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Hyperbolic manifolds are geodesically rigid

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Abstract. We show that if all geodesics of two non-proportional metrics on a closed manifold coincide (as unparameterized curves), then the manifold has a finite fundamental group or admits a local-product structure. This implies that, if the manifold admits a metric of negative sectional curvature, then two metrics on the manifold have the same geodesics if and only if they are proportional.

1. Introduction

1.1. Results

Definition 1. *Let g be a Riemannian metric on a manifold M^n of dimension $n \geq 2$. A Riemannian metric $\bar{g}$ on M^n is called* **geodesically equivalent** *to g, if any geodesic of $\bar{g}$, considered as an unparameterized curve, is a geodesic of g.*

Trivial examples of geodesically equivalent metrics can be obtained by considering proportional metrics g and $C \cdot g$, where C is a positive constant.

Definition 2. *A manifold M^n is called* **geodesically rigid**, *if any two geodesically equivalent Riemannian metrics on M^n are proportional. M^n is called* **hyperbolic**, *if it admits a Riemannian metric of negative sectional curvature.*

Theorem 1. *Every hyperbolic closed connected manifold is geodesically rigid.*

For dimensions two and three, Theorem 1 has been proven in [20,21].

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Recall that hyperbolic manifolds are rigid in different senses. Probably the most famous result in this direction is the Mostow Rigidity Theorem [26]: if two closed manifolds (of dimension greater than two) of constant sectional curvature (-1) are diffeomorphic, then they are isometric.

Different rigidity results have been proven by Benoist, Foulon, Gromov, Hamenstadt, Kanai, A. Katok, Labourie, Ledrappier, Margulis and other mathematicians. Here we recall some results related to rigidities with respect to orbital diffeomorphisms of geodesic flows.

Locally-symmetric closed manifolds of negative curvature are rigid with respect to conjugations of the geodesic flows [2] ([10] for dimension two): Suppose g is an arbitrary metric, and h is a locally symmetric metric of negative sectional curvature. If there exists a diffeomorphism of the tangent bundle of M^n that takes the orbits of the geodesic flow of g to the orbits of the geodesic flow of h (preserving the parameter on the orbits), then there exists a diffeomorphism $\phi : M^n \to M^n$ such that $\phi^* g = C \cdot h$.

For two dimensional manifolds, the above result holds without the assumption that h is locally-symmetric: for the two-dimensional case, the marked length spectrum uniquely determines the metric of nonpositive curvature [28,5,6].

Negatively curved manifolds of arbitrary dimension are spectrally rigid in a slightly weaker sense: there is no isospectral deformation of a closed manifold of negative sectional curvature [7] ([11] for dimension two).

Comparing Theorem 1 with the results listed above, we see that if we add the assumption that the orbital diffeomorphism commutes with the projection, the rigidity results remain true even if we do not require that the metrics are negatively curved and that the parameter of the geodesics is preserved.

As explained below, Theorem 1 is a special case of

Theorem 2. *Let M^n be a closed connected manifold. Suppose two non-proportional Riemannian metrics g, $\bar g$ on M^n are geodesically equivalent. If the fundamental group of M^n is infinite, then there exist $r \in \{1, 2, \dots, n-1\}$, a Riemannian metric $\tilde g$ and foliations B_r (of dimension r) and B_{n-r} (of dimension $n-r$) such that, in a neighborhood $U(p)$ of every point $p \in M^n$, there exist coordinates*

$$(\bar x, \bar y) = \big((x_1, x_2, \dots, x_r), (y_{r+1}, y_{r+2}, \dots, y_n)\big)$$

such that the x-coordinates are constant on every fiber of the foliation $B_{n-r} \cap U(p)$, the y-coordinates are constant on every fiber of the foliation $B_r \cap U(p)$, and the metric $\tilde g$ has the block-diagonal form

$$ds^2 = \sum_{i,j=1}^{r} G_{ij}(\bar x) dx_i dx_j + \sum_{i,j=r+1}^{n} H_{ij}(\bar y) dy_i dy_j, \tag{1}$$

where the first block depends on the first r coordinates and the second block depends on the remaining $n-r$ coordinates. Moreover, the universal cover *of the manifold (with the lifted metric) is isometric to the direct product of two simply-connected Riemannian manifolds* (M_1^r, g_1) *(of dimension* r*) and* (M_2^{n-r}, g_2) *(of dimension* $(n-r)$*) with the product metric* $g_1 + g_2$.

It is known [29], that the fundamental group of a closed hyperbolic manifold is infinite. There are many ways to show that the lift of any metric (not necessarily of negative curvature) of any closed hyperbolic manifold can not be the product metric $g_1 + g_2$. For example, since the universal cover of a hyperbolic manifold is contractible, the components M_1^r and M_2^{n-r} are contractible as well and, hence, the diameters of M_1^r and M_2^{n-r} are infinite. Then, for every fixed number $D \in R$, we can find a geodesic triangle such that the union of the D-neighborhoods of the first two edges does not contain the third edge. For example, we can take the vertices $(x_1, x_2), (x_1, y_2), (y_1, x_2) \in M_1^r \times M_2^{n-r}$ of the triangle such that the distance between x_1 and y_1 (on M_1^r) and the distance between x_2 and y_2 (on M_2^{n-r}) are greater than $4D$. Then, the product of the Riemannian manifolds (M_1^r, g_1) and (M_2^{n-r}, g_2) is not quasi-isometric to a CAT(-1) space. Thus M^n is not hyperbolic.

1.2. History

The theory of geodesically equivalent metrics has long and fascinating history that goes back to the works of Beltrami, Dini and Levi-Civita.

Beltrami [1] was the first to observe that two different metrics can have the same geodesics. Let us describe a natural multi-dimensional generalization of his example: the metric g is the restriction of the Euclidean metrics $dx_1^2 + \ldots + dx_{n+1}^2$ to the sphere

$$S^n \stackrel{\text{def}}{=} \left\{(x_1, x_2, \ldots, x_{n+1}) \in R^{n+1} : x_1^2 + x_2^2 + \ldots + x_{n+1}^2 = 1\right\}.$$

The metric $\bar g$ is the pull-back $l^* g$, where the mapping $l : S^n \to S^n$ is given by $l : v \mapsto \frac{A(v)}{\|A(v)\|}$, where A is an arbitrary linear nondegenerate transformation of R^{n+1}.

The metrics g and $\bar g$ are geodesically equivalent. Indeed, the geodesics of g are great circles (the intersections of planes that go through the origin with the sphere). The mapping A is linear and, hence, takes planes to planes. Since the normalization $w \mapsto \frac{w}{\|w\|}$ takes planes to their intersections with the sphere, the mapping l takes great circles to great circles. Thus the metrics g and $\bar g$ are geodesically equivalent. Evidently, if A is not proportional to an orthogonal transformation, then g and $\bar g$ are non-proportional.

At the end of his paper [1], Beltrami formulated the problem of describing all geodesically equivalent metrics (for surfaces.) It is not clear from the text whether he assumed the local or the global description; actually, his motivation came from a certain problem of cartography, which requires the global setting. Nevertheless, partially because of strong results of Dini, Levi-Civita, Weyl, E. Cartan and Eisenhart, the theory of geodesically equivalent metrics was mostly a local geometry.

Locally, in a neighborhood of almost every point, a complete description of geodesically equivalent metrics has been given by Dini [8] for surfaces and Levi-Civita [18] for manifolds of arbitrary dimension. One of the results of Levi-Civita states that, locally, the set of metrics admitting (nontrivial) geodesic equivalence is quite rich: in the most general case, it is controlled by $n = dim(M)$ parameters, each of them being a function of one variable. For dimensions two and three, every closed manifold has a metric such that in a neighborhood of every point the metric admits a non-proportional geodesically equivalent metric.

Later, geodesically equivalent metrics were considered by Weyl, E. Cartan and Eisenhart. Weyl studied geodesically equivalent metrics on the tensor level and found a few tensor reformulations of geodesic equivalence. One of his most remarkable results is the construction of the projective Weyl tensor [33]: if two metrics are geodesically equivalent, then their projective Weyl tensors coincide. E. Cartan [4] studied geodesic equivalence on the level of affine connections. He introduced the so-called projective connection, which allows reconstruction of geodesics as unparameterized curves. In his book [9], Eisenhart systematically applied both methods and obtained a series of local results.

However, our knowledge of the global (when the manifold is closed or complete) behavior of geodesically equivalent metrics is not satisfactory. Global aspects have been intensively studied by French, Soviet and Japanese geometry schools. But, probably because of the influence of earlier researchers, all known global results require fairly strong additional geometrical assumptions.

Roughly speaking, one takes some geometric assumption written in tensor form, combines it with one of the tensor reformulations of geodesic equivalence and deduces some new object with global geometric properties, see the survey paper [25].

New methods for global investigation of geodesically equivalent metrics have been suggested in [19, 32]. The main observation of [19, 32] is that the existence of $\bar{g}$ geodesically equivalent to g allows one to construct commuting integrals for the geodesic flow of g, see Theorem 4 below.

This observation can be used most efficiently when the number of functionally independent integrals equals the dimension of the manifold. This corresponds to the case when there exists a point on the manifold where the number of different eigenvalues of g with respect to $\bar{g}$ is equal to the dimension of the manifold. Then the geodesic flow of the metric g is Liouville-integrable, and we can apply the well-developed machinery of integrable systems. The following theorem has been obtained in [21, 22] by combining ideas of [31] with technique from [15]:

Theorem 3 ([21, 22]). *Let M^n be a connected closed manifold and g, $\bar{g}$ be geodesically equivalent Riemannian metrics on M^n. Suppose there exists* *a point of the manifold where the number of different eigenvalues of g with respect to $\bar{g}$ equals n. Then the following holds:*

1. *The first Betti number $b_1(M^n)$ is not greater than n.*
2. *The fundamental group of the manifold is virtually Abelian.*
3. *If, in addition, there exists a point where the number of different eigenvalues of g with respect to $\bar{g}$ is less than n, then $b_1(M^n) < n$.*
4. *If $b_1(M^n) = n$, then M^n is homeomorphic to the torus T^n.*

Under the additional assumption that there exists a point where the number of different eigenvalues of one metric with respect to the other is equal to the dimension of the manifold, Theorem 1 follows immediately from Theorem 3: by [29], the fundamental group of a hyperbolic manifold is not virtually Abelian.

If $n = 2$, this additional assumption is equivalent to the non-proportionality of the metrics, and Theorems 1, 2, 3 are equivalent: each of them tells us that a closed surface of genus greater than one is geodesically rigid. A self-contained proof of the last statement can be found in [20].

Note that for dimension three, under the additional assumption that there exists a point where the number of different eigenvalues of one metric with respect to the other is equal to the dimension of the manifold, Theorem 2 follows from Theorem 3 modulo the Poincare conjecture [23].

Theorems 1, 2 are the first multidimensional global results on geodesically equivalent metrics with no additional local assumptions.

1.3. Are Theorems 1 and 2 sharp?

Theorems 1 and 2 are sharp for dimension two: it follows from Beltrami's example that the sphere and the projective plane are not geodesically rigid. Since the geodesics of any flat metric are straight lines, any two flat metrics on the torus (or on the Klein bottle) are geodesically related (that is, there exists a diffeomorphism that takes the geodesics of the first metric to the geodesics of the second.)

Theorem 2 shows that the converse of Theorem 1 is not true in general: for dimension three, it can be shown that only Seifert manifolds with zero Euler number admit a metric $\tilde{g}$ and foliations B_r, B_{n-r} as in Theorem 2, see [23]. Thus every Seifert manifold with a torus base and with non-zero Euler number is geodesically rigid. It is known that Seifert manifolds are not hyperbolic.

The conclusion of Theorem 2 has two conditions: the finiteness of the fundamental group and the existence of the metrics $\tilde{g}$ and the foliations B_r, B_{n-r}.

The second condition is sharp: if M^n admits a metric $\tilde{g}$ and foliations B_r, B_{n-r} as in Theorem 2, it admits a pair of non-proportional geodesically equivalent metrics. Indeed, denote by g_1, g_2 the restrictions of $\tilde{g}$ to the leaves of the foliations B_r, B_{n-r}, respectively. Consider the metric $\bar{g} \stackrel{\text{def}}{=} g_1 + 2 \cdot g_2$.

