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Singularity of mean curvature flow of Lagrangian submanifolds

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Abstract. In this article we study the tangent cones at first time singularity of a Lagrangian mean curvature flow. If the initial compact submanifold Σ_0 is Lagrangian and almost calibrated by $\operatorname{Re}\Omega$ in a Calabi-Yau n-fold (M, Ω), and $T > 0$ is the first blow-up time of the mean curvature flow, then the tangent cone of the mean curvature flow at a singular point (X_0, T) is a stationary Lagrangian integer multiplicity current in $\mathbf{R}^{2n}$ with volume density greater than one at X_0. When $n = 2$, the tangent cone is a finite union of at least two 2-planes in $\mathbf{R}^4$ which are complex in a complex structure on $\mathbf{R}^4$.

1. Introduction

Let M be a compact Calabi-Yau manifold of complex dimension n with a Kähler form ω, a complex structure J, a Kähler metric g and a parallel holomorphic $(n, 0)$-form Ω of unit length. An immersed submanifold Σ in M is Lagrangian if $\omega|_\Sigma = 0$. The induced volume form $d\mu_\Sigma$ on a Lagrangian submanifold Σ from the Ricci-flat metric g is related to Ω by

$$\Omega|_\Sigma = e^{i\theta} d\mu_\Sigma = \cos\theta d\mu_\Sigma + i \sin\theta d\mu_\Sigma, \tag{1}$$

where the phase function θ is multi-valued and is well-defined up to an additive constant $2k\pi$, $k \in \mathbf{Z}$. Nevertheless, $\cos\theta$ and $\sin\theta$ are single valued

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functions on Σ. For any tangent vector X to M a straightforward calculation shows

$$X\theta = -g(\mathbf{H}, JX) \tag{2}$$

where $\mathbf{H}$ is the mean curvature vector of Σ in M (cf. [HL], [TY]). Equivalently, $\mathbf{H} = J\nabla\theta$. The Lagrangian submanifold Σ is *special*, i.e. it is a minimal submanifold, if and only if θ is constant. When θ is constant on a Lagrangian submanifold Σ, the real part of $e^{-i\theta}\Omega$ is a calibration of M with comass one and Σ is a volume minimizer in its homology class [HL]. Let $\mathrm{R}e\Omega$ be the real part of Ω. A Lagrangian submanifold is called *almost calibrated* by $\mathrm{R}e\Omega$ if $\cos\theta > 0$.

Constructing minimal Lagrangian submanifolds is an important but very challenging task. In a compact Kähler-Einstein surface, Schoen and Wolfson [ScW] have shown the existence of a branched surface which minimizes area among Lagrangian competitors in each Lagrangian homology class, by variational method.

For a one-parameter family of immersions $F_t = F(\cdot, t) : \Sigma \to M$, we denote the image submanifolds by $\Sigma_t = F_t(\Sigma)$. If Σ_t evolves along the gradient flow of the volume functional, the first variation of the volume functional asserts that Σ_t satisfy a mean curvature flow equation:

$$\begin{cases} \dfrac{d}{dt}F(x,t) = \mathbf{H}(x,t) \\ F(x,0) = F_0(x). \end{cases} \tag{3}$$

When Σ is compact the mean curvature flow (3) has a smooth solution for short time $[0, T)$ by the standard parabolic theory. If Σ_0 is Lagrangian in a Kähler-Einstein ambient space M, Smoczyk has shown that Σ_t remains Lagrangian for $t < T$ and the phase function θ evolves by

$$\frac{d\theta}{dt} = \Delta\theta \tag{4}$$

where Δ is the Laplacian of the induced metric on Σ_t ([Sm1-3], also see [TY] for a derivation of (4)). It then follows that

$$\frac{\partial \cos\theta}{\partial t} = \Delta\cos\theta + |\mathbf{H}|^2\cos\theta. \tag{5}$$

If the initial Lagrangian submanifold Σ_0 is almost calibrated, Σ_t is almost calibrated, i.e. $\cos\theta > 0$, along a smooth mean curvature flow by the parabolic maximum principle.

It is well-known that if $|\mathbf{A}|^2$, where $\mathbf{A}$ is the second fundamental form on Σ_t, is bounded uniformly as $t \to T > 0$ then (3) admits a smooth solution over $[0, T+\epsilon)$ for some $\epsilon > 0$. When $\max_{\Sigma_t}|\mathbf{A}|^2$ becomes unbounded as $t \to T$, we say that the mean curvature flow develops a singularity at T. A lot of work has been devoted to understand these singularities (cf. [CL1-2], [E1-2], [H1-3], [HS1-2], [I1], [Wa], [Wh1-3].)

In this paper, we shall study the tangent cones at singularities of the mean curvature flow of a compact Lagrangian submanifold in a compact Calabi-Yau manifold. Especially, we shall focus on the structure of tangent cones of the mean curvature flow where a singularity occurs at the first singular time $T < \infty$.

To describe the tangent cones, suppose that (X_0, T) is a singular point of the flow (3), i.e. $|\mathbf{A}(x, t)|$ becomes unbounded when $(x, t) \to (X_0, T)$. For an arbitrary sequence of numbers $\lambda \to \infty$ and any $t < 0$, if $T+\lambda^{-2}t > 0$ we set

$$F_\lambda(x, t) = \lambda\big(F(x, T + \lambda^{-2}t) - X_0\big).$$

We denote the scaled submanifold by $\big(\Sigma_t^\lambda, d\mu_t^\lambda\big)$. If the initial submanifold is Lagrangian and almost calibrated by $\mathrm{Re}\,\Omega$, it is proved in Proposition 2.3 that there is a subsequence $\lambda_i \to \infty$ such that for any $t < 0$, $\big(\Sigma_t^{\lambda_i}, d\mu_t^{\lambda_i}\big)$ converges to $(\Sigma^\infty, d\mu^\infty)$ in the sense of measures; the limit Σ^∞ is called a *tangent cone arising from the rescaling* λ, or simply a λ *tangent cone at* (X_0, T). This tangent cone is independent of t as shown in Proposition 2.3.

There is also a time dependent scaling which we would like to consider

$$\widetilde{F}(\cdot, s) = \frac{1}{\sqrt{2(T-t)}} F(\cdot, t), \tag{6}$$

where $s = -\frac{1}{2}\log(T-t)$, $c_0 \le s < \infty$. Here we have chosen the coordinates so that $X_0 = 0$. Rescaling of this type arises naturally in classification of singularities of mean curvature flows [H2]: assume $\lim_{t\to T^-}\max_{\Sigma_t}|\mathbf{A}|^2 = \infty$, if there exists a positive constant C such that $\limsup_{t\to T^-}\big((T-t)\max_{\Sigma_t}|\mathbf{A}|^2\big) \le C$, the mean curvature flow F has a Type I singularity at T; otherwise it has a Type II singularity at T. Denote $\widetilde{\Sigma}_s$ the rescaled submanifold by $\widetilde{F}(\cdot, s)$. If a subsequence of $\widetilde{\Sigma}_s$ converges in measures to a limit $\widetilde{\Sigma}_\infty$, then the limit is called a *tangent cone arising from the time dependent scaling at* (X_0, T), or simply a *t tangent cone*. In this paper, a *tangent cone* of the mean curvature flow at (X_0, T) means either a λ tangent cone or a t tangent cone at (X_0, T).

The main result of this paper is

Theorem 1.1. *Let (M, Ω) be a compact Calabi-Yau manifold of complex dimension n. If the initial compact submanifold Σ_0 is Lagrangian and almost calibrated by $\mathrm{Re}\,\Omega$, and $T > 0$ is the first blow-up time of the mean curvature flow (3), and (X_0, T) is a singular point, then the tangent cone of the mean curvature flow at (X_0, T) is a stationary Lagrangian integer multiplicity current in $\mathbf{R}^{2n}$ with volume density greater than one at X_0. When $n = 2$, the tangent cone is a finite union of at least two 2-planes in $\mathbf{R}^4$ which are complex in a complex structure on $\mathbf{R}^4$.*

For symplectic mean curvature flow in Kähler-Einstein surfaces, results similar to Theorem 1.1 were obtained in [CL1]. The authors are grateful to Professor Gang Tian for stimulating conversation. The authors thank the referee for useful comments.

2. Existence of λ tangent cones

This section contains basic formulas and estimates which are essential for this article. First, we will derive a monotonicity formula which has a weight function introduced by the n-form $\operatorname{Re}\Omega$. Second, we use the monotonicity formula to derive three integral estimates, which roughly say that when averaged over any time interval the mean curvature vector $\mathbf{H}_\lambda$ and the derivative of the phase function $\cos\theta_\lambda$ both tend to 0 in the L^2 norm over a fixed ball near the singularity, as $\lambda \to \infty$. Another direct consequence of the monotonicity formula is that there is an upper bound of the volume density of the rescaled submanifolds Σ_t^λ, which allows us to extract converging subsequence in measure.

2.1. A weighted monotonicity formula

Let $H(\mathbf{X}, \mathbf{X}_0, t_0, t)$ be the backward heat kernel on $\mathbf{R}^k$. Let N_t be a smooth family of submanifolds of dimension n in $\mathbf{R}^k$ defined by $F_t : N \to \mathbf{R}^k$. Define

$$\begin{aligned}\rho(\mathbf{X}, t) &= (4\pi(t_0 - t))^{(k-n)/2} H(\mathbf{X}, \mathbf{X}_0, t_0, t)\\ &= \frac{1}{(4\pi(t_0 - t))^{n/2}} \exp\left(-\frac{|\mathbf{X} - \mathbf{X}_0|^2}{4(t_0 - t)}\right)\end{aligned} \tag{7}$$

for $t < t_0$.

A straightforward calculation (cf. [CL1], [H1], [Wa]) shows

$$\frac{\partial}{\partial t}\rho = \left(\frac{n}{2(t_0 - t)} - \frac{\mathbf{H} \cdot (\mathbf{X} - \mathbf{X}_0)}{2(t_0 - t)} - \frac{|\mathbf{X} - \mathbf{X}_0|^2}{4(t_0 - t)^2}\right)\rho$$

and along N_t

$$\Delta\rho = \left(\frac{\langle \mathbf{X} - \mathbf{X}_0, \nabla\mathbf{X}\rangle^2}{4(t_0 - t)^2} - \frac{\langle \mathbf{X} - \mathbf{X}_0, \Delta\mathbf{X}\rangle}{2(t_0 - t)} - \frac{|\nabla\mathbf{X}|^2}{2(t_0 - t)}\right)\rho$$

where Δ, ∇ are on N_t in the induced metric. Let $N_t = \Sigma_t$ be a smooth 1-parameter family of compact Lagrangian submanifolds in a compact Calabi-Yau manifold (M, Ω) of complex dimension n. Note that in the induced metric on Σ_t

$$|\nabla F|^2 = n \quad \text{and} \quad \Delta F = \mathbf{H}.$$

Therefore

$$\left(\frac{\partial}{\partial t} + \Delta\right)\rho = -\left(\left|\mathbf{H} + \frac{(F - \mathbf{X}_0)^\perp}{2(t_0 - t)}\right|^2 - |\mathbf{H}|^2\right)\rho. \tag{8}$$

On Σ_t we set

$$v = \cos\theta.$$

Denote the injectivity radius of (M, g) by i_M. For $\mathbf{X}_0 \in M$, take a normal coordinate neighborhood U and let $\phi \in C_0^\infty(B_{2r}(\mathbf{X}_0))$ be a cut-off function with $\phi \equiv 1$ in $B_r(\mathbf{X}_0)$, $0 < 2r < i_M$. Using the local coordinates in U we may regard $F(x, t)$ as a point in $\mathbf{R}^{2n}$ whenever $F(x, t)$ lies in U. We define

$$\Psi(\mathbf{X}_0, t_0, t) = \int_{\Sigma_t} \frac{1}{v}\phi(F)\rho(F, \mathbf{X}_0, t, t_0)d\mu_t$$

where ρ is defined by (7) by taking $k = 2n$.

