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On the metric properties of multimodal interval maps and C^2 density of Axiom A

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Abstract. In this paper, we shall prove that Axiom A maps are dense in the space of C^2 interval maps (endowed with the C^2 topology). As a step of the proof, we shall prove real and complex a priori bounds for (first return maps to certain small neighborhoods of the critical points of) real analytic multimodal interval maps with non-degenerate critical points. We shall also discuss rigidity for interval maps without large bounds.

Contents

1. Introduction

It is a basic problem in the theory of dynamical systems to describe typical systems. In the one-dimensional case, it is conjectured that generic systems are structurally stable, and even more, they are Axiom A systems.

For any $r \in \mathbb{N}$, let $C^r([0, 1], [0, 1])$ be the space of all C^r maps from $[0, 1]$ into itself, endowed with the C^r topology. A map f in this space is

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called C^r-*structurally stable* if there is a C^r neighborhood $\mathcal{U}$ of f such that every $g \in \mathcal{U}$ is topologically conjugate to f. A C^1 map f satisfies *Axiom A* if the following hold:

(A1) all periodic points are hyperbolic;

(A2) letting $B(f)$ denote the union of basins of attracting periodic points of f and letting $\Omega = [0, 1] - B(f)$, then Ω is a *hyperbolic* set, that is, there are constants $C > 0$ and $\lambda > 1$, such that

$$\left|(f^k)'(x)\right| \geq C\lambda^k$$

holds for all $x \in \Omega$ and $k \in \mathbb{N}$.

Conjecture 1. For any $r \in \mathbb{N}$, any $f \in C^r([0, 1], [0, 1])$ can be approximated in the C^r topology by maps $g \in C^r([0, 1], [0, 1])$ satisfying Axiom A.

An Axiom A map has very simple dynamics. In fact, it is easy to see that the set Ω as in (A2) is a nowhere dense compact set with zero Lebesgue measure. It is also rather easy to show that for any $r \geq 1$, C^r Axiom A maps form an open subset of $C^r([0, 1], [0, 1])$. Moreover, it is well known that if $r \geq 2$, then Axiom A, together with a few other mild conditions, implies C^r-structurally stability. So an affirmative answer to the above conjecture implies the following: for any $r \geq 1$, C^r-structurally stable maps are Axiom A. See [35] for more details.

Conjecture 1 has been studied by many authors. For $r = 1$, it was proved by Jakobson [18] using a purely real method. His approach was based on a closing lemma, and thus seemed irremediably tied to the C^1 topology. In [5], Blokh and Misiurewicz noticed that one can do a better C^2 closing perturbation under a geometric assumption: arbitrarily big space for certain first return maps (*large bounds*, see Definition 2.5). From this observation they derived that $C^2([0, 1], [0, 1])$ contains a dense subset consisting of maps for which each critical point has a minimal ω-limit set. (For an Axiom A map f, the ω-limit set of each critical point is a periodic orbit, and hence obviously minimal.) The gain from C^1 to C^2 comes from the critical point, and does not seem possible to improve using these methods.

There was, however, much progress in the unimodal case in 1990s, due to successful application of "complex" tools. First, through a rigidity approach, Lyubich [27] and Graczyk and Świątek [11] proved that Axiom A maps are dense in the family of real quadratic polynomials. Using Sullivan's deformation trick [46], one observes that it is enough to prove that for any two topologically conjugate real quadratic maps (without attracting cycles) are quasisymmetrically conjugate on their postcritical sets. Later, Kozlovski [20, 21] solved the conjecture for general smooth unimodal maps. He first developed some real tools to address the problem of lack of negative Schwarzian derivative. This allows him to extend earlier work by Lyubich [25] and Levin and van Strien [22] to obtain suitable complex extensions for certain first return maps in the real analytic case. Then he introduced a new deformation trick, which essentially shows that a good complex extension, rigidity and absence of invariant line fields, imply density of Axiom A in any C^r topology. See also [24,2].

The goal of this paper is to prove the following

Main Theorem. *Axiom A is open and dense in the space* $C^2([0, 1], [0, 1])$.

Main Corollary. *C^2-structurally stable maps satisfy Axiom A, and form an open and dense subset of* $C^2([0, 1], [0, 1])$.

The strategy of this paper is as follows. We shall first study the real geometry of a smooth interval map f with non-degenerate critical points, and show that at any non-periodic recurrent critical point, f either has large bounds or *essentially bounded geometry* (Theorem 1). Based on a careful analysis of the macrostructure of the postcritical set, we shall show that the geometry of this (Cantor) set is quasisymmetrically rigid for maps without large bounds (Theorem 2). Furthermore, modifying the method of [31], we shall show that if f is real analytic, then for any non-periodic recurrent critical point c with a minimal ω-limit set, there exists a generalized-renormalization with respect to c which can be extended to a generalized polynomial-like mapping (Theorem 3). For precise statements of these theorems, see Sect. 2.

The main theorem will then be derived from these results. Essentially it suffices to show that a smooth interval map f cannot be C^2-structurally stable if it has a non-periodic recurrent critical point, c. In the case that f has large bounds at c, this follows from the argument in [5]. In the remaining case, we shall apply Kozlovski's deformation trick.

It should be noted that in [26,27], Lyubich already noticed that lack of large bounds gives severe restrictions on the geometry of unimodal interval maps. It turns out that a unimodal interval map with a non-degenerate critical point has large bounds unless it is infinitely renormalizable (in the classical sense) of "essentially bounded type". The situation is quite different when we consider more general interval maps. In particular, the so-called Fibonacci unimodal map, while non-renormalizable, has bounded geometry, provided that the critical order is greater than 2, see [19,6]. A concrete example of a non-renormalizable bimodal cubic polynomial with bounded geometry has also been constructed in [48].

The proof of Theorem 1 will be given in Sect. 4, where we shall prove a slightly stronger result (Theorem 1'), which asserts that in the case of lack of large bounds, a smaller nice interval can not be geometrically deep inside a bigger one unless it is combinatorially deep inside as well. A proof of this result for unimodal maps can be found in [26], but the combinatorial arguments (return graph, ranks, essential periods, and so on) extensively used therein seem difficult to generalize to the multimodal case. We shall prove this result by an induction on a certain object. This argument provides an example showing how we deal with the combinatorial complexity of multimodal maps, and very similar ideas will also be used in the proof of Theorems 2 and 3.

Theorem 3 essentially reduces problems on the dynamics of real analytic interval maps to the polynomial-like case. Polynomial-like maps have many good properties which general interval maps themselves do not share. In particular, the good external structure of a polynomial-like mapping is of importance in this paper (to apply Kozlovski's deformation trick). There have been a lot of papers written in the literature on the polynomial-like extension property for interval maps. For all unimodal maps f in the so-called *Epstein class*, Levin and van Strien [22] obtained this property, by improving earlier work of Sullivan [46] and Lyubich [26]. See also Graczyk and Świątek [12], and Lyubich and Yampolsky [26, 31] for alternative proofs in this case. In the real analytic unimodal case, this property was obtained by Kozlovski [21]. However, for multimodal maps, only a very special case has been treated before: infinitely renormalizable maps of bounded type, see [15, 45]. Our approach to Theorem 3 is based on a careful analysis of the real geometry and motivated by Lyubich and Yampolsky's argument, and also borrows an idea from [24] (see Sect. 8.1). We shall deal with all possible combinatorics, except when $\omega(c)$ is non-minimal, in which case the argument has to be slightly different since there are infinitely many branches in a generalized renormalization. This case will be done elsewhere. We should note that as in all the papers cited above, our approach also gives the *"complex bounds"* property, which asserts that for certain extensions, each domain is universally well inside the range. This property, which will be proved in the appendix, should be useful for further development on the dynamics of multimodal interval maps, although we shall not make use of it here.

The rigidity conjecture asserts that the quasisymmetric and topological conjugacy classes of an interval map (without periodic attractors) are equal. An affirmative answer to this conjecture would imply Conjecture 1 together with our Theorem 3. Theorem 2 can be considered as a weaker version of this conjecture. Currently, the rigidity conjecture was only proved for real quadratic polynomials, in the papers [11, 27] cited above. See also [44] for an alternative proof by considering iteration on the universal Teichmüller space. All these proofs use complex analysis, quasiconformal mapping theory, and complex a priori bounds. Unfortunately, all of them rely heavily on the fact that the map has a unique non-degenerate critical point, and do not admit trivial generalization to the multimodal case. In fact we do not even know if the large bounds property is topological.

Let us outline the structure of this paper. In Sect. 2, we shall review basic concepts in real one-dimensional dynamics which will be used in our arguments, and give precise statements of Theorems 1, 2 and 3. In Sect. 3, we shall prove several lemmas concerning the real dynamics. The next two sections, Sect. 4 and Sect. 5 are due to the analysis of the geometry of the postcritical set. In Sect. 4, we shall prove Theorem 1, where the main step is to prove a more technical result Proposition 4.1, through an induction argument. In Sect. 5 we shall give a more detailed analysis for the case of essentially bounded geometry. We shall see that in this case, the geometry is close to being bounded, and the only exception is caused by the presence of a long central cascade of saddle node type. In Sect. 6, we shall prove Theorem 2. In Sect. 7, we collect a few known facts in complex analysis and complex dynamics, which will be used in Sect. 8, where we prove Theorem 3. The proof of the main theorem will be completed in Sect. 9. In the appendix, we shall show how to obtain "complex bounds".

General notation. We use $\mathbb{N}$ to denote the set of positive integers.

We use $\mathrm{dom}(f)$ to denote the domain of a map f.

For any topological space A and a connected subset A_0, we use $\mathrm{Comp}_{A_0}(A)$ to denote the connected component of A which contains A_0. Moreover, for $x \in A$, $\mathrm{Comp}_x(A) := \mathrm{Comp}_{\{x\}}(A)$.

For any subset $X \subset \mathbb{C}$, we use $\mathrm{Cl}(X)$ to denote the closure of X in $\mathbb{C}$.

For any bounded open interval I and any $\lambda > 0$, we use λI to denote the open interval which has the same middle point as I and length $\lambda|I|$. The interval $(1 + 2\delta)I$ is often referred to as the *δ-neighborhood of I*.

A map f from an open set $U \subset \mathbb{C}$ into $\mathbb{C}$ is called *real symmetric* if for any $z \in U$, we have $\bar{z} \in U$, and $\overline{f(z)} = f(\bar{z})$.

$A \Subset B$ means that A is compactly contained in B.

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2. Some definitions and statements of results

The central part of this work is to gain control on the geometry of a multimodal interval map. Our analysis is inspired by recent research on unimodal interval maps, and exploits the powerful renormalization idea.

2.1. Nice open sets and first return maps. Unless otherwise stated, f stands for a smooth map $f : [0, 1] \to [0, 1]$ such that $f(\{0, 1\}) \subset \{0, 1\}$, and such that all the critical points of f are contained in $(0, 1)$ and non-degenerate. Let $\mathcal{N}$ denote the collection of all such maps f, and let $\mathrm{Crit}(f)$ denote the set of critical points of f. (In fact, many of the following arguments also work for C^3 interval maps with C^3 non-flat critical points which are all of turning type.)

We say an interval I is *symmetric* if it contains exactly one critical point of f and $f(\partial I)$ consists of a single point.

By a *chain* we mean a sequence of open intervals $\{G_i\}_{i=0}^n$ contained in $(0, 1)$ such that G_i is a component of $f^{-1}(G_{i+1})$ for every $0 \le i \le n - 1$.

Note that $f(\partial G_i) \subset \partial G_{i+1}$. The *order* of the chain is the number of the integers i with $0 \le i < n$, such that G_i contains a critical point, and the *intersection multiplicity* is the maximal number of the intervals G_i, $0 \le i \le n$, which have a non-empty intersection. We shall also say that G_0 is a *pull back* of G_n.

Definition 2.1. Let $f \in \mathcal{N}$. An open set $T \subset [0, 1]$ is called *nice* if for any $x \in \partial T$, and any $n \in \mathbb{N}$ we have $f^n(x) \notin T$.

This concept was introduced by Martens [34]. We shall use D_T to denote the set of points which enter T under forward iterates of f, that is,

$$D_T = \left\{x \in \mathrm{dom}(f) : \text{ for some } k \in \mathbb{N},\ f^k(x) \in T\right\}.$$

For any $x \in D_T$, the minimal positive integer $k = k(x)$ such that $f^k(x) \in T$ will be called *the entry time* of x to T. The map

$$R_T : D_T \to T$$

defined by $R_T(x) = f^{k(x)}(x)$ is *the first entry map* to T. The *first return map* to T is the restriction of R_T to $D_T \cap T$. A component of D_T is called an *entry domain* to T, and also a *return domain* if it is contained in T.

We shall repeatedly use the following (well-known) properties of a nice open set T:

- any two pull backs of components of T are disjoint or nested, i.e., one is contained in the other;
- the entry time $k(x)$ is constant in any entry domain J, and the first entry map R_T is proper: $R_T(\partial J) \subset \partial T$;
- a finite union of components of D_T is again a nice open set.

Note. For any $x \in D_T$, we shall use $\mathcal{L}_x(T)$ to denote the component of D_T which contains x, and define inductively, $\mathcal{L}_x^{i+1}(T) = \mathcal{L}_x(\mathcal{L}_x^i(T))$ for any i.

Given a nice open interval $T \subset (0, 1)$, and $x \in D_T$, if $k \in \mathbb{N}$ is the (first) entry time of x to T, then there is a unique chain $\{T_i\}_{i=0}^k$ with $T_k = T$ and $T_0 \ni x$, which will be referred to as the *chain corresponding to the first entry of x to T*. As the intervals T_i, $0 \le i < k$, are pairwise disjoint, the intersection multiplicity of this chain is at most 2, and the order is bounded by the number of critical points of f.

2.2. Generalized renormalization and real box mappings. Let c be a non-periodic recurrent critical point of f. Note that there exist arbitrarily small nice intervals which contain this point. Indeed, it is well-known that c is accumulated by periodic points of f. Given a periodic point p close to c, let I be the maximal symmetric open interval which contains c and is disjoint from the orbit of p. Then I is a nice interval.

For a symmetric nice interval $I \ni c$, we denote by $\mathbf{D}_I$ the union of all components of D_I which intersect $\omega(c) \cap I$, and define the *generalized* *renormalization* $\mathbf{R}_I$ to be the first return map to I restricted to $\mathbf{D}_I$. Here $\omega(c)$ denotes the ω-limit set of c.

By a *(principal) nest* (around c), we mean a sequence of nice intervals

(1) $$\mathbf{I} = \{I^0 \supset I^1 \supset I^2 \supset \cdots\},$$

where $I^0 \ni c$ is a symmetric nice interval, and $I^{n+1} = \mathcal{L}_c(I^n)$ for all $n \geq 0$. Given such a nest $\mathbf{I}$, consider the generalized renormalizations,

$$\mathbf{R}_{I^n} : \mathbf{D}_{I^n} \to I^n.$$

The configuration $(I^n, \mathbf{D}_{I^n})$ provides a parameter to describe the geometric property of $\omega(c)$. As every branch of $R_{I^{n+1}}$ is an iterate of R_{I^n}, the generalized renormalizations also contain fruitful combinatorial information.

Note. Once a symmetric nice interval I is given, we define the corresponding nest as $I^0 = I$ and $I^i = \mathcal{L}_c(I^{i-1})$ for all $i \geq 1$, where c is the critical point in I.

Definition 2.2. An interval I is called *properly periodic* if there is a positive integer s such that the interiors of $I, f(I), \ldots, f^{s-1}(I)$ are pairwise disjoint, $f^s(I) \subset I$, and $f^s(\partial I) \subset \partial I$. The integer s is called the *period* of I. We say that f is *infinitely renormalizable* at a critical point c, if there is an arbitrarily small properly periodic interval $I \ni c$. Otherwise, we say that f is *only finitely renormalizable* at c.

If f is only finitely renormalizable at c, then for each small symmetric nice interval I containing c, $I^n \to \{c\}$ as $n \to \infty$. However, if f is infinitely renormalizable at c, then for any such an interval I, $\bigcap_n I^n$ is a properly periodic interval.

In the unimodal case, a generalized renormalization defined as above has a unique critical point, but in the multimodal case, it may have a large number of critical branches, and each branch may have many critical points. In the following we shall introduce the notion *real box mapping*, which can be considered as a different type of generalized renormalization. Compared to the generalized renormalization introduced above, a real box mapping has less critical points and simpler branches, and so serves as a more convenient notion for us to apply Kozlovski's "deformation trick".

Definition 2.3. Let c be a non-periodic recurrent critical point of f. Let $[c] = \{d \in \mathrm{Crit}(f) : \omega(d) = \omega(c) \ni c, d\}$, and $b = \#[c]$. Let $I \ni c$ be a symmetric nice interval. For any $c' \in [c]$, let $I(c') = \mathrm{Comp}_{c'}(D_I \cup I)$. Denote by $I = I_0, I_1, I_2, \ldots, I_{b-1}$ these intervals $I(c'), c' \in [c]$. Notice that $\bigcup_{i=0}^{b-1} I_i$ is a nice open set. The *real box mapping associated to* I, denoted by $\mathbf{B}_I$, is defined to be the first return map to this nice open set, restricted to those return domains which intersect $\omega(c)$.

We shall only use real box mappings in the case that $\omega(c)$ is minimal, and I is a small *strictly nice* or properly periodic interval, where a strictly nice interval is, by definition, a nice interval for which the endpoints stay away from its closure under forward iterates of f. In this case, the real box mapping $\mathbf{B}_I$ falls into the class $\mathcal{F}_b$ defined below, see Lemma 3.2.

Definition 2.4. For any $b \in \mathbb{N}$, let $\mathcal{E}_b$ be the collection of smooth maps

$$f : \Big(\bigcup_{j=0}^{m} J_j\Big) \cup \Big(\bigcup_{i=1}^{b-1} I_i\Big) \to \bigcup_{i=0}^{b-1} I_i \tag{2}$$

with the following properties:

- I_i's are open intervals with pairwise disjoint closures;
- m is a non-negative integer;
- J_j's are open intervals contained in I_0, and the closures of J_j's are pairwise disjoint and contained in I_0 unless $m = 0$ in which case we also allow $J_0 = I_0$;
- f is a proper map;
- f extends to a smooth map defined on the closure of its domain such that f' does not vanish at the boundary;
- for each $1 \leq j \leq m$, $f|J_j$ is a diffeomorphism;
- for any $U \in \{J_0, I_1, I_2, \dots, I_{b-1}\}$, $f|U$ has a unique critical point c_U, which is non-degenerate, i.e., $f''(c_U) \neq 0$.

Moreover, let $\mathcal{F}_b$ be the subspace of $\mathcal{E}_b$ consisting of maps f for which the following hold:

- all the critical points do not escape under forward iterates of f,
- all the critical points are non-periodic and recurrent, and they have the same ω-limit set which is a minimal set.

A map $f \in \mathcal{E}_b$ extends naturally to a smooth map defined on the closure of dom(f). Let $\mathcal{G}_b$ denote the collection of maps in $\mathcal{F}_b$ whose natural extensions have only hyperbolic repelling periodic points.

The set $\bigcup_{i=0}^{b-1} I_i$ is called the *range* of f.

Remark 2.1. For a map f in the class $\mathcal{E}_b$, the concepts of nice open sets, properly periodic intervals, renormalizations, etc, can be formulated similarly as in the case that f is an interval endomorphism. For example, an open set T contained in the range of f is called *nice*, if for any $x \in \partial T$ and for any $n \in \mathbb{N}$, we have $f^n(x) \notin T$ as long as $x, f(x), \dots, f^{n-1}(x)$ are defined.

Note. We use $\mathcal{N}'$ to denote the family $\mathcal{N} \cup (\bigcup_{b=1}^{\infty} \mathcal{E}_b)$.

2.3. Scaling factors and large bounds. In what follows, f is a map in the class $\mathcal{N}'$. Let c be a non-periodic recurrent critical point of f, and let $I^0 \supset I^1 \supset \cdots$ be a nest around c. Among many geometric parameters, the *scaling factors* $\lambda_{I^n} = |I^n|/|I^{n+1}|$ are of particular interest. In general, they can be arbitrarily close to 1 or arbitrarily large. But a remarkable fact in one-dimensional dynamics (*real bounds*) is that they can only be close to 1 in special situations. To be more precise we need a few definitions. Let us say that R_{I^n} displays a *central return*, if $R_{I^n}(c) \in I^{n+1}$. By a *central cascade*, we mean a subnest

$$I^n \supset I^{n+1} \supset \cdots \supset I^{n+N}, \tag{3}$$

such that $R_{I^{n+i}}$ display central returns for all $0 \leq i \leq N-2$. In other words, the return times of c to these intervals $I^n, I^{n+1}, \ldots, I^{n+N-1}$ are all the same. The following theorem of van Strien and Vargas says that the scaling factor λ_{I^n} is uniformly bounded away from 1 unless I^n is contained in a long central cascade.

Theorem 2.1 ([47], see also [49,42]).

1. *There exists* $\lambda = \lambda(\#Crit(f)) > 1$ *with the following property. Let us consider a nest* $I^0 \supset I^1 \supset I^2 \supset \cdots$ *with* $|I^0|$ *sufficiently small. If* R_{I^n} *does not display a central return, then*

$$\lambda_{I^{n+1}} = \frac{|I^{n+1}|}{|I^{n+2}|} \geq \lambda.$$

2. *For any* $\xi > 0$*, there exists* $\xi' = \xi'(\xi, \#Crit(f)) > 0$ *with the following property. If* $(1+2\xi)I^{n+1} \subset I^n$*, then for each return domain* J *to* I^{n+1}*, we have*

$$(1+2\xi')J \subset I^{n+1}.$$

Real bounds were proved for S-unimodal maps earlier by Martens [34]. These bounds will play an important role in our consideration. But clearly they do not yet give a satisfactory description of the geometry of $\omega(c)$. Most of our analysis will be done under the assumption that the scaling factors are uniformly bounded from above. It turns out that this assumption gives severe restrictions on the geometric properties, in a similar way to that noticed by Lyubich [26,27] in the unimodal case.

Remark 2.2. In the unimodal case, the scaling factors give distortion control of the generalized renormalizations: if λ_{I^n} is bounded away from 1, then each branch of $\mathbf{R}_{I^{n+1}}$ can be expressed as $L \circ f$, where L is a diffeomorphism with uniformly bounded distortion. This can be seen as follows: if J is a return domain to I^{n+1} with return time s, and if $\{G_j\}_{j=0}^{s}$ is the chain with $G_s = I^n$ and $G_0 \supset J$, then for every $0 \leq j \leq s-1$, G_j is disjoint from I^{n+1}, and thus does not contain the critical point, which implies that $f^{s-1}|f(J)$ has bounded distortion, by the real Koebe principle (Lemma 3.5). However, this argument fails in the multimodal case, because there are critical points outside I^{n+1} which may enter the intervals G_j arbitrarily many times.

Our first theorem says that if the scaling factors are not uniformly bounded, then f will have *large bounds*, defined as follows.

Definition 2.5. Let c be a non-periodic recurrent critical point of f. A symmetric nice interval $I \ni c$ is called *C-nice* if for any $x \in \omega(c) \cap I$, we have

$$(1 + 2C)\mathcal{L}_x(I) \subset I.$$

We say that f has *large bounds* at c if for any $C > 0$, there is an arbitrarily small symmetric C-nice interval which contains c. We say that f has *essentially bounded geometry* at c if there is a constant $C > 1$ such that for any symmetric nice interval $I \ni c$, we have

$$|I| \leq C|\mathcal{L}_c(I)|.$$

Theorem 1. *Let f be a map in $\mathcal{N}'$ and let c be a non-periodic recurrent critical point of f. If f does not have large bounds at c, then it has essentially bounded geometry at c.*

Note that the large bounds property is exactly the geometric condition needed in Blokh and Misiurewicz's C^2 closing lemma.

Theorem 1 follows immediately from the usual Koebe principle in the unimodal case, but the situation is much more complicated in the multimodal case. In fact, we shall prove the following more general result, from which Theorem 1 follows immediately.

Theorem 1'. *Let f and c be as above. For any $d \in \mathbb{N}$ and any $\xi > 0$, there exists $\xi' = \xi'(\xi, d, \#Crit(f)) > 0$ such that the following holds. Let I be a sufficiently small symmetric nice interval which contains c. Let $x \in \omega(c) \cap I$ and $J = \mathcal{L}_x^d(I)$. If $I \supset (1 + 2\xi)J$, then for any $y \in orb(c)$, we have*

$$\mathcal{L}_y(I) \supset (1 + 2\xi')\mathcal{L}_y(J),$$

Moreover, for fixed d and $\#Crit(f)$, we have

$$\xi' \to \infty \text{ as } \xi \to \infty. \tag{4}$$

The first part of this theorem is a special case of Theorem B.1 in [47], which asserts that this part holds even without the assumption that $J = \mathcal{L}_x^d(I)$. To us, the second part, the dependence (4) is very important. Note that without the combinatorial hypothesis, it is impossible to obtain this dependence. A typical example where (4) does not hold is when J is deep inside a saddle node central cascade.

Remark 2.3. It is easy to see that if f has large bounds at a non-periodic recurrent critical point c, then it also has large bounds at any other recurrent critical point d with the same ω-limit set. Thus for a map f in the class $\mathcal{F}_b$, it makes sense to say whether f has large bounds or not, without referring to a specified critical point.

Essentially bounded geometry. The definition of "essentially bounded geometry" does not tell us much, but in Sect. 5, we shall justify the name by showing that the geometry of a smooth interval map with "essentially bounded geometry" is close to being bounded, and the only exception is caused by the presence of a long central cascade of saddle node type.

To be more precisely, let us consider a nest $I^0 \supset I^1 \supset \cdots$. As a consequence of Theorem 1', we shall prove that a return domain to I^{n+1} which intersects $\omega(c)$ cannot be deep inside I^n, see Corollary 5.3. So if the scaling factors $\lambda_{I^n} = |I^n|/|I^{n+1}|$ are bounded away from 1, then for any n, any return domain to I^{n+1} must be commensurable to I^{n+1} (for otherwise it would be deep inside I^n). Together with Theorem 2.1, it follows that if there are no long central cascades in the nest, then the whole nest has bounded geometry.

If a long central cascade does exist, then the geometry is no longer bounded: the scaling factors can be close to 1. We shall show, however, that a long *maximal* central cascade can only be essentially of *saddle node* type, and thus the geometry is still under satisfactory control, see Proposition 5.1 and Theorem 5.4. Here we say that a central cascade as in (3) is maximal if $R_{I^{n+N-1}}$ displays a non-central return.

In Sect. 5.5, we consider the case that f is infinitely renormalizable at c, and show that the geometry of an appropriate initial partition at each renormalization level is also uniformly bounded. See Theorem 5.6.

2.4. Rigidity.

Definition 2.6. Two maps $f : (\bigcup_{j=0}^{m} J_j) \cup (\bigcup_{i=1}^{b-1} I_i) \to \bigcup_{i=0}^{b-1} I_i$ and $\tilde{f} : (\bigcup_{j=0}^{m} \tilde{J}_j) \cup (\bigcup_{i=1}^{b-1} \tilde{I}_i) \to \bigcup_{i=0}^{b-1} \tilde{I}_i$ in the class $\mathcal{F}_b$ are *combinatorially equivalent* if there is a homeomorphism $h : \mathbb{R} \to \mathbb{R}$ such that for each $0 \le i \le b-1$ and $0 \le j \le m$, we have

- $h(J_j) = \tilde{J}_j$, $h(I_i) = \tilde{I}_i$;
- if c_i is the critical point of f in I_i, then $\tilde{c}_i = h(c_i)$ is a critical point of $\tilde{f}$;
- for any $k \in \mathbb{N}$, $h(f^k(c_i)) = \tilde{f}^k(\tilde{c}_i)$;
- for each $z \in \partial(\mathrm{dom}(f))$, $h(f(z)) = \tilde{f}(h(z))$.

Such a map h will be called a *combinatorial equivalence* between f and $\tilde{f}$.

Definition 2.7. A homeomorphism $h : \mathbb{R} \to \mathbb{R}$ is called *quasisymmetric* (*qs* in short) if there exists $M > 1$, such that for any $x, y, z \in \mathbb{R}$ with $y - x = z - y \neq 0$, we have

$$\frac{1}{M} \le \frac{h(z) - h(y)}{h(y) - h(x)} \le M.$$

Theorem 2. *Let $f, \tilde{f} \in \mathcal{G}_b$ be two combinatorially equivalent maps which have essentially bounded geometry. Then they are quasisymmetrically conjugate on the postcritical sets, i.e., the combinatorial equivalence can be realized by a quasisymmetric map.*

We should remark that in case that both of the maps f and $\tilde{f}$ have good complex extensions (polynomial-like box mappings, see the next subsection), then by a well-known pull-back argument due to Thurston, the qs partial conjugacy between the postcritical sets can be promoted to a qs global conjugacy. See [9] and also Sect. 9.

As we have noted before, it is conjectured that this theorem is true even without the assumption that f and $\tilde{f}$ have essentially bounded geometry (*the rigidity conjecture*). This conjecture has been verified in the unimodal case in [11,27] using complex tools, but the multimodal case is still essentially open. A main property, "*the linear growth of the principal moduli*", which holds for a unimodal map (with a non-degenerate critical point) fails in the multimodal case. Theorem 2 is a weaker version of this conjecture, and will be proved using a purely real argument, motivated by Sullivan's proof in [46] for the quadratic Feigenbaum polynomials.

2.5. Polynomial-like extension.

Definition 2.8. Let V_i, $0 \leq i \leq b-1$ be pairwise disjoint topological disks. Let U_j, $0 \leq j \leq m$ be topological disks with pairwise disjoint closures which are contained in V_0. A map

$$F : \Big(\bigcup_{j=0}^{m} U_j\Big) \cup \Big(\bigcup_{i=1}^{b-1} V_i\Big) \to \bigcup_{i=0}^{b-1} V_i \tag{5}$$

is called *a polynomial-like mapping* if for each component U of $\mathrm{dom}(F)$, $F|U$ is a holomorphic proper map onto a component of $\bigcup_{i=0}^{b-1} V_i$. The *filled Julia set* of F is defined to be the set of non-escaping points. The boundary of this set is called the *Julia set* of F.

Polynomial-like mappings first appeared in Douady and Hubbard's work [8] on quadratic polynomials, and the notion has been generalized several times since then. Douady and Hubbard's original definition requires $b = 1$ and $m = 0$, in which case, we say that F is a *DH-polynomial-like mapping*.

Definition 2.9. A polynomial-like map F is called a (holomorphic) polynomial-like box mapping if the following hold:

- for each $1 \leq j \leq m$, $F|U_j$ is a conformal map onto some V_i and
- for any $U \in \{U_0, V_1, V_2, \dots, V_{b-1}\}$, $F|U$ has a unique critical point of order 2.

Theorem 3. *Let f be a real analytic map in the class $\mathcal{F}_b$ and let c be a critical point of f. If f has essentially bounded geometry at c, then there exists an arbitrarily small nice interval I containing c such that the real box mapping associated to I extends to a polynomial-like box mapping.*

In fact, if f has large bounds at c, the same result holds as well, and the proof is even much simpler, although it is unnecessary for our main theorem. Our approach is motivated by previous works in the unimodal case, specially [31]. As is expected, the proof gives *complex bounds* for appropriate extensions, see Theorem 3' in the appendix.

3. Background in real dynamics

In this section, we recall some known results and prove a few lemmas in real one-dimensional dynamics which will be used later. Recall that $\mathcal{N}' = \mathcal{N} \cup (\bigcup_{b=1}^{\infty} \mathcal{E}_b)$.

3.1. No wandering interval theorem. For an interval map f, a *wandering interval* is an interval J such that $f^s|J$ is well-defined for all $s \in \mathbb{N}$, and such that the intervals J, $f(J)$, $f^2(J), \ldots$ are pairwise disjoint, and such that J is not contained in the basin of a periodic attractor.

Theorem 3.1. *A map $f \in \mathcal{N}'$ has no wandering interval. Equivalently, if J is an interval such that $f^s|J$ is well defined for all $s \in \mathbb{N}$ and such that* $\liminf_{s\to\infty} |f^s(J)| = 0$*, then there is a periodic point p such that*

$$\lim_{s\to\infty} \sup_{x\in J} d\big(f^s(x), f^s(p)\big) = 0.$$

This theorem follows from the work by Guckenheimer, de Melo and van Strien, Lyubich, Blokh and Lyubich, and Martens, de Melo and van Strien. See [14,35,25,3,37]. In particular, this implies

Lemma 3.1. *Let $f \in \mathcal{N}'$ and let c be a non-periodic critical point of f. Then for any $\delta > 0$, there is a $\delta' > 0$ such that the following hold.*

(1) *For any interval J and any non-negative integer s, if $f^s|J$ is well-defined and $f^s(J) \subset (c-\delta', c+\delta')$, then $|J| < \delta$;*
(2) *Assume that c is recurrent and that $\omega(c)$ is minimal. For any interval J and any non-negative integer s, if $f^s|J$ is well-defined, and if there is a point $x \in \omega(c)$ such that $f^s(J) \subset (x-\delta', x+\delta')$, then $|J| < \delta$.*

Proof. We shall only prove the first statement. The second one can be done in a similar way. Arguing by contradiction, assume that this statement is false. Then for any $k = 1, 2, \ldots$, there exist an interval J_k and a non-negative integer s_k, such that $f^{s_k}|J_k$ is well-defined, $\lim_k |J_k| > 0$ and $\lim_k \sup_{x\in J_k} d(c, f^{s_k}(x)) = 0$. By passing to a subsequence, we may assume that $\bigcap_k J_k$ contains a non-degenerate interval J. Then

$$\lim_{k\to\infty} \sup_{x\in J} d\big(c, f^{s_k}(x)\big) = 0.$$

It follows immediately that $s_k \to \infty$ as k tends to ∞. So $f^s|J$ is well-define for all $s \in \mathbb{N}$. By Theorem 3.1, there is a periodic point p such that

$$\lim_{s\to\infty} \sup_{x\in J} d\big(f^s(p), f^s(x)\big) = 0.$$

But this implies that c is contained in the orbit of p, and hence periodic, a contradiction. □

Lemma 3.2. *Let f be a map in $\mathcal{N}'$, and let c be a non-periodic recurrent critical point with a minimal ω-limit set. There exists $\delta > 0$, such that for any symmetric interval $I \ni c$ with $|I| < \delta$, if it is strictly nice or properly periodic, then the real box mapping $\mathbf{B}_I$ belongs to the class $\mathcal{F}_b$, where b is the number of critical points contained in $\omega(c)$.*

Proof. Let $\mathbf{B}_I : (\bigcup_{j=0}^m J_j) \cup (\bigcup_{i=1}^{b-1} I_i) \to \bigcup_{i=0}^b I_i$ be the real box mapping associated to I.

Since $\omega(c)$ is a compact set contained in D_I, there are only finitely many entry domains to I which intersect $\omega(c)$. Since each J_j is an entry domain to I, $m < \infty$.

If I is a properly periodic interval, then we have $J_0 = I_0$. If I is strictly nice, then for each $0 \le j \le m$, J_j is compactly contained in I, and the closures of the J_j's are pairwise disjoint.

Let $[c]$ denote the set of critical points which are contained in $\omega(c)$. Let $\delta_1 = d(\omega(c), \mathrm{Crit}(f) - \omega(c)) > 0$. By Lemma 3.1, if $|I|$ is sufficiently small, then each pull back of I has length less than δ_1. For any component M of $\mathrm{dom}(\mathbf{B}_I)$, let s be the entry time of M to $\bigcup_i I_i$, and let $0 \le i' \le b-1$ be such that $f^s(M) \subset I_{i'}$. Let us consider the chain $\{G_k\}_{k=0}^s$ with $G_s = I_{i'}$ and $G_0 \supset M$. For each $0 \le k \le s$, $|G_k| < \delta_1$. Since $G_k \cap \omega(c) \ne \emptyset$, $G_k \cap (\mathrm{Crit}(f) - [c]) = \emptyset$. Since s is the entry time of M to $\bigcup_i I_i$, $G_k \cap \bigcup_i I_i = \emptyset$ for each $1 \le k \le s-1$, and in particular, $G_k \cap [c] = \emptyset$. Consequently, $f^{s-1}|G_1$ is a diffeomorphism. Therefore, $\mathbf{B}_I$ has exactly b critical branches $J_0, I_1, \ldots, I_{b-1}$ each of which has a unique critical point. The orbits of these critical points under iterates of $\mathbf{B}_I$ are contained in $\omega(c) \cap (\bigcup_i I_i)$, and hence compactly contained in $\bigcup_i I_i$. □

We shall also use the following theorem of [37].

Theorem 3.2. *For any map $f \in \mathcal{N}'$, there exist $\rho > 1$ and $N \in \mathbb{N}$, such that if p is a periodic point of period $n \ge N$, then we have*

$$\big|(f^n)'(p)\big| \ge \rho.$$

3.2. The real Koebe principle. Let $J \Subset I$ be two intervals, and let L and R be the components of $I - J$. The *cross ratios* $A(I, J)$, $B(I, J)$ are defined to be

$$A(I, J) = \frac{|I||J|}{|L \cup J||R \cup J|}, \quad B(I, J) = \frac{|I||J|}{|L||R|}.$$

For any C^3 maps $h : I \to \mathbb{R}$ with $h' \neq 0$, the *Schwarzian derivative* is defined as

$$Sh = \frac{h'''}{h'} - \frac{3}{2}\left(\frac{h''}{h'}\right)^2 .$$

As is well known, a diffeomorphism with negative Schwarzian derivative expands the cross ratios ([35]). For a general smooth map, the distortion of cross ratios is described in the following lemma.

Lemma 3.3. (de Melo, van Strien, [36]). *Let $h : [0,1] \to [0,1]$ be a C^3 diffeomorphism. There exists a constant $C > 0$ such that for any $J \Subset I \subset [0,1]$, we have*

$$\exp\big(C|L||R|\big) \geq \frac{A\big(h(I), h(J)\big)}{A(I,J)} \geq \exp\big(-C|L||R|\big), \tag{6}$$

and

$$\exp\big(C|I|^2\big) \geq \frac{B\big(h(I), h(J)\big)}{B(I,J)} \geq \exp\big(-C|I|^2\big), \tag{7}$$

where L, R are the components of $I - J$.

Lemma 3.4. *Let $\delta > 0$ and $C \in (0,1]$ be constants. Let $h : T \to (-\delta, 1+\delta)$ be a C^1 diffeomorphism. Assume that for any intervals $J \Subset I \subset T$, we have*

$$\frac{B\big(h(I), h(J)\big)}{B(I,J)} \geq C.$$

Then for any $x, y \in T$ with $h(x), h(y) \in [0,1]$, we have

$$C^6\left(\frac{1+\delta}{\delta}\right)^2 \leq \frac{h'(x)}{h'(y)} \leq \frac{1}{C^6}\left(\frac{1+\delta}{\delta}\right)^2 .$$

Proof. This is a well-known lemma. For a proof, see [35] Theorem IV.1.2. □

Lemma 3.5. *For any map $f \in \mathcal{N}'$, and for any $\delta > 0$ and $N \in \mathbb{N}$, there are constants $\varepsilon = \varepsilon(\delta, N, f) > 0$, $\delta_1 = \delta_1(\delta) > 0$ and $K = K(\delta) > 1$ with the following property. Let $n \in \mathbb{N}$, and let $I \Supset J$ be intervals contained in $dom(f^n)$ such that $f^n|I$ is a diffeomorphism and $f^n(I) \supset (1+2\delta) f^n(J)$. Assume that*

$$\#\big\{0 \leq i \leq n-1 : f^i(I) \Supset f^n(J)\big\} \leq N \tag{8}$$

and that $\max_{0\le i\le n-1}|f^i(I)|\le\varepsilon$. *Then*

(1) $I\supset(1+2\delta_1)J$;
(2) *for any* $x, y\in J$,

$$\frac{(f^n)'(x)}{(f^n)'(y)}\le K.$$

Moreover, $\delta_1\to\infty$ *as* $\delta\to\infty$.

Proof. See [20] or Lemma 2.3 in [42]. □

Lemma 3.6. *Let* f *be a map in* $\mathcal{N}'$. *For any* $p, q\in\mathbb{N}$ *and any* $\delta>0$, *there exist constants* $\varepsilon=\varepsilon(f,\delta,p,q)>0$ *and* $\delta_1=\delta_1(\delta,p)>0$ *such that the following holds. Let* $\mathbb{G}=\{G_j\}_{j=0}^s$ *and* $\mathbb{G}'=\{G'_j\}_{j=0}^s$ *be chains such that* $\max_{0\le j\le s}|G'_j|\le\varepsilon$ *and* $G_j\subset G'_j$ *for any* $0\le j\le s$. *Assume that the order of the chain* $\mathbb{G}'$ *is at most* p *and that*

$$\#\left\{0\le j\le s : G'_j\Supset G_s\right\}\le q. \tag{9}$$

If $(1+2\delta)G_s\subset G'_s$, *then* $(1+2\delta_1)G_0\subset G'_0$. *Moreover, for a fixed* p, $\delta_1(\delta,p)\to\infty$ *as* $\delta\to\infty$.

Proof. See Proposition 2.2 in [42]. □

Remark 3.1. Before the work [20], to apply the real Koebe principle for smooth interval maps, one usually had to estimate the intersection multiplicity of the chains involved, which in practice is not always straightforward. As we shall see, the conditions (8) and (9) in the above lemmas are much easier to check in many cases. For instance, let $\{G'_j\}_{j=0}^s$ be a chain such that G'_s is a nice interval, and let G_s be a subinterval of G'_s. Assume that for some $x\in G_s$, we have $G_s\supset\mathcal{L}_x^r(G'_s)$. Then

$$\#\left\{0\le j\le s-1 : G'_j\Supset G_s\right\}\le r.$$

Indeed, if $0\le j_1<j_2<\cdots<j_n\le s-1$ be all the integers such that $G'_{j_i}\Supset G_s$, then $G_s\Subset G'_{j_1}\subset\mathcal{L}_x(G'_{j_2})\subset\cdots\subset\mathcal{L}_x^n(G'_s)$ and hence $n\le r$.

Remark 3.2. Keep the notation in the previous lemma. Let $s_1<s_2<\cdots<s_n$ be all the integers between 1 and $s-1$ such that G'_{s_i} contains a critical point, and let $s_0=0$, $s_{n+1}=s$. Then for any $0\le i\le n$, the map $f^{s_{i+1}-s_i-1}: G_{s_i+1}\to G_{s_{i+1}}$ is a diffeomorphism with uniformly bounded distortion (which depends only on δ and p). Thus $f^s|G_0$ can be written as the composition of at most p functions of the form $L\circ f$, with L being a diffeomorphism with uniformly bounded distortion. In this sense, we shall say that the map $f^n|G_0$ has (p,δ)-*uniformly good distortion,* or just *uniformly good distortion* when we do not specify the constants p and δ.

In [20], Kozlovski proved that for a smooth unimodal map with a non-flat critical point, the first entry map to a small neighborhood of the critical value has negative Schwarzian. This result has been extended to the multimodal case in [47].

Lemma 3.7. *Let f be an interval map in $\mathcal{N}'$. There exist neighborhoods U_i of the critical points so that whenever $f^n(x) \in U_i$ for some $x \in [0,1]$ and some $n \in \mathbb{N}$, then the Schwarzian derivative of f^{n+1} at x is negative:*

$$Sf^{n+1}(x) < 0.$$

Proof. See Theorem B.3 in [47]. □

As we shall work on neighborhoods of critical points rather than the critical values, it is often more convenient for us to use Lemmas 3.5 and 3.6 rather than the previous one. Actually, in Sect. 5.6, we shall show that if f has essentially bounded geometry at a critical point c, then the real box mapping associated to a small nice interval $I \ni c$ has negative Schwarzian derivative.

Lemma 3.8. *Let $f \in \mathcal{F}_b$ and let $c \in Crit(f)$. Let I be a nice interval, and let $I' \supset I$ be an open interval with $(I' - I) \cap \omega(c) = \emptyset$. Let J be a component of D_I with $J \cap \omega(c) \neq \emptyset$, and let s be the entry time of J to I. Then the chain $\mathbb{G}' = \{G'_j\}_{j=0}^s$ with $G'_s = I'$ and $G'_0 \supset J$ has order $\leq b$, and intersection multiplicity ≤ 4.*

Proof. Let $\mathbb{G} = \{G_j\}_{j=0}^s$ be the chain with $G_s = I$ and $G_0 = J$. Then these intervals G_j, $0 \leq j \leq s-1$, are pairwise disjoint, and thus the chain $\mathbb{G}$ has order at most b. Since $I' - I$ is disjoint from $\omega(c)$, so is $G'_j - G_j$ for any $0 \leq j \leq s$. As $\omega(c) \supset \mathrm{Crit}(f)$, it follows that the chain $\mathbb{G}'$ has the same order as $\mathbb{G}$.

