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Cluster algebras II: Finite type classification

Sergey Fomin[1], **Andrei Zelevinsky**[2]

[1] Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA (e-mail: fomin@umich.edu)
[2] Department of Mathematics, Northeastern University, Boston, MA 02115, USA (e-mail: andrei@neu.edu)

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1. Introduction and main results

1.1. Introduction. The origins of cluster algebras, first introduced in [9], lie in the desire to understand, in concrete algebraic and combinatorial terms, the structure of "dual canonical bases" in (homogeneous) coordinate rings of various algebraic varieties related to semisimple groups. Several classes of such varieties—among them Grassmann and Schubert varieties, base affine spaces, and double Bruhat cells—are expected (and in many cases proved)

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to carry a cluster algebra structure. This structure includes the description of the ring in question as a commutative ring generated inside its ambient field by a distinguished family of generators called cluster variables. Even though most of the rings of interest to us are finitely generated, their set of cluster variables may well be infinite. A cluster algebra has *finite type* if it has a finite number of cluster variables.

The main result of this paper (Theorem 1.4) provides a complete classification of the cluster algebras of finite type. This classification turns out to be identical to the Cartan-Killing classification of semisimple Lie algebras and finite root systems. This result is particularly intriguing since in most cases, the symmetry exhibited by the Cartan-Killing type of a cluster algebra is not apparent at all from its geometric realization. For instance, the coordinate ring of the base affine space of the group SL_5 turns out to be a cluster algebra of the Cartan-Killing type D_6. Other examples of similar nature can be found in Sect. 12, in which we show how cluster algebras of types $ABCD$ arise as coordinate rings of some classical algebraic varieties.

In order to understand a cluster algebra of finite type, one needs to study the combinatorial structure behind it, which is captured by its *cluster complex*. Roughly speaking, it is defined as follows. The cluster variables for a given cluster algebra are not given from the outset but are obtained from some initial "seed" by an explicit process of "mutations"; each mutation exchanges a cluster variable in the current seed by a new cluster variable according to a particular set of rules. In a cluster algebra of finite type, this process "folds" to produce a finite set of seeds, each containing the same number n of cluster variables (along with some extra information needed to perform mutations). These n-element collections of cluster variables, called *clusters*, are the maximal faces of the (simplicial) cluster complex.

In Theorem 1.13, we identify this complex as the dual simplicial complex of a *generalized associahedron* associated with the corresponding root system. These complexes (indeed, convex polytopes [7]) were introduced in [11] in relation to our proof of Zamolodchikov's periodicity conjecture for algebraic Y-systems. A generalized associahedron of type A is the usual associahedron, or the Stasheff polytope [25]; in types B and C, it is the cyclohedron, or the Bott-Taubes polytope [5, 24].

One of the crucial steps in our proof of the classification theorem is a new combinatorial characterization of Dynkin diagrams. In Sect. 8, we introduce an equivalence relation, called *mutation equivalence*, on finite directed graphs with weighted edges. We then prove that a connected graph Γ is mutation equivalent to an orientation of a Dynkin diagram if and only if every graph that is mutation equivalent to Γ has all edge weights ≤ 3. We do not see a direct way to relate this description to any previously known characterization of the Dynkin diagrams.

We already mentioned that the initial motivation for the study of cluster algebras came from representation theory; see [27] for a more detailed discussion of the representation-theoretic context. Another source of inspiration was Lusztig's theory of total positivity in semisimple Lie groups, which was further developed in a series of papers of the present authors and their collaborators (see, e.g., [19, 8] and references therein). The mutation mechanism used for creating new cluster variables from the initial "seed" was designed to ensure that in concrete geometric realizations, these variables become regular functions taking positive values on all totally positive elements of a variety in question, a property that the elements of the dual canonical basis are known to possess [18].

Following the foundational paper [9], several unexpected connections and appearances of cluster algebras have been discovered and explored. They included: Laurent phenomena in number theory and combinatorics [10], Y-systems and thermodynamic Bethe ansatz [11], quiver representations [21], and Poisson geometry [12].

While this text belongs to an ongoing series of papers devoted to cluster algebras and related topics, it is designed to be read independently of any other publications on the subject. Thus, all definitions and results from [9, 11, 7] that we need are presented "from scratch", in the form most suitable for our current purposes. In particular, the core concept of a normalized cluster algebra [9] is defined anew in Sect. 1.2, while Sect. 3 provides the relevant background on generalized associahedra [11, 7].

The main new results (Theorems 1.4–1.13) are stated in Sects. 1.3–1.5. The organization of the rest of the paper is outlined in Sect. 1.6.

1.2. Basic definitions. We start with the definition of a (normalized) cluster algebra $\mathcal{A}$ (cf. [9, Sects. 2 and 5]). This is a commutative ring embedded in an ambient field $\mathcal{F}$ defined as follows. Let $(\mathbb{P}, \oplus, \cdot)$ be a *semifield*, i.e., an abelian multiplicative group supplied with an *auxiliary addition* $\oplus$ which is commutative, associative, and distributive with respect to the multiplication in $\mathbb{P}$. The following example (see [9, Example 5.6]) will be of particular importance to us: let $\mathbb{P}$ be a free abelian group, written multiplicatively, with a finite set of generators p_j $(j \in J)$, and with auxiliary addition $\oplus$ given by

$$\prod_j p_j^{a_j} \oplus \prod_j p_j^{b_j} = \prod_j p_j^{\min(a_j, b_j)} . \tag{1.1}$$

We denote this semifield by $\mathrm{Trop}(p_j : j \in J)$. The multiplicative group of any semifield $\mathbb{P}$ is torsion-free [9, Sect. 5], hence its group ring $\mathbb{Z}\mathbb{P}$ is a domain. As an *ambient field* for $\mathcal{A}$, we take a field $\mathcal{F}$ isomorphic to the field of rational functions in n independent variables (here n is the *rank* of $\mathcal{A}$), with coefficients in $\mathbb{Z}\mathbb{P}$.

A *seed* in $\mathcal{F}$ is a triple $\Sigma = (\mathbf{x}, \mathbf{p}, B)$, where

- $\mathbf{x}$ is an n-element subset of $\mathcal{F}$ which is a transcendence basis over the field of fractions of $\mathbb{Z}\mathbb{P}$.
- $\mathbf{p} = (p_x^{\pm})_{x \in \mathbf{x}}$ is a $2n$-tuple of elements of $\mathbb{P}$ satisfying the *normalization condition* $p_x^+ \oplus p_x^- = 1$ for all $x \in \mathbf{x}$.
- $B = (b_{xy})_{x,y \in \mathbf{x}}$ is an $n \times n$ integer matrix with rows and columns indexed by $\mathbf{x}$, which is *sign-skew-symmetric* [9, Definition 4.1]; that is,

(1.2) for every $x, y \in \mathbf{x}$, either $b_{xy} = b_{yx} = 0$, or $b_{xy}b_{yx} < 0$.

We will need to recall the notion of *matrix mutation* [9, Definition 4.2]. Let $B = (b_{ij})$ and $B' = (b'_{ij})$ be real square matrices of the same size. We say that B' is obtained from B by a *matrix mutation* in direction k if

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k; \\ b_{ij} + \dfrac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & \text{otherwise.} \end{cases} \tag{1.3}$$

Definition 1.1. (Seed mutations) *Let* $\Sigma = (\mathbf{x}, \mathbf{p}, B)$ *be a seed in* $\mathcal{F}$*, as above, and let* $z \in \mathbf{x}$*. Define the triple* $\overline{\Sigma} = (\overline{\mathbf{x}}, \overline{\mathbf{p}}, \overline{B})$ *as follows:*

- $\overline{\mathbf{x}} = \mathbf{x} - \{z\} \cup \{\overline{z}\}$*, where* $\overline{z} \in \mathcal{F}$ *is determined by the* exchange relation

$$z\,\overline{z} = p_z^+ \prod_{\substack{x \in \mathbf{x} \\ b_{xz} > 0}} x^{b_{xz}} + p_z^- \prod_{\substack{x \in \mathbf{x} \\ b_{xz} < 0}} x^{-b_{xz}} \tag{1.4}$$

- *the* $2n$*-tuple* $\overline{\mathbf{p}} = (\overline{p}_x^{\pm})_{x \in \overline{\mathbf{x}}}$ *is uniquely determined by the normalization conditions* $\overline{p}_x^+ \oplus \overline{p}_x^- = 1$ *together with*

$$\overline{p}_x^+ / \overline{p}_x^- = \begin{cases} p_z^- / p_z^+ & \text{if } x = \overline{z}; \\ \left(p_z^+\right)^{b_{zx}} p_x^+ / p_x^- & \text{if } b_{zx} \geq 0; \\ \left(p_z^-\right)^{b_{zx}} p_x^+ / p_x^- & \text{if } b_{zx} \leq 0. \end{cases} \tag{1.5}$$

- *the matrix* $\overline{B}$ *is obtained from* B *by applying the matrix mutation in direction* z *and then relabeling one row and one column by replacing* z *by* $\overline{z}$*.*

If the triple $\overline{\Sigma}$ *is again a seed in* $\mathcal{F}$ *(i.e., if the matrix* $\overline{B}$ *is sign-skew-symmetric), then we say that* Σ *admits a* mutation *in the direction* z *that results in* $\overline{\Sigma}$*.*

We note that the exchange relation (1.4) is a reformulation of [9, (2.2),(4.2)], while the rule (1.5) is a rewrite of [9, (5.4), (5.5)]. The elements $\overline{p}_x^{\pm}$ are determined by (1.5) uniquely since $p \oplus q = 1$ and $p/q = u$ imply $p = u/(1 \oplus u)$ and $q = 1/(1 \oplus u)$. In particular, the first case in (1.5) yields $\overline{p}_{\overline{z}}^{\pm} = p_z^{\mp}$.

It is easy to check that the mutation of $\overline{\Sigma}$ in direction $\overline{z}$ recovers Σ.

Definition 1.2. (Normalized cluster algebra) *Let* $\mathcal{S}$ *be a set of seeds in* $\mathcal{F}$ *with the following properties:*

- *every seed* $\Sigma \in \mathcal{S}$ *admits mutations in all* n *conceivable directions, and the results of all these mutations belong to* $\mathcal{S}$*;*

- *any two seeds in $\mathcal{S}$ are obtained from each other by a sequence of mutations.*

The sets $\mathbf{x}$*, for* $\Sigma = (\mathbf{x}, \mathbf{p}, B) \in \mathcal{S}$*, are called* clusters; *their elements are the* cluster variables*; the set of all cluster variables is denoted by* $\mathcal{X}$*. The set of all elements* $p_x^\pm \in \mathbf{p}$*, for all seeds* $\Sigma = (\mathbf{x}, \mathbf{p}, B) \in \mathcal{S}$*, is denoted by* $\mathcal{P}$*. The* ground ring $\mathbb{Z}[\mathcal{P}]$ *is the subring of* $\mathcal{F}$ *generated by* $\mathcal{P}$*. The (normalized)* cluster algebra $\mathcal{A} = \mathcal{A}(\mathcal{S})$ *is the* $\mathbb{Z}[\mathcal{P}]$*-subalgebra of* $\mathcal{F}$ *generated by* $\mathcal{X}$*. The* exchange graph *of* $\mathcal{A}(\mathcal{S})$ *is the* n*-regular graph whose vertices are labeled by the seeds in* $\mathcal{S}$*, and whose edges correspond to mutations. (This is easily seen to be equivalent to [9, Definition 7.4].)*

Definition 1.2 is a bit more restrictive than the one given in [9], where we allowed to use any subring with unit in $\mathbb{ZP}$ containing $\mathcal{P}$ as a ground ring. Some concrete examples of cluster algebras will be given in Sect. 12.2 below.

Remark 1.3. There is an involution $\Sigma \mapsto \Sigma^\vee$ on the set of seeds in $\mathcal{F}$ acting by $(\mathbf{x}, \mathbf{p}, B) \mapsto (\mathbf{x}, \mathbf{p}^\vee, -B)$, where $(p_x^\vee)^\pm = p_x^\mp$. An easy check show that this involution commutes with seed mutations. Therefore, if a collection of seeds $\mathcal{S}$ satisfies the conditions in Definition 1.2, then so does the collection $\mathcal{S}^\vee$. The corresponding cluster algebras $\mathcal{A}(\mathcal{S})$ and $\mathcal{A}(\mathcal{S}^\vee)$ are canonically identified with each other. (This is a reformulation of [9, (2.8)].)

Two cluster algebras $\mathcal{A}(\mathcal{S}) \subset \mathcal{F}$ and $\mathcal{A}(\mathcal{S}') \subset \mathcal{F}'$ over the same semifield $\mathbb{P}$ are called *strongly isomorphic* if there exists a $\mathbb{ZP}$-algebra isomorphism $\mathcal{F} \to \mathcal{F}'$ that transports some (equivalently, any) seed in $\mathcal{S}$ into a seed in $\mathcal{S}'$, thus inducing a bijection $\mathcal{S} \to \mathcal{S}'$ and an algebra isomorphism $\mathcal{A}(\mathcal{S}) \to \mathcal{A}(\mathcal{S}')$.

The set of seeds $\mathcal{S}$ for a cluster algebra $\mathcal{A} = \mathcal{A}(\mathcal{S})$ (hence the algebra itself) is uniquely determined by any single seed $\Sigma = (\mathbf{x}, \mathbf{p}, B) \in \mathcal{S}$. Thus, $\mathcal{A}$ is determined by B and $\mathbf{p}$ up to a strong isomorphism, justifying the notation $\mathcal{A} = \mathcal{A}(B, \mathbf{p})$. In general, an $n \times n$ matrix B and a $2n$-tuple $\mathbf{p}$ satisfying the normalization conditions define a cluster algebra $\mathcal{A}(B, \mathbf{p})$ if and only if any matrix obtained from B by a sequence of mutations is sign-skew-symmetric. This condition is in particular satisfied whenever B is *skew-symmetrizable* [9, Definition 4.4], i.e., there exists a diagonal matrix D with positive diagonal entries such that DB is skew-symmetric. Indeed, matrix mutations preserve skew-symmetrizability [9, Proposition 4.5], and any skew-symmetrizable matrix is sign-skew-symmetric.

Every cluster algebra over a fixed semifield $\mathbb{P}$ belongs to a *series* $\mathcal{A}(B, -)$ consisting of all cluster algebras of the form $\mathcal{A}(B, \mathbf{p})$, where B is fixed, and $\mathbf{p}$ is allowed to vary. Two series $\mathcal{A}(B, -)$ and $\mathcal{A}(B', -)$ are *strongly isomorphic* if there is a bijection sending each cluster algebra $\mathcal{A}(B, \mathbf{p})$ to a strongly isomorphic cluster algebra $\mathcal{A}(B', \mathbf{p}')$. (This amounts to requiring that B and B' can be obtained from each other by a sequence of matrix mutations, modulo simultaneous relabeling of rows and columns.)

1.3. Finite type classification. A cluster algebra $\mathcal{A}(\mathcal{S})$ is said to be of *finite type* if the set of seeds $\mathcal{S}$ is finite.

Let $B = (b_{ij})$ be an integer square matrix. Its *Cartan counterpart* is a generalized Cartan matrix $A = A(B) = (a_{ij})$ of the same size defined by

$$a_{ij} = \begin{cases} 2 & \text{if } i = j; \\ -|b_{ij}| & \text{if } i \neq j. \end{cases} \tag{1.6}$$

The following classification theorem is our main result.

Theorem 1.4. *All cluster algebras in any series $\mathcal{A}(B, -)$ are simultaneously of finite or infinite type. There is a canonical bijection between the Cartan matrices of finite type and the strong isomorphism classes of series of cluster algebras of finite type. Under this bijection, a Cartan matrix A of finite type corresponds to the series $\mathcal{A}(B, -)$, where B is an arbitrary sign-skew-symmetric matrix with $A(B) = A$.*

We note that in the last claim of Theorem 1.4, the series $\mathcal{A}(B, -)$ is well defined since A is symmetrizable and therefore B must be skew-symmetrizable.

By Theorem 1.4, each cluster algebra of finite type has a well-defined *type* (e.g., $A_n, B_n, \ldots$), mirroring the Cartan-Killing classification.

We prove Theorem 1.4 by splitting it into the following three statements (Theorems 1.5–1.7 below).

Theorem 1.5. *Suppose that*

(1.7) *A is a Cartan matrix of finite type;*

(1.8) *$B_\circ = (b_{ij})$ is a sign-skew-symmetric matrix such that $A = A(B_\circ)$ and $b_{ij}b_{ik} \geq 0$ for all i, j, k;*

(1.9) *$\mathbf{p}_\circ$ is a $2n$-tuple of elements in $\mathbb{P}$ satisfying the normalization conditions.*

Then $\mathcal{A}(B_\circ, \mathbf{p}_\circ)$ is a cluster algebra of finite type.

It is easy to see that for any Cartan matrix A of finite type, there is a matrix $B_\circ$ satisfying (1.8). Indeed, the sign-skew-symmetric matrices B with $A(B) = A$ are in a bijection with orientations of the Coxeter graph of A (recall that this graph has I as the set of vertices, with i and j joined by an edge whenever $a_{ij} \neq 0$): under this bijection, $b_{ij} > 0$ if and only if the edge $\{i, j\}$ is oriented from i to j. Condition (1.8) means that $B_\circ$ corresponds to an orientation such that every vertex is a source or a sink; since the Coxeter graph is a tree, hence a bipartite graph, such an orientation exists.

Theorem 1.6. *Any cluster algebra $\mathcal{A}$ of finite type is strongly isomorphic to a cluster algebra $\mathcal{A}(B_\circ, \mathbf{p}_\circ)$ for some data of the form* (1.7)–(1.9).

Theorem 1.7. *Let B and B' be sign-skew-symmetric matrices such that $A(B)$ and $A(B')$ are Cartan matrices of finite type. Then the series $\mathcal{A}(B,-)$ and $\mathcal{A}(B',-)$ are strongly isomorphic if and only if $A(B)$ and $A(B')$ are of the same Cartan-Killing type.*

In the process of proving these theorems, we obtain the following characterizations of the cluster algebras of finite type.

Theorem 1.8. *For a cluster algebra $\mathcal{A}$, the following are equivalent:*

(i) *$\mathcal{A}$ is of finite type;*
(ii) *the set $\mathcal{X}$ of all cluster variables is finite;*
(iii) *for every seed $(\mathbf{x}, \mathbf{p}, B)$ in $\mathcal{A}$, the entries of the matrix $B = (b_{xy})$ satisfy the inequalities $|b_{xy}b_{yx}| \le 3$, for all $x, y \in \mathbf{x}$.*
(iv) *$\mathcal{A} = \mathcal{A}(B_\circ, \mathbf{p}_\circ)$ for some data of the form* (1.7)–(1.9).

The equivalence (i) $\Longleftrightarrow$ (iv) in Theorem 1.8 is tantamount to Theorems 1.5–1.6.

1.4. Cluster variables in the finite type. The techniques in our proof of Theorem 1.5 allow us to enunciate the basic properties of cluster algebras of finite type. We begin by providing an explicit description of the set of cluster variables in terms of the corresponding root system.

For the remainder of Sect. 1, $A = (a_{ij})_{i,j\in I}$ is a Cartan matrix of finite type and $\mathcal{A} = \mathcal{A}(B_\circ, \mathbf{p}_\circ)$ a cluster algebra (of finite type) related to A as in Theorem 1.5. Let Φ be the root system associated with A, with the set of simple roots $\Pi = \{\alpha_i : i \in I\}$ and the set of positive roots $\Phi_{>0}$. (Our convention on the correspondence between A and Φ is that the simple reflections s_i act on simple roots by $s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$.) Let $\mathbf{x}_\circ = \{x_i : i \in I\}$ be the cluster for the initial seed $(\mathbf{x}_\circ, \mathbf{p}_\circ, B_\circ)$. (By an abuse of notation, we label the rows and columns of $B_\circ$ by the elements of I rather than by the variables x_i, for $i \in I$.) We will use the shorthand $x^\alpha = \prod_{i\in I} x_i^{a_i}$ for any vector $\alpha = \sum_{i\in I} a_i\alpha_i$ in the root lattice.

The following result shows that the cluster variables of $\mathcal{A}$ are naturally parameterized by the set $\Phi_{\ge -1} = \Phi_{>0} \cup (-\Pi)$ of *almost positive roots.*

Theorem 1.9. *There is a unique bijection $\alpha \mapsto x[\alpha]$ between the almost positive roots in Φ and the cluster variables in $\mathcal{A}$ such that, for any $\alpha \in \Phi_{\ge -1}$, the cluster variable $x[\alpha]$ is expressed in terms of the initial cluster $\mathbf{x}_\circ = \{x_i : i \in I\}$ as*

$$x[\alpha] = \frac{P_\alpha(\mathbf{x}_\circ)}{x^\alpha}, \tag{1.10}$$

where P_α is a polynomial over $\mathbb{ZP}$ with nonzero constant term. Under this bijection, $x[-\alpha_i] = x_i$.

Formula (1.10) is an example of the *Laurent phenomenon* established in [9] for arbitrary cluster algebras: every cluster variable can be written as a Laurent polynomial in the variables of an arbitrary fixed cluster and the elements of $\mathbb{P}$. In [9], we conjectured that the coefficients of these Laurent polynomials are always nonnegative. Our next result establishes this conjecture (indeed, strengthens it) in the special case of the distinguished cluster $\mathbf{x}_\circ$ in a cluster algebra of finite type.

Theorem 1.10. *Every coefficient of each polynomial P_α (see* (1.10)*) can be written as a polynomial in the elements of $\mathcal{P}$ (see Definition 1.2) with positive integer coefficients.*

1.5. Cluster complexes. We next focus on the combinatorics of clusters. As before, $\mathcal{A}$ is a cluster algebra of finite type associated with a root system Φ.

Theorem 1.11. *The exact form of each exchange relation* (1.4) *in $\mathcal{A}$ (that is, the cluster variables, exponents, and coefficients appearing in the right-hand side) depends only on the ordered pair $(z, \overline{z})$ of cluster variables, and not on the particular choice of clusters (or seeds) containing them.*

In fact, we do more: we describe in concrete root-theoretic terms all pairs (β, β') of almost positive roots such that the product $x[\beta]x[\beta']$ appears as a left-hand side of an exchange relation, and for every such pair, we describe the exponents appearing on the right. See Definition 4.2 and formula (5.1).

Theorem 1.12. *Every seed $(\mathbf{x}, \mathbf{p}, B)$ in $\mathcal{A}$ is uniquely determined by its cluster $\mathbf{x}$. For any cluster $\mathbf{x}$ and any $x \in \mathbf{x}$, there is a unique cluster $\mathbf{x}'$ with $\mathbf{x} \cap \mathbf{x}' = \mathbf{x} - \{x\}$.*

We conjecture that the requirement of finite type in Theorems 1.11 and 1.12 can be dropped; that is, *any* cluster algebra conjecturally has these properties.

