Minimal entropy and collapsing with curvature bounded from below

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Abstract. We show that if a closed manifold M admits an $\mathcal{F}$-structure (not necessarily polarized, possibly of rank zero) then its minimal entropy vanishes. In particular, this is the case if M admits a non-trivial S^1-action. As a corollary we obtain that the simplicial volume of a manifold admitting an $\mathcal{F}$-structure is zero.

We also show that if M admits an $\mathcal{F}$-structure then it collapses with curvature bounded from below. This in turn implies that M collapses with bounded scalar curvature or, equivalently, its Yamabe invariant is non-negative.

We show that $\mathcal{F}$-structures of rank zero appear rather frequently: every compact complex elliptic surface admits one as well as any simply connected closed 5-manifold.

We use these results to study the minimal entropy problem. We show the following two theorems: suppose that M is a closed manifold obtained by taking connected sums of copies of S^4, $\mathbb{C}P^2$, $\overline{\mathbb{C}P}^2$, $S^2 \times S^2$ and the $K3$ surface. Then M has zero minimal entropy. Moreover, M admits a smooth Riemannian metric with zero topological entropy if and only if M is diffeomorphic to S^4, $\mathbb{C}P^2$, $S^2 \times S^2$, $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ or $\mathbb{C}P^2 \# \mathbb{C}P^2$. Finally, suppose that M is a closed simply connected 5-manifold. Then M has zero minimal entropy. Moreover, M admits a smooth Riemannian metric with zero topological entropy if and only if M is diffeomorphic to S^5, $S^3 \times S^2$, the nontrivial S^3-bundle over S^2 or the Wu-manifold $SU(3)/SO(3)$.

1. Introduction

Let M^n be a closed orientable connected smooth manifold. Given a Riemannian metric g, let ϕ_t be the geodesic flow of g.

Perhaps the simplest dynamical invariant that one can associate to ϕ_t to roughly measure its orbit structure complexity is the *topological entropy*, which we denote by $\mathrm{h}_{\mathrm{top}}(g)$. Positive entropy means in general, that the geodesic flow presents somewhere in the phase space (the unit sphere bundle of the manifold) a complicated dynamical behaviour. There are various equivalent ways of defining entropy (see Subsect. 2.3) and among them there is a formula, known as Mañé's formula, that gives a nice Riemannian description of $\mathrm{h}_{\mathrm{top}}(g)$. Given points p and q in M and $T > 0$, define $n_T(p, q)$ to be the number of geodesic arcs joining p and q with length $\leq T$. R. Mañé [30] showed that

$$\mathrm{h}_{\mathrm{top}}(g) = \lim_{T \to \infty} \frac{1}{T} \log \int_{M \times M} n_T(p, q)\, dp\, dq.$$

One of the main goals in this paper will be the study of the variational theory of the functional $g \mapsto \mathrm{h}_{\mathrm{top}}(g)$. In general this functional is only upper semicontinuous in the C^∞ topology ([32,46]) and it has a simple behaviour under scaling of the metric: if c is any positive constant, then $\mathrm{h}_{\mathrm{top}}(cg) = \frac{\mathrm{h}_{\mathrm{top}}(g)}{\sqrt{c}}$. Hence if we want to extract interesting extremal metrics from this functional a normalization is required. The Riemannian invariant that we will use for this normalization is the volume $\mathrm{Vol}(M, g)$.