Proposition 2.1. *Let $F_t : \Sigma \to M$ be a smooth mean curvature flow of a compact Lagrangian submanifold Σ_0 in a compact Calabi-Yau manifold M of complex dimension n. Suppose that Σ_0 is almost calibrated by $Re\Omega$. Then there are positive constants c_1 and c_2 depending only on M, F_0 and r which is the constant in the definition of ϕ, such that*

$$\begin{aligned}\frac{\partial}{\partial t}&\left(e^{c_1\sqrt{t_0-t}}\int_{\Sigma_t}\frac{1}{v}\phi\rho d\mu_t\right)\\ &\leq -e^{c_1\sqrt{t_0-t}}\int_{\Sigma_t}\frac{1}{v}\phi\rho\left(\frac{2|\nabla v|^2}{v^2}+\left|\mathbf{H}+\frac{(F-\mathbf{X}_0)^\perp}{2(t_0-t)}\right|^2+\frac{|\mathbf{H}|^2}{2}\right)d\mu_t \quad (9)\\ &\quad + c_2e^{c_1\sqrt{t_0-t}}.\end{aligned}$$

Proof. Notice that

$$\Delta F = \mathbf{H} + g^{ij}\Gamma^\alpha_{ij}v_\alpha$$

where v_α, $\alpha = 1, ..., n$ is a basis of $T^\perp\Sigma_t$, g^{ij} is the induced metric on Σ_t and Γ^α_{ij} is the Christoffel symbol on M. Equation (8) reads as

$$\left(\frac{\partial}{\partial t}+\Delta\right)\rho = -\left(\left|\mathbf{H}+\frac{(F-\mathbf{X}_0)^\perp}{2(t_0-t)}\right|^2 - |\mathbf{H}|^2 + \frac{g^{ij}\Gamma^\alpha_{ij}v_\alpha\cdot(F-\mathbf{X}_0)}{t_0-t}\right)\rho. \quad (10)$$

From (5) we have

$$\frac{\partial}{\partial t}\frac{1}{v} = \Delta\frac{1}{v} - \frac{|\mathbf{H}|^2}{v} - \frac{2|\nabla v|^2}{v^3}.$$

Using the equation above and the generalized monotonicity formula in [EH], we can derive our weighted monotonicity formula (9). For completeness we give a detailed proof here, due to higher codimension and non-Euclidean ambient space.

Recall that

$$\frac{d}{dt}d\mu_t = -|\mathbf{H}|^2 d\mu_t$$

and

$$\frac{\partial \phi(F)}{\partial t} = \nabla\phi \cdot \mathbf{H}.$$

Now we have

$$\begin{aligned}
&\frac{d}{dt}\int_{\Sigma_t}\frac{1}{v}\phi\rho \\
&= \int_{\Sigma_t}\phi\rho\Delta\frac{1}{v} - \int_{\Sigma_t}\left(\frac{|\mathbf{H}|^2}{v} + \frac{2}{v^3}|\nabla v|^2\right)\phi\rho + \int_{\Sigma_t}\frac{1}{v}\nabla\phi\cdot\mathbf{H}\rho \\
&\quad - \int_{\Sigma_t}\frac{1}{v}\phi\left(\Delta\rho + \left(\left|\mathbf{H} + \frac{(F-\mathbf{X}_0)^\perp}{2(t_0-t)}\right|^2 - |\mathbf{H}|^2 + \frac{g^{ij}\Gamma_{ij}^\alpha v_\alpha\cdot(F-\mathbf{X}_0)}{t_0-t}\right)\rho\right) \\
&\quad - \int_{\Sigma_t}\frac{1}{v}\phi\rho|\mathbf{H}|^2 \\
&\le -\int_{\Sigma_t}\phi\rho\left(\frac{2}{v^3}|\nabla v|^2 + \frac{1}{v}\left|\mathbf{H} + \frac{(F-\mathbf{X}_0)^\perp}{2(t_0-t)}\right|^2 + \frac{|\mathbf{H}|^2}{v}\right) \\
&\quad + \int_{\Sigma_t}\left(\phi\rho\Delta\frac{1}{v} - \frac{1}{v}\phi\Delta\rho\right) - \int_{\Sigma_t}\frac{1}{v}\phi\rho\frac{g^{ij}\Gamma_{ij}^\alpha v_\alpha\cdot(F-\mathbf{X}_0)}{t_0-t} \\
&\quad + \int_{\Sigma_t}\frac{1}{v}\rho\left(\epsilon^2\phi|\mathbf{H}|^2 + \frac{1}{4\epsilon^2}\frac{|\nabla\phi|^2}{\phi}\right) \qquad (11)
\end{aligned}$$

where we used Cauchy-Schwartz inequality for $\nabla\phi\cdot\mathbf{H}$. By Stokes formula

$$\int_{\Sigma_t}\left(\phi\rho\Delta\frac{1}{v} - \frac{1}{v}\phi\Delta\rho\right) = 2\int_{\Sigma_t}\frac{1}{v}\nabla\phi\nabla\rho + \int_{\Sigma_t}\frac{1}{v}\rho\Delta\phi.$$

Since $\phi\in C_0^\infty(B_{2r}(\mathbf{X}_0), \mathbf{R}^+)$, we have (cf. [B] and Lemma 6.6 in [Il])

$$\frac{|\nabla\phi|^2}{\phi} \le 2\max_{\phi>0}|\nabla^2\phi|.$$

Note that $\nabla\phi\equiv 0$ in $B_r(\mathbf{X}_0)$, so $|\rho\Delta\phi|$ and $|\nabla\phi\cdot\nabla\rho|$ are bounded in $B_{2r}(\mathbf{X}_0)$. Hence

$$\int_{\Sigma_t}\left|\frac{1}{v}\rho\Delta\phi\right| + \int_{\Sigma_t}\left|\frac{1}{v}\nabla\phi\cdot\nabla\rho\right| \le C\int_{\Sigma_t}\frac{1}{v}d\mu_t \le \frac{C}{\min_{\Sigma_0}v}\mathrm{vol}(\Sigma_0) \qquad (12)$$

where C depends only on r and $\max(|\nabla^2\phi| + |\nabla\phi|)$.

Since $\Gamma_{ij}^\alpha(\mathbf{X}_0) = 0$, we may choose r sufficiently small such that

$$\left|g^{ij}\Gamma_{ij}^\alpha(F)\right| \le C|F-\mathbf{X}_0|$$

in $B_{2r}(\mathbf{X}_0)$ for some constant C depending on M. We claim

$$\frac{\left|g^{ij}\Gamma_{ij}^\alpha v_\alpha\cdot(F-\mathbf{X}_0)\right|}{t_0-t}\rho(F,t) \le c_1\frac{\rho(F,t)}{\sqrt{t_0-t}} + C. \qquad (13)$$

In fact it suffices to show for any x and $s > 0$

$$\frac{x^2}{s}\frac{e^{-x^2/s}}{s^{n/2}} \leq C\left(1 + \frac{1}{s^{1/2}}\frac{e^{-x^2/s}}{s^{n/2}}\right).$$

To see this, let $y = x^2/s$ and then it is easy to verify that

$$y \leq C\left(s^{n/2}e^y + \frac{1}{s^{1/2}}\right)$$

holds trivially if $y \leq 1/s^{1/2}$ and follows from $y^{n+1} \leq Ce^y$ if $y > 1/s^{1/2}$ for some C. So (13) is established.

Letting $\epsilon^2 = 1/2$ in (11) and applying (12), (13) to (11) we have

$$\frac{\partial}{\partial t}\Psi \leq -\int_{\Sigma_t}\frac{1}{v}\phi\rho\left(\frac{2|\nabla v|^2}{v^2} + \left|\mathbf{H} + \frac{(F-\mathbf{X}_0)^\perp}{2(t_0-t)}\right|^2 + \frac{|\mathbf{H}|^2}{2}\right) + \frac{c_1}{\sqrt{t_0-t}}\Psi + c_2.$$

The proposition follows. □

Suppose that (X_0, T) is a singular point of the mean curvature flow (3). We now describe the rescaling process around (X_0, T). For any $t < 0$, we set

$$F_\lambda(x,t) = \lambda\big(F(x, T + \lambda^{-2}t) - X_0\big) \tag{14}$$

where λ are positive constants which go to infinity. The scaled submanifold is denoted by $\Sigma_t^\lambda = F_\lambda(\Sigma, t)$ on which $d\mu_t^\lambda$ is the area element obtained from $d\mu_t$. If g^λ is the metric on Σ_t^λ, it is clear that

$$g_{ij}^\lambda = \lambda^2 g_{ij}, \quad (g^\lambda)^{ij} = \lambda^{-2}g^{ij}.$$

We therefore have

$$\frac{\partial F_\lambda}{\partial t} = \lambda^{-1}\frac{\partial F}{\partial t}$$
$$\mathbf{H}_\lambda = \lambda^{-1}\mathbf{H}$$
$$|\mathbf{A}_\lambda|^2 = \lambda^{-2}|\mathbf{A}|^2.$$

It follows that the scaled submanifold also evolves by a mean curvature flow

$$\frac{\partial F_\lambda}{\partial t} = \mathbf{H}_\lambda. \tag{15}$$

Moreover, since

$$d\mu_t^\lambda(F_\lambda(x,t)) = \lambda^n d\mu_t(F(x, T + \lambda^{-2}t))$$
$$\Omega|_{\Sigma_t^\lambda}(F_\lambda(x,t)) = \lambda^n \Omega|_{\Sigma_t}(F(x, T + \lambda^{-2}t))$$

we have

$$\cos\theta_\lambda(F_\lambda(x,t)) = \cos\theta(F(x, T + \lambda^{-2}t)).$$

2.2. Integral estimates

Proposition 2.2. *Let (M, Ω) be a Calabi-Yau manifold of complex dimension n. If the initial compact submanifold is Lagrangian and is almost calibrated by ReΩ, then for any $R > 0$ and any $-\infty < s_1 < s_2 < 0$, we have*

$$\int_{s_1}^{s_2} \int_{\Sigma_t^\lambda \cap B_R(0)} |\nabla \cos\theta_\lambda|^2 d\mu_t^\lambda dt \to 0 \text{ as } \lambda \to \infty, \tag{16}$$

$$\int_{s_1}^{s_2} \int_{\Sigma_t^\lambda \cap B_R(0)} |\mathbf{H}_\lambda|^2 d\mu_t^\lambda dt \to 0 \text{ as } \lambda \to \infty, \tag{17}$$

and

$$\int_{s_1}^{s_2} \int_{\Sigma_t^\lambda \cap B_R(0)} \left|F_\lambda^\perp\right|^2 d\mu_t^\lambda dt \to 0 \text{ as } \lambda \to \infty. \tag{18}$$