We define the *cluster complex* $\Delta(\mathcal{A})$ as the simplicial complex whose ground set is $\mathcal{X}$ (the set of all cluster variables) and whose maximal simplices are the clusters. By Theorem 1.12, the cluster complex encodes the combinatorics of seed mutations. Thus, the dual graph of $\Delta(\mathcal{A})$ is precisely the exchange graph of $\mathcal{A}$.

Our next result identifies the cluster complex $\Delta(\mathcal{A})$ with the dual complex $\Delta(\Phi)$ of the *generalized associahedron* of the corresponding type. The simplicial complexes $\Delta(\Phi)$ were introduced and studied in [11]; see also [7] and Sect. 3 below.

Theorem 1.13. *Under the bijection $\Phi_{\geq -1} \to \mathcal{X}$ of Theorem 1.9, the cluster complex $\Delta(\mathcal{A})$ is identified with the simplicial complex $\Delta(\Phi)$. In particular, the cluster complex does not depend on the coefficient semifield $\mathbb{P}$, or on the choice of the coefficients $\mathbf{p}_\circ$ in the initial seed.*

1.6. Organization of the paper. The bulk of the paper is devoted to the proofs of Theorems 1.4–1.8. We already noted that Theorem 1.4 follows from Theorems 1.5–1.7, and that the implications (iv) $\Longrightarrow$ (i) and (i) $\Longrightarrow$ (iv) in Theorem 1.8 are essentially Theorems 1.5 and 1.6, respectively. Furthermore, (i) $\Longrightarrow$ (ii) is trivial, while (ii) $\Longrightarrow$ (iii) follows from [9, Theorem 6.1]. Thus, we need to prove the following:

- Theorem 1.5;
- Theorem 1.6 via the implication (iii) $\Longrightarrow$ (iv) in Theorem 1.8;
- Theorem 1.7.

Figure 1 shows the logical dependences between these proofs, and the sections containing them. Theorems 1.9–1.13, which only rely on Theorem 1.5, are proved in Sects. 5– 6, following the completion of the proof of Theorem 1.5.

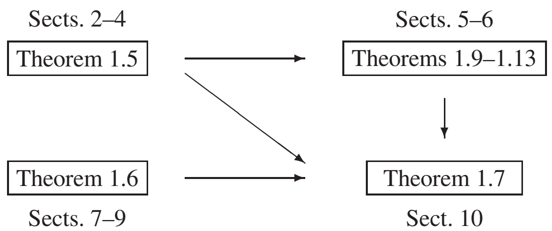


Fig. 1. Logical dependences among the proofs of Theorems 1.5–1.13

The concluding Sect. 12 provides explicit geometric realizations for some special cluster algebras of the classical types $ABCD$. These examples are based on a general criterion given in Sect. 11 for a cluster algebra to be isomorphic to a $\mathbb{Z}$-form of the coordinate ring of some algebraic variety. In particular, we show that a $\mathbb{Z}$-form of the homogeneous coordinate ring of the Grassmannian $\mathrm{Gr}_{2,m}$ ($m \geq 5$) in its Plücker embedding carries two different cluster algebra structures of types A_{m-3} and B_{m-2} , respectively.

2. Cluster algebras via pseudomanifolds

2.1. Pseudomanifolds and geodesic loops. This section begins our proof of Theorem 1.5. Its main result (Proposition 2.3) provides sufficient conditions ensuring that a cluster algebra that arises from a particular combinatorial construction is of finite type.

The first ingredient of this construction is an $(n-1)$-dimensional pure simplicial complex Δ (finite or infinite) on the ground set Ψ. Thus, every maximal simplex in Δ is an n-element subset of Ψ (a "cluster"). A simplex of codimension 1 (i.e., an $(n-1)$-element subset of Ψ) is called a *wall*. The vertices of the *dual graph* Γ are the clusters in Δ; two clusters are connected by an edge in Γ if they share a wall.

We assume that Δ is a *pseudomanifold*, i.e.,

(2.1) every wall is contained in precisely two maximal simplices (clusters);

(2.2) the dual graph Γ is connected.

In view of (2.1), the graph Γ is n-regular, i.e., there are precisely n edges incident to every vertex $C \in \Gamma$.

Example 2.1. Let $n = 1$. Then (2.1) is saying that the empty simplex is contained in precisely two 0-dimensional simplices (points in Ψ). Thus a 0-dimensional pseudomanifold must be a disjoint union of two points (a 0-dimensional sphere). The dual graph Γ has these points as vertices, with an edge connecting them.

For $n = 2$, a 1-dimensional pseudomanifold is nothing but a 2-regular connected graph—thus, either an infinite chain or a cycle. Ditto for its dual graph.

For a non-maximal simplex $D \in \Delta$, we denote by Δ_D the *link* of D. This is the simplicial complex on the ground set $\Psi_D = \{\alpha \in \Psi - D : D \cup \{\alpha\} \in \Delta\}$ such that D' is a simplex in Δ_D if and only if $D \cup D'$ is a simplex in Δ. The link Δ_D is a pure simplicial complex of dimension $n - |D| - 1$ satisfying property (2.1) of a pseudomanifold.

We will assume that Δ satisfies the following additional condition:

(2.3) the link of every non-maximal simplex D in Δ is a pseudomanifold.

Equivalently, the dual graph Γ_D of Δ_D is connected.

Conditions (2.1)–(2.3) can be restated as saying that Δ is a (possibly infinite, simplicial) *abstract polytope* in the sense of [1] or [23]; another terminology is that Δ is a *thin, residually connected* complex (see, e.g., [2]).

We identify the graph Γ_D with an induced subgraph in Γ whose vertices are the maximal simplices in Δ that contain D. In particular, for $|D| = n-2$, the pseudomanifold Δ_D is 1-dimensional, so Γ_D is either an infinite chain or a finite cycle in Γ. In the latter case, we call Γ_D a *geodesic loop*. (This is a *geodesic* in Γ with respect to the canonical *connection* on Γ, in the sense of [4, 14].)

We assume that

(2.4) the fundamental group of Γ is generated by the geodesic loops pinned down to a fixed basepoint.

More precisely, by (2.4) we mean that the fundamental group of Γ is generated by all the loops of the form $PL\bar{P}$, where L is a geodesic loop, P is a path originating at the basepoint, and $\bar{P}$ is the inverse path to P.

Lemma 2.2. *Let Δ be the boundary complex of an n-dimensional simplicial convex polytope. Then conditions* (2.1)–(2.4) *are satisfied.*

Equivalently, conditions (2.1)–(2.4) hold if Δ is (the simplicial complex of) the normal fan of a simple n-dimensional convex polytope Δ^*.

Proof. Statements (2.1)–(2.2) are trivial. The link Δ_D of each non-maximal face D in Δ is again a simplicial polytope, implying (2.3). Specifically, Δ_D is canonically identified (see, e.g., [3, Problem VI.1.4.4]) with the dual polytope for the dual face D^* in the dual simple polytope Δ^*. Thus, the graph Γ_D is the 1-skeleton of D^*.

It remains to check (2.4). The case $n = 2$ is trivial, so let us assume that $n \geq 3$. Each geodesic loop Γ_D (for an $(n-2)$-dimensional face D) is identified with the 1-skeleton (i.e, the boundary) of the dual 2-dimensional face D^* in Δ^*. The boundary cell complex of Δ^* is spherical, hence simply connected. On the other hand, the fundamental group of this complex can be obtained as a quotient of the fundamental group of its 1-skeleton Γ by the normal subgroup generated by the boundaries of its 2-dimensional cells pinned down to a basepoint (see, e.g., [22, Theorems VII.2.1, VII.4.1]), or, equivalently, by the subgroup generated by all pinned-down geodesic loops. This proves (2.4). □

2.2. Sufficient conditions for finite type. We next describe the second ingredient of our construction. Suppose that we have a family of integer matrices $B(C) = (b_{\alpha\beta}(C))_{\alpha,\beta\in C}$, for all vertices C in Γ, satisfying the following conditions:

(2.5) all the matrices $B(C)$ are sign-skew-symmetric.

(2.6) for every edge $(C, \overline{C})$ in Γ, with $\overline{C} = C - \{\gamma\} \cup \{\overline{\gamma}\}$, the matrix $B(\overline{C})$ is obtained from $B(C)$ by a matrix mutation in direction γ followed by relabeling one row and one column by replacing γ by $\overline{\gamma}$.

We need one more assumption concerning the matrices $B(C)$, which will require a little preparation. Fix a geodesic Γ_D and associate to its every vertex $C = D \cup \{\alpha, \beta\}$ the integer $b_{\alpha\beta}(C)b_{\beta\alpha}(C)$. It is trivial to check, using (2.6), that this integer depends only on D, not on the particular choice of α and β. We say that Γ_D is of *finite type* if $b_{\alpha\beta}(C)b_{\beta\alpha}(C) \in \{0, -1, -2, -3\}$ for some (equivalently, any) vertex $C = D \cup \{\alpha, \beta\}$ on Γ_D. If this is the case, then we associate to Γ_D the *Coxeter number* $h \in \mathbb{Z}_{>0}$ defined by

$$2\cos(\pi/h) = \sqrt{|b_{\alpha\beta}(C)b_{\beta\alpha}(C)|}\,,$$

or, equivalently, by the table

$\lvert b_{\alpha\beta}(C)b_{\beta\alpha}(C)\rvert$	0	1	2	3
$h(C, x, y)$	2	3	4	6

.

Our last condition is:

(2.7) every geodesic loop in Γ is of finite type, and has length $h + 2$, where h is the corresponding Coxeter number.

Proposition 2.3. *Assume that a simplicial complex Δ and a family of matrices $(B(C))$ satisfy the assumptions* (2.1)–(2.7) *above. Let $B = B(C)$ for some vertex C, and let $\mathcal{A} = \mathcal{A}(B, \mathbf{p})$ be the cluster algebra associated with B and some coefficient tuple $\mathbf{p}$. There exists a surjection from the set of vertices of Γ onto the set of all seeds for $\mathcal{A}$. In particular, if Δ is finite, then $\mathcal{A}$ is of finite type.*

We will prove this proposition by showing that, whether Δ is finite or infinite, its dual graph Γ is always a covering graph for the exchange graph of $\mathcal{A}(B, \mathbf{p})$. To formulate this more precisely, we need some preparation.

Let C be a vertex of Γ. A *seed attachment* at C consists of a choice of a seed $\Sigma = (\mathbf{x}, \mathbf{p}, B)$ and a bijection $\alpha \mapsto x[C, \alpha]$ between C and $\mathbf{x}$ identifying the matrices $B(C)$ and B, so that $b_{\alpha\beta}(C) = b_{x[C,\alpha],x[C,\beta]}$. The *transport* of a seed attachment along an edge $(C, \overline{C})$ with $\overline{C} = C - \{\gamma\} \cup \{\overline{\gamma}\}$ is defined as follows: the seed $\overline{\Sigma} = (\overline{\mathbf{x}}, \overline{\mathbf{p}}, \overline{B})$ attached to $\overline{C}$ is obtained from Σ by the mutation in direction $x[C, \gamma]$, and the corresponding bijection $\overline{C} \to \overline{\mathbf{x}}$ is uniquely determined by $x[\overline{C}, \alpha] = x[C, \alpha]$ for all $\alpha \in C \cap \overline{C}$. (The remaining cluster variable $x[\overline{C}, \overline{\gamma}]$ is obtained from $x[C, \gamma]$ by the corresponding exchange relation (1.4).) We note that transporting the resulting seed attachment backwards from $\overline{C}$ to C recovers the original seed attachment.

Proposition 2.3 is an immediate consequence of the following lemma.

Lemma 2.4. *Let Δ and $(B(C))$ satisfy* (2.1)–(2.7). *Suppose we are given a vertex $C_\circ$ in Γ together with a seed attachment involving a seed $\Sigma_\circ$ for a cluster algebra $\mathcal{A}$.*

1. *The given seed attachment at $C_\circ$ extends uniquely to a family of seed attachments at all vertices in Γ such that, for every edge $(C, \overline{C})$, the seed attachment at $\overline{C}$ is obtained from that at C by transport along $(C, \overline{C})$.*
2. *Let $\Sigma(C)$ denote the seed attached to a vertex C. The map $C \mapsto \Sigma(C)$ is a surjection onto the set of all seeds for $\mathcal{A}$.*
3. *For every vertex C and every $\alpha \in C$, the cluster variable $x[C, \alpha]$ attached to α at C depends only on α (so can be denoted by $x[\alpha]$).*
4. *The map $\alpha \mapsto x[\alpha]$ is a surjection from the ground set Ψ onto the set of all cluster variables for $\mathcal{A}$.*

Proof. 1. Since Γ is connected, we can transport the initial seed attachment at $C_\circ$ to an arbitrary vertex C along a path from $C_\circ$ to C. We need to show that the resulting seed attachment at C is independent of the choice of a path. For that, it suffices to prove that transporting a seed attachment along a loop in Γ brings it back unchanged. By (2.4), it is enough to show this for the geodesic loops. Then the claim follows from (2.7) and [9, Theorem 7.7].

2. Take an arbitrary seed Σ for $\mathcal{A}$. By Definition 1.2, Σ can be obtained from the initial seed $\Sigma_\circ$ by a sequence of mutations. This sequence is uniquely lifted to a path $(C_\circ, \ldots, C)$ in Γ such that transporting the initial

seed attachment at $C_\circ$ along the edges of this path produces the chosen sequence of mutations. Hence $\Sigma(C) = \Sigma$, as desired.

3. Let $\alpha \in \Psi$, and let C and C' be two vertices of Γ such that $\alpha \in C \cap C'$. By (2.3), C and C' can be joined by a path $(C_1 = C, C_2, \ldots, C_\ell = C')$ such that $\alpha \in C_i$ for all i. Hence $x[C_1, \alpha] = x[C_2, \alpha] = \cdots = x[C_\ell, \alpha]$, as needed.

4. Follows from Part 2. □

Remark 2.5. Parts 1 and 2 in Lemma 2.4 imply that the map $C \mapsto \Sigma(C)$ induces a covering of the exchange graph of $\mathcal{A}$ by the graph Γ. If, in addition, the map $\alpha \mapsto x[\alpha]$ in Lemma 2.4 is a bijection, then the map $C \mapsto \Sigma(C)$ is also a bijection. Thus, the latter map establishes an isomorphism between Γ and the exchange graph of $\mathcal{A}$, and between Δ and the cluster complex of $\mathcal{A}$.

3. Generalized associahedra

This section contains an exposition of the results in [11] and [7] that will be used later in our proof of Theorem 1.5.

Let $A = (a_{ij})_{i,j \in I}$ be an indecomposable Cartan matrix of finite type, and Φ the corresponding irreducible root system of rank $n = |I|$. We retain the notation introduced in Sect. 1.4. In particular, $\Phi_{\geq -1} = \Phi_{>0} \cup (-\Pi)$ denotes the set of almost positive roots.

The Coxeter graph associated to Φ is a tree; recall that this graph has the index set I as the set of vertices, with i and j joined by an edge whenever $a_{ij} \neq 0$. In particular, the Coxeter graph is bipartite; the two parts $I_+, I_- \subset I$ are determined uniquely up to renaming. The *sign function* $\varepsilon : I \to \{+, -\}$ is defined by

$$\varepsilon(i) = \begin{cases} + & \text{if } i \in I_+ ; \\ - & \text{if } i \in I_- . \end{cases} \tag{3.1}$$

Let $Q = \mathbb{Z}\Pi$ denote the root lattice, and $Q_\mathbb{R}$ the ambient real vector space. For $\gamma \in Q_\mathbb{R}$, we denote by $[\gamma : \alpha_i]$ the coefficient of α_i in the expansion of γ in the basis Π. Let τ_+ and τ_- denote the piecewise-linear automorphisms of $Q_\mathbb{R}$ given by

$$[\tau_\varepsilon \gamma : \alpha_i] = \begin{cases} -[\gamma : \alpha_i] - \sum_{j \neq i} a_{ij} \max([\gamma : \alpha_j], 0) & \text{if } i \in I_\varepsilon; \\ [\gamma : \alpha_i] & \text{otherwise.} \end{cases} \tag{3.2}$$

It is easy to see that each of τ_+ and τ_- is an involution that preserves the set $\Phi_{\geq -1}$. More specifically, the action of τ_+ and τ_- on $\Phi_{\geq -1}$ can be described as follows:

$$\tau_\varepsilon(\alpha) = \begin{cases} \alpha & \text{if } \alpha = -\alpha_i,\ i \in I_{-\varepsilon}; \\ \left(\prod_{i \in I_\varepsilon} s_i\right)(\alpha) & \text{otherwise.} \end{cases} \tag{3.3}$$

(The product of simple reflections $\prod_{i\in I_\varepsilon} s_i$ is well-defined since its factors commute). To illustrate, consider the type A_2, with $I_+ = \{1\}$ and $I_- = \{2\}$. Then

$$\begin{array}{ccccccccc} -\alpha_1 & \overset{\tau_+}{\longleftrightarrow} & \alpha_1 & \overset{\tau_-}{\longleftrightarrow} & \alpha_1+\alpha_2 & \overset{\tau_+}{\longleftrightarrow} & \alpha_2 & \overset{\tau_-}{\longleftrightarrow} & -\alpha_2\,. \\ \circlearrowright & & & & & & & & \circlearrowright \\ \tau_- & & & & & & & & \tau_+ \end{array} \tag{3.4}$$

We denote by $\langle \tau_-, \tau_+\rangle$ the group generated by τ_- and τ_+ .

The Weyl group of Φ is denoted by W, its longest element by $w_\circ$, and its Coxeter number by h.

Theorem 3.1. [11, Theorems 1.2, 2.6]

1. *The order of $\tau_-\tau_+$ is equal to $(h+2)/2$ if $w_\circ = -1$, and to $h+2$ otherwise. Accordingly, $\langle \tau_-, \tau_+\rangle$ is a dihedral group of order $(h+2)$ or $2(h+2)$.*
2. *The correspondence $\Omega \mapsto \Omega\cap(-\Pi)$ is a bijection between the $\langle \tau_-, \tau_+\rangle$-orbits in $\Phi_{\geq -1}$ and the $\langle -w_\circ\rangle$-orbits in $(-\Pi)$.*

We note that Theorem 3.1 is stronger than [11, Theorem 2.6], since in the latter, τ_- and τ_+ are treated as permutations of the set $\Phi_{\geq -1}$, rather than as transformations of the entire space $\mathbb{Q}_\mathbb{R}$. This stronger version follows from [11, Theorem 1.2] by "tropical specialization" (see [11, (1.8)]).

According to [11, Sect. 3.1], there exists a unique function $(\alpha, \beta) \mapsto (\alpha\|\beta)$ on $\Phi_{\geq -1} \times \Phi_{\geq -1}$ with nonnegative integer values, called the *compatibility degree,* such that

$$(-\alpha_i\|\alpha) = \max\,([\alpha : \alpha_i], 0) \tag{3.5}$$

for any $i \in I$ and $\alpha \in \Phi_{\geq -1}$, and

$$(\tau_\varepsilon\alpha\|\tau_\varepsilon\beta) = (\alpha\|\beta) \tag{3.6}$$

for any $\alpha, \beta \in \Phi_{\geq -1}$ and any sign ε. We say that α and β are *compatible* if $(\alpha\|\beta) = 0$. (This is equivalent to $(\beta\|\alpha) = 0$ by [11, Proposition 3.3, Part 2].)

Let $\Delta(\Phi)$ be the simplicial complex on the ground set $\Phi_{\geq -1}$ whose simplices are the subsets of mutually compatible roots. As in Sect. 2 above, the maximal simplices of $\Delta(\Phi)$ are called *clusters*.

Theorem 3.2. [11, Theorems 1.8, 1.10] [7, Theorem 1.4]

1. *Each cluster in $\Delta(\Phi)$ is a $\mathbb{Z}$-basis of the root lattice Q; in particular, all clusters are of the same size n.*
2. *The cones spanned by the simplices in $\Delta(\Phi)$ form a complete simplicial fan in $Q_\mathbb{R}$.*
3. *This simplicial fan is the normal fan of a simple n-dimensional convex polytope, the* generalized associahedron *of the corresponding type.*

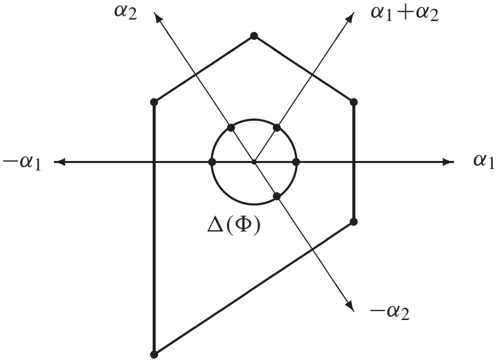


Fig. 2. The complex $\Delta(\Phi)$ and the corresponding polytope in type A_2

Generalized associahedra of types ABC are: in type A, the Stasheff polytope, or ordinary *associahedron* (see, e.g., [25, 17] or [13, Chap. 7]); in types B and C, the Bott-Taubes polytope, or *cyclohedron* (see [5, 20, 24]). Explicit combinatorial descriptions of generalized associahedra of types $ABCD$ in relation to the root system framework are discussed in [11, 7]; see also Sect. 12 below.

Proposition 3.3. [11, Theorem 3.11] *Every vector $\gamma \in Q$ has a unique* cluster expansion, *that is, γ can be expressed uniquely as a nonnegative linear combination of mutually compatible roots from $\Phi_{\geq -1}$ (the* cluster components *of γ).*

Proposition 3.4. [7, Proposition 1.13] *Let $[\gamma : \alpha]_{\text{clus}}$ denote the coefficient of an almost positive root α in the cluster expansion of a vector $\gamma \in Q$. Then we have $[\sigma(\gamma) : \sigma(\alpha)]_{\text{clus}} = [\gamma : \alpha]_{\text{clus}}$ for $\sigma \in \langle \tau_+, \tau_- \rangle$.*

We call two roots $\beta, \beta' \in \Phi_{\geq -1}$ *exchangeable* if $(\beta \| \beta') = (\beta' \| \beta) = 1$. The choice of terminology is motivated by the following proposition.

Proposition 3.5. [7, Lemma 2.2] *Let C and $C' = C - \{\beta\} \cup \{\beta'\}$ be two adjacent clusters. Then the roots β and β' are exchangeable.*

The converse of Proposition 3.5 is also true: see Corollary 4.4 below.

