$\begin{aligned}\sum_{|\alpha|\le M-41, |\beta|\le M/2} \|\partial^\alpha u'(t,\,\cdot\,)\partial^\beta u'(t,\,\cdot\,)\|_2 \\ \le C \sum_{|\gamma|\le \max(M-39, 2+M/2)} \|\langle x\rangle^{-1/2} Z^\gamma u'(t,\,\cdot\,)\|_2^2,\end{aligned}$$

which means that for $40 < M \le 100$, (5.21) and Gronwall's inequality yield

(5.23)
$$E_M^{1/2}(u)(t) \le C(1+t)^{C\varepsilon}\Big[\varepsilon + \sum_{|\alpha|\le \max(M-39, 2+M/2)} \|\langle x\rangle^{-1/2} Z^\alpha u'\|_{L^2(S_t)}^2\Big],$$

if, as before, $S_t = [0,t] \times \mathbb{R}^3 \backslash \mathcal{K}$.

If we use (5.22) and (5.23) along with a simple induction argument we conclude that we would have the desired bounds

$$(5.24)\qquad E_{100}^{1/2}(u)(t) \le C\varepsilon(1+t)^{C\varepsilon+\sigma}$$

for arbitrarily small $\sigma > 0$ if we could prove the following

Lemma 5.2. *Under the above assumptions if $M \le 100$ and*

$$\sum_{|\alpha|\le M} \|\partial^\alpha u'(t,\,\cdot\,)\|_2 + \sum_{|\alpha|\le M-3} \|\langle x\rangle^{-1/2}\partial^\alpha u'\|_{L^2(S_t)} + \sum_{|\alpha|\le M-4} \|Z^\alpha u'(t,\,\cdot\,)\|_2 + \sum_{|\alpha|\le M-6} \|\langle x\rangle^{-1/2} Z^\alpha u'\|_{L^2(S_t)} \le C\varepsilon(1+t)^{C\varepsilon+\sigma}, \tag{5.25}$$

with $\sigma > 0$, then there is a constant C' so that

$$\sum_{|\alpha|\le M-2} \|\langle x\rangle^{-1/2}\partial^\alpha u'\|_{L^2(S_t)} + \sum_{|\alpha|\le M-3} \|Z^\alpha u'(t,\,\cdot\,)\|_2 + \sum_{|\alpha|\le M-5} \|\langle x\rangle^{-1/2} Z^\alpha u'\|_{L^2(S_t)} \le C'\varepsilon(1+t)^{C'\varepsilon+C'\sigma}. \tag{5.26}$$

Proof of Lemma 5.2. Let us start out by estimating the first term in the right side of (5.26). By (5.4) and (2.22) we have

$$\begin{aligned}(\log(2+t))^{-1/2} &\sum_{|\alpha|\le M-2} \|\langle x\rangle^{-1/2}\partial^\alpha u'\|_{L^2(S_t)} \\ &\le C\varepsilon + (\log(2+t))^{-1/2} \sum_{|\alpha|\le M-2} \|\langle x\rangle^{-1/2}\partial^\alpha w'\|_{L^2(S_t)} \\ &\le C\varepsilon + C\sum_{|\alpha|\le M-1}\int_0^t \|\partial^\alpha \Box w(s,\,\cdot\,)\|_2\, ds + C\sum_{|\alpha|\le M-2} \|\partial^\alpha \Box w\|_{L^2(S_t)}.\end{aligned} \tag{5.27}$$

Since $\partial^\alpha \Box w = \partial^\alpha \Box u - \partial^\alpha \Box u_0$, (5.4) implies that the right side is

$$\le C\varepsilon + C\sum_{|\alpha|\le M-1}\int_0^t \|\partial^\alpha \Box u(s,\,\cdot\,)\|_2 ds + C\sum_{|\alpha|\le M-2} \|\partial^\alpha \Box u\|_{L^2(S_t)}.$$