Proof. For any $R > 0$, we choose a cut-off function $\phi_R \in C_0^\infty(B_{2R}(0))$ with $\phi_R \equiv 1$ in $B_R(0)$, where $B_r(0)$ is the metric ball centered at 0 with radius r in $\mathbf{R}^{2n}$. For any fixed $t < 0$, the mean curvature flow (3) has a smooth solution near $T + \lambda^{-2}t < T$ for sufficiently large λ, since $T > 0$ is the first blow-up time of the flow. Let $v_\lambda = \cos\theta_\lambda$. It is clear

$$\begin{aligned}
&\int_{\Sigma_t^\lambda} \frac{1}{v_\lambda} \frac{1}{(0-t)^{n/2}} \phi_R(F_\lambda) \exp\left(-\frac{|F_\lambda|^2}{4(0-t)}\right) d\mu_t^\lambda \\
&= \int_{\Sigma_{T+\lambda^{-2}t}} \frac{1}{v_\lambda} \phi(F_\lambda) \frac{1}{(T-(T+\lambda^{-2}t))^{n/2}} \exp\left(-\frac{|F(x, T+\lambda^{-2}t) - X_0|^2}{4(T-(T+\lambda^{-2}t))}\right) d\mu_t,
\end{aligned}$$

where ϕ is the function defined in the definition of Φ. Note that $T+\lambda^{-2}t \to T$ for any fixed t as $\lambda \to \infty$. By the weighted monotonicity formula (9),

$$\frac{\partial}{\partial t}\left(e^{c_1\sqrt{t_0-t}}\Psi\right) \le c_2 e^{c_1\sqrt{t_0-t}},$$

and it then follows that $\lim_{t\to t_0} e^{c_1\sqrt{t_0-t}}\Psi$ exists. This implies, by taking $t_0 = T$ and $t = T + \lambda^{-2}s$, that for any fixed s_1 and s_2 with $-\infty < s_1 < s_2 < 0$,

$$\begin{aligned}
&e^{c_1\sqrt{T-(T+\lambda^{-2}s_2)}} \int_{\Sigma_{s_2}^\lambda} \frac{1}{v_\lambda} \phi_R \frac{1}{(0-s_2)^{n/2}} \exp\left(-\frac{|F_\lambda|^2}{4(0-s_2)}\right) d\mu_{s_2}^\lambda \\
&-e^{c_1\sqrt{T-(T+\lambda^{-2}s_1)}} \int_{\Sigma_{s_1}^\lambda} \frac{1}{v_\lambda} \phi_R \frac{1}{(0-s_1)^{n/2}} \exp\left(-\frac{|F_\lambda|^2}{4(0-s_1)}\right) d\mu_{s_1}^\lambda \\
&\to 0 \text{ as } \lambda \to \infty.
\end{aligned} \tag{19}$$

Integrating (9) from $T+\lambda^{-2}s_1$ to $T+\lambda^{-2}s_2$, and letting $T+\lambda^{-2}s = t$, we get

$$\begin{aligned}
&-e^{c_1\sqrt{-\lambda^{-2}s_2}}\int_{\Sigma^\lambda_{s_2}}\frac{1}{v_\lambda}\phi_R\frac{1}{(0-s_2)^{n/2}}\exp\left(-\frac{|F_\lambda|^2}{4(0-s_2)}\right)d\mu^\lambda_{s_2}\\
&\quad+e^{c_1\sqrt{-\lambda^{-2}s_1}}\int_{\Sigma^\lambda_{s_1}}\frac{1}{v_\lambda}\phi_R\frac{1}{(0-s_1)^{n/2}}\exp\left(-\frac{|F_\lambda|^2}{4(0-s_1)}\right)d\mu^\lambda_{s_1}\\
&\geq\int_{T+\lambda^{-2}s_1}^{T+\lambda^{-2}s_2}e^{c_1\sqrt{T-t}}\int_{\Sigma_t}\frac{1}{v}\phi\rho\left(\frac{2|\nabla v|^2}{v^2}+\left|\mathbf{H}+\frac{(F-\mathbf{X}_0)^\perp}{2(T-t)}\right|^2+\frac{|\mathbf{H}|^2}{2}\right)d\mu_t\\
&\quad-c_2\lambda^{-2}(s_2-s_1)\\
&\geq\int_{s_1}^{s_2}e^{c_1\sqrt{-\lambda^{-2}s}}\int_{\Sigma^\lambda_s}\frac{1}{v_\lambda}\phi_R\rho(F_\lambda,s)\left|\mathbf{H}_\lambda+\frac{(F_\lambda)^\perp}{2(-s)}\right|^2d\mu^\lambda_s\\
&\quad+\int_{s_1}^{s_2}e^{c_1\sqrt{-\lambda^{-2}s}}\int_{\Sigma^\lambda_s}\frac{1}{v_\lambda}\phi_R\rho(F_\lambda,s)\frac{|\mathbf{H}_\lambda|^2}{2}d\mu^\lambda_s\\
&\quad+\int_{s_1}^{s_2}e^{c_1\sqrt{-\lambda^{-2}s}}\int_{\Sigma^\lambda_s}\frac{2}{v^3_\lambda}|\nabla v_\lambda|^2\phi_R\rho(F_\lambda,s)d\mu^\lambda_s-c_2\lambda^{-2}(s_2-s_1).
\end{aligned}\tag{20}$$

From (19) and (20) the proposition follows. □

2.3. *Upper bound on volume density*

Now we show the existence of the λ tangent cones by deriving an finite upper bound for the volume density. These cones are independent of t, but may depend on the blowing up sequence λ. Some of the arguments below were used in [I1] and [E1], and an analogue of the area estimate (21) for hypersurfaces was obtained in [E1].

Proposition 2.3. *Suppose that Σ_t evolves along mean curvature flow and Σ_0 is a compact Lagrangian submanifold in (M,Ω) and is almost calibrated by $Re\,\Omega$. For any λ, $R>0$ and any $t<0$,*

$$\mu^\lambda_t\left(\Sigma^\lambda_t\cap B_R(0)\right)\leq CR^n,\tag{21}$$

where $B_R(0)$ is a metric ball in $\mathbf{R}^{2n}$ and $C>0$ is independent of λ. For any sequence $\lambda_i\to\infty$, there is a subsequence $\lambda_k\to\infty$ such that $\left(\Sigma^{\lambda_k}_t,\mu^{\lambda_k}_t\right)\to(\Sigma^\infty,\mu^\infty)$ in the sense of measures, for any fixed $t<0$, where $(\Sigma^\infty,\mu^\infty)$ is independent of t. The multiplicity of Σ^∞ is finite.

Proof. We shall first prove the inequality (21). We shall use C below for uniform positive constants which are independent of R and λ. Straightforward computation shows

$$\begin{aligned}\mu_t^\lambda\big(\Sigma_t^\lambda \cap B_R(0)\big) &= \lambda^n \int_{\Sigma_{T+\lambda^{-2}t}\cap B_{\lambda^{-1}R}(X_0)} d\mu_t \\ &= R^n(\lambda^{-1}R)^{-n}\int_{\Sigma_{T+\lambda^{-2}t}\cap B_{\lambda^{-1}R}(X_0)} d\mu_t \\ &\le CR^n \int_{\Sigma_{T+\lambda^{-2}t}\cap B_{\lambda^{-1}R}(X_0)} \frac{1}{v_\lambda}\frac{1}{(4\pi)^{n/2}(\lambda^{-1}R)^n} e^{-\frac{|X-X_0|^2}{4(\lambda^{-1}R)^2}} d\mu_t \\ &= CR^n\Psi\big(X_0, T+(\lambda^{-1}R)^2+\lambda^{-2}t, T+\lambda^{-2}t\big).\end{aligned}$$

By the weighted monotonicity inequality (9), we have

$$\begin{aligned}\mu_t^\lambda\big(\Sigma_t^\lambda \cap B_R(0)\big) &\le CR^n\Psi\big(X_0, T+(\lambda^{-1}R)^2+\lambda^{-2}t, T/2\big)+CR^n \\ &\le \frac{\mu_{T/2}(\Sigma_{T/2})}{T^{n/2}\min_{\Sigma_0} v} CR^n + CR^n.\end{aligned}$$

Since volume is non-increasing along mean curvature flow:

$$\frac{\partial}{\partial t}\mu_t(\Sigma_t) = -\int_{\Sigma_t} |\mathbf{H}|^2 d\mu_t,$$

we have therefore established (21):

$$\mu_t^\lambda\big(\Sigma_t^\lambda \cap B_R(0)\big) \le CR^n.$$

By (21), the compactness theorem for the measures (c.f. [Si1], 4.4) and a diagonal subsequence argument, we conclude that there is a subsequence $\lambda_k \to \infty$ such that $\big(\Sigma_{t_0}^{\lambda_k}, \mu_{t_0}^{\lambda_k}\big) \to \big(\Sigma_{t_0}^\infty, \mu_{t_0}^\infty\big)$ in the sense of measures for a fixed $t_0 < 0$.

We now show that, for any $t < 0$, the subsequence λ_k which we have chosen above satisfies $\big(\Sigma_t^{\lambda_k}, \mu_t^{\lambda_k}\big) \to \big(\Sigma_{t_0}^\infty, \mu_{t_0}^\infty\big)$ in the sense of measures. And consequently the limit $\big(\Sigma_{t_0}^\infty, \mu_{t_0}^\infty\big)$ is independent of t_0, here $\Sigma_{t_0}^\infty$ is the support of the limiting Radon measure. Recall that the following standard formula for mean curvature flow

$$\frac{d}{dt}\int_{\Sigma_t^\lambda} \phi d\mu_t^\lambda = -\int_{\Sigma_t^\lambda}\big(\phi|\mathbf{H}_\lambda|^2 + \nabla\phi\cdot\mathbf{H}_\lambda\big) d\mu_t^\lambda \tag{22}$$

is valid for any test function $\phi \in C_0^\infty(M)$ (cf. (1) in Sect. 6 in [I2] and [B] in the varifold setting).

Then for any given $t < 0$ integrating (22) yields

$$\begin{aligned}\int_{\Sigma_t^{\lambda_k}} \phi d\mu_t^{\lambda_k} - \int_{\Sigma_{t_0}^{\lambda_k}} \phi d\mu_{t_0}^{\lambda_k} &= \int_t^{t_0}\int_{\Sigma_t^{\lambda_k}} \big(\phi|\mathbf{H}_{\lambda_k}|^2 + \nabla\phi\cdot\mathbf{H}_{\lambda_k}\big) d\mu_t^{\lambda_k} dt \\ &\to 0 \text{ as } k\to\infty \text{ by } (17).\end{aligned}$$

So, for any fixed $t < 0$, $\left(\Sigma_t^{\lambda_k}, \mu_t^{\lambda_k}\right) \to \left(\Sigma_{t_0}^{\infty}, \mu_{t_0}^{\infty}\right)$ in the sense of measures as $k \to \infty$. We denote $\left(\Sigma_{t_0}^{\infty}, \mu_{t_0}^{\infty}\right)$ by $(\Sigma^{\infty}, \mu^{\infty})$, which is independent of t_0.