Since the metric $\tilde{g}$ is equal to $g_1 + g_2$, the metrics $\tilde{g}$ and $\bar{g}$ are geodesically equivalent (they even have the same Christoffel symbols!).

The first condition is not sharp in general: by [24], a closed three-manifold is geodesically rigid if and only if it is homeomorphic neither to a lens space nor to a Seifert manifold with zero Euler number. Seifert manifolds with zero Euler number have infinite fundamental groups. It is known that there exist closed three-manifolds (other than the lens spaces) whose fundamental group is finite. The Poincare homology sphere is probably the most famous example, see [13].

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2. Integrability of geodesic flows for geodesically equivalent metrics

Let $g = (g_{ij})$ and $\bar{g} = (\bar{g}_{ij})$ be Riemannian metrics on a manifold M^n. Consider the (1,1)-tensor L given by the formula

$$L^i_j \stackrel{\text{def}}{=} \left(\frac{\det(\bar{g})}{\det(g)}\right)^{\frac{1}{n+1}} \bar{g}^{i\alpha} g_{\alpha j}. \tag{2}$$

Then, L determines the family S_t, $t \in R$, of (1, 1)-tensors

$$S_t \stackrel{\text{def}}{=} \det(L - t\,\mathrm{Id})\,(L - t\,\mathrm{Id})^{-1}. \tag{3}$$

Remark 1. Although $(L - t\,\mathrm{Id})^{-1}$ is not defined for t lying in the spectrum of L, the tensor S_t is well-defined for every t. Moreover, S_t is a polynomial in t of degree $n - 1$ with coefficients being (1,1)-tensors.

We will identify the tangent and cotangent bundles of M^n by g. This identification allows us to transfer the natural Poisson structure from T^*M^n to TM^n.

Theorem 4 ([19,32]). *If g, $\bar{g}$ are geodesically equivalent, then, for every $t_1, t_2 \in R$, the functions*

$$I_{t_i} : TM^n \to R, \quad I_{t_i}(v) \stackrel{\text{def}}{=} g(S_{t_i}(v), v) \tag{4}$$

are commuting integrals for the geodesic flow of g.

Since L is clearly self-adjoint with respect to $\bar{g}$, the eigenvalues of L are real. At every point $x \in M^n$, let us denote by $\lambda_1(x) \leq \ldots \leq \lambda_n(x)$ the eigenvalues of L at the point.

Corollary 1. *Let (M^n, g) be a geodesically complete connected Riemannian manifold. Let a Riemannian metric $\bar{g}$ on M^n be geodesically equivalent to g. Then, for every $i \in \{1, \ldots, n-1\}$, for every $x, y \in M^n$, the following holds:*

1. $\lambda_i(x) \leq \lambda_{i+1}(y)$.
2. *If $\lambda_i(x) < \lambda_{i+1}(x)$, then $\lambda_i(z) < \lambda_{i+1}(z)$ for almost every point $z \in M^n$.*

In order to prove Corollary 1, we need the following technical lemma. For every fixed $v = (\xi_1, \xi_2, \ldots, \xi_n) \in T_x M^n$, the function (4) is a polynomial in t. Consider the roots of this polynomial. From the proof of Lemma 1, it will be clear that they are real. We denote them by

$$t_1(x, v) \leq t_2(x, v) \leq \ldots \leq t_{n-1}(x, v).$$

Lemma 1. *The following holds for every $i \in \{1, \ldots, n-1\}$:*

1. *For every $v \in T_x M^n$,*

$$\lambda_i(x) \leq t_i(x, v) \leq \lambda_{i+1}(x).$$

   In particular, if $\lambda_i(x) = \lambda_{i+1}(x)$, then $t_i(x, v) = \lambda_i(x) = \lambda_{i+1}(x)$.

2. *If $\lambda_i(x) < \lambda_{i+1}(x)$, then for every $\tau \in R$ the Lebesgue measure of the set*

$$V_\tau \subset T_x M^n, \quad V_\tau \stackrel{\text{def}}{=} \left\{ v \in T_x M^n : t_i(x, v) = \tau \right\},$$

   is zero.

Proof of Lemma 1. By definition, the tensor L is self-adjoint with respect to g. Then, for every $x \in M^n$, there exists "diagonal" coordinates in $T_x M^n$ where the metric g is given by the diagonal matrix $\mathrm{diag}(1, 1, \ldots, 1)$ and the tensor L is given by the diagonal matrix $\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Then, the tensor (3) reads:

$$\begin{aligned} S_t &= \det(L - t\mathrm{Id})(L - t\mathrm{Id})^{(-1)} \\ &= \mathrm{diag}(\Pi_1(t), \Pi_2(t), \ldots, \Pi_n(t)), \end{aligned}$$

where the polynomials $\Pi_i(t)$ are given by the formula

$$\Pi_i(t) \stackrel{\text{def}}{=} (\lambda_1 - t)(\lambda_2 - t) \ldots (\lambda_{i-1} - t)(\lambda_{i+1} - t) \ldots (\lambda_{n-1} - t)(\lambda_n - t).$$

Hence, for every $v = (\xi_1, \ldots, \xi_n) \in T_x M^n$, the polynomial $I_t(x, v)$ is given by

$$I_t = \xi_1^2 \Pi_1(t) + \xi_2^2 \Pi_2(t) + \ldots + \xi_n^2 \Pi_n(t). \tag{5}$$

Evidently, the coefficients of the polynomial I_t depend continuously on the eigenvalues λ_i and on the components ξ_i. Then, it is sufficient to prove the first statement of the lemma assuming that the eigenvalues λ_i are all different and that ξ_i are non-zero. For every $\alpha \neq i$, we evidently have $\Pi_\alpha(\lambda_i) \equiv 0$. Then,

$$I_{\lambda_i} = \sum_{\alpha=1}^{n} \Pi_\alpha(\lambda_i)\xi_\alpha^2 = \Pi_i(\lambda_i)\xi_i^2.$$

Hence $I_{\lambda_i}(x, v)$ and $I_{\lambda_{i+1}}(x, v)$ have different signs. Hence, the open interval $]\lambda_i, \lambda_{i+1}[$ contains a root of the polynomial $I_t(x, v)$. The degree of the polynomial I_t is equal $n - 1$; we have $n - 1$ disjoint intervals; each of these intervals contains at least one root so that all roots are real and the ith root lies between λ_i and λ_{i+1}. The first statement of the lemma is proved.

Let us prove the second statement of Lemma 1. Suppose $\lambda_i < \lambda_{i+1}$. Let first $\lambda_i < \tau < \lambda_{i+1}$. Then, the set

$$V_\tau \stackrel{\text{def}}{=} \left\{v \in T_xM^n : t_i(x, v) = \tau\right\},$$

consists of the points v where the function $I_\tau(x, v) \stackrel{\text{def}}{=} (I_t(x, v))_{|t=\tau}$ is zero; then it is a nontrivial quadric in $T_xM^n \equiv R^n$ and its measure is zero.

Let τ be one of the endpoints of the interval $[\lambda_i, \lambda_{i+1}]$. Without loss of generality, we can suppose $\tau = \lambda_i$. Let k be the multiplicity of the eigenvalue λ_i. Then, every coefficient $\Pi_\alpha(t)$ of the quadratic form (5) has the factor $(\lambda_i - t)^{k-1}$. Hence,

$$\hat{I}_t \stackrel{\text{def}}{=} \frac{I_t}{(\lambda_i - t)^{k-1}}$$

is a polynomial in t and $\hat{I}_\tau$ is a nontrivial quadratic form. Evidently, for every point $v \in V_\tau$, we have $\hat{I}_\tau(v) = 0$ so that the set V_τ is a subset of a nontrivial quadric in T_xM^n and its measure is zero. Lemma 1 is proved.

Proof of Corollary 1. The first statement of Corollary 1 follows immediately from the first statement of Lemma 1: Let us join the points $x, y \in M^n$ by a geodesic $\gamma : R \to M^n, \gamma(0) = x,\ \gamma(1) = y$. Consider the one-parametric family of integrals $I_t(x, v)$ and the roots

$$t_1(x, v) \le t_2(x, v) \le \ldots \le t_{n-1}(x, v).$$

By Theorem 4, each root t_i is constant on every orbit $(\gamma, \dot{\gamma})$ of the geodesic flow of g so that

$$t_i(\gamma(0), \dot{\gamma}(0)) = t_i(\gamma(1), \dot{\gamma}(1)).$$

Using Lemma 1, we obtain

$$\lambda_i(\gamma(0)) \le t_i(\gamma(0), \dot{\gamma}(0)), \quad \text{and} \quad t_i(\gamma(1), \dot{\gamma}(1)) \le \lambda_{i+1}(\gamma(1)).$$

Thus $\lambda_i(\gamma(0)) \le \lambda_{i+1}(\gamma(1))$ and the first statement of Corollary 1 is proved.

Let us prove the second statement of Corollary 1. Suppose $\lambda_i(y) = \lambda_{i+1}(y)$ for every point y of some subset $V \subset M^n$. Then, the value of λ_i is a constant (independent of $y \in V$). Indeed, by the first statement of Corollary 1,

$$\lambda_i(y_0) \leq \lambda_{i+1}(y_1) \text{ and } \lambda_i(y_1) \leq \lambda_{i+1}(y_0),$$

so that $\lambda_i(y_0) = \lambda_i(y_1) = \lambda_{i+1}(y_1) = \lambda_{i+1}(y_0)$ for every $y_0, y_1 \in V$.

We denote this constant by τ. Let us join the point x with every point of V by all possible geodesics. Consider the set $V_\tau \subset T_xM^n$ of the initial velocity vectors (at the point x) of these geodesics.

By the first statement of Lemma 1, for every geodesic γ passing through at least one point of V, the value $t_i(\gamma, \dot{\gamma})$ is equal to τ. By the second statement of Lemma 1, the measure of the set V_τ is zero. Since the set V lies in the image of the exponential mapping of the set V_τ, the measure of the set V is also zero. Corollary 1 is proved.

3. Plan of proof for Theorem 2

Let M^n be closed and connected. Suppose non-proportional Riemannian metrics g and $\bar{g}$ on M^n are geodesically equivalent. As in the previous section, consider the eigenvalues $\lambda_1 \leq \ldots \leq \lambda_n$ of the tensor L given by (2). By Corollary 1, the following two cases are possible.

Case 1: There exists $r \in \{1, \ldots, n-1\}$ and a constant $\lambda \in R$ such that, for every $x \in M^n$

$$\lambda_r(x) < \lambda < \lambda_{r+1}(x).$$

Case 2: The following two conditions hold:

(i) For every $r \in \{1, \ldots, n-1\}$, the maximum $\max_{x \in M^n}(\lambda_r(x))$ is equal to the minimum $\min_{x \in M^n}(\lambda_{r+1}(x))$.

(ii) At least one of the eigenvalues of L is not constant.

In Sect. 4, we show that, in Case 1, we can canonically construct a metric $\tilde{g}$ and foliations B_r, B_{n-r} as in Theorem 2. This result follows from Levi-Civita's Theorem (Theorem 5). Levi-Civita's Theorem ensures that the distribution spanned over the eigenspaces of the first r eigenvalues of L is integrable, (Corollary 2) so that it generates an $r-$dimensional foliation B_r. Similarly, the distribution spanned over the eigenspaces of the last $n-r$ eigenvalues of L generates an $(n-r)-$dimensional foliation B_{n-r}. Levi-Civita's Theorem also gives us a local description of geodesically equivalent metrics in a neighborhood of almost every point. From this description, it is easy to see how one can change g in order to obtain $\tilde{g}$ (as in Theorem 2). In the proof of Corollary 4, we show that the change can be done invariantly and globally.

A manifold equipped with a metric $\tilde{g}$ and foliations B_r and B_{n-r} as in Theorem 2 is called a locally product manifold [30]. The local and global theory of such manifolds is quite well developed (see, for example, [27]). The universal cover of such manifolds (with the lifted metric and foliations) is isomorphic to the direct product of two simply-connected Riemannian manifolds (M_1^r, g_1) and (M_2^{n-r}, g_2), see [16], Chapter 4, Sects. 5,6.