Proposition 3.6. [7, Theorem 1.14] *If $n > 1$ and $\beta, \beta' \in \Phi_{\geq -1}$ are exchangeable, then the set*

$$\{\sigma^{-1}(\sigma(\beta) + \sigma(\beta')) : \sigma \in \langle \tau_+, \tau_- \rangle\}$$

consists of two elements of Q, one of which is $\beta + \beta'$, and the other will be denoted by $\beta \uplus \beta'$. In the special case where $\beta' = -\alpha_j$, $j \in I$, we have

$$(-\alpha_j) \uplus \beta = \tau_{-\varepsilon(j)}(-\alpha_j + \tau_{-\varepsilon(j)}(\beta)) = \beta - \alpha_j + \textstyle\sum_{i \neq j} a_{ij} \alpha_i \,. \qquad (3.7)$$

A precise rule for deciding whether an element $\sigma^{-1}(\sigma(\beta) + \sigma(\beta'))$ is equal to $\beta + \beta'$ or $\beta \uplus \beta'$ is given in Lemma 4.7 below.

Remark 3.7. If $n = 1$, i.e., Φ is of type A_1 with a unique simple root α_1, then $\{\beta, \beta'\} = \{-\alpha_1, \alpha_1\}$, and the group $\langle \tau_+, \tau_- \rangle$ is just the Weyl group $W = \langle s_1 \rangle$. Thus, in this case, the set in Proposition 3.6 consists of a single element $\beta + \beta' = 0$. It is then natural to set $\beta \uplus \beta' = 0$ as well.

Remark 3.8. All results in this section extend in an obvious way to the case of an arbitrary Cartan matrix of finite type (not necessarily indecomposable). In that generality, Φ is a disjoint union of irreducible root systems $\Phi^{(1)}, \dots, \Phi^{(m)}$, and the clusters for Φ are the unions $C_1 \cup \cdots \cup C_m$, where each C_k is a cluster for $\Phi^{(k)}$.

4. Proof of Theorem 1.5

In this section, we complete the proof of Theorem 1.5. The plan is as follows. Without loss of generality, we can assume that the Cartan matrix A is indecomposable, so the corresponding root system Φ is irreducible. By Theorem 3.2 and Lemma 2.2, conditions (2.1)–(2.4) are satisfied. By Proposition 2.3, to prove Theorem 1.5 it suffices to define a family of matrices $B(C)$, for each cluster C in $\Delta(\Phi)$, such that (2.5)–(2.7) hold, together with

(4.1) for some cluster $C_\circ$, the matrix $B_\circ = B(C_\circ)$ is as in (1.8).

Defining the matrices $B(C)$ requires a little preparation. Throughout this section, all roots are presumed to belong to the set $\Phi_{\geq -1}$.

Lemma 4.1. *There exists a sign function $(\beta, \beta') \mapsto \varepsilon(\beta, \beta') \in \{\pm 1\}$ on pairs of exchangeable roots, uniquely determined by the following properties:*

(4.2) $\varepsilon(-\alpha_j, \beta') = -\varepsilon(j)$;

(4.3) $\varepsilon(\tau\beta, \tau\beta') = -\varepsilon(\beta, \beta')$ *for* $\tau \in \{\tau_+, \tau_-\}$ *and* β, $\beta' \notin \{-\alpha_j : \tau(-\alpha_j) = -\alpha_j\}$.

Moreover, this function is skew-symmetric:

(4.4) $$\varepsilon(\beta', \beta) = -\varepsilon(\beta, \beta').$$

Proof. The uniqueness of $\varepsilon(\beta, \beta')$ is an easy consequence of Theorem 3.1, Part 2. Let us prove the existence. For a root $\beta \in \Phi_{\geq -1}$ and a sign ε, let $k_\varepsilon(\beta)$ denote the smallest nonnegative integer k such that $\tau_\varepsilon^{(k+1)}(\beta) = \tau_\varepsilon^{(k)}(\beta) \in -\Pi$, where we abbreviate

$$\tau_\varepsilon^{(k)} = \underbrace{\tau_\pm \cdots \tau_{-\varepsilon}\tau_\varepsilon}_{k \text{ factors}}$$

(cf. Theorem 3.1). In view of [7, Theorem 3.1], we always have

$$k_+(\beta) + k_-(\beta) = h + 1; \tag{4.5}$$

in particular, $k_{\varepsilon(j)}(-\alpha_j) = h + 1$ and $k_{-\varepsilon(j)}(-\alpha_j) = 0$. It follows from (4.5) that if β and β' are incompatible (in particular, exchangeable), then $k_\varepsilon(\beta) < k_\varepsilon(\beta')$ for precisely one choice of a sign ε. Let us define $\varepsilon(\beta, \beta')$ by the condition

$$k_{\varepsilon(\beta,\beta')}(\beta) < k_{\varepsilon(\beta,\beta')}(\beta') \,. \tag{4.6}$$

The properties (4.2)–(4.4) are immediately checked from this definition. □

We are now prepared to define the matrices $B(C) = (b_{\alpha\beta}(C))$.

Definition 4.2. *Let C be a cluster in $\Delta(\Phi)$, that is, a $\mathbb{Z}$-basis of the root lattice Q consisting of n mutually compatible roots. Let $C' = C - \{\beta\} \cup \{\beta'\}$ be an adjacent cluster obtained from C by exchanging a root $\beta \in C$ with some other root β'. The entries $b_{\alpha\beta}(C)$, $\alpha \in C$, of the matrix $B(C)$ are defined by*

$$b_{\alpha\beta}(C) = \varepsilon(\beta, \beta') \cdot [\beta + \beta' - (\beta \uplus \beta') : \alpha]_C, \tag{4.7}$$

where $[\gamma : \alpha]_C$ denotes the coefficient of α in the expansion of a vector $\gamma \in Q$ in the basis C.

To complete the proof of Theorem 1.5, all we need to show is that the matrices $B(C)$ described in Definition 4.2 satisfy (4.1) and (2.5)–(2.7).

Proof of (4.1). Let $C_\circ = -\Pi$ be the cluster consisting of all the negative simple roots. Applying (4.7) and using (4.2) and (3.7), we obtain

$$\begin{aligned} b_{-\alpha_i,-\alpha_j}(C_\circ) &= -\varepsilon(j) \cdot \Big[-\sum_{k\neq j} a_{kj}\alpha_k : -\alpha_i \Big]_{C_\circ} \\ &= \begin{cases} 0 & \text{if } i = j; \\ \varepsilon(j) a_{ij} & \text{if } i \neq j, \end{cases} \end{aligned} \tag{4.8}$$

establishing (4.1). □

Proof of (2.5). We start by summarizing the basic properties of cluster expansions of $\beta + \beta'$ and $\beta \uplus \beta'$ for an exchangeable pair of roots.

Lemma 4.3. *Let $\beta, \beta' \in \Phi_{\geq -1}$ be exchangeable.*

1. *No negative simple root can be a cluster component of $\beta + \beta'$.*
2. *The vectors $\beta + \beta'$ and $\beta \uplus \beta'$ have no common cluster components. That is, no root in $\Phi_{\geq -1}$ can contribute, with nonzero coefficient, to the cluster expansions of both $\beta + \beta'$ and $\beta \uplus \beta'$.*
3. *All cluster components of $\beta + \beta'$ and $\beta \uplus \beta'$ are compatible with both β and β'.*

4. *A root* $\alpha \in \Phi_{\geq -1}$, $\alpha \notin \{\beta, \beta'\}$, *is compatible with both* β *and* β' *if and only if it is compatible with all cluster components of* $\beta + \beta'$ *and* $\beta \uplus \beta'$.
5. *If* $\alpha \in -\Pi$ *is compatible with all cluster components of* $\beta + \beta'$, *then it is compatible with all cluster components of* $\beta \uplus \beta'$.

Proof. 1. Suppose $[\beta + \beta' : -\alpha_i]_{\mathrm{clus}} > 0$ for some $i \in I$. (Here we use the notation from Proposition 3.4.) Since all roots compatible with $-\alpha_i$ and different from $-\alpha_i$ do not contain α_i in their simple root expansion, it follows that $[\beta + \beta' : \alpha_i] < 0$. This can only happen if one of the roots β and β', say β, is equal to $-\alpha_i$; but since β' is incompatible with β, we will still have $[\beta + \beta' : \alpha_i] \geq 0$, a contradiction.

2. Suppose α is a common cluster component of $\beta + \beta'$ and $\beta \uplus \beta'$. Applying if necessary a transformation from $\langle \tau_+, \tau_- \rangle$, we can assume that α is negative simple (see Proposition 3.4, and Theorem 3.1, Part 2). But this is impossible by Part 1.

3. The claim for $\beta + \beta'$ is proved in [7, Lemma 2.3]. Since $\beta \uplus \beta' = \sigma^{-1}(\sigma(\beta) + \sigma(\beta'))$ for some $\sigma \in \langle \tau_+, \tau_- \rangle$, the claim for $\beta \uplus \beta'$ follows from Proposition 3.4.

4. First suppose that α is compatible with both β and β'. The fact that α is compatible with all cluster components of $\beta + \beta'$ is proved in [7, Lemma 2.3]. The fact that α is compatible with all cluster components of $\beta \uplus \beta'$ now follows in the same way as in Part 3.

To prove the converse, suppose that α is incompatible with β. As in Part 2 above, we can assume that $\alpha = -\alpha_i$ for some i. Thus, we have $[\beta : \alpha_i] > 0$. Since $\alpha \neq \beta'$, it follows that $[\beta + \beta' : \alpha_i] > 0$ as well. Therefore, $[\gamma : \alpha_i] > 0$ for some cluster component γ of $\beta + \beta'$, and we are done.

5. This follows from [7, Theorem 1.17]. □

As a corollary of Lemma 4.3, we obtain the converse of Proposition 3.5.

Corollary 4.4. *Let* β *and* β' *be exchangeable almost positive roots. Then there exist two adjacent clusters* C *and* C' *such that* $C' = C - \{\beta\} \cup \{\beta'\}$.

Proof. By Lemma 4.3, Parts 3 and 4, the set consisting of β and all cluster components of $\beta + \beta'$ and $\beta \uplus \beta'$ is compatible (i.e., all its elements are mutually compatible). Thus, there exists a cluster C containing this set. Again using Lemma 4.3, Part 4, we conclude that every element of $C - \{\beta\}$ is compatible with β'. Hence $C - \{\beta\} \cup \{\beta'\}$ is a cluster, as desired. □

Corollary 4.5. *Let* C *and* $C' = C - \{\beta\} \cup \{\beta'\}$ *be adjacent clusters . Then all cluster components of* $\beta + \beta'$ *and* $\beta \uplus \beta'$ *belong to* $C \cap C' = C - \{\beta\}$.

Proof. By Lemma 4.3, Parts 3 and 4, any cluster component of $\beta + \beta'$ or $\beta \uplus \beta'$ is compatible with every element of C, hence must belong to C. □

We next derive a useful alternative description of the matrix entries of $B(C)$.

Lemma 4.6. *In the situation of Definition 4.2, we have*

$$b_{\alpha\beta}(C) = \varepsilon(\beta, \beta') \cdot \big([\beta + \beta' : \alpha]_{\text{clus}} - [\beta \uplus \beta' : \alpha]_{\text{clus}}\big)\ . \tag{4.9}$$

In particular, the entry $b_{\alpha\beta}(C)$ is uniquely determined by α, β, and β'.

Proof. By Corollary 4.5, the cluster expansions of the vectors $\beta + \beta'$ and $\beta \uplus \beta'$ are the same as their basis C expansions. Thus, (4.7) is equivalent to (4.9). □

We will need the following lemma (cf. Proposition 3.6).

Lemma 4.7. *For a pair of exchangeable roots β and β' and a sign ε, we have*

$$(\tau_\varepsilon^{(k)})^{-1}(\tau_\varepsilon^{(k)}(\beta) + \tau_\varepsilon^{(k)}(\beta')) = \begin{cases} \beta + \beta' & \textit{if } 0 \le k \le \min(k_\varepsilon(\beta), k_\varepsilon(\beta'));\\ \beta \uplus \beta' & \textit{if } \min(k_\varepsilon(\beta), k_\varepsilon(\beta')) < k \\ & \le \max(k_\varepsilon(\beta), k_\varepsilon(\beta')). \end{cases}$$

Proof. This is a consequence of [7, Lemma 3.2]. □

We next establish the following symmetry property.

Lemma 4.8. *For $\tau \in \{\tau_+, \tau_-\}$, we have $b_{\tau\alpha,\tau\beta}(\tau C) = -b_{\alpha\beta}(C)$.*

Proof. Let us introduce some notation. For every pair of exchangeable roots (β, β'), we denote by $S_+(\beta, \beta')$ and $S_-(\beta, \beta')$ the two elements of Q given by

$$\begin{aligned} S_{\varepsilon(\beta,\beta')}(\beta, \beta') &= \beta + \beta',\\ S_{-\varepsilon(\beta,\beta')}(\beta, \beta') &= \beta \uplus \beta'. \end{aligned} \tag{4.10}$$

In this notation, (4.9) takes the form

$$b_{\alpha\beta}(C) = [S_+(\beta, \beta') : \alpha]_{\text{clus}} - [S_-(\beta, \beta') : \alpha]_{\text{clus}}\ . \tag{4.11}$$

The functions $S_\pm$ satisfy the following symmetry property:

$$\tau\, S_\varepsilon(\beta, \beta') = S_{-\varepsilon}(\tau\beta, \tau\beta') \ \text{ for } \tau \in \{\tau_+, \tau_-\}; \tag{4.12}$$

this follows by comparing (4.3) with Lemma 4.7. The lemma follows by combining (4.11) and (4.12) with Proposition 3.4. □

We are finally ready for the task at hand: verifying that each matrix $B(C)$ is sign-skew-symmetric. In view of (4.9) and Lemma 4.3, Part 2, the signs of its entries are given as follows:

$$\operatorname{sgn}(b_{\alpha\beta}(C)) = \begin{cases} \varepsilon(\beta, \beta') & \text{if } \alpha \text{ is a cluster component of } \beta + \beta';\\ -\varepsilon(\beta, \beta') & \text{if } \alpha \text{ is a cluster component of } \beta \uplus \beta';\\ 0 & \text{otherwise.} \end{cases} \tag{4.13}$$

Since β is incompatible with β', Lemma 4.3, Part 3, ensures that β cannot be a cluster component of $\beta + \beta'$ or $\beta \uplus \beta'$, so all diagonal entries of $B(C)$ are equal to 0.

Lemma 4.9. *Let α and β be two different elements of a cluster C, and let the corresponding adjacent clusters be $C - \{\alpha\} \cup \{\alpha'\}$ and $C - \{\beta\} \cup \{\beta'\}$. Then we have $b_{\alpha\beta}(C) = 0$ if and only if α' is compatible with β'.*

Proof. By (4.13), condition $b_{\alpha\beta}(C) = 0$ is equivalent to

$$[\beta + \beta' : \alpha]_{\text{clus}} = [\beta \uplus \beta' : \alpha]_{\text{clus}} = 0. \tag{4.14}$$

Since all cluster components of $\beta + \beta'$ and $\beta \uplus \beta'$ belong to C (see Corollary 4.5), and since α' is compatible with all elements of $C - \{\alpha\}$, condition (4.14) holds if and only if α' is compatible with all cluster components of $\beta + \beta'$ and $\beta \uplus \beta'$. By Lemma 4.3, Part 4, this is in turn equivalent to α' being compatible with β'. □

Since the compatibility relation is symmetric, Lemma 4.9 implies that $b_{\alpha\beta}(C) = 0$ is equivalent to $b_{\beta\alpha}(C) = 0$, as needed.

It remains to consider the case where both $b_{\alpha\beta}(C)$ and $b_{\beta\alpha}(C)$ are nonzero. By (4.13), this means that α is a cluster component of $\beta + \beta'$ or $\beta \uplus \beta'$, while β is a cluster component of $\alpha + \alpha'$ or $\alpha \uplus \alpha'$. By Lemma 4.8, it is enough to consider the special case $\alpha = -\alpha_j \in -\Pi$. Let us abbreviate $\varepsilon_\circ = \varepsilon(\beta, \beta')$. In view of Lemma 4.3, Part 1, (4.13) yields $\text{sgn}(b_{\alpha\beta}(C)) = -\varepsilon_\circ$. Interchanging α and β, and applying the same rule (4.13), we obtain, taking into account (4.2), that

$$\text{sgn}(b_{\beta\alpha}(C)) = \begin{cases} -\varepsilon(j) & \text{if } \beta \text{ is a cluster component of } \alpha + \alpha'; \\ \varepsilon(j) & \text{if } \beta \text{ is a cluster component of } \alpha \uplus \alpha'. \end{cases}$$

By Lemma 4.3, Part 1, and Proposition 3.4, the element of the set $\{\alpha + \alpha', \alpha \uplus \alpha'\}$ that does not have β as a cluster component is equal to

$$\big(\tau_{\varepsilon_\circ}^{(k_\circ)}\big)^{-1}\big(\tau_{\varepsilon_\circ}^{(k_\circ)}(\alpha) + \tau_{\varepsilon_\circ}^{(k_\circ)}(\alpha')\big),$$

where $k_\circ = k_{\varepsilon_\circ}(\beta)$. Our goal is to prove that $\text{sgn}(b_{\beta\alpha}(C)) = \varepsilon_\circ$, which can now be restated as follows:

$$\big(\tau_{\varepsilon_\circ}^{(k_\circ)}\big)^{-1}\big(\tau_{\varepsilon_\circ}^{(k_\circ)}(\alpha) + \tau_{\varepsilon_\circ}^{(k_\circ)}(\alpha')\big) = \begin{cases} \alpha + \alpha' & \text{if } \varepsilon_\circ = \varepsilon(j); \\ \alpha \uplus \alpha' & \text{if } \varepsilon_\circ = -\varepsilon(j). \end{cases} \tag{4.15}$$

It was shown in the proof of Lemma 4.1 that $k_\circ < k_{\varepsilon_\circ}(\beta')$. As a main step towards proving (4.15), we show that $k_\circ < k_{\varepsilon_\circ}(\alpha')$. Suppose on the contrary that $k_{\varepsilon_\circ}(\alpha') \le k_\circ$. Applying Lemma 4.7 with $k = k_{\varepsilon_\circ}(\alpha') \le \min(k_{\varepsilon_\circ}(\beta), k_{\varepsilon_\circ}(\beta'))$, we see that

$$\tau_{\varepsilon_\circ}^{(k)}(\beta + \beta') = \tau_{\varepsilon_\circ}^{(k)}(\beta) + \tau_{\varepsilon_\circ}^{(k)}(\beta')\ .$$

To arrive at a contradiction, notice that α' is compatible with all cluster components of $\beta + \beta'$ but is incompatible with β' (see Lemma 4.9). It follows that the root $\tau_{\varepsilon_\circ}^{(k)}(\alpha') \in -\Pi$ is compatible with all cluster components of $\tau_{\varepsilon_\circ}^{(k)}(\beta) + \tau_{\varepsilon_\circ}^{(k)}(\beta')$ but incompatible with $\tau_{\varepsilon_\circ}^{(k)}(\beta')$. But this contradicts Lemma 4.3, Parts 4–5.

Having established the inequality $k_\circ < k_{\varepsilon_\circ}(\alpha')$, we see that (4.15) becomes a special case of Lemma 4.7. Indeed, if $\varepsilon_\circ = \varepsilon(j)$ then $k_{\varepsilon_\circ}(\alpha) = k_{\varepsilon_\circ}(-\alpha_j) = h+1$, so $k_\circ < \min(k_{\varepsilon_\circ}(\alpha), k_{\varepsilon_\circ}(\alpha'))$; and if $\varepsilon_\circ = -\varepsilon(j)$ then $k_{\varepsilon_\circ}(\alpha) = k_{\varepsilon_\circ}(-\alpha_j) = 0$, so $\min(k_{\varepsilon_\circ}(\alpha), k_{\varepsilon_\circ}(\alpha')) \le k_\circ < \max(k_{\varepsilon_\circ}(\alpha), k_{\varepsilon_\circ}(\alpha'))$. This concludes our proof of (2.5). □

Proof of (2.6). Let C and $\overline{C} = C - \{\gamma\} \cup \{\overline{\gamma}\}$ be adjacent clusters. Our task is to show that every matrix entry $b_{\alpha\beta}(\overline{C})$ is obtained from the entries of $B(C)$ by the procedure described in (2.6). We already know that all diagonal entries in $B(C)$ and $B(\overline{C})$ are equal to 0 which is consistent with the matrix mutation rule (1.3) (since we have already proved that these matrices are sign-skew-symmetric). So let us assume that $\alpha \ne \beta$. If $\beta = \overline{\gamma}$ then the desired equality $b_{\alpha\overline{\gamma}}(\overline{C}) = -b_{\alpha\gamma}(C)$ is immediate from (4.9) and (4.4). Thus, it remains to treat the case where $\alpha \ne \beta$ and $\beta \ne \overline{\gamma}$. By Lemma 4.8, it is enough to consider the special case $\beta = -\alpha_j \in -\Pi$.

In view of (4.7), (3.7) and (4.2), we have

$$\begin{aligned} b_{\alpha\beta}(C) &= -\varepsilon(j)[\delta : \alpha]_C\,, \\ b_{\alpha\beta}(\overline{C}) &= -\varepsilon(j)[\delta : \alpha]_{\overline{C}}\,, \end{aligned} \tag{4.16}$$

where we use the notation

$$\delta = \delta_j = -\textstyle\sum_{i \ne j} a_{ij}\alpha_i\,.$$

Since the basis $\overline{C}$ is obtained from C by replacing γ by $\overline{\gamma}$, we can use the expansion

$$\gamma + \overline{\gamma} = \sum_{\alpha \in C \cap \overline{C}} [\gamma + \overline{\gamma} : \alpha]_{\text{clus}} \cdot \alpha$$

(cf. Corollary 4.5) to obtain:

$$\begin{aligned} &[\delta : \overline{\gamma}]_{\overline{C}} = -[\delta : \gamma]_C \\ &[\delta : \alpha]_{\overline{C}} = [\delta : \alpha]_C + [\gamma + \overline{\gamma} : \alpha]_{\text{clus}} \cdot [\delta : \gamma]_C \quad \text{for } \alpha \in C \cap \overline{C}\,. \end{aligned} \tag{4.17}$$

Combining (4.16) and (4.17), we get

$$b_{\overline{\gamma}\beta}(\overline{C}) = -b_{\gamma\beta}(C) \tag{4.18}$$

$$b_{\alpha\beta}(\overline{C}) = b_{\alpha\beta}(C) + [\gamma + \overline{\gamma} : \alpha]_{\text{clus}} \cdot b_{\gamma\beta}(C) \text{ for } \alpha \in C \cap \overline{C}. \tag{4.19}$$

Formula (4.18) takes care of the first case in the mutation rule (1.3). If $b_{\gamma\beta}(C) = 0$, then (4.19) again agrees with (1.3). So let us assume that $b_{\gamma\beta}(C) \ne 0$. Applying (4.9) to $b_{\alpha\gamma}(C)$ and $b_{\beta\gamma}(C)$, we obtain:

$$[\gamma + \overline{\gamma} : \alpha]_{\text{clus}} = \max(\varepsilon(\gamma, \overline{\gamma}) b_{\alpha\gamma}(C), 0), \tag{4.20}$$

$$[\gamma + \overline{\gamma} : \beta]_{\text{clus}} = \max(\varepsilon(\gamma, \overline{\gamma}) b_{\beta\gamma}(C), 0). \tag{4.21}$$

By Lemma 4.3, Part 1, the left-hand side of (4.21) equals 0. Hence

$$\varepsilon(\gamma, \overline{\gamma}) = -\mathrm{sgn}(b_{\beta\gamma}(C)) = \mathrm{sgn}(b_{\gamma\beta}(C)) \tag{4.22}$$

(using that $B(C)$ is sign-skew-symmetric). Now (4.19), (4.20), and (4.22) give

$$b_{\alpha\beta}(\overline{C}) = b_{\alpha\beta}(C) + \max(\mathrm{sgn}(b_{\gamma\beta}(C)) \cdot b_{\alpha\gamma}(C), 0) \cdot b_{\gamma\beta}(C),$$

which is easily seen to be equivalent to the second case in (1.3). This completes the verification that the matrices $B(C)$ satisfy (2.6). □

Proof of (2.7). We proceed by induction on the rank n of a root system Φ. For induction purposes, we need to allow Φ to be reducible; this is possible in view of Remark 3.8. For $n = 2$, the generalized associahedra of types $A_1 \times A_1$, A_2, B_2 and G_2 are convex polygons with 4, 5, 6, and 8 sides, respectively, matching the claim.