Assume that the intersection multiplicity of $\mathbb{G}'$ is ≥ 5. Then we can find $0 \leq j_1 < j_2 < j_3 \leq s-1$ such that there is a point $x \in \bigcap_{i=1}^3 G'_{j_i} - \bigcup_{i=1}^3 G_{j_i}$. Choose $1 \leq k < m \leq 3$ such that G_{j_k} and G_{j_m} are on the same side of x. Then $G'_{j_k} \supset G_{j_m}$ or $G'_{j_m} \supset G_{j_k}$. Since $G_0 = J$ intersects $\omega(c)$, it follows that $(G'_{j_i} - G_{j_i}) \cap \omega(c) \neq \emptyset$ for $i = k$ or m, a contradiction. □

3.3. An improved macroscopic Koebe principle. In [47], the following result was derived from the real bounds:

Theorem 3.3 (Improved macroscopic Koebe principle). *Let f be a map in $\mathcal{N}'$. For each $\xi > 0$, there exists $\xi' > 0$ with the following property. Let I be a nice interval and J a subinterval of I such that $(1+2\xi)J \subset I$. Then for any $x \in [0,1]$ and any $k \in \mathbb{N}$ with $f^k(x) \in J$,*

$$(1+2\xi')Comp_x\big(f^{-k}(J)\big) \subset \mathcal{L}_x(I).$$

Proof. This is exactly Theorem B.1 in [47] if $x \in I$. In the case that $x \notin I$, let $k' \in \mathbb{N}$ be the entry time of x to I. Then by that theorem, $\mathrm{Comp}_{f^{k'}(x)}(f^{-(k-k')}(J))$ is well inside I. Applying Lemma 3.6 to the chain corresponding to the first entry of x to I, we complete the proof. □

We should remark that the constant ξ' depends only on ξ and the number of critical points of f, provided that each interval in D_I is sufficiently small.

We shall need the following result in Sect. 9, which was proved in [42] by a Yoccoz type τ-function argument. Using the previous theorem, we can now give it a very short proof.

Theorem 3.4. *Let f be a map in $\mathcal{N}'$ and let c be a non-periodic recurrent critical point. If $\omega(c)$ is non-minimal, then f has large bounds at c.*

Proof. It is not difficult to see that f is not infinitely renormalizable at c, using the no wandering interval theorem (Theorem 3.1). Let $I := I^0 \ni c$ be a small symmetric nice interval such that I does not contain any properly periodic interval of f, and let $I^{n+1} = \mathcal{L}_c(I^n)$ for all $n \geq 0$. Then $|I^n| \to 0$ as $n \to \infty$.

Moreover, by replacing I with a smaller symmetric nice interval if necessary, we may assume that D_I have infinitely many components $J_0 \ni c$, $J_1, J_2, \ldots$ intersecting $orb(c) \cap I$. For any $i \geq 1$, let $r_i \in \mathbb{N}$ be the entry time of J_i to I, let $n_i \in \mathbb{N}$ be the entry time of c to J_i, and let $A_i = \mathcal{L}_c(J_i)$. Then $|A_i| \to 0$ as $i \to \infty$.

For any $x \in orb(c) \cap A_i$, $f^j(x) \notin A_i$ for all $1 \leq j \leq n_i$. Thus $f^{n_i}(\mathcal{L}_x(A_i)) \subset \mathcal{L}_{f^{n_i}(x)}(A_i)$. By Lemma 3.6, if $\mathcal{L}_{f^{n_i}(x)}(A_i)$ is deep inside J_i, then so is $\mathcal{L}_x(A_i)$ in A_i.

Let $m(1) < m(2) < \cdots$ be all the positive integers such that $R_{I^{m(k)-1}}(c) \notin I^{m(k)}$. By Theorem 2.1, for any k, $|I^{m(k)+1}|/|I^{m(k)}|$ is uniformly bounded away from 1. Take a large $i \in \mathbb{N}$. Then there are many n's such that $I^{n+1} \supset A_i$ and $|I^n|/|I^{n+1}|$ is uniformly bounded away from 1. Applying Theorem 3.3, we see that $\mathcal{L}_{f^{n_i}(x)}(A_i)$ is deep inside $\mathcal{L}_{f^{n_i}(x)}(I) = J_i$. So A_i is a C-nice interval for a large C. The proof is completed. □

3.4. Measure of the postcritical set. Later on we shall need to estimate the total length of a sequence of intervals, where the following result will be convenient to us.

Proposition 3.5. *Let $f \in \mathcal{F}_b$ and $c \in Crit(f)$. Then $\omega(c)$ has (one-dimensional) Lebesgue measure zero.*

For a proof of this proposition, see [47]. The infinitely renormalizable case was proved earlier in [4]. The finitely renormalizable case was also claimed in [49], but the proof seems incomplete. The proposition will be used in the following form:

Corollary 3.6. *Let $f \in \mathcal{F}_b$ and $c \in Crit(f)$. For any $\varepsilon > 0$, there is a $\delta > 0$ with the following property. Let $\{G_i\}_{i=0}^s$ be a chain with intersection multiplicity N. If $G_0 \cap \omega(c) \neq \emptyset$ and if $|G_s| \leq \delta$, then*

$$\sum_{i=0}^{s} |G_i| < N\varepsilon.$$

Proof. Since G_s is contained in a small neighborhood of a point in $\omega(c)$, any pull back of this interval has a small length, by Lemma 3.1. Thus, $\max_j |G_j|$ is small, and so $\bigcup_j G_j$ is contained in a small neighborhood of $\omega(c)$. Since $\omega(c)$ has measure zero, a small neighborhood of $\omega(c)$ has a small measure. The corollary follows. □

4. Essentially bounded geometry and large bounds

In this section, we begin to analyze the geometry of the postcritical set of a smooth interval map f. The goal is to prove Theorems 1 and 1'. As we have mentioned before, Theorem 1' will also play a crucial role in our further geometrical analysis for maps without large bounds in the next section.

For a nice open set K, let $\mathcal{M}(K)$ be the collection of all intervals which are pull backs of components of K. Then any two intervals in $\mathcal{M}(K)$ are either disjoint, or nested, i.e., one is contained in the other.

Definition 4.1. Let $J \subset I$ be intervals in $\mathcal{M}(K)$. The *(combinatorial) depth of J in I*, denoted by $\mathrm{Dep}(I, J)$, is the minimal non-negative integer d such that there exists a point $x \in J$ with $J \supset \mathcal{L}_x^d(I)$.

For instance, $\mathrm{Dep}(I, I) = 0$, and if J is a return domain to I then $\mathrm{Dep}(I, J) = 1$ as long as $J \neq I$. Note that if $\mathrm{Dep}(I, J) = d$, then for any $y \in J$, we have $\mathcal{L}_y^d(I) \subset J$. By Theorem 3.3, Theorem 1' follows from the following:

Proposition 4.1. *Let $d \in \mathbb{N}$ and let $C > 0$. For any $f \in \mathcal{N}'$, there exist constants $\xi = \xi(\#Crit(f), C, d) > 0$ and $\varepsilon = \varepsilon(f, C, d) > 0$ such that the following holds. Let K be a nice open set such that every interval in $\mathcal{M}(K)$ has length less than ε. Let $I \supset J$ be intervals belonging to $\mathcal{M}(K)$ such that $Dep(I, J) \leq d$ and $I \supset (1 + 2\xi)J$. Then for any $x \in D_J$, we have*

$$\mathcal{L}_x(I) \supset (1 + 2C)\mathcal{L}_x(J). \tag{10}$$

If the entry time s of x to J coincides with that to I, then the chain $\{I_i\}_{i=0}^s$ with $I_s = I$ and $I_0 \ni x$ has bounded order, and thus the proposition follows immediately from the real Koebe principle (Lemma 3.6). In general, the entry times can be different, and we shall consider all the entries of x to I before the first one to J. If the pull backs of I corresponding to these earlier entries are monotone, then we are again able to use the real Koebe principle directly to conclude the proof. To deal with the critical return branches, we shall show that we can pull back a big space along an arbitrarily long central cascade without too much loss, while at each time passing a central cascade we get a definite amount of bound by results on real bounds. The formal proof of this proposition will be organized by induction on the cardinality of the critical points involved in the pull backs of I corresponding to the earlier entries. It is rather complicated, and will occupy most of the rest of this section. Let us first show how Theorem 1 follows.

Proof of Theorem 1. Assume that f does not have essentially bounded geometry at a non-periodic recurrent critical point c. Then by definition, there is a sequence $\{T_i\}_{i=1}^{\infty}$ of symmetric nice intervals which contain c, such that $|T_i|/|T_i^1| \to \infty$ as $i \to \infty$, where $T_i^1 = \mathcal{L}_c(T_i)$. Note that $|T_i| \to 0$ as $i \to \infty$. By Proposition 4.1, for any $C > 0$, T_i^1 is a C-nice interval for i sufficiently large. This proves that f has large bounds. □

4.1. Preparatory lemmas.

Lemma 4.1. *Let $d \in \mathbb{N}$ and let K be an arbitrary nice open set. Let I, J, H be intervals in $\mathcal{M}(K)$ such that $J \subset I$ and $H \subset D_I$. Let $x \in H \cap D_J$, and let s be the entry time of x to J. Assume that $f^i(x) \notin H$ for all $1 \leq i \leq s-1$, and that $J \supset \mathcal{L}^d_{f^s(x)}(I)$. Then*

$$\mathcal{L}_x(J) \supset \mathcal{L}_x^d(H).$$

Proof. We first prove the lemma for $d = 1$. Let $\{J_i\}_{i=0}^{s}$ be the chain with $J_s = J$ and $J_0 \ni x$. Then $J_i \cap H = \emptyset$ for all $1 \leq i \leq s-1$. Indeed, for $1 \leq i \leq s-1$, $J_i \subset H$ implies $f^i(x) \in H$, and if $H \subset J_i$ then $f^{s-i}(H) \subset J$, which implies $f^{s-i}(x) \in J$. Both cases contradict the hypothesis. Now assume $\mathcal{L}_x(H) \not\subset \mathcal{L}_x(J)$, and consider $z \in (\partial \mathcal{L}_x(J)) \cap \mathcal{L}_x(H)$. Since $f^i(z) \notin H$ for all $1 \leq i \leq s-1$, $f^s(z)$ is contained in $H \cup D_H \subset D_I$. But $f^s(z) \in \partial J$, contradicting the hypothesis $J \supset \mathcal{L}_{f^s(x)}(I)$.

Now let us consider the general case. Let $I_i = \mathcal{L}^i_{f^s(x)}(I)$ for each $i \geq 0$. Let $H_0 = H$ and $H_i = \mathcal{L}_x(I_i)$ for each $i \geq 1$. Let $0 \leq d' \leq d-1$ be an integer such that

$$I \supset I_1 \supset I_2 \supset \cdots \supset I_{d'} \supset J \supset I_{d'+1}.$$

Then we have $H_{i+1} \supset \mathcal{L}_x(H_i)$ for all $i \geq 0$ and also $\mathcal{L}_x(J) \supset H_{d'+1} \supset \mathcal{L}_x(H_{d'})$. Therefore,

$$\mathcal{L}_x(J) \supset \mathcal{L}_x^{d'+1}(H) \supset \mathcal{L}_x^d(H).$$

□

Lemma 4.2. *There exist $\delta_1 > 0$ and N which depend only on $\#Crit(f)$ with the following property. Let I be a small symmetric nice interval containing $c \in Crit(f)$ and let s be the return time of c to I. Then one of the following holds:*

(1) *I contains the* δ_1*-neighborhood of* I^1*;*
(2) *the intersection multiplicity of the chain* $\{T_i\}_{i=0}^s$ *with* $T_s = (1+2\delta_1)I$ *and* $T_0 \supset I^1$ *is at most* N*.*

Proof. This follows from Lemmas 2 and 3 of [47]. □

Lemma 4.3. *There exist constants* $\delta_2 > 0$ *and* $C > 1$ *depending only on* $\#Crit(f)$ *with the following property. Let* I *be a small symmetric nice interval containing* $c \in Crit(f)$*. If* $|I|/|I^1| < 1 + 2\delta_2$*, then we have*

$$\left|(f^s)'(x)\right| \le C \tag{11}$$

for all $x \in I^1$*, where* s *is the return time of* c *to* I*.*

Proof. Let $\delta_2 = \delta_1/2$, where $\delta_1 > 0$ is as in the previous lemma. Then we are in the latter case of that lemma, and thus $f^s|I^1$ can be written as the composition of a bounded number of functions of the form $L \circ f$, where L has uniformly bounded distortion. Since

$$\frac{\left|f^s(I^1)\right|}{|I^1|} \le \frac{|I|}{|I^1|} < 1 + 2\delta_2,$$

the lemma follows. □

4.2. Proof of Proposition 4.1. The proof will be done by induction on the cardinality of the critical set $\mathrm{Crit}(I, J, x)$ defined as follows.

Let $s \in \mathbb{N}$ be the entry time of x to J, let $n_0 = 0$ and let $0 < n_1 < \cdots < n_k = s$ be all the positive integers such that $f^{n_i}(x) \in I$, $1 \le i \le k$. For any $0 \le i \le k-1$, consider the chain $\{G_j^i\}_{j=0}^{n_{i+1}-n_i}$ with $G^i_{n_{i+1}-n_i} = I$ and $G_0^i \ni f^{n_i}(x)$. Let

$$\mathrm{Crit}(I, J, x) = \mathrm{Crit}(f) \cap \left(\bigcup_{i=0}^{k-1} \bigcup_{j=0}^{n_{i+1}-n_i-1} G_j^i \right).$$

We are going to prove the following two inductive statements.

Statement N**.** *For any* $C > 0$ *and* $d \in \mathbb{N}$*, there exist* $\xi_N = \xi_N(C, d) > 0$ *and* $\varepsilon = \varepsilon(f, N, C, d)$ *with the following property. Let* K *be a nice open set, let* I, J *be intervals in* $\mathcal{M}(K)$ *with* $I \supset J$ *and* $Dep(I, J) \le d$*, and let* $x \in D_J$*. Assume that the following hold:*

- *every interval in* $\mathcal{M}(K)$ *has length less than* ε*;*
- $(1+2\xi_N)J \subset I$*, and* $\#Crit(I, J, x) \le N$*.*

Then (10) holds.

Statement N'**.** *For any* $C > 0$ *and* $d \in \mathbb{N}$*, there exist* $\xi'_N = \xi'_N(C, d) > 0$ *and* $\varepsilon' = \varepsilon'(N, f, C, d) > 0$ *with the following property. Let* K, I, J, x *be as above. Assume that the following hold:*

- *every interval in $\mathcal{M}(K)$ has length less than ε';*
- $(1+2\xi'_N)J \subset I$*, and* $\#Crit(I, J, x) = N$*;*
- *there exists* $c \in Crit(I, J, x)$ *such that I is symmetric and* $I \ni c$*.*

Then (10) holds.

Obviously, Statement N' follows from Statement N. We shall prove Statement 0, and (Statement N'+ Statement $N-1 \Longrightarrow$ Statement N) in this subsection, and prove (Statement $(N-1) \Longrightarrow$ Statement N') in the next subsection.

Proof of Statement 0. Since $\mathrm{Crit}(I, J, x) = \emptyset$, the chain $\{G_j\}_{j=0}^s$ with $G_s = I$ and $G_0 \ni x$ is monotone. Since $\#\{0 \le j < s : G_j \supsetneq J\} \le d$ (c.f. Remark 3.1), by Lemma 3.5, $\mathcal{L}_x(J)$ is deep inside $G_0 \subset \mathcal{L}_x(I)$, provided that J is deep inside I. □

Proof of (Statement N'+ Statement $(N-1) \Longrightarrow$ Statement N). Let I, J, x be as in Statement N. Let $n_0 = 0, n_1, \ldots, n_k = s$ be as above. Then there exist $0 \le i \le k-1$ and $0 \le j \le n_{i+1} - n_i - 1$ such that G^i_j contains a critical point of f, say c'. Let us choose i, j with this property and such that (i, j) is maximal in the lexicographical order. Let $I' = G^i_j$ and $J' = \mathcal{L}_{f^{n_i+j}(x)}(J)$. Since $\{G_m\}_{m=n_i+j+1}^s$ with $G_s = I$ and $G_{n_i+j+1} \ni f^{n_i+j+1}(x)$ is a monotone chain, by Lemma 3.5, J' is deep inside $G_0 \subset I'$, provided that J is sufficiently deep inside I. If $n_i + j = 0$, then this proves what we want. So assume that $n_i + j \neq 0$. To prove that $\mathcal{L}_x(J)$ is deep inside $\mathcal{L}_x(I)$, it suffices to prove that $\mathcal{L}_x(J')$ is deep inside $\mathcal{L}_x(I')$. By Lemma 4.1, $\mathrm{Dep}(I', J') \le \mathrm{Dep}(I, J) = d$. Clearly we have $\mathrm{Crit}(I', J', x) \subset \mathrm{Crit}(I, J, x)$. If these two sets do not coincide, then $\#\mathrm{Crit}(I', J', x) \le N-1$, and so Statement $(N-1)$ applies. Otherwise, $\mathrm{Crit}(I', J', x) \ni c'$, and Statement N' applies. □

4.3. Induction step. In this subsection, we shall prove

$$\text{Statement } N-1 \implies \text{Statement } N'.$$

So let us assume that I is a symmetric nice interval, $\#\mathrm{Crit}(I, J, x) = N$, and $\mathrm{Crit}(I, J, x) \ni c$, where c is the critical point in I.

By definition, the *height* χ_I of I is the number of positive integers m such that $R_{I^{m-1}}$ displays a non-central return, i.e., $R_{I^{m-1}}(c) \notin I^m$. Note that if $\chi_I = 0$, then the first return of c to I^0 enters I^j for all $j \ge 0$, which implies that $\omega(c) \cap (I^0 - I^j) = \emptyset$, and that $\bigcap_j I^j$ is a periodic interval.

If $J \ni c$, we define $e(I, J) = \infty$. Otherwise, we define $e(I, J)$ to be the non-negative integer such that $J \subset I^{e(I,J)} - I^{e(I,J)+1}$. Note that in the latter case,

$$e(I, J) \le \mathrm{Dep}(I, J) - 1.$$

Furthermore, define $m(1)$ to be the minimal positive integer such that $R_{I^{m(1)-1}}(c) \notin I^{m(1)}$ if $\chi_I \ge 1$, and define $m(1) = \infty$ otherwise.

We shall fix a constant $C > 0$ throughout this subsection.

Lemma 4.4. *There exists $C_1 = C_1(C) > 0$, and for any $\Sigma > 0$ there exists $\varphi_1(\Sigma) > \Sigma$ with the following property. Assume that $(1 + 2\varphi_1(\Sigma))J \subset I$, and that (10) fails. Then the following hold:*

- $|I^0| \le C_1|I^1|$;
- *there exists $1 \le s_1 < s$ such that $f^{s_1}(x) \in I^1$, and such that*

$$Dep\big(I^1, \mathcal{L}_{f^{s_1}(x)}(J)\big) \le d,\ Crit\big(I^1, \mathcal{L}_{f^{s_1}(x)}(J), x\big) = Crit(I, J, x);$$

- $(1 + 2\Sigma)\mathcal{L}_{f^{s_1}(x)}(J) \subset I^1$.

Proof. Let $0 \le s_1 < s$ be maximal such that $f^{s_1}(x) \in I^1$. (Such an integer exists because $\mathrm{Crit}(I, J, x) \ni c$.) Then $\mathrm{Crit}(I, J, f^{s_1+1}(x)) \subset \mathrm{Crit}(I, J, x) - \{c\}$. Assume that I contains a large neighborhood of J. Then it follows from Statement $N - 1$ that $\mathcal{L}_{f^{s_1+1}(x)}(I)$ contains a large neighborhood of $\mathcal{L}_{f^{s_1+1}(x)}(J)$, and thus, $I^1 = \mathcal{L}_{f^{k_1}(x)}(I)$ contains a large neighborhood of $J^1 := \mathcal{L}_{f^{s_1}(x)}(J)$. If $s_1 = 0$, then we have (10). So assume that $s_1 > 0$.

Note that $\mathrm{Crit}(I, I^1, x) \subset \mathrm{Crit}(I, J, x) - \{c\}$. If $I^0 \supset (1 + 2\xi_{N-1})I^1$, where ξ_{N-1} is as in Statement $N - 1$, then by that statement, $(1+2C)\mathcal{L}_x(I^1) \subset \mathcal{L}_x(I)$. Since $\mathcal{L}_x(J) \subset \mathcal{L}_x(I^1)$, (10) follows, which contradicts the hypothesis. Thus, $|I^0|/|I^1|$ is bounded from above.

By Lemma 4.1, $\mathrm{Dep}(I^1, J^1) \le \mathrm{Dep}(I, J) \le d$. Clearly, $\mathrm{Crit}(I^1, J^1, x) \subset \mathrm{Crit}(I, J, x)$. If these two sets do not coincide, then $\#\mathrm{Crit}(I^1, J^1, x) < N$, and so by Statement $N - 1$, $\mathcal{L}_x(I) \supset \mathcal{L}_x(I^1)$ contains a large neighborhood of $\mathcal{L}_x(J^1) = \mathcal{L}_x(J)$, which is ruled out by the assumption. So $\mathrm{Crit}(I^1, J^1, x) = \mathrm{Crit}(I, J, x)$. □

Lemma 4.5. *For any $\Sigma > 0$, there exists $\varphi_2(\Sigma) > \Sigma$ such that if $(1 + 2\varphi_2(\Sigma))J \subset I$ and if (10) fails, then for any $1 \le i \le d + 1$, the following hold:*

1. $|I^{i-1}| \le C_1|I^i|$, *where $C_1 = C_1(C)$ is as in Lemma 4.4;*
2. *there exists $1 \le s_i < s$ such that $f^{s_i}(x) \in I^i$, and such that*

$$Dep\big(I^i, \mathcal{L}_{f^{s_i}(x)}(J)\big) \le d,\ Crit\big(I^i, \mathcal{L}_{f^{s_i}(x)}(J), x\big) = Crit(I, J, x);$$

3. $(1 + 2\Sigma)\mathcal{L}_{f^{s_i}(x)}(J) \subset I^i$.

Proof. Define $\varphi_2(\Sigma) = \varphi_1^{d+1}(\Sigma)$. Then by Lemma 4.4, we can prove inductively that for any $1 \le i \le d + 1$, there exists $0 < s_i < s$ such that

1. $|I^{i-1}| \le C_1|I^i|$,
2. $f^{s_i}(x) \in I^i$, $\mathrm{Dep}(I^i, \mathcal{L}_{f^{s_i}(x)}(J)) \le d$ and $Crit(I^i, \mathcal{L}_{f^{s_i}(x)}(J), x) = Crit(I, J, x)$;
3. $(1 + 2\varphi_1^{d+1-i}(\Sigma))\mathcal{L}_{f^{s_i}(x)}(J) \subset I^i$.

This proves the lemma. □

Lemma 4.6. *There exist $C_2, C_3 > 0$ depending on C such that the following holds. Assume that $(1+2C_3)J \subset I$ and that (10) is false. If $m(1) > d$, then for any $0 \le i \le d$,*

$$|I^i - I^{i+1}| \ge |I^0 - I^i|/C_2.$$

Proof. If $|I^0|/|I^1| < 1 + 2\delta_2$, then by Lemma 4.3, $|(f^t)'(y)|$ is bounded from above on I^1, where t is the return time of c to I. Since $f^{jt}(I^i - I^{i+1})$ contains a component of $I^{i-j} - I^{i-j+1}$ for any $j \le i \le m(1) - 1$, this lemma follows.

Now assume that $|I^0|/|I^1| \ge 1 + 2\delta_2$. Then by Theorem 2.1, I^i contains a definite neighborhood of I^{i+1} for all $i \le d$. In particular, $|I^i - I^{i+1}|$ is comparable to $|I^i|$. Taking $C_3 = \varphi_2(1)$ and applying Lemma 4.5, we see that all these $|I_i|$ are comparable to $|I^0|$. The lemma follows. □

Lemma 4.7. *For any $\Sigma > 0$, there exists $\varphi_3(\Sigma) > \Sigma$ with the following property. Assume that $(1 + 2\varphi_3(\Sigma))J \subset I$ and that (10) is false. Then there exists $p_1 < s$ such that one of the following holds.*

(1) *$m(1) < \infty$, $(1 + 2\Sigma)\mathcal{L}_{f^{p_1}(x)}(J) \subset I^{m(1)}$, and $Dep\big(I^{m(1)}, \mathcal{L}_{f^{p_1}(x)}(J)\big) \le d$;*
(2) *$e(I, J) < \infty$, and there exists $k' \ge 1$ such that $f^{p_1}(x) \in I^{k'} - I^{k'+e(I,J)}$, and*

$$(1 + 2\Sigma)\mathcal{L}_{f^{p_1}(x)}(J) \subset I^{k'}, \text{ and } Dep\big(I^{k'}, \mathcal{L}_{f^{p_1}(x)}(J)\big) \le d.$$

Proof. Let $M > C_2^d$ be a large constant, and assume that $I \supset (1 + 2M)J$.

First let us show that $J \not\ni c$. Otherwise, from $\text{Dep}(I, J) \le d$ we obtain that $J \supset I^d$. But Lemma 4.6 asserts that $|I|/|I^d| \le C_2^d$; a contradiction. It follows that $e := e(I, J) \le d - 1$ and $J \subset I^e - I^{e+1}$.

If $m(1) \le d$, then taking p_1 to be $s_{m(1)}$ as in Lemma 4.5, we see that the first alternative of this lemma holds. So let us assume that $m(1) > d$. Let $k \ge k_0 \ge \max\{1, k - m(1) + e\}$ be the minimal integer such that

$$f^{n_{k-1}}(x) \in I^{e+1}, \cdots, f^{n_{k_0}}(x) \in I^{k-k_0+e}.$$

Let $J_0 = J$ and let $J_i = \mathcal{L}_{f^{n_{k-i}}(x)}(J)$ for each $0 < i \le k$. Note that for $0 \le i \le k - k_0 - 1$, we have

$$J_i \subset I^{i+e} - I^{i+e+1}.$$

Let T be the component of $I - I^{e+2}$ which contains J. Then provided that we have chosen the constant M sufficiently large, T contains a large neighborhood of J because by Lemma 4.6, $I^{e+1} - I^{e+2}$ is not much smaller than $I^0 - I^{e+1}$. Consider the chain $\{T_i\}_{i=0}^{s-n_{k_0}}$ with $T_{s-n_{k_0}} = T$ and $T_0 \supset J_{k-k_0}$. Obviously, the chain has intersection multiplicity $e + 2$, and hence its order is at most $(e + 2)\#\text{Crit}(f) \le (d + 1)\#Crit(f)$. By Lemma 3.6, T_0

contains a large neighborhood of J_{k-k_0}. In particular, I^{k-k_0} contains a large neighborhood of J_{k-k_0}. Note that $\mathrm{Dep}(I^{k-k_0}, J_{k-k_0}) \le d$ by Lemma 4.1.

By the definition of k_0, we have the following three possibilities:

1) $k_0 = 1$. Let us show that this case cannot happen if M has been chosen sufficiently large. In fact, in this case, the first entry of x to I^{k-k_0} coincides with that to J_{k-k_0}, and so by Lemma 3.6, $\mathcal{L}_x(I) \supset \mathcal{L}_x(I^{k-k_0})$ contains the C-neighborhood of $\mathcal{L}_x(J) = \mathcal{L}_x(J_{s-s_0})$, a contradiction.

2) $k_0 = k - m(1) + e$. Then $k - k_0 = m(1) - e$ and $m(1) < \infty$. As $e < d$, applying Lemma 4.5 to the triple $(I^{m(1)-e}, J_{m(1)-e}, x)$, we conclude that the first alternative of this lemma holds.

3) $k_0 > \max(1, k-m(1)+e)$ and $f^{n_{k_0-1}}(x) \notin I^{k-k_0+e+1}$. Let $k_1 \le k-k_0+e$ be such that

$$f^{n_{k_0-1}}(x) \in I^{k_1} - I^{k_1+1}.$$

Let us first deal with the case

$$k_1 \le k - k_0. \tag{12}$$

In this case, $I' := \mathcal{L}_{f^{n_{k_0-1}}(x)}(I^{k-k_0}) \subset I^{k_1} - I^{k_1+1}$. Let $J' = \mathcal{L}_{f^{n_{k_0-1}}(x)}(J_{k-k_0})$. Then by Lemma 3.6, I' contains a large neighborhood of J'. If $\mathrm{Crit}(I', J', x) \not\ni c$, then $\#\mathrm{Crit}(I', J', x) < N$, and hence by Statement $N-1$, $\mathcal{L}_x(I) \supset \mathcal{L}_x(I')$ contains a large neighborhood of $\mathcal{L}_x(J) = \mathcal{L}_x(J')$, a contradiction. Thus $\mathrm{Crit}(I', J', x) \ni c$. Note that this implies that $c \in D_{I'}$. Since $R_I^i(c) \notin I^{k_1} - I^{k_1+1}$ for any $i < m(1) - k_1$, we have $m(1) < \infty$ and $\mathcal{L}_c(I') \subset I^{m(1)}$. By the definition of $\mathrm{Crit}(I', J', x)$, there exists a maximal integer p_1 with $0 \le p_1 < n_{k_0-1}$ and such that $f^{p_1}(x) \in I^{m(1)}$. Applying Statement $N-1$ to the triple $(I', J', f^{p_1+1}(x))$, we see that $\mathcal{L}_{f^{p_1+1}(x)}(J')$ is deep inside $\mathcal{L}_{f^{p_1+1}(x)}(I')$, from which it follows that $\mathcal{L}_{f^{p_1}(x)}(J) = \mathcal{L}_{f^{p_1}(x)}(J')$ is deep inside $\mathcal{L}_{f^{p_1}(x)}(I')$. As $\mathcal{L}_{f^{p_1}(x)}(I') \subset I^{m(1)}$, the first alternative of this lemma holds. This proves the lemma under the assumption (12).

If (12) is false, then $k - k_0 + e \ge k_1 \ge k - k_0 + 1$, and thus

$$f^{n_{k_0-1}}(x) \in I^{k-k_0+1} - I^{k-k_0+e(I,J)+1}.$$

Setting $p_1 = n_{k_0-1}$ and $k' = k-k_0+1$, the second alternative of this lemma holds. □

Lemma 4.8. *For any $\Sigma > 0$, there exists $\varphi_4(\Sigma) > \Sigma$ with the following property. Assume that $(1 + 2\varphi_4(\Sigma))J \subset I$ and that (10) is false. Then $m(1) < \infty$ and there exists $q_1 < s$ such that*

$$(1 + 2\Sigma)\mathcal{L}_{f^{q_1}(x)}(J) \subset I^{m(1)}, \text{ and } Dep\big(I^{m(1)}, \mathcal{L}_{f^{q_1}(x)}(J)\big) \le d.$$

Proof. This statement follows from Lemma 4.7 by induction on $e(I, J)$. Note that if $e(I, J) = 0$, then certainly the second case of that lemma cannot happen. This proves the starting step. For the induction step, we observe that in the second case of that lemma, $\mathrm{Crit}(I^{k'}, \mathcal{L}_{f^{p_1}(x)}(J), x) \subset \mathrm{Crit}(I, J, x)$, and $e(I^{k'}, \mathcal{L}_{f^{p_1}(x)}(J))$ is either ∞ or $< e(I, J)$. □

Proof of (Statement $N - 1 \Longrightarrow$ Statement N'). Let l be a large positive integer (to be determined below). Let us assume that I contains the $\varphi_4^l(C)$-neighborhood of J, and prove that (10) holds. Arguing by contradiction, assume that (10) fails. Then by Lemma 4.8, we can find positive integers $m(1)$ and $q_1 < s$ such that

- $R_{I^{m(1)-1}}(c) \notin I^{m(1)}$,
- $(1 + 2\varphi_4^{l-1}(C))\mathcal{L}_{f^{q_1}(x)}(J) \subset I^{m(1)}$,
- $\mathrm{Dep}(I^{m(1)}, \mathcal{L}_{f^{q_1}(x)}(J)) \le d$.

Clearly, $\mathrm{Crit}(I^{m(1)}, \mathcal{L}_{f^{q_1}(x)}(J), x) \subset \mathrm{Crit}(I, J, x)$. We may assume that these two sets coincide, for otherwise, $\#\mathrm{Crit}(I^{m(1)}, \mathcal{L}_{f^{q_1}(x)}(J), x) < N$ and Statement $N - 1$ implies that the C-neighborhood of $\mathcal{L}_x(J) = \mathcal{L}_x(\mathcal{L}_{f^{q_1}(x)}(J))$ is contained in $\mathcal{L}_x(I^{m(1)}) \subset \mathcal{L}_x(I)$; a contradiction. So we can apply Lemma 4.8 to the triple $(I^{m(1)}, \mathcal{L}_{f^{q_1}(x)}(J), x)$, and obtain positive integers $m(2) > m(1)$ and $q_2 < q_1$ such that

- $R_{I^{m(2)-1}}(c) \notin I^{m(2)}$;
- $(1 + 2\varphi_4^{l-2}(\Sigma))\mathcal{L}_{f^{q_2}(x)}(J) \subset I^{m(2)}$;
- $\mathrm{Dep}(I^{m(2)}, \mathcal{L}_{f^{q_2}(x)}(J)) \le d$.

Applying the argument to the triple $(I^{m(2)}, \mathcal{L}_{f^{q_2}(x)}(J), x)$, and so on, we find $m(l) > m(l-1) > \cdots > m(1) > 0$ and $q_1 > q_2 > \cdots q_l > 0$ such that for all $1 \le i \le l$,

$$R_{I^{m(i)-1}}(c) \notin I^{m(i)}, \text{ and } \mathcal{L}_{f^{q_i}(x)}(J) \subset I^{m(i)}.$$

By Theorem 2.1, for each $1 \le i \le l - 1$, $|I^{m(i)}|/|I^{m(i)+1}|$ is uniformly bounded away from 1, and thus by the Theorem 3.3, $\mathcal{L}_x(I^{m(i)})$ contains a definite neighborhood of $\mathcal{L}_x(I^{m(i)+1})$. In particular, if l has been chosen large enough, then

$$(1 + 2C)\mathcal{L}_x(J) \subset (1 + 2C)\mathcal{L}_x(I^{m(l)}) \subset \mathcal{L}_x(I),$$

a contradiction. The proof is completed. □

5. Essentially bounded geometry

The goal of this section is to give a more detailed analysis of the geometry of a map $f \in \mathcal{F}_b$ which does not have large bounds, and justify the name of "essentially bounded geometry".

We first study the properties of a long central cascade. Recall that a *central cascade* is a finite sequence of symmetric nice intervals

$$I^0 \supset I^1 \supset \cdots \supset I^m, \tag{13}$$

such that $I^{i+1} = \mathcal{L}_c(I^i)$ for each $0 \le i \le m-1$, where c is the critical point contained in I^0, and such that $R_{I^i}(c) \in I^{i+1}$ for each $0 \le i \le m-2$. Such a central cascade is called *maximal* if $R_{I^{m-1}}$ displays a non-central return, i.e., $R_{I^{m-1}}(c) \notin I^m$. We say that the central cascade is of *saddle node type* if $R_{I^0}|I^1$ has all the critical points in I^m, and does not have a fixed point.

Definition 5.1. For a symmetric nice interval I containing $c \in \mathrm{Crit}(f)$, we define the *scaling factor* to be

$$\lambda_I = \lambda_I^f = \frac{|I|}{\left|\mathcal{L}_c^f(I)\right|}.$$

For a nice open set K with $K \cap \omega(c) \neq \emptyset$, we define the *limit scaling factor* to be

$$\Lambda_K = \sup_I \lambda_I^f,$$

where the supremum is taken over all symmetric nice intervals in $\mathcal{M}(K)$. The limit scaling factor may be infinity.

For a map without large bounds, it turns out that a maximal long central cascade must be essentially of saddle node type. More precisely,

Proposition 5.1. *For any $b \in \mathbb{N}$, $\delta > 0$ and $\rho > 1$, there exists $l = l(\delta, \rho, b) \in \mathbb{N}$ with the following property. For any $f \in \mathcal{F}_b$, there exists $\varepsilon = \varepsilon(f, \delta, \rho) > 0$ such that the following holds. Consider a maximal central cascade $I^0 \supset I^1 \supset \cdots \supset I^m$. Assume that $|I^0| < \varepsilon$, $\Lambda_{I^0} \le \rho$, $I^0 \supset (1+2\delta)I^1$, and $m > 3l$. Then*

(1) *the central cascade $I^l \supset I^{l+1} \supset \cdots \supset I^{m-l}$ is of saddle node type;*
(2) *for any $x \in I^{m-l}$, we have*

$$\left|R_{I^0}(x) - x\right| \ge \frac{1}{l(\delta, \rho, n)}|I^0|. \tag{14}$$

The proof is based on an idea of Sullivan: the first return map $R_{I^1}|I^2$ is C^1 close to a map in the *Epstein class*, considered up to rescaling. The conclusion is drawn by a limit argument: if it takes a long time for a critical point to escape the central cascade, then the orbit of the first escaping critical point will create a fixed point in the limit, which is forced to be parabolic by the upper bounds on the scaling factors. In Sect. 5.1, we shall review this principle of Sullivan in more details, and the proof of this proposition will be done in Sect. 5.2.

Another important ingredient of our analysis is the following

Proposition 5.2. *Let $f \in \mathcal{F}_b$ and let $c \in Crit(f)$. Then for any $\rho > 1$ and any $d \in \mathbb{N}$, there exist $\varepsilon = \varepsilon(f, \rho, d) > 0$ and $C = C(b, \rho, d) > 0$ with the following property. Let $K \subset (c - \varepsilon, c + \varepsilon)$ be a nice open set with $\Lambda_K < \rho$. Let $J \subset I$ be intervals in $\mathcal{M}(K)$ intersecting $\omega(c)$ such that $Dep(I, J) \leq d$. (See Definition 4.1.) Then*

$$\left|\mathcal{L}_c(I)\right| \leq C\left|\mathcal{L}_c(J)\right|.$$

This proposition will be proved in Sect. 5.3. Together with Proposition 4.1, it implies that a nice interval can not be geometrically deep inside another one unless it is combinatorially deep inside as well.

Corollary 5.3. *Under the circumstances of Proposition 5.2, $(1+2C')J \not\subset I$, where $C' > 0$ is a constant depending only on ρ, d and b.*

Proof. By Proposition 4.1, if I contains a large neighborhood of J, then $\mathcal{L}_c(J)$ is deep inside $\mathcal{L}_c(I)$. □

With these results in hand, we then proceed to describe the geometric properties of f. We shall need one more definition.

Definition 5.2. Let $c \in \text{Crit}(f)$. For each nice interval I, let $\mathbf{D}_I$ denote the union of all components of D_I which intersect $I \cap \omega(c)$. We say that I has *C-bounded geometry* if the following hold:

(i) $\left((1 + 1/C)I - (1 + 1/C)^{-1}I\right) \cap \omega(c) = \emptyset$;
(ii) for each component J of $I - \partial\mathbf{D}_I$, we have $|J| \geq |I|/C$.

Note that by Lemma 3.8 and Lemma 3.6, the condition (i) implies that the first return map $R_I|\mathbf{D}_I$ has uniformly good distortion when I is small. (See also Proposition 5.8 for a more detailed description of the distortion.) The following two theorems, which will be proved in Sect. 5.4, describe the properties of "essentially bounded geometry" in a nest: the second one shows that the geometry is bounded unless we are deep in a long central cascade; while the first one, complementary to Proposition 5.1, gives more detailed control of the geometry of a long maximal central cascade.

Theorem 5.4. *For any $\delta > 0$, $\rho > 1$ and $b \in \mathbb{N}$, there exist $C > 0$ and $q \in \mathbb{N}$ with the following property. Let $f \in \mathcal{F}_b$ and let $c \in Crit(f)$. Let us consider a maximal central cascade $I^0 \supset I^1 \supset \cdots \supset I^m$ with $m > 3l$, where $l = l(\delta, \rho, b)$ is as in Proposition 5.1. Assume that $\Lambda_{I^0} \leq \rho$ and $I^0 \supset (1 + 2\delta)I^1$. If $|I^0|$ is sufficiently small, then the following hold:*

(1) *for each $0 \leq i \leq m - 1$, we have*

$$\frac{1}{C}\frac{1}{(k+1)^2} \leq \frac{|I^i - I^{i+1}|}{|I^0|} \leq C\frac{1}{(k+1)^2},$$

where $k = \min(i, m - i)$;

(2) *for any* $0 \le i \le l$, *and any* $x \in (I^i - I^{i+1}) \cap \omega(c)$, *we have*

$$R_{I^i}(x) \notin I^q - I^{m-q}.$$

Theorem 5.5. *Let* $f \in \mathcal{F}_b$ *and let* $c \in Crit(f)$. *For any* $\delta > 0$, $\rho > 0$ *and* $p \in \mathbb{N}$, *there are* $\varepsilon = \varepsilon(\delta, \rho, f) > 0$ *and* $C = \eta(\delta, b, \rho, p) > 0$ *with the following property. Let us consider a nest* $I := I^0 \supset I^1 \supset \cdots$. *Assume that*

$$|I| < \varepsilon, \quad \Lambda_I < \rho, \text{ and } I \supset (1 + 2\delta)I^1.$$

Let $m(0) = 0$ *and let* $m(1) < m(2) < \cdots$ *be all the non-central return moments. Then for any* $n \ge 2$ *with* $\inf_{k \ge 0} |n - m(k)| \le p$, I^n *has* C*-bounded geometry.*

In Sect. 5.5, we shall consider the case that f is infinitely renormalizable, and show that the initial geometry of a sufficiently deep renormalization is uniformly bounded. To be more precise, we need to introduce some notation. Let c be a critical point of f, and let T be the component of the domain of f containing c. Let $\tau : T \to T$ be the involution with the property that $f \circ \tau = f$ holds on T. Let

$$B_1 \supset B_2 \supset \cdots \tag{15}$$

be all the symmetric open properly periodic intervals which contain c, and let $1 \le s_1 < s_2 < \cdots$ be the corresponding periods. For any $n \in \mathbb{N}$, f^{s_n} has a fixed point in ∂B_n which we denote by β_n. (So $B_n = (\beta_n, \tau(\beta_n))$.) Let α_n be the innermost fixed point of $f^{s_n}|B_n$, i.e., α_n is the fixed point of $f^{s_n}|B_n$ such that f^{s_n} does not have a fixed point in $A_n := (\alpha_n, \tau(\alpha_n))$. Note that f^{s_n} reverses the orientation at α_n. Let x_n be the point in $(f^{s_n}|B_n)^{-1}(\alpha_n)$ which is closest to β_n, and let $E_n = B_n - \{x_n, \tau(x_n)\} \cup \{\alpha_n, \tau(\alpha_n)\}$.

Theorem 5.6. *Let* $f \in \mathcal{F}_b$ *be an infinitely renormalizable map with essentially bounded geometry. Then there exists* $C > 1$ *such that for any* $n \in \mathbb{N}$, *and for any component* J *of* E_n, *the following hold.*

- $|J| \ge |B_n|/C$;
- J *has* C*-bounded geometry.*

5.1. Extension to a quasi-regular function. Sullivan [46] observed that the first return map to a small interval extends to a quasi-regular map with a small dilatation, from which various geometric estimates are deduced by limit arguments. In this subsection, we review this subtle observation of Sullivan, which we shall also use several times.

Lemma 5.1. *Let* $h : [0, 1] \to [0, 1]$ *be a* C^1 *diffeomorphism. Then* h *extends to a real symmetric* K*-qc map* $H : \mathbb{C} \to \mathbb{C}$, *where*

$$K = \exp\left\{ \sup_{t \in [0,1]} \left| \frac{h''(t)}{h'(t)} \right| \right\}.$$

Proof. Without loss of generality, let us assume that h is orientation-preserving. Let us first extend h to a homeomorphism of the real line by defining $h(x) = x$ for all $x \notin [0, 1]$, and then define a homeomorphism H of the complex plane by $H(x, y) = (h(x), y)$. Direct computation shows that H is a K-qc map with

$$K = \sup_{x\in[0,1]} \left\{h'(x), h'(x)^{-1}\right\}.$$

Take $x_0 \in [0, 1]$ such that $h'(x_0) = 1$. Then we have

$$\left|\log h'(x) - \log h'(x_0)\right| = \left|\int_{x_0}^{x} \frac{h''(t)}{h'(t)}\right| \leq |x - x_0| \sup \left|\frac{h''}{h'}\right| \leq \sup \left|\frac{h''}{h'}\right|,$$

and thus the lemma holds. □

Proposition 5.7. *Let $f \in \mathcal{F}_b$. Let $J \subset dom(f)$ be an open interval which does not contain a critical point of f. Then $f^{-1} : f(J) \to J$ extends to a real symmetric $O(|J|)$-qc map from $\mathbb{C}_{f(J)}$ to an open set of $\mathbb{C}_J$.*

Proof. Let $A = \mathrm{Comp}_J(\mathrm{dom}(f))$ and let $B = \mathrm{Comp}_{f(J)}(\mathrm{range}(f))$.