If $M \le 40$ we can use (5.7) and (5.25) to see that the last two terms are $\le C\varepsilon(1+t)^{C\varepsilon+\sigma}$. If $40 < M \le 100$ we can repeat the proof of (5.23) to conclude that they are

$$\begin{aligned}&\le C\varepsilon(1+t)^{2C\varepsilon+2\sigma} + C\sum_{|\alpha|\le \max(M-39,2+M/2)} \|\langle x\rangle^{-1/2} Z^\alpha u'\|^2_{L^2(S_t)} \\ &\quad + C\sup_{0\le s\le t}\Big(\sum_{|\alpha|\le M-6} \|Z^\alpha u'(s,\,\cdot\,)\|_2\Big) \sum_{|\alpha|\le \max(M-39,2+M/2)} \|\langle x\rangle^{-1/2} Z^\alpha u'\|_{L^2(S_t)} \\ &\le C\varepsilon(1+t)^{2C\varepsilon+2\sigma},\end{aligned}$$

using the induction hypothesis (5.25) and the fact that $\max(M-39, 2+M/2) \le M-6$ if $M \ge 40$. Thus, the left side of (5.27) is $\le C\varepsilon(1+t)^{2C\varepsilon+2\sigma}$, which by (5.4) means that

$$\sum_{|\alpha|\le M-2} \|\langle x\rangle^{-1/2}\partial^\alpha u'\|_{L^2(S_t)} \le C\varepsilon(1+t)^{2C\varepsilon+2\sigma}\log(2+t).$$

Thus, we have the desired bounds for the first term in the left side of (5.26).

We need to control the second term in the left side of (5.26). Here we need to use (2.20). In order to do so, we need to estimate the first term in its right side. We note that if $Y_{M-3,0}(t)$ is as in (2.20), then

$$\begin{aligned}
&\sum_{|\alpha|\le M-3} \left\|\Box_\gamma Z^\alpha u(t,\,\cdot\,)\right\|_2 \\
&\qquad \le C \sum_{|\beta|+|\gamma|\le M-3} \|Z^\beta u'(t,\,\cdot\,)Z^\gamma u'(t,\,\cdot\,)\|_2 \\
&\qquad \le C \sum_{|\beta|\le M-3, |\gamma|\le 40} \|Z^\beta u'(t,\,\cdot\,)\|_2\|Z^\gamma u'(t,\,\cdot\,)\|_\infty \\
&\qquad\quad + C \sum_{|\beta|,|\gamma|\le M-43} \|Z^\beta u'(t,\,\cdot\,)Z^\gamma u'(t,\,\cdot\,)\|_2 \\
&\qquad \le \frac{C\varepsilon}{1+t} Y^{1/2}_{M-3,0}(t) + C \sum_{|\beta|\le M-41} \|\langle x\rangle^{-1/2} Z^\beta u'(t,\,\cdot\,)\|_2^2.
\end{aligned}$$

In the last step, we used (5.7) and Lemma 4.5. By plugging this into (2.20), we conclude that

$$\begin{aligned}
\partial_t Y_{M-3,0}(t) \le \frac{C\varepsilon}{1+t} Y_{M-3,0}(t) &+ C \sum_{|\beta|\le M-41} \|\langle x\rangle^{-1/2} Z^\beta u'(t,\,\cdot\,)\|_2^2 Y^{1/2}_{M-3,0}(t) \\
&+ C \sum_{|\alpha|\le M-2} \|\langle x\rangle^{-1/2}\partial^\alpha u'(t,\,\cdot\,)\|_2^2.
\end{aligned}$$

Therefore, by Gronwall's inequality, we have

$$\begin{aligned}
&\sum_{|\alpha|\le M-3} \|Z^\alpha u'(t,\,\cdot\,)\|_2^2 \le C Y_{M-3,0}(t) \\
&\le C(1+t)^{C\varepsilon}\Big(\varepsilon^2 + \sum_{|\beta|\le M-41} \|\langle x\rangle^{-1/2} Z^\beta u'(t,\,\cdot\,)\|^2_{L^2(S_t)} \sup_{0<s<t} Y^{1/2}_{M-3,0}(s) \\
&\qquad + \sum_{|\alpha|\le M-2} \|\langle x\rangle^{-1/2}\partial^\alpha u'(t,\,\cdot\,)\|^2_{L^2(S_t)}\Big).
\end{aligned}$$