For the induction step, consider a geodesic loop L in the dual graph $\Gamma = \Gamma_{\Delta(\Phi)}$ for a root system Φ of rank $n \geq 3$. According to the definition of a geodesic, the clusters lying on L are obtained from some initial cluster C by fixing $n - 2$ of the n roots and alternately exchanging the remaining two roots. By (3.6), every transformation from $\langle \tau_+, \tau_- \rangle$ sends geodesics to geodesics. Furthermore, the Coxeter number associated to a geodesic does not change, by Lemma 4.8. Using Theorem 3.1, Part 2, we may therefore assume, without loss of generality, that one of the $n - 2$ fixed roots in the initial cluster C is $-\alpha_j \in -\Pi$.

Lemma 4.10. *Let Φ' be the rank $n - 1$ root subsystem of Φ spanned by the simple roots α_i for $i \neq j$.*

1. *The correspondence $C' \mapsto \{-\alpha_j\} \cup C'$ is a bijection between the clusters in $\Delta(\Phi')$ and the clusters in $\Delta(\Phi)$ that contain $-\alpha_j$. Thus, it identifies the cluster complex $\Delta(\Phi')$ with the link $\Delta_{\{-\alpha_j\}}$ of $\{-\alpha_j\}$ in $\Delta(\Phi)$ (see Sect. 2.1).*
2. *Assume that the sign function* (3.1) *for Φ' is a restriction of the sign function for Φ. The matrix $B(C')$ associated to a cluster $C' \subset \Phi'_{\geq -1}$ can be obtained from the matrix $B(\{-\alpha_j\} \cup C')$ by crossing out the row and column corresponding to the root $-\alpha_j$.*

Proof. Part 1 follows from [11, Proposition 3.5 (3)]. The assertion in Part 2 is immediately checked in the special case $C' = \{-\alpha_i : i \neq j\}$ (see (4.8)). It is then extended to an arbitrary cluster C' because the graph $\Gamma' = \Gamma_{\Delta(\Phi')}$ is connected, and the propagation rules for the matrices $B(C')$ and $B(\{-\alpha_j\} \cup C')$ are easily seen to be exactly the same. (Here we use the fact that conditions (2.1), (2.5), and (2.6) have already been checked for Γ and Γ' alike.) □

By Lemma 4.10, Part 1, L can be viewed as a geodesic loop in $\Gamma' = \Gamma_{\Delta(\Phi')}$. To complete the induction step, it remains to notice that, in view of Lemma 4.10, Part 2, the Coxeter number associated with L in Γ' coincides with the original value in Γ. This completes the proof of (2.7). □

Theorem 1.5 is proved.

5. Proofs of Theorems 1.9 and 1.11–1.13

Proof of Theorems 1.11–1.13 modulo Theorem 1.9. To take Theorems 1.11–1.13 out of the way, we begin by deducing them from Theorem 1.9. We adopt all the conventions and notation of Sect. 4. In particular, we assume, without loss of generality, that the Cartan matrix A is indecomposable, so the corresponding (finite) root system Φ is irreducible. We have proved that the complex $\Delta(\Phi)$ on the ground set $\Phi_{\geq -1}$ of almost positive roots, together with the family of matrices $B(C)$ introduced in Definition 4.2, satisfy conditions (2.1)–(2.7).

Let $\mathcal{A} = \mathcal{A}(B_\circ, \mathbf{p}_\circ)$ be the cluster algebra of finite type appearing in Theorem 1.5. Here we choose an initial seed $\Sigma_\circ = (\mathbf{x}_\circ, \mathbf{p}_\circ, B_\circ)$ for $\mathcal{A}$ by identifying the matrix $B_\circ$ with the matrix $B(C_\circ)$ at the cluster $C_\circ = -\Pi$ in $\Delta(\Phi)$ (see (4.8)). This gives us a seed attachment at $C_\circ$. Applying Lemma 2.4, we obtain a surjection $\alpha \mapsto x[\alpha]$ from $\Phi_{\geq -1}$ onto the set of all cluster variables in $\mathcal{A}$. Note that at this point, we have not yet proved that the variables $x[\alpha]$ are all distinct.

Assume for a moment that Theorem 1.9 has been established. Then the map $\alpha \mapsto x[\alpha]$ is a bijection, and Theorems 1.12 and 1.13 follow by Remark 2.5.

As for Theorem 1.11, it becomes a consequence of Lemma 4.6. To be more precise, let us associate to every lattice vector $\gamma \in Q$ a monomial in the cluster variables by setting

$$x[\gamma] = \prod_\alpha x[\alpha]^{m_\alpha}, \quad m_\alpha = [\gamma : \alpha]_{\text{clus}} .$$

In view of (4.9), every exchange relation (1.4) corresponding to adjacent clusters C and $C - \{\beta\} \cup \{\beta'\}$ can be written in the form

$$x[\beta]\, x[\beta'] = p_\beta^{\varepsilon(\beta,\beta')}(C)\, x[\beta + \beta'] + p_\beta^{-\varepsilon(\beta,\beta')}(C)\, x[\beta \uplus \beta'], \tag{5.1}$$

for some coefficients $p_\beta^\pm(C) \in \mathbb{P}$. Thus, the set of cluster variables and the respective nonzero exponents that appear in the right-hand side of (5.1) are uniquely determined by β and β'. The same holds for the coefficients $p_\beta^\pm(C)$, since the cluster variables appearing in the right-hand side are algebraically independent. □

We denote $p^\pm_{\beta,\beta'} = p^\pm_\beta(C)$. This notation is justified in view of Theorem 1.11.

Remark 5.1. In view of Corollary 4.4, the exchange relation (5.1) holds for *every pair* (β, β') of exchangeable roots. Also note that, in view of (3.7) and (4.2), the exchange relation (5.1) takes the following more explicit form if β' is negative simple:

$$\begin{aligned} x[\beta]\, x[-\alpha_j] &= p^{\varepsilon(j)}_{\beta,-\alpha_j}\, x[\beta - \alpha_j] + p^{-\varepsilon(j)}_{\beta,-\alpha_j}\, x[\beta \uplus (-\alpha_j)] \\ &= p^{\varepsilon(j)}_{\beta,-\alpha_j}\, x[\beta - \alpha_j] + p^{-\varepsilon(j)}_{\beta,-\alpha_j}\, x[\beta - \alpha_j + \textstyle\sum_{i\neq j} a_{ij}\alpha_i] . \end{aligned} \tag{5.2}$$

For the classical types, the list of all exchangeable pairs $(\beta, -\alpha_j)$, together with the explicitly given cluster expansions for $\beta - \alpha_j$ and $\beta \uplus (-\alpha_j) = \beta - \alpha_j + \sum_{i \neq j} a_{ij}\alpha_i$, was given in [7, Sect. 4].

Proof of Theorem 1.9. We prove (1.10) by induction on

$$k(\alpha) = \min(k_+(\alpha), k_-(\alpha)) \geq 0$$

(see the proof of Lemma 4.1). If $k(\alpha) = 0$, then α is a negative simple root, and there is nothing to prove. So we assume that $k(\alpha) = k \geq 1$, and that (1.10) holds for all roots α' with $k(\alpha') < k$.

By the definition of $k(\alpha)$, we have

$$\alpha = \tau_{\varepsilon(j)}^{(k)}(-\alpha_j) = \tau_{-\varepsilon(j)}^{(k-1)}(\alpha_j)$$

for some $j \in I$. Since α_j and $-\alpha_j$ are exchangeable, so are α and $\tau(-\alpha_j)$, where we abbreviate $\tau = \tau_{-\varepsilon(j)}^{(k-1)}$. Let us write the corresponding exchange relation. Using the $\langle \tau_\pm \rangle$-invariance of the exponents appearing in exchange relations (Lemma 4.8), together with (4.9) and (3.7), we obtain:

$$x[\alpha]\, x[\tau(-\alpha_j)] = q \prod_{i \neq j} x[\tau(-\alpha_i)]^{-a_{ij}} + r, \tag{5.3}$$

where $q, r \in \mathbb{P}$. For $k = 1$, we have $\alpha = \alpha_j$, and (5.3) yields

$$x[\alpha_j] = \frac{q \prod_{i \neq j} x_i^{-a_{ij}} + r}{x_j},$$

establishing (1.10). Thus, we may assume that $k \geq 2$. In this case, all the roots $\alpha' \neq \alpha$ that appear in (5.3) are positive with $k(\alpha') < k$. Abbreviating

$$\gamma = \sum_{i \neq j} (-a_{ij}) \cdot \tau(-\alpha_i)$$

and applying the induction assumption, we can rewrite (5.3) as

$$x[\alpha] = x^{\tau(-\alpha_j) - \gamma} \cdot \frac{q \prod_{i \neq j} P_{\tau(-\alpha_i)}^{-a_{ij}} + r x^\gamma}{P_{\tau(-\alpha_j)}}, \tag{5.4}$$

where all $P_{\alpha'}$ are polynomials over $\mathbb{ZP}$ in the variables from the initial cluster $\mathbf{x}_\circ$ with nonzero constant terms. The next step of the proof relies on the following trivial lemma.

Lemma 5.2. *Let P and Q be two polynomials (in any number of variables) with coefficients in a domain S, and with nonzero constant terms a and b, respectively. If the ratio P/Q is a Laurent polynomial over S, then it is in fact a polynomial over S with the constant term a/b.*

By [9, Theorem 3.1], $x[\alpha]$ is a Laurent polynomial. Hence, by Lemma 5.2, the second factor in (5.4) is a polynomial over $\mathbb{ZP}$ with nonzero constant term. To complete the proof of Theorem 1.9, it remains to compare (5.4) with (1.10), and to observe that

$$\begin{aligned}\gamma &= \tau\Big(\textstyle\sum_{i\neq j} a_{ij}\alpha_i\Big) && \text{(by Proposition 3.4)}\\ &= \tau(\alpha_j \uplus (-\alpha_j)) && \text{(by (3.7))}\\ &= \tau(\alpha_j) + \tau(-\alpha_j) && \text{(by Lemma 4.7)}\\ &= \alpha + \tau(-\alpha_j). && \square\end{aligned}$$

Remark 5.3. Unfortunately, the argument above does not establish Theorem 1.10 because there is no guarantee that the second factor in (5.4) is a polynomial with coefficients in $\mathbb{Z}_{\geq 0}[\mathcal{P}]$ even if we assume that all the polynomials $P_{\alpha'}$ appearing there have this property. (Recall from Definition 1.2 that $\mathcal{P}$ denotes the set of all coefficients $p^{\pm}_{\beta,\beta'}$ appearing in various exchange relations (5.1); we denote by $\mathbb{Z}_{\geq 0}[\mathcal{P}]$ the set of polynomials with nonnegative integer coefficients in the elements of $\mathcal{P}$.) The proof of Theorem 1.10 given in Sect. 6 below does not rely on Theorem 1.9, thus providing an alternative proof of the latter.

6. Proof of Theorem 1.10

We use the nomenclature of root systems given in Bourbaki [6], including the labeling of the simple roots in Φ by the indices $1, \ldots, n$. On the other hand, our convention on associating a Cartan matrix A to a root system Φ, as described in Sect. 1.4, is *transposed* to that in [6]—and the same as that in Kac [15].

We abbreviate $x_i = x[-\alpha_i]$ for $i = 1, \ldots, n$. Our goal is to prove that, for every almost positive root α, we can write $x[\alpha]$ as a Laurent polynomial in $x_1, \ldots, x_n$ with coefficients in $\mathbb{Z}_{\geq 0}[\mathcal{P}]$. This time we will proceed by induction on the height of α (recall that $\mathrm{ht}(\alpha) = \sum_i [\alpha : \alpha_i]$). The base case $\alpha \in -\Pi$ is trivial. The induction step will follow from the lemma below.

Lemma 6.1. *For every positive root α, there exists an index $j \in I$ such that*

(6.1) $$x_j x[\alpha] = F(x[\beta_1], \ldots, x[\beta_m]),$$

where F is a polynomial with coefficients in $\mathbb{Z}_{\geq 0}[\mathcal{P}]$ in some cluster variables $x[\beta_1], \ldots, x[\beta_m]$ such that $\mathrm{ht}(\beta_i) < \mathrm{ht}(\alpha)$ *for all i.*

The rest of this section is devoted to the proof of Lemma 6.1.

We call a positive root α *non-exceptional* if there exists a negative simple root $-\alpha_j$ exchangeable with α; otherwise, α will be called *exceptional*. If the root α in Lemma 6.1 is non-exceptional, and $-\alpha_j$ is a negative simple root exchangeable with α, then one easily sees that all cluster components of the vectors $\alpha - \alpha_j$ and $\alpha \uplus (-\alpha_j)$ appearing in the right-hand side of (5.2) are of smaller height than α, and we are done. Thus, it remains to prove Lemma 6.1 for the exceptional roots. First, we identify them explicitly.

Lemma 6.2. *The complete list of all exceptional positive roots is as follows:*

(6.2) Φ *is of type* E_8, *and* $\alpha = \alpha_{\max}$ *is the highest root in* Φ*;*

(6.3) Φ *is of type* F_4, *and* $\alpha = \alpha_{\max} = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$*;*

(6.4) Φ *is of type* F_4, *and* $\alpha = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_2$*;*

(6.5) Φ *is of type* G_2, *and* $\alpha = \alpha_{\max} = 3\alpha_1 + 2\alpha_2$*;*

(6.6) Φ *is of type* G_2, *and* $\alpha = 2\alpha_1 + \alpha_2$.

Proof. As noted in [7, Remark 1.16], α and $-\alpha_j$ are exchangeable if and only if

$$[\alpha : \alpha_j] = \left[\alpha^\vee : \alpha_j^\vee\right] = 1\,, \tag{6.7}$$

where $\alpha^\vee$ is the coroot corresponding to α under the natural bijection between Φ and the dual system $\Phi^\vee$. Let (α, β) denote a W-invariant scalar product on the root lattice Q. Then $[\alpha^\vee : \alpha_j^\vee] = \frac{(\alpha_j,\alpha_j)}{(\alpha,\alpha)}[\alpha : \alpha_j]$, so (6.7) is equivalent to

$$[\alpha : \alpha_j] = 1\,, \quad (\alpha, \alpha) = (\alpha_j, \alpha_j)\,. \tag{6.8}$$

Thus, we need to verify that for every positive root α, there exists a simple root α_j satisfying (6.8), unless α appears on the list (6.2)–(6.6), in which case there is no such simple root. This is checked by direct inspection using, e.g., the tables in [6]. (In all classical types, the list of all pairs $(\alpha, -\alpha_j)$ satisfying (6.7) was given in [7].) □

Proof of Lemma 6.1 for the type E_8 *and* $\alpha = \alpha_{\max}$. This case is by far the hardest among (6.2)–(6.6), so we will treat it in detail. We will prove that in this special case, Lemma 6.1 holds with $j = 8$, in the standard numeration of simple roots (see Fig. 3).

We will need the following construction. In view of Lemma 4.8, any transformation $\sigma \in \langle \tau_+, \tau_- \rangle$ gives rise to a "twisted" cluster algebra $\sigma(\mathcal{A})$ whose seeds are the transfers by σ of the seeds of $\mathcal{A}$; if σ is written in terms of τ_+ and τ_- as a product of an odd number of factors, this transfer involves the change of signs for the matrices B and the corresponding interchange of p^+ and p^- for the coefficients, as in Remark 1.3. This twist preserves the Cartan-Killing type.

Direct computation shows that for $\sigma = (\tau_-\tau_+)^8 = (\tau_+\tau_-)^8$ (cf. Theorem 3.1), we have $\sigma(\alpha_{\max}) = -\alpha_4$ and $\sigma(-\alpha_8) = \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5$. To prove Lemma 6.1 for the type E_8 and $\alpha = \alpha_{\max}$, it is therefore sufficient to show that, in the twisted cluster algebra $\sigma(\mathcal{A})$, we have

(6.9) $$x_4 x[\alpha_2+\alpha_3+2\alpha_4+\alpha_5] = \tilde{F}(x[\beta_1],\dots,x[\beta_m])\,,$$

where $\tilde{F}$ is a polynomial with coefficients in $\mathbb{Z}_{\geq 0}[\mathcal{P}]$, and each β_i is different from $\alpha_2+\alpha_3+2\alpha_4+\alpha_5$.

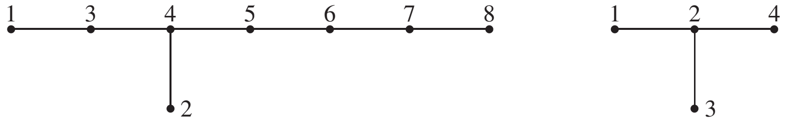


Fig. 3. Dynkin diagrams of types E_8 and D_4

Let $J=\{2,3,4,5\}\subset I$, and let $\Phi(J)$ denote the type D_4 root subsystem of Φ spanned by the simple roots α_j with $j\in J$. Applying Lemma 4.10 four times, we conclude that the correspondence

$$C' \mapsto (-\Pi(I-J))\cup C'$$

identifies the cluster complex $\Delta(\Phi(J))$ with the link of $-\Pi(I-J)$ in the cluster complex $\Delta(\Phi)$; here we use the notation

$$-\Pi(I-J)=\{-\alpha_i : i\in I-J\}.$$

The exchange graph $\Gamma(J)=\Gamma_{\Delta(\Phi(J))}$ is therefore identified with the induced subgraph in the exchange graph of $\mathcal{A}$ whose vertices are all the clusters containing $-\Pi(I-J)$. Let $\mathcal{A}'$ denote the subring in $\mathcal{A}$ generated by the cluster variables $x[\alpha]$ for $\alpha\in\Phi(J)_{\geq -1}$, together with the "coefficients" in all exchange relations corresponding to the edges in $\Gamma(J)$, where by a "coefficient" we mean the part of a monomial that does *not* involve the variables $x[\alpha]$ for $\alpha\in\Phi(J)_{\geq -1}$. (Thus, each "coefficient" is a product of an element of $\mathcal{P}$ and a monomial in the variables x_i for $i\in I-J$.) By Lemma 4.10, Part 2, the ring $\mathcal{A}'$ is a normalized cluster algebra of type D_4 (cf. [9, Proposition 2.6]). The claim (6.9) now becomes a consequence of the following lemma.

Lemma 6.3. *In the case of type D_4, with the notation as in Fig. 3, we have*

$$x_2 x[\alpha_{\max}] = G(x[\gamma_1],\dots,x[\gamma_k])\,.$$

where $\alpha_{\max}=\alpha_1+2\alpha_2+\alpha_3+\alpha_4$, the almost positive roots $\gamma_1,\dots,\gamma_k$ are different from $\alpha_{\max}$, and G is a polynomial with coefficients in $\mathbb{Z}_{\geq 0}[\mathcal{P}]$.

We note that the roots $\beta_1,\dots,\beta_m$ appearing in (6.9) are of two kinds: first, the images of $\gamma_1,\dots,\gamma_k$ under the embedding $D_4\to E_8$, and second, (some of) the "frozen" roots $-\alpha_1,-\alpha_6,-\alpha_7,-\alpha_8$.

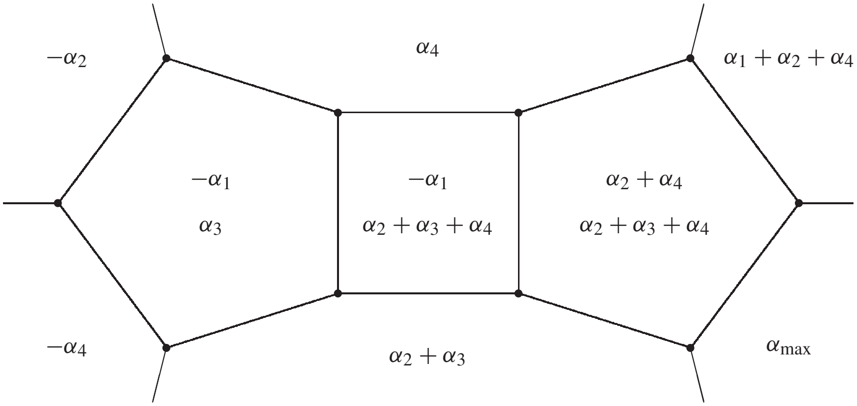


Fig. 4. Fragment of the exchange graph in the type D_4

Proof. Figure 4 shows a fragment of the exchange graph in type D_4, with each vertex C representing a cluster containing the 4 roots written into the regions adjacent to C. The mutual compatibility of the roots in each of these quadruples is easily checked from the definitions.

We next write the exchange relations for some pairs of adjacent clusters shown in Fig. 4. In doing so, we use:

- (implicitly) the combinatorial interpretation of almost positive roots of type D_n given in [11, Sect. 3.5] and reproduced in Sect. 12.4 below; see specifically [11, Fig. 7] for the type D_4;
- the resulting explicit expressions for the exchange relations which are consequences of [7, Lemma 4.6] (see Proposition 12.14 below);
- the monomial relations among the coefficients of exchange relations along a geodesic of type A_2, as given in [9, (6.11)]; our notation is patterned after [9, Fig. 3].