Assume first that A does not contain a critical point. Let γ_0 and γ_1 be the orientation-preserving affine homeomorphisms of the real line such that $\gamma_0(J) = \gamma_1(f(J)) = (0, 1)$. Consider the map $F = \gamma_0 \circ f^{-1} \circ \gamma_1^{-1}$: $[0, 1] \to [0, 1]$. It follows from the previous lemma that F extends to a real symmetric K-qc map from $\mathbb{C}$ onto itself with

$$K = \exp\left\{\sup_{t\in[0,1]} \left|\frac{F''(t)}{F'(t)}\right|\right\}.$$

Direct computation shows that

$$\frac{F''(t)}{F'(t)} = \frac{1}{\gamma_1'} \frac{(f^{-1})''\big(\gamma_1^{-1}(t)\big)}{(f^{-1})'\big(\gamma_1^{-1}(t)\big)} = O\big(|J|\big),$$

and thus the proposition holds.

Now assume that A contains a critical point c of f. Then $f|A = Q \circ \phi$, where Q is a quadratic map, and ϕ is a smooth diffeomorphism defined on A. Similarly as above, we show that $\phi^{-1}|\phi(J)$ extends to a real symmetric $O(|J|)$-qc map from $\mathbb{C}$ onto itself. As the square root function extends naturally to a conformal map from $\mathbb{C}_{f(J)}$ onto an open set of $\mathbb{C}_{\phi(J)}$, this proposition follows. □

For any $C_1, C_2 \geq 1$, we say that a diffeomorphism $\phi : [-1, 1] \to [-1, 1]$ is in the class $\mathcal{K}(C_1, C_2)$ if the $C^{1+1/2}$ norm of ϕ is at most C_1, and $\phi^{-1}|(-1, 1)$ extends to a real symmetric C_2-qc map from $\mathbb{C}_{(-1,1)}$ into itself. For any $u \in [-1/2, 1/2]$, let $Q_u(z) = u(z^2 - 1) + z$. For any $v \in (0, 2]$, let $P_v(z) = v(z^2 - 1) + 1$. Let $\mathcal{SE}(C_1, C_2, k)$ denote the set of all functions $\Phi : [-1, 1] \to [-1, 1]$ which can be written as

$$\Phi = \psi_m \circ \phi_m \circ \cdots \circ \psi_2 \circ \phi_2 \circ \psi_1 \circ \phi_1,$$

for some $m \le k$, where for each $1 \le i \le m$, $\phi_i \in \mathcal{K}(C_1, C_2)$; and $\psi_i = Q_{u_i}$ for some $u_i \in [-1/2, 1/2]$, or $\psi_i = P_{v_i}$ for some $v_i \in [1/C_1, 2]$.

Proposition 5.8. *For any $\delta > 0$ and $b, N \in \mathbb{N}$, there is a constant $C > 1$ with the following property. For any $f \in \mathcal{F}_b$, and any $\eta > 0$, there is an $\varepsilon > 0$ such that the following holds. Let $c \in Crit(f)$. Let $\mathbb{G}' = \{G_i'\}_{i=0}^s$ and $\mathbb{G} = \{G_i\}_{i=0}^s$ be chains such that $G_i \subset G_i'$, $G_0 \cap \omega(c) \neq \emptyset$. Assume that the intersection multiplicity of $\mathbb{G}'$ is at most N, and that $|G_s'| < \varepsilon$. Moreover, assume that*

$$(1 + 2\delta)G_s \subset G_s', \ \text{and} \left|f^s(G_0)\right| \ge \delta|G_s|.$$

For any $0 \le i \le s$, let γ_i be the orientation-preserving homeomorphism of $\mathbb{R}$ such that $\gamma_i(G_i) = (-1, 1)$. Then the map

$$\Phi = \gamma_s \circ f^s \circ \gamma_0^{-1} : [-1, 1] \to [-1, 1]$$

belongs to the class $\mathcal{SE}(C, 1 + \eta, 2Nb)$.

Proof. Let us fix a constant $\eta > 0$ and assume that $|G_s'|$ is small. By Corollary 3.6, $\sum_{i=0}^s |G_s'|$ is small. We first consider the case that $\mathbb{G}'$ is a monotone chain, and prove that $\Phi \in \mathcal{K}(C, 1 + \eta)$ for some constant $C = C(\delta) > 1$. By Proposition 5.7, Φ^{-1} extends to a real symmetric K-qc map from $\mathbb{C}_{(-1,1)}$ into itself with $K = 1 + O(\sum_{i=0}^{s-1} |G_i|)$ which is close to 1. It remains to estimate the $C^{1+1/2}$-norm of Φ. To this end, take $x, y \in G_0$, and let $T \subset G_s'$ be the $\delta\sqrt{|G_s|/(4|f^s(x) - f^s(y)|)}$-neighborhood of $(f^s(x), f^s(y))$. Let $\{T_i\}_{i=0}^s$ be the chain with $T_s = T$ and $T_0 \ni x, y$. Since G_s' contains a definite neighborhood of T_s, the map $f^s|T_0$ has uniformly bounded distortion by Lemma 3.5. Thus for all $0 \le i \le s - 1$, we have that

$$|T_i|/|G_i| \asymp |T|/|G_s|.$$

Therefore

$$\sum_{i=0}^{s-1} |T_i| \asymp \left(\sum_{i=0}^{s-1} |G_i|\right) \frac{|T|}{|G_s|} = O\left(\sqrt{\frac{\left|f^s(x) - f^s(y)\right|}{|G_s|}}\right).$$

So for any intervals $U \Subset V \subset T_0$,

$$\frac{B\big(f^s(V), f^s(U)\big)}{B(U, V)} - 1 \geq -O\left(\sum_{i=0}^{s-1} |f^i(V)|\right) = -O\left(\sum_{i=0}^{s-1} |T_i|\right) = -O\left(\sqrt{\frac{\big|f^s(x) - f^s(y)\big|}{|G_s|}}\right).$$

Applying Lemma 3.4, we obtain

$$\frac{(f^s)'(x)}{(f^s)'(y)} \leq 1 + C\sqrt{\frac{\big|f^s(x) - f^s(y)\big|}{|G_s|}},$$

where $C > 1$ is a constant depending only on $\delta > 0$. It follows that the $1/2$-Hölder norm of Φ' is uniformly bounded.

Now assume that the chain $\mathbb{G}'$ is not monotone. Let $s_1 < s$ be maximal such that G'_{s_1} contains a critical point, say c. Then by what we have proved, the map $\gamma_s \circ f^{s-s_1-1} \circ \gamma_{s_1+1}^{-1}$ belongs to the class $\mathcal{K}(C, 1+\eta)$. By Lemma 3.6, $f^{s-s_1-1}|G_{s_1+1}$ has bounded distortion. Since $|f^s(G_0)|/|G_s| \geq \delta$, it follows that $|f^{s_1+1}(G_0)|/|G_{s_1+1}|$ can not be too small. Since $f(G_{s_1}) \supset f^{s_1+1}(G_0)$, this implies that $|f(G_{s_1})|/|G_{s_1+1}|$ is bounded away from zero, and so $\gamma_{s_1+1} \circ f \circ \gamma_{s_1}^{-1} \in \mathcal{SE}(C, 1+\eta, 1)$. Moreover, $|f^{s_1}(G_0)|/|G_{s_1}|$ is bounded away from zero, and so the proof of the proposition is completed by induction on the order of $\mathbb{G}'$. □

A map $\Phi : [-1, 1] \to [-1, 1]$ is in the *Epstein class* if $\Phi \in \mathcal{SE}(C, 1, N)$ for some $C > 1$ and $N \in \mathbb{N}$. For such a map Φ and for any affine homeomorphisms $\gamma, \hat{\gamma}$ of the real axis, we shall also say that the map $\hat{\gamma} \circ \Phi \circ \gamma$ is in the Epstein class.

Remark that for any $C \geq 1$ and $N \in \mathbb{N}$, the family $\mathcal{SE}(C, \infty, N) = \bigcup_{n=1}^{\infty} \mathcal{SE}(C, n, N)$ is compact in the C^1 topology. Moreover, for any $\delta > 0$, the family $\mathcal{SE}(C, 1+\delta, N)$ is also compact in the C^1 topology, and any possible limit of a sequence $f_n \in \mathcal{SE}(C, 1+1/n, N)$ is contained in the Epstein class. We shall use the following two lemmas frequently.

Lemma 5.2. *For any* $C > 1$, $N \in \mathbb{N}$, *there is a constant* $\delta > 0$ *with the following property. Let* $\Phi : [-1, 1] \to [-1, 1]$ *be a map in the class* $\mathcal{SE}(C, \infty, N)$ *and let* $F = t\Phi$ *for some* $t \in [1, C]$. *Assume that* F *does not have a hyperbolic attracting fixed point in* $(-1, 1)$. *Let* $c \in (-1, 1)$ *be a critical point of* F, *i.e.,* $F'(c) = 0$. *Then* $d(F(c), c) \geq \delta$. *Moreover, if* $x \in [-1, 1]$ *is a fixed point of* F, *then* $d(x, c) \geq \delta$.

Proof. The latter statement follows from the former one since F has bounded $C^{1+1/2}$ norm. So it suffices to prove the former one. Arguing by contradiction, assume that the statement fails. Then for any $n \in \mathbb{N}$, we can find $\Phi_n \in \mathcal{SE}(C, \infty, N)$, $t_n \in [1, C]$, and $c_n \in (-1, 1)$ with $F_n'(c_n) = 0$, and $d(F_n(c_n), c_n) \leq 1/n$, where $F_n = t_n \Phi_n$. As $\mathcal{SE}(C, \infty, N)$ is a compact family in the C^1 topology, by passing to a subsequence we may assume that Φ_n converges to a map $\Phi \in \mathcal{SE}(C, \infty, N)$ in the C^1 topology. We may also assume that $t_n \to t$ and $c_n \to c$. Then $F_n \to F = t\Phi$ and $F(c) = c$. So c is a hyperbolic attracting fixed point of F. Note that $d(c_n, \{-1, 1\})$ is bounded away from zero and thus $c \in (-1, 1)$. It follows that for n sufficiently large, F_n has a hyperbolic attracting fixed point, which contradicts our assumption. □

Lemma 5.3. *Let $J \subset I$ be intervals and let $P : J \to I$ be a map in the Epstein class with $P \neq id$. If $u, v \in J$ are distinct fixed points of P such that $P|(u, v)$ is monotone, then either* $0 \leq P'(u) < 1$ *or* $0 \leq P'(v) < 1$.

Proof. By an observation of Sullivan ([46]), P has non-positive Schwarzian derivative. Thus the map $P'|[u, v]$ takes its minimum at either u or v. The lemma follows easily. □

5.2. All central cascades are essentially of saddle node type.

Proof of Proposition 5.1. Let $0 < m_1 \leq m$ be the minimal integer such that $R_{I^0}|I^1$ has a critical value which is not contained in I^{m_1}. Below we shall prove:

(∗) if m_1 is sufficiently large, then for some $l(\delta, \rho, b)$, (14) holds for all $x \in I^{m_1}$.

Let us first show how (∗) implies the proposition. Assume first that m_1 is large. Note that (∗) implies that R_{I^0} does not have a fixed point in I^{m_1}. By definition of m_1, R_{I^1} does not have a fixed point in $I^0 - I^{m_1}$ either. So $I^0 \supset I^1 \supset \cdots \supset I^{m_1}$ is a saddle node central cascade. Now take an arbitrarily point $x \in I^m$. Note that $x, R_{I^0}(x), R_{I^0}^2(x), \ldots, R_{I^0}^{m-m_1}(x) \in I^{m_1}$ lie in order, and hence $|x - R_{I^0}^{m-m_1}(x)| \geq (m - m_1) l(\delta, \rho, b)^{-1} |I^0|$. Consequently,

$$m - m_1 \leq l(\delta, \rho, b) \frac{\left|x - R_{I^0}^{m-m_1}(x)\right|}{|I^0|} \leq l(\delta, \rho, b).$$

This proves the proposition in the case that m_1 is sufficiently large. If m_1 is not large, then by Theorem 2.1, $|I^{m_1}|/|I^{m_1+1}|$ is bounded away from 1, and so we may apply (∗) to the central cascade $I^{m_1} \supset I^{m_1+1} \supset \cdots \supset I^m$. As $R_{I^0}|I^1$ has at most b critical values, the proposition is proved by repeating this argument at most b times.

The statement (∗) will be shown by contradiction. Assume that (∗) fails. Then for each $n = 1, 2, \ldots$, there exists a map $f_n \in \mathcal{F}_b$ such that for any $\varepsilon_n > 0$, there exists a maximal central cascade $\{I^j(n)\}_{j=0}^{m(n)}$ with respect to f_n around $c_n \in \mathrm{Crit}(f_n)$ with the following properties:

(i) $|I^0(n)| < \varepsilon_n$;
(ii) $I^0(n) \supset (1+2\delta)I^1(n)$, $\Lambda^{f_n}_{I^0(n)} \le \rho$;
(iii) $m_1(n) \to \infty$ as $n \to \infty$; and
(iv) there exists $x_n \in I^{m_1(n)}(n)$ such that

$$\frac{\left|R^{f_n}_{I^0(n)}(x_n) - x_n\right|}{|I^0(n)|} \to 0, \text{ as } n \to \infty,$$

where $m_1(n)$ is the minimal positive integer such that $R^{f_n}_{I^0(n)}$ has a critical value not contained in $I^{m_1(n)}(n)$.

Let $\varphi_n : I^2(n) \to I^1(n)$ denote the first return map (under f_n) of $I^2(n)$ to $I^1(n)$. Let h_n be the orientation-preserving affine homeomorphism of $\mathbb{R}$ such that $\tilde{I}_2(n) := h_n(I_2(n)) = (-1,1)$, and let $\Phi_n = h_n \circ \varphi_n \circ h_n^{-1}$. By assuming ε_n sufficiently small, it follows from Proposition 5.8 that

$$\frac{\left|I^2(n)\right|}{\left|I^1(n)\right|}\Phi_n \in \mathcal{SE}\left(C, 1+\frac{1}{n}, b\right),$$

where $C = C(b,\delta)$ is a constant. Thus after passing to a subsequence, we may assume that Φ_n converges in the C^1 topology to a map $\Phi : \tilde{I}^1 \to \tilde{I}^0$, which is in the Epstein class. The map Φ is even: $\Phi(z) = \Phi(-z)$, and it does not have a hyperbolic attracting cycle in $(-1,1)$ because Φ_n does not. Since $\Phi'(0) = 0$, $\Phi(0) \neq 0$. Therefore $|I^{m_1(n)-1}(n)|/|I^1(n)|$ and hence $|I^{m_1(n)}(n)|/|I^1(n)|$ is bounded away from 0. It follows that $|\varphi_n(I^{m_1(n)}(n))|/|I^1(n)|$ is bounded away from zero as well.

Claim 1. There is a constant $\sigma > 0$, such that for any $n \in \mathbb{N}$, we have

$$\left|I^{m_1(n)-1}(n)\right| \ge (1+2\sigma)\left|I^{m_1(n)}(n)\right|.$$

We may assume that $m_1(n) > 1$. Let u_n be a critical point of φ_n with $\varphi_n(u_n) \in I^{m_1(n)-1}(n) - I^{m_1(n)}(n)$. Let s be the return time of $I^1(n)$ to $I^0(n)$, and let $0 \le k \le s-1$ be the minimal integer such that $f_n^k(u_n) = p \in \mathrm{Crit}(f_n)$. Let $J = \mathcal{L}^{f_n}_p(I^{m_1(n)-1}(n))$. Then

$$f_n^{s-k}\left(\mathcal{L}^{f_n}_p(J)\right) \subset I^{m_1(n)-1}(n) - I^{m_1(n)}(n).$$

Assume that $|I^{m_1(n)-1}(n)|/|I^{m_1(n)}(n)|$ is close to 1. Then $I^{m_1(n)-1}(n) - I^{m_1(n)}(n)$ is tiny compared to $I^1(n)$. Since $f_n^{s-k}(J) \supset \varphi_n(I^{m_1(n)}(n))$, its length is comparable to that of $I^1(n)$. Since the map $f_n^{s-k}|J$ has uniformly good distortion, it follows that $\mathcal{L}^{f_n}_p(J)$ is deep inside J, which contradicts the hypothesis that $\Lambda_{I^0(n)} \le \rho$. The proof of the claim is completed.

Let us continue the proof of $(*)$. To fix the notation, let us assume that φ_n maps $\partial I^2(n)$ to the left endpoint of $I^1(n)$ for all n. Let $K^-(n)$ and $K^+(n)$ denote the left and right component of $\tilde{I}^1(n) - \tilde{I}^{m_1(n)}(n)$ respectively. Let $u_n \in I^2(n)$ be a critical point of φ_n such that $\varphi_n(u_n) \notin I^{m_1(n)}(n)$ and let

$v_n \in I^2(n)$ be the critical point of φ_n which is closest to the left endpoint of $I^2(n)$. Let $\tilde{u}_n$ and $\tilde{v}_n$ be the corresponding critical points of Φ_n. We may assume that $\tilde{u}_n \to u$ and $\tilde{v}_n \to v$ as $n \to \infty$. Note that $\Phi'(u) = \Phi'(v) = 0$.

Claim 2. For n sufficiently large, we have

$$\Phi_n(\tilde{u}_n) \in K^-(n).$$

Arguing by contradiction, assume that the claim fails. By passing to a subsequence, we may assume that $\Phi_n(\tilde{u}_n) \in K^+(n)$ for all n. Notice that $\Phi_n(\tilde{v}_n) \geq \Phi_n(x)$ for all $x \in I^1$. So $\Phi_n(\tilde{v}_n) \in K^+(n)$. By a straightforward combinatorial argument, we obtain

$$\cdots \leq \Phi^{j+1}(v) \leq \Phi^j(v) \leq \cdots \leq \Phi^2(v) \leq -\Phi(v) \leq v \leq -v \leq \Phi(v).$$

Let $y = \lim_{j\to\infty} \Phi^j(v)$. Then $\Phi(y) = y$. Since Φ does not have a hyperbolic attracting fixed point, $\Phi(v) > v$. Let us show that $y = -\Phi(v)$. Arguing by contradiction, assume that $y < -\Phi(v)$. Then $\Phi(-\Phi(v)) = \Phi(\Phi(v)) = \Phi^2(v) < -\Phi(v)$. Since $\Phi(v) > v$, it follows that Φ has a fixed point y' in the open interval $(-\Phi(v), v)$. Notice that Φ is monotone on (y, y'). Since Φ is in the Epstein class, by Lemma 5.3 we know that either y or y' is a hyperbolic attracting fixed point of Φ, which is absurd. This proves that $y = -\Phi(v)$. Therefore $-\Phi(v) = \Phi^2(v) = \Phi^3(v)$ is a fixed point of Φ. Since $(-\Phi_n(v_n), \Phi_n^3(v_n))$ contains a component of $\tilde{I}^{m_1(n)-2}(n) - \tilde{I}^{m_1(n)-1}(n)$, it follows that $|\tilde{I}^{m_1(n)-2}(n)|/|\tilde{I}^{m_1(n)-1}(n)|$ is close to 1 for all large n. Consequently, $|\tilde{I}^{m_1(n)-1}(n)|/|\tilde{I}^{m_1(n)}(n)|$ is close to 1 for all large n, which, however, is ruled out by Claim 1. The proof of Claim 2 is completed.

Therefore, $\Phi_n(\tilde{u}_n) \in K^-(n)$ for all large n. From this it follows that

$$u \geq \Phi(u) \geq \Phi^2(u) \geq \cdots .$$

Let $y = \lim_{j\to\infty} \Phi^j(u)$. Then $\Phi(y) = y$.

Let us define a point $\tilde{x}_n'$ for each $n \in \mathbb{N}$, such that this point is contained in the left component of $\tilde{I}^{m_1(n)}(n) - \{\tilde{v}_n\}$ and such that $\Phi_n(\tilde{x}_n') - \tilde{x}_n' \to 0$ as $n \to \infty$. This can be done as follows. If Φ_n has a fixed point in the left component of $\tilde{I}^{m_1(n)}(n) - \{\tilde{v}_n\}$, then we take $\tilde{x}_n'$ to be any of such fixed points. Otherwise, $\Phi_n(\tilde{v}_n) < \tilde{v}_n$. If $\tilde{v}_n - \Phi_n(\tilde{v}_n) \geq |\tilde{x}_n - \Phi_n(\tilde{x}_n)|$ ($\tilde{x}_n = h_n(x_n)$ and x_n is as in (iv)), then $\tilde{x}_n$ belongs to the left component of $\tilde{I}^{m_1(n)}(n) - \{\tilde{v}_n\}$ and we take $\tilde{x}_n' = \tilde{x}_n$. If $\tilde{v}_n - \Phi_n(\tilde{v}_n) < |\tilde{x}_n - \Phi_n(\tilde{x}_n)|$, then we take $\tilde{x}_n' = \tilde{v}_n$. It is easy to check that such defined points $\tilde{x}_n'$ satisfy the requirements stated at the beginning of this paragraph.

Passing to a subsequence, let us assume that $\lim_{n\to\infty} \tilde{x}_n'$ exists, and denote it by x. Then $\Phi(x) = x \geq \Phi(u)$. Assume that $\Phi^2(u) < \Phi(u)$. Then we obtain a non-degenerate interval $[y, x]$ which is mapped by Φ onto itself monotonically, which is ruled out by Lemma 5.3 again. So $\Phi(u) = \Phi^2(u) = y$. As in the proof of Claim 2, this implies that as $n \to \infty$, $|I^{m_1(n)-1}(n)|/|I^{m_1(n)}| \to 1$, contradicting Claim 1. The proof of $(*)$ is completed. □

5.3. "Geometrically deep" implies "combinatorially deep". Let us prove Proposition 5.2. We need the following lemma.

Lemma 5.4. *Let $f \in \mathcal{F}_b$ and let $c \in Crit(f)$. Let $I^0 \supset I^1 \supset \cdots \supset I^m$ be a central cascade such that $I^0 \ni c$ is a small nice interval with $\Lambda_{I^0} \le \rho$. Then*

$$|I^0| \le \rho' |I^m|,$$

where ρ' is constant depending only on ρ and b.

Proof. Let $\delta_1 > 0$ be the constant as in Lemma 4.2. If $I^0 \supset (1+\delta_1/2)I^1$, then the statement follows from Proposition 5.1. So let us assume $I^0 \subset (1+\delta_1)I^1$. Let s be the return time of c to I^1. By Lemma 4.2, the chain $\{G'_j\}_{j=0}^s$ with $G'_s = (1+2\delta_1)I^0$ and $G'_0 \supset I^1$ has uniformly bounded intersection multiplicity. Since $f^s(c) \in I^{m-1}$ and since $|I^{m-1}| \le \rho|I^m|$, it suffices to show that $d(f^s(c), c)/|I^0|$ is bounded away from zero. So we may assume that $d(f^s(c), c)/|I^0| \le 1/4$. Then $|f^s(I^1)|/|I^0|$ is bounded away from zero because $f^s(I^1)$ contains a component of $I^0 - \{f^s(c)\}$. Let γ_0 and γ_s be affine homeomorphisms of $\mathbb{R}$ such that $\gamma_0(I^1) = \gamma_s(I^0) = (-1, 1)$. By Proposition 5.8, $\gamma_s \circ f^s \circ \gamma_0^{-1} : [-1, 1] \to [-1, 1]$ belongs to $\mathcal{SE}(C, 2, N)$, where $C > 1$ and $N \in \mathbb{N}$ depend only on b (and δ_1). As $|I^0| \le \rho|I^1|$, applying Lemma 5.2, we obtain that $d(f^s(c), c)/|I^0|$ is bounded away from zero. □

Proof of Proposition 5.2. Let $\hat{I} = \mathcal{L}_c(I)$ and $\hat{J} = \mathcal{L}_c(J)$. Let us first prove that if $|\hat{I}|/|\hat{J}|$ is sufficiently large, then for any $x \in \omega(c)$, $\mathcal{L}_x(\hat{I})$ contains a large neighborhood of $\mathcal{L}_x(\hat{J})$. As before, let us write $\hat{I}^0 = \hat{I}$, and $\hat{I}^{n+1} = \mathcal{L}_c(\hat{I}^n)$ for all $n \ge 0$. Let $m(0) = 0$, and let $m(1) < m(2) < \cdots$ be all the positive integers such that $R_{\hat{I}^{m(i)-1}}(c) \notin \hat{I}^{m(i)}$. By the previous lemma, $|\hat{I}^{m(i)}|/|\hat{I}^{m(i+1)}|$ is uniformly bounded from above. Let k be the maximal non-negative integer such that $\hat{I}^{m(k)} \supsetneq \hat{J}$. Then k is large provided that $|\hat{I}|/|\hat{J}|$ is sufficiently large. By Theorem 2.1, $\hat{I}^{m(i)}$ contains a definite neighborhood of $\hat{I}^{m(i)+1}$. By Theorem 3.3, $\mathcal{L}_x(\hat{I}^{m(i)+1})$ is well inside $\mathcal{L}_x(\hat{I}^{m(i)})$ for all i. Thus $\mathcal{L}_x(\hat{J}) \subset \mathcal{L}_x(\hat{I}^{m(k)})$ is deep inside $\mathcal{L}_x(\hat{I})$.

Let us now prove the proposition in the case $d = 1$. Of course, we may assume that J does not contain c. Let y be an arbitrary point in $\omega(c) \cap \hat{J}$, and s the entry time of $\hat{J}$ to J. Then $f^s(\mathcal{L}_y(\hat{J})) \subset \mathcal{L}_{f^s(y)}(\hat{J})$ since $f^i(y) \notin \hat{J}$ for all $1 \le i \le s$. Arguing by contradiction, assume that $|\hat{I}|/|\hat{J}|$ is large. Then $\mathcal{L}_{f^s(y)}(\hat{J})$ is deep inside $\mathcal{L}_{f^s(y)}(\hat{I})$. As

$$\mathcal{L}_{f^s(y)}(\hat{I}) \subset \mathcal{L}_{f^s(y)}(I) \subset J,$$

it follows that $\mathcal{L}_{f^s(y)}(\hat{J})$ is deep inside J. By Lemma 3.6, $\mathcal{L}_y(\hat{J})$ is deep inside $\hat{J}$. In particular, $\lambda_{\hat{J}}$ is large, which contradicts the hypothesis that $\Lambda_K < \rho$.

To deal with the case $d > 1$, let y_0 be a point in $\omega(c) \cap J$, and define $T_i = \mathcal{L}^i_{y_0}(I)$ for all $i \geq 0$. Let $d' < d$ be maximal such that $J \subset T_{d'}$. Then by what we have proved above, $|\mathcal{L}_c(T_i)|/|\mathcal{L}_c(T_{i+1})|$ is uniformly bounded from above for any i, and so is $|\mathcal{L}_c(T_{d'})|/|\mathcal{L}_c(J)|$ as well. Thus $|\hat{I}|/|\hat{J}|$ is bounded from above. □

5.4. Geometry of a nest. In this subsection, we prove Theorems 5.4 and 5.5.

Proof of Theorem 5.4. By Proposition 5.1, $I^l \supset I^{l+1} \supset \cdots \supset I^{m-l}$ is a saddle node central cascade. Without loss of generality, assume that $R_{I^l}|I^{l+1}$ contains the left component of $I^l - I^{l+1}$. For each i, let a_i be the left endpoint of I^i and let $a'_i = f(a_i)$. Consider the diffeomorphism $f^s : (a'_{m-l}, a'_{l+1}) \to (a'_{m-l-1}, a'_l)$. By Lemma 3.7, this map has negative Schwarzian. By Proposition 5.1, $|a_{m-l} - a_{m-l-1}|/|I^0|$ is bounded away from zero, and thus so is $|a'_{m-l} - a'_{m-l-1}|/|a'_{m-l} - a'_l|$. By Theorem 2.1 and by the assumption $\Lambda_I \leq \rho$, $|a_l - a_{l+1}|/|I^0|$ is bounded away from zero, and thus so is $|a'_l - a'_{l+1}|/|a'_l - a'_{m-l}|$. By Yoccoz's lemma (see [10] Sect. 4.1 and Appendix B), it follows that $|a'_i - a'_{i+1}|/|a'_l - a'_{m-l}|$, and thus $|a_i - a_{i+1}|/|a_l - a_{m-l}|$ is comparable to $(\min(i - l, m - l - i) + 1)^{-2}$, for all $l \leq i \leq m - l - 1$. This proves the first statement.

Now let us prove the second statement. Take a point $x \in (I^i - I^{i+1}) \cap \omega(c)$ for some $0 \leq i \leq l$, and let P be the component of $I^i - \mathrm{Cl}(I^{i+1})$ containing x. Then P is a nice interval. Arguing by contradiction, assume that $R_{I^i}(x) \in I^q - I^{m-q}$ for a large q. Since $(\partial I^q \cup \partial I^{m-q}) \cap D_P = \emptyset$, we have $R_{I^i}(\mathcal{L}_x(P)) \subset \mathcal{L}_{R_{I^i}(x)}(P) \subset I^q - I^{m-q}$. By the first statement of this theorem, this implies that $\mathcal{L}_{R_{I^i}(x)}(P)$ is deep inside I^i. By Lemma 3.6, it follows that $\mathcal{L}_x(P)$ is deep inside $\mathcal{L}_x(I^i) \subset P$, which is ruled out by Corollary 5.3. □

Proposition 5.9. *Let $f \in \mathcal{F}_b$ and let $c \in Crit(f)$. For any $\delta > 0$ and $\rho > 1$, there are $\varepsilon = \varepsilon(\delta, \rho, f) > 0$ and $C = \eta(\delta, \rho, b) > 0$ with the following property. Let $I \ni c$ be a symmetric nice interval such that $|I| < \varepsilon$, $\Lambda_I < \rho$, and $I \supset (1 + 2\delta)I^1$. Then*

(1) *each component of $\mathbf{D}_{I^1}$ is C-commensurable to $|I^1|$;*
(2) *I^2 has C-bounded geometry.*

Proof. (1) Let J be a component of $\mathbf{D}_{I^1}$. If $|J|/|I^1|$ were small, then J would be deep inside in $I^0 := I$, which is ruled out by Corollary 5.3.

(2) Since $\bigcup_{n=0}^{\infty} f^n(\partial I^1) \cap I^0 = \emptyset$, each component of $\mathbf{D}_{I^1}$ is compactly contained in I^1. Let us first prove that each component J of $I^1 - \mathbf{D}_{I^1}$ with $\partial J \cap \partial I^1 \neq \emptyset$ is commensurable to I^1. To see this, let J_1 be the component of $\mathbf{D}_{I^1}$ which has a common endpoint with J. By the first statement of this proposition, $|J_1|/|I^1|$ is bounded away from zero. By Theorem 2.1, so is $|J|/|J_1|$. Therefore $|J|/|I^1|$ is bounded away from zero.

It follows that there exists a constant $\sigma = \sigma(\delta, \rho, b) > 0$ such that

$$\left(I^1 - (1-2\sigma)I^1\right) \cap \omega(c) = \emptyset.$$

By choosing σ smaller, Lemma 3.6 implies that for any component J of $\mathbf{D}_{I^1}$,

$$\left(J - (1-2\sigma)J\right) \cap \omega(c) = \emptyset.$$

In particular, this holds for $J = I^2$. Moreover, this implies that a definite neighborhood of I^2 is disjoint from $\omega(c) - I^2$.

Now let J be a component of $I^2 - \partial\mathbf{D}_{I^2}$. We need to prove that $|J|/|I^2|$ is bounded away from zero. If $J \subset \mathbf{D}_{I^2}$, then this follows from the first statement of this proposition since $|I^1|/|I^2|$ is also bounded away from 1 by Theorem 2.1. If J is a component of $I^2 - \mathbf{D}_{I^2}$ with $\partial J \cap \partial I^2 \neq \emptyset$, then this follows from a similar argument as above.

So let us assume that $\partial J \cap \partial I^2 = \emptyset$. Let J_1 and J_2 be the components of $\mathbf{D}_{I^2}$ which have common boundary points with J. Let s_i be the return times of J_i to I^2. Without loss of generality, assume $s_1 \leq s_2$. Let $K \subset I^1$ be a definite neighborhood of I^2 such that $K - I^2$ is disjoint from $\omega(c)$. Let $\{K_j\}_{j=0}^{s_2}$ be the chain with $K_{s_2} = K$ and $K_0 \supset J_2$. Then by Lemma 3.8, the order and the intersection multiplicity of this chain are both bounded from above. Thus, by Lemma 3.6, K_0 contains a definite neighborhood of J_2. It suffices to show that K_0 is disjoint from J_1. Arguing by contradiction, assume that there exists $x \in K_0 \cap J_1$. Notice that for any $0 \leq j \leq s_2 - 1$, $K_j \subset D_{I^1}$, and hence $K_j \cap I^2 = \emptyset$ since $f^j(J_2) \cap I^2 = \emptyset$. It follows that $s_1 = s_2$ and $f^{s_2}(x) \in I^2$. But this is absurd because $f^{s_2}|K_0$ has all its critical points in J_2. □

Proof of Theorem 5.5. By Theorems 2.1 and 5.4, $|I^{n-2}|/|I^{n-1}|$ is bounded away from 1, which implies that I^n has uniformly bounded geometry by Proposition 5.9. □

5.5. Initial geometry of infinitely renormalizable maps. Let us now consider a map $f \in \mathcal{F}_b$ which is infinitely renormalizable. Let c be a critical point of f, and let B_n, A_n, E_n, etc. be defined as before (above Sect. 5.1).

The goal of this subsection is to prove Theorem 5.6. Together with the control of the geometry of the nests $A_n^0 \supset A_n^1 \supset \cdots$ which connect two consecutive renormalization levels, this will give us a satisfactory description of the geometric properties of f.

If $\mathcal{L}_c(A_n) = A_n$, then $A_n = B_{n+1}$, and $s_{n+1} = 2s_n$. In this case, we say that the n-th renormalization of f is *immediately renormalizable*, and define $\chi_n = -1$. In all other cases, let χ_n be the height of A_n, that is, the number of positive integers m such that $R_{A_n^{m-1}}(c) \notin A_n^m$. Moreover, let $m_n(0) = 0$ and let $m_n(1) < m_n(2) < \cdots < m_n(\chi_n)$ be all the positive integers such that

$$R_{A_n^{m_n(i)-1}}(c) \notin A_n^{m_n(i)}.$$

The integer χ_n will be referred to as *the height of the n-th renormalization* of f.

Lemma 5.5. *There is a constant $\delta = \delta(b) > 0$ such that for all n sufficiently large, the δ-neighborhood of B_n does not contain $f^i(B_n)$ for any $1 \le i \le s_n - 1$. Moreover, if $\mathbb{G} = \{G_j\}_{j=0}^{s_n}$ is the chain with $G_{s_k} = (1+2\delta)B_n$ and $G_0 \supset B_n$, then*

- *$\mathbb{G}$ has intersection multiplicity at most 4; and*
- *f^{s_n} does not have a critical point in $G_0 - B_n$.*

Proof. (cf. Proposition 3.2 in [42]) The first statement follows from the "shortest interval argument". Let $1 \le k \le s_n$ be such that $|f^k(B_n)| = \min_{i=1}^{s_n} |f^i(B_n)|$. Let M be the 1/2-neighborhood of $f^k(B_n)$. Then for any $1 \le i \le s_n$ with $i \ne k$, $f^i(B_n) \not\subset M$. Let $\mathbb{M} = \{M_j\}_{j=0}^k$ be the chain with $M_k = M$ and $M_0 \supset B_n$. Let us prove that the order of this chain is uniformly bounded from above. Let $\mathbb{M}' = \{M'_j\}_{j=0}^k$ be the chain with $M'_k = f^k(B_n)$ and $M'_0 \supset B_0$. As the intervals M'_j, $1 \le j \le k$ are pairwise disjoint, the order of $\mathbb{M}'$ is uniformly bounded from above. If there is a critical point $c' \in M_j - M'_j$ for some $0 \le j \le k-1$, then $M_j - M'_j$ contains a component of $\mathcal{L}_{c'}(B_n) - \{c'\}$, and thus $M_k - M'_k$ contains $f^{k-j}(\mathcal{L}_{c'}(B_n))$ which is of the form $f^i(B_n)$, contradicting the choice of M_k. This proves that the oder of $\mathbb{M}$ coincides with that of $\mathbb{M}'$, and thus it is uniformly bounded from above. By Lemma 3.6, $M_0 \supset (1+2\delta)B_n$ for some $\delta = \delta(b) > 0$. Obviously, for any $1 \le i \le s_n - 1$, $f^i(B_n) \not\subset M_0$.

Let us consider the chain $\mathbb{G}$. Using a similar argument as above, we prove that f^{s_n} does not have a critical point in $G_0 - B_n$. To show the intersection multiplicity is at most 4, first note that for any $1 \le i, j \le s_n$ with $i \ne j$, $G_j \not\supset f^i(B_n)$. Assume that there are $1 \le j_0 < j_1 < j_2 < j_3 \le s_n$ such that $G_{j_0} \cap G_{j_1} \cap G_{j_2} \cap G_{j_3}$ contains a point, say x. Then there are $0 \le l < m \le 3$ such that $x \notin f^{j_l}(B_n) \cup f^{j_m}(B_n)$ and such that $f^{j_l}(B_n)$ and $f^{j_m}(B_n)$ are on the same side of x. It follows that $G_{j_l} \supset f^{j_m}(B_n)$ or $G_{j_m} \supset f^{j_l}(B_n)$; a contradiction. □

Lemma 5.6. *For any $k \in \mathbb{N}$, there exists a constant $C = C(k, b) > 1$ such that for all n sufficiently large, if u is a critical point of $f^{ks_n}|Cl(B_n)$ and x is a fixed point of $f^{ks_n}|Cl(B_n)$, then*

$$\left|f^{ks_n}(u) - u\right| \ge |B_n|/C, \text{ and } |u - x| \ge |B_n|/C.$$

Proof. Let us fix a large positive integer n. Let $\delta > 0$ is as in the previous lemma, and let $T_n = (1+2\delta)B_n$, $S_n = \mathrm{Comp}_{B_n}(f^{-s_n}(T_n))$. Then $f^{s_n}|S_n$ does not have a critical point in $S_n - B_n$. It follows that $S_n \subset T_n$ because otherwise f^{s_n} would have an attracting fixed point in $S_n - B_n$, which contradicts Theorem 3.2.

Let $\mathbb{G}' = \{G'_j\}_{j=0}^{ks_n}$ and $\mathbb{G} = \{G_j\}_{j=0}^{ks_n}$ be the chains with $G'_{ks_n} = T_n$, $G_{ks_n} = B_n$ and $G'_0 \supset G_0 = B_n$. Then $G'_{ks_n} \supset G'_{(k-1)s_n} \supset \cdots G'_{s_n} \supset G'_0$.

By the previous lemma, for each $1 \le i \le k$, the subchain $\{G'_j\}_{j=(i-1)s_n}^{is_n}$ has intersection multiplicity ≤ 4. Thus the intersection multiplicity of $\mathbb{G}'$ is at most $4k$.

Let $h : \mathbb{R} \to \mathbb{R}$ be an affine homeomorphism so that $h(B_n) = (-1, 1)$, and let $\Phi = h \circ f^{s_n} \circ h^{-1}|[-1, 1]$. By Proposition 5.8, there exist $C' > 1$ and $N_0 \in \mathbb{N}$ such that $\Phi \in \mathcal{SE}(C', 2, N_0)$. Applying Lemma 5.2 completes the proof. □

Lemma 5.7. *There exists a constant* $\delta = \delta(b) \in (0, 1)$*, such that for all n sufficiently large, if* $\chi_n \ge 0$*, then each component of* $A_n - A_n^1$ *has length at least* $\delta|B_n|$*. Moreover,* $(1 + 2\delta)B_{n+1} - B_{n+1}$ *is disjoint from* $\omega(c)$*.*

Proof. Let U be the component of $A_n - A_n^1$ which contains α_n in its closure. To prove the former statement, it suffices to show that $|U|/|B_n|$ is bounded away from zero. Note that the return time of U to A_n is $2s_n$. If $f^{2s_n}|U$ is not monotone, then $\mathrm{Cl}(U)$ contains a critical point of $f^{2s_n}|B_n$, as well as a fixed point, α_n, of this map, so the statement follows from Lemma 5.6. If $f^{2s_n}|U$ is monotone, then $f^{2s_n}(U) = A_n$. By Lemma 5.6, $|A_n|/|B_n|$ is bounded away from zero. By Lemma 4.3, $f^{s_n}|B_n$ has uniformly bounded derivative. The statement follows.

Now let us show that a definite neighborhood of B_{n+1} is disjoint from $\omega(c) - B_{n+1}$. Since $A_n^{m_n(\chi_n)} - B_{n+1}$ is disjoint from $\omega(c)$, it suffices to show that $A_n^{m_n(\chi_n)}$ contains a definite neighborhood of $A_n^{m_n(\chi_n)+1}$. If $\chi_n = 0$, this is just the former statement which we have proved. If $\chi_n > 0$, then this follows from Theorem 2.1. □

Lemma 5.8. *For any* $\rho > 1$ *and any non-negative integer k, there exists a constant* $C = C(\rho, k, b) > 0$ *such that if n is a sufficiently large positive integer and if* $\Lambda_{E_n} < \rho$*, then each component of* $B_n - f^{-ks_n}(\alpha_n)$ *has length at least* $|B_n|/C$*.*

Proof. Note that inbetween α_n and β_n, f^{s_n} has a critical point, and thus by Lemma 5.6, $|\beta_n - \alpha_n|/|B_n|$ is bounded from zero. Since $f^{s_n}((\alpha_n, \tau(\beta_n))) \supset (\alpha_n, \beta_n)$, and since $f^{s_n}|B_n$ has uniformly bounded derivative, $|\alpha_n - \tau(\beta_n)|/|B_n|$ is also bounded away from zero.

Let (x, y) be a component of $B_n - f^{-ks_n}(\alpha_n)$. If $f^{ks_n}|(x, y)$ is monotone, then $f^{ks_n}((x, y))$ is a component of $B_n - \{\alpha_n\}$, which has length comparable to $|B_n|$. Again because $f^{s_n}|B_n$ has uniformly bounded derivative, $|x - y|/|B_n|$ is not very small. Now assume that $f^{ks_n}|(x, y)$ is not monotone. Let $0 \le i \le ks_n - 1$ be minimal such that $f^i([x, y])$ contains a critical point c' of f.

If $i = 0$, then $c' = c$. Let $I = \mathrm{Comp}_c(B_n - \{\alpha_n\})$. Notice that both I and (x, y) belong to $\mathcal{M}(B_n - \{\alpha_n\})$, and that $\mathrm{Dep}(I, (x, y)) \le k$. By Corollary 5.3, (x, y) cannot be deep inside I. As both components of $I - \{c\}$ have length comparable to $|B_n|$, it follows that $|y - x|/|B_n|$ is bounded away from zero. If $i > 0$, let $B'_n = \mathcal{L}_{c'}(B_n)$ and $\alpha'_n = f^i(\alpha_n)$. Let $x' = f^i(x)$, $y' = f^i(y)$, and let $I' = \mathrm{Comp}_{c'}(B'_n - \alpha'_n)$. Then the same argument as above shows that $|y' - x'|/|B'_n|$ is bounded away from zero, which implies that so is $|y - x|/|B_n|$. □

Proposition 5.10. *Let $f \in \mathcal{F}_b$ and $c \in Crit(f)$ be as above. For any $\rho > 0$, there exist $\delta = \delta(\rho, b) > 0$ and $N = N(\rho, b) \in \mathbb{N}$ such that for any sufficiently large $n \in \mathbb{N}$ with $\Lambda_{E_n} < \rho$, the following hold.*

(i) For any $x \in \omega(c) \cap (B_n - (x_n, \tau(x_n)))$, there is $m \leq N$ with $f^m(x) \in (x_n, \tau(x_n))$.
(ii) $((1 + 2\delta)B_n - (1 - 2\delta)B_n) \cap \omega(c) = \emptyset$.
(iii) If moreover $\chi_n \geq 0$ or $\Lambda_{E_{n+1}} < \rho$, then $((1 + 2\delta)A_n - (1 - 2\delta)A_n) \cap \omega(c) = \emptyset$.

Proof. (i) Notice that f^{s_n} maps (β_n, x_n) diffeomorphically onto the interval (β_n, α_n). Define $y_0 = \alpha_n$, $y_1 = x_n$, and $y_i \in (\beta_n, x_n)$ to be such that $f^{s_n}(y_i) = y_{i-1}$ for all $i \geq 2$. Let us first prove the following

Claim. There exists $\mu = \mu(b) \in (0, 1)$ such that

$$|y_i - y_{i-1}| \leq \mu^i |B_n|,$$

provided that n is large enough.