In Sects. 5 and 6, we show that, in Case 2, the fundamental group of the manifold is finite. The key result of Sect. 5 is Theorem 6 which, roughly speaking, tells us that every (closed) manifold with two geodesically equivalent metrics satisfying (i), (ii), has a closed submanifold U with two geodesically equivalent metrics satisfying (i), (ii) such that the natural homomorphism $\mathrm{Id}_* : \pi_1(U) \to \pi_1(M^n)$ is a surjection. Consequently applying Theorem 6, we come to one of the following subcases:

Subcase 1: The dimension n of the manifold M^n is $q+1$, where $q \geq 1$. The eigenvalues $\lambda_1 = \ldots = \lambda_q \stackrel{\text{def}}{=} \lambda$ are constant, the eigenvalue λ_{q+1} is not constant and there exists $z \in M^{q+1}$ such that $\lambda_{q+1}(z) = \lambda$.

Subcase 2: The dimension n of the manifold M^n is 2. The eigenvalues λ_1 and λ_2 are not constant and there exists a point $z \in M^2$ such that $\lambda_1(z) = \lambda_2(z)$.

Subcase 3: The dimension n of the manifold M^n is $q+2$, where $q \geq 1$, the eigenvalues λ_1 and λ_{q+2} are not constant and there exist $z_1, z_2 \in M^n$ such that $\lambda_1(z_1) = \lambda_{q+2}(z_2)$.

Subcase 4: The dimension n of the manifold M^n is $n = q + r + 1$; $q > 0$, $r > 0$. The eigenvalues $\lambda_1 = \lambda_2 = \ldots = \lambda_r$ and $\lambda_{r+2} = \lambda_{r+3} = \ldots = \lambda_n$ are constant. The eigenvalue λ_{r+1} is not constant. There exist points $z_0, z_1 \in M^n$ such that $\lambda_{r+1}(z_0) = \lambda_1$ and $\lambda_{r+1}(z_1) = \lambda_n$.

Remark 2. Although the metrics g and $\bar{g}$ are symmetrically related by their assumed geodesic equivalence, the tensor L is not invariant with respect to the permutation of the metrics. More precisely, the tensor (2) constructed for $\bar{g}, g$ is the inverse of (2) constructed for $g, \bar{g}$. In particular, this permutation transforms Subcase 1 to the following subcase which is not listed above: $\lambda_1 \neq const$, $\lambda_2 = \ldots = \lambda_n \stackrel{\text{def}}{=} \lambda = const$, $\max_{x \in M^n}(\lambda_1(x)) = \lambda$.

Also, unless otherwise stated, the metrics will be treated unequally, with precedence being afforded to g. In particular, the word "orthogonal" will always mean "orthogonal with respect to g".

In Sect. 6 we deal with these subcases. We show that, in all these subcases, the fundamental group of the manifold is finite. This completes the proof of Theorem 2.

4. Levi-Civita's Theorem, vanishing of the Nijenhuis torsion for L and the proof for Case 1

Let g, $\bar{g}$ be Riemannian metrics. Consider the tensor L given by (2). At every point $x \in M^n$, consider the different eigenvalues $\phi_1(x) < \phi_2(x) <$

$\ldots < \phi_m(x)$ of L. Let $k_i(x)$ be the multiplicity of the eigenvalue $\phi_i(x)$ so that $k_1(x)+\ldots+k_m(x) = n$. Consider the ordered set $K(x) \stackrel{\text{def}}{=} \{k_1(x), k_2(x), \ldots \ldots, k_m(x)\}$.

Definition 3. *A point $x \in M^n$ is called* stable *(with respect to the metrics g, $\bar{g}$), if it has a neighborhood $U(x)$ such that $K(x) = K(y)$ for every $y \in U(x)$.*

The following theorem is due to Levi-Civita 1896.

Theorem 5 ([18]). *Let g, $\bar{g}$ be Riemannian metrics on M^n. Let a point $x \in M^n$ be stable; let $K(x)$ be equal to $\{k_1, k_2, \ldots, k_m\}$. The metrics are geodesically equivalent in some sufficiently small neighborhood $U(x)$ of the point x, if and only if there exists a coordinate system $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_m)$ (in $U(x)$), where $\bar{x}_i = (x_i^1, \ldots, x_i^{k_i})$, $(1 \le i \le m)$, such that the quadratic forms of the metrics g and $\bar{g}$ have the following form:*

$$\begin{aligned} g(\dot{\bar{x}}, \dot{\bar{x}}) &= P_1(\bar{x})A_1(\bar{x}_1, \dot{\bar{x}}_1) + P_2(\bar{x})A_2(\bar{x}_2, \dot{\bar{x}}_2) + \cdots + \\ &\quad + P_m(\bar{x})A_m(\bar{x}_m, \dot{\bar{x}}_m), \\ \bar{g}(\dot{\bar{x}}, \dot{\bar{x}}) &= \rho_1 P_1(\bar{x})A_1(\bar{x}_1, \dot{\bar{x}}_1) + \rho_2 P_2(\bar{x})A_2(\bar{x}_2, \dot{\bar{x}}_2) + \cdots + \\ &\quad + \rho_m P_m(\bar{x})A_m(\bar{x}_m, \dot{\bar{x}}_m), \end{aligned}$$

where $A_i(\bar{x}_i, \dot{\bar{x}}_i)$ are positive-definite quadratic forms in the velocities $\dot{\bar{x}}_i$ with coefficients depending on $\bar{x}_i$,

$$\begin{aligned} P_i &\stackrel{\text{def}}{=} (\phi_i - \phi_1)\cdots(\phi_i - \phi_{i-1})(\phi_{i+1} - \phi_i)\cdots(\phi_m - \phi_i), \\ \rho_i &= \frac{1}{\phi_1 \ldots \phi_m}\frac{1}{\phi_i} \end{aligned}$$

and $0 < \phi_1 < \phi_2 < \ldots < \phi_m$ *are smooth functions such that*

$$\phi_i = \begin{cases} \phi_i(\bar{x}_i), & \textit{if} \quad k_i = 1 \\ \text{constant}, & \textit{otherwise}. \end{cases}$$

Remark 3. In Levi-Civita coordinates from Theorem 5, the tensor L is given by the diagonal matrix

$$\operatorname{diag}(\underbrace{\phi_1, \ldots, \phi_1}_{k_1}, \ldots, \underbrace{\phi_m, \ldots, \phi_m}_{k_m}),$$

so that the notation "ϕ" for the different eigenvalues of L used before Levi-Civita's Theorem is compatible with the notations inside the theorem.

Corollary 2. *Suppose the Riemannian metrics g, $\bar{g}$ are geodesically equivalent. Then the Nijenhuis torsion of the tensor L vanishes.*

Proof. Nijenhuis torsion is a tensor, so it is sufficient to check its vanishing at almost every point. By Corollary 1, almost every point of M^n is stable. In the Levi-Civita coordinates from Theorem 5, the tensor L is given by the diagonal matrix

$$\mathrm{diag}(\underbrace{\phi_1, \dots, \phi_1}_{k_1}, \dots .., \underbrace{\phi_m, \dots, \phi_m}_{k_m}).$$

Since the eigenvalue ϕ_i can depend on $\bar{x}_i$ only, the Nijenhuis torsion of L is zero [12]. Corollary 2 is proved.

Remark 4. A self-contained proof of Corollary 2 which does not require Levi-Civita's Theorem can be found in [3].

Below we assume that the Riemannian metrics $g, \bar{g}$ are geodesically equivalent and that M^n is closed. Suppose the eigenvalue λ_r is not a constant. Consider a number λ such that

$$\min_{x \in M^n}(\lambda_r(x)) < \lambda < \max_{x \in M^n}(\lambda_r(x)).$$

Consider a point x_0 where $\lambda_r(x_0) = \lambda$. Suppose the differential $d\lambda_r$ is not zero at the point x_0. Denote by W_0 the connected component of the set

$$\left\{x \in M^n : \ \lambda_r(x) = \lambda\right\}$$

containing the point x_0.

Corollary 3. *The set W_0 is a closed submanifold of codimension* 1. *At every point $x \in W_0$, the tangent space $T_x W_0$ is orthogonal to the eigenvector of L corresponding to the eigenvalue λ.*

Proof. By Corollary 1, at every point of W_0, the eigenvalue λ_r has multiplicity one. Then, the distribution orthogonal to its eigenvector is well-defined in a neighborhood of W_0. Since L is self-adjoint with respect to g, the orthogonal distribution is spanned over all other eigenspaces. Since the Nijenhuis torsion of L vanishes, the distribution is integrable. We denote by W_1 the integral hypersurface of the distribution passing through the point x_0. By Corollary 2, the eigenvalue λ_r is constant along the integral hypersurfaces of the distribution [12]. Hence, $W_1 \subset W_0$. Let us show that W_1 coincides with W_0.

By Corollary 2, in a small neighborhood of every point of W_1, there exist coordinates $x, y_1, \dots, y_{n-1}$ such that W_0 is given by the equation $x = 0$ and λ_r is independent of the coordinates $y_1, \dots, y_{n-1}$. Then, the set of the points of W_1 where $d\lambda_r = 0$ is open (in W_1). Since it is evidently closed, and since it does not coincide with the whole W_1, the set is empty. Thus $d\lambda_r \neq 0$ at every point of W_1. Then, by implicit function theorem, W_0 coincides with W_1. Corollary 3 is proved.

Corollary 4. *Let the eigenvalues of the tensor L for geodesically equivalent Riemannian metrics $g, \bar g$ satisfy the assumptions of Case 1: that is, for a certain $r \in \{1, \dots, n-1\}$ and $\lambda \in R$, for every $x \in M^n$, we have*

$$\begin{cases} \lambda_i(x) < \lambda & \text{if} \quad i \le r \\ \lambda_i(x) > \lambda & \text{if} \quad i > r. \end{cases}$$

Then, there exists a Riemannian metric $\tilde g$ on M^n and foliations B_r (of dimension r) and B_{n-r} (of dimension $n-r$) such that in a neighborhood of any point $x \in M^n$ there exist coordinates $(\bar x, \bar y) = \big((x_1, x_2, \dots x_r), (y_{r+1}, y_{r+2}, \dots, y_n)\big)$ such that the x-coordinates are constant on every leaf of the foliation B_{n-r}, the y-coordinates are constant on every leaf of the foliation B_r, and the metric $\tilde g$ is block-diagonal such that the first $(r \times r)$ block depends on the x-coordinates and the second $((n-r) \times (n-r))$ block depends on the y-coordinates.

Proof. We will explicitly construct the metric $\tilde g$ and the foliations B_r, B_{n-r}.

At every tangent space T_xM^n, denote by V_r (by V_{n-r}, respectively) the vectorspace spanned over the eigenspaces of L corresponding to the eigenvalues $\lambda_1, \dots, \lambda_r$ (the eigenvalues $\lambda_{r+1}, \dots, \lambda_n$, respectively). Since there is no point where $\lambda_r = \lambda_{r+1}$, we have that V_r and V_{n-r} are smooth distributions on M^n invariant with respect to L. Since the Nijenhuis torsion of L is zero, the distributions are integrable so that they generate two transversal foliations B_r, B_{n-r} of dimensions r and $n-r$.

Denote by χ_r, χ_{n-r} the characteristic polynomials of the restriction of L to V_r, V_{n-r}, respectively. Consider the (1,1)-tensor

$$C \stackrel{\text{def}}{=} \big((-1)^r \chi_r(L) + \chi_{n-r}(L)\big)^{-1}$$

and the metric $\tilde g$ given by the relation

$$\tilde g(u, v) \stackrel{\text{def}}{=} g(C(u), v)$$

for any vectors u, v. (In the tensor notations, the metric $\tilde g$ is given by $g_{i\alpha} C^{\alpha}_{j}$.)