In the previous step we estimated the last term in the right. Since the inductive hypothesis handles the first factor of the second term, we conclude that the second term in (5.26) also satisfies the desired bounds. Using (2.23), this in turn implies that the third term satisfies the bounds, which completes the proof. □

This proves (5.24). By elliptic regularity, we get

$$\sum_{|\alpha|\le 100} \|\partial^\alpha u'(t,\,\cdot\,)\|_2 \le C\varepsilon(1+t)^{C\varepsilon+\sigma},$$

which in turn yields (5.9). We also get from Lemma 5.2 that

$$\sum_{|\alpha|\le 98} \|\langle x\rangle^{-1/2}\partial^\alpha w'\|_{L^2(S_t)} + \sum_{|\alpha|\le 97} \|Z^\alpha w'(t,\,\cdot\,)\|_2 + \sum_{|\alpha|\le 95} \|\langle x\rangle^{-1/2} Z^\alpha w'\|_{L^2(S_t)} \le C'\varepsilon(1+t)^{C'\varepsilon+C'\sigma}, \tag{5.28}$$

since the same sort of bounds hold when u is replaced by w.

Here and in what follows σ denotes a small constant that must be taken to be larger and larger at each occurrence. Note that in terms of the number of Z derivatives (5.26) is considerably stronger than the variants of (5.10) and (5.11) where one just takes the terms with $\nu = 0$. This is because just as in going from (5.9) to (5.28) there is a loss of derivatives, there will be a loss of derivatives in going from L^2 bounds for terms of the form $L^\nu Z^\alpha w'$ to those of the form $L^{\nu+1} Z^\alpha w'$.

The proof of the estimates involving powers of L is a bit more complicated. Still we shall follow the above strategy. First we shall estimate $L^\nu \partial^\alpha u'$ in L^2 when α is small using (5.7). Then we shall estimate the remaining parts of (5.10) and (5.11) for this value of ν by an inductive argument that is similar to the one in Lemma 5.2.

The main part of the next step will be to show that

$$\sum_{\substack{|\alpha|+\mu\le 92\\ \mu\le 1}} \|L^\mu \partial^\alpha u'(t,\,\cdot\,)\|_2 \le C\varepsilon(1+t)^{C\varepsilon+\sigma}. \tag{5.29}$$

For this we shall want to use (2.17). We first must establish appropriate versions of (2.16) for $N_0+\nu_0 \le 92$, $\nu_0 = 1$. For this we note that for $M \le 92$

$$\begin{aligned}
&\sum_{\substack{j+\mu\le M\\ \mu\le 1}} \Big(\big|\tilde L^\mu \partial_t^j \Box_\gamma u\big| + \big|[\tilde L^\mu \partial_t^j, \Box - \Box_\gamma]u\big| \Big) \\
&\quad \le C\Big(\sum_{j\le M-1} \big|\tilde L \partial_t^j u'\big| + \sum_{j\le M-2} \big|\tilde L \partial_t^j \partial^2 u\big| \Big) \sum_{|\alpha|\le 40} |\partial^\alpha u'| \\
&\quad + C \sum_{|\alpha|\le M-41} |L\partial^\alpha u'| \sum_{|\alpha|\le M} |\partial^\alpha u'| + C \sum_{|\alpha|\le M} |\partial^\alpha u'| \sum_{|\alpha|\le \max(M/2,M-40)} |\partial^\alpha u'|.
\end{aligned}$$