Let T_n^+ be the component of $(1 + 2\delta)B_n - B_n$ which contains β_n, where δ is as in Lemma 5.5. Let $H_0 = T_n^+ \cup [\beta_n, \alpha_n) \subset (1 + 2\delta)B_n$. Let H_i be the component of $f^{-is_n}(H_0)$ which contains β_n. Consider the chain $\{G_j\}_{j=0}^{s_n}$ with $G_{s_n} = H_0$ and $G_0 = H_1$. This is a monotone chain with intersection multiplicity bounded by 4. By Lemma 5.6, H_0 contains a definite neighborhood of $[\beta_n, y_1]$, and so by Lemma 3.5, H_1 contains a definite neighborhood of $[\beta_n, y_2]$. For $i \geq 1$, consider the map $f^{is_n} : f(H_{i+1}) \to f(H_1)$, which is a diffeomorphism with negative Schwarzian by Lemma 3.7. Since $f^{is_n}(f([\beta_n, y_{i+2}])) \subset [\beta_n, f(y_2)]$ is well inside $f(H_1)$, it follows that $f(H_{i+1})$ contains a definite neighborhood of $f(\beta_n, y_{i+2})$, and thus $|\beta_n - y_{i+1}|/|\beta_n - y_i|$ is bounded away from 1. The claim follows.

Let m be the maximal positive integer (if it exists) such that $(y_m, y_{m+1}) \cup (\tau(y_m), \tau(y_{m+1}))$ intersects $\omega(c)$. (If such an integer does not exist, then we have nothing to show.) It suffices to show that m is not very large. Arguing by contradiction, assume that m is large. Let us show that this contradicts the assumption $\Lambda_{E_n} \leq \rho$. To this end, let $k \in \mathbb{N}$ be such that $f^{ks_n}(c) \in (y_m, y_{m+1}) \cup (\tau(y_m), \tau(y_{m+1}))$. Then by the maximality of m, $z := f^{(k-1)s_n}(c) \in (x_n, \tau(x_n))$. Let $K = \mathrm{Comp}_z(E_n)$. Notice that

$$f^{s_n}\big(\mathcal{L}_z(K)\big) \subset (y_m, y_{m+1}) \cup \big(\tau(y_m), \tau(y_{m+1})\big).$$

It follows that $|\mathcal{L}_z(K)|/|B_n|$ is small. On the other hand, for each component L of $K - \{z\}$, $f^{s_n}(L) \supset (f^{s_n}(z), \alpha_n)$ is commensurable to B_n, and thus $|L|/|B_n|$ is bounded away from zero. Therefore $\mathcal{L}_z(K)$ is deep inside K, which is a contradiction by Corollary 5.3.

(ii) The statement $(B_n-(1-2\delta)B_n)\cap\omega(c)=\emptyset$ follows from (i). Let us prove that $\omega(c)$ cannot be too close to B_n from outside either. If $\chi_{n-1}\geq 0$, then this follows from Lemma 5.7. Assume that $\chi_{n-1}=-1$, then $B_n=A_{n-1}$. Let $B_n'=\mathcal{L}_{f^{s_{n-1}}(c)}(B_n)$. Then $\omega(c)\cap B_{n-1}\subset B_n\cup B_n'$. Since $|B_n|\asymp|B_n'|$ and since the convex hull of $B_n'\cap\omega(c)$ is well inside B_n', the statement follows.

(iii) If $\chi_n=-1$, then $A_n=B_{n+1}$, so the statement follows from (ii). Now let us turn to the case that $\chi_n\geq 0$.

Let $U_0=A_n$, and for $j\geq 0$, let U_{j+1} be the return domain to U_j which has α_n in its closure. Then the return time of U_{j+1} to U_j is always $2s_n$. Notice that $U_1\not\ni c$ since $\chi_n\geq 0$. We claim that $f^{2s_n}|U_2$ is monotone. To see this, assume that $f^{2s_n}|U_1$ is not monotone and let u be the critical point of $f^{2s_n}|U_1$ which is closest to α_n. Then we have $f^{2s_n}(U_1)=[f^{2s_n}(u),\alpha_n)$. Since $\omega(u)\ni c$, $f^{2s_n}(u)\notin U_1$, and thus $U_2\subset(\alpha_n,u)$, which implies the claim.

To complete the proof, we shall prove that there exists a positive integer $N=N(\rho,b)$ such that $U_N\cap\omega(c)=\emptyset$. This is enough because $|U_N|/|B_n|$ is bounded from below by a positive constant (depending on N) and because f^{s_n} reverses the orientation at α_n.

First let us show that there exists $\mu=\mu(b)\in(0,1)$ such that $|U_j|/|B_n|\leq\mu^j$ holds for all $j\geq 0$. To see this, we observe that $f^{2s_n}|B_n$ has a a critical point on the opposite side of α_n to c. Let a be such a critical point which is closest to α_n. By Lemma 5.6, $|a-\alpha_n|/|B_n|$ is uniformly bounded away from zero. Note that $(f^{2s_n}(a),\alpha_n)\ni a$ for otherwise $f^{s_n}|B_n$ would have a periodic attractor. Let $H_0=(a,\alpha_n]\cup U_0$ and for $i\geq 1$, let H_i denote the component of $f^{-2is_n}(H_0)$ which contains α_n. Clearly, H_0 contains a definite neighborhood of U_2. Arguing as in the proof of the claim in (i), we obtain that $|U_j|/|B_n|$ decreases exponentially fast.

Let m be the maximal non-negative integer such that $U_m\cap\omega(c)\neq\emptyset$. Arguing by contradiction, assume that m is very large. Then U_m is contained in a tiny neighborhood of α_n. Let

$$P=\bigcup_{i=0}^{\infty}f^{is_n}(\partial U_2),\ \text{ and } X=B_n-P.$$

Then $P\ni\alpha_n$ is an f^{s_n}-invariant finite set and X is a nice open set. Let $X_0=U_2\cup f^{s_n}(U_2)$ which is a union of components of X. Let $r\geq 2$ be a positive integer such that $f^{rs_n}(c)\in U_m$. If both $f^{(r-1)s_n}(c)$ and $f^{(r-2)s_n}(c)$ are contained in X_0, then we have $f^{(r-2)s_n}(c)\in U_{m+1}$, which is ruled out by the maximality of m. So there is a maximal integer r' such that $r-2\leq r'<r$ and such that $y=f^{r's_n}(c)$ is contained in a component X_1 of $X-X_0$. Note that $f^{s_n}(\mathcal{L}_y(X_1))$ is contained in U_m or $f^{s_n}(U_{m+1})$ according to $r-r'=1$ or not. This implies that $\mathcal{L}_y(X_1)$ is contained in a small neighborhood of a point in $(f^{s_n}|B_n)^{-1}(\alpha_n)$. Finally, let $X_1'=\mathrm{Comp}_{X_1}(B_n-X_0)$. Noting $\partial X_1'\cap(f^{s_n}|B_n)^{-1}(\alpha_n)=\emptyset$, and $\mathcal{L}_x^5(X_1')\subset\mathcal{L}_x(X_1)$, we obtain that $\mathcal{L}_x^5(X_1')$ is deep inside X_1'; a contradiction. □

Proof of Theorem 5.6. Let I be a component of E_n. Since Λ_{E_n} and $\Lambda_{E_{n+1}}$ are both uniformly bounded, it follows from Lemma 5.7 and the previous proposition that there exists $\delta > 0$ such that

$$|I| \geq \delta |B_n|, \text{ and } \big((1+2\delta)I - (1-2\delta)I\big) \cap \omega(c) = \emptyset.$$

By choosing δ smaller, we may assume that $I' := (1+2\delta)I$ satisfies the following property: $\partial E_n \cap I' \subset \partial I$.

Let J be a component of $I - \partial \mathbf{D}_I$. We need to show that $|J|/|I|$ is bounded away from 0. If $J \subset \mathbf{D}_I$, then this is true for otherwise J would be deep inside I, which is ruled out by Corollary 5.3. Now let us assume that J is a component of $I - \mathbf{D}_I$.

Case 1. $\partial J \cap \partial I \neq \emptyset$. Let z be the common endpoint of I and J. Assume first that $z \in \{\alpha_n, \beta_n\}$. Then D_I has a component J_0 with $\partial J_0 \ni z$. Moreover, the return time of J_0 to I is either s_n or $2s_n$. Note that $|R_I(J_0)|/|I|$ is not small, because if $R_I|J_0$ is monotone, then $R_I(J_0) = I$, and otherwise, $R_I(J)$ contains a point of $\omega(c)$. Since $f^{s_n}|B_n$ has uniformly bounded derivative, it follows that $|J_0|/|I|$ is bounded away from zero. Since $J \supset J_0$, $|J|/|I|$ is bounded away from zero. Now we assume that $z \notin \{\alpha_n, \beta_n\}$. Let J_1 be the component of $\mathbf{D}_I$ which has a common endpoint with J. Let r be the return time of J_1 to I. Consider the chain $\{I'_i\}_{i=0}^{r}$ with $I'_r = I'$ as above and $I'_0 \supset J_1$. By Lemma 3.8, this chain has intersection multiplicity at most 4, and thus by Lemma 3.6, I'_0 contains a definite neighborhood of J_1. If $I'_0 \not\supset \mathrm{Cl}(J)$, then $|J|/|J_1|$ is bounded away from zero, and thus we are done. Assume $I'_0 \supset \mathrm{Cl}(J)$. Then $z \in I'_0$, and hence $f^r(z) \in I' = I'_r$. Let $\zeta := f^{s_n}(z) \in \{\alpha_n, \beta_n\}$. Note that r is a multiple of s_n, and hence $f^r(z) = \zeta$, which implies that $\zeta \in I'$. By our choice of I', $\zeta \in \partial I$. As β_n and $\tau(\beta_n)$ belong to different components of E_n, we have $\zeta = \alpha_n$ and $I = (\alpha_n, z)$. Therefore D_I has a component J_0 with $\partial J_0 \cap \partial I = \{z\}$, and the return time of J_0 to I is either s_n or $2s_n$. As above, we show that $|J|/|I| \geq |J_0|/|I|$ is bounded away from zero.

Case 2. $\partial J \cap \partial I = \emptyset$. Then there are two components J_1 and J_2 of $\mathbf{D}_I$ such that $\partial J \cap \partial J_i \neq \emptyset$, $i = 1, 2$. Let r_i be the return time of J_i to I. Without loss of generality, assume that $r_1 \leq r_2$. Let I' be as above, and consider the chain $\{I'_i\}_{i=0}^{r_2}$ with $I'_{r_2} = I'$ and $I'_0 \supset J_2$. Then as before, I'_0 contains a definite neighborhood of J_2. It suffices to show that $J_1 \cap I'_0 = \emptyset$. Arguing by contradiction, assume that there is a point $y \in \partial J_1 \cap I'_0$. Let $y' \neq y$ be the other endpoint of J. Since f^{r_2} does not have a critical point in $I'_0 - J_2$, we must have $r_1 < r_2$. Note that $r_2 - r_1$ is a multiple of s_n. Since $f^{r_1}(y) \in \partial I$, $f^{r_2}(y) = \alpha_n$ or β_n. Since $f^{r_2}(y) \in I'$, by the choice of I', it is contained in the boundary of I. But $f^{r_2}(y') \in \partial I$, and $f^{r_2}(J_2) \subset I$, so $f^{r_2}(y) \notin \mathrm{Cl}(I)$, a contradiction. □

For later use, let us include the following lemma to conclude this subsection.

Lemma 5.9. *Assume $\chi_n \geq 0$. Then f^{3s_n} has a critical point in $A_n - A_n^1$.*

Proof. Let $k \in \mathbb{N}$ be such that $R_{A_n}|A_n^1 = f^{ks_n}|A_n^1$. If $k = 1$, then f^{s_n} has a critical point in $A_n - A_n^1$, because $f^{s_n}(\partial A_n^1) \subset \partial A_n$ and because f^{s_n} reverses the orientation at α_n. Now assume that $k \geq 2$. Let U be the component of D_{A_n} such that $U \subset A_n$, and $\partial U \ni \alpha_n$. The return time of U to A_n under f is $2s_n$. Since $\chi_n \geq 0$, $U \cap A_n^1 = \emptyset$. If $f^{2s_n}|U$ is monotone, then $f^{2s_n}(U) = A_n \ni c$, and thus $f^{3s_n}|U$ has a critical point. □

5.6. Negative Schwarzian derivative property of real box mappings.

Proposition 5.11. *Let $f \in \mathcal{F}_b$. If f has essentially bounded geometry, or if f is infinitely renormalizable, then the real box mapping $\mathbf{B}_I$ associated to a sufficiently small symmetric nice interval I has negative Schwarzian derivative.*

Proof. Let $c \in \mathrm{Crit}(f)$. Let us first show that there exist $C > 1$ and $\delta > 0$ such that we can find an arbitrarily small symmetric nice interval $I \ni c$ with the following properties:

- each component of $\mathrm{dom}(\mathbf{B}_I) \cap I$ has length at least $|I|/C$;
- $((1 + 2\delta)I - I) \cap \omega(c) = \emptyset$.

If f is finitely renormalizable, then for a sufficiently small symmetric nice interval K, there are infinitely many positive integers n such that R_{K^n} display non-central returns. By Theorem 2.1, K^{n+1} contains a definite neighborhood of K^{n+2}, and so by Proposition 5.9, the statement holds. Let us now consider the case that f is infinitely renormalizable. In this case, it suffices to show that there are infinitely many properly periodic intervals B_{n+1} which have definite neighborhoods disjoint from $\omega(c) - B_{n+1}$. If there are infinitely many n such that $\chi_n \geq 0$, then this follows from Lemma 5.7. If $\chi_n = -1$ for all large n, then $\Lambda_{E_n} = 1$ for all large E_n, and so the assertion follows from Proposition 5.10 (ii).

Now fix a small symmetric nice interval I with these properties. Let J be a component of $\mathrm{dom}(\mathbf{B}_I)$ and let V be the component of $D_I \cup I$ which contains $\mathbf{B}_I(J)$. Write $\mathbf{B}_I|J = f^k|J$. Then for any $x \in J$, we have

$$Sf^{k-1}\big(f(x)\big) = \sum_{i=0}^{k-2} Sf\big(f^{i+1}(x)\big)\Big((f^i)'\big(f(x)\big)\Big)^2.$$

Let $\{J_i\}_{i=0}^k$ be the chain with $J_k = V$ and $J_0 = J$. Provided that I is sufficiently small, $f^i : J_1 \to J_{i+1}$ has distortion bounded by a constant $C_1 = C_1(\delta) > 1$ for each $0 \leq i \leq k - 1$. Therefore,

$$Sf^{k-1}\big(f(x)\big) \leq C_1^2 |J_1|^{-2} \sup_{y \in \mathrm{dom}(f)} Sf(y) \sum_{i=1}^{k} |J_i|^2.$$

Since the J_i's are pairwise disjoint, $\sum_{i=1}^{k} |J_i|^2$ is small (provided that $|I|$ is small). As Sf is uniformly bounded from above, this implies that

$$Sf^{k-1}\big(f(x)\big) \le o(1)\frac{1}{|J_1|^2}.$$

Let $a \in \mathrm{Crit}(f)$ be the critical point closest to J. Then $d(a, J) \le C|J|$, and so $Sf(x) \asymp -\frac{1}{(x-a)^2}$ for all $x \in J$. Thus

$$Sf^k(x) = Sf^{k-1}\big(f(x)\big)f'(x)^2 + Sf(x) < 0.$$

So far we have proved that f has a small symmetric nice interval $I \ni c$ such that the associated real box mapping $\mathbf{B}_I$ has negative Schwarzian derivative. For any symmetric nice interval $I' \subset I$, each branch of $\mathbf{B}_{I'}$ is a restriction of iterates of $\mathbf{B}_I$, and hence has negative Schwarzian derivative as well. □

6. A rigidity theorem

In this section, we shall prove the following rigidity theorem:

Theorem 2. *If $f : (\bigcup_{j=0}^{m} J_j) \cup (\bigcup_{i=1}^{b-1} I_i) \to \bigcup_{i=0}^{b-1} I_i$ and $\tilde{f} : (\bigcup_{j=0}^{m} \tilde{J}_j) \cup (\bigcup_{i=1}^{b-1} \tilde{I}_i) \to \bigcup_{i=0}^{b-1} \tilde{I}_i$ are two combinatorially equivalent maps in $\mathcal{G}_b$ which have essentially bounded geometry, then they are qs conjugate between the postcritical sets, that is, there is a qs map $h : \mathbb{R} \to \mathbb{R}$ such that for any $0 \le i \le b-1$ and any $n \in \mathbb{N} \cup \{0\}$, we have $h(f^n(c_i)) = \tilde{f}^n(\tilde{c}_i)$, where c_i ($\tilde{c}_i$, respectively) is the critical point of f ($\tilde{f}$, respectively) in I_i ($\tilde{I}_i$, respectively).*

The proof of the theorem uses a purely real argument. The idea goes back to Sullivan's proof of rigidity for real quadratic Feigenbaum maps, where the maps automatically have bounded geometry, [46]. Roughly speaking, the postcritical set of f can be written as the intersection of a nested sequence $E_1 \supset E_2 \supset \cdots$ of combinatorially defined open sets. The essentially bounded geometry condition will enable us to find a C-qs map $\varphi : J \to \tilde{J}$ for each k and each component J of E_k, which maps $J \cap E_{k+1}$ to $\tilde{E}_{k+1} \cap \tilde{J}$, where C is a constant independent of J. These maps can in fact be chosen appropriately to satisfy additional conditions, and can then be "glued" to provide a qs conjugacy between the postcritical sets. However, the presence of long central cascades makes the argument somewhat complicated.

By Proposition 5.11, we may assume that both of f and $\tilde{f}$ have negative Schwarzian derivative. Let Λ be an upper bound for the scaling factors of all symmetric nice intervals with respect to either f or $\tilde{f}$.

We shall first show that these two maps are topologically conjugate, by a well-known pull back argument. See Sect. 6.1.

Lemma 6.1. *Let f and $\tilde{f}$ be two maps in $\mathcal{G}_b$ which are combinatorially equivalent. Then they are topologically equivalent.*

Then we fix a topological conjugacy between f and $\tilde{f}$, and denote it by h. For any $X \subset \bigcup_{i=0}^{b-1} I_i$, set $\tilde{X} = h(X)$. Similarly, for any $x \in \bigcup_{i=0}^{b-1} I_i$, set $\tilde{x} = h(x)$.

For any interval $I \subset \text{range}(f)$ and any $U \subset I$, define

$$\Delta_0(I, U) = \max \left\{ \frac{|I|}{|J|} : J \text{ is a component of } I - \partial U, \text{ and } \partial J \cap \partial I \neq \emptyset \right\},$$

and let $\mathbb{Q}(I, U)$ be the set of all qs maps $\phi : I \to \tilde{I}$, such that $\phi = h$ on $\partial I \cup \partial U$. For any $\phi \in \mathbb{Q}(I, U)$, let $\alpha_\phi(I, U)$ be the minimal number $Q \geq 1$ such that

- for any $u, v, w \in I$ with $u < v < w$ and $v - u = w - v$,

$$\frac{1}{Q} \leq \frac{\phi(w) - \phi(v)}{\phi(v) - \phi(u)} \leq Q; \tag{16}$$

- for any $a \in \partial I$, and any $x \in I$,

$$\frac{1}{Q} \frac{|\phi(I)|}{|I|} \leq \frac{\phi(x) - \phi(a)}{x - a} \leq Q \frac{|\phi(I)|}{|I|}; \tag{17}$$

- if J is a component of U consisting of more than one point (i.e., J is a non-degenerate interval), then for any $a \in \partial J$, and any $x \in int(J)$,

$$\frac{1}{Q} \frac{|\phi(J)|}{|J|} \leq \frac{\phi(x) - \phi(a)}{x - a} \leq Q \frac{|\phi(J)|}{|J|}. \tag{18}$$

Moreover, let

$$\Delta_1(I, U) = \inf_{\phi \in \mathbb{Q}(I,U)} \alpha_\phi(I, U),$$

and

$$\Delta(I, U) = \max \big(\Delta_0(I, U), \Delta_1(I, U) \big).$$

Given a nice interval I with respect to f, define

$$\mathbf{Q}(I) = \bigcap_{n=1}^{\infty} \bigcup_{x \in \omega(c) \cap I} \mathcal{L}_x^n(I).$$

Note that if I does not contain a properly periodic interval, then $\mathbf{Q}_I = \omega(c) \cap I$; otherwise, $\mathbf{Q}_I$ is the union of maximal properly periodic intervals which are contained in I. The main step is to prove the following proposition, see Sect. 6.4.

Proposition 6.1. *For any $C > 1$, there exists $C' = C(C, f, \tilde{f}) > 1$, such that if both I and $\tilde{I}$ have C-bounded geometry with respect to f and $\tilde{f}$ respectively (see Definition 5.2), then $\Delta(I, \mathbf{Q}_I) \leq C'$.*

This will complete the proof of Theorem 2 in the case that f is finitely renormalizable. In the infinitely renormalizable case, together with the control of the initial geometry for renormalizations, as proved in Sect. 5.5, this implies the following corollary.

Corollary 6.2. *Assume that f (and so $\tilde{f}$) is infinitely renormalizable. Then there exists $C > 1$ such that for any n,*

$$\Delta(B_n, \mathbf{E}_{B_{n+1}}) \leq C,$$

where B_n is a symmetric properly periodic interval as in Sect. 5.5, and $\mathbf{E}_{B_{n+1}}$ is the union of all components of $D_{B_{n+1}}$ which intersect the critical orbits of f.

Then the proof of Theorem 2 is completed by the following gluing lemma, which will be proved in Sect. 6.2.

Lemma 6.2. ("Gluing" lemma). *Let $C > 1$ be a constant. Let I be an open interval in the range of f. Let $I = T_0 \supset T_1 \supset \cdots T_n$ be a sequence of subsets of I such that each T_i is a finite union of disjoint open intervals. Assume that for any $0 \leq i < n$, and any component P of T_i, the following hold: (i) $\Delta(P, T_{i+1} \cap P) \leq C$; (ii) if J is a component of $P - \partial T_{i+1}$ with $\partial J \cap \partial P \neq \emptyset$, then either $J \cap T_{i+1} = \emptyset$ or $J \subset T_n$. Then*

$$\Delta(I, T_n) \leq C',$$

where C' is a constant depending only on C (independent of n).

To show Proposition 6.1, we introduce an object, called an *admissible triple*, defined as follows:

Definition 6.1. Let $f \in \mathcal{G}_b$ and let $c \in \mathrm{Crit}(f)$. A triple $\mathbb{I} = (I, U, V)$ of open sets is called *admissible (with respect to f)* if the following hold:

(1) I is a nice interval;
(2) $U \subsetneq I$ is a nice open set with $U \cap \omega(c) \neq \emptyset$ and $\partial U \cap D_I = \emptyset$;
(3) V is the union of all components of $D_I \cap I$ which intersect $\omega(c) - U$.

The *critical set* $\mathrm{Crit}(\mathbb{I})$ of the admissible triple $\mathbb{I}$ is defined as follows. For each component J of V, let $\{G_j^J\}_{j=0}^{s(J)}$ denote the f-chain with $G_{s(J)}^J = I$ and $G_0^J = J$, where $s(J)$ is the return time of J to I (under f). Then

$$\mathrm{Crit}(\mathbb{I}) = \mathrm{Crit}(f) \cap \Big(\bigcup_J \bigcup_{j=0}^{s(J)-1} G_j^J \Big), \tag{19}$$

where J runs over all components of V.

For any $C > 1$, we say that the triple $\mathbb{I}$ has *C-bounded geometry* if $((1 + 1/C)I - (1 + 1/C)^{-1}I) \cap \omega(c) = \emptyset$, and if each component of $I - \partial U \cup \partial V$ has length at least $|I|/C$.

Furthermore, for any admissible triple $\mathbb{I} = (I, U, V)$, we define $\mathbb{W}(\mathbb{I})$ to be the union of all components of D_U which intersect $\omega(c) \cap V$.

In Sect. 6.4, we shall prove the following proposition by induction on $\#\mathrm{Crit}(\mathbb{I})$, which is the key step to Proposition 6.1. Some preparation for the proof is done in Sect. 6.3.

Proposition 6.3. *Let f and $\tilde{f}$ be as in Theorem 2. For any $C > 1$, there exists $C_1 = C_1(C, f, \tilde{f}) > 1$ such that the following holds. Let $\mathbb{I} = (I, U, V)$ be an admissible triple with respect to f. Assume that $\mathbb{I}$ and $\tilde{\mathbb{I}}$ have C-bounded geometry with respect to f and $\tilde{f}$ respectively. Then $\Delta(I, U \cup \mathbb{W}(\mathbb{I})) \le C_1$.*

6.1. Topological conjugacy.

Proof of Lemma 6.1. Let $A = \mathbb{R} - \mathrm{dom}(f)$, $C = \bigcup_{c \in \mathrm{Crit}(f)} \bigcup_{n=0}^{\infty} \{f^n(c)\}$, and let $\tilde{A}, \tilde{C}$ be the corresponding objects for $\tilde{f}$. Let $h_0 : \mathbb{R} \to \mathbb{R}$ be a combinatorial equivalence between f and $\tilde{f}$. Then there exists a homeomorphism $h_1 : \mathbb{R} \to \mathbb{R}$ such that $h_1 = h_0$ on A and such that $\tilde{f} \circ h_1 = h_0 \circ f$ holds on $\mathrm{dom}(f)$. Moreover, $h_1 = h_0$ on C, and so it is again a combinatorial equivalence between f and $\tilde{f}$. Repeating the argument by replacing h_0 with h_1, and so on, we obtain a sequence of homeomorphisms $h_k : \mathbb{R} \to \mathbb{R}$ such that for any $k \ge 0$, the following hold:

- $\tilde{f} \circ h_{k+1} = h_k \circ f$ holds on $\mathrm{dom}(f)$;
- $h_{k+1} = h_k$ on $\bigcup_{i=0}^{k} f^{-i}(A \cup C)$.

Since f does not have no periodic attractor, the set $X = \bigcup_{i=0}^{b-1} I_i - \bigcup_{k=1}^{\infty} f^{-k}(A \cup C)$ does not contain an interval. Similarly, $\tilde{X} = \bigcup_{i=1}^{b-1} \tilde{I}_i - \bigcup_{k=1}^{\infty} \tilde{f}^{-k}(\tilde{A} \cup \tilde{C})$ does not contain an interval either. Thus there exists a homeomorphism $h : \mathbb{R} \to \mathbb{R}$ which coincides with h_k on $f^{-k}(A \cup C)$ for all $k \in \mathbb{N}$, which gives a topological conjugacy between f and $\tilde{f}$. □

6.2. Gluing lemma.

Proof of Lemma 6.2. For each $0 \le i \le n$ and each component P of T_i, let h_P be a map in $\mathbb{Q}(P, P \cap T_{i+1})$ with $\alpha_{h_P}(P, T_{i+1} \cap P) \le C$, where $T_{n+1} = \emptyset$; moreover, in the case that $P \subset T_{i+1}$ or $i = n$, we take h_P to be affine. Let $\phi : I \to \tilde{I}$ be the homeomorphism such that for any $0 \le i \le n$ and any $x \in T_i - T_{i+1}$, $\phi(x) = h_P(x)$, where $P = \mathrm{Comp}_x(T_i)$. Let us prove that $\alpha_\phi(I, T_n)$ is bounded from above. Notice that (17) and (18) obviously hold for an appropriate choice of the constant Q. So it suffices to prove that ϕ is a qs map with a bound on the dilatation.

Claim. There exists a constant $C_1 = C_1(C) > 1$ with the following property. For any $0 \le i \le n-1$ and any component P of T_i, if $u, v \in P$ and $[u, v] \not\subset T_{i+1}$, then

$$\frac{1}{C_1} \le \frac{\big|\phi(v) - \phi(u)\big|}{\big|h_P(v) - h_P(u)\big|} \le C_1. \tag{20}$$

To prove this claim, we only need to consider the case that either u or v is contained in the boundary of T_{i+1} and $(u, v) \subset P - \partial T_{i+1}$, since the general case can be reduced to this one. Let K be the component of $P - \partial T_{i+1}$ which contains (u, v). If $K \not\subset T_{i+1}$, then $\phi(u) = h_P(u)$ and $\phi(v) = h_P(v)$, and thus the inequality (20) holds. So we may assume $K \subset T_{i+1}$. By the property of the map h_P, we have

$$\frac{\big|h_P(v) - h_P(u)\big|}{|v-u|} \asymp \frac{\big|h(K)\big|}{|K|}.$$

So it suffices to show that

$$\frac{\big|\phi(v) - \phi(u)\big|}{|v-u|} \asymp \frac{\big|h(K)\big|}{|K|}. \tag{21}$$

To see this, let us distinguish two cases.

Case 1. $|v-u| \le |K|/C$. Then $(u, v) \subset K - \partial T_{i+2}$ by the hypothesis (i). Let L be the component of $K - \partial T_{i+2}$ which contains u, v. If $L \not\subset T_{i+2}$, then $\phi = h_K$ on L, and hence (21) follows from the property of h_K. If $L \subset T_{i+2}$, then by condition (ii), either $i+2 = n$ or $L \subset T_{i+3}$. By construction, $\phi|L = h_L$ is affine, and so

$$|\phi(v) - \phi(u)| = |v-u||h(L)|/|L|.$$

Since $|L| \asymp |K|$ and $|h(L)| \asymp |h(K)|$, (21) follows.

Case 2. $|v-u|/|K| \ge C^{-1}$. For definiteness, assume that $u \in \partial K$ and $u < v$. Let $v' = u + C^{-1}|K|$. Then by what we have proved in Case 1,

$$|\phi(v') - \phi(u)| \asymp |v'-u||h(K)|/|K| = |h(K)|/C.$$

Since $|\phi(v') - \phi(u)| \le |\phi(v) - \phi(u)| \le |h(K)|$, (21) follows again. The proof of Claim 1 is completed.

Now let $u < v < w$ be three points in I with $w - v = v - u$. Let us show that

$$A := \frac{\phi(w) - \phi(v)}{\phi(v) - \phi(u)}$$

is bounded from above by a constant.

To this end, let $0 \le i \le n$ be maximal such that u, v, w are contained in a component of T_i. Without loss of generality, let us assume $i = 0$. Let

$h_0 = h_{T_0}$. By Claim 1, we may assume that one of the intervals $[u, v]$ and $[v, w]$ is contained in T_1. To fix the notation, let us assume that $[u, v]$ is contained in a component P of T_1. Then $w \notin P$. By Claim 1, $|\phi(w) - \phi(v)| \asymp |h_0(w) - h_0(v)|$. So it suffices to show that

$$\big|\phi(v) - \phi(u)\big| \asymp \big|h_0(v) - h_0(u)\big|. \tag{22}$$

Let $v' > v$ be an endpoint of P. We first assume that $(v' - v)/|P|$ is very small. If $|v' - v|/|v - u|$ is also small, then $|h_0(v') - h_0(v)|/|h_0(v') - h_0(u)|$ is very small. By Claim 1, $|\phi(v') - \phi(u)| \asymp |h_0(v') - h_0(u)|$, and $|\phi(v') - \phi(v)| \asymp |h_0(v') - h_0(v)|$. The inequality (22) follows. If $|v' - v|/|v - u|$ is not small, then we have $|v' - u| \le |P|/C$. Let P_2 be the component of $P - \partial T_2$ which contains (u, v'). If $P_2 \not\subset T_2$, then

$$\begin{aligned} \big|\phi(u) - \phi(v)\big| = \big|h_P(u) - h_P(v)\big| &\asymp \big|h_P(v) - h_P(v')\big| \\ &\asymp \big|h_0(v) - h_0(v')\big| \\ &\asymp \big|h_0(u) - h_0(v)\big|. \end{aligned}$$

In the case $P_2 \subset T_2$, we have $|\phi(u) - \phi(v)| = |h_{P_2}(u) - h_{P_2}(v)|$, and so (22) can be proved in a similar way.

Now let us assume that $|v' - v| \asymp |P|$. Since $|v - u| = |w - v| \ge |v' - v|$, it follows that $|v - u| \asymp |P|$. Let $1 \le k \le n$ be the minimal integer such that $[u, v] \not\subset T_{k+1}$ and denote by P_i the component of T_i which contains $[u, v]$ for all $0 \le i \le k$. Let $1 \le m \le k$ be minimal such that $P_k = P_m$. By the assumption (i), $|P_i|/|P_{i+1}| \ge 1 + C^{-1}$ for all $1 \le i < m$. Since $P_m = P_k \supset [u, v]$ which is commensurable to $P = P_1$, m is uniformly bounded from above. If $m = k$, then $[u, v] \not\subset T_{m+1}$, and thus by the claim above,

$$\big|\phi(v) - \phi(u)\big| \asymp \big|h_{P_m}(v) - h_{P_m}(u)\big|. \tag{23}$$

If $m < k$, then by the assumption (ii), $P_m = P_{m+1}$, which implies that $\phi|P_m = h_{P_m}$, and hence (23) holds as well. Since $|P_1| \asymp |P_2| \asymp \cdots \asymp |P_m| \asymp |v - u|$, since all the maps h_{P_i} are C-qs, (22) follows easily. □

6.3. Creating new triples. In this subsection, let $f \in \mathcal{G}_b$ and let $c \in \mathrm{Crit}(f)$.

Let $\mathbb{I} = (I, U, V)$ be an admissible triple. For any component I' of D_I with $I' \cap \omega(c) \neq \emptyset$, $I' \cap U = \emptyset$, there exists a new triple $\mathcal{P}(\mathbb{I}, I') = (I', U', V')$, where U' is the union of components J of $D_{U \cup I'}$ which intersect $I' \cap \omega(c)$ and satisfy $R_{U \cup I'}(J) \subset U$ and V' is the union of all other components of $D_{U \cup I'}$ intersecting $I' \cap \omega(c)$. (Notice that $U \cup I'$ is a nice open set.) It is clear that $\mathcal{P}(\mathbb{I}, I')$ is again an admissible triple.

Lemma 6.3. *For any $C > 1$, there is $C' > 1$ such that if the triple $\mathbb{I} = (I, U, V)$ is admissible and has C-bounded geometry, and if I' is a component of D_I with $I' \cap \omega(c) \neq \emptyset$, $I' \cap U = \emptyset$, then $\mathbb{I}' = \mathcal{P}(\mathbb{I}, I')$ has*

C'-bounded geometry. Moreover, if r is the entry time of I' to I under f, and $\{G_i\}_{i=0}^{r}$ is the f-chain with $G_r = I$ and $G_0 = I'$, then

$$Crit(\mathbb{I}') \subset Crit(\mathbb{I}) \cup \left(\bigcup_{i=0}^{r-1} G_i\right). \tag{24}$$

Remark 6.1. If I' appears as one of the intervals $G_j^J, 0 \le j \le s_J - 1$ in (19), then $\bigcup_{i=0}^{r-1} G_i \subset \bigcup_{j=0}^{s_J-1} G_j^J$, and hence $\text{Crit}(\mathbb{I}') \subset \text{Crit}(\mathbb{I})$. In particular, it is the case if $I' = \mathcal{L}_{c'}(I)$ for some $c' \in \text{Crit}(\mathbb{I})$.

Proof of Lemma 6.3. Let us write $\mathbb{I}' = (I', U', V')$. Let T be the union of components of V which are components of $D_{I'} \cup I'$. For any $x \in \omega(c) \cap I'$, let $k = k(x)$ be the minimal positive integer such that $R_I^k(x)$ belongs to a component A of $U \cup T$. Then the pull back of A along the orbit $\{x, f(x), \ldots, R_I^k(x)\}$ is exactly $\text{Comp}_x(U' \cup V')$. It follows easily from this observation that (24) holds.

By assumption, $(1 + 1/C)I - (1 + 1/C)^{-1}I$ is disjoint from $\omega(c)$. By Lemmas 3.8 and 3.6, there exists $\delta = \delta(C) > 0$, such that for any entry domain M to I, we have

$$\big((1 + 2\delta)M - (1 - 2\delta)M\big) \cap \omega(c) = \emptyset. \tag{25}$$

It follows that for any $x \in M \cap \omega(c)$, we have $|\mathcal{L}_x(M)| \asymp |M|$, because otherwise $\mathcal{L}_x(M)$ would be deep inside M, which contradicts the assumption that f has essentially bounded geometry by Corollary 5.3. In particular, the length of each component of $U' \cup V'$ is comparable to $|I'|$ since it contains an interval of the form $\mathcal{L}_x(I')$ with $x \in I' \cap \omega(c)$. To complete the proof of this lemma, it remains to show that for any non-degenerate component J of $I' - U' \cup V'$, $|J|/|I'|$ is not so small. To this end, we first prove

Claim. For any component M of D_I, and any $x \in \omega(c) \cap M$, if L is a component of $M - \mathcal{L}_x(U \cup T)$, then $|L|/|M|$ is uniformly bounded away from zero.

As above, let $k = k(x)$ be the minimal positive integer such that $R_I^k(x) \in U \cup T$. Let H_k denote the component of $U \cup T$ which contains $R_I^k(x)$, and let $H_i = \mathcal{L}_{R^i(x)}(U \cup T) = \mathcal{L}_{R^i(x)}(H_k)$ for any $1 \le i \le k - 1$.

Let us prove that for any $\xi > 0$ there exists $\xi' > 0$ such that if $(1 + 2\xi)H_i \subset I$ holds for some $1 \le i \le k$, then each component of $M - \mathcal{L}_x(U \cup T)$ has length $\ge \xi'|M|$. In fact, by Theorem 3.3, there exists $\xi'' > 0$ such that $M \supset (1 + 2\xi'')\mathcal{L}_x(U \cup T)$. Together with (25), this implies the statement.

If $\partial H_k \cap \partial I = \emptyset$, then it follows from the bounded geometry property of $\mathbb{I}$ that I contains a definite neighborhood of H_k, and thus the claim holds. Assume $\partial H_k \cap \partial I \neq \emptyset$. Let $Y_{k-1} = \mathcal{L}_{R_I^{k-1}(x)}(I)$. Let us prove that each component P of $Y_{k-1} - H_{k-1}$ is commensurable to Y_{k-1}. In fact, since the first entry map $R_I|Y_{k-1}$ has uniformly good distortion, it suffices to show that $|R_I(P)|/|I|$ is bounded away from zero. But $R_I(P)$ is either equal to I or it is bounded by an endpoint of I and a point in $\omega(c) \cap I$, and so this follows from the bounded geometry property of $\mathbb{I}$.

In particular, the claim holds if $k = 1$. Assume $k \geq 2$. In this case, by what we have proved above, either $\partial H_{k-1} \cap \partial I \neq \emptyset$ or I contains a definite neighborhood of H_{k-1}. As we have already proved, the claim holds in the latter case. Let us consider the former case. Let $Y_{k-2} = \mathcal{L}_{R_I^{k-2}(x)}(I)$. Then by a similar argument as above, we prove that each component of $Y_{k-2} - H_{k-2}$ is commensurable to Y_{k-2}, and in particular, the claim holds if $k = 2$. If $k \geq 3$ and $H_{k-i} \cap \partial I \neq \emptyset$ for $i = 0, 1$, then H_{k-2} is well inside I since it is disjoint from $H_k \cup H_{k-1}$, and thus the claim follows again. The proof of the claim is completed.

Let us continue the proof of this lemma. Let J be a non-degenerate component of $I' - U' \cup V'$ with $\partial J \cap \partial I' \neq \emptyset$. Let J_1 be the component of $U' \cup V'$ with $\partial J_1 \cap \partial J \neq \emptyset$. Then J is a component of $I' - J_1$, and so by the claim above, $|J|/|J_1|$ is bounded away from zero. Since $|J_1|/|I'|$ is also bounded away from zero, so is $|J|/|I'|$.

Now let J be a non-degenerate component of $I' - U' \cup V'$ with $\partial J \cap \partial I' = \emptyset$. Then there are two distinct components J_1, J_2 of $U' \cup V'$ such that $\partial J \cap \partial J_i \neq \emptyset$, $i = 1, 2$. Let $n_i \in \mathbb{N}$ be the entry time of J_i to $U \cup I'$ under f, $i = 1, 2$. Let us assume $n_1 \leq n_2$. To show that $|J|/|I'|$ is bounded away from zero, it suffices to show that $|J|/|J_2|$ is not small.

Let $m \in \mathbb{N}$ be the minimal positive integer such that $R_I^m(J_2) \subset U \cup T$. Let P_i be the component of $U \cup V$ which contains $R_I^i(J_2)$ for any $1 \leq i \leq m$. Then for all $1 \leq i \leq m - 1$, we have $P_i \subset V - T$. Let $1 \leq m_1 \leq m$ be minimal such that

$$J_1 \not\subset \mathrm{Comp}_{J_2}\big((R_I^{m_1})^{-1}(P_{m_1})\big). \tag{26}$$

Such an integer m_1 exists since $J_2 = \mathrm{Comp}_{J_2}((R_I^m)^{-1}(P_m))$ is disjoint from J_1. Let q be the positive integer such that $R_I^{m_1}|J_2 = f^q|J_2$. By the minimality of m_1, we have

$$f^q(J_1) = R_I^{m_1}(J_1) \subset I.$$

Let $Q_{m_1} = \mathrm{Comp}_{f^q(J_1)}(U \cup V)$. Let $P = P_{m_1}$ if $m_1 = m$ and $\mathcal{L}_{f^q(J_2)}(U \cup T)$ otherwise. Let $Q = Q_{m_1}$ if $Q_{m_1} \subset U \cup T$, and $\mathcal{L}_{f^q(J_1)}(U \cup T)$ otherwise.

From the bounded geometry property of $\mathbb{I}$, it follows that there is a definite neighborhood P' of P which is disjoint from $\omega(c) - P$. Consider the chain $\{G'_j\}_{j=0}^q$ with $G'_q = P'$ and $G'_0 \supset J_2$. Then by Lemmas 3.8 and 3.6, G'_0 contains a definite neighborhood of J_2. So we may assume that $G'_0 \cap J_1 \neq \emptyset$. Since $f^q|G'_0$ has all its critical points in J_2 and since $\partial J_1 \cap \partial J_2 = \emptyset$, it follows that $\partial P \cap \partial Q = \emptyset$. By the claim above, this implies that $dist(P, Q)/|P|$ is bounded from zero. Thus there is a definite neighborhood P'' of P which is contained in P' and disjoint from Q. Applying Lemmas 3.8 and 3.6 to the chain $\{G''_j\}_{j=0}^q$ with $G_q = P''$ and $G''_0 \supset J_2$, we see that G''_0 contains a definite neighborhood of J_2. As $G''_0 \cap J_1 = \emptyset$, it follows that $|J|/|J_2|$ is not small. □

Corollary 6.4. *If I is a nice interval which has C-bounded geometry, and if I' is a component of D_I which intersects $\omega(c)$, then I' has C'-bounded geometry.*

Proof. Let U be a component of $\mathbf{D}_I$ which is also contained in $D_{I'}$. Such a component exists since $\omega(c) \cap I' \neq \emptyset$. Let V be all other components of $D_I \cap I$ intersecting $\omega(c)$. Applying the previous proposition to the admissible triple (I, U, V), we see that $\mathcal{P}(\mathbb{I}, I') = (I', U', V')$ has bounded geometry. Noticing that $U' \cup V'$ is exactly $\mathbf{D}_{I'}$, the corollary follows. □

6.4. The main step. Throughout this subsection, let f and $\tilde{f}$ be as in Theorem 2. Our goal is to prove Proposition 6.3. We are going to use the following lemma frequently.

For any $C > 1$, let $\mathcal{U}_C$ denote the collection of open intervals I contained in the range of f with the property that $(1+1/C)I - (1+1/C)^{-1}I$ is disjoint from the critical orbits of f. Similarly we define $\tilde{\mathcal{U}}_C$.

Lemma 6.4. *For each $C > 1$ and $p \in \mathbb{N}$, there exists a constant $C' = C'(C, p) > 1$ with the following property. Let I be an open interval, and let $U \subset I$ be an open set. Let $\{I_i\}_{i=0}^s$ be a chain with $I_s = I$, and let $U_0 = f^{-s}(U) \cap I_0$. Assume that I and each component of U belong to $\mathcal{U}_C$, and that $\tilde{I}$ and each component of $\tilde{U}$ belong to $\tilde{\mathcal{U}}_C$. If the order of the chain $\{I_i\}_{i=0}^s$ is not greater than p, and if $\Delta(I, U) \leq C$, then*

(i) I_0 and each component of U_0 belong to $\mathcal{U}_{C'}$;
(ii) $\tilde{I}_0$ and each component of $\tilde{U}_0$ belong to $\tilde{\mathcal{U}}_{C'}$;
(iii) $\Delta(I_0, U_0) \leq C'$.

Proof. The first and second statements follow from Lemma 3.6. An upper bound on $\Delta_0(I_0, U_0)$ also follows from that lemma.