Let us show that the metric $\tilde g$ and the foliations B_r, B_{n-r} are as we need. First of all, $\tilde g$ is an everywhere defined Riemannian metric. Indeed, take an arbitrary point $x \in M^n$. At the tangent space T_xM^n, we can find a coordinate system where the tensor L and the metric g are diagonal. In this coordinate system, the characteristic polynomials χ_r, χ_{n-r} are given by

$$\begin{aligned} (-1)^r \chi_r &= (t-\lambda_1)(t-\lambda_2)\dots(t-\lambda_r) \\ \chi_{n-r} &= (\lambda_{r+1}-t)(\lambda_{r+2}-t)\dots(\lambda_n-t). \end{aligned}$$

Then, the (1,1)-tensor $((-1)^r\chi_r(L) + \chi_{n-r}(L))$ is given by the diagonal matrix

$$\operatorname{diag}\left(\prod_{j=r+1}^{n}(\lambda_j-\lambda_1), \dots, \prod_{j=r+1}^{n}(\lambda_j-\lambda_r), \prod_{j=1}^{r}(\lambda_{r+1}-\lambda_j), \dots, \prod_{j=1}^{r}(\lambda_n-\lambda_j)\right). \quad (6)$$

We see that the tensor is diagonal and all its diagonal components are positive. Then, the tensor C is well-defined and $\tilde{g}$ is a Riemannian metrics.

Let us show that at every point there exists a coordinate system

$$(\bar{x}, \bar{y}) = \big((x_1, x_2, \dots, x_r), (y_{r+1}, y_{r+2}, \dots, y_n)\big)$$

such that the x-coordinates are constant on every leaf of the foliation B_{n-r}, the y-coordinates are constant on every leaf of the foliation B_r, and the metric g is block-diagonal with the first $(r \times r)$ block depending on the first r coordinates and the second $((n-r) \times (n-r))$ block depending on the last $n-r$ coordinates.

Since the foliations B_r, B_{n-r} and the metric $\tilde{g}$ are globally defined, it is sufficient to verify this condition at almost every point. Indeed, this condition is equivalent to the condition that, locally, for every leaf of the foliations B_r, B_{n-r}, the result of the parallel translation (in the sense of $\tilde{g}$) of any vector orthogonal (here it does not matter which metric: g, $\bar{g}$ or $\tilde{g}$ we take) to the leaf along any curve on the leaf does not depend on the curve [30]. This is a differential condition and it is sufficient to verify it at almost every point.

By Corollary 1, almost every point of M^n is stable. Consider the Levi-Civita coordinates $\bar{x}_1, \dots, \bar{x}_m$ from Theorem 5. There exists q such that $k_1 + k_2 + \dots + k_q = r$. By definition, the coordinates $\bar{x}_1, \dots, \bar{x}_q$ are constant on every leaf of the foliation B_{n-r}, the coordinates $\bar{x}_{q+1}, \dots, \bar{x}_m$ are constant on every leaf of the foliation B_r.

Using (6), we see that, in the Levi-Civita coordinates, $\tilde{g}$ is given by

$$\tilde{g}(\dot{\bar{x}}, \dot{\bar{x}}) = \tilde{P}_1(\bar{x}) A_1(\bar{x}_1, \dot{\bar{x}}_1) + \tilde{P}_2(\bar{x}) A_2(\bar{x}_2, \dot{\bar{x}}_2) + \dots + \\ + \tilde{P}_m(\bar{x}) A_m(\bar{x}_m, \dot{\bar{x}}_m),$$

where the functions $\tilde{P}$ are as follows: for $i \le q$, the function $\tilde{P}_i$ is given by

$$\tilde{P}_i \stackrel{\text{def}}{=} \frac{(\phi_i - \phi_1) \dots (\phi_i - \phi_{i-1})(\phi_{i+1} - \phi_i) \dots (\phi_q - \phi_i)}{(\phi_{q+1} - \phi_i)^{k_{q+1}-1} \dots (\phi_m - \phi_i)^{k_m - 1}}.$$

For $i > q$, the function $\tilde{P}_i$ is given by

$$\tilde{P}_i \stackrel{\text{def}}{=} \frac{(\phi_i - \phi_{q+1}) \dots (\phi_i - \phi_{i-1})(\phi_{i+1} - \phi_i) \dots (\phi_m - \phi_i)}{(\phi_i - \phi_1)^{k_1 - 1} \dots (\phi_i - \phi_q)^{k_q - 1}}.$$

Since ϕ_j is a constant whenever $j \ne 1$, the function $(\phi_i - \phi_j)^{k_j - 1}$ does not depend on the variables $\bar{x}_j$. Then, the metric $\tilde{g}$ is as we need. Corollary 4 is proved.

5. Proof for Case 2

Within this section, we assume that the manifold is closed and connected, and that the Riemannian metrics g, $\bar{g}$ are geodesically equivalent. We need the following technical lemmas.

Lemma 2. *Consider $r \in \{1, \dots, n-1\}$. Take $x_0 \in M^n$ such that*

$$\lambda_r(x_0) = \max_{x \in M^n}(\lambda_r(x)) \stackrel{\text{def}}{=} \lambda.$$

Consider $q \in \{0, \dots, n-r\}$ such that the eigenvalues $\lambda_{r+1}, \lambda_{r+2}, \dots, \lambda_{r+q}$ are constant and are equal to λ.

Denote by $V \subset T_{x_0}M^n$ the eigenspace of L corresponding to the eigenvalue λ. Let $v_0 \neq 0 \in T_{x_0}M^n$ be orthogonal to V. Consider the geodesic passing through the point x_0 with the velocity vector v_0.

Then, at every point of this geodesic, the number λ is an eigenvalue of L of multiplicity at least $q+1$.

Proof. First we consider the case $\lambda_{r+q+1}(x_0) = \lambda$. In this case, λ is a root of $I_t(x_0, v_0)$ of multiplicity at least $q+2$. More precisely, as in the proof of Lemma 1, consider diagonal coordinates on $T_{x_0}M^n$ such that the metric g is given by the diagonal matrix $\text{diag}(1, 1, \dots, 1)$ and the tensor L is given by the diagonal matrix $\text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$. In these coordinates, the function I_t is given by (5). Since by assumptions the vector v_0 is orthogonal to the eigenspace of L corresponding to λ, the components $\xi_r, \dots, \xi_{r+q+1}$ of the vector v_0 are zero. Clearly, for every $i \notin \{r, r+1, \dots, r+q+1\}$, the number λ is a root of the polynomial $\Pi_i(t)$ of multiplicity at least $q+2$. Thus λ is a root of the polynomial $I_t(x_0, v_0)$ of multiplicity at least $q+2$. Denote by γ the geodesic passing through x_0 with the velocity vector v_0. By Theorem 4, for every $\tau \in R$, the eigenvalue λ is a root of multiplicity at least $q+2$ of the polynomial $I_t(\gamma(\tau), \dot{\gamma}(\tau))$. Then, by Lemma 1, at least $q+1$ eigenvalues of L are equal to λ. Lemma 2 is proved under the additional assumption $\lambda_{r+q+1}(x_0) = \lambda$.

Note that if $\lambda_{r-1}(x_0) = \lambda$, then, by Corollary 1, the eigenvalue λ_r is a constant so that at every point of the manifold $\lambda_r = \lambda_{r+1} = \dots = \lambda_{r+q} = \lambda$ so that λ is an eigenvalue of L of multiplicity at least $q+1$.

Below we assume that either $r = 1$ or $\lambda > \lambda_{r-1}(x_0)$; either $r + q = n$ or $\lambda_{r+q+1}(x_0) > \lambda$.

We will consider the cases $q = 0$ and $q > 0$ separately, although the proofs use the same ideas.

Suppose $q = 0$. Consider the polynomial $I_t(x_0, v_0)$ and its roots $t_1(x_0, v_0) \le t_2(x_0, v_0) \le \dots \le t_{n-1}(x_0, v_0)$. If λ is a root of the polynomial of multiplicity at least 2, the lemma obviously follows from Theorem 4 and Lemma 1.

Evidently, λ is a root of the polynomial $I_t(x_0, v_0)$. More precisely, in the diagonal coordinates from the proof of Lemma 1, the component ξ_r is zero. All remaining terms of the sum (5) have the factor $(\lambda - t)$ and, therefore, vanish when $t = \lambda$. Below we assume that λ is a simple root of the polynomial $I_t(x_0, v_0)$.

Consider the functions

$$I_\lambda : TM^n \to R, \;\; I_\lambda(x,v) \stackrel{\text{def}}{=} (I_t(x,v))_{|t=\lambda}\,,$$
$$I_{\lambda_r} : TM^n \to R, \;\; I_{\lambda_r}(x,v) \stackrel{\text{def}}{=} (I_t(x,v))_{|t=\lambda_r(x)}\,.$$

By assumptions, the eigenvalue λ_r has multiplicity one in a small neighborhood of x_0. Then, it is a smooth function on this neighborhood so that the function $I_{\lambda_r}(x,v)$ is also smooth on the tangent bundle to this neighborhood. Consider the function $I_\lambda - I_{\lambda_r}$. Its differential vanishes at the point (x_0,v_0). More precisely, by assumptions, λ is a simple root of the polynomial $I_t(x_0,v_0)$ so that in a neighborhood of the point $\big(\lambda,(x_0,v_0)\big) \in R\times TM^n$ the function $I_t(x,v)$ is a monotone function in t. Since λ_r is no greater than λ, the difference $I_\lambda(x,v) - I_{\lambda_r}(x,v)$ is either always non-positive or always non-negative in a small neighborhood of (x_0,v_0). By assumptions, $\lambda_r(x_0) = \lambda$ so that $I_\lambda(x_0,v_0) - I_{\lambda_r}(x_0,v_0) = 0$. Hence, the function $I_\lambda - I_{\lambda_r}$ has a local extremum at the point (x_0,v_0) and its differential vanishes at this point.

Now, the differential of the function I_{λ_r} also vanishes at the point (x_0,v_0). More precisely, as we have shown in the proof of Lemma 1, the function I_{λ_r} is either always non-positive or always non-negative. Since $I_{\lambda_r}(x_0,v_0)$ is zero, the point (x_0,v_0) is an extremum of the function I_{λ_r}. Hence, the differential of I_{λ_r} vanishes at the point (x_0,v_0).

Thus the differential of the function I_λ is zero at the point (x_0,v_0). Consider the geodesic γ such that $(\gamma(0),\dot\gamma(0)) = (x_0,v_0)$. Evidently, the differential of any integral is preserved by the geodesic flow so that the differential dI_λ vanishes at every point $(\gamma(\tau),\dot\gamma(\tau))$. Let us prove that λ is an eigenvalue of L at the point $\gamma(\tau)$ of this geodesic.

Consider the diagonal coordinate system at the tangent space to the point $\gamma(\tau)$. In this coordinate system, the restriction of the function I_λ to the tangent space $T_{\gamma(\tau)}M^n$ is equal to

$$\sum_{\alpha=1}^{n} \xi_\alpha^2 \Pi_\alpha(\lambda).$$

The partial derivatives $\frac{\partial I_\lambda}{\partial \xi_\alpha}$ are

$$\frac{\partial I_\lambda}{\partial \xi_\alpha} = 2\xi_\alpha \Pi_\alpha(\lambda).$$

Since all of them are zero, at least one of the functions $\Pi_\alpha(\lambda)$ is zero. It can happen if and only if λ is one of the eigenvalues $\lambda_1,\ldots,\lambda_n$. Thus λ is an eigenvalue of L at the point $\gamma(\tau)$. The lemma is proved for $q=0$.

Let us prove the lemma assuming $q \geq 1$.

Consider the functions

$$\begin{aligned}
\tilde I_t : R \times TM^n \to R,\ \tilde I_t(x,v) &\stackrel{\text{def}}{=} \tfrac{I_t(x,v)}{(\lambda-t)^{q-1}},\\
\tilde I'_t : R \times TM^n \to R,\ \tilde I'_t(x,v) &\stackrel{\text{def}}{=} \tfrac{d}{dt}\left(\tilde I_t(x,v)\right),\\
\tilde I'_\lambda : TM^n \to R,\ \tilde I'_\lambda(x,v) &\stackrel{\text{def}}{=} \left(\tilde I'_t(x,v)\right)_{|t=\lambda},\\
\tilde I'_{\lambda_r} : TM^n \to R,\ \tilde I'_{\lambda_r}(x,v) &\stackrel{\text{def}}{=} \left(\tilde I'_t(x,v)\right)_{|t=\lambda_r(x)}.
\end{aligned}$$

By Lemma 1, at every point (x, v) of TM^n, the number λ is a root of I_t of multiplicity at least $q-1$. Then, $\tilde I'_t$ is indeed a polynomial of degree $(n-q-1)$ and, therefore, the functions $\tilde I'_\lambda$, $\tilde I'_{\lambda_r}$, are well-defined.