The inequality (21) yields a uniform upper bound on $R^{-n}\mu_t^{\lambda_k}(\Sigma_t^{\lambda_k} \cap B_R(0))$, which yields finiteness of the multiplicity of Σ^{∞}. □

Definition 2.4. Let (X_0, T) be a first time singular point of the mean curvature flow of a compact Lagrangian submanifold Σ_0 in a compact Calabi-Yau manifold M. We call $(\Sigma^{\infty}, d\mu^{\infty})$ obtained in Proposition 2.3 *a* λ *tangent cone of the mean curvature flow* Σ_t *at* (X_0, T).

3. Rectifiability of λ tangent cones

In this section we shall show that the λ tangent cone Σ^{∞} is $\mathcal{H}^n$-rectifiable, where $\mathcal{H}^n$ is the n-dimensional Hausdorff measure.

Proposition 3.1. *Let M be a compact Calabi-Yau manifold of complex dimension n. If the initial compact submanifold* Σ_0 *is Lagrangian and almost calibrated by Re* Ω*, then the* λ *tangent cone* $(\Sigma^{\infty}, d\mu^{\infty})$ *of the mean curvature flow at* (X_0, T) *is* $\mathcal{H}^n$*-rectifiable.*

Proof. Let $\left(\Sigma_t^k, d\mu_t^k\right) = \left(\Sigma_t^{\lambda_k}, d\mu_t^{\lambda_k}\right)$. We set

$$A_R = \left\{ t \in (-\infty, 0) \,\middle|\, \lim_{k\to\infty} \int_{\Sigma_t^k \cap B_R(0)} |\mathbf{H}_k|^2 d\mu_t^k \neq 0 \right\},$$

and

$$A = \bigcup_{R>0} A_R.$$

Denote the measures of A_R and A by $|A_R|$ and $|A|$ respectively. It is clear from (17) that $|A_R| = 0$ for any $R > 0$. So $|A| = 0$.

For any $\xi \in \Sigma^{\infty}$, choose $\xi_k \in \Sigma_t^k$ with $\xi_k \to \xi$ as $k \to \infty$. By the monotonicity identity (17.4) in [Si1], we have

$$\sigma^{-n}\mu_t^k(B_\sigma(\xi_k)) = \rho^{-n}\mu_t^k(B_\rho(\xi_k)) - \int_{B_\rho(\xi_k)\setminus B_\sigma(\xi_k)} \frac{|D^{\perp}r|^2}{r^n} d\mu_t^k$$
$$-\frac{1}{n}\int_{B_\rho(\xi_k)} (x - \xi_k)\cdot \mathbf{H}_k \left(\frac{1}{r_\sigma^n} - \frac{1}{\rho^n}\right) d\mu_t^k, \qquad (23)$$

for all $0 < \sigma \leq \rho$, where $\mu_t^k(B_\sigma(\xi_k))$ is the measure of $\Sigma_t^k \cap B_\sigma((\xi_k))$, $r = r(x)$ is the distance from ξ_k to x, $r_\sigma = \max\{r, \sigma\}$, and $D^{\perp}r$ denotes the orthogonal projection of Dr (which is a vector of length 1) onto $\left(T_{\xi_k}\Sigma_t^k\right)^{\perp}$. Choosing $t \notin A$, we have

$$\lim_{k\to\infty} \int_{B_\rho(\xi_k)} |\mathbf{H}_k|^2 d\mu_t^k = 0.$$

Hölder's inequality and (21) then lead to

$$\begin{aligned}
&\lim_{k\to\infty}\left|\int_{B_\rho(\xi_k)}(x-\xi_k)\cdot\mathbf{H}_k\left(\frac{1}{r_\sigma^n}-\frac{1}{\rho^n}\right)d\mu_t^k\right| \\
&\quad\le C\rho\left(\frac{1}{\sigma^n}-\frac{1}{\rho^n}\right)\lim_{k\to\infty}\left(\sqrt{\mu_t^k(B_\rho(\xi_k))}\sqrt{\int_{B_\rho(\xi_k)}|\mathbf{H}_k|^2 d\mu_t^k}\right) \\
&\quad\le C\rho^{1+n/2}\left(\frac{1}{\sigma^n}-\frac{1}{\rho^n}\right)\lim_{k\to\infty}\sqrt{\int_{B_\rho(\xi_k)}|\mathbf{H}_k|^2 d\mu_t^k} \\
&\quad= 0.
\end{aligned} \tag{24}$$

Letting $k \to \infty$ in (23) and using (24), we obtain

$$\sigma^{-n}\mu^\infty(B_\sigma(\xi)) \le \rho^{-n}\mu^\infty(B_\rho(\xi)),$$

for all $0 < \sigma \le \rho$. By (21) we know that

$$\lim_{\rho\to 0}\rho^{-n}\mu^\infty(B_\rho(\xi)) < C < \infty.$$

Therefore, $\lim_{\rho\to 0}\rho^{-n}\mu^\infty(B_\rho(\xi))$ exists.

We shall show that the following density estimate holds

$$\lim_{\rho\to 0}\rho^{-n}\mu^\infty(B_\rho(\xi)) \ge \frac{1}{4c(n)+4} > 0 \tag{25}$$

for some positive constant $c(n)$ which will be determined below. Assume (25) fails to hold. Then there is $\rho_0 > 0$ such that

$$(2\rho_0)^{-n}\mu^\infty(B_{2\rho_0}(\xi)) < \frac{1}{4c(n)+4}.$$

By the monotonicity formula (23) and that μ_t^k converges to μ^∞ as measures, there exists $k_0 > 0$ such that, for all $0 < \rho < 2\rho_0$ and $k > k_0$, we have

$$\rho^{-n}\mu_t^k(B_\rho(\xi)) < \frac{1}{2c(n)+2}. \tag{26}$$

Take a cut-off function $\phi_\rho \in C_0^\infty(B_\rho(\xi))$ on the $2n$-dimensional ball $B_\rho(\xi_k)$ so that

$$\begin{aligned}
&\phi_\rho \equiv 1 \ \text{ in } \ B_{\frac{\rho}{2}}(\xi) \\
&0 \le \phi_\rho \le 1, \ \text{ and } \ |\nabla\phi_\rho| \le \frac{C}{\rho}, \ \text{ in } B_\rho(\xi).
\end{aligned}$$

From (22), we have

$$
\begin{aligned}
&\rho^{-n}\int_{B_\rho(\xi)}\phi_\rho d\mu^k_{t-r^2}-\rho^{-n}\int_{B_\rho(\xi)}\phi_\rho d\mu^k_t\\
&\le C\rho^{-n}\int_{t-r^2}^t\int_{B_\rho(\xi)}|\mathbf{H}_k|^2 d\mu^k_s ds+C\rho^{-n-1}\int_{t-r^2}^t\int_{B_\rho(\xi)}|\mathbf{H}_k| d\mu^k_s ds\\
&\le C\rho^{-n}\int_{t-r^2}^t\int_{B_\rho(\xi)}|\mathbf{H}_k|^2 d\mu^k_s ds\\
&\quad +C\rho^{-n-1}\int_{t-r^2}^t\left(\int_{B_\rho(\xi)}|\mathbf{H}_k|^2 d\mu^k_s\right)^{1/2}\mu^k_s(B_\rho(\xi))^{1/2}ds\\
&\le C\rho^{-n}\int_{t-r^2}^t\int_{B_\rho(\xi)}|\mathbf{H}_k|^2 d\mu^k_s ds\\
&\quad +C\rho^{-n/2-1}\int_{t-r^2}^t\left(\int_{B_\rho(\xi)}|\mathbf{H}_k|^2 d\mu^k_s\right)^{1/2} ds \ \text{ by (21)}\\
&\to 0,\quad \text{as } k\to\infty \ \text{ by (17).}
\end{aligned}
$$

Here we have used C for uniform positive constants which are independent of k and ρ. Therefore, there are constants $\delta_1 > 0$ and $k_1 > 0$ such that for all ρ and k with $0 < \rho < \delta_1$, $0 < r < 1$, and $k > k_1$ the estimate

$$
\rho^{-n}\mu^k_{t-r^2}(B_\rho(\xi)) < \frac{1}{c(n)+1} < 1 \tag{27}
$$

holds. Let $d\sigma^k_{t-r^2}$ be the area element of $\partial B_\rho(\xi)\cap\Sigma^k_{t-r^2}$. By the co-area formula, for $0 < r \ll 1$, for a smooth cut-off function ϕ with support in the $2n$-dimensional ball $B_{\delta_1}(0)$ in $\mathbf{R}^{2n}$ with $0 \le \phi \le 1$, $\phi \equiv 1$ in $B_{\delta_1/2}(0)$, we have

$$
\begin{aligned}
\Phi_k(\xi,t,t-r^2) &= \frac{1}{(4\pi r^2)^{n/2}}\int_{\Sigma^k_{t-r^2}}\phi\, e^{-\frac{|F_k-\xi|^2}{4r^2}}d\mu^k_{t-r^2}\\
&\le \frac{1}{(4\pi)^{n/2}r^n}\int_0^{\delta_1}\int_{\partial B_\rho(\xi)\cap\Sigma^k_{t-r^2}} e^{-\frac{\rho^2}{4r^2}}d\sigma^k_{t-r^2}d\rho\\
&\le \frac{1}{(4\pi)^{n/2}r^n}\int_0^{\delta_1} e^{-\frac{\rho^2}{4r^2}}\frac{d}{d\rho}\mathrm{Vol}\big(B_\rho(\xi)\cap\Sigma^k_{t-r^2}\big)d\rho\\
&\le \frac{1}{\pi^{n/2}(2r)^n}\mathrm{Vol}\big(B_{\delta_1}(\xi)\cap\Sigma^k_{t-r^2}\big)e^{-\frac{\delta_1^2}{4r^2}}\\
&\quad +\frac{1}{(c(n)+1)\pi^{n/2}}\int_0^{\delta_1} e^{-\frac{\rho^2}{4r^2}}\frac{\rho^n}{2^n r^n}d\frac{\rho^2}{4r^2}
\end{aligned}
$$

by integration by parts and (27). By (21),

$$\frac{1}{\pi^{n/2}(2r)^n}\mathrm{Vol}\big(B_{\delta_1}(\xi)\cap\Sigma^k_{t-r^2}\big)e^{-\frac{\delta_1^2}{4r^2}}\le C\left(\frac{\delta_1}{2r}\right)^n e^{-\frac{\delta_1^2}{4r^2}}=o(r).$$

Letting $y=\rho/2r$ we have

$$\int_0^{\delta_1} e^{-\frac{\rho^2}{4r^2}}\left(\frac{\rho}{2r}\right)^n d\left(\frac{\rho}{2r}\right)^2\le 2\int_0^\infty e^{-y^2}y^{n+1}dy=c(n)<\infty,$$

and there is an explicit formula for $c(n)$ depends on whether n is odd or even. Thus we conclude

$$\Phi_k(\xi,t,t-r^2)\le 1+o(r).$$

For any classical mean curvature flow Γ_t in a compact Riemannian manifold which is isometrically embedded in $\mathbf{R}^N$, White proves a local regularity theorem (Theorem 3.1 and Theorem 4.1 in [Wh1]): When $\dim\Gamma_t=n$, there is a constant $\epsilon>0$ such that if the Gaussian density satisfies

$$\lim_{r\to 0}\int_{\Gamma_{t-r^2}}\frac{1}{(4\pi r^2)^{n/2}}\exp\left(-\frac{|y-x|^2}{4r^2}\right)d\mu(y)<1+\epsilon,$$

then the mean curvature flow is smooth in a neighborhood of x. Combining this regularity result with (28), we are led to choose $r>0$ sufficiently small and then conclude that

$$\sup_{B_r(\xi)\cap\Sigma^k_t}|\mathbf{A}_k|\le C$$

and consequently Σ^k_t converges strongly in $B_r(\xi)\cap\Sigma^k_t$ to $\Sigma^\infty_t\cap B_r(\xi)$, as $k\to\infty$. So $\Sigma^\infty\cap B_r(\xi)$ is smooth. Smoothness of $\Sigma^\infty\cap B_r(\xi)$ immediately implies

$$\lim_{\rho\to 0}\rho^{-n}\mu^\infty(B_\rho(\xi))=1.$$

This contradicts (26). Hence we have established (25).