To check that $\Delta_1(I_0, U_0)$ is bounded, let $\phi : (I, U) \to (\tilde{I}, \tilde{U})$ be a map in $\mathbb{Q}(I, U)$, with $\alpha_\phi(I, U) = \Delta_1(I, U)$. We only need to treat the case where the chain $\{I_i\}_{i=0}^s$ is monotone and the case where $s = 1$ and I_0 contains a critical point, since the general case then follows by induction. If the chain $\{I_i\}$ is monotone, then $\phi_0 = (\tilde{f}^s|\tilde{I}_0)^{-1} \circ \phi \circ f^s|I_0 \in \mathbb{Q}(I_0, U_0)$. Since the diffeomorphisms $f^s|I_0$ and $\tilde{f}^s|\tilde{I}_0$ have bounded distortion, $\alpha_{\phi_0}(I_0, U_0)$ is of order $\alpha_\phi(I, U)$. Thus $\Delta_1(I_0, U_0)$ is bounded from above. Now assume that $s = 1$ and I_0 contains a critical point, say c. Let J be the component of U which contains $f(c)$. By assumption, $f(c)$ ($\tilde{f}(\tilde{c})$, respectively) divides J ($\tilde{J}$, respectively) into two commensurable parts. Since ϕ is a C-qs map, $\phi(f(c))$ divides $\tilde{J}$ into commensurable parts as well. Thus there is a diffeomorphism ψ of $\tilde{I}$, with bounded distortion, such that $\psi(\phi(f(c))) = \tilde{f}(\tilde{c})$, and such that $\psi = id$ outside $\tilde{J}$. The map $\psi \circ \phi$ belongs to $\mathbb{Q}(I, U)$, and $\alpha_{\psi\circ\phi}$ is again bounded. By pulling back this map, we define a map $\phi_0 \in \mathbb{Q}(I_0, U_0)$. It is not difficult to check that $\alpha_{\phi_0}(I_0, U_0)$ is bounded, and thus so is $\Delta_1(I_0, U_0)$. □

Proof of Proposition 6.3. We shall prove this proposition by induction on $\#\mathrm{Crit}(\mathbb{I})$.

Starting step. Assume that $\#\mathrm{Crit}(\mathbb{I}) = 0$. Then for each component J of V, $R_I : J \to I$ is a diffeomorphism, where R_I is the first return map under f to I. For any $n \in \mathbb{N}$, let

$$S_n = \left\{x \in I : R_I^k(x) \in V \text{ for all } 0 \le k \le n\right\},$$

and

$$S_n' = \left\{x \in I : R_I^k(x) \in V \text{ for all } 0 \le k \le n-1, \text{ and } R_I^n(x) \in U\right\}.$$

For each component J of S_n, $R_I^{n+1}|J : J \to I$ is a diffeomorphism, with bounded distortion (since it extends to a diffeomorphism onto $(1 + 1/C)I$ with negative Schwarzian). By the bounded geometry of the configuration $(I, U \cup V)$, there exist constants $0 < \mu_1 < \mu_2 < 1$ depending only on C, such that for any component J' of $S_n \cup S_n'$,

$$\mu_1^{n+1} \le |J'|/|I| \le \mu_2^{n+1}.$$

Thus, for each n, $\Delta(I, S_n \cup (\bigcup_{i=0}^n S_n'))$ is bounded by a constant depending on C and n. So it suffices to show that there exists $p = p(C, f) \in \mathbb{N}$ such that $S_p \cap \omega(c) = \emptyset$.

To this end, let n be the maximal non-negative integer such that S_n has a component J which intersects $\omega(c)$. Arguing by contradiction, assume that n is large. Then $|J|/|I|$ is small and thus J is deep inside I, since $I - (1 + 1/C)^{-1}I$ is disjoint from $\omega(c)$.

Let us show that this contradicts the hypothesis that f has essentially bounded geometry. Let x be an arbitrary point in $U \cap \omega(c)$. Let $0 = s_0 < s_1 < s_2 < \cdots$ be all the nonnegative integers such that $R_I^{s_k}(x) \in U$. Let s be the minimal positive integer such that $R_I^s(x) \in J$. Such an s exists because $\omega(x) = \omega(c) \ni x$. Let k be maximal such that $s_k < s$, and write $z = R_I^{s_k}(x)$, $w = R_I^{s_k+1}(x)$. Note that $R_I(\mathcal{L}_z(U)) \subset \mathcal{L}_w(U)$, and $R_I^{s-s_k-1}(\mathcal{L}_w(U)) \subset J$. Since the pull back of I along the orbit $\{w, f(w), \dots, R_I^s(x)\}$ is monotone, $\mathcal{L}_w(U)$ is deep inside I. The chain corresponding to $R_I|\mathcal{L}_z(I)$ has bounded order, and thus $\mathcal{L}_z(U)$ is deep inside $\mathcal{L}_z(I) \subset U$, which is ruled out by Corollary 5.3.

Induction step. Let N be a positive integer and assume that the proposition holds in the case $\#\mathrm{Crit}(\mathbb{I}) < N$. We shall prove it in the case $\#\mathrm{Crit}(\mathbb{I}) = N$. For that purpose, we shall first prove the following proposition.

Proposition 6.5. *For any $C > 1$, there exists $C' > 1$ with the following property. Let $\mathbb{I} = (I, U, V)$ be an admissible triple such that I is a symmetric nice interval which contains a critical point c, #Crit$(\mathbb{I}) = N$, and Crit$(\mathbb{I}) \ni c$. Assume that $\mathbb{I}$ and $\tilde{\mathbb{I}}$ have C-bounded geometry with respect to f and $\tilde{f}$ respectively. Then $\Delta(I, U \cup \mathbb{W}(\mathbb{I})) \leq C'$.*

Proof. Let $L = \mathcal{L}_c(U)$. Since U contains a component of $\mathbf{D}_I$, L contains an interval from $\mathcal{M}(I)$. Therefore, there exists a maximal non-negative integer $n \geq 0$ such that $I^n \supsetneq L$. Note that $L \subset I^1$, so if $n = 0$, then $I^1 = L$. In this case, the proposition follows from the induction hypothesis, applying to the triple $(I, U \cup I^1, V - I^1)$. So let us assume that $n \geq 1$. Let $m(0) = 0$ and let $m(1) < m(2) < \cdots < m(k)$ be all the positive integers which are not greater than n and satisfy $R_{I^{m(i)-1}}(c) \notin I^{m(i)}$, $1 \leq i \leq k$. Note that $m(k) = n$, and let $m(k+1) = n + 1$.

Lemma 6.5. *There exists $N = N(f)$ such that $k \leq N(f)$.*

Proof. By Theorems 2.1 and 3.3, $\mathcal{L}_c(I^{m(j)+1})$ is uniformly well inside $\mathcal{L}_c(I^{m(j)})$ for each $1 \leq j \leq k - 1$. Let J be an arbitrary component of U. By Proposition 5.2, $\mathcal{L}_c(J) \subset L$ is commensurable to $\mathcal{L}_c(I) = I^1$, and so k is bounded. □

Let $L_i = I^i$ for each $0 \leq i \leq n$, and let $L_{n+1} = L$. For each $1 \leq i \leq n + 1$, let R_i denote the first entry map to $U \cup L_i$ under f, and let S_i denote the union of all the components of $U \cup L_i \cup D_{U \cup L_i}$ which intersect $\omega(c) \cap I$. Let $U_0 := U$, $V_0 := V$ and $\mathbb{I}_0 := \mathbb{I}$. For each $0 \leq i \leq n - 1$, inductively define

$$\mathbb{I}_{i+1} := \mathcal{P}(\mathbb{I}_i, I^{i+1}) = (I^{i+1}, U_{i+1}, V_{i+1}).$$

By maximality of n, $L_{i+1} \subset V_i$ for all $0 \leq i \leq n - 1$ and $L \subset U_n$. For each $0 \leq i \leq n$, let $U_i' = U_i \cup L_{i+1}$ and $V_i' = V_i - L_{i+1}$, let $\mathbb{I}_i' = (I^i, U_i', V_i')$ and let $W_i = \mathbb{W}(\mathbb{I}_i')$. Then $U_i' \cup W_i = S_{i+1} \cap I^i$. By Lemma 6.3 (see also Remark 6.1), for each $0 \leq i \leq n-1$, we have Crit$(\mathbb{I}_{i+1}) \subset$ Crit$(\mathbb{I}_i)$. Thus for each $0 \leq i \leq n$, Crit$(\mathbb{I}_i') \subset$ Crit$(\mathbb{I}_i) - \{c\} \subset$ Crit$(\mathbb{I}) - \{c\}$, and in particular,

$$\#\text{Crit}(\mathbb{I}_i') \leq N - 1.$$

Lemma 6.6. *For any $q \in \mathbb{N}$, there exists $C_q > 1$ such that for any $0 \leq i \leq \min\{q, n\}$, the following hold:*

(1) $\Delta(I, S_{i+1}) \leq C_q$;
(2) *the triple $\mathbb{I}_i$ has C_q-bounded geometry.*

Proof. The second statement follows from Lemma 6.3 by induction on i. Moreover by induction hypothesis applying to the triples $\mathbb{I}_i'$, it follows that for each $0 \leq i \leq \min(q, n)$, $\Delta(I^i, S_{i+1} \cap I^i) = \Delta(I^i, U_i' \cup W_i)$ is bounded from above by a constant depending only on C and q.

To prove that $\Delta(I, S_{i+1})$ is bounded from above we shall apply Lemma 6.2. Set $T_0 = I$, and $T_i = S_i$ for $1 \le i \le \min(q, n) + 1$. To check the conditions in that lemma, fix an $i \le \min(q, n)$, and let P be a component of S_i. Let us first show that $\Delta(P, S_{i+1})$ is bounded from above. This is true for $P = I^i$ as checked above, and also true if $R_i(P) \subset U$ since in this case, $P \subset S_{i+1}$. The only remaining case is that P is an entry domain to I^i, when the claim follows from Lemma 6.4. Now let us check the second condition. To this end, let J be an outermost component of $P - \partial S_{i+1}$, and assume that it is a component of S_{i+1}. Note that L_{i+1} is compactly contained in L_i, and hence $R_{i+1}(J) \subset U$. It follows that $J \subset S_j$ for all $j \ge i + 1$. So both of the conditions in Lemma 6.2 are satisfied, and the proof of this lemma is completed. □

Lemma 6.7. *There exists* $C' > 1$ *such that*

(1) $\Delta(I, S_{m(1)}) \le C'$;
(2) *the triple* $\mathbb{I}_{m(1)}$ *has* C'*-bounded geometry.*

Proof. Let $l = l(C^{-1}, \Lambda, b)$ be as in Proposition 5.1. By Lemma 6.6, we may assume that $m(1)$ is large. In particular, we may assume that $m(1) > 3l$ and that I is so small that Proposition 5.1 applies. Then we have a saddle node central cascade

$$I^l \supset I^{l+1} \supset \cdots I^{m(1)-l}.$$

It suffices to show that

(i) $\Delta(I^l, S_{m(1)-l} \cap I^l)$ is uniformly bounded;
(ii) the triple $\mathbb{I}_{m(1)-l}$ has uniformly bounded geometry.

In fact, assume that these two statements are true, then by the previous lemma, we see that both of $\mathbb{I}_{m(1)}$ and $\Delta(I^l, S_{m(1)} \cap I^l)$ are bounded. Set $T_0 = I$, $T_1 = S_l$, and $T^2 = S_{m(1)}$. Arguing similarly as in the proof of the previous lemma, we see that these open sets satisfy the conditions in Lemma 6.2, and thus $\Delta(I, S^{m(1)})$ is uniformly bounded.

To prove (i) and (ii), let P be the interior of the component of $I^l - I^{l+1}$ which is contained in $R_{I^0}(I^{l+1})$. Note that $P \cup I^{m(1)-l}$ is a nice open set. Let $U' = P \cap U_l$, and $V' = P \cap V_l$. Let X_0 be the union of the components J of $D_{P \cup I^{m(1)-l}}$ which intersect $V' \cap \omega(c)$ such that $R(J) \subset I^{m(1)-l}$, where R denotes the first entry map (under f) to $I^{m(1)-l} \cup P$ and $D_{P \cup I^{m(1)-l}}$ is the domain of R. Let $X = X_0 \cup U'$, and let Y be the union of components J of $D_{P \cup I^{m(1)-l}}$ which intersect $V' \cap \omega(c)$ such that $R(J) \subset P$.

Notice that for each $x \in \omega(c) \cap V'$, if $R_{I^l}(x) \in I^i - I^{i+1}$ for some $l \le i \le m(1) - l - 1$, then $\mathrm{Comp}_x(X_0 \cup Y) \subset Y$, and it is a component of $(R_{I^l})^{-1}(I^i - I^{i+1})$; if $R_{I^l}(x) \in I^{m(1)-l}$, then $\mathrm{Comp}_x(X_0 \cup Y) \subset X_0$, and it is a component of $(R_{I^l})^{-1}(I^{m(1)-l})$. Moreover, if $R_{I^l}(x) \in I^i - I^{i+1}$, then $\max(l - i, m(1) - l - i)$ can not be too large by Theorem 5.4. It follows easily from these observations that (P, X, Y) is an admissible triple with uniformly bounded geometry and

$$\mathrm{Crit}(P, X, Y) \subset \mathrm{Crit}(\mathbb{I}_l').$$

Similarly, we show that $(\tilde{P}, \tilde{X}, \tilde{Y})$ is an admissible triple with uniformly bounded geometry for $\tilde{f}$. Let $Z = \mathbb{W}((P, X, Y))$. Then by the induction hypothesis, $\Delta(P, X \cup Z) = \Delta(P, S_{m(1)-l})$ is uniformly bounded.

Let J be a component of $I^{i-1} - I^i$ for any $l+1 \leq i \leq m(1) - l$. Let us show that $\Delta(J, J \cap S_{m(1)-l})$ is uniformly bounded. For $J = P$ this has been verified above. If J is the other component of $I^l - I^{l+1}$, then this follows from symmetry. In all other cases, this follows from the fact that $R_0^{i-l-1} : J \to P$ and $\tilde{R}_0^{i-l-1} : \tilde{J} \to \tilde{P}$ are diffeomorphisms with uniformly bounded distortion.

Let $T_0 = I^l$, $T_1 = T_0 - \bigcup_{i=l}^{m-l} \partial I^i$, and $T_2 = S_{m(1)-l} \cap I^l$. By Theorem 5.4, $\Delta(T_0, T_1)$ is uniformly bounded. By Lemma 6.2, it follows that $\Delta(I^l, I^l \cap S_{m(1)-l})$ is bounded. This completes the proof of (i).

Finally, let us prove (ii). Let A be the union of components K of $D_{I^{m(1)-l} \cup P}$ such that $K \cap I^{m(1)-l} \cap \omega(c) \neq \emptyset$ and $R_{I^{m(1)-l} \cup P}(K) \subset I^{m(1)-l}$, and let B be the union of all other components of $D_{I^{m(1)-l} \cup P}$ which intersect $I^{m(1)-l} \cap \omega(c)$. Notice that each component of A is a component of $(R_I)^{-1}(I^{m(1)-l})$, and each component of B is a component of $(R_I)^{-1}(I^{m(1)-l-1} - I^{m(1)-l})$. Thus $(I^{m(1)-l}, B, A)$ is an admissible triple with uniformly bounded geometry. By Lemma 6.3, for any component K of B, $\mathcal{P}((P, X, Y), K)$ has uniformly bounded geometry. By construction, neither component of Y can be an entry domain to K, and so $\mathcal{P}((P, X, Y), K) = (K, (U_{m(1)-l} \cup V_{m(1)-l}) \cap K, \emptyset)$. The statement (ii) follows. □

Applying Lemma 6.7 to the triple $\mathbb{I}_{m(1)}$, and so on, we prove that $\Delta(I_{m(i)-1}, S_{m(i)})$ is uniformly bounded for all $0 \leq i \leq k+1$. Here we use the fact that k is uniformly bounded. Applying Lemma 6.2 once again, we conclude that $\Delta(I, U \cup \mathbb{W}(\mathbb{I})) = \Delta(I, S_{n+1})$ is bounded. The proof of Proposition 6.5 is completed. □

Completion of the induction step. Let us consider a triple $\mathbb{I} = (I, U, V)$ with $\mathrm{Crit}(\mathbb{I}) = N$. Let c be a critical point of f which is contained in $\mathrm{Crit}(\mathbb{I})$, let $I' = \mathcal{L}_c(I)$, and let $\mathbb{I}' := \mathcal{P}(\mathbb{I}, I') = (I', U', V')$. By Remark 6.1, $\mathrm{Crit}(\mathbb{I}') \subset \mathrm{Crit}(\mathbb{I})$. Note that for any $x \in \omega(c) \cap I'$, $\mathrm{Comp}_x(U' \cup \mathbb{W}(\mathbb{I}')) = \mathcal{L}_x(U)$. Let $\mathbf{E}_U$ denote the union of components of D_U which intersect $\omega(c)$. By Proposition 6.5, $\Delta(I', \mathbf{E}_U \cap I')$ is bounded.

For each component J of V, there is a minimal non-negative integer j such that $f^j(J)$ is a component of D_I which contains a critical point in $\mathrm{Crit}(\mathbb{I})$. Moreover, the diffeomorphisms $f^j|J$ and $\tilde{f}^j|\tilde{J}$ have uniformly bounded distortion. Since f^j maps each component of $\mathbf{E}_U \cap J$ onto a component of $\mathbf{E}_U \cap f^j(J)$, it follows that $\Delta(J, \mathbf{E}_U \cap J)$ is bounded. Finally, setting $T_0 = I$, $T_1 = U \cup V$, and $T_2 = \mathbb{W}(\mathbb{I}) \cup U$, and applying Lemma 6.2, we conclude that $\Delta(I, U \cup \mathbb{W}(\mathbb{I})$ is uniformly bounded. The proof of the induction step, and thus the proof of Proposition 6.3 is completed. □

Proof of Proposition 6.1. Let us first consider the case that I is a symmetric nice interval. Let $m(0) = 0$, and $m(1) < m(2) < \cdots$ be all the positive integers such that $R_{I^{m(i)-1}}$ displays a non-central return. Let $\chi = \chi_I$ be the height of I, i.e., the number of all the positive integers $m(i)$. The height χ is finite if and only if $\mathbf{T}(I) := \bigcap_n I^n$ is a properly periodic interval. For $n = 0, 1, \ldots,$ let S_n denote the union of all the components of $I^n \cup D_{I^n}$ which intersect $\omega(c) \cap I$. Then $D_0 \supset D_1 \supset \cdots$.

By Theorem 5.5, the intervals $I^{m(i)}$, $0 \le i \le \chi$, have uniformly bounded geometry. Using the same argument as in the proof of Lemma 6.7, but replacing the induction hypothesis with Proposition 6.3, we prove that there exists a constant C' such that for any $0 \le i \le \chi - 1$, $\Delta(I^{m(i)}, S_{m(i+1)} \cap I^{m(i)}) \le C'$. By Lemma 6.4, there exists a constant C'' such that for any other component P of $S_{m(i)}$, $\Delta(P, S_{m(i+1)} \cap P) \le C''$. By Lemma 6.2, there exists a C-qs map $\phi : I \to \tilde{I}$, which coincides with h on $\bigcup_n \partial S_n$.

If $\mathbf{T}(I) = \{c\}$, then for any $x \in \omega(c) \cap I$, we have $\bigcap_{n=0}^{\infty} \mathrm{Comp}_x(S_n) = \{x\}$. So ϕ is a map in the class $\mathbb{Q}(I, \mathbf{Q}_I)$ and $\alpha_\psi(I, \mathbf{Q}_I)$ is bounded.

Assume now that $\mathbf{T}(I)$ is a properly periodic interval of f. Note that each component of $\mathbf{Q}_I$ is well inside a component of $D_{m(\chi)}$. So by redefining the map ϕ in the open set $S_{m(\chi)}$ appropriately, we can obtain a map $\psi \in \mathbb{Q}(I, \mathbf{Q}_I)$ with a bound on $\alpha_\psi(I, \mathbf{Q}_I)$.

Now let us consider the general case. Let U be the union of components of $\mathbf{D}_I$ on which the first return map R_I is not monotone, and let V be the other components of $\mathbf{D}_I$. Note that $U \ne \emptyset$. Applying Proposition 6.3 to the triple (I, U, V), we see that $\Delta(I, W)$ is uniformly bounded, where W is the union of components of $U \cup D_U$ intersecting $\omega(c) \cap I$. For each component J of U, there exists a minimal integer $j \ge 0$ such that $J' = f^j(J)$ is a critical return domain to I. Then J' is a symmetric nice interval, so by what we have proved above, $\Delta(J', \mathbf{Q}_{J'})$ is uniformly bounded. As $f^j|J$ and $\tilde{f}^j|\tilde{J}$ are diffeomorphisms with bounded distortion, this implies that $\Delta(J, \mathbf{Q}_J)$ is uniformly bounded. Note that $\mathbf{Q}_J = \mathbf{Q}_I \cap J$, and so $\Delta(J, \mathbf{Q}_I \cap J)$ is uniformly bounded. For each component of $W - U$, the corresponding statement remains true by Lemma 6.4. By Lemma 6.2, the proposition follows. □

6.5. Completion of the proof of Theorem 2.

Proof of Theorem 2 in the finitely renormalizable case. Let c be a critical point of f, and let $I \ni c$ be a sufficiently small symmetric nice interval such that $\mathbf{T}(I) = \bigcap_{i=0}^{\infty} I^i = \{c\}$. By Proposition 6.1, for each component U of $D_I \cup I$ which intersects $\omega(c)$, there exists a qs map $\phi_U : U \to \tilde{U}$ such that for each $n \in \mathbb{N}$ with $f^n(c) \in U$, we have $\phi_U(f^n(c)) = \tilde{f}^n(\tilde{c})$. As there are only finitely many such components U, there exists a qs homeomorphism ϕ of the real line which coincides with ϕ_U on U for each U. This map ϕ is the partial conjugacy as desired. □

Now we turn to the proof of Theorem 2 in the case that f is infinitely renormalizable. Let c be a critical point of f, and let $s_n, \alpha_n, B_n, A_n, E_n$ be as defined in Sect. 5.5.

Proposition 6.6. *Assume that f is infinitely renormalizable. Then there is a $C > 1$ such that for any n, $\Delta(B_n, \mathbf{E}_{B_{n+1}} \cap B_n) \leq C$.*

Proof. We distinguish two cases.

Case 1. A_n is a properly periodic interval of f. That is, the n-th renormalization $f^{s_n} : B_n \to B_n$ is immediately renormalizable. In this case, $\mathbf{E}_{B_{n+1}} \cap B_n$ consists of two adjacent intervals. Each of them is commensurable to, and well inside in I. The same holds for the objects with tilde. So the proposition holds.

Case 2. A_n is not properly periodic. Then by Theorem 5.6, each component J of E_n has uniformly bounded geometry. By Proposition 6.1, it follows that $\Delta(J, \mathbf{E}_{B_{n+1}} \cap J)$ is uniformly bounded. Thus $\Delta(B_n, \mathbf{E}_{B_{n+1}} \cap B_n)$ is uniformly bounded. □

Proof of Theorem 2 in the infinitely renormalizable case. By Lemma 6.2, it suffices to prove that for any n and any component P of $\mathbf{E}_{B_n}$, $\Delta(P, P \cap \mathbf{E}_{B_{n+1}})$ is uniformly bounded. If $P = B_n$, then this has been proved by Proposition 6.6. For any other component, this follows from Theorem 5.6 and Lemma 6.4. □

7. Background in complex analysis

7.1. Poincaré disks and the Schwarz lemma. For an open interval I, we define $\mathbb{C}_I = \mathbb{C} - (\mathbb{R} - I)$. This is a simply connected Riemann surface conformally equivalent to the upper half plane $\mathbb{H}$, and thus carries a hyperbolic metric. For any $\theta \in (0, \pi)$, we define the *Poincaré disk* $D_\theta(I)$ by

$$D_\theta(I) = \left\{z \in \mathbb{C}_I : d(z, I) < \log\tan(\pi/2 - \theta/4)\right\},$$

where d denote the hyperbolic distance in $\mathbb{C}_I$. As noticed in [46], if $\theta \leq \pi/2$, then $D_\theta(I)$ is the union of two Euclidean disks which are symmetric with respect to $\mathbb{R}$ and intersect $\mathbb{R}$ on I with external angle θ at each intersection point. We shall also use the notation $D_*(I) := D_{\pi/2}(I)$.

For any $a \in \mathbb{R}$ and any $\varepsilon \in (0, \pi)$, define

$$S^+(a, \varepsilon) = \left\{z = a + re^{i\theta} : r \geq 0, |\theta| \leq \varepsilon\right\},$$
$$S^-(a, \varepsilon) = \left\{z = a + re^{i\theta} : r \geq 0, \pi - \varepsilon \leq \theta \leq \pi + \varepsilon\right\}.$$

Moreover, for any bounded interval $I = (a, b)$ with $a < b$ and any $\varepsilon \in (0, \pi)$, let

$$S(I, \varepsilon) = S^-(a, \varepsilon) \cup S^+(b, \varepsilon).$$

Lemma 7.1. (Schwarz). *Let* $0 \le \varepsilon < \theta < \pi$ *and let* h *be a univalent function defined on* $D_\varepsilon((0,1))$ *which maps* $(0,1)$ *onto itself. Then*

$$h\Big(D_\theta\big((0,1)\big)\Big) \subset D_{\frac{\pi}{\pi-\varepsilon}(\theta-\varepsilon)}\big((0,1)\big).$$

Proof. Choosing an appropriate branch, we define a conformal map ϕ : $D_\varepsilon(0,1) \to \mathbb{H}$ by the following formula

$$\phi(z) = i\left(\sqrt{\frac{z}{1-z}}\right)^{\frac{\pi}{\pi-\varepsilon}}.$$

Notice that $\phi(D_\theta((0,1)))$ is a domain bounded by two radial lines through the origin which have angle

$$\alpha = \frac{\pi-\theta}{2}\frac{\pi}{\pi-\varepsilon}$$

with the imaginary axis $\mathbb{I}$. Thus

$$\phi\Big(D_\theta\big((0,1)\big)\Big) = \left\{z \in \mathbb{H} : d(z,\mathbb{I}) < \log\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)\right\},$$

where d is the hyperbolic distance in $\mathbb{H}$. Considering the holomorphic map $h \circ \phi^{-1} : \mathbb{H} \to \mathbb{C}_{(0,1)}$ and applying the Schwarz lemma, we see that for any $z \in D_\theta((0,1))$,

$$\begin{aligned} d'\big(h(z),(0,1)\big) &= d'\Big(h\circ\phi^{-1}\big(\phi(z)\big),(0,1)\Big) \\ &\le d\big(\phi(z),\mathbb{I}\big) \\ &< \log\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right), \end{aligned}$$

where d' is the hyperbolic distance in $\mathbb{C}_{(0,1)}$. The lemma follows. □

Lemma 7.2. *For any* $\varepsilon \in (0,\pi)$ *and* $\delta > 0$*, there exist* $\theta_0 = \theta_0(\varepsilon) \in (0,\pi/2)$ *and* $C = C(\varepsilon,\delta) > 1$ *such that the following holds. Let* $\theta \in (0,\theta_0)$ *and let* h *be a univalent function defined on* $D_\theta((-\delta,1+\delta))$ *which maps* $(0,1)$ *onto itself. Then for any* $z \in D_{2\theta}((0,1)) - S((0,1),\varepsilon)$*, we have*

$$d\big(\phi(z),(0,1)\big) \le Cd\big(z,(0,1)\big), \tag{27}$$

where d denotes the Euclidean distance.

Proof. Let $A = \{z \in \mathbb{C} - S((0,1),\varepsilon) : d(z,(0,1)) \le 1\}$. Choose θ_0 sufficiently small so that $A \subset \mathrm{Cl}(D_{\theta_0}((0,1)))$. Then for $z \in A$, (27) follows from the Koebe distortion theorem. For $z \in D_{2\theta}((0,1)) - (S((0,1),\varepsilon)\cup A)$, let $t \ge 2\theta$ be the maximal positive number such that $z \in \mathrm{Cl}(D_t((0,1)))$. Then there exists $C' = C'(\varepsilon) > 1$ such that

$$\frac{1}{C't} \le d\big(z,(0,1)\big) \le \frac{C'}{t}.$$

By Lemma 7.1, $\phi(z) \in D_\alpha((0,1))$ for $\alpha = \frac{\pi}{\pi-\theta}(t-\theta) \ge t/2$. So $d(\phi(z),(0,1))$ is of order $1/t$, and (27) follows. □

Lemma 7.3. *Let $\theta \in (0, \pi/2)$ and $\delta > 0$. Let $\psi : D_{\theta/2}((-\delta, 1+\delta)) \to \mathbb{C}$ be a real symmetric univalent map whose restriction to the real line is monotone increasing. Let $0 < a < b < 1$. Then for any $\varepsilon \in (0, \pi)$, there are constants $\varepsilon' = \varepsilon'_{\delta,\theta}(\varepsilon) \in (0, \pi)$ and $\varepsilon'' = \varepsilon''_{\delta,\theta}(\varepsilon) \in (0, \pi)$ such that*

- *for any $z \in D_\theta((0, 1)) \cap S^+_\varepsilon(a)$, we have*
$$\psi(z) \in S^+_{\varepsilon'}\big(\psi(a)\big);$$
- *for any $z \in D_\theta((0, 1)) - S_\varepsilon((a, b))$, we have*
$$\psi(z) \notin S_{\varepsilon''}\big(\psi(a), \psi(b)\big).$$

Moreover, for fixed θ and δ, $\varepsilon' \to 0$ as $\varepsilon \to 0$.

Proof. We may assume that ψ fixes 0, 1. Then the maps ψ with these properties form a compact family in the topology of uniform convergence on compact sets. The lemma follows easily from compactness arguments. □

7.2. The quadratic map.

Lemma 7.4. *Let $K > 1$ be a constant and $P(z) = z^2$. Then there exists a constant $C = C(K) > 1$ such that for any $\theta \in (0, \pi)$, we have*

$$P^{-1}\Big(D_\theta\big((-K, 1)\big)\Big) \subset D_{C^{-1}\theta}\big((-1, 1)\big).$$

Proof. Arguing by contradiction, assume that the lemma fails. Then for every $n \in \mathbb{N}$, there exist $\theta_n \in (0, \pi)$ and $z_n \in \mathbb{C}$ such that

$$z_n \notin D_{\theta_n/n}\big((-1, 1)\big) \text{ and } w_n := z_n^2 \in D_{\theta_n}\big((-K, 1)\big).$$

After passing to a subsequence, we may assume that θ_n converges as $n \to \infty$. Obviously, the limit is zero. We may assume that z_n and w_n are contained in the upper half plane $\mathbb{H}$ since $D_{\theta_n/n}((-1, 1))$ is symmetric with respect to both the real axis and the origin. Write $z_n = r_n e^{it_n}$, with $r_n > 0$ and $t_n \in (0, \pi/2)$.

Let us prove that $\{r_n\}$ are uniformly bounded from above. First notice that, if r_n is big, then

$$Re\left(\frac{w_n - 1}{w_n + K}\right) = r_n^4 - K + r_n^2(K - 1)\cos 2t_n > 0.$$

From $z_n \notin D_{\theta_n/n}((-1, 1))$, we obtain

$$\tan\arg\frac{z_n - 1}{z_n + 1} \leq \tan\theta_n/n; \tag{28}$$

and from $w_n \in D_{\theta_n}((-K, 1))$, we obtain

$$\tan \arg \frac{w_n - 1}{w_n + K} \geq \tan \theta_n. \tag{29}$$

Substituting $z_n = r_n e^{it_n}$ in (28) and $w_n = r_n^2 e^{i2t_n}$ in (29), we obtain

$$\frac{2r_n \sin t_n}{r_n^2 - 1} \leq \tan \frac{\theta_n}{n}, \tag{30}$$

and

$$\frac{r_n^2(K+1)\sin 2t_n}{r_n^4 - K + r_n^2(K-1)\cos 2t_n} \geq \tan \theta_n. \tag{31}$$

Dividing (31) by (30), we obtain

$$\frac{r_n\left(r_n^2 - 1\right)}{r_n^4 - K + r_n^2(K-1)\cos 2t_n} \frac{\sin 2t_n}{\sin t_n} \to \infty, \quad \text{as } n \to \infty. \tag{32}$$

Thus,

$$\frac{r_n\left(r_n^2 - 1\right)}{r_n^4 - K + r_n^2(K-1)\cos 2t_n} \to \infty, \quad \text{as } n \to \infty. \tag{33}$$

It follows that r_n are uniformly bounded. Consequently, for large n, $\arg z_n = t_n$ is close to 0, and so $\arg(w_n - 1)/(w_n + K) \asymp \arg(w_n - 1)$ and $\arg(z_n - 1)/(z_n + 1) \asymp \arg(z_n - 1)$, which implies that $\arg(w_n - 1)/\arg(z_n - 1)$ is very large. On the other hand,

$$\arg(w_n - 1) = \arg\left(z_n^2 - 1\right) = \arg(z_n - 1) + \arg(z_n + 1) \leq 2\arg(z_n - 1),$$

a contradiction. □

Lemma 7.5. *Let $P(z) = z^2$, $a \geq 0$ and $\varepsilon \in (0, \pi/2)$. Then*

$$S^+_{\frac{\varepsilon}{2}}(a) \subset P^{-1}\left(S^+_\varepsilon(a^2)\right) \cap \{z : \mathrm{Re} z \geq 0\} \subset S^+_\varepsilon(a);$$

$$S^-_\varepsilon(a) \cap \{z : \mathrm{Re} z \geq 0\} \subset P^{-1}\left(S^-_\varepsilon(a^2)\right) \cap \{z : \mathrm{Re} z \geq 0\} \subset S^-_{\frac{\pi-\varepsilon}{2}}(a).$$

Proof. Let z be a point in the first quadrant. Then $\arg(z-a) \geq \arg(z+a) \geq 0$. Together with $\arg(z^2 - a^2) = \arg(z + a) + \arg(z - a)$, this implies the first formula. The proof of the second one is similar. □

7.3. Proper maps. We shall also use the following lemma.

Lemma 7.6. *Let $U \subset V$ be topological disks, and let $F : U \to V$ be a holomorphic proper map of degree N. Assume that*

$$K = \left\{ z \in U : F^k(z) \in U \text{ for all } k \in \mathbb{N} \right\}$$

is a connected compact set. Then there exist $U' \subset U$ and $V' \subset V$, such that $F : U' \to V'$ is a DH-polynomial-like map of degree N. Moreover, if $\mathrm{mod}(V - K) \geq \delta$, then we can choose U' and V' appropriately so that $\mathrm{mod}(V' - U') \geq \delta'$, where $\delta' > 0$ is a constant depending only on $\delta > 0$ and N.

For a proof, see Lemma 2.4 in [31].

8. Polynomial-like extension properties of the first return maps

In this section we shall prove Theorem 3. The proof is based on the analysis on the real geometry of f which we have done in Sects. 4 and 5, and uses many ideas coming from Lyubich and Yampolsky [31].

Theorem 3. *Let f be a real analytic map in the class $\mathcal{F}_b$ which has essentially bounded geometry, and let c be a critical point of f. Then for any $\varepsilon > 0$, there is a symmetric nice interval $I \ni c$ with $|I| < \varepsilon$, such that the real box mapping associated to I extends to a real symmetric polynomial-like box mapping.*

As usual let us say that f is *non-renormalizable* if it does not have any properly periodic interval. Since the finitely renormalizable case can be reduced to the non-renormalizable case (by considering the real box mapping associated to a small symmetric nice interval), we shall assume throughout this section that the map f is either non-renormalizable or infinitely renormalizable. Moreover, by Proposition 5.11, we may assume that f has negative Schwarzian.

Recall that the height of a symmetric nice interval I is the number of positive integers m such that $R_{I^{m-1}}$ displays a non-central return.

In Sect. 8.1, we follow an idea of Levin & van Strien [24] to extend f to a *smooth polynomial-like mapping F* which is holomorphic near the real line. This extension is proper, but not necessarily holomorphic on the whole domain. We shall explain there how to get a polynomial-like extension from a quasi-polynomial-like one.

Most of our effort is put into looking for a small symmetric nice interval I for which the associated real box mapping has a quasi-polynomial-like extension. To this end, following an idea of [31], we shall prove that there exists a constant $C > 1$ which depends only on f with the following property. For any $\Delta > 0$, there exists a small symmetric nice interval I such that each branch of the generalized renormalization $\mathbf{R}_I|J = f^s|J$ extends to a holomorphic branched covering $F^s : U \to V$ with

$V = D_*((1+2\Delta)I) \cap \mathbb{C}_I$, and such that for all $z \in U$, the following Lyubich-Yampolsky type inequality holds:

$$\left(\frac{d(z,J)}{|J|}\right)^2 \le C \max\left\{1, \frac{d\big(F^s(z), I\big)}{|I|}\right\}. \tag{34}$$

In Sect. 8.2, we shall prove several geometric estimates on complex extensions of first return maps to nice intervals, by applying the well-known Schwarz lemma and Koebe distortion theorem in complex analysis, coupled with the essentially bounded nest geometry. In particular, we shall study the complex pull back of a truncated Poincaré disk of the form $D_\theta(K) \cap \mathbb{C}_I$ corresponding to the first return map to I, where $K \supset I$ are symmetric nice intervals, and I is a real pull back of K; and obtain a priori control on this complex pull back. See Propositions 8.2 and 8.3.

In Sect. 8.3, we shall prove Theorem 3 under the assumption that f has an arbitrarily small symmetric nice interval with a sufficiently large height. This assumption is satisfied by all non-renormalizable maps as well as some infinitely renormalizable maps. More precisely, let I be a small symmetric nice interval which contains a critical point c, let $m(0) = 0$, and let $m(1) < m(2) < \cdots$ be all the positive integers such that $R_{I^{m(i)-1}}(c) \notin I^{m(i)}$. Combining Propositions 8.2 and 8.3, with a *jump* argument introduced in [31], we shall prove that for any $k \ge 3$, each branch of the first return map $\mathbf{R}_{I^{m(k)}}|J = f^s|J$ extends to a holomorphic map $F^s : U \to V$ with $V = D_*(I^{m(3)}) \cap \mathbb{C}_{I^{m(k)}}$, and that the extension satisfies the inequality (34) for some C which depends only on f. If $I^{m(k)}$ is well defined for a large k, then it follows that the real box mapping $B_{I^{m(k)}}$ has a quasi-polynomial-like extension.

In Sect. 8.4, we shall deal with the remaining situation. In this case, for any symmetric nice interval I, and a large constant Δ, $(1+2\Delta)I$ contains properly periodic intervals larger than I, and so we have to investigate the property of complex extensions corresponding to the renormalization levels. By adopting another *jump* argument introduced in [31] (Lemma 8.10), we shall prove another a priori estimate, Lemma 8.13. Using this lemma instead of Propositions 8.2 and 8.3, and arguing in the same way as in Sect. 8.3, we show that the real box mapping associated to a small properly periodic interval has a quasi-polynomial-like extension.

8.1. Extension to a smooth polynomial-like box mapping. Let us consider a real analytic map

$$f : \Big(\bigcup_{j=0}^{m} \mathcal{J}_j\Big) \cup \Big(\bigcup_{i=1}^{b-1} \mathcal{I}_i\Big) \to \bigcup_{i=0}^{b-1} \mathcal{I}_i$$

in the class $\mathcal{F}_b$ which is either non-renormalizable or infinitely renormalizable.

Following Levin and van Strien [24], let us first extend f to a real symmetric *smooth polynomial-like box mapping* $F : (\bigcup_{j=0}^{m} \mathcal{U}_j) \cup (\bigcup_{i=1}^{b-1} \mathcal{V}_i) \to \bigcup_{i=0}^{b-1} \mathcal{V}_i$ such that F is holomorphic in a neighborhood of $\mathrm{dom}(f)$. Here, by a smooth polynomial-like box mapping, we mean that

- $\mathcal{V}_i$'s, $0 \le i \le b-1$ are topological disks with disjoint closures;
- $\mathcal{U}_j$'s, $0 \le j \le m$ are topological disks with disjoint closures which are compactly contained in $\mathcal{V}_0$;
- for each $\mathcal{U} \in \{\mathcal{U}_0, \mathcal{V}_i : 1 \le i \le b-1\}$, there exists $0 \le i' \le b-1$ such that $F|\mathcal{U} : \mathcal{U} \to \mathcal{V}_{i'}$ is a C^1 double branched covering;
- for each $1 \le j \le m$, $F|\mathcal{U}_j$ is a C^1 diffeomorphism onto some $\mathcal{V}_i$.

This kind of extension obviously exists and is certainly not unique. By choosing the extension appropriately, we may assume that the $\mathcal{U}_j$'s and $\mathcal{V}_i$'s are Jordan domains with C^1 boundary. We may also assume that for any component $\mathcal{U}$ of $\mathrm{dom}(F)$, $F|\mathcal{U}$ extends to a C^1 map defined on $\mathrm{Cl}(\mathcal{U})$ such that the derivative DF is non-degenerate on $\partial\mathcal{U}$. In the following, we shall fix such an extension F. Note that F and f have the same critical points.

Given a topological disk V contained in the range of F, and an F-orbit $\{F^i(z)\}_{i=0}^{n}$ with $F^n(z) \in V$, by considering the *complex pull back* of V along this orbit, we obtain a sequence of topological disks $U_0 \ni z$, $U_1 \ni F(z)$, $\cdots$, $U_n = V \ni F^n(z)$, with the property that for each $0 \le i \le n-1$, U_i is a component of $F^{-1}(U_{i+1})$. We say that this *complex pull back is holomorphic* if $F|U_i$ is holomorphic for each $0 \le i \le n-1$. We shall reserve the notion "chain" for a sequence of intervals obtained by (real) pull back of the (real) map f. To avoid confusion, we shall only talk of a complex pull back of a topological disk and a real pull back of an interval.

Given a symmetric nice interval I and a real symmetric topological disk V with $I \subset V \subset \mathbb{C}_I \cap (\bigcup_{i=0}^{b-1} \mathcal{V}_i)$, the associated real box mapping $\mathbf{B}_I : (\bigcup_{j=0}^{r} J_j) \cup (\bigcup_{i=1}^{b-1} I_i) \to \bigcup_{i=0}^{b-1} I_i$ admits the following extension to the complex plane. For any $1 \le i \le b-1$, let s_i be the entry time of I_i to I, and let $V_i = \mathrm{Comp}_{I_i}(F^{-s_i}(V))$. For any $0 \le j \le r$, let p_j be the return time of J_j to I, and let $U_j = \mathrm{Comp}_{J_j}(F^{-p_j}(V))$. Then it is easy to check that for each $0 \le j \le r$ ($1 \le i \le b-1$, respectively), the map $\mathbf{B}_I|J_j$ ($\mathbf{B}|I_i$, respectively) extends to a proper map from U_j (V_i, respectively) onto some $V_{i'}$ without increasing the number of critical points. Remark that if V is a Jordan domain with C^1 boundary, then so are the topological disks U_j are V_i.

To show that $\mathbf{B}_I$ extends to a polynomial-like box mapping, it suffices to find an appropriate topological disk V such that the objects defined above satisfy the following:

(E1) the closures of U_j's are pairwise disjoint;
(E2) $U_j \Subset V$;
(E3) the extended maps $F^{s_i}|V_i$, $F^{p_j}|U_j$ are holomorphic.

Note that (E3) implies that the corresponding extension of $\mathbf{B}_I$ is holomorphic.

If the objects defined above satisfy the conditions (E1) and (E2) (but maybe not (E3)), then we shall say that $\mathbf{B}_I$ has a *smooth polynomial-like extension determined by* V. If they satisfy (E3) (but maybe not (E1) or (E2)) and

(E2') $U_j \subset V$, and $\mathrm{Cl}(U_j) - V \subset \mathbb{R}$,

then we say that $\mathbf{B}_I$ has a *quasi-polynomial-like extension determined by* V. The following is a useful observation in [24]:

Lemma 8.1. *Let I be a symmetric nice interval, and let V and V' be two real symmetric topological disks in $\bigcup_{i=0}^{b-1} \mathcal{V}_i$ which contain I. Assume that V' is a Jordan domain with $Cl(V') \cap \mathbb{R} = Cl(I)$. If $\mathbf{B}_I$ has a smooth polynomial-like extension determined by V' and a quasi-polynomial-like extension determined by V, then $\mathbf{B}_I$ has a polynomial-like extension determined by $V'' = Comp_I(V \cap V')$.*

Proof. Let J_j, I_i and s_i, p_j be as above. Let U_j, V_i, and U'_j, V'_i be defined as above for V and V' respectively. Let $U''_j = \mathrm{Comp}_{J_j}(U_j \cap U'_j)$ and $V''_i = \mathrm{Comp}_{I_i}(V_i \cap V'_i)$. Then these topological disks are the corresponding objects as defined above for V''. It is obvious that these objects satisfy (E1) and (E3). To complete the proof, we need to check that $U''_j \Subset V''$ for each $0 \leq j \leq r$.