First we will show that the differential of the function $\tilde I'_\lambda$ vanishes at the point (x_0, v_0).

At the tangent space to every point of the manifold, consider the diagonal coordinate system from the proof of Lemma 1. In view of (5), the function $\tilde I_t$ is given by

$$\begin{aligned}
&\tilde\Pi_1(t)\xi_1^2 + \ldots + \tilde\Pi_r(t)\xi_r^2 + \tilde\Pi(t)\left[\xi_{r+1}^2 + \ldots + \xi_{r+q}^2\right]\\
&\qquad + \tilde\Pi_{r+q+1}(t)\xi_{r+q+1}^2 + \ldots + \tilde\Pi_n(t)\xi_n^2.
\end{aligned} \tag{7}$$

Here the functions $\tilde\Pi_\alpha$ and $\tilde\Pi$ are defined as follows:

$$\begin{aligned}
\tilde\Pi &\stackrel{\text{def}}{=} \prod_{j\neq r+1,\ldots,r+q} (\lambda_j - t),\\
\tilde\Pi_\alpha &\stackrel{\text{def}}{=} \prod_{\substack{j\neq r+1,\ldots,r+q-1\\ j\neq\alpha}} (\lambda_j - t).
\end{aligned}$$

Although the diagonal coordinate system is not uniquely defined, the sum of the first $(r-1)$ terms (respectively, the sum of the last $(n-r-q)$ terms) of (7) is a well-defined smooth function on the tangent bundle to a sufficiently small neighborhood of the point x_0. Indeed, by assumptions, in a small neighborhood of x_0, either $r = 1$ or $\lambda_{r-1} < \lambda_r$ and either $r+q = n$ or $\lambda_{r+q+1} > \lambda$. At every point of this neighborhood, consider the subspace $V_{first} \in T_xM^n$ (respectively, V_{last}) spanned over the eigenspaces of L corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_{r-1}$ (respectively, $\lambda_{r+q+1}, \ldots, \lambda_n$). The distributions V_{first}, V_{last} are smooth so that the orthogonal projections Pr_{first}, Pr_{last} in the metric g to these distributions are smooth as well. Hence, the functions

$$\begin{aligned}
\tilde F_t : R \times TM^n \to R,\ \tilde F_t(x,v) &\stackrel{\text{def}}{=} \tilde I_t(x, \mathrm{Pr}_{first}(v)),\\
\tilde F'_t : R \times TM^n \to R,\ \tilde F'_t(x,v) &\stackrel{\text{def}}{=} \tilde I'_t(x, \mathrm{Pr}_{first}(v)),\\
\tilde F'_\lambda : TM^n \to R,\quad \tilde F'_\lambda(x,v) &\stackrel{\text{def}}{=} \tilde I'_\lambda(x, \mathrm{Pr}_{first}(v)),\\
\tilde F'_{\lambda_r} : TM^n \to R,\quad \tilde F'_{\lambda_r}(x,v) &\stackrel{\text{def}}{=} \tilde I'_{\lambda_r}(x, \mathrm{Pr}_{first}(v)).
\end{aligned}$$

are well-defined smooth functions on the tangent bundle to the small neighborhood of x_0. It is easy to see that the function $\tilde{F}_t$ is the sum of the first $r-1$ terms of (7). We can similarly obtain the sum of the last $(n-r-q)$ terms by changing Pr_{first} to Pr_{last}.

Let us show that the differential of the function $\tilde{F}'_\lambda$ vanishes at the point (x_0, v_0). We will show that the differential of the function $\tilde{F}'_\lambda$ vanishes at every point (x_0, v) of $T_{x_0}M^n$. If the projection $Pr_{first}(v)$ is zero, there is nothing to prove. Suppose $Pr_{first}(v) \neq 0$. Then, the function $\tilde{F}_t$ is a polynomial in t of degree $n-q$. Clearly,

$$\tilde{F}'_\lambda = \left[\tilde{F}'_\lambda - \tilde{F}'_{\lambda_r}\right] + \tilde{F}'_{\lambda_r}.$$

Let us show that λ is a simple root of the polynomial $\tilde{F}'_t(x_0, v)$. The polynomial $\tilde{F}_t(x_0, v)$ has $n-q$ roots. Clearly,

$$\lambda = \lambda_r, \lambda = \lambda_{r+q}, \lambda_{r+q+1}, \lambda_{r+q+2}, \dots, \lambda_n$$

give us $n-r-q+2$ roots of $\tilde{F}_t(x_0, v)$. Arguing as in the proof of Lemma 1, one can show that there exist $r-2$ roots smaller than λ. Then, λ is a root of $\tilde{F}_t(x_0, v)$ of multiplicity precisely two. Hence, λ is a root of $\tilde{F}'_t(x_0, v)$ of multiplicity precisely one.

Then, in a small neighborhood of of the point $\big(\lambda, (x_0, v)\big) \in R \times TM^n$, the function $\tilde{F}'_t$ is a monotone function in t. Using that λ_r is no greater than λ, we obtain that the difference $\tilde{F}'_\lambda - \tilde{F}'_{\lambda_r}$ is either always non-positive or always non-negative in a small neighborhood of (x_0, v). By assumptions, $\lambda_r(x_0) = \lambda$ so that $\tilde{F}'_\lambda(x_0, v) - \tilde{F}'_{\lambda_r}(x_0, v)$ is zero and, hence, the function $\tilde{F}'_\lambda - \tilde{F}'_{\lambda_r}$ has a local extremum at the point (x_0, v) and its differential vanishes.

The differential of the function $\tilde{F}'_{\lambda_r}$ also vanishes at the point (x_0, v). Indeed, one can check directly that all its components in the diagonal coordinate system have the same sign: for $\alpha < r$ the coefficient of ξ_α^2 is equal to

$$-\prod_{\substack{j\neq r,\dots,r+q-1\\ j\neq\alpha}} (\lambda_j - \lambda_r).$$

Since $\tilde{F}'_{\lambda_r}(x_0, v) = 0$, the point (x_0, v) is a local extremum of the function $\tilde{F}'_{\lambda_r}$. Then, the differential of the function $\tilde{F}'_{\lambda_r}$ vanishes at the point (x_0, v).

Thus the differential of the function $\tilde{F}'_\lambda$ vanishes at the point (x_0, v_0). Similarly, the differential of the function $\tilde{H}'_\lambda(x, v) \stackrel{\text{def}}{=} \tilde{I}'_\lambda(x, Pr_{last}(v))$ vanishes at the point (x_0, v_0). Since the components $\xi_r, \xi_{r+1}, \dots, \xi_{r+q}$ of the vector v_0 are zero, the differential of the quadratic in velocities function

$$\tilde{I}'_\lambda - \tilde{H}'_\lambda - \tilde{F}'_\lambda$$

also vanishes at the point (x_0, v_0). Thus the differential of the function $\tilde{I}'_\lambda$ vanishes at the point (x_0, v_0).

The function $\tilde{I}'_\lambda$ is an integral of the geodesic flow. Then, its differential is preserved by the geodesic flow. Then, it is zero at every point $(\gamma(\tau), \dot\gamma(\tau))$ of the geodesic orbit $(\gamma, \dot\gamma)$ such that $(\gamma(0), \dot\gamma(0)) = (x_0, v_0)$. Let us prove that, at the point $\gamma(\tau)$, the multiplicity of the eigenvalue λ is at least $q + 1$.

Suppose λ is not an eigenvalue of L of multiplicity at least $q + 1$. Then, $\lambda_r(\gamma(\tau)) < \lambda$ and either $r + q = n$ or $\lambda < \lambda_{r+q+1}(\gamma(\tau))$. Consider the diagonal coordinates at the tangent space to the point $\gamma(\tau)$. In these coordinates, the components $\xi_{r+1}, \dots, \xi_{r+q}$ of $\dot\gamma(\tau)$ are zero. More precisely, as we have shown above, λ is a root of the polynomial $\tilde{I}_t(x_0, v_0)$. Then, it is a root of the polynomial $\tilde{I}_t(\gamma(\tau), \dot\gamma(\tau))$. In the diagonal coordinates, $\tilde{I}_t$ is given by (7). We see that the polynomials $\tilde\Pi_\alpha(t)$ vanish when $t = \lambda$ and that polynomial $\tilde\Pi(t)$ does not vanish when $t = \lambda$. Then, the sum $\xi_{r+1}^2 + \dots + \xi_{r+q}^2$ vanishes so that the components $\xi_{r+1}, \dots, \xi_{r+q}$ are zero.

For $\alpha \neq r + 1, \dots, r + q$, the partial derivatives $\frac{\partial \tilde{I}'_\lambda}{\partial \xi_\alpha}$ are equal to

$$-2\xi_\alpha \prod_{\substack{i \neq r+1, \dots, r+q \\ i \neq \alpha}} (\lambda_i - \lambda).$$

Since all these derivatives are zero, and since at least one of the components ξ_α is not zero, λ is an eigenvalue of L of multiplicity at least $q+1$. Lemma 2 is proved.

Lemma 3. *Consider $r \in \{1, \dots, n - 1\}$, $q \in \{0, \dots, n - r - 1\}$. Suppose there exist $x_0, x_1 \in M^n$ such that*

$$\lambda_r(x_0) = \lambda_{r+q+1}(x_1) \stackrel{\text{def}}{=} \lambda < \lambda_{r+q+1}(x_0).$$

Consider a geodesic γ such that $\gamma(0) = x_0$, $\gamma(1) = x_1$.

Then, the eigenvalues $\lambda_{r+1}, \lambda_{r+2}, \dots, \lambda_{r+q}$ are constant and are equal to λ, and, at the point x_0, the vector $\dot\gamma(0)$ is orthogonal to the eigenspace of L corresponding to the eigenvalue λ.

Proof. By Corollary 1, the eigenvalues $\lambda_{r+1}, \lambda_{r+2}, \dots, \lambda_{r+q}$ are constant. Let us prove that the eigenspace of L corresponding to the eigenvalue λ is orthogonal to $\dot\gamma(0)$.

First of all, λ is a root of multiplicity at least $q + 1$ of the polynomial $I_t(\gamma, \dot\gamma)$. Indeed, by Lemma 1, we have

$$\begin{aligned}
\lambda = \lambda_r(\gamma(0)) &\le t_r(\gamma(0), \dot\gamma(0)) \le \dots \\
&\le t_{r+q-1}(\gamma(0), \dot\gamma(0)) \le t_{r+q}(\gamma(0), \dot\gamma(0)), \\
t_r(\gamma(1), \dot\gamma(1)) &\le t_{r+1}(\gamma(1), \dot\gamma(1)) \le \dots \\
&\le t_{r+q}(\gamma(1), \dot\gamma(1)) \le \lambda_{r+q+1}(\gamma(1)) = \lambda.
\end{aligned}$$

Since by Theorem 4, $t_i(\gamma(0), \dot\gamma(0)) = t_i(\gamma(1), \dot\gamma(1))$ for every i, we obtain

$$t_r(\gamma(0), \dot\gamma(0)) = t_{r+1}(\gamma(0), \dot\gamma(0)) = \dots \\ = t_{r+q-1}(\gamma(0), \dot\gamma(0)) = t_{r+q}(\gamma(0), \dot\gamma(0)) = \lambda.$$

Thus λ is a root of the polynomial $I_t(\gamma(0), \dot\gamma(0))$ of multiplicity at least $q+1$.

Without loss of generality, we can assume either $r = 1$ or $\lambda_{r-1}(x_0) < \lambda$. Consider the polynomial $\tilde I_t$ from the proof of Lemma 2.