In summary, we have shown that $\lim_{\rho\to 0}\rho^{-n}\mu^\infty(B_\rho(\xi))$ exists and for $\mathcal{H}^n$ almost all $\xi\in\Sigma^\infty$,

$$\frac{1}{4c(n)+4}\le\lim_{\rho\to 0}\rho^{-n}\mu^\infty(B_\rho(\xi))<\infty. \tag{28}$$

Finally, we recall a fundamental theorem of Priess in [P]: if $0\le m\le p$ are integers and Ω is a Borel measure on $\mathbf{R}^p$ such that

$$0<\lim_{r\to 0}\frac{\Omega(B_r(x))}{r^m}<\infty,$$

for almost all $x\in\Omega$, then Ω is m-rectifiable. Now we conclude from (28) that $(\Sigma^\infty,\mu^\infty)$ is $\mathcal{H}^n$-rectifiable. □

Remark 3.2. For the λ tangent cones, one can show $\lim_{\rho\to 0}\rho^{-n}\mu^{\infty}(B_\rho(\xi)) > 0$ by using Brakke's clearing out lemma. The argument in the proof of Proposition 3.1 works for the t tangent cones in Sect. 6 (44) as well, and it provides a uniform lower volume density bound. One may generalize the clearing out lemma to equation (36) to prove (44) for the time dependent scaling.

4. Minimality of the λ tangent cones

In this section, we will show that the λ tangent cone Σ^∞ is a stationary integer multiplicity rectifiable current in $\mathbf{R}^{2n}$.

Theorem 4.1. *Let M be a compact Calabi-Yau manifold. If the initial compact submanifold is Lagrangian and is almost calibrated by $\mathrm{Re}\,\Omega$, then the λ tangent cone Σ^∞ is a stationary rectifiable Lagrangian current in $\mathbf{R}^{2n}$ with volume density greater than one at X_0.*

Proof. Let V_t^k be the varifold defined by Σ_t^k. By the definition of varifolds, we have

$$V_t^k(\psi) = \int_{\Sigma_t^k} \psi\big(x, T\Sigma_t^k\big) d\mu_t^k$$

for any $\psi \in C_0^0(G^2(\mathbf{R}^{2n}), R)$, where $G^2(\mathbf{R}^{2n})$ is the Grassmanian bundle of all n-dimensional planes tangent to Σ_t^∞ in $\mathbf{R}^{2n}$. For each smooth submanifold Σ_t^k, the first variation δV_t^k of V_t^k (cf. [A], (39.4) in [Si1] and (1.7) in [I2]) is

$$\delta V_t^k = -\mu_t^k \lfloor \mathbf{H}_k.$$

By Proposition 2.2, we have that $\delta V_t^k \to 0$ at $t \notin A$ as $k \to \infty$, where A is defined in the proof of Proposition 3.1.

Recall that a k-varifold is a Radon measure on $G^k(M)$, where $G^k(M)$ is the Grassmann bundle of all k-planes tangent to M. Allard's compactness theorem for rectifiable varifolds (6.4 in [A], also see 1.9 in [I2] and Theorem 42.7 in [Si1]) asserts the following: let (V_i, μ_i) be a sequence of rectifiable k-varifolds in M with

$$\sup_{i\geq 1}(\mu_i(U) + |\delta V_i|(U)) < \infty \ \text{ for each } U \subset\subset M.$$

Then there is a varifold (V, μ) of locally bounded first variation and a subsequence, which we also denote by (V_i, μ_i), such that (i) Convergence of measures: $\mu_i \to \mu$ as Radon measures on M, (ii) Convergence of tangent planes: $V_i \to V$ as Radon measures on $G^k(M)$, (iii) Convergence of first variations: $\delta V_i \to \delta V$ as TM-valued Radon measures, (iv) Lower semicontinuity of total first variations: $|\delta V| \leq \liminf_{i\to\infty} |\delta V_i|$ as Radon measures.

By (iii) in Allard's compactness theorem, we have

$$-\mu^\infty \lfloor \mathbf{H}_\infty = \delta V^\infty = \lim_{k\to\infty} \delta V_t^k = 0.$$

Therefore Σ^∞ is stationary. The rescaling process in a neighborhood of X_0 in M implies that the metrics g^λ tends to the flat metric on $\mathbf{R}^{2n}$ and the Kähler 2-form ω^λ tends to a constant closed 2-form ω_0 which is determined by $\omega_0(0) = \omega(X_0)$. The tangent spaces to Σ_t^k converge to that of Σ^∞ as measures by (ii) in Allard's compactness theorem. Hence $\omega^{\lambda_k}|_{\Sigma_t^k} \to \omega_0|_{\Sigma^\infty}$. But Σ_t^k is Lagrangian, it follows $\omega^{\lambda_k}|_{\Sigma_t^k} = 0$ therefore $\omega_0|_{\Sigma^\infty} = 0$. Therefore, Σ^∞ is a Lagrangian.

On the other hand, as $\lambda \to \infty$ in the blow-up precess, the holomorphic $(n, 0)$-form Ω converges to a constant holomorphic $(n, 0)$-form Ω_0 on $\mathbf{R}^{2n}$ determined by $\Omega_0(0) = \Omega(X_0)$. We write

$$\operatorname{Re}\Omega_0|_{\Sigma^\infty} = \theta_0 d\mu^\infty,$$

$$\operatorname{Re}\Omega^{\lambda_k}|_{\Sigma_t^k} = \cos\theta_{\lambda_k} d\mu_t^k$$

and from Allard's compactness theorem

$$\operatorname{Re}\Omega^{\lambda_k}|_{\Sigma_t^k} \to \operatorname{Re}\Omega_0|_{\Sigma^\infty},$$

and the tangent cone Σ^∞ is of integer multiplicity by the integral compactness theorem of Allard ([A] and [Si1] 42.8). It follows that $\operatorname{Re}\Omega_0|_{\Sigma^\infty} > 0$, which implies that the tangent cone Σ^∞ is orientable. Since Σ^∞ is of integer multiplicity, we have that $d\mu^\infty = \eta(x)\mathcal{H}^n$ where $\eta(x)$ is a locally $\mathcal{H}^n$-integrable positive integer-valued function. So the cone is an integral current (see Definition 27.1 in [Si1]).

We now show that the volume density of Σ^∞ at X_0 is greater than 1. Otherwise, we would have

$$\lim_{\rho\to 0} \frac{1}{\omega_n \rho^n} \mu^\infty(B_\rho(0)) \le 1$$

where ω_n is the volume of the unit n-ball in $\mathbf{R}^n$:

$$\omega_n = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}.$$

It then follows from (23) that for any $\epsilon > 0$, there are $\delta > 0$ and $k_0 > 0$ such that for any $0 < \rho < 2\delta$ and $k > k_0$,

$$\rho^{-n}\mu^k_{0-r^2}(B_\rho(\xi)) < \omega_n(1+\epsilon) \tag{29}$$

for any fixed $r > 0$. The choice of r will be based on the following observation. Set

$$\Phi(F, \mathbf{X}_0, t_0, t) = \int_{\Sigma_t} \phi(F) \frac{1}{(4\pi(t_0 - t))^{n/2}} e^{-\frac{|F-\mathbf{X}_0|^2}{4(t_0-t)}} d\mu_t$$

where ϕ is supported in $B_\delta(0)$ and $0 \le \phi \le 1, \phi \equiv 1$ in $B_{\delta/2}(0)$. Then we have

$$\begin{aligned}
\Phi(F_k, 0, 0, 0 - r^2) &\le \frac{1}{(4\pi r^2)^{n/2}} \int_0^\delta \int_{\partial B_\rho(0)\cap\Sigma^k_{0-r^2}} e^{-\frac{\rho^2}{4r^2}} d\mu^k_{0-r^2} d\rho \\
&\le \frac{1}{(4\pi r^2)^{n/2}} \int_0^\delta e^{-\frac{\rho^2}{4r^2}} \int_{\partial B_\rho(0)\cap\Sigma^k_{0-r^2}} d\mu^k_{0-r^2} d\rho \\
&\le \frac{1}{(4\pi r^2)^{n/2}} \int_0^\delta e^{-\frac{\rho^2}{4r^2}} \frac{\rho}{2r^2} \mathrm{Vol}\big(B_\rho(0) \cap \Sigma^k_{0-r^2}\big) d\rho \\
&\quad + \frac{1}{(4\pi r^2)^{n/2}} e^{-\frac{\delta^2}{4r^2}} \mathrm{Vol}\big(B_\rho(0) \cap \Sigma^k_{0-r^2}\big) \\
&\le \frac{(1+\epsilon)\omega_n}{(4\pi r^2)^{n/2}} \int_0^\delta e^{-\frac{\rho^2}{4r^2}} \frac{\rho^{n+1}}{2r^2} d\rho + o(r) \text{ by (29) and (21)} \\
&= \frac{(1+\epsilon)\omega_n}{\pi^{n/2}} \int_0^{\frac{\delta^2}{4r^2}} e^{-x} x^{\frac{n}{2}} dx + o(r) \\
&\le 1 + \epsilon + o(r)
\end{aligned}$$

because $\Gamma\big(\frac{n}{2}+1\big) = \int_0^\infty e^{-x} x^{\frac{n}{2}} dx$. Choosing $r > 0$ sufficiently small, we therefore have

$$\Phi\big(F, X_0, T, T - \lambda_k^{-2} r^2\big) = \Phi(F_k, 0, 0, 0 - r^2) \le 1 + \epsilon.$$

Now by White's local regularity theorem ([Wh1] Theorem 3.1 and Theorem 4.1, also see [E2]), (X_0, T) could not be a singular point of the mean curvature flow. This is a contradiction. □

5. Flatness of λ-cone in dimension 2

Regularity of the λ tangent cone can be greatly improved in the 2-dimensional case: $\dim_{\mathbf{C}} M = 2$.