Take a point $z \in \mathrm{Cl}(U''_j)$, and let us show that $z \in V''$. Note that $z \in \mathrm{Cl}(U'_j) \subset V'$. So it suffices to prove that $z \in V$. Arguing by contradiction, assume that $z \notin V$. Then $z \in \mathrm{Cl}(U_j) - V \subset \mathbb{R}$. But this implies that $z \in \mathrm{Cl}(U'_j) \cap \mathbb{R} = \mathrm{Cl}(J_j) \subset V$, a contradiction. □

Recall that $\mathcal{M}(\mathcal{I}_0)$ is the collection of all (nice) intervals which are (real) pull backs of $\mathcal{I}_0$. For any $I \in \mathcal{M}(\mathcal{I}_0)$, there is a natural way as described in the following, to construct a smooth polynomial-like extension of $\mathbf{B}_I$. Let $\{G_i\}_{i=0}^k$ be the chain with $G_k = \mathcal{I}_0$ and $G_0 = I$, and let $P = \mathrm{Comp}_I(F^{-k}(\mathcal{V}_0))$. Then $\mathbf{B}_I$ has a smooth polynomial-like extension determined by P. To see this, we observe that for any non-negative integer n, any component U of $F^{-n}(\mathcal{V}_0)$ is a Jordan domain. Moreover, if $n_1 < n_2$, and U_1 and U_2 are components of $F^{-n_1}(\mathcal{V}_0)$ and $F^{-n_2}(\mathcal{V}_0)$ respectively, then either $U_1 \cap U_2 = \emptyset$ or $U_2 \Subset U_1$. Note also that P has C^1 boundary, and hence $\mathrm{Cl}(P) \cap \mathbb{R} = \mathrm{Cl}(P \cap \mathbb{R}) = \mathrm{Cl}(I)$. Together with Lemma 8.1, this implies

Lemma 8.2. *For any symmetric nice interval $I \in \mathcal{M}(\mathcal{I}_0)$, if the real box mapping $\mathbf{B}_I$ has a quasi-polynomial-like extension determined by a real symmetric topological disk V, then it has a polynomial-like extension determined by a real symmetric topological disk $V' \subset V$.*

The lemma will only be used in the non-renormalizable case. If f is infinitely renormalizable, then we shall use a more geometric method to construct a polynomial-like extension from a quasi-polynomial-like one.

Lemma 8.3. *Assume that f is infinitely renormalizable. Let B be a symmetric properly periodic interval and let s be its period. Assume that $\mathbf{B}_B$ has a quasi-polynomial-like extension determined by a real symmetric topological disk V, and that there exists $\nu \in \mathbb{N}$ such that*

$$Comp_B\big(F^{-\nu s}(V)\big) \Subset V.$$

Then there exists a real symmetric topological disk $V' \subset V$, which determines a polynomial-like extension of $\mathbf{B}_B$.

Proof. Let $U = \mathrm{Comp}_B(F^{-s}(V))$, and let

$$\mathcal{K} = \big\{z \in U : F^{ks}(z) \in U \text{ for all } k \in \mathbb{N}\big\}.$$

Note that $\mathcal{K}$ is the filled Julia set of the DH-polynomial-like map

$$F^{\nu s} : \mathrm{Comp}_B\big(F^{-\nu s}(V)\big) \to V,$$

and thus it is compact. This DH-polynomial-like mapping, as an extension of $f^{\nu s} : B \to B$, has all critical points contained in its filled Julia set $\mathcal{K}$, and thus $\mathcal{K}$ is connected. By Lemma 7.6, there exist topological disks $V' \subset V$ and $U' \subset U$, such that $F^s : U' \to V'$ is a DH-polynomial-like mapping. Clearly, V' determines a polynomial-like extension of $\mathbf{B}_B$. □

8.2. Control of complex pull backs by real geometry. Given a (real) chain $\{G_j\}_{j=0}^{s}$ and a topological disk $V \supset G_s$, information on the complex pull back of V along $\{G_j\}$ can often be read from the real axis. In this subsection, we collect lemmas on this kind of control. We begin with a few little lemmas.

Lemma 8.4. *For any $\theta_0 \in (0, \pi)$, there is an $\eta > 0$ with the following property. Let I be an open interval in $\bigcup_{i=0}^{b-1} \mathcal{I}_i$, and let J be a component of $f^{-1}(I)$. If $J \cap \omega(c) \neq \emptyset$, $|I| < \eta$, and $f|J$ is monotone, then for any $\theta \in (\theta_0, \pi)$, we have*

$$U := Comp_J\Big(F^{-1}\big(D_\theta(I)\big)\Big) \subset D_{\theta - M|J|}(J),$$

where $M > 0$ is a constant depending only on F. In particular, $F : U \to D_\theta(I)$ is a conformal map.

Proof. This follows from Lemma 7.1 (the Schwarz lemma). To be more precise, let $K = \mathrm{Comp}_J(\mathrm{dom}(f))$ and $L = \mathrm{Comp}_I(\mathrm{range}(f))$. We first consider the case that $f|K$ is a diffeomorphism. Then the corresponding branch of F^{-1} is holomorphic and univalent on a neighborhood of L, which contains $D_\varepsilon(I)$ for $\varepsilon = M|J|$, where $M > 0$ is a constant depending only on F. Thus by Lemma 7.1, $U \subset D_\alpha(J)$, where

$$\alpha = \frac{\pi}{\pi - \varepsilon}(\theta - \varepsilon) \geq \theta - \varepsilon \geq \theta - M|J|.$$

So the lemma holds in this case. Now assume that $f|K$ has a critical point. Then F can be written in the form $Q \circ \phi$, where Q is a quadratic map. Note that Q^{-1} extends to a univalent map from $\mathbb{C}_I$ into $\mathbb{C}_{\phi(J)}$, thus $\phi(U) \subset D_\theta(\phi(J))$. As ϕ^{-1} is holomorphic and univalent on a neighborhood of $\phi(K)$, arguing as above gives us the desired estimate. □

Lemma 8.5. *For any $\delta > 0$, $N \in \mathbb{N}$ and any $\theta \in (0, \pi)$, there is an $\eta > 0$ such that the following holds. Let $\{G_j\}_{j=0}^s$ be a chain of order $\leq N$, such that $G_0 \cap \omega(c) \neq \emptyset$. Let $\{G'_j\}_{j=0}^s$ be another chain with $G'_s = (1 + 2\delta)G_s$ and $G'_0 \supset G_0$. Assume that*

(i) for any $0 \leq j < s$, we have $(G'_j - G_j) \cap Crit(f) = \emptyset$;
(ii) $|f^s(G_0)| \geq \delta |G_s|$;
(iii) $\sum_{j=0}^s |G'_j| < \eta$.

Let $V = D_\theta(G_s)$ and $U = Comp_{G_0}(F^{-s}(V))$. Then for each $0 \leq j \leq s-1$, we have

$$F^j(U) \subset D_{\theta/C}(G_j),$$

where $C = C(N, \delta) > 1$ is a constant (independent of θ). In particular, $F^s : U \to V$ is holomorphic.

Proof. If $\{G_j\}_{j=0}^s$ is a monotone chain, then by the previous lemma, we have $F^j(U) \subset D_{\theta - M \sum_{i=j}^{s-1} |G_i|}(G_j)$, for any $0 \leq j \leq s-1$, where $M > 0$ is a constant depending only on F. Provided that $\eta < \theta/2M$, it follows that $F^j(U) \subset D_{\theta/2}(G_j)$ for any $0 \leq j \leq s-1$.

Assume now that $\{G_j\}_{j=0}^s$ is not monotone. Let $s_1 < s$ be maximal such that G_{s_1} contains a critical point. Then by the above argument, $F^{s_1+1}(U) \subset D_{\theta/2}(G_{s_1+1})$. Since $f^{s-s_1-1}|G_{s_1+1}$ has bounded distortion, and since $f^{s-s_1-1}(f(G_{s_1})) \supset f^s(G_0)$ is not so small compared to G_s, $|f(G_{s_1})|/|G_{s_1+1}|$ is bounded away from zero. Therefore by Lemma 7.1 and Lemma 7.4, $F^{s_1}(U) \subset D_{\theta/C_1}(G_{s_1})$ for some constant $C_1 = C_1(\delta) > 1$.

Note that the assumptions (i)–(iii) are true for the shorter chains $\{G_j\}_{j=0}^{s_1}$ and $\{G'_j\}_{j=0}^{s_1}$ if we replace $\delta > 0$ with a smaller constant $\delta' > 0$, and that $\{G_j\}_{j=0}^{s_1}$ has order $\leq N - 1$. Thus, the lemma follows by induction. □

Remark 8.1. The assumptions (i) and (ii) are true provided that $(1+2\delta)G_s - (1 - 2\delta)G_s$ is disjoint from $\omega(c)$. This lemma will often be used in the case that G_s is a small nice interval which intersects $\omega(c)$, and $\{G_j\}_{j=0}^s$ is a chain corresponding to the first entry of some $x \in \omega(c)$ to G_s. We remind the reader that in this case, the order of this chain is bounded from above by the number of critical points of f.

Proposition 8.1. *For any $\varepsilon \in (0, \pi)$, $\delta > 0$ and $\theta \in (0, \pi)$, there are $\eta = \eta(\varepsilon, \delta, \theta) > 0$ and $C = C(\varepsilon, \delta) > 1$ with the following property. Let I be a nice interval intersecting $\omega(c)$ with $|I| < \eta$ and $((1 + 2\delta)I -$*

$(1-2\delta)I) \cap \omega(c) = \emptyset$. Let J be a component of D_I intersecting $\omega(c)$ and s the entry time of J to I. Let $V = D_\theta(I)$ and $U = Comp_J(F^{-s}(V))$. Then for any $w \in V - S(I, \varepsilon)$, and any $z \in U$ with $F^s(z) = w$, we have

$$\frac{d(z, J)}{|J|} \le C \max\left\{1, \frac{d(w, I)}{|I|}\right\}.$$

Moreover, if there exists a critical point c' of f such that $d(J, c') \le |J|/\delta$, then

$$\frac{d(z, J)}{|J|} \le C \max\left\{1, \sqrt{\frac{d(w, I)}{|I|}}\right\}.$$

(In particular, this holds in the case $J \ni c'$.)

Proof. Let $V' = D_{\theta/2}((1+\delta)I)$, and let $U' = \mathrm{Comp}_J(F^{-s}V')$. Let us first prove that $F^s : U' \to V'$ is holomorphic provided that η is small enough. To this end, let $\{G'_j\}_{j=0}^s$ and $\{G_j\}_{j=0}^s$ be the chains with $G'_s = (1+2\delta)I$, $G_s = (1+\delta)I$, and $G'_0 \supset G_0 \supset J$. By Lemma 3.8, $\{G'_j\}_{j=0}^s$ has intersection multiplicity bounded by 4, and thus by Corollary 3.6, $\sum_j |G'_j|$ is small provided that η is sufficiently small. Applying Lemma 8.5, we obtain that $F^s|U'$ is holomorphic. Similarly, for any $K > 1$, there exists $\eta > 0$ such that if $|I| \le \eta$, then for any $0 \le s_1 < s_2 \le s$, the pull back of $D_{\theta/K}(G_{s_2})$ along the chain $\{G_j\}_{j=s_1}^{s_2}$ is holomorphic.

Let us now prove the first inequality. If the chain $\{G_j\}_{j=0}^s$ is monotone, then this inequality follows from Lemma 7.2. So let us assume that the chain $\{G_j\}_{j=0}^s$ is not monotone. Let $0 \le n_1 < n_2 < \cdots < n_k < s$ be all the integers such that G_{n_i} contains a critical point. Let $A = \{w \in \mathbb{C} : d(w, I) = 1\}$. If $w \in A - S(I, \varepsilon)$, then $d(z, J)/|J|$ is bounded from above by a constant and so the inequality holds. Assume $w \in V - (A \cup S(I, \varepsilon))$. Let $\{I_j\}_{j=0}^s$ be the chain with $I_s = I$ and $I_0 = J$. Let $t = t_s \in (\theta, \pi)$ be the positive number such that $w \in \partial D_t(I)$ and for each $0 \le j \le s-1$, let $t_j \in (0, \pi)$ be such that $F^j(z) \in \partial D_{t_j}(I_j)$. Arguing as in the proof of Lemma 7.2, we obtain $t_{n_k+1}/t \ge 1/2$. By Lemma 7.4, t_{n_k}/t_{n_k+1} is bounded away from zero. As $k \le \#\mathrm{Crit}(f)$, by repeating this argument a few times, we conclude that t_0/t is bounded away from zero. Since $d(w, I)/|I| \asymp 1/t$, the first inequality follows.

To prove the second inequality, just notice that $f|J$ is of the form $Q \circ \phi$, where Q is a quadratic map and ϕ is a real symmetric conformal map from a neighborhood of c' to a neighborhood of 0. By the argument above, $d(F^{s-1}(z), f(J))/|f(J)|$ is of order $d(w, I)/|I|$. Considering the behavior of the quadratic polynomial $z \mapsto z^2$ near 0, the second inequality follows easily. □

Let us now state a lemma concerning pull back along a long central cascade. This lemma can be shown in the same way as Lemma 6.2 in [31], but we shall give a more elementary proof here.

Lemma 8.6. *For any $\delta > 0$ and $\theta \in (0, \pi)$, there exists $\theta_1 \in (0, \pi)$ with the following property. Let*

$$I^0 \supset I^1 \supset I^2 \supset \cdots \supset I^m$$

be a central cascade such that R_{I^0} does not have a critical point in $I^1 - I^m$, and let r be the return time of I^1 to I^0. Let $J \subset I^1 - I^m$ be an interval such that $J_{ir} = f^{ir}(J) \subset I^1$ for each $i = 0, 1, \dots, p-1$. Let $V = D_\theta(I^0) \cap \mathbb{C}_{J_{pr}}$, and $U = Comp_J(F^{-pr}(V))$. Assume that

$$I^0 \supset (1 + 2\delta)I^1, \ \ \big((1 + 2\delta)I^0 - (1 - 2\delta)I^0\big) \cap \omega(c) = \emptyset,$$

and $|I^0|$ is sufficiently small. Then

$$U \subset D_{\theta_1}(I^0).$$

Remark 8.2. Applying this lemma to the interval $f^{ir}(J)$ instead of J, $0 \le i \le p$, we obtain that $F^{ir}(U) \subset D_{\theta_1}(I^0)$. By choosing θ_1 smaller if necessary, we have $F^j(U) \subset D_{\theta_1}(\mathcal{L}_{J_j}(I^0))$ for all $0 \le j \le pr - 1$, by Lemma 8.5.

Proof. By Lemma 8.5, there is a $\theta' > 0$ such that

$$W := \mathrm{Comp}_{I^1}\Big(F^{-r}\big(D_\theta(I^0)\big)\Big) \subset D_{\theta'}(I^1).$$

Take a point $z \in U$, and let us prove that $z \in D_{\theta_1}(I^0)$ for an appropriate choice of θ_1. For each $0 \le i \le p$, write $\zeta_i = F^{ir}(z)$. We may assume that there is $0 < q < p$ such that $\zeta_q \notin D_\theta(I^0)$, for otherwise $z \in D_{\theta'}(I^1) \cup D_\theta(I^0)$. Let q be maximal with this property. Then

$$\zeta_q \in W - D_\theta(I^0).$$

Note that $J \subset I^q - I^m$. Let T_q be the component of $I^q - I^m$ which contains J and for each $0 \le i < q$, let $T_i = f^{(q-i)r}(T_q)$, which is the component of $I^i - I^{m-q+i}$ containing $f^{(q-i)r}(J)$. Moreover, for each $0 \le i \le q$, let P_i be the component of $I^i - \mathrm{Cl}(I^{i+1})$ which is contained in T_i.

Since $I^0 \supset (1 + 2\delta)I^1$, there exists a constant $\theta'' \in (0, \pi)$ such that $W - V \subset D_{\theta''}(P_0) \subset D_{\theta''}(T_0)$. Let $Y_0 = D_{\theta''}(T_0)$, and let $Y_i = \mathrm{Comp}_{J_{(q-i)r}}(F^{-ir}(Y_0))$ for each $0 \le i \le q$. Then $F^{qr} : Y_q \to Y_0$ is a diffeomorphism, and $\zeta_i \in Y_{q-i}$ for each $0 \le i \le q$. Moreover, since $\zeta_q \in D_{\theta''}(P_0)$, we have $z = \zeta_0 \in Y_q \cap (F^{-qr}(D_{\theta''}(P_0)))$. Note that P_0 is a nice interval and $f^{qr} : P_q \to P_0$ is a first entry to P_0. By Corollary 3.6, $\sum_{j=0}^{qr} |f^j(P_q)|$ is small. Applying Lemma 8.4, we obtain that

$$z \in D_{\theta_1}(P_q) \subset D_{\theta_1}(I^0),$$

where θ_1 is a constant. ∎

We are going to prove two propositions in a more complicated situation. For any nice interval I which intersects $\omega(c)$, as in Sect. 6, define

$$\mathbf{Q}(I) = \bigcap_{k=1}^{\infty} \bigcup_{x \in \omega(c) \cap I} \mathcal{L}_x^k(I).$$

Recall that if that I does not contain a properly periodic interval of f, then $\mathbf{Q}(I)$ is just the Cantor set $\omega(c) \cap I$, and otherwise, it is the union of all maximal properly periodic intervals contained in I.

Proposition 8.2. *For any $\theta \in (0, \pi)$, $\delta > 0$, $\rho > 1$, $C > 1$ and $q \in \mathbb{N}$, there exist $\eta = \eta(\theta, \delta, \rho, C, q, F) > 0$ and $\theta_1 = \theta_1(\theta, \delta, \rho, C, q, b) \in (0, \pi)$ with the following property. Let I be a nice interval intersecting $\omega(c)$ such that $|I| < \eta$, $\Lambda_I < \rho$, and*

$$\big((1 + 2\delta)I - (1 - 2\delta)I\big) \cap \omega(c) = \emptyset.$$

Let $x \in \omega(c) \cap I$, and let $x, f^{n_1}(x), \ldots, f^{n_k}(x)$ be successive returns to I. Let $\mathbb{I} = \{I_j\}_{j=0}^{n_k}$ be the chain with $I_{n_k} = I$ and $I_0 \ni x$, and let $\mathbb{J} = \{J_j\}_{j=0}^{n_k}$ be a chain with $J = J_{n_k} \subset I$, and $J_0 \ni x$. Assume that the following hold:

(i) the maximal number of elements of $\{x, f^{n_1}(x), \ldots, f^{n_k}(x)\}$ contained in the same component of $\mathbf{Q}(I)$ is at most q;
(ii) the chain $\mathbb{J}$ has intersection multiplicity at most q; and
(iii) $|I_0| > |I|/C$.

Let $V = D_\theta(I) \cap \mathbb{C}_J$ and $U = Comp_x(F^{-n_k}(V))$. Then for any $0 \le j \le n_k - 1$, we have

$$F^j(U) \subset D_{\theta_1}\big(\mathcal{L}_{f^j(x)}(I)\big) \cap \mathbb{C}_{J_j}.$$

Remark 8.3. We shall often apply this proposition in the following situation: J is a pull back of I, and $f^{n_k}(x)$ is the first entry of x to J. In this case, both of the conditions (i) and (ii) are clearly satisfied for $q = 2$. Condition (iii) is usually very easy to check. For instance, if the orbit $\{f^j(x)\}_{j=1}^{n_k}$ does not enter $\mathcal{L}_x^r(I)$ at all, then I_0 contains $\mathcal{L}_x^{r+1}(I)$, and thus is commensurable to I by Corollary 5.3. Similarly, for any nice interval K which has bounded geometry, if $x \in K \subset I$ and if the sequence $\{f^j(x)\}_{j=1}^{n_k}$ does not enter $\mathcal{L}_x^r(K)$, then I_0 is commensurable to K.

Proof of Proposition 8.2. Let $\{G_j^i\}_{j=0}^{n_{i+1}-n_i}$ be the chain with $G_{n_{i+1}-n_i}^i = I$ and $G_0^i \ni f^{n_i}(x)$ for all $0 \le i \le k-1$, where $n_0 = 0$. Let

$$\begin{aligned} \text{Crit} &= \text{Crit}\Big(I, \big\{x, f^{n_1}(x), \ldots, f^{n_k}(x)\big\}\Big) \\ &:= \bigcup_{i=0}^{k-1} \bigcup_{j=0}^{n_{i+1}-n_i-1} G_j^i \cap \text{Crit}(f). \end{aligned} \tag{35}$$

We shall use induction on #Crit. Let $U_j = F^j(U)$ for each $0 \le j \le n_k$.

Starting step. Assume #Crit $= 0$. Then $\{G_j^i\}_{j=0}^{n_{i+1}-n_i}$ is a monotone chain for each $0 \le i \le k-1$. Let us prove that k is uniformly bounded from above by a constant depending only on C, δ and ρ, from which the proposition follows by Lemma 8.4.

To this end we first observe that D_I has a component intersecting $\omega(c) \cap I$ on which R_I has a critical point, since $\omega(c)$ contains all critical points of f. Thus, D_I has at least two components intersecting $\omega(c) \cap I$. Each of these components is commensurable to I for otherwise it would be deep inside I since $(I - (1-2\delta)I) \cap \omega(c) = \emptyset$, which is ruled out by the assumption $\Lambda_I \le \rho$ (Corollary 5.3). For any $0 \le i \le k$, let $T_i \ni x$ be a subinterval of I such that $R_I^i(T_i) = I$. Then $R_I^i : (T_i, T_{i+1}) \to (I, \mathcal{L}_{f^{n_i}(x)}(I))$ is a diffeomorphism, which extends to a diffeomorphism onto $(1+2\delta)I$. Since f has negative Schwarzian derivative, it follows that the maps $R_I^i|T_i$, $0 \le i \le k-1$, have uniformly bounded distortion. Thus, $|T_{i+1}|/|T_i|$ is uniformly bounded from 1. Since $T_s = I_0$, and since $|I_0|/|I|$ is bounded away from zero, we conclude that s is uniformly bounded.

Induction step. Assuming that the proposition holds when #Crit $< N$, let us consider the case #Crit $= N$.

For the same reason as stated in the proof of Proposition 4.1, we may assume that I is a symmetric nice interval and that Crit $=$ Crit$(I, \{x, f^{n_1}(x), \ldots, f^{n_k}(x)\})$ contains the critical point in I, say c. If $\chi_I = 0$, then for any $i \ge 0$, $R_{I^i}(c) \in I^{i+1}$. So $\mathbf{T}(I) := \bigcap I^i$ is a periodic interval, and $I - \mathbf{T}(I)$ is disjoint from $\omega(c)$. Thus all these points $f^{n_i}(x)$ are contained in $\mathbf{T}(I) = \mathbf{Q}(I)$. By assumption (i), this implies that $k \le q$. So the proposition follows easily from Lemma 8.5. Now let us assume that $\chi_I \ge 1$. Let $m = m(1)$ be the minimal positive integer with $R_{I^{m-1}}(c) \notin I^m$. Let $0 \le k_1 < k$ be the maximal integer with $f^{n_{k_1}}(x) \in I^1$. We first prove

Claim 1. For any $n_{k_1} \le j \le n_k - 1$, we have

$$F^j(U) \subset D_{\sigma_1}(\mathcal{L}_{f^j(x)}(I)),$$

where $\sigma_1 > 0$ is a constant.

To prove this claim, we first notice that

$$\mathrm{Crit}\Big(I, \big\{f^{n_{k_1}+1}(x), f^{n_{k_1}+2}(x), \ldots, f^{n_k}(x)\big\}\Big) \subset \mathrm{Crit}\Big(I, \big\{x, f^{n_1}(x), \ldots, f^{n_k}(x)\big\}\Big) - \{c\}.$$

Next let us show that $|I_{n_{k_1}+1}|/|I|$ is uniformly bounded away from zero. Arguing by contradiction, assume that $|I_{n_{k_1}+1}|/|I|$ is very small. Then $|I_{n_{k_1}}|/|I^1|$ is also very small. Since $I_{n_{k_1}} \supset \mathcal{L}_{f^{n_{k_1}}(x)}(I^1)$, $|\mathcal{L}_{f^{n_{k_1}}(x)}(I^1)|/|I|$ is small, and so $\mathcal{L}_{f^{n_{k_1}}(x)}(I^1)$ is deep inside I, which is absurd by Corollary 5.3. Applying the induction hypothesis gives us the desired estimate for all $n_{k_1+1} \leq j < n_k$. Finally applying Lemma 8.5, we obtain the desired estimate for $n_{k_1} \leq j < n_{k_1+1}$.

Claim 2. There exists a constant $\theta' \in (0, \pi)$, such that one of the following holds:

- for all $0 \leq j < n_k$, we have

$$F^j(U) \subset D_{\theta'}\big(\mathcal{L}_{f^j(x)}(I)\big) \cap \mathbb{C}_{J_j}; \tag{36}$$

- there exists an integer k' with $1 \leq k' < k$, such that $I_{n_{k'}} \subset I^m$, $F^{n_{k'}}(U) \subset D_{\theta'}(I^m)$, and such that (36) holds for all $n_{k'} \leq j < n_k$.

Of course we may assume $k_1 > 0$. Let $0 \leq k_1' \leq k_1$ be minimal with $f^{n_{k_1'}}(x) \in I^1$. Remark that if (36) holds for $n_{k_1'} \leq j < n_k$, then it holds for all $0 \leq j < n_{k_1'}$ as well (with a smaller θ'). Indeed, if $k_1' > 0$, then $n_{k_1'}$ is the first entry time of x to I^1, and so this follows from Lemma 8.5.

Note that $I_{n_{k_1}} \subset I^1$. For a similar reason as in the proof of Claim 1, the pull back of I^1 along the orbit $\{f^i(x)\}_{i=n_{k_1'}}^{n_{k_1}}$ is commensurable to I^1. Since

$$\mathrm{Crit}\Big(I^1, \big\{f^{n_i}(x) : k_1' \leq i \leq k_1,\ f^{n_i}(x) \in I^1\big\}\Big) \subset \mathrm{Crit}\Big(I, \big\{f^{n_i}(x) : 0 \leq i \leq k\big\}\Big),$$

we may assume that these two sets coincide, for otherwise the induction hypothesis applies. In particular, there is a maximal integer $k_2 < k_1$, with $f^{n_{k_2}}(x) \in I^2$. Similarly as in Claim 1, we find a constant $\sigma_2 > 0$ such that for all $n_{k_2} \leq j < n_{k_1}$, $F^j(U) \subset D_{\sigma_2}(\mathcal{L}_{f^j(x)}(I^1)) \cap \mathbb{C}_{J_j}$.

Since $\chi_I \geq 1$, $(I - I^1) \cap \omega(c) \neq \emptyset$, and thus $I \supset (1+2\delta)I^1$ provided that $|I|$ is sufficiently small. Here we use $1/(1-2\delta) > 1+2\delta$. Let $l = l(\delta, \rho, b)$ be as in Proposition 5.1. Repeating the above argument, we conclude that for an appropriately chosen constant $\theta' > 0$, either (36) holds for all $0 \leq j < n_k$, or there is a sequence of integers

$$k_0 := k > k_1 > k_2 > \cdots > k_{\min(m,3l)} > 0$$

such that $I_{n_{k_i}} \subset I^i$, and such that $F^j(U) \subset D_{\theta'}(\mathcal{L}_{f^j(x)}(I^{i-1}))$ holds for $n_{k_i} \leq j < n_{k_{i-1}}$. In particular, it follows that Claim 2 holds if $m \leq 3l$.

So assume that $m > 3l$ and that we are in the latter case. Then $I^l \subset I^{l+1} \subset \cdots I^{m-l}$ is a saddle node central cascade. Let $k_l' \leq k_l$ be minimal with $f^{n_{k_l'}}(x) \in I^l$. Similarly as above, it suffices to prove that (36) holds for all $j \geq n_{k_l'}$ (with an appropriately chosen positive constant θ'). Let $k'' \geq k_l'$ be minimal such that

$$f^{n_{k''+i}}(x) \notin I^0 - I^{l+1}$$

for all $1 \le i < k_l - k''$. Then either $f^{n_{k''}}(x) \in I^0 - I^{l+1}$ or $k'' = k_l'$. Let us show that there exists a constant $\theta'' > 0$ such that

$$F^j(U) \subset D_{\theta''}\big(\mathcal{L}_{f^j(x)}(I^l)\big) \tag{37}$$

holds for all $n_{k''} \le j < n_{k_l}$. To this end, we distinguish two cases:

Case 1. $f^{n_{k_l}}(x) \in I^{m-l}$. Then for each $k'' < t \le k_l$, we have $f^{n_t}(x) \in I^{m-l}$ since

$$R_{I^0}(I^{l+1} - I^{m-l}) \subset I^l - I^{m-l-1}$$

is disjoint from I^{m-l}. By Proposition 5.1, $k_l - k'' \le l$ and so (37) follows from Lemma 8.5.

Case 2. $f^{n_{k_l}}(x) \in I^t - I^{t+1}$ for some $l \le t \le m-l-1$. Let $0 \le r \le m-l-t$ be maximal such that

$$f^{n_{k_l-1}}(x) \in I^{t+1},\ f^{n_{k_l-2}}(x) \in I^{t+2}, \cdots, f^{n_{k_l-r}}(x) \in I^{t+r}.$$

Since the chain $\{J_j\}_{j=0}^{n_k}$ has intersection multiplicity $\le q$, we have $J_{n_{k_l-j}} \subset I^l - I^{m-l}$ for all $q \le j \le r - q$. Thus by Lemma 8.6, and Lemma 8.5 we conclude that (37) holds for all $j \ge n_{k_l-r}$ (for some $\theta'' > 0$). Moreover, by Lemma 8.5 again, to prove that (37) holds for $n_{k''} \le j < n_{k_l-r}$, it suffices to show that $k_l - r - k''$ is uniformly bounded from above. If $f^{n_{k_l-r}}(x) \in I^{m-l}$, then as in the proof of Case 1, we know that $k_l - r - k'' \le l$. If $f^{n_{k_l-r}}(x) \notin I^{m-l}$ and $k'' < k_l - r$, then by the combinatorics of a saddle node central cascade, we have $k'' = k_l - r - 1$. This proves that (37) holds for all $n_{k''} \le j < n_{k_l}$.

Now let us complete the proof of Claim 2. We may assume that $k'' > k_l'$. Then $f^{n_{k''}}(x) \in I^{l_1} - I^{l_1+1}$ for some $0 \le l_1 \le l$, and so

$$K := \mathcal{L}_{f^{n_{k''}}(x)}(I^l) \subset I^{l_1} - I^{l_1+1},$$

which implies $\mathcal{L}_c(K) \subset I^m$. Note that $I_{n_{k''}} \subset K$.

Let $k''' \le k''$ be the minimal non-negative integer such that $f^{n_{k'''}}(x) \in K$. Let us consider

$$\mathbb{K} = \Big(K, \big\{f^{n_j}(x) : k''' \le j \le k'', f^{n_j}(x) \in K\big\}\Big).$$

Obviously, $\mathrm{Crit}(\mathbb{K}) \subset \mathrm{Crit}(I, \{x, f^{n_1}(x), \ldots, f^{n_k}(x)\})$.

Let $\{G_j\}_{j=n_{k'''}}^{n_{k''}}$ be the chain with $G_{n_{k''}} = K$, and $G_{n_{k'''}} \ni f^{n_{k'''}}(x)$. By Theorem 5.5, there exists a constant $\delta' > 0$ such that $(1+2\delta')K - (1-2\delta')K$ is disjoint from $\omega(c)$. Let us show that $|G_{n_{k'''}}|/|K|$ is uniformly bounded away from zero. Assume not. Then $G_{n_{k'''}}$ is deep inside K. Since $n_{k'''}$ is the entry time of x to K, it follows that $I_0 \subset \mathcal{L}_x(G_{n_{k'''}})$ is deep inside $\mathcal{L}_x(K) \subset I$, which contradicts the assumption (iii).

Assume $\mathrm{Crit}(\mathbb{K}) \not\ni c$. Then it follows from the induction hypothesis that $F^j(U) \subset D_{\theta'''}(\mathcal{L}_{f^j(x)}(K))$ for some $\theta''' \in (0, \pi)$ and for all $n_{k'''} \le j < n_{k''}$.

Since $n_{k'''}$ is the entry time of x to K, it follows from Lemma 8.5 that (36) holds for all $0 \le j < n_{k'''}$, and thus Claim 2 holds in this case.

Let us assume $\mathrm{Crit}(\mathbb{K}) \ni c$. Then there exists a maximal positive integer k' with $k''' < k' < k''$ and $f^{n_{k'}}(x) \in I^m$. From $\mathcal{L}_c(K) \subset I^m$ and $I_{n_{k''}} \subset K$, we obtain $I_{n_{k'}} \subset I^m$. Moreover, it follows from the induction hypothesis and Lemma 8.5 in the same way as in the proof of Claim 1, that $F^j(U) \subset D_{\theta'}(\mathcal{L}_{f^j(x)}(K))$ for some $\theta' > 0$, and for all $n_{k'} \le j < n_{k''}$. In particular, $F^{n_{k'}}(U) \subset D_{\theta'}(I^m)$. So Claim 2 holds in this case as well. We have completed the proof of Claim 2.

In other words, we have found a constant $\nu_1 \in (0, \pi)$, such that either of the following holds:

- for any $0 \le j < n_k$, we have

$$F^j(U) \subset D_{\nu_1}\big(\mathcal{L}_{f^j(x)}(I)\big); \tag{38}$$

- $\chi_I \ge 1$, and there exists $p_1 < p_0 := k$ such that (38) holds for all $n_{p_1} \le j < n_k$, and

$$I_{n_{p_1}} \subset I^{m(1)}, \;\; F^{n_{p_1}}(U) \subset D_{\nu_1}(I^{m(1)}).$$

Let $0 \le p_1' < p_1$ be minimal such that $f^{n_{p_1'}}(x) \in I^{m(1)}$. Note that the pull back of $I^{m(1)}$ along the orbit $\{f^j(x)\}_{j=n_{p_1'}}^{n_{p_1}}$ is commensurable to $I^{m(1)}$. Then by the same argument as above, we have a constant $\nu_2 \in (0, \pi)$, such that either of the following holds:

- for any $n_{p_1'} \le j < n_{p_1}$, we have

$$F^j(U) \subset D_{\nu_2}\big(\mathcal{L}_{f^j(x)}(I^{m(1)})\big); \tag{39}$$

- $\chi_I \ge 2$, and there is $p_2 < p_1$ such that (39) holds for all $n_{p_2} \le j < n_{p_1}$, and

$$I_{n_{p_2}} \subset I^{m(2)}, \;\; F^{n_{p_2}}(U) \subset D_{\nu_2}(I^{m(2)}).$$

If the former case happens, then by Lemma 8.5, there exists $\nu_2' \in (0, \nu_2)$ such that $F^j(U) \subset D_{\nu_2'}(\mathcal{L}_{f^j(x)}(I^{m(1)}))$ holds for all $0 \le j < n_{p_1'}$. So the proposition holds for $\theta_1 = \nu_2'$ in this case. Similarly, for any $t \in \mathbb{N}$ there exist constants $\nu_t, \nu_t' \in (0, \pi)$ such that either the proposition holds for $\theta_1 = \nu_t'$, or $\chi_I \ge t$ and there is a $p_t < k$ such that $I_{p_t} \subset I^{m(t)}$. The latter case cannot happen for a large t, for otherwise $I_0 \subset \mathcal{L}_x(I_{p_t}) \subset \mathcal{L}_x(I^{m(t)})$ is deep inside I by Theorems 2.1 and 3.3, which contradicts the assumption (iii). This completes the proof of the induction step. □

We shall also need the following estimate:

Proposition 8.3. *In the situation of the previous proposition, if I is a small symmetric nice interval with $\chi_I \geq 1$, and if the orbit $\{x, \ldots, f^{n_k}(x)\}$ enters $I^{m(1)}$ at least $3l(\delta, \rho, b)$ times, where $m(1) \in \mathbb{N}$ is minimal such that $R_{I^{m(1)-1}}$ displays a non-central return, and $l(\delta, \rho, b)$ is the positive integer as in Proposition 5.1, then there exists $k' < k$ such that*

- $\#\{k' \leq i \leq k : f^{n_i}(x) \in I^{m(1)}\} \leq 3l(\delta, \rho, b)$*;*
- *for any $z \in Comp_x(F^{-n_k}(D_\theta(I) \cap \mathbb{C}_J))$, we have $f^{n_{k'}}(z) \in D_{\theta_2}(I^{m(1)})$, where $\theta_2 \in (0, \pi)$ is a constant depending on δ, ρ, q, C and b.*

Proof. Let $l = l(\delta, \rho, b)$. If $m(1) \leq 3l$, then let $k_0 = k$, and for each $0 \leq i \leq m(1) - 1$, define inductively k_{i+1} to be the maximal integer such that $k_{i+1} < k_i$ and such that $f^{n_{k_{i+1}}}(x) \in I^{i+1}$. By assumption, these integers are well defined. Applying the previous proposition, we prove inductively that for each $0 \leq i \leq m(1)$, $F^{n_{k_i}}(z) \in D_{\sigma_i}(I^i)$ for some $\sigma_i \in (0, \pi)$. Setting $k' = k_{m(1)}$ and $\theta_2 = \min_{i=0}^{m(1)} \sigma_i$, we conclude the proof.

Now assume that $m(1) > 3l$. Then define inductively $k = k_0 > k_1 > \cdots > k_l$ to be such that k_{i+1} is the maximal integer less than k_i with the property that $f^{n_{k_{i+1}}}(x) \in I^{i+1}$. Then as above, we obtain $F^{n_{k_l}}(z) \in D_\sigma(I^l)$ for some $\sigma \in (0, \pi)$. Note that $\{x, \ldots, f^{n_{k_l}}(x)\}$ enters $I^{m(1)}$ at least $2l$ times. Let $k'' < k_l$ be minimal such that $f^{n_{k''+i}}(x) \notin I^0 - I^{l+1}$ for all $1 \leq i < k_l - k''$. Proceeding as in the induction step of the previous proposition, by distinguishing two cases according to whether $f^{n_{k_l}}(x) \in I^{m(1)-l}$ or not, we find that k'' is positive, and that $\#\{k'' \leq i \leq k_l : f^{n_i}(x) \in I^{m(1)}\} \geq l$. Let $k' < k''$ be maximal such that $f^{n_{k'}}(x) \in I^{m(1)}$. Then using the same argument as in the proof of the previous proposition, we obtain $F^{n_{k'}}(z) \in D_{\theta_2}(I^{m(1)})$. The proof is completed. □

8.3. Large height case. In this subsection, we shall prove Theorem 3 for maps satisfying a further assumption that there is an arbitrarily small symmetric interval with an arbitrarily large height. More precisely, we shall show

Proposition 8.4. *For each $\rho > 1$, there exists $N_0 = N_0(\rho, b) \in \mathbb{N}$ such that if there is an arbitrarily small symmetric nice interval I with $\Lambda_I \leq \rho$ and $\chi_I > N_0$, then Theorem 3 holds.*

Throughout this subsection, we shall use the following notation:

- I is a symmetric nice interval with $\Lambda_I \leq \rho$, and $\chi = \chi_I$ is the height of I;
- $m(0) = 0$, and $m(1) < m(2) < \cdots$ are all the positive integers such that $R_{I^{m(i)-1}}(c) \notin I^{m(i)}$;
- $3 \leq k_1 < k \leq \chi$ are positive integers;
- J is a component of $D_{I^{m(k)}}$ which intersects $\omega(c) \cap I^{m(k)}$, s is the return time of J to $I^{m(k)}$, and $\{J_j\}_{j=0}^{s}$ is the chain with $J_s = I^{m(k)}$ and $J_0 = J$;
- $V = D_*(I^{m(k_1)}) \cap \mathbb{C}_{I^{m(k)}}$, and $U = \mathrm{Comp}_J(F^{-s}(V))$.

By Theorems 2.1 and 5.5, there exists $\delta = \delta(b, \rho) > 0$ such that for all $i \geq 3$, $((1+2\delta)I^{m(i)} - (1-2\delta)I^{m(i)}) \cap \omega(c) = \emptyset$, and $|I^{m(i)}|/|I^{m(i)+1}| > 1 + 2\delta$. Let $l = l(\delta, \rho, b)$ be as in Proposition 5.1.

The main step is to prove the following proposition.

Proposition 8.5. *For any $N \in \mathbb{N}$, $\rho > 1$, there exist constants $C = C(b, \rho) > 1$ and $\theta = \theta(b, \rho, N) \in (0, \pi)$ such that if $k - k_1 \leq N$, and if $|I|$ is sufficiently small, then for each $0 \leq j < s$, we have*

$$F^j(U) \subset D_\theta\big(\mathcal{L}_{J_j}(I^{m(k_1)})\big). \tag{40}$$

Moreover, for any $z \in U$, we have

$$\left(\frac{d(z, J)}{|J|}\right)^2 \leq C \max\left\{1, \frac{d\big(F^s(z), I^{m(k)}\big)}{|I^{m(k)}|}\right\}. \tag{41}$$

Proof. Note that the pull back of $I^{m(k_1)}$ along the orbit $\{J, f(J), \dots, f^s(J)\}$ contains J, and thus is not so small compared to $I^{m(k_1)}$. Applying Proposition 8.2 we obtain (40). (The constant θ does, however, depend on N.) Let us now turn to the proof of (41).

Take a point $z \in U$, and let $w = F^s(z)$. If $w \in D_*(I^{m(k-3l-10)})$, then as above, Proposition 8.2 implies $z \in D_\theta(I^{m(k-3l-10)})$ for some constant $\theta \in (0, \pi)$. In particular, (41) holds for an appropriately chosen constant C. So let us assume that $w \notin D_*(I^{m(k-3l-10)})$. Let $k_1 \leq k' \leq k - 3l - 11$ be the maximal positive integer such that $w \in D_*(I^{m(k')})$. Let $z_j = F^j(z)$ for all $0 \leq j \leq s$.

Definition. Let $\varepsilon > 0$ be a small quantifier. We say that z_j *ε-jumps* if $z_j \notin S(J_j, \varepsilon)$, and that j is an *ε-good* time if $|J_j|/|J_s| \geq \varepsilon$.

Statement 1. *For any $q \in \mathbb{N}$, there is a constant $\nu_q = \nu(\delta, \rho, q) > 0$ such that for any $k_1 \leq p \leq k$ and any $i < s$, if $J_i \subset I^{m(p)}$, and if*

$$\#\big\{i \leq j < s : J_j \subset I^{m(p)}\big\} \leq q,$$

then i is a ν_q-good time, i.e., $|J_i|/|I^{m(k)}| \geq \nu_q$.

We first observe that the generalized renormalization $\mathbf{R}_{I^{m(p)}}$, i.e., the first return map restricted to the return domains intersecting $I^{m(p)} \cap \omega(c)$, has uniformly bounded derivative. As $f^{s-i} : J_i \to I^{m(k)}$ is a restriction of $\mathbf{R}^{q'}_{I^{m(p)}}$ for some $q' \leq q$, its derivative is also uniformly bounded. Since $f^{s-i}(J_i)$ contains a point in $\omega(c)$ as well as an endpoint of $I^{m(k)}$, it is commensurable to $I^{m(k)}$. Thus $|J_i|/|I^{m(k)}|$ is bounded away from zero.

Let $t_0 = s$ and $t_1 < s$ be maximal such that $J_{t_1} \subset I^{m(k'+3l+1)}$. For $1 \leq i \leq k - k' - 3l - 11$, let $t_{i+1} < t_i$ be the maximal non-negative integer such that $J_{t_{i+1}} \subset I^{m(k'+3l+i+1)}$, if it exists. Let

$$\mathcal{A}_i = \big\{j : s > j \geq t_i,\ J_j \subset I^{m(k'+3l+i)}\big\}, \text{ and } q_i = \#\mathcal{A}_i.$$

Statement 2. *For each* $1 \leq i \leq k - k' - 3l - 10$*,* t_i *exists and*

$$q_i \leq \max\{q_1, q_2, \ldots, q_{i-1}, 10\}. \tag{42}$$

We shall prove this statement by induction on i. For $i = 1$, we have $q_1 = 1$, and so (42) is obviously true. Now assuming that the statement holds for some $1 \leq i \leq k - k' - 3l - 11$, let us prove it for $i + 1$. For the existence of t_{i+1}, it suffices to show that $t_i > 0$. To this end, remark that for any $p \in \mathbb{N}$ and any $y \in \omega(c) \cap I^{m(p+2)}$, we have $R_{I^{m(p)}}(y) \notin I^{m(p+2)}$. Thus

$$\#\left\{0 \leq j \leq s : J_j \subset I^{m(k'+3l+i)}\right\} \geq 2^{[\frac{k-k'-3l-i}{2}]} \geq 32 > 10,$$

and hence $t_i > 0$.