Since λ is a root of the polynomial $I_t(\gamma(0), \dot\gamma(0))$ of multiplicity at least $q+1$, by definition, λ is a root of multiplicity at least two of the polynomial $\tilde I_t(x_0, \gamma_0)$. In the diagonal coordinates at the tangent space to the point $\gamma(0)$, the polynomial $\tilde I_t$ is given by (7). We see that λ is a double-root of all polynomials $\tilde\Pi_\alpha(t)$, and is a simple root of the polynomial $\tilde\Pi(t)$. Then, the sum $\xi_{r+1}^2 + \dots + \xi_{r+q}^2$ vanishes so that the components $\xi_{r+1}, \dots, \xi_{r+q}$ are zero. Hence, the eigenspace of L corresponding to the eigenvalue λ is orthogonal to $\dot\gamma(0)$. Lemma 3 is proved.

Let the eigenvalues $\lambda_1 \le \dots \le \lambda_n$ of the tensor L satisfy the assumptions (i), (ii) of Case 2. Take $r \in \{1, \dots, n-1\}$, $q \in \{0, \dots, n-r-1\}$. Suppose the eigenvalues λ_r, λ_{r+q+1} are not constant. Suppose that the maximum $\max_{x\in M^n}(\lambda_r(x))$ coincides with the minimum $\lambda \stackrel{\text{def}}{=} \min_{x\in M^n}(\lambda_{r+q+1}(x))$. Consider the set

$$U \stackrel{\text{def}}{=} \left\{x \in M^n : (\lambda_r(x) - \lambda)(\lambda_{r+q+1}(x) - \lambda) = 0\right\}.$$

Theorem 6. *The following holds:*

1. *U is a closed connected submanifold of codimension $q+1$.*
2. *The standard homomorphism $\mathrm{Id}_* : \pi_1(U) \to \pi_1(M^n)$ of the fundamental group of U to the fundamental group of M^n is a surjection.*
3. *U is totally geodesic. In particular, the restrictions of the metrics g, $\bar g$ to U have the same (unparameterized) geodesics. We denote by $\tilde L$ the tensor (2) (on U) for these restrictions.*
4. *The eigenvalues of $\tilde L$ satisfy the assumptions (i), (ii) of Case 2.*

Proof. There exists $z \in M^n$ such that $\lambda_r(z) = \lambda_{r+q+1}(z) = \lambda$. More precisely, since M^n is compact, there exist $x_0, x_1 \in M^n$ such that $\lambda_r(x_0) = \lambda_{r+q+1}(x_1) = \lambda$. Suppose $\lambda_{r+q+1}(x_0) > \lambda$. Consider a geodesic γ connecting these points: $\gamma(0) = x_0$, $\gamma(1) = x_1$. By Lemma 3, the initial velocity vector $\dot\gamma(0)$ satisfies the assumptions of Lemma 2. Hence, by Lemma 2, at every point of the geodesic, $\lambda_r = \lambda$ or $\lambda_{r+q+1} = \lambda$. Then, there exists $0 \le \tau \le 1$ such that $\lambda_r(\gamma(\tau)) = \lambda_{r+q+1}(\gamma(\tau)) = \lambda$.

Now let us show that we can choose a point $x_0 \in M^n$ such that $\lambda_r(x_0) = \lambda < \lambda_{r+q+1}(x_0)$.

By assumption, the eigenvalue λ_{r+q+1} is not constant. Then, there exists a point y_1 such that $\lambda < \lambda_{r+q+1}(y_1) < \max_{x\in M^n}(\lambda_{r+q+1}(x))$ and such that the differential $d\lambda_{r+q+1}(y_1)$ is not zero. Denote by W_0 the connected component of the set

$$\{x \in M^n : \ \lambda_{r+q+1}(x) = \lambda_{r+q+1}(y_1)\}$$

containing the point y_1. By Corollary 3, W_0 is a submanifold of M^n of codimension 1, and the eigenvector of L corresponding to λ_{r+q+1} is orthogonal to W_0 at every point.

Consider the shortest curve γ connecting the submanifold W_0 with the point z. The curve γ is an (unparameterized) geodesic. Let $x_0 \in W_0$ be its starting point and $v_0 \neq 0$ be the initial velocity vector. Consider the roots $t_1(x_0, v_0) \leq t_2(x_0, v_0) \leq \ldots \leq t_{n-1}(x_0, v_0)$ of $I_t(x_0, v_0)$.

Since v_0 is orthogonal to W_0, in the diagonal coordinates, the polynomial $I_t(x_0, v_0)$ is equal to (we assume $v_0 = (\xi_1, \ldots, \xi_n)$)

$$\xi^2_{r+q+1} \prod_{i \neq r+q+1} (\lambda_i - t).$$

Then, $t_r(x_0, v_0) = \lambda_r(x_0)$. Since the geodesic γ passes through the point z where $\lambda_r = \lambda_{r+1} = \lambda$, we have $t_r(x_0, v_0) = \lambda$ so that $\lambda_r(x_0) = \lambda$. Thus $\lambda_r(x_0) = \lambda < \lambda_{r+q+1}(x_0)$.

Similarly, there exists $x_1 \in M^n$ such that $\lambda_r(x_1) < \lambda = \lambda_{r+q+1}(x_1)$.

Let us show that U is indeed a submanifold of codimension $q + 1$. Take a point $x \in U$. At this point, $\lambda_r = \lambda$ or $\lambda_{r+q+1} = \lambda$. Suppose $\lambda_{r+q+1}(x) = \lambda > \lambda_r(x)$. Consider a geodesic γ such that $\gamma(0) = x_0$, $\gamma(1) = x$.

By Lemma 3, the point $(\gamma(0), \dot{\gamma}(0)) \in TM^n$ satisfies the assumptions of Lemma 2. Then, every point of the geodesic γ is a point of U. Then, without loss of generality, we can assume that the points x, x_0 are not conjugate along γ (otherwise, instead of the point x_0 consider a new point $\tilde{x}_0 \in \gamma$ close enough to the point x_0).

Consider the exponential mapping $exp : T_{x_0}M^n \to M^n$. Denote by $O_{x_0} \subset T_{x_0}M^n$ a sufficiently small neighborhood of $\dot{\gamma}(0)$. Denote by $V^{\perp}_{x_0} \subset T_{x_0}$ the subspace orthogonal (in g) to the eigenspace of L corresponding to λ. The space $V^{\perp}_{x_0}$ has codimension $q + 1$. By Lemma 3, every point of U from a sufficiently small neighborhood of x is an image of a point from $O_{x_0} \cap V^{\perp}_{x_0}$. By Lemma 2, image of every point of $O_{x_0} \cap V^{\perp}_{x_0}$ is a point of U. Since the points x_0, x are not conjugate, the restriction of the the exponential mapping to O_{x_0} is a diffeomorphism. Thus in a neighborhood of every point where $\lambda_{r+q+1} = \lambda > \lambda_r$, the set U is a submanifold of codimension $q + 1$.

Similarly, U is a submanifold of codimension $q + 1$ in a neighborhood of every point where $\lambda_{r+q+1} > \lambda = \lambda_r$. Let us prove that U is a submanifold of codimension $q + 1$ in a neighborhood of a point z where $\lambda_{r+q+1}(z) = \lambda = \lambda_r(z)$.

Without loss of generality, we can assume that z is not conjugate (along the chosen geodesics) to the points x_0, x_1. Arguing as above, considering the exponential mapping from the tangent space to x_0, we obtain that, in a small neighborhood of z, the image $exp_{x_0}(O_{x_0} \cap V_{x_0}^{\perp})$ belongs to U and is a submanifold of codimension $q+1$. Considering the exponential mapping from the tangent space to x_1, we obtain that, in a small neighborhood of z, the image $exp_{x_1}(O_{x_1} \cap V_{x_1}^{\perp})$ belongs to U and is a submanifold of codimension $q+1$. By Lemmas 2,3, every point of U lying close to z lies in the union of the images. In order to prove that U is a submanifold of codimension $q+1$, we show that these images coincide in a small neighborhood of z.

Before proving this, we will show that the above constructed images $exp_x(O_x \cap V_x^{\perp})$, where $O_x \subset T_xM^n$ is a sufficiently small neighborhood of a regular point of the exponential mapping and $V_x^{\perp} \subset T_xM^n$ is the space orthogonal to the eigenspace of L corresponding to λ, are totally geodesic.

Clearly, they are totally geodesic in the neighborhood of x such that $\lambda_r(x) < \lambda_{r+q+1}(x)$. More precisely, the tangent space T_xU coincides with $V_x^{\perp}$. Indeed, by Lemma 2, the subspace $V_x^{\perp}$ lies in T_xU. Since $V_x^{\perp}$ and T_xU have the same dimension, they coincide. Since, by Lemma 2, every geodesic with initial velocity vector from $V_x^{\perp} = T_xU$ lies in U, the submanifold U is totally geodesic near the points where $\lambda_r < \lambda_{r+q+1}$.

In order to prove that U is totally geodesic near the points where $\lambda_r = \lambda_{r+q+1}$, we show that the set of these points is nowhere dense in U. Suppose that the set of points where $\lambda_r = \lambda_{r+q+1}$ is not nowhere dense in U. Then, there exists a small smooth disk $D_{n-q-1} \in M^n$ of dimension $n-q-1$ such that $\lambda_r = \lambda_{r+q+1} = \lambda$ at every point of the disk. Consider a convex $n-$ball containing this disk and a point y_0 of the ball such that, for every point of the disk, the shortest geodesic connecting the point y_0 with the point of the disk is transversal to the disk. Since at almost every point of the manifold $\lambda_r < \lambda < \lambda_{r+q+1}$, without loss of generality, we can assume $\lambda_r(y_0) < \lambda < \lambda_{r+q+1}(y_0)$. Consider the restriction of the function $\tilde{I}_t$ from the proof of Lemma 2 to $T_{y_0}M^n$. In the diagonal coordinates, it is given by (7). Denote by $Z \in T_{y_0}M^n$ the set of the initial vectors of geodesic that start at the point y_0 and go through at least one point of the disk. Since the disk has codimension $n-q-1$, the set U contains a disk of dimension $n-q$. By Theorem 4 and Lemma 1, at every point $(y_0, v) \in Z$, we have the equality $t_r(y_0, v) = \ldots = t_{r+q}(y_0, v) = \lambda$. Then, the components $\xi_{r+1} \ldots \xi_{r+q}$ of the vector v must be zero, and Z is a subset of a nondegenerate quadric in the coordinates $\xi_1, \ldots, \xi_r, \xi_{r+q+1}, \ldots, \xi_n$. Thus the set has codimension $q+1$ and, therefore, can not contain a disk of dimension $n-q$. The contradiction shows that the set of the points where $\lambda_r = \lambda_{r+q+1} = \lambda$ is nowhere dense in U. Thus the above constructed images $exp_{x_0}(O_{x_0} \cap V_{x_0}^{\perp})$, $exp_{x_1}(O_{x_1} \cap V_{x_1}^{\perp})$ are totally geodesic.

Let us show that they coincide in a small neighborhood of the point z. Indeed, by construction, the points where $\lambda_r = \lambda_{r+q+1} = \lambda$ lie in the intersection of the images. Since the images are smooth and have the same dimension, if they do not coincide, in an arbitrary small neighborhood of z there exist points x, y such that $x \in exp_{x_0}(O_0 \cap V_{x_0}^{\perp})$, $x \notin exp_{x_1}(O_1 \cap V_{x_1}^{\perp})$, $y \in exp_{x_1}(O_1 \cap V_{x_1}^{\perp})$, $y \notin exp_{x_0}(O_0 \cap V_{x_0}^{\perp})$. Clearly, $\lambda_r(x) < \lambda = \lambda_{r+q+1}(x)$

and $\lambda_r(y) = \lambda < \lambda_{r+q+1}(y)$. By Lemma 3, the initial vector of the shortest geodesic connecting these points is orthogonal to the eigenspace of L corresponding to λ. Then, it is tangent to the image $exp_{x_0}(O_{x_0} \cap V_{x_0}^{\perp})$ at the point x. Since $exp_{x_0}(O_{x_0} \cap V_{x_0}^{\perp})$ is totally geodesic, the point y lies in the the image $exp_{x_0}(O_{x_0} \cap V_{x_0}^{\perp})$ as well. Thus the images coincide in a small neighborhood of z. Finally, U is a submanifold of codimension $q+1$. As we have already proved, U is totally geodesic. It is evidently connected.