Theorem 5.1. *Let (M, Ω) be a compact Calabi-Yau surface and let Σ_0 be a compact Lagrangian surface in M which is almost calibrated by ReΩ. If $0 < T < \infty$ is the first blow-up time of a mean curvature flow of Σ_0 in M, then the λ tangent cone at (X_0, T) consists of a finite union (but more than one) of 2-planes in $\mathbf{R}^4$ which are complex in a complex structure on $\mathbf{R}^4$.*

Proof. We use the same notation as that in the proof of Theorem 4.1, we shall show that θ_0 is constant $\mathcal{H}^2$ a.e. on Σ^∞. To do so, we claim that for any $r > 0$, $\xi_1, \xi_2 \in \Sigma^k_t \cap B_{R/2}(0)$ where $t \notin A$ the following holds

$$\left|\frac{1}{\mathrm{Vol}\big(B_r(\xi_1)\cap\Sigma_t^k\big)}\int\limits_{B_r(\xi_1)\cap\Sigma_t^k}\cos\theta_k d\mu_t^k-\frac{1}{\mathrm{Vol}\big(B_r(\xi_2)\cap\Sigma_t^k\big)}\int\limits_{B_r(\xi_2)\cap\Sigma_t^k}\cos\theta_k d\mu_t^k\right|$$

$$\leq\frac{C_1(r)}{\mathrm{Vol}\big(B_r(\xi_1)\cap\Sigma_t^k\big)}\cdot\frac{C_2(r)}{\mathrm{Vol}\big(B_r(\xi_2)\cap\Sigma_t^k\big)}\int_{B_R(0)\cap\Sigma_t^k}|\nabla\cos\theta_k|\,d\mu_t^k,\quad(30)$$

where $B_r(\xi_i)$, $i=1,2$, are the 4-dimensional balls in M. To prove (30), let us first recall the isoperimetric inequality on Σ_t^k (c.f. [HSp] and [MS]): let $B_\rho^k(p)$ be the geodesic ball in Σ_t^k, with radius ρ and center p, then

$$\begin{aligned}&\mathrm{Vol}\big(B_\rho^k(p)\big)\\&\quad\leq C\left(\mathrm{length}\big(\partial\big(B_\rho^k(p)\big)\big)+\int_{B_\rho^k(p)}|\mathbf{H}_k|d\mu_t^k\right)^2\\&\quad\leq C\left(\mathrm{length}\big(\partial\big(B_\rho^k(p)\big)\big)+\left(\int_{B_\rho^k(p)}|\mathbf{H}_k|^2d\mu_t^k\right)^{1/2}\mathrm{Vol}^{1/2}\big(B_\rho^k(p)\big)\right)^2,\end{aligned}$$

for any $p\in\Sigma_t^k$, and almost every $\rho>0$, where C does not depend on k, ρ, and p. By Proposition 2.2, since $t\notin A$ we have

$$\int_{B_\rho^k(p)}|\mathbf{H}_k|^2d\mu_t^k\to 0\text{ as }k\to\infty.$$

So, for k sufficiently large, we obtain:

$$\mathrm{Vol}\big(B_\rho^k(p)\big)\leq C\left(\mathrm{length}\big(\partial\big(B_\rho^k(p)\big)\big)\right)^2.$$

In particular, for k sufficiently large, the isoperimetric inequality implies

$$\mathrm{Vol}\big(B_\rho^k(p)\big)\geq C\rho^2,\quad(31)$$

where C is a positive constant independent of k, ρ and p.

Suppose that the diameter of $B_r(\xi)\cap\Sigma_t^k$ is $d_k(\xi)$. Then

$$\begin{aligned}Cr^2&\geq\int_{B_r(\xi)\cap\Sigma_t^k}d\mu_t^k\quad\text{by (21)}\\&=\int_0^{d_k(\xi)/2}\int_{\partial B_\rho^k(p)}d\sigma d\rho\quad\text{for some }p\in\Sigma_t^k\\&\geq c\int_0^{d_k(\xi)/2}\mathrm{Vol}^{1/2}\big(B_\rho^k(p)\big)d\rho\\&\geq c\int_0^{d_k(\xi)/2}C\rho d\rho\ \text{ by (31)}\\&\geq cd_k(\xi)^2.\end{aligned}$$

We therefore have, for any ξ,

$$d_k(\xi) \leq Cr \tag{32}$$

where the constant C is independent of ξ and k.

For any fixed $\eta \in B_r(\xi_2) \cap \Sigma_t^k$ and any $\xi \in B_r(\xi_1) \cap \Sigma_t^k$, we choose a geodesic $l_{\eta\xi}$ connecting η and ξ, call it a ray from η to ξ. Take an open tubular neighborhood $U(l_{\eta\xi})$ of $l_{\eta\xi}$ in Σ_t^k. Within this neighborhood $U(l_{\eta\xi})$, we call the line in the normal direction of the ray $l_{\eta\xi}$ the normal line which we denote by $n(l_{\eta\xi})$. It is clear that

$$\cos\theta_k(\xi) - \cos\theta_k(\eta) = \int_{l_{\eta\xi}} \partial_l \cos\theta_k dl \tag{33}$$

where dl is the arc-length element of $l_{\eta\xi}$. Choose r small enough so that $B_r(\xi_1) \cap \Sigma_t^k$ is contained in $U(l_{\eta\xi_1})$. Keeping η fixed and integrating (33) with respect to the variable ξ, first along the normal direction $n(l_{\eta\xi_1})$ and then on the ray direction $l_{\eta\xi_1}$, we have

$$\begin{aligned}
&\left| \frac{1}{\mathrm{Vol}\big(B_r(\xi_1) \cap \Sigma_t^k\big)} \int_{B_r(\xi_1)\cap\Sigma_t^k} \cos\theta_k(\xi) d\mu_t^k - \cos\theta_k(\eta) \right| \\
&\quad \leq \frac{1}{\mathrm{Vol}\big(B_r(\xi_1) \cap \Sigma_t^k\big)} \int_0^{d_k(\xi_1)} \int_{n(l_{\eta\xi_1})} \int_{l_{\eta\xi}} |\nabla \cos\theta_k| \, dl dn(\xi) d\rho \\
&\quad \leq \frac{1}{\mathrm{Vol}\big(B_r(\xi_1) \cap \Sigma_t^k\big)} \int_0^{d_k(\xi_1)} \int_{B_R(0)} |\nabla \cos\theta_k| \, d\mu_t^k d\rho \\
&\quad \leq \frac{Cr}{\mathrm{Vol}\big(B_r(\xi_1) \cap \Sigma_t^k\big)} \int_{B_R(0)} |\nabla \cos\theta_k| \, d\mu_t^k,
\end{aligned} \tag{34}$$

here in the last step we have used (32). From (34), integrating with respect to η in $B_r(\xi_2) \cap \Sigma_t^k$ and dividing by $\mathrm{Vol}\big(B_r(\xi_2) \cap \Sigma_t^k\big)$, we get the desired inequality (30).

For $i = 1, 2$ Hölder's inequality and (21) lead to

$$\int_{B_r(\xi_i)\cap\Sigma_t^k} |\nabla \cos\theta_k| \, d\mu_t^k \leq Cr \left(\int_{B_r(\xi_i)\cap\Sigma_t^k} |\nabla \cos\theta_k|^2 \, d\mu_t^k \right)^{1/2}.$$

The triangle inequality implies $B_r^k(\xi_i) \subset B_r(\xi_i) \cap \Sigma_t^k$ for $i = 1, 2$; therefore by (31)

$$\mathrm{Vol}\big(B_r(\xi_i) \cap \Sigma_t^k\big) \geq \mathrm{Vol}\big(B_r^k(\xi_i)\big) \geq Cr^2.$$

Now first letting $k \to \infty$ in (30) and using that the right hand side of (30) tends to 0 by Proposition 2.2, and then letting $r \to 0$, we conclude that $\cos\theta$ is constant $\mathcal{H}^2$ a.e. on Σ^∞.

The $(2,0)$-form Ω_0 is fixed by $\Omega(\mathbf{X}_0)$ hence it has unit length. In the complex structure $J_{\mathbf{X}_0}$ on $\mathbf{R}^4$, $\Omega_0 = dz_1 \wedge dz_2$. We define a new complex structure J^* on $\mathbf{R}^4$:

$$J^*(\partial/\partial x_1) = \theta_0(\partial/\partial y_1), \quad J^*(\partial/\partial y_1) = -1/\theta_0(\partial/\partial x_1),$$

$$J^*(\partial/\partial x_2) = 1/\theta_0(\partial/\partial y_2), \quad J^*(\partial/\partial y_2) = -\theta_0(\partial/\partial x_2).$$

In J^*, the complex coordinates are: $z_1^* = x_1 + \sqrt{-1}\theta_0^{-1} y_1$, $z_2^* = \theta_0^{-1} x_2 + \sqrt{-1} y_2$. Then $\Omega_0^* = dz_1^* \wedge dz_2^*$ satisfies that $\operatorname{Re} \Omega_0^*|_{\Sigma^\infty} = d\mu^\infty$.

We can further choose a new complex structure J' on $\mathbf{R}^4$ such that Ω_0^* is of type $(1,1)$ in J'. In fact, if we express J^* in the local coordinates $x_1, \theta_0^{-1} y_1, \theta_0^{-1} x_2, y_2$ by

$$J^* = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \quad \text{with } I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

then we can take

$$J' = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

Therefore Σ^∞ is a stationary rectifiable current of type $(1,1)$ with respect to the complex structure J'. By Harvey-Shiffman's Theorem 2.1 in [HS], Σ^∞ is a J'-holomorphic subvariety of complex dimension one. It then follows that the singular locus $\mathcal{S}$ of Σ^∞ consists of isolated points.

Without loss of any generality, we may assume $0 \in \Sigma^\infty$ where 0 is the origin of $\mathbf{R}^4$. In fact, if not, Σ^∞ would move to infinity, then we would have

$$\Phi\big(F, X_0, T, T - \lambda_k^{-2} r^2\big) = \Phi\big(F_k, 0, 0, 0 - r^2\big) \to 0 \text{ as } k \to \infty.$$

But White's regularity theorem then implies that (X_0, T) is a regular point. This is impossible.

There is a sequence of points $X_k \in \Sigma_t^k$ satisfying $X_k \to 0$ as $k \to \infty$. By Proposition 2.2, for any s_1 and s_2 with $-\infty < s_1 < s_2 < 0$ and any $R > 0$, we have

$$\int_{s_1}^{s_2} \int_{\Sigma_t^k \cap B_R(0)} \left|F_k^\perp\right|^2 d\mu_t^k dt \to 0 \text{ as } k \to \infty.$$

Thus, by (21)

$$\begin{aligned} &\lim_{k\to\infty} \int_{s_1}^{s_2} \int_{\Sigma_t^k \cap B_R(0)} \left|(F_k - X_k)^\perp\right|^2 d\mu_t^k dt \\ &\leq 2 \lim_{k\to\infty} \int_{s_1}^{s_2} \int_{\Sigma_t^k \cap B_R(0)} \left|F_k^\perp\right|^2 d\mu_t^k dt + C(s_2 - s_1) R^2 \lim_{k\to\infty} |X_k|^2 \\ &= 0. \end{aligned}$$