From this, (5.7), Lemma 4.5 and elliptic regularity we get that for $M \le 92$

$$\sum_{\substack{j+\mu\le M\\ \mu\le 1}} \Big(\big\| \tilde L^\mu \partial_t^j \Box_\gamma u(t,\,\cdot\,)\big\|_2 + \big\| \big[\tilde L^\mu \partial_t^j, \Box - \Box_\gamma\big] u(t,\,\cdot\,)\big\|_2 \Big)$$
$$\le \frac{C\varepsilon}{1+t} \sum_{\substack{j+\mu\le M\\ \mu\le 1}} \big\| \tilde L^\mu \partial_t^j u'(t,\,\cdot\,)\big\|_2$$
$$+ C \sum_{|\alpha|\le M-41} \|\langle x\rangle^{-1/2} L\partial^\alpha u'(t,\,\cdot\,)\|_2 \sum_{|\alpha|\le 94} \|\langle x\rangle^{-1/2} Z^\alpha u'(t,\,\cdot\,)\|_2$$
$$+ C \sum_{|\alpha|\le \max(M,2+M/2)} \|\langle x\rangle^{-1/2} Z^\alpha u'(t,\,\cdot\,)\|_2^2.$$

Based on this if ε is small then (2.16) holds with $\delta = C\varepsilon$ and

$$H_{1,M-1}(t) = C \sum_{|\alpha|\le M-41} \|\langle x\rangle^{-1/2} L\partial^\alpha u'(t,\,\cdot\,)\|_2^2 + C \sum_{|\alpha|\le 94} \|\langle x\rangle^{-1/2} Z^\alpha u'(t,\,\cdot\,)\|_2^2.$$

Therefore since the conditions on the data give $X_{\mu,j}(0) \le C\varepsilon$ if $\mu + j \le 100$ it follows from (2.17) and (5.28) that for $M \le 92$

$$\sum_{\substack{|\alpha|+\mu\le M\\ \mu\le 1}} \|L^\mu \partial^\alpha u'(t,\,\cdot\,)\|_2 \le C\varepsilon(1+t)^{C\varepsilon+\sigma}$$
$$+ C(1+t)^{C\varepsilon} \sum_{|\alpha|\le M-41} \|\langle x\rangle^{-1/2} L\partial^\alpha u'\|^2_{L^2(S_t)}$$
$$+ C(1+t)^{C\varepsilon} \int_0^t \sum_{|\alpha|\le M+1} \|\partial^\alpha u'(s,\,\cdot\,)\|_{L^2(|x|<1)}\, ds. \tag{5.30}$$

If we apply (2.33) and (5.4) we get that the last integral is dominated by $\varepsilon \log(2+t)$ plus

$$\int_0^t \sum_{|\alpha|\le M+1} \|\partial^\alpha w'(s,\,\cdot\,)\|_{L^2(|x|<1)} ds$$
$$\le C \sum_{|\alpha|\le M+2} \int_0^t \Big(\int_0^s \|\partial^\alpha \Box w(\tau,\,\cdot\,)\|_{L^2(|\,|x|-(s-\tau)\,|<10)} d\tau \Big) ds.$$

By (5.4) if we replace w by u_0 we see that the analog of the last term is $O(\log(2+t)\varepsilon)$. We therefore conclude that

$$\sum_{|\alpha|\le M+1}\int_0^t \|\partial^\alpha u'(s,\,\cdot\,)\|_{L^2(|x|<1)}ds \le C\log(2+t)\varepsilon$$
$$+\,C\sum_{|\alpha|\le M+2}\int_0^t\Big(\int_0^s \|\partial^\alpha \Box u(\tau,\,\cdot\,)\|_{L^2(|\,|x|-(s-\tau)\,|<10)}d\tau\Big)ds.$$

Since

$$\sum_{|\alpha|\le M+2}|\partial^\alpha \Box u| \le C\sum_{|\alpha|\le M+3}|\partial^\alpha u'|\sum_{|\alpha|\le 1+M/2}|\partial^\alpha u'|,$$

an application of Lemma 4.5 yields

$$\sum_{|\alpha|\le M+2}\|\partial^\alpha \Box u(\tau,\,\cdot\,)\|_{L^2(|\,|x|-(s-\tau)\,|<10)}$$
$$\le C\sum_{|\alpha|\le 95}\|\langle x\rangle^{-1/2}Z^\alpha u'\|^2_{L^2(|\,|x|-(s-\tau)\,|<20)},$$