Let us prove that the standard homomorphism $\mathrm{Id}_* : \pi_1(U, x_0) \to \pi_1(M^n, x_0)$ of the fundamental groups is a surjection. We have to show that any element of the fundamental group $\pi_1(M^n, x_0)$ can be realized by a loop on U. Consider a geodesic segment γ_{base} connecting the points x_0 and z. By Hopf-Rinow Theorem, for every element of $\pi_1(M^n, x_0)$, there exist a geodesic segment γ connecting the points z and x_0 such that the loop made of the segments γ, γ_{base} realizes this element of the fundamental group. By Lemmas 2, 3, the geodesics γ, γ_{base} lie on U. Thus any element of the fundamental group $\pi_1(M^n, x_0)$ can be realized by a loop on U.

Consider the restriction of the metrics g, $\bar{g}$ to U. Denote by $\tilde{\lambda}_1 \le \ldots \le \tilde{\lambda}_{n-q-1}$ the eigenvalues of the tensor (2) corresponding to these metrics. By direct calculations, using the definition (2), and since the eigenspace of L corresponding to λ is orthogonal to U at almost every point of U, it is easy to verify that the eigenvalues $\tilde{\lambda}_i$ are given by

$$\begin{cases} \tilde{\lambda}_i = \lambda_i \cdot \lambda^{\frac{q+1}{n-q}} & \text{for } \ i < r \\ \tilde{\lambda}_r = \lambda_r \cdot \lambda^{\frac{q+1}{n-q}} & \text{if } \ \ \lambda_r < \lambda \\ \tilde{\lambda}_r = \lambda_{r+q+1} \cdot \lambda^{\frac{q+1}{n-q}} & \text{if } \ \ \lambda_r = \lambda \\ \tilde{\lambda}_i = \lambda_{i+q+1} \cdot \lambda^{\frac{q+1}{n-q}} & \text{for } \ i > r \end{cases}$$

Thus in order to show that the eigenvalues of $\tilde{L}$ satisfy conditions (i), (ii), we have to show that for every i there exists points $y_{max}, y_{min} \in U$ such that

$$\lambda_i(y_{max}) = \max_{x \in M^n}(\lambda_i(x)), \quad \lambda_i(y_{min}) = \min_{x \in M^n}(\lambda_i(x)).$$

We assume $i \le r$, the case $i > r+q$ can be treated similarly. If the eigenvalue λ_i is constant, the statement is trivial. Suppose λ_i is not a constant. Then, for almost every τ such that $\min_{x \in M^n}(\lambda_i(x)) < \tau < \max_{x \in M^n}(\lambda_i(x))$, there exists $y_0 \in M^n$ such that $\lambda_i(y_0) = \tau$, $d\lambda_i(y_0) \ne 0$. Denote by W_0 the connected component of the set

$$\left\{x \in M^n : \ \lambda_i(x) = \tau\right\}$$

containing the point y_0.

By Corollary 3, W_0 is a submanifold and its normals are eigenvectors of L corresponding to the eigenvalue τ. Take a point $z \in U$ such that $\lambda_r(z) = \lambda_{r+q+1}(z)$. If $z \in W_0$, there exists a point of U where $\lambda_i = \tau$. If $z \notin W_0$, consider the shortest curve γ connecting the submanifold W_0 and the point z. Let $y \in W_0$ be the starting point of the curve and $v \neq 0$ be the initial velocity vector. The vector v is orthogonal to W_0. Then, in the diagonal coordinates from the proof of Lemma 1, the polynomial $I_t(y, v)$ is equal to $\xi_i^2 \Pi_i(t)$. Hence, its roots are

$$t_1(y, v) = \lambda_1(y), \dots, t_{i-1}(y, v) = \lambda_{i-1}(y),$$
$$t_i(y, v) = \lambda_{i+1}(y), \dots, t_{n-1}(y, v) = \lambda_n(y).$$

Since the shortest curve γ is an (unparameterized) geodesic passing through the point where $\lambda_{r+q} = \lambda_{r+q+1} = \lambda$, the root $t_{r+q}(y, v) = \lambda$. Hence, $\lambda_{r+q+1}(y) = \lambda$ so that $y \in U$. Thus for almost every τ such that $\min_{x \in M^n}(\lambda_i(x)) < \tau < \max_{x \in M^n}(\lambda_i(x))$ there exist a point of U where $\lambda_i = \tau$. Since U is closed, there exists $y_{max} \in U$ where $\lambda_i = \max_{x \in M^n}(\lambda_i(x))$ and $y_{min} \in U$ where $\lambda_i = \min_{x \in M^n}(\lambda_i(x))$. Finally, the eigenvalues of $\tilde{\lambda}_1, \dots, \tilde{\lambda}_{n-q-1}$ satisfy the assumptions (i), (ii) of Case 2. Theorem 6 is proved.

6. Geodesically equivalent metrics with one or two non-constant eigenvalues of L

Within this section, we assume that M^n is closed, the Riemannian metrics g and $\bar{g}$ are geodesically equivalent, and that the eigenvalues of L satisfy the assumptions (i), (ii) of Case 2. Our goal is to prove that in all four subcases listed in Sect. 3, the fundamental group of M^n is finite.

Remark 5. Actually, it is possible to prove that, under the assumptions of Case 2, Subcases 1,2,3,4, the manifold is covered by the sphere. The proof is quite lengthy. Since this statement is not necessary for Theorem 2, we will not do it.

Proof for Subcase 1. Suppose the dimension of the manifold is $n = q + 1$, where $q \geq 1$, the eigenvalues $\lambda_1 = \dots = \lambda_q \stackrel{\text{def}}{=} \lambda$ are constant, the eigenvalue λ_{q+1} is not constant, and there exists $z \in M^{q+1}$ such that $\lambda_{q+1}(z) = \lambda$.

We will show that the fundamental group of the manifold is either trivial or homeomorphic to Z_2. In order to do this, we show that the number of the points where $\lambda_{q+1} = \lambda$ is at most two.

By Lemma 1, for every geodesic γ passing through at least one such point we have $t_1(\gamma, \dot{\gamma}) = \dots = t_q(\gamma, \dot{\gamma}) = \lambda$. If $v_1, v_2 \in T_{x_0}M^n$ are two tangent vectors at an arbitrary point, x_0, such that $t_1(x_0, v_1) = \dots = t_q(x_0, v_2) = \lambda$, then either they are proportional or $\lambda_1(x_0) = \dots = \lambda_{q+1}(x_0) = \lambda$. Indeed, λ is a root of the polynomial $\tilde{I}_t(x_0, v_i)$ so that $\tilde{I}_\lambda(x_0, v_i) = 0$. In the diagonal coordinates, the polynomial $\tilde{I}_t(x_0, v_i)$ is given by (7). Since $\tilde{\Pi}(\lambda) = 0$, we have $\Pi_{q+1}(\lambda) = 0$ (so that $\lambda_{q+1}(x_0) = \lambda$) or $\xi_1 = \xi_2 = \dots = \xi_q = 0$ (so that the vectors v_1, v_2 are proportional).

Thus any two geodesics passing through points where $\lambda_{q+1} = \lambda$ can transversally intersect only at the points where $\lambda_{q+1} = \lambda$. Clearly, if there exists three such points, then there is a neighborhood such that every point of this neighborhood is a point of transversal intersection of such geodesics. Then, the set of points where $\lambda_{q+1} = \lambda$ is not nowhere dense which contradicts Corollary 1. Theorem 2 is proved under assumptions of Case 2, Subcase 1.

Proof for Subcase 2. Suppose the dimension of the manifold is 2, the eigenvalues λ_1, λ_2 are not constant, and there exists $z \in M^2$ where $\lambda_1 = \lambda_2$.

By definition, the integral I_0 of the geodesic flow of g is quadratic in velocities. At the tangent space to the points where $\lambda_1 < \lambda_2$, the integral is not proportional to the Hamiltonian. At the tangent space to the point z the integral is proportional to the Hamiltonian. By [17] (or [14]), this can happen only on the sphere and on the projective plane. Thus the fundamental group of M^2 is either trivial or homeomorphic to Z_2. Theorem 2 is proved under assumptions of Case 2, Subcase 2.

Proof for Subcase 3. Suppose the dimension of the manifold is $n = q + 2$, where $q \geq 1$, the eigenvalues λ_1 and λ_n are not constant, and there exists $z_1, z_2 \in M^{q+2}$ such that $\lambda_1(z_1) = \lambda_n(z_2) \stackrel{\text{def}}{=} \lambda$.

We will show that the fundamental group of the manifold is either trivial or homeomorphic to Z_2. First of all, by Corollary 1, the eigenvalues $\lambda_2, \dots, \lambda_{q+1}$ are constant and are equal to λ. By Theorem 6, the set

$$U \stackrel{\text{def}}{=} \left\{x \in M^n : \ (\lambda_1(x) - \lambda)(\lambda_n(x) - \lambda) = 0\right\}.$$

is homeomorphic to the circle and the fundamental group of M^n is cyclic. Consider the generator G of the fundamental group. Since the codimension of U is at least two, there exists $\tau < \lambda$ such that the element G can be realized by a curve lying in the set

$$\left\{x \in M^n : \ \lambda_1(x) \leq \tau\right\}.$$

Denote by W_+ the connected component of the set containing the curve.

Let us show that every element of the fundamental group of W_+ or of a double-cover of W_+ can be realized on its boundary.

There exists a small ϵ such that at every point of W_+ we have $\lambda_1 < \tau + \epsilon < \lambda$. On W_+, consider the foliations B_1, B_{q+1} and the metric $\tilde{g}$ from Corollary 4. At every point of W_+, there exist precisely two vectors v tangent to B_1 such that $\tilde{g}(v, v) = 1$. Hence, on W_+ or on a certain double-covering of W_+, there exists a vector field v tangent to B_1 such that $\tilde{g}(v, v) = 1$. By Corollary 4, the flow of the vector field v takes the leaves of B_{q+1} to the leaves. Then, all the leaves are diffeomorphic. By Corollary 3, all leaves are closed. Then, the foliation is indeed a fibration. The base of the fibration is one-dimensional. Since the set W_+ has boundary, the base can not be homeomorphic to the circle. Thus the base is homomorphic to the interval, and every element of the fundamental group (of W_+ or of the double-covering of W_+) can be realized on any fiber.

Denote by W_- the connected component of the set

$$\{x \in M^n : \lambda_1(x) \geq \tau\}$$

having points from W_+. Denote by W_0 the connected component of the boundary of W_- having points of W_+. Clearly, $\lambda_1 = \tau$ at every point of W_0. By Corollary 2, every point of W_0 is a point of W_+; moreover, W_0 is a leaf of the foliation B_{q+1}.

Let us show that every closed curve on W_0 is homotopic to zero in M^n. First of all, the set W_- is (weakly) geodesically convex. Indeed, by Corollary 3, the vector field v is orthogonal to the boundary of the set W_-: we can approach any point of the boundary by a sequence of points where $d\lambda_1 \neq 0$. From (5), it follows that for every geodesic γ tangent to the boundary of W_-, the root $t_1(\gamma, \dot\gamma))$ is equal to τ. Hence, by Lemma 1, the geodesic γ has no points where $\lambda_1 > \tau$. If $\lambda_1 = \tau$ at some point of the geodesic γ, the condition $t_1(\gamma, \dot\gamma)) = \tau$ ensures that $\dot\gamma$ is orthogonal to v at that point. By Corollary 2, the distribution orthogonal to v is integrable. Thus the geodesic γ can not pass through interior points of W_-, and W_- is (weakly) geodesically convex.

Using the convexity, arguing similarly to the proof of the second statement of Theorem 6, we obtain that every element of the fundamental group of W_- can be realized on the intersection $W_- \cap U$. Every connected component of the intersection $W_- \cap U$ is homeomorphic to the interval. Hence, the fundamental group of W_- is trivial. Thus every closed curve lying on W_0 is homotopic to zero. Finally, the order of the generator G is at most two so that the fundamental group of M^n is at most Z_2. Theorem 2 is proved under the assumptions of Case 2, Subcase 3.