Let us denote the tangent spaces of Σ_t^k at the point $F_k(x,t)$ and of Σ^∞ at the point $F^\infty(x,t)$ by $T\Sigma_t^k$ and $T\Sigma^\infty$ respectively. It is clear that

$$(F_k - X_k)^\perp = \text{dist}\left(X_k, T\Sigma_t^k\right),$$

and

$$(F_\infty)^\perp = \text{dist}\,(0, T\Sigma^\infty).$$

By Allard's compactness theorem, we have

$$\begin{aligned}
&\int_{s_1}^{s_2}\int_{\Sigma^\infty\cap B_R(0)} \left|(F_\infty)^\perp\right|^2 d\mu^\infty dt \\
&\qquad = \int_{s_1}^{s_2}\int_{\Sigma^\infty\cap B_R(0)} \left|\text{dist}\,(0, T\Sigma^\infty)\right|^2 d\mu^\infty dt \\
&\qquad = \lim_{k\to\infty}\int_{s_1}^{s_2}\int_{\Sigma_t^k\cap B_R(0)} \left|\text{dist}\left(X_k, T\Sigma_t^k\right)\right|^2 d\mu_t^k dt \\
&\qquad = \lim_{k\to\infty}\int_{s_1}^{s_2}\int_{\Sigma_t^k\cap B_R(0)} \left|(F_k - X_k)^\perp\right|^2 d\mu_t^k dt \\
&\qquad = 0.
\end{aligned}$$

Therefore $F_\infty^\perp \equiv 0$. Differentiating $\langle F_\infty, \nu_\alpha\rangle = 0$, inner product is taken in $\mathbf{R}^4$, leads to

$$0 = \langle \partial_i F_\infty, \nu_\alpha\rangle + \langle F_\infty, \partial_i \nu_\alpha\rangle = \langle F_\infty, \partial_i \nu_\alpha\rangle.$$

Because $\partial_i F_\infty$ is tangential to Σ^∞, by Weingarten's equation we observe

$$(h_\infty)_{ij}^\alpha \langle F_\infty, e_j\rangle = 0 \ \text{ for all } \ \alpha,\ i = 1, 2.$$

Since either $\langle F_\infty, e_1\rangle \neq 0$ or $\langle F_\infty, e_2\rangle \neq 0$, we conclude $\det\left(h_{ij}^\alpha\right) = 0$. Recall $h_{11}^\alpha + h_{22}^\alpha = 0$. It then follows $h_{ij}^\alpha = 0$, for $i, j, \alpha = 1, 2$. Now we conclude that Σ^∞ consists of flat 2-planes. □

6. Tangent cones from a time dependent scaling

In this section, we consider the tangent cones which arise from the rescaled submanifold $\widetilde{\Sigma}_s$ defined by

$$\widetilde{F}(\cdot, s) = \frac{1}{\sqrt{2(T-t)}} F(\cdot, t), \tag{35}$$

where $s = -\frac{1}{2}\log(T-t)$, $c_0 \leq s < \infty$. Here we choose the coordinates so that $X_0 = 0$. Rescaling of this type was used by Huisken [H2] to distinguish Type I and Type II singularities for mean curvature flows. Denote the rescaled submanifold by $\widetilde{\Sigma}_s$. From the evolution equation of F we derive the flow equation for $\widetilde{F}$

$$\frac{\partial}{\partial s}\widetilde{F}(x,s) = \widetilde{\mathbf{H}}(x,s) + \widetilde{F}(x,s). \tag{36}$$

It is clear that

$$\begin{aligned} \cos\widetilde{\theta}(x,s) &= \cos\theta(x,t), \\ |\widetilde{\mathbf{H}}|^2(x,s) &= 2(T-t)|\mathbf{H}|^2(x,t), \\ |\widetilde{\mathbf{A}}|^2(x,s) &= 2(T-t)|\mathbf{A}|^2(x,t). \end{aligned}$$

We set $\widetilde{v}(x,t) = \cos\widetilde{\theta}(x,s)$.

Lemma 6.1. *Assume that* (M,Ω) *is a compact Calabi-Yau manifold and* Σ_t *evolves by a mean curvature flow in* M *with the initial submanifold* Σ_0 *being Lagrangian and almost calibrated by* $Re\,\Omega$*. Then*

$$\left(\frac{\partial}{\partial s} - \widetilde{\Delta}\right)\widetilde{v}(x,s) = |\widetilde{\mathbf{H}}|^2\,\widetilde{v}(x,s). \tag{37}$$

Proof. One can check directly that

$$\left(\frac{\partial}{\partial s} - \widetilde{\Delta}\right)\cos\widetilde{\alpha}(x,s) = 2(T-t)\left(\frac{\partial}{\partial t} - \Delta\right)\cos\alpha(x,t).$$

It follows that

$$\begin{aligned} \left(\frac{\partial}{\partial s} - \widetilde{\Delta}\right)\widetilde{v}(x,s) &= 2(T-t)\left(\frac{\partial}{\partial t} - \Delta\right)v(x,t) \\ &\geq 2(T-t)\,|\mathbf{H}|^2\,v(x,t) \\ &= |\widetilde{\mathbf{H}}|^2\,\widetilde{v}(x,s). \end{aligned}$$

This proves the lemma. □

Next, we shall derive the corresponding weighted monotonicity formula for the scaled flow. By (37), we have

$$\left(\frac{\partial}{\partial s} - \widetilde{\Delta}\right)\frac{1}{\widetilde{v}} = -\frac{|\widetilde{\mathbf{H}}|^2}{\widetilde{v}} - \frac{2|\widetilde{\nabla}\widetilde{v}|^2}{\widetilde{v}^3}.$$

Let

$$\widetilde{\rho}(X) = \exp\left(-\frac{1}{2}|X|^2\right),$$

$$\Psi(s) = \int_{\widetilde{\Sigma}_s}\frac{1}{\widetilde{v}}\phi\widetilde{\rho}(\widetilde{F})d\widetilde{\mu}_s.$$

Lemma 6.2. *There are positive constants c_1 and c_2 which depend on M, F_0 and r which is the constant in the definition of ϕ, so that the following monotonicity formula holds*

$$\begin{aligned}\frac{\partial}{\partial s}\exp(c_1e^{-s})\Psi(s) \leq & -\exp(c_1e^{-s})\left(\int_{\widetilde{\Sigma}_s}\frac{1}{\widetilde{v}}\phi\widetilde{\rho}(\widetilde{F})\left|\widetilde{\mathbf{H}}+\widetilde{F}^{\perp}\right|^2 d\widetilde{\mu}_s\right.\\ & \left.+\int_{\widetilde{\Sigma}_s}\frac{1}{\widetilde{v}}\phi\widetilde{\rho}(\widetilde{F})\frac{|\widetilde{\mathbf{H}}|^2}{2}d\widetilde{\mu}_s+\int_{\widetilde{\Sigma}_s}\frac{2}{\widetilde{v}^3}\left|\widetilde{\nabla}\widetilde{v}\right|^2\phi\widetilde{\rho}(\widetilde{F})d\widetilde{\mu}_s\right)\\ & +c_2\exp(c_1e^{-s}). \qquad (38)\end{aligned}$$

Proof. Note that

$$\begin{aligned}\widetilde{F}(x,s) &= \frac{F(x,t)}{\sqrt{2(T-t)}},\\ \widetilde{\mathbf{H}}(x,s) &= \sqrt{2(T-t)}\mathbf{H}(x,t),\\ |\widetilde{\nabla}\widetilde{v}|^2(x,s) &= 2(T-t)|\nabla v|^2(x,t).\end{aligned}$$

By the chain rule

$$\frac{\partial}{\partial s} = 2e^{-2s}\frac{\partial}{\partial t}$$

and the monotonicity inequality (9) for the unscaled submanifold, we obtain the desired inequality. □

Lemma 6.3. *Let (M,Ω) be a compact Calabi-Yau manifold. If the initial compact submanifold Σ_0 is Lagrangian and almost calibrated by Re Ω, then there is a sequence $s_k\to\infty$ such that, for any $R>0$,*

$$\int_{\widetilde{\Sigma}_{s_k}\cap B_R(0)}|\widetilde{\nabla}\cos\widetilde{\theta}|^2 d\widetilde{\mu}_{s_k}\to 0 \text{ as } k\to\infty, \qquad (39)$$

$$\int_{\widetilde{\Sigma}_{s_k}\cap B_R(0)}|\widetilde{\mathbf{H}}|^2 d\widetilde{\mu}_{s_k}\to 0 \text{ as } k\to\infty, \qquad (40)$$

and

$$\int_{\widetilde{\Sigma}_{s_k}\cap B_R(0)}|\widetilde{F}^{\perp}|^2 d\widetilde{\mu}_{s_k}\to 0 \text{ as } k\to\infty. \qquad (41)$$

Proof. Integrating (38), we have

$$\begin{aligned}\infty > & \int_{s_0}^{\infty}\int_{\widetilde{\Sigma}_s}\frac{1}{\widetilde{v}}\phi\widetilde{\rho}(\widetilde{F})\left|\widetilde{\mathbf{H}}+\widetilde{F}^{\perp}\right|^2 d\widetilde{\mu}_s ds\\ & +\int_{s_0}^{\infty}\left(\int_{\widetilde{\Sigma}_s}\frac{1}{\widetilde{v}}\phi\widetilde{\rho}(\widetilde{F})\frac{|\widetilde{\mathbf{H}}|^2}{2}d\widetilde{\mu}_s+\int_{\widetilde{\mu}_s}\frac{2}{\widetilde{v}^3}|\widetilde{\nabla}\widetilde{v}|^2\phi\widetilde{\rho}(\widetilde{F})d\widetilde{\mu}_s\right)ds.\end{aligned}$$

Hence there is a sequence $s_k \to \infty$, such that as $k \to \infty$

$$\int_{\widetilde{\Sigma}_{s_k}} \frac{1}{\widetilde{v}} \phi \widetilde{\rho}(\widetilde{F}) \frac{|\widetilde{\mathbf{H}}|^2}{2} d\widetilde{\mu}_{s_k} \to 0,$$

$$\int_{\widetilde{\Sigma}_{s_k}} \frac{2}{\widetilde{v}^3} |\widetilde{\nabla} \widetilde{v}|^2 \phi \widetilde{\rho}(\widetilde{F}) d\widetilde{\mu}_{s_k} \to 0,$$

and

$$\int_{\widetilde{\Sigma}_{s_k}} \frac{1}{\widetilde{v}} \phi \widetilde{\rho}(\widetilde{F}) \left| \widetilde{\mathbf{H}} + \widetilde{F}^{\perp} \right|^2 d\widetilde{\mu}_{s_k} \to 0.$$

Since $\widetilde{v}$ has a positive lower bound, the proposition now follows. □

The proof of the following lemma is essentially the same as the one for Proposition 3.1, except there are two parameters λ, t for the λ tangent cones but only one parameter t for the time dependent tangent cones. Note that the alternative proof given in [CL1] using the isoperimetric inequality only works in dimension 2.