It remains to prove that (42) holds for $i + 1$. Arguing by contradiction, assume that this is false. Notice that each integer in $\mathcal{A}_{i+1}$ can be written as $t_{i'}$ for some $1 \leq i' \leq i + 1$. Let $1 \leq i_1 < i_2 \leq i + 1$ be the smallest two integers such that $t_{i_1}, t_{i_2} \in A_{i+1}$. Then $q_{i+1} \leq i - i_2 + 3$. On the other hand, $R_{I^{m(k'+3l+i+1)}}(J_{t_{i_2}}) \subset J_{t_{i_1}}$. Let $r \in \mathbb{N}$ be such that

$$R_{I^{m(k'+3l+i+1)}}|J_{t_{i_2}} = R^r_{I^{m(k'+3l+i_2)}}|J_{t_{i_2}}.$$

Then

$$r \geq 2^{[(i+1-i_2)/2]}.$$

In particular, $q_{i_2} \geq r \geq 2^{[i+1-i_2]/2}$. Since $q_{i+1} > q_{i_2}$,

$$2^{[i+1-i_2]/2} \leq i - i_2 + 3,$$

and hence $q_{i+1} \leq i - i_2 + 3 \leq 10$, which contradicts the assumption that $q_{i+1} > 10$. This completes the proof of the induction step, and hence that of Statement 2.

It follows that for each $1 \leq i \leq k - k' - 3l - 10$, $q_i \leq 10$, and thus t_i is a ν_{10}-good time.

Statement 3. *There is a constant* $\theta_1 = \theta_1(\rho) \in (0, \pi)$ (independent of N)*, such that the following holds. For each* $1 \leq i \leq k - k' - 3l - 10$*, if* $z_{t_{i-1}} \in D_*(I^{m(k'+i-1)})$*, then*

$$z_{t_i} \in D_{\theta_1}(I^{m(k'+i)+1}).$$

Notice that $\{J_j\}_{j=t_i}^{t_{i-1}}$ enters $I^{m(k'+i)}$ more than $3l$ times. By Proposition 8.3, there is an integer $t_i < t_i' \leq t_{i-1}$ such that $J_{t_i'} \subset I^{m(k'+i)}$, and $z_{t_i'} \in D_{\theta_2}(I^{m(k'+i)})$. Let $t_i < t_i'' \leq t_i'$ be the minimal integer such that $J_{t_i''} \subset I^{m(k'+i)}$. It suffices to show that there exists a constant $\theta_2 \in (0, \pi)$ such that $z_{t_i''} \in D_{\theta_2}(I^{m(k'+i)})$. If $t_i'' = t_i'$, then this is true. Assume $t_i'' < t_i'$. Consider the pull back $\{G_j\}_{j=t_i''}^{t_i'}$ with $G_{t_i'} = I^{m(k'+i)}$, and $G_{t_i''} \supset J_{t_i''}$. Then

$|G_{t_i''}|/|G_{t_i'}|$ cannot too small. In fact, $R_{I^{m(k'+i)}}$ maps $\mathrm{Comp}_{J_{t_i}}(D_{I^{m(k'+3l+i)}})$ into $G_{t_i''}$, and so if $|G_{t_i''}|/|G_{t_i'}|$ were very small, then $\mathrm{Comp}_{J_{t_i}}(D_{I^{m(k'+3l+i)}})$ would be much smaller than $I^{m(k'+i)}$, and hence much smaller than $I^{m(k'+3l+i)}$, which contradicts $\Lambda_I \le \rho$ by Corollary 5.3. Applying Proposition 8.2, we obtain the desired estimate.

It follows that for some constant $\varepsilon = \varepsilon(\rho) > 0$, either z_{t_1} ε-jumps, or $z_{t_1} \in D_*(I^{m(k'+1)})$.

In the former case, by Proposition 8.1, we obtain

$$\frac{d(z, J)}{|J|} \le C \max\left\{1, \sqrt{\frac{d(z_{t_1}, J_{t_1})}{|J_{t_1}|}}\right\},$$

for some $C = C(\delta, \varepsilon) > 1$. Since $d(F^{t_1}(z), J_{t_1})/|I^{m(k'+1)}|$ is bounded from above by a constant depending only on θ_1, and since $|J_{t_1}|/|I^{m(k)}| \ge \nu_{10}$, we have

$$\frac{d\big(F^{t_1}(z), J_{t_1}\big)}{|J_{t_1}|} \le C\frac{d(w, I^{m(k)})}{|I^{m(k)}|}.$$

The inequality (41) follows.

In the latter case, by Statement 3 again, either z_{t_2} ε-jumps, or $z_{t_2} \in D_*(I^{m(k'+2)})$. If z_{t_2} ε-jumps, then we are done again. Repeating this argument, we reach at either (41), or $z_{t_{k-k'-3l-10}} \in D_*(I^{m(k-3l-10)})$. In the latter case, applying Proposition 8.2 once again gives us $z \in D_\theta(I^{m(k-3l-10)})$ for some constant $\theta \in (0, \pi)$, and so the inequality (41) holds as well. The proof of this proposition is completed. □

Corollary 8.6. *There exists $N_0' = N_0'(\rho, b) \in \mathbb{N}$ such that in the situation of the previous proposition, if $k - k_1 = N_0'$, then $U \subset V$ and $Cl(U) - V \subset \mathbb{R}$.*

Proof. It follows easily from (41) that $\mathrm{Cl}(U) \Subset \mathrm{Cl}(V)$. Noticing also that $U \cap \mathbb{R} = J \subset V$, the corollary follows. □

Proof of Proposition 8.4. Let ρ be a constant such that for any sufficiently small symmetric nice interval I, we have $\lambda_I = |I|/|I^1| \le \rho$. We shall prove the proposition for $N_0 = N_0' + 3$, where $N_0' = N_0'(\rho, b)$ is as in Corollary 8.6. First let us assume that f is non-renormalizable. Let $I \in \mathcal{M}(\mathcal{I}_0)$ be a small symmetric nice interval. Let $k_1 = 3$ and let $k = N_0$. By Proposition 8.5, $V = D_*(I^{m(k_1)}) \cap \mathbb{C}_{I^{m(k)}}$ determines a quasi-polynomial-like extension of the real box mapping $\mathbf{B}$ associated to $I^{m(k)}$. Thus by Lemma 8.2, $\mathbf{B}$ can be extended to a real symmetric polynomial-like box mapping.

Now assume that f is infinitely renormalizable. Let I be a small symmetric nice interval so that $N = \chi_I \ge N_0$. Let $Y = I^{m(N)}$, and let B be the largest symmetric properly periodic interval contained in I. Let s be the return time of c to B. Note that this is the return time of c to Y as well. Set

$V = D_*(I^{m(N-N_0')}) \cap \mathbb{C}_Y$, and $U = \mathrm{Comp}_{Y^1}(F^{-s}V)$. Then as in the previous case, we have $U \subset V$. Let us show that

$$\mathrm{Comp}_B\big(F^{-(3lN_0'+1)s}(V)\big) \Subset V,$$

which then implies the existence of a polynomial-like extension of $\mathbf{B}_B$ by Lemma 8.3.

To this end, let us consider the chain $\{J_j\}_{j=0}^{3ls}$ with $J_{3ls} = Y$ and $J_0 \ni c$. This chain enters $I^{m(N-N_0'+1)}$ at least $3l$ times, but enters Y only $3l+1$ times. Applying Propositions 8.3 and 8.2, we obtain that

$$(F^s|U)^{-3l}(V) \subset D_{\theta_1}\big(I^{m(N-N_0'+1)}\big),$$

for some $\theta_1 \in (0, \pi)$. Similarly, for each $1 \le i \le N_0' - 1$, we find some $\theta_i > 0$ such that

$$(F^s|U)^{-3l}\big(D_{\theta_i}(I^{m(N-N_0'+i)}) \cap \mathbb{C}_Y\big) \subset D_{\theta_{i+1}}\big(I^{m(N-N_0'+i+1)}\big).$$

In particular, $(F^s|U)^{-3lN_0'}(V) \subset D_\theta(Y)$ for some $\theta > 0$, and thus

$$\mathrm{Comp}_B\big(F^{-(3lN_0'+1)s}(V)\big) \subset D_{\theta'}(Y^1).$$

The closure of the former set is contained in $\mathrm{Cl}(U) \subset V \cup \mathbb{R}$, and thus it is contained $(V \cup \mathbb{R}) \cap \mathrm{Cl}(D_{\theta'}(Y^1)) \subset V$. This completes the proof. □

8.4. Bounded height case. In this subsection, we assume that f is infinitely renormalizable. Let c be the critical point of f in $\mathcal{I}_0$. Let

$$B_1 \supset B_2 \supset B_3 \supset \cdots$$

be all the properly periodic intervals containing c, and let $1 \le s_1 < s_2 < s_3 < \cdots$ be the periods. We continue to use the notation introduced in Sect. 5.5: β_n is the endpoint of ∂B_n satisfying $f^{s_n}(\beta_n) = \beta_n$, α_n is the innermost fixed point of $f^{s_n}|B_n$, x_n is the preimage of α_n under $f^{s_n}|B_n$ which is closest to β_n, $A_n = (\alpha_n, \tau(\alpha_n))$, and $E_n = B_n - \{\alpha_n, x_n, \tau(\alpha_n), \tau(x_n)\}$. (Here $\tau : \mathcal{I}_0 \to \mathcal{I}_0$ is the involution with $f \circ \tau = f$.) Moreover, for each n, let $m_n(0) = 0$, and let $0 < m_n(1) < m_n(2) < \cdots < m_n(\chi_n)$ be all the integers such that $R_{A_n^{m_n(j)-1}}(c) \notin A_n^{m_n(j)}$. The goal of this subsection is to prove the following proposition, which implies Theorem 3 together with Proposition 8.4.

Proposition 8.7. *Assume that* $\limsup \chi_n < \infty$ *and that* $\Lambda_{E_n} < \rho$ *for all sufficiently large* n. *Then for* n *sufficiently large, the real box mapping associated to* B_n *extends to a real symmetric polynomial-like box mapping.*

The proof of this proposition is very similar to that of Proposition 8.4 which we have done in the previous subsection, except that Propositions 8.2 and 8.3 will be replaced by Lemma 8.13 below. More precisely, we first associate to each interval B_n ($n \geq 2$) two definite neighborhoods $S_n \subset T_n$ with the property that S_n is well inside T_n, and set

$$\Omega_n = D_{\pi/4}(T_n), \text{ and } \Omega'_n = \mathrm{Comp}_c(F^{-s_n}\Omega_n).$$

For sufficiently large $n' < n$, we shall prove that $f^{s_n} : B_n \to B_n$ extends to a holomorphic branched covering onto $\Omega_{n'} \cap \mathbb{C}_{T_n}$ which satisfies the Lyubich–Yampolsky type inequality with C independent of n' and n. The proof is again a combination of the a priori estimate given by Lemma 8.13 with the jump argument we have used before.

To prove Lemma 8.13, we shall modify another "jump" argument introduced in [31], formulated in Lemma 8.10.

The intervals T_n and S_n are defined as follows. If $\chi_{n-1} = -1$, then T_n is defined to be the largest open interval such that $T_n \supset B_n = A_{n-1}$ and such that $f^{s_n}|T_n$ has no critical point in $T_n - B_n$. Otherwise, $T_n := A_{n-1}^{m_{n-1}(\chi_{n-1})+1}$. In both cases, $S_n := \mathrm{Comp}_{B_n}(f^{-s_n}(T_n))$.

Lemma 8.7. *There is a constant $\sigma > 0$, such that for all sufficiently large n, the following hold:*

- $(1 + 2\sigma)T_n$ *does not contain* $f^j(B_n)$ *for any* $1 \leq j \leq s_n - 1$;
- $(1 + 2\sigma)^2 B_n \subset (1 + 2\sigma)S_n \subset T_n$.

Proof. Assume first that $\chi_{n-1} \geq 0$. Then by Lemma 5.7, $T_n = A_{n-1}^{m(\chi_{n-1})+1}$ is well inside $A_{n-1}^{m_{n-1}(\chi_{n-1})}$, and thus by Theorem 3.3 the second statement holds. Noting that $A_{n-1}^{m_{n-1}(\chi_{n-1})} - T_n$ is disjoint from $\omega(c)$, the first statement follows.

Assume now that $\chi_{n-1} = -1$. Then, $s_n = 2s_{n-1}$ and $\alpha_{n-1} = \beta_n$. By Lemma 5.6, for any critical point c' of $f^{s_n}|B_{n-1}$, both of $d(c', f^{s_n}(c'))$ and $d(c', \alpha_{n-1})$ are comparable to $|B_n|$. Noting also that $T_n \supset S_n$, the statements follow. □

Lemma 8.8. *Let $\{G_j\}_{j=0}^{s_n}$ be a chain such that G_{s_n} is a symmetric open interval with $B_n \subset G_{s_n} \subset (1 + \sigma)T_n$ and $G_0 \supset B_n$. Let $\theta \in (0, \pi)$, and let $V = D_\theta(G_{s_n})$, $U = Comp_{B_n}(F^{-s_n}V)$. Then provided that n is sufficiently large, we have*

$$F^j(U) \subset D_{\theta/C}(G_j) \text{ for all } 0 \leq j \leq s_n,$$

where $\sigma > 0$ is as in the previous lemma, and $C > 1$ is a constant independent of θ.

Proof. Let $\{G'_j\}_{j=0}^{s_n}$ be the chain with $G'_{s_n} = (1 + 2\sigma)T_n$, and $G'_0 \supset B_n$. Since $(1 + 2\sigma)T_n$ does not contain $f^i(B_n)$ for any $1 \leq i \leq s_n - 1$, we have

- $G'_j - \mathrm{Comp}_{f^j(B_n)} f^{-s_n+j}(B_n)$ does not contain a critical point of f. In particular, $G'_j - G_j$ is disjoint from the critical set of f;
- the chain $\{G'_j\}_{i=0}^{s_n}$ has intersection multiplicity ≤ 4. (See the proof of Lemma 5.5.) In particular, $\sum |G'_j|$ is small provided that n is large (Corollary 3.6);
- $|f^{s_n}(G_0)|/|G_{s_n}|$ is uniformly bounded from zero. Indeed, $f^{s_n}(G_0)$ contains a component of $G_{s_n} - \{c\}$.

By Lemma 8.5 the lemma follows. □

Lemma 8.9. *Let $\theta \in (0, \pi)$ be a constant. Assume that n is sufficiently large and let $\{J_j\}_{j=0}^{s_n}$ be a chain such that $J_{s_n} \subset B_n$ and $J_0 \subset B_n$. Let $z \in Comp_c(F^{-s_n} D_\theta(T_n))$ and $z_j = F^j(z)$. Then the following hold.*

(1) *For any $\varepsilon \in (0, \pi/2)$ there exists $\xi = \xi_\theta(\varepsilon) \in (0, \pi/2)$ such that if $z_{j_0} \notin S(J_{j_0}, \varepsilon)$ for some $1 \leq j_0 \leq s_n$, then for any $0 \leq j \leq j_0$, we have*
$$z_j \notin S(J_j, \xi).$$
(2) *There is a constant $\varepsilon_0 = \varepsilon_0(\theta) \in (0, \pi/10)$, such that if $z_j \in S_{\varepsilon_0}(J_j)$ for all $0 \leq j \leq s_n$, and if a_j is the endpoint of J_j with the property that z_j is contained in the component of $S_{\varepsilon_0}(J_j)$ which contains a_j in its boundary, then $F(a_j) = a_{j+1}$ for any $0 \leq j \leq s_n - 1$.*

Proof. Let $\{G_j\}_{j=0}^{s_n}$ be the chain with $G_{s_n} = T_n$ and $G_0 = S_n$. Then by Lemma 8.8, there is a constant $C > 1$ such that $z_j \in D_{\theta/C}(G_j)$ for any $0 \leq j \leq s_n$.

Let $\{G'_j\}_{j=0}^{s_n}$ be the chain with $G'_{s_n} = (1 + \sigma)T_n$, and $G'_0 \ni c$. Then similarly as in the proof of Lemma 8.8, we can show that for any $\gamma \in (0, \pi)$, provided that n is sufficiently large, the following holds: For any $1 \leq j \leq s_n$,
$$F^j : \mathrm{Comp}_{B_n}\big(F^{-j} D_\gamma(G'_j)\big) \to D_\gamma(G'_j)$$
is a holomorphic proper map. Moreover, it can be written in the form $Q_k \circ \phi_k \circ \cdots \circ Q_1 \circ \phi_1$, with $k \leq b$, where each Q_i is a real quadratic map, and each ϕ_i is a real symmetric univalent map onto $D_\gamma(K_i)$ for some interval $K_i \ni 0$.

By Lemmas 7.3 and 7.5, the lemma follows. □

Lemma 8.10. *For any $\theta \in (0, \pi)$ and $r \in \mathbb{N}$, there exist $\varepsilon_1 = \varepsilon_1(\theta, r) > 0$ and $n_0 = n_0(\theta, r, f) \in \mathbb{N}$ with the following property. Let $n \geq n_0$ and let $\{J_j\}_{j=0}^{rs_n}$ be a chain such that J_{is_n}'s, $0 \leq i \leq r$ are open intervals compactly contained in B_n. Let z be a point in $Comp_{J_0}(F^{-rs_n}(D_\theta(T_n) \cap \mathbb{C}_{J_{rs_n}}))$ and let $z_j = F^j(z)$ for any $0 \leq j \leq rs_n$. Assume that f^{rs_n} has a critical point in each component of $B_n - J_0$, and let $J'_0 \supset J_0$ be the maximal open interval such that $f^{rs_n}|J'_0$ has no critical point in $J'_0 - J_0$. Assume that $z_{is_n} \in S(J_{is_n}, \varepsilon_1)$ for each $0 \leq i \leq r$. Then*
$$z_0 \in D_{\frac{\pi}{4}}(J'_0).$$

Proof. If n is sufficiently large, then by Lemma 8.8, there exists a constant $\theta' = \theta'(r, \theta) \in (0, \pi)$ such that all these points z_{is_n}, $0 \le i \le r$ are contained in $D_{\theta'}(T_n)$. Let $\varepsilon_0 = \varepsilon_0(\theta')$ and $\xi = \xi_{\theta'}(\varepsilon_0)$ be constants as in Lemma 8.9, and let $\varepsilon_1 = \min(\varepsilon_0, \xi)$. For constants chosen in this way, if $z_{is_n} \in S(J_{is_n}, \varepsilon_1)$ for all $0 \le i \le r$, then for each $0 \le j \le rs_n$, we have $z_j \in S(J_j, \varepsilon_0)$. Moreover, if $a_j \in \partial J_j$ is such that z_j belongs to the component of $S(J_j, \varepsilon_0)$ whose closure contains a_j, then $F(a_j) = a_{j+1}$ for each $0 \le j \le rs_n - 1$. Let x_0 be the endpoint of J_0' closer to a_0, and $x_j = F^j(x_0)$. Then there exists a maximal integer q with $0 \le q \le rs_n - 1$, such that x_q is a critical point, say c, of f.

In a neighborhood of c, F can be written as $F(z) = \Phi(z)^2 + F(c)$, where Φ is a real symmetric conformal map defined in a neighborhood of c. Note that $\Phi(z_q)$ is contained in the triangle bounded by the vertical line through $\Phi(c)(= 0)$, and the radial lines through $\Phi(a_q)$ which are in the boundary of $S(\Phi(J_q), \varepsilon_0)$. Thus $\Phi(z_q) \in D_{2\pi/5}((\Phi(c), \Phi(a_q)))$, which implies that

$$z_q \in D_{\pi/3}\big((c, a_q)\big),$$

provided that n is sufficiently large. Since $\sum_{j=0}^{q} |f^j(a_0) - f^j(x_0)|$ is small, Lemma 8.4 implies that $z_0 \in D_{\pi/4}((a_0, x_0)) \subset D_{\pi/4}(J_0')$. □

Lemma 8.11. *For any $\theta \in (0, \pi)$, there exists $\varepsilon_2 = \varepsilon_2(\theta) > 0$, such that if n is sufficiently large, then for any*

$$w \in Comp_{B_n}\Big(F^{-s_n}\big(D_\theta(T_n)\big)\Big) - D_\theta(T_n),$$

and any interval $J \subset B_n$, we have

$$w \notin S(J, \varepsilon_2).$$

Proof. By Lemma 8.8, there exists $C = C(\sigma, b) > 1$ such that if n is sufficiently large, then $\mathrm{Comp}_{B_n}(F^{-s_n}(D_\theta(T_n))) \subset D_{\theta/C}(S_n)$. As T_n contains a definite neighborhood of S_n, the lemma follows easily. □

Lemma 8.12. *For any $\theta \in (0, \pi)$, $\rho > 1$, $q \in \mathbb{N}$, there exists $\theta' \in (0, \pi)$ with the following property. Let n be a large positive integer such that $\chi_n \ge 0$, $\Lambda_{E_{n-1}} \le \rho$, and $\Lambda_{E_n} \le \rho$. Let $\mathbb{J} = \{J_j\}_{j=0}^{s}$ be a chain with $J_s \subset E_n$, $J_0 \subset A_n^1$, and $J_0 \cap \omega(c) \ne \emptyset$. Assume that the chain $\mathbb{J}$ has intersection multiplicity at most q, and that*

$$\#\{0 < j \le s : J_j \subset B_n\} \ge 3, \quad \#\big\{0 \le j \le s : J_j \subset A_n^1\big\} \le q.$$

Let $V = D_\theta(T_n) \cap \mathbb{C}_{J_s}$, and $U = Comp_{J_0}(F^{-s}V)$. Then for any $0 \le j < s$, $F^j(U) \subset D_{\theta'}(\mathcal{L}_{J_j}(B_{n-1}))$. Moreover, we have

$$U \subset D_{\theta'}(A_n). \tag{43}$$

Proof. Clearly $a := s/s_n \in \mathbb{N}$. Let z be a point in U, and let $z_j = F^j(z)$ for any $0 \le j \le s$. By Lemma 8.8, it suffices to prove the following:

Claim. There exists a constant $\theta' \in (0, \pi)$ (independent of z), such that

$$z \in D_{\theta'}(A_n). \tag{44}$$

Moreover, for each $0 \le i \le a$, $z_{is_n} \in D_{\theta'}(T_n)$.

Let $\varepsilon_1 = \varepsilon_1(\theta, 3) > 0$ be as in Lemma 8.10, let $\varepsilon_2 = \varepsilon_2(\theta)$ be as in the previous lemma, and let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. To prove the claim, we distinguish two cases.

Case 1. $z_{is_n} \in S(J_{is_n}, \varepsilon)$ for all $0 \le i \le a$. In particular, this implies that $z_{is_n} \in D_\theta(T_n)$. To show (44), first recall that by Lemma 5.9, f^{3s_n} has a critical point in each component of $A_n^0 - A_n^1$. Next, applying Lemma 8.10 to the chain $\{J_j\}_{j=0}^{3s_n}$, we obtain that $z \in D_{\pi/4}(A_n)$.

Case 2. $z_{i_0 s_n} \notin S(J_{i_0 s_n}, \varepsilon)$ for some $0 \le i_0 \le a$. Let $i_0 \le a$ be maximal with this property. Then for any $i_0 < i \le a$, $z_{is_n} \in D_\theta(T_n)$. Thus, $z_{i_0} \in D_{\theta/C}(T_n)$, where $C > 1$ is a constant as in Lemma 8.8. Let K be the component of E_n which contains $J_{i_0 s_n}$. Note that $z_{i_0 s_n} \in D_{\theta_2}(K)$ for some $\theta_2 > 0$ depending only on θ, ε and ρ. Here we use the fact that $|K|$ is comparable to $|T_n|$: from $\Lambda_{E_{n-1}} \le \rho$ and from the construction of T_n it follows that $|T_n|$ is comparable to $|B_n|$; while from $\Lambda_{E_n} \le \rho$, by Lemma 5.8, it follows that $|K|$ is comparable to $|B_n|$.

If $K = A_n$, then let $i_1 = i_0$. If $K \ne A_n$, then $i_0 > 0$ and let $i_1 < i_0$ be maximal such that $J_{i_1 s_n} \subset A_n$. By Proposition 8.2, to prove the claim it suffices to prove that there exists a constant $\theta_3 \in (0, \pi)$ such that the following hold.

(1) $z_{is_n} \in D_{\theta_3}(T_n)$ for each $i_1 \le i < i_0$;
(2) $z_{i_1 s_n} \in D_{\theta_3}(A_n)$.

To this end, of course we may assume that $K \ne A_n$. Let $i_1 < i_2 \le i_0$ be minimal with $J_{i_2 s_n} \subset K$. Consider the chain $\{G_j\}_{j=i_2 s_n}^{i_0 s_n}$ with $G_{i_0 s_n} = K$ and $G_{i_2 s_n} \supset J_{i_2 s_n}$. Let us show that the ratio $|G_{i_2 s_n}|/|K|$ cannot be too small. Arguing by contradiction, assume that $|G_{i_2 s_n}|/|K|$ is small. By Theorem 5.6, there exists a constant $\delta = \delta(\rho, b) > 0$ such that

$$\big((1+2\delta)K - (1-2\delta)K\big) \cap \omega(c) = \emptyset. \tag{45}$$

So $G_{i_2 s_n}$ is deep inside K. By maximality of i_1, $f^{(i_2 - i_1)s_n}(\mathcal{L}_{J_{i_1 s_n}}(A_n)) \subset G_{i_2 s_n}$. It follows that $\mathcal{L}_{J_{i_1 s_n}}(A_n)$ is deep inside A_n, which contradicts the hypothesis that $\Lambda_{E_n} \le \rho$ by Corollary 5.3. This proves that $|G_{i_2 s_n}|/|K|$ is bounded away from zero. Applying Proposition 8.2, we obtain

- $F^{is_n}(z) \in D_{\theta_4}(\mathcal{L}_{J_{is_n}}(K))$ for $i_2 \le s_n < i_0$;
- $z_{i_2 s_n} \in D_{\theta_4}(K)$ for some $\theta_4 > 0$.

Since $(i_2 - i_1)s_n$ is the entry time of $J_{i_1 s_n}$ to K, and since (45) holds, Lemma 8.5 implies that

$$z_{is_n} \in D_{\theta_3}\big(\mathcal{L}_{J_{is_n}}(K)\big),$$

for each $i_1 \le i < i_2$. This proves (1) and (2), where we use $\mathcal{L}_{J_{i_1 s_n}}(K) \subset A_n$. The proof of the lemma is completed. □

Lemma 8.13. *For any $\theta \in (0, \pi)$, $\rho > 1$, $q \in \mathbb{N}$, $\chi \in \mathbb{N}$, there exist $\theta' \in (0, \pi)$ and $l_0 = l_0(\rho, \chi)$ with the following property. Let n be a large positive integer such that $\Lambda_{E_{n-1}} \le \rho$, $\Lambda_{E_n} \le \rho$ and $\chi_n \le \chi$. Let $\mathbb{J} = \{J_j\}_{j=0}^s$ be a chain such that $J_s \subset E_n \cap D_{B_{n+1}}$, $J_0 \subset B_{n+1}$ and $J_0 \cap \omega(c) \neq \emptyset$. Assume that the intersection multiplicity of $\mathbb{J}$ is at most q. Let $V = D_\theta(T_n) \cap \mathbb{C}_{J_s}$ and $U = Comp_{J_0}(F^{-s}(V))$. Then the following hold.*

(1) *If $\#\{0 \le j \le s : J_j \subset B_{n+1}\} \le q$, then $U \subset D_{\theta'}(T_n)$, and for any $1 \le j \le s-1$,*

$$F^j(U) \subset D_{\theta'}\big(\mathcal{L}_{J_j}(B_{n-1})\big). \tag{46}$$

(2) *If $\#\{0 \le j \le s : J_j \subset B_{n+1}\} = l_0$, then*

$$U \subset D_{\theta'}(S_{n+1}).$$

Proof. Let $a := s/s_n \in \mathbb{N} \cup \{0\}$. First notice that if $a \le 2$, then the first statement follows easily from Lemma 8.5, and the second one is null if we take $l_0 > 3$. So let us assume $a \ge 3$. We shall distinguish two cases.

Case 1. $\chi_n \ge 0$. Let $0 \le p < s$ be maximal such that $J_p \subset A_n^1$ and $(s-p)/s_n \ge 3$. Such an integer p exists because we are assuming $a \ge 3$ and because $J_0 \subset B_{n+1} \subset A_n^1$. By Lemma 8.12, there exists a constant $\theta' > 0$, such that $F^j(U) \subset D_{\theta'}(\mathcal{L}_{J_j}(B_{n-1}))$ for all $p \le j < s$, and such that $F^p(U) \subset D_{\theta'}(A_n)$. Note that the pull back of A_n along the orbit $\{J_j\}_{j=0}^p$ contains B_{n+1}, and hence is commensurable to A_n. By Proposition 8.2, $F^j(U) \subset D_{\theta'}(\mathcal{L}_{J_j}(A_n)) \subset D_{\theta'}(\mathcal{L}_{J_j}(B_{n-1}))$ for all $0 \le j < p$ as well (with a smaller θ'). This proves the first statement of the lemma.

Let us prove the second statement. By Theorem 5.6, there is a $\delta = \delta(\rho) > 0$ such that $A_n^{m_n(i)} \supset (1+2\delta)A_n^{m_n(i)+1}$. Let $l = l(\delta, \rho, b)$ be as in Proposition 5.1, and let $l_0 = 10 l \chi$. Then $\{J_j\}_{j=0}^p$ enters $A_n^{m_n(\chi_n)}$ at least $l_0 - 2 \ge 3 l \chi_n$ times, and thus by Proposition 8.3, we can inductively define $p = j_0 > j_1 > \cdots > j_{\chi_n}$, such that for each $1 \le i \le \chi_n$, we have

- $J_{j_i} \subset A_n^{m_n(i)}$, and $\#\{j_i \le j \le j_{i-1} : J_j \subset A_n^{m_n(i)}\} \le 3l$;
- $F^{j_i}(U) \subset D_{\sigma_i}(A_n^{m_n(i)})$, where $\sigma_i > 0$ is a constant.

In particular, there exists an integer $0 < s' < s$ such that $J_{s'} \subset T_{n+1}$ and $F^{s'}(U) \subset D_\sigma(T_{n+1})$. Since $s'/s_{n+1} \le q$, applying Lemma 8.8 gives us $U \subset D_{\theta''}(S_{n+1})$.

Case 2. $\chi_n = -1$. Then $s_{n+1} = 2s_n$, and so $s/s_n \le 2\#\{0 \le j \le s : J_j \subset B_{n+1}\} \le 2q$. Applying Lemma 8.8 gives us the first statement of this lemma. Next let us prove that the second statement holds if $l_0 \ge 3$. Let $j_0 \le s$ be the maximal positive integer with $J_{j_0} \subset B_{n+1}$, and let $j_1 = j_0 - s_{n+1}$. Note that $j_0 = s$ or $j_0 = s - s_n$. By Lemma 8.8, it suffices to prove $F^{j_1}(U) \subset D_{\theta_2}(T_{n+1})$ for some $\theta_2 \in (0, \pi)$. To this end, take a point $z \in U$, and write $z_j = F^j(z)$ for all $0 \le j \le s$. As $(s - j_1)/s_n \le 3$, there exists a constant $\sigma \in (0, \pi)$, such that $z_{j_0}, z_{j_1} \in D_\sigma(T_n)$. Let $\varepsilon = \varepsilon_1(\sigma, 2)$ be as in Lemma 8.10. If $z_{j_1+is_n} \in S(J_{j_1+is_n}, \varepsilon)$ for all $0 \le i \le 2$, then that lemma implies that $z_{j_1} \in D_{\pi/4}(T_{n+1})$, since ∂T_{n+1} are critical points of f^{2s_n}. Otherwise, by Lemma 8.9, $z_{j_1} \notin S(J_{j_1}, \varepsilon')$ for some definite $\varepsilon' > 0$, which implies $z_{j_1} \in D_{\theta_2}(T_{n+1})$ as well. Here we use the fact that $|T_n|$ and $|T_{n+1}|$ are comparable to each other. The proof of the lemma is completed. □

Lemma 8.14. *For any $m \in \mathbb{N} \cup \{0\}$, $\theta \in (0, \pi)$, $\rho > 1$, and $q, \chi \in \mathbb{N}$, there exists $\theta' \in (0, \pi)$ with the following property. Let n be a large positive integer such that $\chi_{n-i} \le \chi$ for all $0 \le i \le m$ and such that $\Lambda_{E_{n-i}} \le \rho$ for any $0 \le i \le m + 1$. Let $\{J_j\}_{j=0}^s$ be a chain such that $J_s \subset E_{n-m} \cap D_{B_{n+1}}$, $J_0 \subset B_{n+1}$, and $J_0 \cap \omega(c) \ne \emptyset$. Assume that the intersection multiplicity of this chain is at most q, and that $\#\{0 \le j \le s : J_j \subset B_{n+1}\} \le q$. Let $V = D_\theta(T_{n-m}) \cap \mathbb{C}_{J_s}$ and $U = Comp_{J_0}(F^{-s}(V))$. Then*

$$U \subset D_{\theta'}(T_{n-m}).$$

Moreover, for any $0 \le j < s$, $F^j(U) \subset D_{\theta'}(\mathcal{L}_{J_j}(B_{n-m-1}))$.

Proof. Let us prove this lemma by induction on m. For $m = 0$, it follows from Lemma 8.13 (1). Now let m_0 be a positive integer. Assuming that this lemma is true for $m < m_0$, let us prove it for $m = m_0$. Let $l_0 = l_0(\rho, \chi)$ be as in Lemma 8.13 (2). If $\#\{0 \le j \le s : J_j \subset B_{n-m_0+1}\} \le l_0$, then the lemma follows again from Lemma 8.13 (1). Otherwise, let $s_1 < s$ be such that $\#\{s_1 \le j \le s : J_j \subset B_{n-m_0+1}\} = l_0$ and $J_{s_1} \subset B_{n-m_0+1}$. Then by Lemma 8.13 (2), there exists $\theta' \in (0, \pi)$, such that $F^j(U) \subset D_{\theta'}(\mathcal{L}_{J_j}(B_{n-m-1}))$ for all $s_1 \le j < s$, and such that $F^{s_1}(U) \subset D_{\theta'}(T_{n-m_0+1})$. Applying the induction hypothesis to the chain $\{J_j\}_{j=0}^{s_1}$, we complete the proof. □

Proposition 8.8. *For any $\rho > 1$ and $\chi \in \mathbb{N}$, there is a constant $C > 1$ with the following property. Fix a positive integer N. Let n be a sufficiently large positive integer such that $\chi_{n-i} \le \chi$ for all $1 \le i \le N$ and such that $\Lambda_{E_{n-i}} \le \rho$ for all $1 \le i \le N + 1$. Let $V = \Omega_{n-N} \cap \mathbb{C}_{T_n}$, and $U = Comp_{B_n}(F^{-s_n}(V))$. Then for any $0 \le j \le s_n - 1$,*

$$F^j(U) \subset D_\theta\big(\mathcal{L}_{f^j(c)}(B_{n-N-1})\big),$$

where $\theta > 0$ is a constant which may depend on N. Moreover, for any $z \in U$,

$$(47)\qquad \left(\frac{d(z, B_n)}{|B_n|}\right)^2 \le C \max\left\{1, \frac{d(F^{s_n}(z), B_n)}{|B_n|}\right\}.$$

Proof. The first statement follows from Lemma 8.14, and so our task is to show (47). Let $V' = \Omega_{n-N} \cap \mathbb{C}_{B_n}$ and $U' = \mathrm{Comp}_{B_n}(F^{-s_n}(V'))$. If $z \in U - U'$ then (47) follows from Lemma 8.8. So let us assume that $z \in U'$. Write $z_j = F^j(z)$ for all $0 \le j \le s_n$. Let $\{J_j\}_{j=0}^{s_n}$ be the chain with $J_s = J_0 = B_n$. As before, for $\varepsilon > 0$ and $0 \le j \le s_n$, we say that z_j *ε-jumps* if $z_j \notin S(J_j, \varepsilon)$, and that j is an *ε-good* time if $|J_j|/|J_{s_n}| \ge \varepsilon$. Let $n' \le n$ be maximal with $z_{s_n} \in \Omega_{n'}$. Let $l_0 = l_0(\rho, \chi)$ be as in Lemma 8.13.

Claim. There exist constants $\varepsilon > 0$ and $C > 1$ which depend only on ρ and χ such that if n is sufficiently large, then either of the following holds:

- there is an ε-good ε-jump time $j \ge 1$ with $d(z_j, B_n) \le C|B_{n'}|$;
- there is $0 \le j \le s_n$ such that $J_j \subset B_{n-l_0}$ and $z_j \in \Omega_{n-l_0}$.

To prove this claim, we may assume that $n' < n - l_0$, since otherwise the second alternation in this claim holds for $j = s_n$. Let $t_0 = s_n$, and for any $1 \le i \le n - n' - l_0$, let $t_i < s_n$ denote the maximal non-negative integer with $J_{t_i} \subset B_{n'+l_0+i}$. Then for $i \ge 1$, $t_i = s_n - s_{n'+l_0+i}$ and thus t_i are pairwise distinct.

For any $0 \le j_1 \le j_2 \le s_n$ and any $n' \le i \le n$, define

$$X(i; j_1, j_2) := \{j_1 \le j \le j_2 : J_j \subset B_i\}.$$

Then,

$$\#X(n'+1; t_1, t_0) = (t_0 - t_1)/s_{n'+1} + 1 > s_{n'+l_0+1}/s_{n'+1} \ge 2^{l_0} > l_0,$$

and for any $1 \le i \le n - n' - l_0 - 1$,

$$\#X(n'+i+1; t_{i+1}, t_i) = (t_i - t_{i+1})/s_{n'+i+1} + 1 \ge s_{n'+l_0+i}/s_{n'+i+1} > l_0.$$

By a similar argument as in the proof of Statement 1 of Proposition 8.5, we find a constant $\nu = \nu(\rho, l_0) > 0$ such that all these t_i's are ν-good times.

Let $t_i < t_i' < t_{i-1}$ be such that $\#X(n'+i; t_i', t_{i-1}) = l_0$ and $J_{t_i'} \subset B_{n'+i}$. By Lemma 8.13 (2), we have $F^{t_1'}(U) \subset D_{\theta_1}(T_{n'+1})$ for a definite constant θ_1. Furthermore by Lemma 8.14, we get $F^{t_1}(U) \subset D_{\theta_2}(S_{n'+1})$ for a definite constant θ_2. Thus, for an appropriately chosen small constant $\varepsilon > 0$, either z_{t_1} ε-jumps or $z_{t_1} \in \Omega_{n'+1}$. In the former case, the first alternation of the claim holds for $j = t_1$. In the latter case, for the same reasoning as above, we obtain $z_{t_2} \in D_{\theta_2}(S_{n'+2})$, so either the first alternation of the claim holds for $j = t_2$ or $z_{t_2} \in \Omega_{n'+2}$. Repeating this argument, we prove the claim.

If the former alternation in the claim holds, then the inequality (47) follows from Proposition 8.1. If the latter holds, then it follows from Lemma 8.14 that $z_0 \in D_\theta(T_{n-l_0})$ for a universal $\theta > 0$, and thus (47) holds as well. ∎

Proof of Proposition 8.7. Choose N appropriately large and then fix it. Let n be a large positive integer, and let $V = \Omega_{n-N} \cap \mathbb{C}_{T_n}$ and $U = \mathrm{Comp}_c(F^{-s_n}(V))$ be as in the previous proposition. Then by the first statement of the previous proposition, $F|F^j(U)$ is holomorphic for each $0 \le j \le s_n - 1$. Moreover, by (47), we have $U \subset V$ and $\mathrm{Cl}(U) - V \subset \mathbb{R}$. Thus V determines a quasi-polynomial-like extension of $\mathbf{B}_{B_n}$. To show that $\mathbf{B}_{B_n}$ has a polynomial-like extension, by Lemma 8.3, it suffices to show that there exists $q \in \mathbb{N}$ such that

$$\mathrm{Comp}_{B_n}\big(F^{-qs_n}(V)\big) \Subset V.$$

Let $V' = \Omega_{n-N} \cap \mathbb{C}_{B_n}$. Let $l_0 = l_0(\rho, \chi)$ be as in Lemma 8.13 (2), and let $q = Nl_0$. Let $\{J_j\}_{j=0}^{qs_n}$ be the chain with $J_{qs_n} = J_0 = B_n$. Since J_j's enter B_{n-N+1} more than l_0 times, there exists $p_1 \le s_n$ such that $J_{p_1} \subset B_{n-N+1}$, and such that

$$\#\{p_1 \le j \le s_n : J_j \subset B_{n-N+1}\} = l_0.$$

By Lemma 8.13 (2), $\mathrm{Comp}_{J_{p_1}}(F^{-(qs_n-p_1)}(V')) \subset D_{\theta_1}(T_{n-N+1})$ for some $\theta_1 \in (0, \pi)$. Note that $\#\{0 \le j \le p_1 - 1 : J_j \supset B_n\} > Nl_0 - l_0$. Thus we can inductively define positive integers $p_N < \cdots < p_2 < p_1$ such that for each $i \le N$, there exists $\theta_i \in (0, \pi)$ such that

$$\mathrm{Comp}_{J_{p_i}}\big(F^{-(qs_n-p_i)}(V')\big) \subset D_{\theta_i}(T_{n-N+i}), \text{ and } J_{p_i} \subset T_{n-N+i}.$$

Furthermore, applying Lemma 8.8 gives us $\mathrm{Comp}_{B_n}(F^{-qs_n}(V')) \subset D_{\theta'}(S_n)$, where $\theta' \in (0, \pi)$ is a constant. Again by Lemma 8.8, $F^{-qs_n}(\Omega_n) \subset D_{\theta''}(S_n)$ for some $\theta'' \in (0, \pi)$. Thus

$$\mathrm{Comp}_{B_n}\big(F^{-qs_n}(V)\big) \subset \mathrm{Comp}_{B_n}\big(F^{-qs_n}(V')\big) \cup \mathrm{Comp}_{B_n}\big(F^{-qs_n}(\Omega_n)\big)$$

is compactly contained in V. The proof is completed. □

9. Proof of main theorem

In this section, we shall complete the proof of our main theorem and corollary. Recall that $C^2([0,1],[0,1])$ is the space of all C^2 maps from $[0,1]$ into itself, endowed with the C^2 topology. This is a complete metric space.

Recall that $\mathcal{N}$ is the family of all smooth maps $f : [0,1] \to [0,1]$ such that $f(\{0,1\}) \subset \{0,1\}$ and such that all critical points are non-degenerate and contained in the open interval $(0,1)$. In the following we shall consider $\mathcal{N}$ as a subspace of $C^2([0,1],[0,1])$, so $\mathcal{N}$ is endowed with the C^2 topology.

For any $f \in \mathcal{N}$, let $\mathrm{Crit}(f)$ denote the set of its critical points, and define

$$\mathrm{Crit}_1(f) := \big\{c \in \mathrm{Crit}(f) : c \text{ is attracted by a hyperbolic attracting cycle}\big\},$$

$$\mathrm{Crit}_2(f) := \big\{c \in \mathrm{Crit}(f) : c \in \mathrm{Crit}_1(f), \text{ or is precritical}\big\},$$

where a critical point c is called *precritical* if $f^k(c) \in \mathrm{Crit}(f)$ for some positive integer k. Let $N_i : \mathcal{N} \to \mathbb{N} \cup \{0\}$, $i = 1, 2$ be the functions defined by $N_i(f) = \#\mathrm{Crit}_i(f)$. We say that f is *(C^2-)locally best* if N_1 is locally constant and N_2 is locally maximal at f. Notice that N_1 is lower semi-continuous and N_2 is locally bounded from above. So locally best maps form a dense subset of $\mathcal{N}$. The main step is to prove the following:

Theorem 9.1. *If $f \in \mathcal{N}$ is locally best, then*

$$N_2(f) = \#Crit(f). \tag{48}$$

We shall prove this theorem by contradiction. Assume that f is a locally best map with $\mathrm{Crit}(f) \neq \mathrm{Crit}_2(f)$. First, it is well-known that any critical point c not in $\mathrm{Crit}_2(f)$ is non-periodic and recurrent, which is a consequence of the non-existence of wandering intervals. Next, applying the C^2 closing lemma of Blokh and Misiurewicz, we show that f cannot have large bounds at c. So by Theorem 1, f must have essentially bounded geometry at c. Finally we use Kozlovski's deformation trick and apply Theorems 2 and 3 to show that this cannot happen either, and then complete the proof.

Lemma 9.1. *Assume that f is locally best. Then each critical point $c \in Crit(f) - Crit_2(f)$ is recurrent.*

Proof. Assume not. Then we can perturb f in C^2 topology (in fact in any C^r topology as long as f is C^r) to get a map $f_1 \in \mathcal{U}$ such that $N_1(f_1) > N_1(f)$ or $N_2(f_1) > N_2(f)$, which contradicts the locally best property of f. For details of the proof, see for example Lemmas 3.10 and 3.12 in [5]. □

Recall that f has *large bounds* at c if for any $C > 0$ and any $\varepsilon > 0$, there is a C-nice symmetric interval I containing c with $|I| < \varepsilon$.