Proof for Subcase 4. Suppose the dimension of the manifold is $n = q+r+1$; $q > 0, r > 0$. Suppose the eigenvalues $\lambda_1 = \lambda_2 = \ldots = \lambda_r$ and the eigenvalues $\lambda_{r+2} = \lambda_{r+3} = \ldots = \lambda_n$ are constant. Suppose the eigenvalue λ_{r+1} is not constant, and there exist $z_0, z_1 \in M^n$ such that $\lambda_{r+1}(z_0) = \lambda_1$ and $\lambda_{r+1}(z_1) = \lambda_n$.

Consider the sets

$$U_r \stackrel{\text{def}}{=} \{x \in M^n : \lambda_{r+1}(x) = \lambda_n\},$$
$$U_q \stackrel{\text{def}}{=} \{x \in M^n : \lambda_{r+1}(x) = \lambda_1\}.$$

Let us show that the sets U_r, U_q are closed submanifold of dimensions r, q, respectively. Indeed, take a point $x_0 \in U_r$. In a small convex ball around x_0, we have $\lambda_1 < \lambda_{r+1}$. In the ball, consider the distribution of the eigenspace of L corresponding to λ_1. By Corollary 2, the distribution is integrable, and the eigenvalue λ_{r+1} is constant along the integrable submanifold of this distribution. Then, the integral submanifold U_r' passing through the point x_0 lies in U_r. Thus the set U_r contains a submanifold of dimension r.

Let us show that, in the small ball, the submanifold U_r' coincides with U_r. Suppose that there exists a point $z \in U_r$, $z \notin U_r'$ of the ball. Denote by U_r'' the integral submanifold of the distribution passing through the point z.

At every point of $U_r' \cup U_r''$, denote by $V^\perp$ the eigenspace of L corresponding to λ_n. $V^\perp$ is orthogonal to $U' \cup U_r''$. Consider all geodesics γ passing through the points of $U_r' \cup U_r''$ with the velocity vectors $\dot\gamma(0) \in V^\perp$. Since the codimension of U_r' and U_r'' is at least two, and since, by Corollary 1, at almost every point of the ball we have $\lambda_1 < \lambda_{r+1} < \lambda_n$, there exists a point y of the ball where the geodesics intersect transversally. Let us show that this is impossible.

By Lemma 1, for every geodesic γ passing through at least one point of U_r, we have

$$t_{r+1}(\gamma, \dot\gamma) = \ldots = t_{n-1}(\gamma, \dot\gamma) = \lambda_n, \tag{8}$$

so that λ_n is a root of I_t of multiplicity at least q. In the diagonal coordinates from the proof of Lemma 1, the function $I_t(\gamma(\alpha), \dot\gamma(\alpha))$ is equal to

$$\begin{aligned}\left(\xi_1^2 + \ldots + \xi_r^2\right)\left(\lambda_1 - t\right)^{r-1}\left(\lambda_{r+1} - t\right)\left(\lambda_n - t\right)^q + \xi_{r+1}^2\left(\lambda_1 - t\right)^r\left(\lambda_n - t\right)^q \\ + \left(\xi_{r+2}^2 + \ldots + \xi_n^2\right)\left(\lambda_1 - t\right)^r\left(\lambda_{r+1} - t\right)\left(\lambda_n - t\right)^{q-1}.\end{aligned} \tag{9}$$

We see that the first and the second terms of the sum (9) have the factor $(\lambda_n - t)^q$. If $\lambda_{r+1}(\gamma(\alpha)) < \lambda_n$, the third term of (9) has the factor $(\lambda_n - t)^{q-1}$ only. Hence, the components $\xi_{r+2}, \ldots, \xi_n$ vanish.

Since the velocity vector $\dot\gamma(0) = (\xi_1, \ldots, \xi_n)$ lies in $V^\perp$, its components $\xi_1, \ldots, \xi_r$ vanish so that

$$t_1(\gamma, \dot\gamma) = \ldots = t_r(\gamma, \dot\gamma) = \lambda_1. \tag{10}$$

Arguing similar to above, at every point $\gamma(\alpha)$ such that $\lambda_{r+1}(\gamma(\alpha)) > \lambda_1$, one can show that the components $\xi_1, \ldots, \xi_r$ of the vector $\dot\gamma(\alpha)$ vanish as well. Thus a point where $\lambda_1 < \lambda_{r+1} < \lambda_n$ can not be a point of transversal intersection of two geodesics γ. The contradictions shows that the sets U_r' and U_r coincide. Thus U_r is a submanifold of dimension r.

Similarly, U_q is a submanifold of dimension q.

Since the submanifolds U_r, U_q have codimension greater than one, every element of the fundamental group can be realized on the connected set

$$M^n \setminus (U_r \cup U_q) = \{x \in M^n : \ \lambda_1 < \lambda_{r+1}(x) < \lambda_n\}.$$

At every point of $M^n \setminus (U_r \cup U_q)$, consider the vector v_2 satisfying

$$\begin{cases} L(v_2) = \lambda_{r+1} \cdot v_2 \\ g(v_2, v_2) = (\lambda_{r+1}(x) - \lambda_1)(\lambda_n - \lambda_{r+1}(x)) \end{cases} \tag{11}$$

The only freedom we have is the sign of the vector. Then, at least on the double-cover of $M^n \setminus (U_r \cup U_q)$, we can find a vector field v_2 satisfying (11). The fundamental group of the double-cover is finite if and only if the fundamental group of the manifold is finite. Since our goal is to prove that the fundamental group of M^n is finite, we can assume that the vector field v_2 is defined already on $M^n \setminus (U_r \cup U_q)$.

By definition, every point of $M^n \setminus (U_r \cup U_q)$ is stable. By Levi-Civita's Theorem, every point of $M^n \setminus (U_r \cup U_q)$ has a neighborhood with coordinates

$$\bar{x} = (\bar{x}_1, \bar{x}_2, \bar{x}_3) = \left(\left(x_1^1, \dots, x_1^r\right), x_2, \left(x_3^1, \dots, x_3^q\right)\right),$$

where the metrics g, $\bar{g}$ are given by

$$\begin{aligned} g(\dot{\bar{x}}, \dot{\bar{x}}) = & (\lambda_{r+1}(x_2) - \lambda_1)(\lambda_n - \lambda_1) A_1(\bar{x}_1, \dot{\bar{x}}_1) + \\ & + (\lambda_{r+1}(x_2) - \lambda_1)(\lambda_n - \lambda_{r+1}(x_2))\, dx_2^2 + \\ & + (\lambda_n - \lambda_{r+1}(x_2))(\lambda_n - \lambda_1) A_3(\bar{x}_3, \dot{\bar{x}}_3), \end{aligned} \tag{12}$$

$$\begin{aligned} \bar{g}(\dot{\bar{x}}, \dot{\bar{x}}) = & \frac{(\lambda_{r+1}(x_2) - \lambda_1)(\lambda_n - \lambda_1)}{\lambda_1^{r+1} \lambda_{r+1}(x_2) \lambda_n^q} A_1(\bar{x}_1, \dot{\bar{x}}_1) + \\ & + \frac{(\lambda_{r+1}(x_2) - \lambda_1)(\lambda_n - \lambda_{r+1}(x_2))}{\lambda_1^r \lambda_{r+1}^2(x_2) \lambda_n^q}\, dx_2^2 + \\ & + \frac{(\lambda_n - \lambda_{r+1}(x_2))(\lambda_n - \lambda_1)}{\lambda_1^r \lambda_{r+1}(x_2) \lambda_n^{q+1}} A_3(\bar{x}_3, \dot{\bar{x}}_3). \end{aligned} \tag{13}$$

In these coordinates, the vector v_2 is equal to $\pm\frac{\partial}{\partial x_2}$. Then, the distribution orthogonal to the vector field v_2 is integrable and, hence, defines a foliation of codimension one on $M^n \setminus (U_r \cup U_q)$. Moreover, the flow of the vector field v_2 takes the leaves to the leaves. Then, all leaves are diffeomorphic. By Corollary 3, all leaves are compact. Hence, the foliation is a fibration. Since the base of these fibration is an interval, the fundamental group of $M^n \setminus (U_r \cup U_q)$ coincides with the fundamental group of each of its fibers. Denote by W_0 one of the fibers.

In the Levi-Civita coordinates $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$, the fiber W_0 is given by the equation $x_2 = const$, and the restriction of L to the tangent space to W_0 is diagonal

$$\mathrm{diag}(\underbrace{\lambda_1, \dots, \lambda_1}_{r}, \underbrace{\lambda_n, \dots, \lambda_n}_{q}).$$

We see that the eigenspaces corresponding to λ_1 and λ_n are integrable (so that they define two foliations B_r and B_q). We see that the $\bar{x}_3$-coordinates are constant on every leaf of the foliation B_r, the $\bar{x}_1$-coordinates are constant on every leaf of the foliation B_q, and the metric g is block-diagonal with the first $(r \times r)$ block depending on the $\bar{x}_1$-coordinates and the second $(q \times q)$ block depending on the last $\bar{x}_3$ coordinates.

Let us show that every leaf of B_q is closed and is diffeomorphic to the sphere S^q. Take a leaf F of B_q. Since the flow of the vector field v_2 preserves the foliations B_r and B_q, without loss of generality, we may assume that at least one point of the leaf F lies closer to U_r than the radius of injectivity of the manifold.

Consider the shortest curve connecting the leaf F with the submanifold U_r. This curve is an unparameterized geodesic and is orthogonal to the submanifold U_r. Suppose the length of this curve is D. Denote by $x_0 \in U_r$ the endpoint of the curve. Denote by $V^\perp \subset T_{x_0}M^n$ the subspace of $T_{x_0}M^n$ orthogonal to U_r. Consider the sphere $S^q \subset V^\perp$ of radius D (in the metric g). Let us show that the restriction of the exponential mapping to the sphere diffeomorphically maps the sphere to the leaf F.

Take a geodesic γ such that $\gamma(0) = x_0$, $\dot\gamma(0) \in V^\perp$. As we have shown above, for every τ such that $\gamma(\tau) \in M^n \setminus (U_r \cup U_q)$ the conditions (8,10) hold so that the components $\xi_1, \dots, \xi_r$ and $\xi_{r+1}, \dots, \xi_n$ of the velocity vector in (9) vanish.

Then, the velocity vector $\dot\gamma(\tau)$ is the eigenvector of L corresponding to λ_{r+1}, and, hence, is orthogonal to W_0. Then, the image of the sphere lies in W_0.

At every point $x \in M^n$, consider the subspace of T_xM^n spanned over the eigenspaces of L corresponding to λ_n and λ_{r+1}. For every τ such that $\gamma(\tau) \in M^n \setminus U_r$, the velocity vector $\dot\gamma(\tau)$ lies in the subspace. By Corollary 2, the distribution of these subspaces is integrable on $M^n \setminus U_r$. In Levi-Civita coordinates, the integral submanifolds of the distribution are given by

$$\bar{x}_3 = \text{const.} \tag{14}$$

Since the image of the sphere S^q lies in the distribution, the points of the image satisfy (14) so that the image of the sphere S^q coincides with the leaf F.

In particular, every curve lying on F is contractible in M^n.

Similarly, every leaf of the foliation B_r is diffeomorphic to the sphere S^r, and every curve lying on any leaf of B_r is contractible in M^n. Since any sphere is compact, a leaf of B_r can have only finitely many points of intersection with a leaf of B_q. Hence, there exists a finite covering $p : S^r \times S^q \to W_0$ such that the images $p(S^r \times (\text{point of } S^q))$ are the leaves of B_r and the images $p((\text{point of } S^r) \times S^q)$ are the leaves of B_q. Then, the fundamental group of W_0 has a subgroup of finite index such that the subgroup is the direct product of two groups such that the elements of the first group can be realized by curves lying on the leaves of B_r and the elements of the second group can be realized by curves lying on the leaves of B_q. Since every curve lying on the leaves of B_q and B_r is contractible in M^n, the fundamental group of M^n is finite. Theorem 2 is proved under the assumptions of Case 2, Subcase 4.

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