since $3+M/2\le 95$ if $M\le 92$. Since the sets $\{(\tau,x)\,:\,|\,|x|-(j-\tau)\,|<20\}$, $j=0,1,2,\ldots$ have finite overlap, we conclude that for $M\le 92$

$$\sum_{|\alpha|\le M+1}\int_0^t \|\partial^\alpha u'(s,\,\cdot\,)\|_{L^2(|x|<1)}ds \le C\varepsilon\log(2+t)$$
$$+\,C\sum_{|\alpha|\le 95}\|\langle x\rangle^{-1/2}Z^\alpha u'\|^2_{L^2(S_t)}$$
$$\le C\varepsilon(1+t)^{C\varepsilon+\sigma}.$$

Therefore, by (5.30) we have that

$$\sum_{\substack{|\alpha|+\mu\le M\\ \mu\le 1}}\|L^\mu\partial^\alpha u'(t,\,\cdot\,)\|_2 \le C\varepsilon(1+t)^{C\varepsilon+\sigma}$$
$$+\,C(1+t)^{C\varepsilon}\sum_{|\alpha|\le M-41}\|\langle x\rangle^{-1/2}L\partial^\alpha u'\|^2_{L^2(S_t)}.$$

This gives the desired bounds when $M\le 40$.

If we now use (2.22) with $\nu_0=1$ and $N_0+\nu_0=92$, then the analog of Lemma 5.2 where $M=100$ is replaced by $M=92$ and u is replaced by Lu is valid. By an induction argument we get (5.29) from this as well as

$$\sum_{\substack{|\alpha|+\mu\le 90\\ \mu\le 1}}\|\langle x\rangle^{-1/2}L^\mu\partial^\alpha w'\|_{L^2(S_t)}+\sum_{\substack{|\alpha|+\mu\le 89\\ \mu\le 1}}\|L^\mu Z^\alpha w'(t,\,\cdot\,)\|_2$$
$$+\sum_{\substack{|\alpha|+\mu\le 87\\ \mu\le 1}}\|\langle x\rangle^{-1/2}L^\mu Z^\alpha w'\|_{L^2(S_t)}\le C\varepsilon(1+t)^{C\varepsilon+C\sigma}. \tag{5.31}$$

If we repeat this argument we can estimate $L^2 Z^\alpha u'$ and $L^3 Z^\alpha u'$ for appropriate Z^α. Using (5.29) and (5.31) and the last argument gives

$$\sum_{\substack{|\alpha|+\mu\le 84\\ \mu\le 2}} \|L^\mu \partial^\alpha w'(t,\,\cdot\,)\|_2 + \sum_{\substack{|\alpha|+\mu\le 81\\ \mu\le 2}} \|L^\mu Z^\alpha w'(t,\,\cdot\,)\|_2 + \sum_{\substack{|\alpha|+\mu\le 79\\ \mu\le 2}} \|\langle x\rangle^{-1/2} L^\mu Z^\alpha w'\|_{L^2(S_t)} \le C\varepsilon(1+t)^{C\varepsilon+C\sigma}.$$

Then using the estimates for $L^\mu Z^\alpha u'$, $\mu \le 2$ we can argue as above to finally get

$$\sum_{\substack{|\alpha|+\mu\le 76\\ \mu\le 3}} \|L^\mu \partial^\alpha w'(t,\,\cdot\,)\|_2 + \sum_{\substack{|\alpha|+\mu\le 73\\ \mu\le 3}} \|L^\mu Z^\alpha w'(t,\,\cdot\,)\|_2 + \sum_{\substack{|\alpha|+\mu\le 71\\ \mu\le 3}} \|\langle x\rangle^{-1/2} L^\mu Z^\alpha w'\|_{L^2(S_t)} \le C\varepsilon(1+t)^{C\varepsilon+C\sigma}.$$

If we combine this with our earlier bounds, we conclude that (5.10) and (5.11) must be valid.

It remains to prove (5.8). This is straightforward. If we use Theorem 3.1 we find that its left side is dominated by the square of that of (5.11). Hence (5.11) implies (5.8), which finishes the proof.

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