Lemma 9.2. *Assume that f is locally best. If $c \in Crit(f) - Crit_2(f)$, then f does not have large bounds at c. In particular, $\omega(c)$ is a minimal set.*

Proof. Arguing by contradiction, assume that f has large bounds at c. Let $C > 0$ be a large number, and $\varepsilon > 0$ a small one. Let $I \ni c$ be a C-nice interval with $|I| < \varepsilon$. Let s be the return time of c to I. Consider the chain $\{J_j\}_{j=0}^s$ with $J_s = I$ and $J_0 \ni c$. Let $K = \mathcal{L}_{f^s(c)}(I)$, and let $K_1 = \mathrm{Comp}_{f(c)}(f^{-(s-1)}(K))$. By Lemma 3.6, K_1 is deep inside J_1. In particular, $|K_1|/|f(J_0)|$ is small.

Let v be an endpoint of K_1. Let $\phi : J_0 \to \mathbb{R}$ be a smooth map defined as follows. Let $h : [0, 1] \to \mathbb{R}$ be a fixed smooth map such that $h(0) = 1$, $h(1) = 0$ and $h^{(i)}(t) = 0$ for $t = 0, 1$ and any $i \in \mathbb{N}$. Write $J_0 = (c - \eta', c + \eta)$, and let

$$\phi(x) = \begin{cases} \big(v - f(c)\big)h\big(\frac{x-c}{\eta}\big) & \text{if } x \in [c, c+\eta) \\ \big(v - f(c)\big)h\big(\frac{c-x}{\eta'}\big) & \text{otherwise.} \end{cases}$$

Then $\phi(c) = v - f(c)$, $\phi'(c) = 0$, and for any $a \in \partial J_0$ and any $i \in \mathbb{N} \cup \{0\}$, $\phi^{(i)}(a) = 0$. Moreover, ϕ has a small C^2 norm because $|v - f(c)|/|J_0|^2$ is small.

Define $f_1 = f$ outside J_0, and $f_1 = f + \phi$ in J_0. Then $f_1 \in \mathcal{N}$ and it is close to f in C^2 topology. Since f_1 and f only differ on a small neighborhood of c, $\mathrm{Crit}_2(f) \subset \mathrm{Crit}_2(f_1)$. By the locally best property of f, these two sets must coincide, and f_1 is also locally best. So $c \notin \mathrm{Crit}_2(f_1)$. But $f_1^s(c) = f^{s-1} \circ f_1(c) = f^{s-1}(v) \in \partial K$, and so c is a non-recurrent critical point of f_1, which contradicts Lemma 9.1.

The latter statement of the lemma follows from the former one by Theorem 3.4. □

Remark 9.1. The argument which we used above comes from [5]. In order to get a small C^2 norm, the perturbation ϕ has to be supported on a neighborhood of a critical point. If we want to keep the map f unchanged near the critical points, then we can only get C^1 closing. Similarly, for f and f_1 constructed as above, if φ and φ_1 are diffeomorphisms defined on J_0 such that $f(x) = \varphi(x)^2 + f(c)$ and $f_1(x) = \varphi_1(x)^2 + f_1(c)$ hold for $x \in J_0$, then these maps φ_1 and φ are only C^1-close to each other (although f_1 and f are C^2 close). This is nevertheless an improvement of Jakobson's closing lemma [18], which only gives C^0 approximation in both cases.

So far we have shown that if a locally best map f has a critical point $c \in \mathrm{Crit}(f) - \mathrm{Crit}_2(f)$, then c is non-periodic, recurrent and has a minimal ω-limit set, and f does not have large bounds at c. In the following, we are going to prove that this is absurd. First we shall prove that the real box mappings associated to certain small symmetric nice intervals are stable in an appropriate sense.

Recall that an open interval I is *strictly nice* if $\inf_{k \in \mathbb{N}, x \in \partial I} d(f^k(x), I) > 0$. Remark that if f is only finitely renormalizable at c, then there exists an arbitrarily small symmetric strictly nice interval containing c. In fact, for any symmetric nice interval $I \ni c$ with $\mathcal{L}_c(I) \neq I$, $\mathcal{L}_c(I)$ is strictly nice.

Let $I \ni c$ be a small symmetric interval which is properly periodic if f is infinitely renormalizable at c and strictly nice otherwise. Let $\mathbf{B}_I : (\bigcup_{j=0}^m J_j) \cup (\bigcup_{i=1}^{b-1} I_i) \to \bigcup_{i=0}^{b-1} I_i$ be the real box mapping associated to I. Let $\eta = \eta(I) > 0$ be a small constant such that

- for any $c' \in \mathrm{Crit}_1(f)$ and any $k \geq 0$, we have $d(f^k(c'), \mathrm{dom}(\mathbf{B}_I)) \geq \eta$;
- for any $c' \in \mathrm{Crit}_2(f) - \mathrm{Crit}_1(f)$, if k is the minimal positive integer such that $f^k(c') \in \mathrm{Crit}(f)$, then $d(f^i(c'), \mathrm{dom}(\mathbf{B}_I)) \geq \eta$ for any $1 \leq i \leq k-1$;
- the η-neighborhoods of any two distinct components of $\mathrm{dom}(\mathbf{B}_I)$ are disjoint.

Remark that for any $c' \in \mathrm{Crit}(\mathbf{B}_I) \cap \mathrm{Crit}_2(f)$, if $k \in \mathbb{N}$ is minimal with $f^k(c') \in \mathrm{Crit}(f)$, then $\mathbf{B}_I(c') = f^k(c') \in \mathrm{Crit}(\mathbf{B}_I)$.

Now let $\mathcal{B}_I$ denote the space consisting of maps B in $\mathcal{E}_b$ such that $\mathrm{dom}(B) = \mathrm{dom}(\mathbf{B}_I)$, $\mathrm{range}(B) = \mathrm{range}(\mathbf{B}_I)$, $\mathrm{Crit}(B) = \mathrm{Crit}(\mathbf{B}_I)$ and such that the following holds: for all $x \in \partial(\mathrm{dom}(\mathbf{B}_I)) \cup (\mathrm{Crit}(\mathbf{B}_I) - \{c\})$, $B(x) = \mathbf{B}_I(x)$. (For $x \in \partial\mathrm{dom}(\mathbf{B}_I)$, $B(x)$ and $\mathbf{B}_I(x)$ are defined by continuation.) For any $B_1, B_2 \in \mathscr{B}_I$, define

$$d_I(B_1, B_2) = \sup_{x \in \mathrm{dom}(\mathbf{B}_I)} \max_{0 \le i \le 2} \left\{ \left| B_1^{(i)}(x) - B_2^{(i)}(x) \right| \right\}.$$

For any $B \in \mathscr{B}_I$, we construct a piecewise smooth, continuous map $f_B : [0, 1] \to [0, 1]$ as follows. For each component J of $\mathrm{dom}(\mathbf{B}_I)$, let $s = s(J)$ be the positive integer such that $\mathbf{B}_I|J = f^s|J$ and let $0 \le i = i(J) \le b - 1$ be such that $\mathbf{B}_I(J) \subset I_i$. Let K be the component of $f^{-s+1}(I_i)$ which contains $f(J)$. As noted in the proof of Lemma 3.2, $f^{s-1} : K \to I_i$ is a diffeomorphism. Define $f_B|J = (f^{s-1}|K)^{-1} \circ B$ for every J and $f_B = f$ on $[0, 1] - \mathrm{dom}(\mathbf{B}_I)$.

Lemma 9.3. *There exists a constant $\delta = \delta(I) > 0$, such that for any $B \in \mathscr{B}_I$ with $d_I(B, \mathbf{B}_I) \le \delta$, the following hold:*

(1) *for any $c' \in \mathrm{Crit}(B)$ and $k \in \mathbb{N}$, $B^k(c') \in \mathrm{dom}(B)$;*
(2) *for any $c', c'' \in \mathrm{Crit}(B)$ and any non-negative integers k, m, $B^k(c') < B^m(c'')$ if and only if $\mathbf{B}_I^k(c') < \mathbf{B}_I^m(c'')$. Furthermore, $B^k(c') = B^m(c'')$ if and only if $\mathbf{B}_I^k(c') = \mathbf{B}_I^m(c'')$;*
(3) *$B \in \mathscr{F}_b$ and B has essentially bounded geometry.*

Proof. Let $B_t = (1 - t)\mathbf{B}_I + tB$, $t \in [0, 1]$ be a one-parameter family of maps defined on $\mathrm{dom}(B)$. When δ is sufficiently small, all these maps B_t are contained in $\mathscr{E}_b$. In particular, B_t does not have a wandering interval for any $t \in [0, 1]$.

(1) In the case that I is a properly periodic interval, this is clear. So let us assume that I is a strictly nice interval. Arguing by contradiction, assume that the assertion fails. Then there exists a minimal $k_0 \in \mathbb{N}$ such that

$$B^{k_0}(c_{i_0}) \notin \mathrm{dom}(B) \tag{49}$$

holds for some $i_0 \in \{0, 1, \ldots, b - 1\}$. By the minimality of k_0, $c_{i_0} \notin \mathrm{Crit}_2(f)$.

By continuity, there exists $t_0 \in [0, 1]$ such that $B_{t_0}(c_{i_0}) \in \partial\mathrm{dom}(B_{t_0})$. Let us show that this contradicts the hypothesis that f is locally best. Let $f_{t_0} = f_{B_{t_0}}$ be defined as above. Note that f_{t_0} has c_{i_0} as a non-recurrent critical point, but may not be smooth. Since I is strictly nice, there exists $\eta_1 = \eta_1(I) > 0$, such that the following hold:

- if J is a component of D_I intersecting $\omega(c) - \mathrm{dom}(B)$, then $d(J, \mathrm{dom}(B)) > \eta_1$;
- for any $x \in \partial I$ and any $k \ge 1$, $d(f^k(x), I) \ge \eta_1$.

Provided that δ is sufficiently small, there exists a map $g_{t_0} \in \mathcal{N}$ such that

- $g_{t_0}(x) = f_{t_0}(x)$ for any $x \in \mathrm{dom}(B)$ and for any x with $d(x, \mathrm{dom}(B)) > \eta_1$; and
- $\sup |g_{t_0}^{(i)} - f_{t_0}^{(i)}|$ is small for all $0 \le i \le 2$, where the supremum is taken over all points where f_{t_0} is C^2.

Then g_{t_0} is close to f in the C^2 topology and $\mathrm{Crit}_2(g_{t_0}) \supset \mathrm{Crit}_2(f)$. By the locally best property of f, it follows that g_{t_0} is also locally best and $\mathrm{Crit}_2(g_{t_0}) = \mathrm{Crit}_2(f)$. In particular, $c_{i_0} \in \mathrm{Crit}(g_{t_0}) - \mathrm{Crit}_2(g_{t_0})$. But $g_{t_0}^n(c_{i_0}) = f_{t_0}^n(c_{i_0})$ for all $n \ge 0$ and so c_{i_0} is a non-recurrent critical point of g_{t_0}, which contradicts Lemma 9.1.

(2) Let us first prove the following

Claim. For any $0 \le i_0, i_1 \le b-1$, any $t \in [0,1]$, and any $k \in \mathbb{N}$, $B_t^k(c_{i_0}) - c_{i_1}$ and $B^k(c_{i_0}) - c_{i_1}$ have the same sign.

In the case that I is strictly nice, this claim can be proved using the same argument as above. Now let us assume that I is a properly periodic interval. The argument above is not valid since D_I may have a component U with $U \cap (orb(c) - I) \neq \emptyset$ and $\mathrm{Cl}(U) \cap \mathrm{Cl}(I) \neq \emptyset$, but can be refined as follows.

Since f is infinitely renormalizable, there is a symmetric properly periodic interval $J = (a, a') \Subset I$. Let s be the positive integer such that $\mathbf{B}_I^s(J) \subset J$. Then $\bigcup_{i=0}^{\infty} \mathbf{B}_I^i(J) = \bigcup_{i=0}^{s-1} \mathbf{B}_I^i(J)$ is compactly contained in $\mathrm{dom}(\mathbf{B}_I)$. When B is sufficiently close to $\mathbf{B}_I$, there is an interval $J = (a(B), a'(B)) = J(B)$ such that $B^s(J) \subset J$, and such that $a(B)$ is close to a and $a'(B)$ is close to a'. In particular, $\bigcup_{i=0}^{\infty} B^i(J(B))$ is uniformly well separated from $\partial \mathrm{dom}(B)$. Note that the critical orbits of B are contained in $\bigcup_{i=0}^{\infty} B^i(J(B))$. It follows that there is a constant $\eta_1 > 0$, such that for any $B \in \mathcal{B}_I$ which is sufficiently close to $\mathbf{B}_I$, and for any $c' \in \mathrm{Crit}(B)$, we have $\inf_{k=0}^{\infty} d(\partial(\mathrm{dom}(B)), B^k(c')) > \eta_1$.

Arguing by contradiction, assume that the claim fails. Then there exists a minimal $k_0 \in \mathbb{N}$ such that for some $0 \le i_0, i_1 \le b-1$, for some $t \in [0,1]$, $B_t^{k_0}(c_{i_0}) - c_{i_1}$ and $B^{k_0}(c_{i_0}) - c_{i_1}$ have different signs. Obviously $k_0 > 1$ if δ is sufficiently small. By the minimality of k_0, $c_{i_0} \notin \mathrm{Crit}_2(f)$. By continuity, there exists $t_0 \in [0,1]$ such that $B_{t_0}^{k_0}(c_{i_0}) = c_{i_1}$. Let $f_{t_0} = f_{B_{t_0}}$ be as defined above the lemma. Then there exists a map $g = g_{t_0} \in \mathcal{N}$ such that $g(x) = f_{t_0}(x)$ for any $x \notin \mathrm{dom}(B)$ and for any x with $d(x, \partial(\mathrm{dom}(B))) \ge \eta_1$, and such that g is close to f in the C^2 topology. But $\mathrm{Crit}_2(g) \supset \mathrm{Crit}_2(f) \cup \{c_{i_0}\}$, which contradicts the assumption that f is locally best. The proof of the claim is completed.

Thus, for any $c' \in \mathrm{Crit}(B)$, $t \in [0,1]$ and $k \in \mathbb{N}$, $B^k(c')$ and $B_t^k(c')$ are contained in the closure of the same component of $\mathrm{dom}(B) - \mathrm{Crit}(B)$. (In the language of [41], the maps B_t's have the same kneading sequences.)

Assume that for some $c', c'' \in \mathrm{Crit}(B)$ and some non-negative integers k, m, we have $B_0^k(c') < B_0^m(c'')$ but $B_1^k(c') \ge B_1^m(c'')$. Then by continuity there exists $t \in [0,1]$ such that $B_t^k(c') = B_t^m(c'')$. For any $i \geq 0$, since $B_t^{k+i}(c') = B_t^{m+i}(c'')$, $B_0^{k+i}(c')$ and $B_0^{m+i}(c'')$ are contained in the closure of the same component of $\mathrm{dom}(B_0) - \mathrm{Crit}(B_0)$. Thus $B_0^i|[B_0^k(c'), B_0^m(c'')]$ is well-defined and monotone for all $i \geq 0$. By Theorem 3.1, this implies that $\omega_f(c')$ is a periodic orbit, which is absurd. Similarly, we can show that $B_0^k(c') = B_0^m(c'')$ implies that $B_1^k(c') = B_1^m(c'')$, and that $B_0^k(c') > B_0^m(c'')$ implies that $B_1^k(c') > B_1^m(c'')$. The proof of the second assertion of this lemma is completed.

(3) We have shown that the closure of the critical orbits of B is compactly contained in the range of B. In fact, if I is strictly nice, then $\mathrm{dom}(B)$ is compactly contained in $\mathrm{range}(B)$, and thus this follows from the first assertion of this lemma; if I is properly periodic, this has been shown in the proof of the second assertion. If B has a non-recurrent critical point $c' \in \mathrm{Crit}(f) - \mathrm{Crit}_2(f)$, then as above we can construct a map g_1 which also has c' as a non-recurrent critical point, which is absurd by Lemma 9.1. Similarly, by Lemma 9.2, we conclude that B has essentially bounded geometry at c'.

Finally, let us show that the critical points of B have the same ω-limit set. Let $c' \neq c''$ be critical points of B. By the construction of $\mathbf{B}_I$, there is a sequence of positive integers $\{n_k\}_{k=1}^{\infty}$, such that $n_k \to \infty$ and $\mathbf{B}_I^{n_k}(c') \to c''$ as $k \to \infty$. For any $m \in \mathbb{N}$, provided that k is sufficiently large, the interval $[c'', \mathbf{B}_I^{n_k}(c')]$ is very small, and thus $\mathbf{B}_I^m$ is monotone on this interval. By the second assertion of this lemma, B^m is monotone on $[c'', B^{n_k}(c')]$. This implies that $B^{n_k}(c') \to c''$ since B does not have a wandering interval. □

It is easy to see that there exists a real analytic map B in $\mathscr{B}_I$ which is arbitrarily close to $\mathbf{B}_I$. So we can find a map g in $\mathcal{N}$, arbitrarily close to f, such that I is a nice interval with respect to g and such that the corresponding real box mapping $\mathbf{B}_I^g$ is real analytic. Moreover, $\mathrm{Crit}(g) = \mathrm{Crit}(f)$ and $\mathrm{Crit}_2(g) \supset \mathrm{Crit}_2(f)$. Since f is locally best, it follows that $\mathrm{Crit}_2(g) = \mathrm{Crit}_2(f)$ and g is also locally best. Thus, we may assume that $\mathbf{B}_I$ is real analytic.

By Theorem 3, there is an arbitrarily small symmetric nice interval $K \ni c$, such that the real box mapping $\mathbf{B}_K$ associated to K can be extended to a real symmetric polynomial-like box mapping. To simplify the notation, let us assume that $\mathbf{B}_I$ itself has such an extension

$$F : \left(\bigcup_{j=0}^{m} U_j\right) \cup \left(\bigcup_{i=1}^{b-1} V_i\right) \to \bigcup_{i=0}^{b-1} V_i.$$

We may assume that the boundaries of U_j's and V_i's are all real analytic curves. Moreover, in the case that I is strictly nice, we may assume that $V_i \cap \mathbb{R} = I_i$ and $U_j \cap \mathbb{R} = J_j$.

Let $i_0 \in \{0, 1, \ldots, b-1\}$ be such that $F(c) = \mathbf{B}_I(c) \in V_{i_0}$. For $t \in V_{i_0} \cap \mathbb{R}$, let α_t be the conformal automorphism of V_{i_0} such that $\alpha_t'(F(c)) > 0$

and $\alpha_t(F(c)) = t$, and define

$$F_t : \left(\bigcup_{j=0}^{m} U_j\right) \cup \left(\bigcup_{i=1}^{b-1} V_i\right) \to \bigcup_{i=0}^{b-1} V_i$$

by declaring that $F_t|U_0 = \alpha_t \circ F|U_0$ and $F_t = F_0$ on $(\bigcup_{j=1}^{m} U_j) \cup (\bigcup_{i=1}^{b-1} V_i)$. In this way we obtain a real analytic family of polynomial-like box mappings. Let $F_t^{\mathbb{R}}$ denote the real trace of F_t, which is a map in $\mathcal{E}_b$. Let Γ be the subset of $V_{i_0} \cap \mathbb{R}$, consisting of those t such that $F_t^{\mathbb{R}}$ is in $\mathcal{F}_b$, is combinatorially equivalent to the real trace of F and has essentially bounded geometry (at c).

Lemma 9.4. *$F(c)$ is in the interior of Γ.*

Proof. Assume first that I is strictly nice. When t is sufficiently close to $F(c)$, α_t is close to the identity map, and thus the real trace of F_t is a map in the class $\mathcal{B}_I$ and close to $\mathbf{B}_I$ in the d_I distance. By Lemma 9.3 it follows that $t \in \Gamma$. Thus $F(c)$ is in the interior of Γ.

Now let I be a properly periodic interval. Let p be the fixed point of F^b in ∂I. Note that p is a repelling fixed point of F^b (provided that I was chosen to be sufficiently small). Thus for all t sufficiently close to $F(c)$, F_t^b has a fixed point p_t in $V_0 \cap \mathbb{R}$ which is close to p. Define $I_1(t) = (\tau_c(p_t), p_t)$. Here τ_c is the involution of $U_0 \cap \mathbb{R}$ such that $f \circ \tau_c = f$. For any $1 \le i \le b-1$, let $0 \le i' \le b-1$ be such that $F(V_i) \subset V_{i'}$, and define $I_i(t) \subset V_i$ to be the preimage of $I_{i'}(t)$ under $F_t^{\mathbb{R}}$. Rescaling $F_t | \bigcup_{i=0}^{b-1} I_i(t)$, we obtain a map $B_t : \bigcup_{i=0}^{b-1} I_i \to \bigcup_{i=0}^{b-1} I_i$. As above, $B_t \in \mathcal{B}_I$ and $d_I(B_t, \mathbf{B}_I)$ is small, which implies that $t \in \Gamma$. The lemma follows. □

We are now ready to apply the deformation trick from [21] to deduce the necessary contradiction to complete the proof of Theorem 9.1. To this end, we need the following.

Proposition 9.2. *A real symmetric polynomial-like box mapping F carries no invariant line field on its Julia set.*

The proposition was proved in [43] for real rational functions (other than Lattés examples), and the proof extends to the set-up of real symmetric polynomial-like box mappings in a straightforward way. Here by an *invariant line field* of F, we mean a measurable F-invariant Beltrami differential. The proposition means that if $\phi : \mathbb{C} \to \mathbb{C}$ is a qc map such that $\phi \circ F \circ \phi^{-1}$ is a holomorphic map on $\phi(\text{dom}(F))$, then $\bar\partial\phi = 0$, a.e. on the Julia set of F.

Completion of Theorem 9.1 Keep the notation as above. Let $X = \bigcup_{i=0}^{b-1} V_i$. For any $t \in V_{i_0} \cap \mathbb{R}$, let $\phi_t : X \to X$ be a real symmetric qc map such that

- $\phi_t | \partial V_0 = id_{\partial V_0}$;
- $\phi_t \circ F = F_t \circ \phi_t$ on $\partial(\text{dom}(F))$.

We choose the maps ϕ_t to satisfy the following conditions:

- For any $z \in X$, $t \mapsto \phi_t(z)$ is a real analytic function;
- The complex dilatation $\mu_t = \bar{\partial}\phi_t/\partial\phi_t$ depends real analytically on t.

Define a Beltrami differential ν_t on X such that $\nu_t(z) = (F^n)^*(\mu_t)(z)$ for a.e. $z \in F^{-n}(V_0 - \bigcup_{j=0}^m U_j)$, and $\nu_t(z) = 0$ on the filled Julia set of F. Then ν_t depends real analytically on t. By the Measurable Riemann Mapping Theorem, there is a family $\{p_t\}_{t \in V_{i_0} \cap \mathbb{R}}$ of qc homeomorphisms of X with the following properties:

- $p_t(V_i) = V_i$ for each $0 \le i \le b-1$;
- $\bar{\partial} p_t = \nu_t \partial p_t$, a.e.;
- p_t fixes the critical points of F and the endpoints of each $V_i \cap \mathbb{R}$;
- for each $z \in X$, $t \mapsto p_t(z)$ is a real analytic function.

Now let t be a point in Γ. Note that the real trace of F_t has only repelling periodic points. By Theorem 2, there is a qs map h_t which is a conjugacy between $F|\omega_F(c)$ and $F_t|\omega_{F_t}(c)$. So we can construct a real symmetric qc map $\phi_t^0 : X \to X$ which coincides with ϕ_t on $(V_0 - \bigcup_{j=0}^m U_j) \cup (\bigcup_{i=1}^{b-1} \partial V_i)$ and coincides with h_t on $\omega_F(c)$. The qc map ϕ_t^0 provides a Thurston equivalence between F and F_t, from which a qc conjugacy ϕ_t^∞ can be constructed as follows. For any $j \ge 0$, define $\phi_t^{j+1} : X \to X$ inductively to be the qc map such that

- $\phi_t^{j+1} = \phi_t^j$ on $\left(V_0 - \bigcup_{j=0}^m U_j\right) \cup \left(\bigcup_{i=1}^{b-1} \partial V_i\right)$;
- $F_t \circ \phi_t^{j+1} = \phi_t^j \circ F$ on $\mathrm{dom}(F)$;
- ϕ_t^{j+1} is real symmetric.

Then ϕ_t^j converges to a qc map $\phi_t^\infty : X \to X$, which coincides with ϕ_t on $(V_0 - \bigcup_{j=0}^m U_j) \cup (\bigcup_{i=1}^{b-1} \partial V_i)$, and satisfies $F_t \circ \phi_t^\infty = \phi_t^\infty \circ F$. Here we use the fact that the filled Julia set of F has no interior point. Moreover, by Proposition 9.2, the complex dilatation of ϕ_t^∞ is equal to ν_t. It follows that $\phi_t^\infty = p_t$ for all $t \in \Gamma$.

For any $z \in \partial(\mathrm{dom}(F))$, $\phi_t(z) = p_t(z)$ for all $t \in \Gamma$. Since Γ contains a neighborhood of $F(c)$, by the real analytical dependence on t, we conclude that $\phi_t(z) = p_t(z)$ for all $z \in \partial(\mathrm{dom}(F))$ and all $t \in V_{i_0} \cap \mathbb{R}$. This implies that p_t maps a component of $\mathrm{dom}(F)$ onto itself for all $t \in V_{i_0} \cap \mathbb{R}$.

Let $\hat{F}_t = p_t \circ F \circ p_t^{-1}$. Then $\hat{F}_t$ is a real symmetric polynomial-like box mapping which has the same domain and image as F. Since $\hat{F}_t = F_t$ for all $t \in \Gamma$, by analytic continuation, $\hat{F}_t = F_t$ for all $t \in V_{i_0} \cap \mathbb{R}$. Thus $\Gamma = V_{i_0} \cap \mathbb{R}$, which is absurd since the combinatorics clearly change in the family F_t, $t \in V_{i_0} \cap \mathbb{R}$. □

Proof of main theorem. Given any $f \in C^2([0, 1], [0, 1])$, we need to approximate it by maps satisfying Axiom A in the C^2 topology.

We first consider the case $f \in \mathcal{N}$. Since locally best maps are dense in $\mathcal{N}$, we may assume that f is locally best. Then all the critical points of f are contained in the basin of a hyperbolic attracting cycle. The last property is also satisfied by maps in a neighborhood $\mathcal{U}$ of f in $C^2([0,1],[0,1])$. It is well known that there is a map $g \in \mathcal{U}$ which is arbitrarily close to f and has no neutral cycles. By a theorem of Mañé [32], g is hyperbolic.

For an arbitrarily $f \in C^2([0,1],[0,1])$, we can do as follows. First note that we may assume that $f([0,1]) \subset (0,1)$, since

$$f_\varepsilon = \frac{\varepsilon + f}{1 + 2\varepsilon}$$

satisfies this property for all $\varepsilon > 0$. We may also assume that f is smooth. Then we extend f to be a smooth map $\tilde{f} : [-1,2] \to [-1,2]$ such that $\tilde{f}(\{-1,2\}) \subset \{-1,2\}$. Let F be the map in $C^2([0,1],[0,1])$ which is affine conjugate to $\tilde{f}$, then F can be approximated by maps in $\mathcal{N}$ in C^2 topology. It follows that F and hence f can be approximated in the C^2 topology by Axiom A maps. □

It is well-known that the main corollary follows from the main theorem. The proof uses the following.

Proposition 9.3. *Let $r \geq 2$ be a positive integer, and let $f \in C^r([0,1],[0,1])$ be a map which satisfies Axiom A. Assume that*

- *for any $c \in Crit(f)$, we have $c \in (0,1)$ and $f''(c) \neq 0$;*
- *both of 0 and 1 are attracted by periodic attractors of f; and*
- *for any $c, c' \in Crit(f) \cup \{0,1\}$, and any non-negative integers m, n, if $f^m(c) \neq f^n(c')$, then $c = c'$ and $m = n$.*

Then f is C^r-structurally stable.

For a proof of this proposition, see Theorem III.2.5 of [35]. (In that book, the authors considered only maps which map $\{0,1\}$ into itself. But the method extends easily to our setting.)

Appendix: Complex bounds for analytic interval maps

This appendix is an elaboration on Sect. 8. The goal is to to show that the real box mappings associated to certain symmetric nice intervals extend to polynomial-like box mappings with "complex bounds".

Theorem 3'. *For any $b \in \mathbb{N}$, there exist $\mu \in (0,\pi)$ and $p \in \mathbb{N}$ with the following property. Let f be a real analytic map in the class $\mathcal{F}_b$, and let c be a critical point of f. Then the following hold.*

(1) *For any* $\varepsilon > 0$*, there is a symmetric nice interval* I *which contains* c*, such that* $|I| < \varepsilon$ *and* $((1+2\mu)I - I) \cap \omega(c) = \emptyset$*, and such that the real box mapping* $\mathbf{B}_I$ *extends to a real symmetric polynomial-like box mapping* Φ*. Moreover, for any* $x \in \omega(c) \cap I$*, we have*

$$Comp_x\big(dom(\Phi^p)\big) \subset D_\mu\big((1-2\mu)I'\big),$$

where $I' \supset I$ *is the maximal symmetric open interval disjoint from* $\omega(c) - I$*.*

(2) *If* f *is infinitely renormalizable, and if* I *is a small properly periodic interval with period* s*, then the first return map* $f^s : I \to I$ *extends to a DH-polynomial-like mapping* $\psi : U \to V$ *of degree* 2^b *such that*

$$\text{mod}\,\big(V - Cl(U)\big) \geq \mu.$$

Both statements have been proved in the unimodal case before, see [22, 12, 31]. In this case, there have also appeared many applications of these bounds. For example, these bounds were used to prove the local connectivity of Julia sets for real unicritical polynomials with connected Julia sets, and to simplify the proof of the non-existence of invariant line fields for such polynomials, see [16, 22, 38, 23]. Moreover, the second statement was an important ingredient in dealing with the rigidity problem [27, 11], and in the renormalization theory [46, 38, 29]. We believe that the complex bounds claimed in our Theorem 3' will be useful in further understanding the dynamics of a multimodal real polynomial.

Remark that for a properly periodic interval I with period s, all DH-polynomial-like extensions of $f^s : I \to I$ of degree 2^b have the same Julia set. See Proposition 2.2 in [22].

We shall continue using the notation introduced in 8. In particular, we have a smooth polynomial-like extension F. We shall first prove Theorem 3' in case that f has large bounds. Then we shall point out how to refine the arguments in Sect. 8 to get "complex bounds" in the non-renormalizable case. Finally, combining these methods we prove "complex bounds" for infinitely renormalizable maps.

A1. The case that f has large bounds.

Proposition A.1. *There exists a constant* $\kappa = \kappa(b) > 0$ *with the following property. Let* $c \in Crit(f)$*, and let* $I \ni c$ *be a sufficiently small nice interval such that*

$$\big((1+2\kappa)^2 I^1 - I^1\big) \cap \omega(c) = \emptyset, \text{ and } (1+2\kappa)^2 I^1 \subset I,$$

where $I^1 = \mathcal{L}_c(I)$*. Let* $x \in \omega(c) \cap I^1$*, and let* s *be the return time of* x *to* I^1*. Let* $V = D_*((1+2\kappa)I^1)$*, and* $U = Comp_x(F^{-s}(V))$*. Then for each* $0 \leq j < s$*, we have*

$$F^j(U) \Subset D_*\big(\mathcal{L}_{f^j(x)}(I)\big).$$

In particular, $U \Subset D_*(I^1) \subset V$*.*

Proof. Let $\kappa > 0$ be a large constant. Let us consider the chains $\{G'_j\}_{j=0}^s$ and $\{G_j\}_{j=0}^s$ with $G'_s = (1+2\kappa)^2 I^1$, $G_s = (1+2\kappa)I^1$, and $G'_0 \supset G_0 \ni x$. By Lemma 3.8 and Corollary 3.6, $\sum_{j=0}^s |G'_j|$ is small provided that $|I|$ is sufficiently small. By Lemma 8.5, there is a universal constant θ, such that for any $0 \le j < s$, $F^j(U) \subset D_\theta(G_j)$. (In fact, $\theta \to \pi/2$ as $\kappa \to \infty$.) By Lemma 3.6, G_j is deep inside G'_j. Since $G'_j \subset D_I$, it follows that $F^j(U) \Subset D_*(\mathcal{L}_{f^j(x)}(I))$ for all $0 \le j \le s-1$. As $\mathcal{L}_x(I) = I^1$, we have $U \Subset D_*(I^1)$. □

Corollary A.2. *There exists $\rho = \rho(b) > 0$ with the following property. Let $c \in Crit(f)$ and let $K \in \mathcal{M}(\mathcal{I}_0)$ be a sufficiently small symmetric nice interval such that the limit scaling factor Λ_K is $\ge \rho$. Then there is a symmetric nice interval I with $c \in I \in \mathcal{M}(K)$, such that*

- $((1+2\kappa)^2 I - I) \cap \omega(c) = \emptyset$;
- $\mathbf{B}_I$ *extends to a polynomial-like box mapping* Φ;
- *for any $x \in \omega(c) \cap I$, we have*

$$Comp_x\big(dom(\Phi)\big) \subset D_*(I),$$

where κ is as in the previous proposition.

Proof. By Proposition 4.1, there is a C-nice symmetric interval in $\mathcal{M}(K)$, where $C = C(\rho, b)$ is a large constant provided that ρ is sufficiently large. If C is sufficiently large, then by Corollary 4.6 in [42], there is a symmetric nice interval $M \ni c$, which is again contained in $\mathcal{M}(K)$ and satisfies $(1+2\kappa)^2 M^1 \subset M$ and $((1+2\kappa)^2 M^1 - M^1) \cap \omega(c) = \emptyset$. Set $I = M^1$. It follows from the previous proposition that $\mathbf{B}_I$ has a quasi-polynomial-like extension determined by $V = D_*((1+2\kappa)I)$. By Lemma 8.2, $\mathbf{B}_I$ has a polynomial-like extension Φ determined by a real symmetric topological disk $V' \subset V$. Obviously, for any $x \in \omega(c) \cap I$, if s is the return time of x to I, then

$$\mathrm{Comp}_x\big(\mathrm{dom}(\Phi)\big) \subset \mathrm{Comp}_x(F^{-s}V) \Subset D_*(I).$$

□

Corollary A.3. *There exists $\rho = \rho(b) > 0$ with the following property. Assume that f is infinitely renormalizable. Let K be a small symmetric nice interval with $\Lambda_K \ge \rho$, and let $B = \bigcap_{k=1}^\infty K^k$ be the maximal properly periodic interval contained in K. Then the real box mapping $\mathbf{B}_B$ has a polynomial-like extension.*

Proof. Let c be the critical point in K, and let s be the period of B. As in the proof of the previous corollary, we find a symmetric nice interval $I \ni c$ which is contained in $\mathcal{M}(K)$, such that $\mathbf{B}_I$ has a quasi-polynomial-like extension determined by a real symmetric topological disk $V = D_*((1+2\kappa)I)$. Moreover, for any $x \in \omega(c) \cap I$, if q is the return time of x to I, then

$$\mathrm{Comp}_x\big(F^{-q}(V)\big) \Subset D_*(I) \subset V.$$

Let m be a positive integer such that $I^m - B$ is disjoint from $\omega(c)$. Such an integer exists because $\bigcap_{k=0}^{\infty} I^k = \bigcap_{k=0}^{\infty} K^k = B$. Let n be such that I^m is a component of $f^{-n}(I)$, let $V' = \mathrm{Comp}_c(F^{-n}(V))$, and let $U' = \mathrm{Comp}_c(F^{-n-s}(V))$. Then $F^s : U' \to V'$ is holomorphic and $U' \Subset V'$. It follows that $\mathbf{B}_B$ has a polynomial-like extension (determined by V'). □

A2. Bounds for non-renormalizable maps.

Proposition A.4. *Assume that f is non-renormalizable. If there is an $\varepsilon > 0$ such that for any symmetric nice interval I with $|I| < \varepsilon$, we have $\Lambda_I \le \rho$, then Theorem 3' holds.*

Proof. Let I be a symmetric nice interval in $\mathcal{M}(\mathcal{I}_0)$, with $|I| < \varepsilon$. Then $\Lambda_I \le \rho$. Let $m(0) = 0$, and let $m(1) < m(2) < \cdots$ be all the positive integers such that $R_{I^{m(i)-1}}(c) \notin I^{m(i)}$, where c is the critical point in I. Let $k_1 < k$ be large positive integers with $k - k_1 = N_0'$, where $N_0' = N_0'(\rho, b)$ is as in Corollary 8.6, and let $V = D_*(I^{m(k_1)}) \cap \mathbb{C}_{I^{m(k)}}$. Then V determines a quasi-polynomial-like extension of $\mathbf{B}_{I^{m(k)}}$. By Lemma 8.2, $\mathbf{B}_{I^{m(k)}}$ has a polynomial-like extension Φ determined by a topological disk $V' \subset V$.

Let $\delta = \delta(\rho) > 0$ and $l = l(\delta, \rho, b) \in \mathbb{N}$ be determined as in Sect. 8.3 (after the statement of Proposition 8.4). Let $p = 3lN_0' + 1$. To complete the proof we shall show that there exists $\theta \in (0, \pi)$ such that for any $x \in I^{m(k)} \cap \omega(c)$, we have $\mathrm{Comp}_x(\mathrm{dom}(\Phi^p)) \subset D_\theta(I^{m(k)})$.

To this end, let $s \in \mathbb{N}$ be such that $R^p_{I^{m(k)}} = f^s$ near x. It suffices to show that $\mathrm{Comp}_x(F^{-s}V) \subset D_\theta(I^{m(k)})$. Let $\{J_j\}_{j=0}^{s}$ be the chain with $J_s = I^{m(k)}$ and $J_0 \ni x$. Then by Propositions 8.2 and 8.3, (arguing similarly as in the proof of Proposition 8.4 for the infinitely renormalizable case,) there exist $s' < s$ and $\theta' \in (0, \pi)$ such that $J_{s'} \subset I^{m(k)}$ and $\mathrm{Comp}_{J_{s'}}(F^{-(s-s')}(V)) \subset D_{\theta'}(I^{m(k)})$. Finally, applying Lemma 8.5 we obtain the desired estimate. □

A3. Bounds for infinitely renormalizable maps. Now let us assume that f is infinitely renormalizable. Let c be the critical point of f in $\mathcal{I}_0$. We shall continue to use the notation introduced in Sect. 8.4. Let $\rho > 0$ be the constant as in Corollary A.3.

Proof of Theorem 3'(2). Let us say that the n-th renormalization *has a complex bound* μ if $f^{s_n} : B_n \to B_n$ extends to a DH-polynomial-like map $\psi : U \to V$ of degree 2^b such that $\mathrm{mod}(V - U) \ge \mu$. By Theorem 3 and Corollary A.3, we may assume that $f^{s_1} : B_1 \to B_1$ has a polynomial-like extension $F^{s_1} : U_1 \to V_1$. Then for any $n \ge 2$, the real box mapping associated to B_n has a quasi-polynomial-like extension determined by $V_n = V_1 \cap \mathbb{C}_{T_n}$. By Lemma 7.6, to prove that $f^{s_n} : B_n \to B_n$ has a definite complex bound, it suffices to show that it has a DH-polynomial-like extension of degree 2^b, and that the filled Julia set of this extension is contained in $D_\theta(S_n)$ for some definite $\theta > 0$. The proof is completed by the following statements.

Statement 1. *If $I \in \mathcal{M}(E_n)$ is a symmetric nice interval with $\Lambda_I \le \rho$ and $\chi_I \ge N_0' + 3$, where $N_0' = N_0'(\rho, b)$ is as in Corollary 8.6, then provided that n is sufficiently large, the $(n+1)$-th renormalization has a complex bound $\mu_1 = \mu_1(b) > 0$.*

In fact, we have proved in Proposition 8.4 that $f^{s_{n+1}} : B_{n+1} \to B_{n+1}$ extends to a DH-polynomial like mapping $\psi : U \to V$ of degree 2^b, and that there exists a constant $\theta' \in (0, \pi)$ such that $\psi^{-(3lN_0'+1)}(V) \subset D_{\theta'}(T_{n+1})$, which implies that the filled Julia set of ψ is contained in $D_\theta(S_{n+1})$ for some constant $\theta \in (0, \pi)$. By the remark above, the $(n+1)$-th renormalization has a definite complex bound.

Statement 2. *There exists a constant $C = C(b) > 1$ such that if $|B_n| > C|B_{n+1}|$, then provided that n is sufficiently large, the $(n+1)$-th renormalization has a complex bound $\mu_2 = \mu_2(b) > 0$.*

First assume that $\Lambda_{E_n} \le \rho$. When C is large, this implies that $\chi_n \ge N_0' + 3$. Applying Statement 1 to $I = A_n$, we conclude the proof. Now assume that $\Lambda_{E_n} > \rho$. Then there exists a symmetric nice interval K in $\mathcal{M}(E_n)$ such that $\Lambda_K > \rho$. By Corollary A.3, $f^{s_{n+1}}|B_{n+1}$ has a polynomial-like extension ψ. Denote by $\mathcal{K}$ the filled Julia set of ψ. In fact, it was proved there that there is a pull back $I \subset K^1$ of K which contains c, such that $(1+2\kappa)I - I$ is disjoint from $\omega(c)$, and such that $\mathcal{K} \subset D_*(I)$. If $\Lambda_I > \rho$, then for the same reason, we can find a symmetric nice interval $J \ni c$, such that $J \subset I^1 \subset K^2$, $((1+2\kappa)J - J) \cap \omega(c) = \emptyset$, and $\mathcal{K} \subset D_*(J)$. If $\Lambda_J > \rho$, then we repeat the argument. Obviously, this argument must stop within finitely many steps, so without loss of generality, let us assume that $\Lambda_I \le \rho$. If $\chi_I \ge N_0'+3$, then the proof is again completed by Statement 1. So assume $\chi_I \le N_0'+3$. By a similar argument as in the proof of Proposition 8.4, there exist $p \in \mathbb{N}$ and $\theta \in (0, \pi)$, such that

$$\mathrm{Comp}_c\Big(F^{-ps_{n+1}}\big(D_*(I) \cap \mathbb{C}_{T_{n+1}}\big)\Big) \subset D_\theta\big(I^{m(\chi_I)+3}\big).$$

In particular, $\mathcal{K} \subset D_\theta(I^{m(\chi_I)+3})$. Note that $A_n^{m_n(\chi_n)-1}$ contains a point in $\omega(c) - B_{n+1}$, while $I^{m(\chi_I)}$ does not. Therefore $I^{m(\chi_I)} \subset A_n^{m_n(\chi_n)-1}$, and hence $I^{m(\chi_I)+3} \subset A_n^{m_n(\chi_n)+2} = S_{n+1}$. So $\mathcal{K}_{n+1} \subset D_\theta(S_{n+1})$. The statement follows.

Statement 3. *There is a positive integer $N = N(b)$, such that if $|B_{n+i}| < C|B_{n+i+1}|$ for all $i = 0, 1, \ldots, N-1$, then provided that n is sufficiently large, the $(n+N)$-th renormalization has a complex bound $\mu_3 = \mu_3(b) > 0$.*

Notice that $\Lambda_{E_{n+i}} < C$ for all $0 \le i \le N-1$. Moreover, there is an upper bound on χ_{n+i}'s. So this statement follows from the argument in Sect. 8.4.

Statement 4. *For each $k \in \mathbb{N}$, there exists $\mu = \mu(k, b) > 0$ such that if $|B_{n-1}| > C|B_n|$ and $|B_{n+i}| \le C|B_{n+i+1}|$ for any $0 \le i \le k-1$, and if n is sufficiently large, then for each $1 \le i \le k$ the $(n+i)$-th renormalization has a complex bound μ.*

By Statement 2, $f^{s_n} : B_n \to B_n$ has a polynomial-like extension $\phi : U \to V$ of degree 2^b. Moreover, the proof shows that there exists an interval $\hat{T}_n$ with $c \in \hat{T}_n \subset T_n$ such that

- $|\hat{T}_n|/|B_n|$ is uniformly bounded from above, and
- we can choose V so that it is contained in a Poincaré disk $D_\theta(\hat{T}_n)$,

Fix $1 \le i \le k$, and let $V' = V \cap \mathbb{C}_{T_{n+i}}$ and $U' = \mathrm{Comp}_c(F^{-s_{n+i}}(V'))$. Then $F^{s_{n+i}} : U' \to V'$ is a holomorphic proper map of degree 2^b. By Lemmas 8.13 and 8.14, arguing in the same away as in the proof of Proposition 8.7, we show that there exist a positive integer p, and a constant $\theta' \in (0, \pi)$ such that

$$\mathrm{Comp}_c\left(F^{-ps_{n+i}}(V')\right) \subset \mathrm{Comp}_c\left(F^{-ps_{n+i}}\left(D_\theta(\hat{T}_n) \cap \mathbb{C}_{T_{n+i}}\right)\right) \subset D_{\theta'}(S_{n+i}),$$

which implies that the renormalization $f^{s_{n+i}}|B_{n+i}$ has a complex bound. We should remark that in Lemma 8.13, the assumption $\Lambda_{E_{n-1}} \le \rho$ was only used to obtain an upper bound on $|T_{n-1}|/|B_n|$. □

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