The exchange relations for the left pentagonal geodesic in Fig. 4 can be written in the following form, with $p_1, \ldots, p_5 \in \mathcal{P}$:

$$\begin{aligned} x_4 x[\alpha_2+\alpha_3+\alpha_4] &= p_1 x[\alpha_2+\alpha_3] + p_3 p_4 x[\alpha_3], \\ x_2 x[\alpha_2+\alpha_3] &= p_2 x_1 x_4 + p_4 p_5 x[\alpha_3], \qquad (6.10) \\ x_2 x[\alpha_2] &= p_3 x_2 + p_5 p_1, \end{aligned}$$

$$x_2 x[\alpha_2+\alpha_3+\alpha_4] = p_4 x[\alpha_3] x[\alpha_4] + p_1 p_2 x_1, \qquad (6.11)$$

$$x[\alpha_4] x[\alpha_2+\alpha_3] = p_5 x[\alpha_2+\alpha_3+\alpha_4] + p_2 p_3 x_1. \qquad (6.12)$$

(Among these relations, only (6.10), (6.11), and (6.12) are needed in the proof; we wrote all five relations for the sake of clarity.) Similarly, the exchange relations for the right pentagonal geodesic can be written as follows, with $q_1, \dots, q_5 \in \mathcal{P}$:

$$\begin{aligned} x_1 x[\alpha_{\max}] &= q_1 x[\alpha_2+\alpha_3]x[\alpha_2+\alpha_4] \\ &\quad + q_3 q_4 x[\alpha_2+\alpha_3+\alpha_4], \\ x[\alpha_2+\alpha_3]x[\alpha_1+\alpha_2+\alpha_4] &= q_2 x[\alpha_{\max}] + q_4 q_5\,, \\ x[\alpha_4]x[\alpha_{\max}] &= q_3 x[\alpha_1+\alpha_2+\alpha_4]x[\alpha_2+\alpha_3+\alpha_4] \\ &\quad + q_5 q_1 x[\alpha_2+\alpha_4], \end{aligned} \tag{6.13}$$

$$x_1 x[\alpha_1+\alpha_2+\alpha_4] = q_4 x[\alpha_4] + q_1 q_2 x[\alpha_2+\alpha_4], \tag{6.14}$$

$$x[\alpha_4]x[\alpha_2+\alpha_3] = q_5 x_1 + q_2 q_3 x[\alpha_2+\alpha_3+\alpha_4]. \tag{6.15}$$

Comparing (6.15) to (6.12), we conclude that

$$p_5 = q_2 q_3\,. \tag{6.16}$$

Successively applying (6.13), (6.10)–(6.11), (6.16), and (6.14), we obtain:

$$\begin{aligned} & x_1 x_2 x[\alpha_{\max}] \\ &= q_1 x_2 x[\alpha_2+\alpha_3]x[\alpha_2+\alpha_4] + q_3 q_4 x_2 x[\alpha_2+\alpha_3+\alpha_4] \\ &= q_1 p_2 x_1 x_4 x[\alpha_2+\alpha_4] + q_1 p_4 p_5 x[\alpha_3]x[\alpha_2+\alpha_4] \\ &\qquad + q_3 q_4 p_4 x[\alpha_3]x[\alpha_4] + q_3 q_4 p_1 p_2 x_1 \\ &= q_1 p_2 x_1 x_4 x[\alpha_2+\alpha_4] + q_1 q_2 q_3 p_4 x[\alpha_3]x[\alpha_2+\alpha_4] \\ &\qquad + q_3 q_4 p_4 x[\alpha_3]x[\alpha_4] + q_3 q_4 p_1 p_2 x_1 \\ &= q_1 p_2 x_1 x_4 x[\alpha_2+\alpha_4] + q_3 p_4 x[\alpha_3]x_1 x[\alpha_1+\alpha_2+\alpha_4] + q_3 q_4 p_1 p_2 x_1\,, \end{aligned}$$

which implies

$$\begin{aligned} x_2 x[\alpha_{\max}] = q_1 p_2 x_4 x[\alpha_2+\alpha_4] + q_3 p_4 x[\alpha_3]x[\alpha_1+\alpha_2+\alpha_4] \\ + q_3 q_4 p_1 p_2\,, \end{aligned} \tag{6.17}$$

and we are done.

Proof of Lemma 6.1 in the types F_4 and G_2. One way of handling the non-simply-laced cases is to deduce them from the simply-laced ones by means of the "folding" technique (see, e.g., [11, Sect. 2.4]). Alternatively, one can perform direct computations, which show that, in the type F_4, we have

$$\begin{aligned} x_1 x[\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4] = P_1(& x[\alpha_3+\alpha_4], x[\alpha_2+\alpha_3], \\ & x[\alpha_2+2\alpha_3+2\alpha_4], x[\alpha_2+2\alpha_3+\alpha_4]), \end{aligned} \tag{6.18}$$

$$\begin{aligned} x_4 x[2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4] = P_2(& x[\alpha_1+2\alpha_2+3\alpha_3+\alpha_4], \\ & x[\alpha_1+2\alpha_2+2\alpha_3], \\ & x[\alpha_1+\alpha_2+\alpha_3+\alpha_4], \\ & x[\alpha_1+\alpha_2+\alpha_3])\,, \end{aligned} \tag{6.19}$$

and, in the type G_2, we have

$$x_2 x[2\alpha_1 + \alpha_2] = P_3(x[\alpha_1], x[\alpha_1 + \alpha_2]), \tag{6.20}$$

$$x_1 x[3\alpha_1 + 2\alpha_2] = P_4(x[\alpha_2], x[3\alpha_1 + \alpha_2]), \tag{6.21}$$

where P_1, P_2, P_3, P_4 are polynomials with coefficients in $\mathbb{Z}_{\geq 0}[\mathcal{P}]$. Details are left to the reader.

This completes our proofs of Lemma 6.1 and Theorem 1.10. □

7. 2-finite matrices

In accordance with the plan outlined in Sect. 1.6, our next task is to prove the implication (iii) $\Longrightarrow$ (iv) in Theorem 1.8, which will in turn imply Theorem 1.6. As a first step, we restate the claim at hand as a purely combinatorial result (see Theorem 7.1 below) on matrix mutations (1.3).

We shall write $B' = \mu_k(B)$ to denote that a matrix B' is obtained from B by a matrix mutation in direction k. Note that μ_k preserves integrality of entries, and is an involution: $\mu_k(\mu_k(B)) = B$. If two matrices can be obtained from each other by a sequence of matrix mutations followed by a simultaneous permutation of rows and columns, we will say that they are *mutation equivalent*.

A real square matrix $B = (b_{ij})$ is *sign-skew-symmetric* (cf. (1.2)) if, for any i and j, either $b_{ij} = b_{ji} = 0$, or else $b_{ij} b_{ji} < 0$; in particular, $b_{ii} = 0$ for all i. Furthermore, we say that B is 2*-finite* if it has integer entries, and any matrix $B' = (b'_{ij})$ mutation equivalent to B is sign-skew-symmetric and satisfies $|b'_{ij} b'_{ji}| \leq 3$ for all i and j.

In the language just introduced, the implication (iii) $\Longrightarrow$ (iv) in Theorem 1.8 can be formulated as follows.

Theorem 7.1. *Every* 2*-finite matrix* B *is mutation equivalent to a matrix* $B_\circ$ *from Theorem 1.5.*

The converse of Theorem 7.1 also holds: by Theorem 1.5 (which has already been proved), $B_\circ$ is 2-finite.

Our proof of Theorem 7.1 occupies the rest of Sects. 7–9 below. The main result of Sect. 7 is the following proposition.

Proposition 7.2. *Every* 2*-finite matrix is skew-symmetrizable.*

(Recall that a square matrix B is skew-symmetrizable if there exists a diagonal matrix D with positive diagonal entries such that DB is skew-symmetric.)

The rest of this section is devoted to the proof of Proposition 7.2.

The crucial role in the sequel will be played by a combinatorial construction that associates with a sign-skew-symmetric matrix its *diagram*, whose role is parallel to that of the Dynkin diagram for a generalized Cartan matrix.

Definition 7.3. *The* diagram *of a sign-skew-symmetric matrix* $B = (b_{ij})_{i,j\in I}$ *is the weighted directed graph* $\Gamma(B)$ *with the vertex set* I *such that there is a directed edge from* i *to* j *if and only if* $b_{ij} > 0$*, and this edge is assigned the weight* $|b_{ij}b_{ji}|$*.*

More generally, we will use the term *diagram* to denote a finite directed graph without loops and multiple edges, whose edges are assigned positive real weights. By some abuse of notation, we denote by the same symbol Γ the underlying directed graph of a diagram. If two vertices of Γ are not joined by an edge, we may also say that they are joined by an edge of weight 0.

The following lemma is an analogue of the well-known symmetrizability criterion [15, Exercise 2.1].

Lemma 7.4. *A matrix* $B = (b_{ij})$ *is skew-symmetrizable if and only if, first, it is sign-skew-symmetric and, second, for all* $k \geq 3$ *and all* $i_1, \dots, i_k$*, it satisfies*

$$b_{i_1i_2}b_{i_2i_3}\cdots b_{i_ki_1} = (-1)^k b_{i_2i_1}b_{i_3i_2}\cdots b_{i_1i_k}\,. \tag{7.1}$$

Proof. The "only if" part is trivial. Thus, let us assume that B is sign-skew-symmetric and satisfies (7.1). Without loss of generality, we also assume that B is indecomposable, i.e., cannot be represented as a direct (block-diagonal) sum of two proper submatrices. It follows that the graph $\Gamma(B)$ is connected. Let T be one of its spanning trees. There exists a diagonal matrix $D = (d_{ij})$ with positive diagonal entries such that $d_{ii}b_{ij} = -d_{jj}b_{ji}$ for every edge (i, j) in T. (Such a matrix can be constructed inductively by setting d_{ii} equal to an arbitrary positive number for some vertex i, and moving within the tree T away from this vertex.) Then DB is skew-symmetric, for the following reason: by definition of a spanning tree, any edge (i, j) of $\Gamma(B)$ which is not in T belongs to a cycle in which the rest of the edges belong to T; then use (7.1). □

Lemma 7.5. *Let* B *be a* 2*-finite matrix. Then the edges of every triangle in* $\Gamma(B)$ *are oriented in a cyclic way.*

Proof. Suppose on the contrary that $b_{ij}, b_{ik}, b_{kj} > 0$ for some distinct i, j, k. Then in the matrix $B' = \mu_k(B)$, we have $b'_{ij} = b_{ij} + b_{ik}b_{kj} \geq 2$ and $b'_{ji} = b_{ji} - b_{jk}b_{ki} \leq -2$, violating 2-finiteness. □

Lemma 7.6. *Let* B *be a* 2*-finite matrix. Then*

$$b_{ij}b_{jk}b_{ki} = -b_{ji}b_{kj}b_{ik} \tag{7.2}$$

for any distinct i, j, k*. Also, in every triangle in* $\Gamma(B)$*, the edge weights are either* $\{1, 1, 1\}$ *or* $\{2, 2, 1\}$*.*

Proof. In view of Lemma 7.5, we may assume without loss of generality that B is a 3×3 matrix

$$\begin{bmatrix} 0 & a_1 & -c_2 \\ -a_2 & 0 & b_1 \\ c_1 & -b_2 & 0 \end{bmatrix}, \tag{7.3}$$

where $a_1, b_1, c_1, a_2, b_2, c_2$ are positive integers. (If one of these entries is 0, then (7.2) is automatic.) Again without loss of generality, we may assume that the entry of maximal absolute value in B is $-c_2$. We claim that, under this assumption,

$$c_1 = a_2 b_2, \ \ c_2 = a_1 b_1 \,, \tag{7.4}$$

implying $a_1 b_1 c_1 = a_2 b_2 c_2$, and hence proving (7.2).

Indeed, we have

$$\mu_2(B) = \begin{bmatrix} 0 & -a_1 & a_1 b_1 - c_2 \\ a_2 & 0 & -b_1 \\ -a_2 b_2 + c_1 & b_2 & 0 \end{bmatrix}. \tag{7.5}$$

Applying Lemma 7.5 to $\mu_2(B)$, we conclude that

$$a_1 b_1 - c_2 \geq 0, \ \ a_2 b_2 - c_1 \geq 0,$$

where either both inequalities are strict, or both are equalities. We need to show that the former case is impossible. Indeed, otherwise we would have had $a_2 b_2 > c_1 \geq 1$, implying $\max(a_2, b_2) \geq 2$; also, $a_1 b_1 > c_2 \geq \max(a_1, b_1)$, implying $a_1 \geq 2$ and $b_1 \geq 2$. But then $\max(a_1 a_2, b_1 b_2) \geq 4$, contradicting the 2-finiteness of B.

It remains to show that the set of edge weights $\{a_1 a_2, b_1 b_2, c_1 c_2\}$ is either $\{1, 1, 1\}$, or $\{2, 2, 1\}$. The only other option consistent with both (7.4) and the inequalities $a_1 a_2 \leq 3$, $b_1 b_2 \leq 3$, $c_1 c_2 \leq 3$ is $c_2 = 3$, $\{a_1, b_1\} = \{3, 1\}$, $c_1 = a_2 = b_2 = 1$. Say $a_1 = 3$ and $b_1 = 1$ (the other case is analogous). Then $B' = \mu_1(B)$ has $|b'_{23} b'_{32}| = 4$, violating 2-finiteness. □

Now everything is ready for the proof of Proposition 7.2. It suffices to check that every 2-finite matrix satisfies the criterion (7.1). Suppose this is not the case. Among all instances where (7.1) is violated for some 2-finite matrix B, pick one with the smallest value of k. Then $b_{i_j, i_m} = 0$ for any pair of subscripts (i_j, i_m) not appearing in (7.1). (Otherwise we could obtain (7.1) as a corollary of its counterparts for two smaller cycles.) In other words, the diagram $\Gamma(B)$ restricted to the vertices $i_1, \ldots, i_k$ must be a cycle. Pick any two consecutive edges on this cycle that form an oriented 2-path (that is, $b_{i_{j-1} i_j} b_{i_j i_{j+1}} > 0$). (If there is no such pair, we will need to first apply a mutation at an arbitrary vertex i_j.) By Lemma 7.6, we have $k \geq 4$, hence $b_{i_{j-1} i_{j+1}} = 0$. Now apply the mutation μ_{i_j}. In the resulting matrix, condition (7.1) for the sequence of indices $i_1, \ldots, i_{j-1}, i_{j+1}, \ldots, i_k$ will be equivalent to (7.1) in the original matrix; hence it must fail, contradicting our choice of k. □

8. Diagram mutations

Let $B = (b_{ij})_{i,j\in I}$ be a skew-symmetrizable matrix. Notice that the diagram $\Gamma(B)$ does *not* determine B: for instance, the matrix $(-B^T)$ has the same diagram as B. However, the following important property holds.

Proposition 8.1. *For a skew-symmetrizable matrix B, the diagram $\Gamma' = \Gamma(\mu_k(B))$ is uniquely determined by the diagram $\Gamma = \Gamma(B)$ and an index $k \in I$. Specifically, Γ' is obtained from Γ as follows:*

- *The orientations of all edges incident to k are reversed, their weights intact.*
- *For any vertices i and j which are connected in Γ via a two-edge oriented path going through k (refer to Fig. 5 for the rest of notation), the direction of the edge (i, j) in Γ' and its weight c' are uniquely determined by the rule*

$$\pm\sqrt{c} \pm \sqrt{c'} = \sqrt{ab}\,, \tag{8.1}$$

  where the sign before $\sqrt{c}$ (resp., before $\sqrt{c'}$) is "+" if i, j, k form an oriented cycle in Γ (resp., in Γ'), and is "−" otherwise. Here either c or c' can be equal to 0.
- *The rest of the edges and their weights in Γ remain unchanged.*

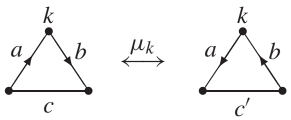


Fig. 5. Diagram mutation

Remark 8.2. If B has integer entries, then all edge weights in Γ are positive integers. The rule (8.1) ensures that the same is true for Γ': indeed, the fact that $c' = (\sqrt{ab} \mp \sqrt{c})^2 = ab + c \mp 2\sqrt{abc}$ is an integer (that is, abc is a perfect square) is an easy consequence of the skew-symmetrizability of B (more specifically, of the identity (7.1) with $k = 3$).

Our proof of Proposition 8.1 is based on the following construction.

Lemma 8.3. *Let B be a skew-symmetrizable matrix. Then there exists a diagonal matrix H with positive diagonal entries such that HBH^{-1} is skew-symmetric. Furthermore, the matrix $S(B) = (s_{ij}) = HBH^{-1}$ is uniquely determined by B. Specifically, the matrix entries of $S(B)$ are given by*

$$s_{ij} = \operatorname{sgn}(b_{ij})\sqrt{|b_{ij}b_{ji}|}\,. \tag{8.2}$$

Proof. Let D be a skew-symmetrizing matrix for B, i.e., a diagonal matrix with positive diagonal entries such that DB is skew-symmetric. Setting $H = D^{1/2}$, we see that $HBH^{-1} = H^{-1}(DB)H^{-1}$ is skew-symmetric. To prove (8.2), note that

$$\operatorname{sgn}(s_{ij}) = \operatorname{sgn}\big(h_i b_{ij} h_j^{-1}\big) = \operatorname{sgn}(b_{ij}),$$
$$s_{ij}^2 = |s_{ij}s_{ji}| = \big|\big(h_i b_{ij} h_j^{-1}\big)\cdot\big(h_j b_{ji} h_i^{-1}\big)\big| = |b_{ij}b_{ji}|,$$

where the h_i are the diagonal entries of H.

Lemma 8.4. *Let B be a skew-symmetrizable matrix. Then, for any $k \in I$, we have $S(\mu_k(B)) = \mu_k(S(B))$.*

Proof. Follows from Lemma 8.3, together with the directly checked fact that the mutation rules are invariant under conjugation by a diagonal matrix with positive entries. □

Proof of Proposition 8.1. Formula (8.2) shows that the diagram $\Gamma(B)$ and the skew-symmetric matrix $S(B)$ encode the same information about a skew-symmetrizable matrix B: having an edge in $\Gamma(B)$ directed from i to j and supplied with weight c is the same as saying that $s_{ij} = \sqrt{c}$ and $s_{ji} = -\sqrt{c}$. Lemma 8.4 asserts that, as B undergoes a mutation μ_k, so does the matrix $S(B)$. Translating this statement into the language of diagrams, we obtain Proposition 8.1. □

In the situation of Proposition 8.1, we write $\Gamma' = \mu_k(\Gamma)$, and call the transformation μ_k a *diagram mutation* in the direction k. Two diagrams Γ and Γ' related by a sequence of diagram mutations are called *mutation equivalent,* and we write $\Gamma \sim \Gamma'$. A diagram Γ is called 2*-finite* if any diagram $\Gamma' \sim \Gamma$ has all edge weights equal to $1, 2$ or 3. Thus a matrix B is 2-finite if and only if its diagram $\Gamma(B)$ is 2-finite. (Here we rely on Proposition 7.2.) Note that a diagram is 2-finite if and only if so are all its connected components.

In the case of 2-finite diagrams, Lemmas 7.5 and 7.6 ensure that every triangle is oriented in a cyclic way, and has edge weights $(1, 1, 1)$ or $(2, 2, 1)$. As a result, the rules of diagram mutations (as given in Proposition 8.1) simplify as follows.

Lemma 8.5. *Let Γ be a* 2*-finite diagram, and k a vertex of Γ. Then the diagram $\mu_k(\Gamma)$ is obtained from Γ as follows:*

- *The orientations of all edges incident to k are reversed, their weights intact.*
- *For any vertices i and j which are connected in Γ via a two-edge oriented path going through k, the diagram mutation μ_k affects the edge connecting i and j in the way shown in Fig. 6, where the weights c and c' are related by*

$$\sqrt{c} + \sqrt{c'} = \sqrt{ab}\ ; \tag{8.3}$$

here either c or c′ can be equal to 0.

- *The rest of the edges and their weights in* Γ *remain unchanged.*

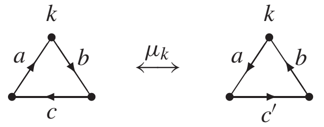


Fig. 6. Mutation of 2-finite diagrams

Taking into account Propositions 7.2 and 8.1, we see that Theorem 7.1 becomes a consequence of the following classification of 2-finite diagrams.

Theorem 8.6. *Any connected* 2*-finite diagram is mutation equivalent to an orientation of a Dynkin diagram. (Cf. Fig. 7, where all unspecified weights are equal to* 1.*) Furthermore, all orientations of the same Dynkin diagram are mutation equivalent to each other.*

As already noted following Theorem 7.1, the converse is true as well: any diagram mutation equivalent to an orientation of a Dynkin diagram is 2-finite.

9. Proof of Theorem 8.6

Throughout this section, all diagrams are presumed connected, and all edge weights are positive integers. With some abuse of notation, we use the same symbol Γ to denote a diagram and the set of its vertices. A diagram that is not 2-finite will be called 2*-infinite.*

Definition 9.1. *A* subdiagram *of a diagram* Γ *is a diagram* Γ′ *obtained from* Γ *by taking an induced directed subgraph on a subset of vertices and keeping all its edge weights the same as in* Γ. *We will denote this by* $\Gamma \supset \Gamma'$.

We will repeatedly use the following obvious fact: any subdiagram of a 2-finite diagram is 2-finite. Equivalently, any diagram that has a 2-infinite subdiagram is 2-infinite.

The proof of Theorem 8.6 will proceed in several steps.

9.1. Shape-preserving diagram mutations. Let *k* be a sink (resp., source) of a diagram Γ, that is, a vertex such that all edges incident to *k* are directed towards *k* (resp., away from *k*). Then a diagram mutation at *k* reverses the orientations of all edges incident to *k*, leaving the rest of the graph and all the edge weights unchanged. We shall refer to such mutations as *shape-preserving*.

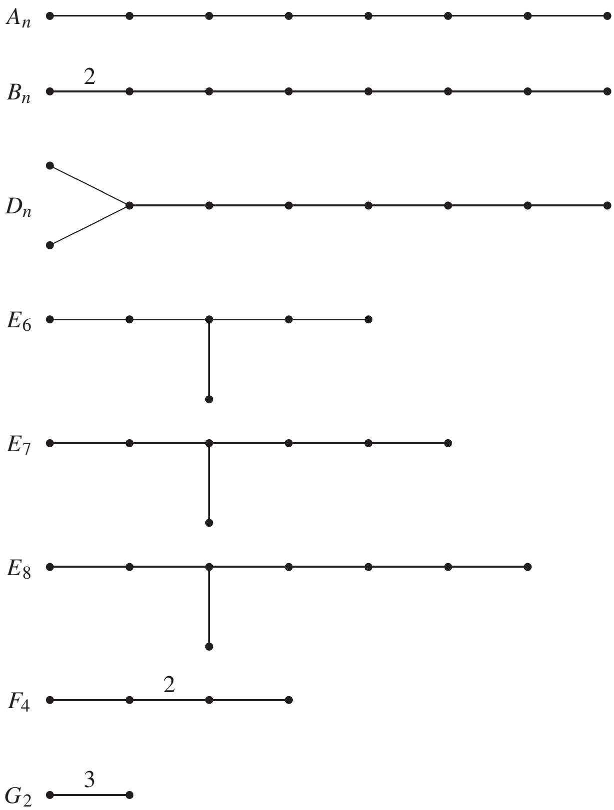


Fig. 7. Dynkin diagrams

Proposition 9.2. *Let T be a subdiagram of a diagram Γ such that:*

(i) *T is a tree.*
(ii) *T is attached to the rest of Γ by a single vertex $v \in T$, i.e., no vertex in $T - \{v\}$ is joined by an edge with a vertex in $\Gamma - T$.*

Then any diagram obtained from Γ by arbitrarily re-orienting the edges of T (while keeping the rest of Γ intact) is mutation equivalent to Γ.