Lemma 6.4. *There is a subsequence of s_k, which we also denote by s_k, such that $\left(\widetilde{\Sigma}_{s_k}, d\widetilde{\mu}_{s_k}\right) \to \left(\widetilde{\Sigma}_\infty, d\widetilde{\mu}_\infty\right)$ in the sense of measures. And $\left(\widetilde{\Sigma}_\infty, d\widetilde{\mu}_\infty\right)$ is $\mathcal{H}^n$-rectifiable.*

Proof. To show the subconvergence, it suffices to show that, for any $R > 0$,

$$\widetilde{\mu}_{s_k}\left(\widetilde{\Sigma}_{s_k} \cap B_R(0)\right) \leq CR^n, \tag{42}$$

where $B_R(0)$ is a metric ball in $\mathbf{R}^{2n}$, $C > 0$ is independent of k. Direct calculation leads to

$$\begin{aligned}
&\widetilde{\mu}_{s_k}\left(\widetilde{\Sigma}_{s_k} \cap B_R(0)\right) \\
&\quad = (2(T-t))^{-n/2} \int_{\Sigma_{T-e^{2s_k}} \cap B_{\sqrt{2(T-t)}R}(0)} d\mu_t \\
&\quad = R^n \left(\sqrt{2} e^{-s_k} R\right)^{-n} \int_{\Sigma_{T-e^{2s_k}} \cap B_{\sqrt{2}e^{-s_k}R}(0)} d\mu_t \\
&\quad \leq CR^n \int_{\Sigma_{T-e^{2s_k}} \cap B_{\sqrt{2(T-t)}R}(0)} \frac{1}{v} \frac{1}{(4\pi)^{n/2} (\sqrt{2} e^{-s_k} R)^n} e^{-\frac{|X-X_0|^2}{4\sqrt{2}e^{-s_k}R}} d\mu_t \\
&\quad \leq CR^n \Psi\left(0, T + (\sqrt{2} e^{-s_k} R)^2 - e^{2s_k}, T - e^{2s_k}\right).
\end{aligned}$$

By the monotonicity inequality (9), we have

$$\begin{aligned}
\widetilde{\mu}_{s_k}\left(\widetilde{\Sigma}_{s_k} \cap B_R(0)\right) &\leq CR^n \Phi\left(0, T + (\sqrt{2} e^{-s_k} R)^2 - e^{2s_k}, T/2\right) + CR^n \\
&\leq \frac{C\mu_{T/2}(\Sigma_{T/2})}{T^{n/2} \min_{\Sigma_0} v} R^n + CR^n.
\end{aligned}$$

Since volume is non-increasing along mean curvature flow, we see

$$\widetilde{\mu}_{s_k}\big(\widetilde{\Sigma}_{s_k} \cap B_R(0)\big) \leq CR^n.$$

We now prove that $\big(\widetilde{\Sigma}_\infty, d\widetilde{\mu}_\infty\big)$ is $\mathcal{H}^n$-rectifiable. For any $\xi \in \widetilde{\Sigma}_\infty$, choose $\xi_k \in \widetilde{\Sigma}_{s_k}$ with $\xi_k \to \xi$ as $k \to \infty$. By the monotonicity identity (17.4) in [Si1], we have

$$\sigma^{-n}\widetilde{\mu}_{s_k}(B_\sigma(\xi_k)) = \rho^{-n}\widetilde{\mu}_{s_k}(B_\rho(\xi_k)) - \int_{B_\rho(\xi_k)\setminus B_\sigma(\xi_k)} \frac{|D^\perp r|^2}{r^n} d\widetilde{\mu}_{s_k}$$
$$-\frac{1}{n}\int_{B_\rho(\xi_k)} (x - \xi_k)\cdot \widetilde{\mathbf{H}}_k \left(\frac{1}{r_\sigma^n} - \frac{1}{\rho^n}\right) d\widetilde{\mu}_{s_k}, \qquad (43)$$

for all $0 < \sigma \leq \rho$, where $\widetilde{\mu}_{s_k}(B_\sigma(\xi_k))$ is the area of $\widetilde{\Sigma}_{s_k} \cap B_\sigma(\xi_k)$, $r_\sigma = \max\{r, \sigma\}$ and $D^\perp r$ denotes the orthogonal projection of Dr (which is a vector of length 1) onto $\big(T_{\xi_k}\widetilde{\Sigma}_{s_k}\big)^\perp$. Letting $k \to \infty$, by Lemma 6.3, we have

$$\sigma^{-n}\widetilde{\mu}_\infty(B_\sigma(\xi)) \leq \rho^{-n}\widetilde{\mu}_\infty(B_\rho(\xi)),$$

for all $0 < \sigma \leq \rho$. Therefore, $\lim_{\rho\to 0}\rho^{-n}\widetilde{\mu}_\infty(B_\rho(\xi))$ exists and is finite by (42).

By converting s to t, the argument for the positive lower bound of the volume density in the proof of Proposition 3.1 carries over to the present situation.

We conclude that $\lim_{\rho\to 0}\rho^{-n}\widetilde{\mu}_\infty(B_\rho(\xi))$ exists and for $\mathcal{H}^n$ almost all $\xi \in \widetilde{\Sigma}_\infty$,

$$0 < C \leq \lim_{\rho\to 0}\rho^{-n}\widetilde{\mu}_\infty(B_\rho(\xi)) < \infty. \qquad (44)$$

Priess's theorem in [P] then asserts the $\mathcal{H}^n$-rectifiability of $\big(\widetilde{\Sigma}_\infty, d\widetilde{\mu}_\infty\big)$. □

Definition 6.5. We call $\big(\widetilde{\Sigma}_\infty, d\widetilde{\mu}_\infty\big)$ obtained in Lemma 6.4 *a tangent cone of the mean curvature flow Σ_t at (X_0, T) in the time dependent scaling.*

With the lemmas established in this section, by using arguments completely similar to those for the λ tangent cones in the previous sections, we can prove

Theorem 6.6. *Let (M, Ω) be a compact Calabi-Yau manifold. If the initial compact submanifold Σ_0 is Lagrangian and almost calibrated by Re Ω and $T > 0$ is the first blow-up time of the mean curvature flow, then the tangent cone $\widetilde{\Sigma}_\infty$ of the mean curvature flow at (X_0, T) coming from time dependent scaling is a rectifiable stationary Lagrangian current with integer multiplicity in $\mathbf{R}^{2n}$. Moreover, if M is of complex 2-dimensional, then $\widetilde{\Sigma}_\infty$ consists of a finitely many (at least two) 2-planes in $\mathbf{R}^4$ which are complex in a complex structure on $\mathbf{R}^4$.*

The result below can also be found in [Wa].

Corollary 6.7. *If the initial compact submanifold* Σ_0 *is Lagrangian and is almost calibrated in a compact Calabi-Yau manifold* (M, Ω)*, then mean curvature flow does not develop Type I singularity.*

Proof. Let X_0 be a Type I singularity at $T < \infty$ and set $\lambda = \max_{\Sigma_t} |\mathbf{A}|^2$. The λ tangent cone Σ_∞ is smooth if T is a Type I singularity. Therefore Σ_∞ is a smooth minimal Lagrangian submanifold in $\mathbf{C}^n$ by Theorem 6.6. Because Σ_∞ is smooth, (18) implies $F_\infty^\perp \equiv 0$ everywhere. The monotonicity identity (23) then implies $\sigma^{-n}\mu(\Sigma_\infty \cap B_\sigma(0))$ is a constant independent of σ, and the volume density ratio at 0 is one due to the smoothness of Σ_∞, so Σ_∞ is a flat linear subspace of $\mathbf{R}^{2n}$. But the second fundamental form of Σ_∞ has length one at 0 according to the blow-up process, and the contradiction rules out any Type I singularities. □

References

[A] Allard, W.: First variation of a varifold. Ann. Math. **95**, 419–491 (1972)

[B] Brakke, K.: The motion of a surface by its mean curvature. Princeton Univ. Press 1978

[CL1] Chen, J., Li, J.: Singularities of codimension two mean curvature flow of symplectic surfaces. Preprint (2002)

[CL2] Chen, J., Li, J.: Mean curvature flow of surface in 4-manifolds. Adv. Math. **163**, 287–309 (2001)

[CLT] Chen, J., Li, J., Tian, G.: Two-dimensional graphs moving by mean curvature flow. Acta Math. Sin., Engl. Ser. **18**, 209–224 (2002)

[CT] Chen, J., Tian, G.: Moving symplectic curves in Kähler-Einstein surfaces. Acta Math. Sin., Engl. Ser. **16**, 541–548 (2000)

[E1] Ecker, K.: On regularity for mean curvature flow of hypersurfaces. Calc. Var. **3**, 107–126 (1995)

[E2] Ecker, K.: A local monotonicity formula for mean curvature flow. Ann. Math. **154**, 503–525 (2001)

[EH] Ecker, K., Huisken, G.: Mean curvature evolution of entire graphs. Ann. Math. **130**, 453–471 (1989)

[HL] Harvey, R., Lawson, H.B.: Calibrated geometries. Acta Math. **148**, 47–157 (1982)

[HS] Harvey, R., Shiffman, B.: A characterization of holomorphic chains. Ann. Math. **99**, 553–587 (1974)

[HSp] Hoffman, D., Spruck, J.: Sobolev and isoperimetric inequalities for Riemannian submanifolds. Commun. Pure Appl. Math. **27**, 715–727 (1974)

[H1] Huisken, G.: Asymptotic behavior for singularities of the mean curvature flow. J. Differ. Geom. **31**, 285–299 (1990)

[H2] Huisken, G.: Flow by mean curvature of convex surfaces into spheres. J. Differ. Geom. **20**, 237–266 (1984)

[H3] Huisken, G.: Contracting convex hypersurfaces in Riemannian manifolds by their mean curvature. Invent. Math. **84**, 463–480 (1986)

[HS1] Huisken, G., Sinestrari, C.: Convexity estimates for mean curvature flow and singularities of mean convex surfaces. Acta Math. **183**, 45–70 (1999)

[HS2] Huisken, G., Sinestrari, C.: Mean curvature flow singularities for mean convex surfaces. Calc. Var. Partial Differ. Equ. **8**, 1–14 (1999)

[I1] Ilmanen, T.: Singularity of mean curvature flow of surfaces. Preprint

[I2] Ilmanen, T.: Elliptic Regularization and Partial Regularity for Motion by Mean Curvature. Mem. Am. Math. Soc. **520** (1994)

[MS] Michael, J.H., Simon, L.: Sobolev and mean-valued inequalities on generalized submanifolds of R^n. Commun. Pure Appl. Math. **26**, 361–379 (1973)

[P] Priess, D.: Geometry of measures in R^n; Distribution. rectifiability, and densities; Ann. Math. **125**, 537–643 (1987)

[ScW] Schoen, R., Wolfson, J.: Minimizing area among Lagrangian surfaces: the mapping problem. J. Differ. Geom. **58**, 1–86 (2001)

[Si1] Simon, L.: Lectures on Geometric Measure Theory. Proc. Center Math. Anal. 3. Australian National Univ. Press 1983

[Sm1] Smoczyk, K.: Der Lagrangesche mittlere Kruemmungsfluss. Univ. Leipzig (Habil.-Schr.), 102 S. 2000

[Sm2] Smoczyk, K.: Harnack inequality for the Lagrangian mean curvature flow. Calc. Var. Partial Differ. Equ. **8**, 247–258 (1999)

[Sm3] Smoczyk, K.: Angle theorems for the Lagrangian mean curvature flow. Math. Z. **240**, 849–883 (2002)

[TY] Thomas, R., Yau, S.T.: Special Lagrangians, stable bundles and mean curvature flow. Comm. Anal. Geom. **10**, 1075–1113 (2002)

[Wa] Wang, M.-T.: Mean curvature flow of surfaces in Einstein four manifolds. J. Differ. Geom. **57**, 301–338 (2001)

[Wh1] White, B.: A local regularity theorem for classical mean curvature flow. Preprint (2000)

[Wh2] White, B.: Stratification of minimal surfaces, mean curvature flows, and harmonic maps. J. Reine Angew. Math. **488**, 1–35 (1997)

[Wh3] White, B.: The size of the singular set in mean curvature flow of mean-convex sets. J. Am. Math. Soc. **13**, 665–695 (2000)