In particular, any two orientations of a tree diagram are mutation equivalent.

(A *tree diagram* is a diagram whose underlying graph is an orientation of a tree.)

Proof. Using induction on the size of T, we will show that one can arbitrarily re-orient the edges of T by applying a sequence of shape-preserving mutations at the vertices of $T - \{v\}$. If T consists of a single vertex v, there is nothing to prove. Otherwise, pick a leaf $l \in T$ different from v, and apply the inductive assumption to the diagram $\Gamma' = \Gamma - \{l\}$ and its subdiagram

$T' = T - \{l\}$. So we are able to arbitrarily re-orient the edges of T' by a sequence of shape-preserving mutations of Γ' at the vertices of $T' - \{v\}$. To do the same for T, we lift this sequence from Γ' to Γ as follows: each time right before we need to perform a mutation at the unique vertex $k \in T'$ adjacent to l, we first mutate at l if necessary to make k a source or sink in T, rather than just in T'. This way, we can achieve an arbitrary re-orientation of the edges of T' by a sequence of shape-preserving mutations of Γ at the vertices of $T - \{v\}$. The remaining edge (k, l) can then be given an arbitrary orientation by a (shape-preserving) mutation at l. □

As a practical consequence of Proposition 9.2, in drawing a diagram Γ, we do not have to specify orientations of edges in any subdiagram $T \subset \Gamma$ satisfying the conditions of the proposition. Figure 7 provides an example of this (see also Fig. 10 below).

Proposition 9.2 also justifies notation of the form $\Gamma \sim A_m$, $\Gamma \supset A_m$, etc.

9.2. Taking care of the trees

Proposition 9.3. *Any 2-finite tree diagram is an orientation of a Dynkin diagram.*

Proof. A diagram Γ is called an *extended Dynkin tree diagram* if

- Γ is a tree diagram with edge weights ≤ 3;
- Γ is not on the Dynkin diagram list;
- every proper subdiagram of Γ is a disjoint union of Dynkin diagrams.

(In this definition, we ignore the orientations of the edges.) To prove the proposition, it is enough to show that any extended Dynkin tree diagram is 2-infinite. Direct inspection shows that Fig. 8 provides a complete list of such diagrams. Here each tree $X_n^{(1)}$ has $n + 1$ vertices. As before, all unspecified edge weights are equal to 1; in the diagram $G_2^{(1)}$, we have $a \in \{1, 2, 3\}$. We note that all these diagrams are associated with untwisted affine Lie algebras and can be found in the tables in [6] or in [15, Chap. 4, Table Aff 1]. The only diagram from those tables that is missing in Fig. 8 is $A_n^{(1)}$, which is an $(n + 1)$-cycle; it will be treated in Sect. 9.3.

In showing that an extended Dynkin tree diagram is 2-infinite, we can choose its orientation arbitrarily, by Proposition 9.2. Let us start with the three infinite series $B_n^{(1)}$, $C_n^{(1)}$, and $D_n^{(1)}$, and in each case let us orient all the edges left to right. Let us denote the diagram in question by $X_n^{(1)}$; thus, if $X = D$ (resp., B, C) then the minimal value of n is equal to 4 (resp., 3, 2). If n is greater than this minimal value, then performing a mutation at the second vertex from the left, and subsequently removing this vertex (together with all incident edges) leaves us with a subdiagram of type $X_{n-1}^{(1)}$. Using induction on n, it remains to check the basic cases $D_4^{(1)}$, $B_3^{(1)}$ and $C_2^{(1)}$. For $C_2^{(1)}$, the mutation at the middle vertex produces a triangle with edge weights (2, 2, 2),

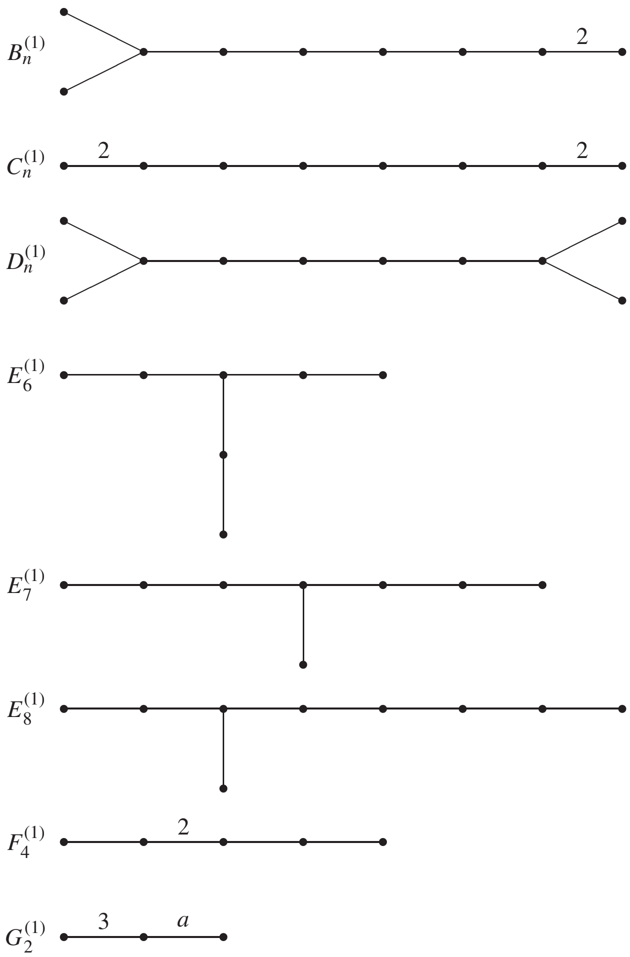


Fig. 8. Extended Dynkin tree diagrams

which is 2-infinite by Lemma 7.6. For $B_3^{(1)}$, mutating at the branching vertex and then removing it leaves us with the subdiagram $C_2^{(1)}$ which was just shown to be 2-infinite. Finally, for $D_4^{(1)}$, let the branching point be labeled by 1, and let it be joined with vertices 2 and 4 by incoming edges, and with 3 and 5 by outgoing edges. Then the composition of mutations $\mu_3 \circ \mu_2 \circ \mu_1$ makes the subdiagram on the vertices 2, 4 and 5 a 2-infinite triangle.

To see that $G_2^{(1)}$ is 2-infinite, orient the two edges left to right and mutate at the middle vertex to obtain a 2-infinite triangle. To see that $F_4^{(1)}$ is 2-infinite, again orient all the edges left to right, label the vertices also left to right, and apply $\mu_1 \circ \mu_2 \circ \mu_3 \circ \mu_4$ to obtain a subdiagram $C_2^{(1)}$ on the vertices 1, 3 and 5.

The remaining three cases $E_6^{(1)}$, $E_7^{(1)}$ and $E_8^{(1)}$ can be treated in a similar manner but we prefer another approach. To describe it, we will need to introduce some notation.

Definition 9.4. *For $p, q, r \in \mathbb{Z}_{\geq 0}$, we denote by $T_{p,q,r}$ the tree diagram (with all edge weights equal to 1) on $p + q + r + 1$ vertices obtained by connecting an endpoint of each of the three chains A_p, A_q and A_r to a single extra vertex (see Fig. 9).*

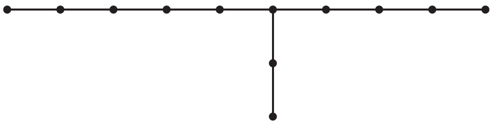


Fig. 9. The tree diagram $T_{5,4,2}$

Definition 9.5. *For $p, q, r \in \mathbb{Z}_{>0}$ and $s \in \mathbb{Z}_{\geq 0}$, let $S^s_{p,q,r}$ denote the diagram (with all edge weights equal to 1) on $p + q + r + s$ vertices obtained by attaching three branches A_{p-1}, A_{q-1}, and A_{r-1} to three consecutive vertices on a* cyclically oriented $(s + 3)$*-cycle (see Fig. 10).*

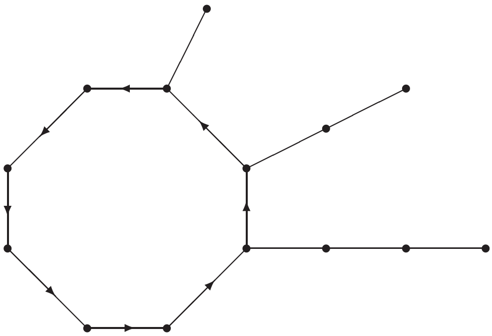
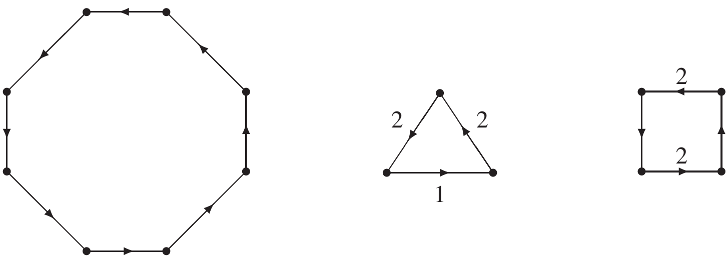


Fig. 10. The diagram $S^5_{4,3,2}$

Lemma 9.6. *The diagram $S^s_{p,q,r}$ is mutation equivalent to $T_{p+r-1,q,s}$.*

Proof. Let us consider the subdiagram of $S^s_{p,q,r}$ obtained by removing the middle branch A_q. This subdiagram is a copy of A_{p+s+r}. We label its vertices consecutively by $1, \ldots, p+s+r$, starting with the endpoint of A_r; and we orient the edges of A_p and A_r so that all the edges of A_{p+s+r} point at the same direction. Now a direct check shows that $\mu_1 \circ \mu_2 \circ \cdots \circ \mu_{s+r}$ transforms $S^s_{p,q,r}$ into $T_{p+r-1,q,s}$. □



(a) (b) (c)

Fig. 11. 2-finite cycles

The proof of Proposition 9.3 can now be completed as follows:

$$\begin{aligned} E_6^{(1)} &= T_{2,2,2} \sim S_{2,2,1}^2 \supset D_5^{(1)}\,; \\ E_7^{(1)} &= T_{3,1,3} \sim S_{3,1,1}^3 \supset E_6^{(1)}\,; \\ E_8^{(1)} &= T_{2,1,5} \sim S_{2,1,1}^5 \supset E_7^{(1)}\,. \end{aligned}$$

□

9.3. Taking care of the cycles

Proposition 9.7. *Let Γ be a 2-finite diagram whose underlying graph is an n-cycle for some $n \geq 3$ (with some orientation of edges). Then Γ must be one of the diagrams shown in Fig. 11. More precisely, one of the following holds:*

(a) *Γ is an oriented cycle with all weights equal to 1.*
In this case, $\Gamma \sim D_n$ (with the understanding that $D_3 = A_3$).
(b) *Γ is an oriented triangle with edge weights 2, 2, 1 shown in Fig. 11(b).*
In this case, $\Gamma \sim B_3$.
(c) *Γ is an oriented 4-cycle with edge weights 2, 1, 2, 1 shown in Fig. 11(c).*
In this case, $\Gamma \sim F_4$.

In particular, the edges in Γ must be cyclically oriented.

Proof. The case $n = 3$ of Proposition 9.7 follows from Lemmas 7.5 and 7.6, so for the rest of the proof we assume that $n \geq 4$.

We begin by proving the last claim of Proposition 9.7 by induction on n. Invoking if necessary a shape-preserving mutation, we may assume that there is a vertex $v \in \Gamma$ that has one incoming and one outgoing edge. Let $\Gamma' = \mu_v(\Gamma)$. Then the subdiagram $\Gamma'' = \Gamma' - \{v\}$ is an $(n-1)$-cycle, which must be cyclically oriented by the induction assumption. Backtracking to Γ, we obtain the desired claim.

Furthermore, observe that the product of edge weights of Γ'' is the same as in Γ. Again using induction together with Lemma 7.6, we conclude that this product is either 1 or 4. In the former case, Γ is an oriented n-cycle, and we apply Lemma 9.6 to obtain $\Gamma = S^{n-3}_{1,1,1} \sim T_{1,1,n-3} = D_n$, as needed. In the latter case, Γ has two edge weights equal to 2, and the rest of them are equal to 1. Then either Γ is one of the two diagrams (b) and (c) in Fig. 11, or else it contains a 2-infinite subdiagram $C^{(1)}_m$ for some $m \geq 2$. It remains to show that the diagrams in Fig. 11(b)–(c) are mutation equivalent to B_3 and F_4, respectively. This is straightforward. □

9.4. Completing the proof of Theorem 8.6. The second claim in Theorem 8.6 follows from Proposition 9.2, so we only need to show that a connected 2-finite diagram Γ is mutation equivalent to some Dynkin diagram. We proceed by induction on n, the number of vertices in Γ. If $n \leq 3$, then Γ is either a tree or a cycle, and the theorem follows by Propositions 9.3 and 9.7. So let us assume that the statement is already known for some $n \geq 3$; we need to show that it holds for a diagram Γ on $n+1$ vertices. Pick a vertex $v \in \Gamma$ such that the subdiagram $\Gamma' = \Gamma - \{v\}$ is connected. Since Γ' is 2-finite, it is mutation equivalent to some Dynkin diagram X_n. Furthermore, we may assume that Γ' is (isomorphic to) our favorite representative of the mutation equivalence class of X_n. For each X_n, we will choose a representative that is most convenient for the purposes of this proof.

Case 1. *Γ' is a Dynkin diagram with no branching point, i.e., is of one of the types A_n, B_n, F_4, or G_2.* Let us orient the edges of Γ' so that they all point in the same direction. If v is adjacent to exactly one vertex of Γ', then Γ is a tree, and we are done by Proposition 9.3. If v is adjacent to more than 2 vertices of Γ', then Γ has a cycle subdiagram whose edges are not cyclically oriented, contradicting Proposition 9.7. Therefore we may assume that v is adjacent to precisely two vertices v_1 and v_2 of Γ'. Thus, Γ has precisely one cycle $\mathcal{C}$, which furthermore must be of one of the types (a)–(c) described in Proposition 9.7 and Fig. 11.

Subcase 1.1. *$\mathcal{C}$ is an oriented cycle with unit edge weights.* If Γ has an edge of weight ≥ 2, then it contains a subdiagram of type $B^{(1)}_m$ or $G^{(1)}_2$, unless $\mathcal{C}$ is a 3-cycle, in which case $\mu_v(\Gamma) \sim B_{n+1}$. On the other hand, if all edges in Γ are of weight 1, then it is one of the diagrams $S^s_{p,q,r}$ in Lemma 9.6 (with $q = 0$). Hence Γ is mutation equivalent to a tree, and we are done by Proposition 9.3.

Subcase 1.2. *$\mathcal{C}$ is as in Fig. 11(b).* If one of the edges (v, v_1) and (v, v_2) has weight 1, then μ_v removes the edge (v_1, v_2), resulting in a tree, and we are done again. So assume that both (v, v_1) and (v, v_2) have weight 2. If at least one edge outside $\mathcal{C}$ has weight ≥ 2, then $\Gamma \supset C^{(1)}_m$ or $\Gamma \supset G^{(1)}_2$. It remains to consider the case shown in Fig. 12 (as before, unspecified edge weights are equal to 1). Direct check shows that $\mu_l \circ \cdots \circ \mu_2 \circ \mu_1 \circ \mu_{v_2} \circ \mu_v(\Gamma) = B_{n+1}$, and we are done.

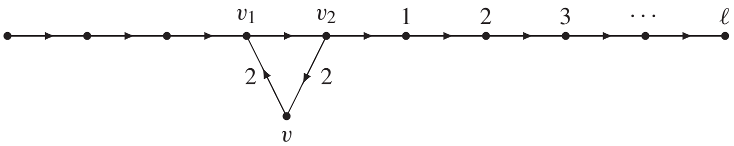


Fig. 12. Subcase 1(b)

Subcase 1.3. $\mathcal{C}$ is as in Fig. 11(c). It suffices to show that any diagram obtained from $\mathcal{C}$ by adjoining a single vertex adjacent to one of its vertices is 2-infinite. If this extra edge has weight 1 (resp., 2, 3), then the resulting 5-vertex diagram has a 2-infinite subdiagram of type $B_3^{(1)}$ (resp., $C_2^{(1)}$, $G_2^{(1)}$), proving the claim.

Case 2. $\Gamma' \sim D_n$ *($n \geq 4$).* By Proposition 9.7(a), we may assume that Γ' is an oriented n-cycle with unit edge weights.

Subcase 2.1. v is adjacent to a single vertex $v_1 \in \Gamma'$. If the edge (v, v_1) has weight ≥ 2, then Γ has a subdiagram $B_3^{(1)}$ or $G_2^{(1)}$. If the edge (v, v_1) has weight 1, then by Lemma 9.6, Γ is mutation equivalent to a tree, and we are done again.

Subcase 2.2. v is adjacent to exactly two vertices v_1 and v_2 of Γ', which are adjacent to each other. Then the triangle (v, v_1, v_2) is either an oriented 3-cycle with unit edge weights or the diagram in Fig. 11(b). In the former case, $\mu_v(\Gamma)$ is an oriented $(n+1)$-cycle, so $\Gamma \sim D_{n+1}$. In the latter case, μ_v reverses the orientation of the edge (v_1, v_2), transforming Γ' into an improperly oriented (hence 2-infinite) cycle (cf. Proposition 9.7).

Subcase 2.3. v is adjacent to two non-adjacent vertices of Γ' (and maybe to some other vertices). In this case, Γ contains a subdiagram which is a non-cyclically-oriented cycle, contradicting Proposition 9.7.

Case 3. $\Gamma' \sim E_n = T_{1,2,n-4}$*, for $n \in \{6, 7, 8\}$.* By Lemma 9.6, we may assume that $\Gamma' = S_{1,2,1}^{n-4}$. In other words, Γ' is a cyclically oriented $(n-1)$-cycle $\mathcal{C}$ with unit edge weights, and an extra edge of weight 1 connecting a vertex in $\mathcal{C}$ to a vertex $v_1 \notin \mathcal{C}$.

Subcase 3.1. v is adjacent to v_1, and to no other vertices in Γ'. If the edge (v, v_1) has weight ≥ 2, then Γ has a 2-infinite subdiagram $B_3^{(1)}$ or $G_2^{(1)}$. If (v, v_1) has weight 1, then by Lemma 9.6, Γ is mutation equivalent to a tree.

Subcase 3.2. v is adjacent to a vertex $v_2 \in \mathcal{C}$, and to no other vertices in Γ'. Then Γ has a subdiagram of type $D_m^{(1)}$ or $B_3^{(1)}$ or $G_2^{(1)}$.

Subcase 3.3. v is adjacent to at least two vertices in $\mathcal{C}$. By the analysis in Subcases 2.2–2.3 (with Γ' replaced by $\mathcal{C}$), we must have $\Gamma - \{v_1\} \sim D_n$, and the problem reduces to Case 2 already treated above (the role of v now played by v_1).

Subcase 3.4. v is adjacent to v_1 and a single vertex $v_2 \in \mathcal{C}$. Let v_0 be the only vertex on $\mathcal{C}$ adjacent to v_1 in Γ'. If v_2 is neither v_0 nor a vertex adjacent to v_0, then the three cycles in Γ cannot be simultaneously oriented. If $v_2 = v_0$, then μ_{v_1} removes the edge (v, v_2), transforming Γ into a diagram mutation equivalent to a tree by Lemma 9.6. If v_2 is adjacent to v_0, then $\Gamma - \{v_0\}$ has no branching point, and the problem reduces to Case 1 already treated above, with the role of v now played by v_0 .

This concludes the proof of Theorem 8.6. As a consequence, we obtain Theorems 7.1 and 1.6. □

10. Proof of Theorem 1.7

Let B and B' be sign-skew-symmetric matrices such that both $A = A(B)$ and $A' = A(B')$ are Cartan matrices of finite type. We already proved that both B and B' are 2-finite. We need to show that B and B' are mutation equivalent if and only if A and A' are of the same type. Without loss of generality, we may assume that A and A' are indecomposable, i.e., the corresponding root systems Φ and Φ' are irreducible.

We first prove the “only if” part. If B and B' are mutation equivalent, then the simplicial complexes $\Delta(\Phi)$ and $\Delta(\Phi')$ are isomorphic to each other, by Theorem 1.13. In particular, Φ and Φ' have the same rank and the same cardinality. A direct check using the tables in [6] shows that the only different Cartan-Killing types with this property are B_n and C_n for all $n \geq 3$, and also E_6, which has the same data as B_6 and C_6. To distinguish between these types, note that mutation-equivalent skew-symmetrizable matrices share the same skew-symmetrizing matrix D. Furthermore, D is skew-symmetrizing for B if and only if it is symmetrizing for A; thus, the diagonal entries of D are given by $d_i = (\alpha_i, \alpha_i)$, where (α, β) is a W-invariant scalar product on the root lattice. Since the root system of type B_n (resp., C_n) has one short simple root and $n-1$ long ones (resp., one long and $n-1$ short), the corresponding matrices B and B' cannot be mutation equivalent. The same is true for E_6 and B_6 (or C_6) since all simple roots for E_6 are of the same length.

To prove the “if” part, suppose that A and A' are of the same Cartan-Killing type. By Proposition 9.2, we may assume without loss of generality that B and B' have the same diagram. By Lemma 8.3, we have $S(B) = S(B')$. Since B and B' share a skew-symmetrizing matrix D, the proof of Lemma 8.3 shows that $S(B) = HBH^{-1}$ and $S(B') = HB'H^{-1}$ for $H = D^{1/2}$. Hence $B = B'$, and we are done. □

11. On cluster algebras of geometric type

In this section we present two general results on cluster algebras of geometric type in the sense of [9, Definition 5.7]. These algebras are *not* assumed to be of finite type, so all the necessary background is contained in Sect. 1.2.

Recall that a cluster algebra is of *geometric type* if it satisfies the following two conditions:

(11.1) The coefficient semifield $\mathbb{P}$ is of the form $\mathrm{Trop}(p_j : j \in J)$. That is, the multiplicative group of $\mathbb{P}$ is a free abelian group with a finite set of generators p_j $(j \in J)$, and the auxiliary addition $\oplus$ is given by (1.1).

(11.2) Every element $p \in \mathcal{P}$, i.e., every coefficient in one of the exchange relations (1.4), is a monomial in the p_j with all exponents nonnegative.

We note a little discrepancy between our choice of the ground ring $\mathbb{Z}[\mathcal{P}]$ in Definition 1.2 and the choice described in [9, Sect. 5], where the ground ring was taken to be the polynomial ring $\mathbb{Z}[p_j : j \in J]$. The following additional assumption guarantees that these two choices coincide:

(11.3) Every generator p_j of $\mathbb{P}$ belongs to $\mathcal{P}$.

11.1. Geometric realization criterion. Our first result gives sufficient conditions under which a cluster algebra of geometric type can be realized as a $\mathbb{Z}$-form of the coordinate ring $\mathbb{C}[X]$ of some algebraic variety X.

We make the following assumptions on X:

(11.4) X is a rational quasi-affine irreducible algebraic variety over $\mathbb{C}$.

Irreducibility implies that the ring of regular functions $\mathbb{C}[X]$ is a domain, so its fraction field is well defined. Quasi-affine means Zariski open in some affine variety; this condition is imposed to ensure that the fraction field of $\mathbb{C}[X]$ coincides with the usual field $\mathbb{C}(X)$ of rational functions on X. Rationality means that X is birationally isomorphic to an affine space, i.e., $\mathbb{C}(X)$ is isomorphic to the field of rational functions over $\mathbb{C}$ in $\dim(X)$ independent variables.

Let $\mathcal{A}$ be a cluster algebra of rank n whose coefficient system satisfies conditions (11.1)–(11.3), and let $\mathcal{X}$ be the set of cluster variables in $\mathcal{A}$. Suppose the variety X satisfies

(11.5) $$\dim(X) = n + |J|;$$

also suppose we are given a family of functions

$$\{\varphi_y : y \in \mathcal{X}\} \cup \{\varphi_j : j \in J\}$$

in $\mathbb{C}[X]$ satisfying the following conditions:

(11.6) the functions φ_y and φ_j generate $\mathbb{C}[X]$;

(11.7) every exchange relation (1.4) becomes an identity in $\mathbb{C}[X]$ if we replace each cluster variable y by φ_y, and each coefficient $p_z^\pm = \prod_{j \in J} p_j^{a_j}$ by $\prod_{j \in J} \varphi_j^{a_j}$.

Proposition 11.1. *Under conditions* (11.1)–(11.7)*, the correspondence*

$$y \mapsto \varphi_y \ (y \in \mathcal{X}), \quad p_j \mapsto \varphi_j \ (j \in J) \tag{11.8}$$

extends uniquely to an algebra isomorphism between the cluster algebra $\mathcal{A}$ and the $\mathbb{Z}$-form of $\mathbb{C}[X]$ generated by all φ_y and φ_j.

Proof. Pick an arbitrary cluster $\mathbf{x}$ of $\mathcal{A}$, and let $\tilde{\mathbf{x}} = \mathbf{x} \cup \{p_j : j \in J\}$. Since $\mathbf{x}$ is a transcendence basis of the ambient field $\mathcal{F}$ over $\mathbb{ZP}$, the set $\tilde{\mathbf{x}}$ is a transcendence basis of $\mathcal{F}$ over $\mathbb{Q}$. Furthermore, every cluster variable is uniquely expressed as a rational function in $\tilde{\mathbf{x}}$ by iterating the exchange relations away from a seed containing $\mathbf{x}$ in the exchange graph of $\mathcal{A}$. In view of (11.7), we can apply the same procedure to express all functions φ_y and φ_j inside the field $\mathbb{C}(X)$ as rational functions in the set

$$\varphi(\tilde{\mathbf{x}}) = \{\varphi_x : x \in \mathbf{x}\} \cup \{\varphi_j : j \in J\}.$$

Furthermore, we have $|\varphi(\tilde{\mathbf{x}})| = \dim(X)$ by (11.5). Since X is rational, we conclude from (11.6) that $\varphi(\tilde{\mathbf{x}})$ is a transcendence basis of the field of rational functions $\mathbb{C}(X)$, and that the correspondence (11.8) extends to an embedding of fields $\mathcal{F} \to \mathbb{C}(X)$, and hence to an embedding of algebras $\mathcal{A} \to \mathbb{C}[X]$. This proves Proposition 11.1. □

11.2. Sharpening the Laurent phenomenon. As mentioned in Sect. 1.4, the *Laurent phenomenon*, established in [9] for arbitrary cluster algebras, says that every cluster variable can be written as a Laurent polynomial in the variables of an arbitrary fixed cluster, with coefficients in $\mathbb{ZP}$. For the cluster algebras of geometric type, this result can be sharpened as follows.

Proposition 11.2. *In any cluster algebra with the coefficient system satisfying conditions* (11.1)–(11.3)*, every cluster variable is expressed in terms of an arbitrary cluster* $\mathbf{x}$ *as a Laurent polynomial with coefficients in* $\mathbb{Z}[\mathcal{P}]$.

Proof. Fix some generator $p = p_{j_\circ}$ of the coefficient semifield $\mathbb{P} = \mathrm{Trop}(p_j : j \in J)$. We will think of any cluster variable z as a Laurent polynomial $z(p)$ whose coefficients are integral Laurent polynomials in the set $\mathbf{x} \cup \{p_j : j \in J, j \neq j_\circ\}$. Our goal is to show that $z(p)$ is in fact a polynomial in p; Proposition 11.2 will then follow by varying a distinguished index $j_\circ$ over the index set J.

Define the *distance* $d(z, \mathbf{x})$ between z and $\mathbf{x}$ as the shortest distance in the exchange graph between a seed containing z and a seed whose cluster is $\mathbf{x}$. We will use induction on $d(z, \mathbf{x})$ to show the following strengthening of the desired statement:

- $z(p)$ is a polynomial in p whose constant term $z(0)$ is a subtraction-free rational expression in $\mathbf{x} \cup \{p_j : j \in J, j \neq j_\circ\}$ (in particular, $z(0) \neq 0$).

If $d(z, \mathbf{x}) = 0$, then $z \in \mathbf{x}$, and there is nothing to prove. If $d(z, \mathbf{x}) > 0$, then, by the definition of the distance, z participates in an exchange relation (1.4) such that all the other participating cluster variables are at a smaller distance from $\mathbf{x}$ than z. Applying the inductive assumption to all these cluster variables and using Lemma 5.2 together with the fact that, by the normalization condition, p appears in at most one of the monomials on the right hand side of (1.4), we obtain our claim for z. □

12. Examples of geometric realizations of cluster algebras

In this section, we present some examples of concrete geometric realizations of cluster algebras $\mathcal{A} = \mathcal{A}(B, \mathbf{p})$ of finite type. In all these examples, the Cartan counterpart of B is a Cartan matrix of one of the classical types A_n, B_n, C_n, D_n, and the coefficient system of $\mathcal{A}$ satisfies conditions (11.1)–(11.3).

12.1. Type A_1. We start by presenting four natural geometric realizations of cluster algebras of type A_1. Such an algebra $\mathcal{A}$ has only two one-element clusters $\{x\}$ and $\{\overline{x}\}$, and a single exchange relation

$$x\overline{x} = p^+ + p^- , \tag{12.1}$$

where p^+ and p^- belong to the coefficient semifield $\mathbb{P}$. By Definition 1.2, $\mathcal{A}$ is a subalgebra of the ambient field $\mathcal{F}$ generated by x, $\overline{x}$, p^+, and p^-.

Example 12.1. Let $\mathcal{A}$ have the coefficient semifield $\mathbb{P} = \mathrm{Trop}(p)$ (the free abelian group with one generator), and let the coefficients in (12.1) be given by $p^+ = p$ and $p^- = 1$. Let $G = SL_2(\mathbb{C})$ be the group of complex matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with $ad - bc = 1$. The correspondence

$$x \mapsto a, \quad \overline{x} \mapsto d, \quad p \mapsto bc$$

identifies $\mathcal{A}$ with the subring of the coordinate ring $\mathbb{C}[G]$ generated by a, d, and bc. It is easy to see that this ring is a $\mathbb{Z}$-form of the ring of invariants $\mathbb{C}[G]^H$, where H is the maximal torus of diagonal matrices in G acting on G by conjugation.

The next three examples give three different realizations of the same cluster algebra $\mathcal{A}$ for which the coefficients in (12.1) are the generators of $\mathbb{P} = \mathrm{Trop}(p^+, p^-)$.

Example 12.2. Let N be the group of complex matrices of the form

$$\begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}.$$

The correspondence

$$x \mapsto a, \quad \overline{x} \mapsto b, \quad p^+ \mapsto c, \quad p^- \mapsto ab - c$$

identifies $\mathcal{A}$ with a $\mathbb{Z}$-form $\mathbb{Z}[N] = \mathbb{Z}[a, b, c]$ of the ring $\mathbb{C}[N]$.

Example 12.3. Let $G = SL_3(\mathbb{C})$, and let $N \subset G$ be the same as in Example 12.2. Let $X = G/N$ be the *base affine space* of G taken in the standard embedding into $\mathbb{C}^3 \times \bigwedge^2 \mathbb{C}^3$. Let $(\Delta_1, \Delta_2, \Delta_3)$ and $(\Delta_{12}, \Delta_{13}, \Delta_{23})$ (here $\Delta_{ij} = \Delta_i \wedge \Delta_j$) be the standard (Plücker) coordinates in $\mathbb{C}^3$ and $\bigwedge^2 \mathbb{C}^3$, respectively. In these coordinates, the coordinate ring of X is given by

$$\mathbb{C}[X] = \mathbb{C}[\Delta_1, \Delta_2, \Delta_3, \Delta_{12}, \Delta_{13}, \Delta_{23}]/\langle \Delta_1\Delta_{23} - \Delta_2\Delta_{13} + \Delta_3\Delta_{12} \rangle.$$

The correspondence

$$x \mapsto \Delta_2, \quad \overline{x} \mapsto \Delta_{13}, \quad p^+ \mapsto \Delta_1\Delta_{23}, \quad p^- \mapsto \Delta_3\Delta_{12}$$

identifies $\mathcal{A}$ with the subring of $\mathbb{C}[X]$ generated by $\Delta_2, \Delta_{13}, \Delta_1\Delta_{23}$, and $\Delta_3\Delta_{12}$. It is easy to see that this ring is a $\mathbb{Z}$-form of the ring of invariants $\mathbb{C}[X]^T$, where $T \subset G$ is the torus of all diagonal matrices of the form

$$\begin{bmatrix} t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t^{-1} \end{bmatrix},$$

acting on X by left translations.

Example 12.4. Let $X \subset \bigwedge^2 \mathbb{C}^4$ be the affine cone over the Grassmannian $\mathrm{Gr}_{2,4}$ taken in its Plücker embedding. In the standard coordinates $(\Delta_{ij} : 1 \le i < j \le 4)$ on $\bigwedge^2 \mathbb{C}^4$, the coordinate ring of X is given by

$$\mathbb{C}[X] = \mathbb{C}[(\Delta_{ij})]/\langle \Delta_{12}\Delta_{34} - \Delta_{13}\Delta_{24} + \Delta_{14}\Delta_{23} \rangle \,.$$

The correspondence

$$x \mapsto \Delta_{13}, \quad \overline{x} \mapsto \Delta_{24}, \quad p^+ \mapsto \Delta_{12}\Delta_{34}, \quad p^- \mapsto \Delta_{14}\Delta_{23}$$

identifies $\mathcal{A}$ with the subring of $\mathbb{C}[X]$ generated by $\Delta_{13}, \Delta_{24}, \Delta_{12}\Delta_{34}$, and $\Delta_{14}\Delta_{23}$. This ring is a $\mathbb{Z}$-form of the ring of invariants $\mathbb{C}[X]^T$, where $T \subset SL_4$ is the torus of all diagonal matrices of the form

$$\begin{bmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & t_1^{-1} & 0 \\ 0 & 0 & 0 & t_2^{-1} \end{bmatrix},$$

naturally acting on X.

12.2. Type A_n $(n \geq 2)$**.** Here we present a geometric realization of a cluster algebra of type A_n for all $n \geq 2$, for a special choice of a coefficient system, to be specified below.

First, we reproduce the concrete description of the cluster complex of type A_n given in [11, Sect. 3.5]. We identify $\Phi_{\geq -1}$ with the set of all diagonals of a regular $(n+3)$-gon $\mathbf{P}_{n+3}$. Under this identification, the roots in $-\Pi$ correspond to the diagonals on the "snake" shown in Fig. 13. Noncrossing diagonals represent compatible roots, while crossing diagonals correspond to roots whose compatibility degree is 1. (Here and in the sequel, two diagonals are called *crossing* if they are distinct and have a common interior point.) Thus, each positive root $\alpha[i, j] = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j$ corresponds to the unique diagonal that crosses precisely the diagonals $-\alpha_i, -\alpha_{i+1}, \ldots, -\alpha_j$ from the snake (see Fig. 14).

The clusters are in bijection with the triangulations of $\mathbf{P}_{n+3}$ by noncrossing diagonals. The cluster complex is the dual complex of the ordinary associahedron. Two triangulations are joined by an edge in the exchange graph if and only if they are obtained from each other by a "flip" that replaces a diagonal in a quadrilateral formed by two triangles of the triangulation by another diagonal of the same quadrilateral. See [11, Sect. 3.5] and [7, Sect. 4.1] for further details.

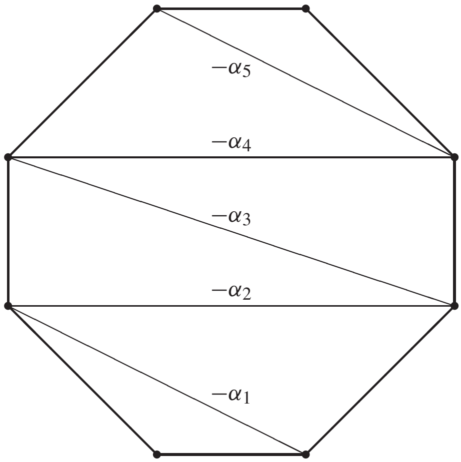


Fig. 13. The "snake" in type A_5

We next describe the cluster variables and the exchange relations in concrete combinatorial terms. For a diagonal $[a, b]$, we denote by x_{ab} the cluster variable $x[\alpha]$ associated to the corresponding root. We adopt the convention that $x_{ab} = 1$ if a and b are two consecutive vertices of $\mathbf{P}_{n+3}$. Comparing (4.9) with [7, Lemma 4.2], we conclude that the matrices $B(C)$ can be described as follows.

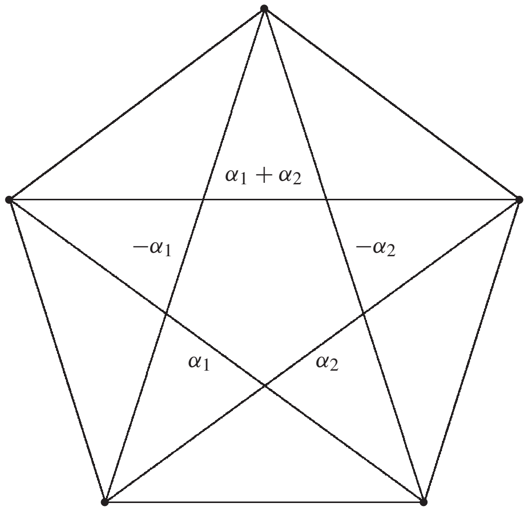


Fig. 14. Labeling of the diagonals in type A_2

Proposition 12.5. *Let C be the cluster corresponding to a triangulation T of $\mathbf{P}_{n+3}$, and let $B(C) = B(T)$ be the corresponding matrix with rows and columns indexed by the diagonals in T. Then each matrix entry $b_{\alpha\beta}$ is equal to 0 unless α and β are two sides of some triangle (a, b, c) in T; in the latter case, $b_{\alpha\beta} = 1$ (resp., -1) if $\alpha = [a, b]$, $\beta = [a, c]$, and the order of points a, b, c is counter-clockwise (resp., clockwise).*

In view of Proposition 12.5, the exchange relations (5.1) in a cluster algebra of type A_n have the form

$$x_{ac}x_{bd} = p^+_{ac,bd}\, x_{ab}\, x_{cd} + p^-_{ac,bd}\, x_{ad}\, x_{bc}\ , \tag{12.2}$$

where a, b, c, d are any four vertices of $\mathbf{P}_{n+3}$ taken in counter-clockwise order, and $p^{\pm}_{ac,bd}$ are elements of the coefficient semifield $\mathbb{P}$. See Fig. 15.

Let $\mathcal{A}_\circ$ denote the cluster algebra of type A_n associated with the following coefficient system. We take

$$\mathbb{P} = \mathrm{Trop}(p_{ab} : [a, b] \text{ is a side of } \mathbf{P}_{n+3})\ , \tag{12.3}$$

and define the coefficients in (12.2) by

$$p^+_{ac,bd} = q_{ab}\, q_{cd}\,, \quad p^-_{ac,bd} = q_{ad}\, q_{bc}\ , \tag{12.4}$$

where

$$q_{ab} = \begin{cases} 1 & \text{if } [a, b] \text{ is a diagonal;} \\ p_{ab} & \text{if } [a, b] \text{ is a side.} \end{cases} \tag{12.5}$$

Since $p^+_{ac,bd}$ and $p^-_{ac,bd}$ have no common factors, they satisfy the normalization condition $p^+_{ac,bd} \oplus p^-_{ac,bd} = 1$; a direct check using Proposition 12.5

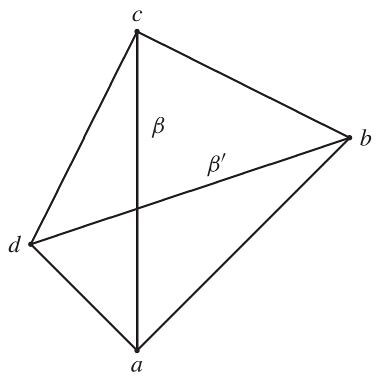


Fig. 15. Exchanges in type A_n

shows that this choice of coefficients also satisfies the mutation rule (1.5), making the cluster algebra $\mathcal{A}_\circ$ well-defined.

*Example 12.6 (*Geometric realization for $\mathcal{A}_\circ$ in type A_n*).* Let $X = X_{n+3}$ be the affine cone over the Grassmannian $\mathrm{Gr}_{2,n+3}$ of 2-dimensional subspaces in $\mathbb{C}^{n+3}$ taken in its Plücker embedding (cf. Example 12.4); simply put, X is the variety of all nonzero decomposable bivectors in $\bigwedge^2 \mathbb{C}^{n+3}$. Let $(\Delta_{ab} : 1 \le a < b \le n+3)$ be the standard Plücker coordinates on X. We identify the indices $1, \ldots, n+3$ with the vertices of $\mathbf{P}_{n+3}$ by numbering these vertices, say, counterclockwise. Thus, we associate the Plücker coordinates with all the sides and diagonals of $\mathbf{P}_{n+3}$. Note that we have previously used the same set $\{(ab) : 1 \le a < b \le n+3)\}$ to label the cluster variables x_{ab} and the coefficients p_{ab} .

Proposition 12.7. *The correspondence sending each cluster variable x_{ab} and each coefficient p_{ab} to the corresponding element Δ_{ab} extends uniquely to an algebra isomorphism between the cluster algebra $\mathcal{A}_\circ$ and the $\mathbb{Z}$-form $\mathbb{Z}[X]$ of $\mathbb{C}[X]$ generated by all Plücker coordinates.*

Proof. This is a special case of Proposition 11.1. To see this, we need to verify the conditions (11.4)–(11.7). The fact that X satisfies (11.4) is well known (for the rationality property, it is enough to note that X has a Zariski open subset isomorphic to an affine space). For the dimension count (11.5), we have

$$\dim(X) = \dim(\mathrm{Gr}_{2,n+3}) + 1 = 2n + 3 = n + |J|,$$

as required. The property (11.6) means that $\mathbb{C}[X]$ is generated by all Plücker coordinates, which is trivial. Finally, (11.7) follows from the standard fact that the Plücker coordinates satisfy the Grassmann-Plücker relations

$$\Delta_{ac}\Delta_{bd} = \Delta_{ab}\,\Delta_{cd} + \Delta_{ad}\,\Delta_{bc}$$

for all $1 \le a < b < c < d \le n+3$. □

We note that the ring $\mathbb{Z}[X]$ is naturally identified with the ring of SL_2-invariant polynomial functions with coefficients in $\mathbb{Z}$ on the space of $(n+3)$-tuples of vectors in $\mathbb{C}^2$. Representing these vectors as columns of a $2\times(n+3)$ matrix $Z=(z_{ij})$, we identify the Plücker coordinates with the 2×2 minors of Z:

$$\Delta_{ab} = z_{1a}z_{2b} - z_{1b}z_{2a} \quad (1 \le a < b \le n+3).$$

Remark 12.8. It is classically known that the monomials in the Plücker coordinates that are not divisible by $\Delta_{ac}\Delta_{bd}$ for any $a<b<c<d$, form a $\mathbb{Z}$-basis in $\mathbb{Z}[X]$ (see [16] or [26] for a proof). Let us translate this fact into the setting of cluster algebras. We shall call a monomial $\prod_\alpha x_\alpha^{m_\alpha}$ in the cluster variables *compatible* if $m_\alpha m_\beta = 0$ whenever the roots α and β are incompatible, i.e., whenever the corresponding diagonals cross each other. (Equivalently, all variables contributing to a compatible monomial belong to a single cluster.) In this terminology, the cluster algebra $\mathcal{A}_\circ$ is a free $\mathbb{Z}[\mathcal{P}]$-module with the basis formed by all compatible monomials. We believe that this property remains true for an arbitrary cluster algebra of finite type (we have checked it for all classical types); we plan to investigate it in a separate publication. We note that linear independence of compatible monomials is an immediate consequence of Theorem 1.9 and the uniqueness of cluster expansions (Proposition 3.3).

12.3. Types B_n and C_n. Let Φ be a root system of type B_n or C_n. We identify the set I in a standard way with $[1,n]$. As in [7, Sect. 4.2], in order to treat both cases at the same time, we set $r=1$ for Φ of type B_n, and $r=2$ for Φ of type C_n. Once again, our convention for the Cartan matrices is different from the one in [6] but agrees with that in [15]: thus, we have $a_{n-1,n}=-r$ and $a_{n,n-1}=-2/r$.

We recall the combinatorial description of the cluster complex of type B_n/C_n from [11, Sect. 3.5]. Let Θ denote the 180° rotation of a regular $(2n+2)$-gon $\mathbf{P}_{2n+2}$. There is a natural action of Θ on the diagonals of $\mathbf{P}_{2n+2}$. Each orbit of this action is either a diameter (i.e., a diagonal connecting antipodal vertices) or an unordered pair of centrally symmetric non-diameter diagonals of $\mathbf{P}_{2n+2}$. Following [11, Sect. 3.5], we identify almost positive roots in Φ with these orbits. Under this identification, each of the roots $-\alpha_i$ for $i=1,\dots,n-1$ is represented by a pair of diagonals on the "snake" shown in Fig. 16, whereas $-\alpha_n$ is identified with the only diameter on the snake. Two Θ-orbits represent compatible roots if and only if the diagonals they involve do not cross each other. More generally, in type B_n (resp., C_n), for $\alpha,\beta\in\Phi_{\ge -1}$, the compatibility degree $(\alpha\|\beta)$ is equal to the number of crossings of one of the diagonals representing α (resp., β) by the diagonals representing β (resp., α). Thus, each positive root $\beta=\sum_i b_i\alpha_i$ is represented by the unique Θ-orbit such that every diagonal representing $-\alpha_i$ (resp., β) crosses the diagonals representing β (resp., $-\alpha_i$) at b_i points.

The clusters are in bijection with the centrally-symmetric (that is, Θ-invariant) triangulations of $\mathbf{P}_{2n+2}$ by non-crossing diagonals. The cluster

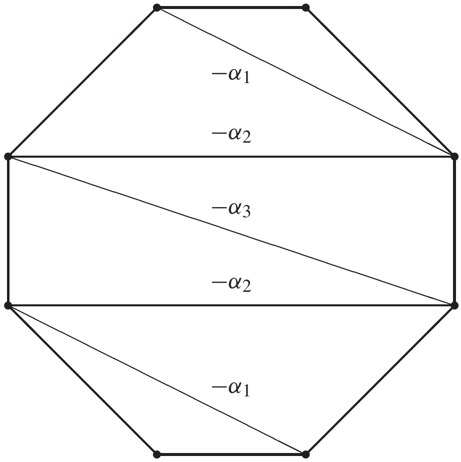


Fig. 16. The “snake” for the types B_3 and C_3

complex is the dual complex for the Bott-Taubes cyclohedron [5]. Two centrally symmetric triangulations are joined by an edge in the exchange graph if and only if they are obtained from each other either by a flip involving two diameters, or by a pair of centrally symmetric flips. See [11, Sect. 3.5] and [7, Sect. 4.2] for details.

For a vertex a of $\mathbf{P}_{2n+2}$, let $\overline{a}$ denote the antipodal vertex $\Theta(a)$. For a diagonal $[a, b]$, we denote by x_{ab} the cluster variable $x[\alpha]$ associated to the root corresponding to the Θ-orbit of $[a, b]$. Thus, we have $x_{ab} = x_{ba} = x_{\overline{a}\,\overline{b}}$. Similarly to the type A_n, we adopt the convention that $x_{ab} = 1$ if a and b are consecutive vertices in $\mathbf{P}_{2n+2}$.

Comparing (5.1) with [7, Lemma 4.4], we obtain the following concrete description of the exchange relations in types B_n and C_n .

Proposition 12.9. *The exchange relations in a cluster algebra of type B_n or C_n have the following form:*

$$x_{ac}x_{bd} = p^{+}_{ac,bd}\, x_{ab}\, x_{cd} + p^{-}_{ac,bd}\, x_{ad}\, x_{bc}\ , \tag{12.6}$$

whenever $a, b, c, d, \overline{a}$ are in counter-clockwise order;

$$x_{ac}x_{a\overline{b}} = p^{+}_{ac,a\overline{b}}\, x_{ab}\, x_{a\overline{c}} + p^{-}_{ac,a\overline{b}}\, x^{2/r}_{a\overline{a}}\, x_{bc}\ , \tag{12.7}$$

whenever $a, b, c, \overline{a}$ are in counter-clockwise order;

$$x_{a\overline{a}}x_{b\overline{b}} = p^{+}_{a\overline{a},b\overline{b}}\, x^{r}_{ab} + p^{-}_{a\overline{a},b\overline{b}}\, x^{r}_{a\overline{b}}\ , \tag{12.8}$$

whenever $a, b, \overline{a}$ are in counter-clockwise order. See Fig. 17.

We will provide an explicit realization of a special cluster algebra $\mathcal{A}_\circ$ of type B_n or C_n similar to its namesake for A_n. The algebra $\mathcal{A}_\circ$ corresponds to

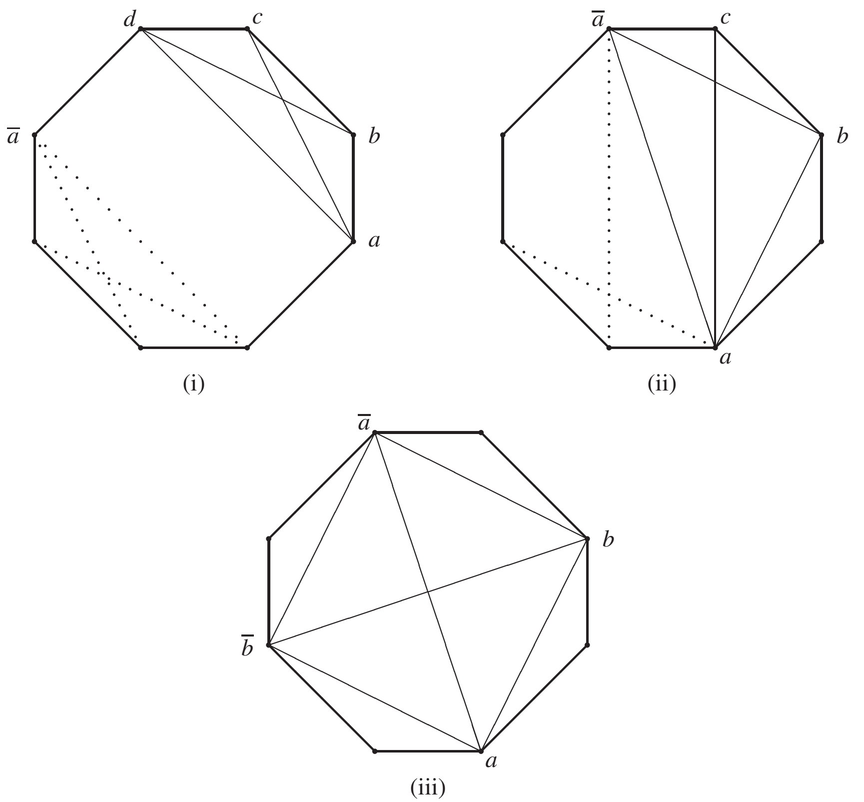


Fig. 17. Exchanges in types B_n and C_n

the following special choice of coefficients. We set $\mathbb{P} = \mathrm{Trop}(\{p_\delta\})$, where δ runs over all centrally-symmetric pairs of sides of the polygon $\mathbf{P}_{2n+2}$. For such a pair $\delta = \{[a,b],[\overline{a},\overline{b}])\}$, we write the corresponding generator of $\mathbb{P}$ as $p_\delta = p_{ab} = p_{\overline{a}\,\overline{b}}$. The coefficients in (12.6)–(12.8) are specified in a similar way to (12.4)–(12.5). More precisely, to obtain a coefficient of some monomial in (12.6)–(12.8), take this monomial and replace each of its cluster variables x_{ab} by 1 (resp., p_{ab}) if $[a,b]$ is a diagonal (resp., a side) of $\mathbf{P}_{2n+2}$. The fact that these coefficients satisfy the normalization condition is again obvious; we leave to the reader a direct check that they also satisfy the mutation rule (1.5).

So far the material for B_n has been completely parallel to that for C_n. However, our geometric realizations for these two types are quite different from each other.

*Example 12.10 (*Geometric realization for $\mathcal{A}_\circ$ in type B_n*).* Somewhat surprisingly, it turns out that the algebra $\mathcal{A}_\circ$ for the type B_n (for $n \geq 3$) is isomorphic, as a ring, to the cluster algebra $\mathcal{A}_\circ$ for the type A_{n-1}. Recall that the latter is naturally identified with the ring $\mathbb{Z}[X_{n+2}]$ generated by the

Plücker coordinates Δ_{ab} (for $1 \le a < b \le n+2$) on the Grassmannian $\mathrm{Gr}_{2,n+2}$ (see Proposition 12.7).

Let us label the vertices of $\mathbf{P}_{2n+2}$ in the counterclockwise order by the indices $1, \dots, n+1, \overline{1}, \dots, \overline{n+1}$. We associate a function from $\mathbb{Z}[X_{n+2}]$ to every Θ-orbit on the set of all diagonals and sides of $\mathbf{P}_{2n+2}$, as follows:

$$\begin{aligned} &[a,\overline{a}] \mapsto \Delta_{a\overline{a}} = \Delta_{a,n+2} && (1 \le a \le n+1),\\ &\{[a,b],[\overline{a},\overline{b}]\} \mapsto \Delta_{ab} && (1 \le a < b \le n+1),\\ &\{[a,\overline{b}],[\overline{a},b]\} \mapsto \Delta_{a\overline{b}} = \Delta_{a,n+2}\Delta_{b,n+2} - \Delta_{ab} && (1 \le a < b \le n+1). \end{aligned} \tag{12.9}$$

Proposition 12.11. *The correspondence sending each cluster variable and each coefficient for the cluster algebra $\mathcal{A}_\circ$ of type B_n to the element in* (12.9) *with the same label extends uniquely to an algebra isomorphism of $\mathcal{A}_\circ$ with* $\mathbb{Z}[X_{n+2}]$.

Proof. The proof is similar to that of Proposition 12.7. It is enough to check that our data satisfy the conditions (11.4)–(11.7). Condition (11.4) was already checked in the proof of Proposition 12.7. The dimension count (11.5) now takes the form:

$$\dim(X_{n+2}) = 2n+1 = n + |J|,$$

as required. The property (11.6) is clear since all the Plücker coordinates Δ_{ab} for $1 \le a < b \le n+2$ are among the functions (12.9). Finally, (11.7) amounts to checking the following six identities (with $r=1$) obtained from the exchange relations (12.6)–(12.8) (we have to take into account possible positions of vertices in Fig. 17 among the vertices $1, \dots, n{+}1, \overline{1}, \dots, \overline{n+1}$):

$$\Delta_{ac}\Delta_{bd} = \Delta_{ab}\Delta_{cd} + \Delta_{ad}\Delta_{bc} \quad (1 \le a < b < c < d \le n+1), \tag{12.10}$$

$$\Delta_{a\overline{c}}\Delta_{bd} = \Delta_{a\overline{b}}\Delta_{cd} + \Delta_{a\overline{d}}\Delta_{bc} \quad (1 \le a < b < c < d \le n+1), \tag{12.11}$$

$$\Delta_{a\overline{c}}\Delta_{b\overline{d}} = \Delta_{ab}\Delta_{cd} + \Delta_{a\overline{d}}\Delta_{b\overline{c}} \quad (1 \le a < b < c < d \le n+1), \tag{12.12}$$

$$\Delta_{ac}\Delta_{a\overline{b}} = \Delta_{ab}\Delta_{a\overline{c}} + \Delta_{a\overline{a}}^{2/r}\Delta_{bc} \quad (1 \le a < b < c \le n+1), \tag{12.13}$$

$$\Delta_{a\overline{b}}\Delta_{b\overline{c}} = \Delta_{ab}\Delta_{bc} + \Delta_{b\overline{b}}^{2/r}\Delta_{a\overline{c}} \quad (1 \le a < b < c \le n+1), \tag{12.14}$$

$$\Delta_{a\overline{a}}\Delta_{b\overline{b}} = \Delta_{ab}^{r} + \Delta_{a\overline{b}}^{r} \quad (1 \le a < b \le n+1). \tag{12.15}$$

Of these identities, (12.10) is a Grassmann-Plücker relation, and the rest are reduced to this relation by simple algebraic manipulations. □

*Example 12.12 (*Geometric realization for $\mathcal{A}_\circ$ in type C_n*).* Let SO_2 be the group of complex matrices

$$\begin{bmatrix} u & -v \\ v & u \end{bmatrix}$$

with $u^2 + v^2 = 1$. Consider the algebra $\mathcal{R} = \mathbb{C}[\mathrm{Mat}_{2,n+1}]^{SO_2}$ of SO_2-invariant polynomial functions on the space of $2 \times (n+1)$ complex matrices, or, equivalently, on the space of $(n+1)$-tuples of vectors in $\mathbb{C}^2$. Alternatively, $\mathcal{R}$ can be identified with the ring of invariants $\mathbb{C}[\mathrm{Mat}_{2,n+1}]^T$, where $T \subset SL_2$ is the torus of all diagonal matrices of the form

$$\begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}.$$

Indeed, we have $g(SO_2)g^{-1} = T$, where

$$g = \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix},$$

so the map $f \mapsto f^g$ defined by $f^g(z) = f(gz)$ is an isomorphism

$$\mathbb{C}[\mathrm{Mat}_{2,n+1}]^T \to \mathbb{C}[\mathrm{Mat}_{2,n+1}]^{SO_2}. \qquad (12.16)$$

The ring $\mathcal{R} = \mathbb{C}[\mathrm{Mat}_{2,n+1}]^T$ can also be viewed as the coordinate ring $\mathbb{C}[X]$ of the variety X of complex $(n+1) \times (n+1)$ matrices of rank ≤ 1 (even more geometrically, $X - \{0\}$ is the affine cone over the product of two copies of the projective space $\mathbb{CP}^n$ taken in the Segre embedding). Specifically, the map

$$y = \begin{bmatrix} y_{11} & \cdots & y_{1,n+1} \\ y_{21} & \cdots & y_{2,n+1} \end{bmatrix} \mapsto (y_{1a}y_{2b})_{a,b=1,\dots,n+1} \in X$$

induces an algebra isomorphism $\mathbb{C}[X] \to \mathbb{C}[\mathrm{Mat}_{2,n+1}]^T$. Combining this with (12.16), we obtain an isomorphism $\mathbb{C}[X] \to \mathbb{C}[\mathrm{Mat}_{2,n+1}]^{SO_2}$ induced by the map

$$\begin{bmatrix} z_{11} & \cdots & z_{1,n+1} \\ z_{21} & \cdots & z_{2,n+1} \end{bmatrix} \mapsto ((z_{1a} - iz_{2a})(z_{1b} + iz_{2b}))_{a,b=1,\dots,n+1} \in X.$$

By analogy with (12.9), we associate an element from $\mathcal{R} = \mathbb{C}[X]$ to every Θ-orbit on the set of all diagonals and sides of $\mathbf{P}_{2n+2}$, as follows:

$$\begin{aligned} \{[a,b],[\overline{a},\overline{b}]\} \mapsto \Delta_{ab} &= z_{1a}z_{2b} - z_{1b}z_{2a} \\ &= \frac{y_{1a}y_{2b} - y_{1b}y_{2a}}{2i}(1 \le a < b \le n+1), \\ \{[a,\overline{b}],[\overline{a},b]\} \mapsto \Delta_{a\overline{b}} &= z_{1a}z_{1b} + z_{2a}z_{2b} \\ &= \frac{y_{1a}y_{2b} + y_{1b}y_{2a}}{2}(1 \le a \le b \le n+1). \end{aligned} \qquad (12.17)$$

The following result is a type C_n counterpart of Proposition 12.11.

Proposition 12.13. *The correspondence sending each cluster variable and each coefficient for the cluster algebra $\mathcal{A}_\circ$ of type C_n to the element in* (12.17) *with the same label extends uniquely to an algebra isomorphism of $\mathcal{A}_\circ$ with a $\mathbb{Z}$-form of $\mathcal{R}$.*

Proof. The proof is analogous to that of Proposition 12.11. The only work involved is to check that the functions in (12.17) satisfy the identities (12.10)–(12.15), this time with $r = 2$. This is completely straightforward; for example, (12.15) becomes

$$\big(z_{1a}^2 + z_{2a}^2\big)\big(z_{1b}^2 + z_{2b}^2\big) = \big(z_{1a}z_{2b} - z_{1b}z_{2a}\big)^2 + \big(z_{1a}z_{1b} + z_{2a}z_{2b}\big)^2.$$

□

12.4. Type D_n. Let Φ be a root system of type D_n (allowing for $n = 3$, in which case Φ is of type A_3). According to [11, Sect. 3.5], the almost positive roots (hence the cluster variables) for the type D_n have a natural surjection onto those for B_{n-1}. This surjection is one-to-one over the roots corresponding to pairs of diagonals of $\mathbf{P}_{2n}$, and two-to-one over those corresponding to diameters. Thus, the roots in $\Phi_{\geq -1}$ are represented by Θ-orbits on the set of diagonals in a regular $2n$-gon, in which each diameter can be of one of two different "colors"; we denote the two different kinds of diameters by $[a, \overline{a}]$ and $\widetilde{[a, \overline{a}]}$. The negative simple roots form a "type D snake" shown in Fig. 18. Two Θ-orbits represent compatible roots if and only if the diagonals they involve do not cross each other; here we use the following convention:

(12.18) diameters of the same color do not cross each other.

More generally, for $\alpha, \beta \in \Phi_{\geq -1}$, the compatibility degree $(\alpha \| \beta)$ is equal to the number of Θ-orbits in the set of crossing points between the diagonals representing α and β (again, with the convention (12.18)). Each positive root $\beta = \sum_i b_i \alpha_i$ is then represented by the unique Θ-orbit such that the diagonals representing β cross the diagonals representing $-\alpha_i$ at b_i pairs of centrally symmetric points (counting an intersection of two diameters of different color and location as one such pair).

Accordingly, the cluster variables for D_n can be denoted as x_α, for all diagonals α in $\mathbf{P}_{2n}$ (with the convention $x_\alpha = x_{\Theta(\alpha)}$), plus n extra variables $\tilde{x}_\beta$ for all diameters $\tilde{\beta}$.

With the help of [7, Lemma 4.6], we obtain the following analogue of Proposition 12.9.

Proposition 12.14. *The exchange relations in a cluster algebra of type D_n have the following form:*

(12.19) $$x_{ac}x_{bd} = p^+_{ac,bd}\, x_{ab}\, x_{cd} + p^-_{ac,bd}\, x_{ad}\, x_{bc}$$

whenever $a, b, c, d, \overline{a}$ are in counter-clockwise order;

(12.20) $$x_{ac}x_{a\overline{b}} = p^+_{ac,a\overline{b}}\, x_{ab}\, x_{a\overline{c}} + p^-_{ac,a\overline{b}}\, x_{a\overline{a}}\, \tilde{x}_{a\overline{a}}\, x_{bc}$$

whenever $a, b, c, \overline{a}$ are in counter-clockwise order;

(12.21) $$x_{a\overline{a}}\tilde{x}_{b\overline{b}} = p^+_{a\overline{a},b\overline{b}}\, x_{ab} + p^-_{a\overline{a},b\overline{b}}\, x_{a\overline{b}}$$

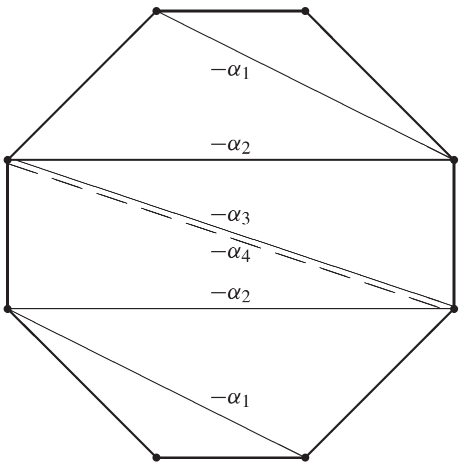


Fig. 18. Representing the roots in $-\Pi$ for the type D_4

whenever $a, b, \overline{a}$ are in counter-clockwise order;

(12.22) $$x_{a\overline{a}}x_{b\overline{c}} = p^{+}_{a\overline{a},b\overline{c}}\, x_{ab}\, x_{c\overline{c}} + p^{-}_{a\overline{a},b\overline{c}}\, x_{a\overline{c}}\, x_{b\overline{b}}$$

and

(12.23) $$\tilde{x}_{a\overline{a}}x_{b\overline{c}} = \tilde{p}^{+}_{a\overline{a},b\overline{c}}\, x_{ab}\, \tilde{x}_{c\overline{c}} + \tilde{p}^{-}_{a\overline{a},b\overline{c}}\, x_{a\overline{c}}\, \tilde{x}_{b\overline{b}}$$

whenever $a, b, c, \overline{a}$ are in counter-clockwise order.

We define a special coefficient system and the corresponding cluster algebra $\mathcal{A}_\circ$ of type D_n in precisely the same way as for the types B_n and C_n above. We conclude this paper with a geometric realization of this algebra similar to the one given in Example 12.10.

Example 12.15 (Geometric realization for $\mathcal{A}_\circ$ in type D_n). Consider the same variety X_{n+2} as in Example 12.10. Let X be the divisor in X_{n+2} given by the equation $\Delta_{n+1,n+2} = 0$; thus, we have

$$\mathbb{C}[X] = \mathbb{C}[X_{n+2}]/\langle \Delta_{n+1,n+2}\rangle\ .$$

(Geometrically, X is the affine cone over the Schubert divisor in the Grassmannian $\mathrm{Gr}_{2,n+2}$.) Let $\mathbb{Z}[X]$ denote the $\mathbb{Z}$-form of $\mathbb{C}[X]$ generated by all Plücker coordinates.

By analogy with (12.9), we introduce the following family of functions from $\mathbb{Z}[X]$:

(12.24) $$\begin{aligned} [a,\overline{a}] &\mapsto \Delta_{a\overline{a}} = \Delta_{a,n+1}, && (1 \le a \le n),\\ \widetilde{[a,\overline{a}]} &\mapsto \tilde{\Delta}_{a\overline{a}} = \Delta_{a,n+2} && (1 \le a \le n),\\ \{[a,b],[\overline{a},\overline{b}]\} &\mapsto \Delta_{ab} && (1 \le a < b \le n),\\ \{[a,\overline{b}],[\overline{a},b]\} &\mapsto \Delta_{a\overline{b}} = \Delta_{a,n+1}\Delta_{b,n+2} - \Delta_{ab} && (1 \le a < b \le n). \end{aligned}$$

(Note that in $\mathbb{Z}[X]$, there is a relation $\Delta_{a,n+1}\Delta_{b,n+2} = \Delta_{a,n+2}\Delta_{b,n+1}$ since we have $\Delta_{n+1,n+2} = 0$.)

Proposition 12.16. *The correspondence sending each cluster variable and each coefficient for the cluster algebra $\mathcal{A}_\circ$ of type D_n to the element in* (12.24) *with the same label extends uniquely to an algebra isomorphism between $\mathcal{A}_\circ$ and $\mathbb{Z}[X]$.*

Proof. The proof is completely analogous to that of Proposition 12.11. Details are left to the reader. □

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