Set the *minimal entropy* of M to be

$$\mathrm{h}(M) := \inf\{\mathrm{h}_{\mathrm{top}}(g) \mid g \text{ is a smooth metric on } M \text{ with } \mathrm{Vol}(M, g) = 1\}.$$

A smooth metric g_0 with $\mathrm{Vol}(M, g_0) = 1$ is *entropy minimizing* if

$$\mathrm{h}_{\mathrm{top}}(g_0) = \mathrm{h}(M).$$

The *minimal entropy problem* for M is whether or not there exists an entropy minimizing metric on M. Say that the *minimal entropy problem can be solved* for M if there exists an entropy minimizing metric on M. Smooth manifolds are hence naturally divided into two classes: those for which the minimal entropy problem can be solved and those for which it cannot. Passing by, we note that we do not obtain a meaningful invariant if we replace the infimum by the supremum. Indeed, Manning proved in [29] that

$$\sup\{\mathrm{h}_{\mathrm{top}}(g) \mid g \text{ is a smooth metric on } M \text{ with } \mathrm{Vol}(M, g) = 1\} = \infty.$$

There are a number of classes of manifolds for which the minimal entropy problem can be solved. For instance, the minimal entropy problem can always be solved for a closed orientable surface M. For the 2-sphere and the 2-torus, this follows from the fact that both a metric with constant positive

curvature and a flat metric have zero topological entropy. For surfaces of higher genus, A. Katok [23] proved that each metric of constant negative curvature minimizes topological entropy, and conversely that any metric that minimizes topological entropy has constant negative curvature.

This result of Katok has been generalized to higher dimensions by G. Besson, G. Courtois and S. Gallot [6], as follows. Suppose that M^n $(n \geq 3)$ admits a locally symmetric metric g_0 of negative curvature, normalized so that $\mathrm{Vol}(M, g_0) = 1$. Then g_0 is the unique entropy minimizing metric up to isometry. Unlike the case of a surface, a locally symmetric metric of negative curvature on a closed n-manifold $(n \geq 3)$ is unique up to isometry, by the rigidity Theorem of Mostow.

A positive solution to the minimal entropy problem appears to single out manifolds that have either a high degree of symmetry or a low topological complexity. What this means in our context will become apparent below. A similar phenomenon is observed for closed 3-manifolds [2].

There is a close relationship between minimal entropy, *minimal volume* and *simplicial volume*. As we shall explain in Subsect. 2.4 there is a positive constant $c(n)$ such that

$$(1) \qquad c(n) \, \|M\| \leq [\mathrm{h}(M)]^n \leq (n-1)^n \mathrm{MinVol}(M).$$

Recall that the minimal volume $\mathrm{MinVol}(M)$ is the infimum of $\mathrm{Vol}(M, g)$ where g runs over all metrics whose sectional curvature is bounded in absolute value by 1. Also recall that the simplicial volume of a closed orientable manifold M, $\|M\|$, is defined as the infimum of $\sum_i |r_i|$ where the r_i are the coefficients of a real cycle that represents the fundamental class of M. This number is a homotopy invariant of M. The minimal volume does depend on the smooth structure of M (see [5]) but we do not know if the same holds true for the minimal entropy.

Computing these invariant is in general a very difficult task. J. Cheeger and M. Gromov introduced in [9,20] the concept of $\mathcal{F}$-structure (see Sect. 5 for the precise definition). An $\mathcal{F}$-*structure* on a manifold M is a natural generalization of an effective torus action on M. The structure partitions M into disjoint orbits which are flat manifolds amenable to collapse. When the dimension of the orbits are, in a certain precise sense, locally constant, the structure is said to be *polarized*. The simplest $\mathcal{F}$-structures are the $\mathcal{T}$-structures, which consist of open coverings of the manifold and torus actions on each of the elements of the covering which commute on overlaps. The simplest example of a polarized $\mathcal{T}$-structure is given by a locally free circle action.

Cheeger and Gromov proved in [9,20] that if M admits a polarized $\mathcal{F}$-structure then the minimal volume of M vanishes. The vanishing of the minimal volume implies in turn that all the characteristic numbers of the manifold are zero. Cheeger and Gromov also proved that if the $\mathcal{F}$-structure has *positive rank*, i.e., all its orbits have positive dimension, then the Euler characteristic of M must be zero. There exist plenty of examples of closed

manifolds which admit $\mathcal{F}$-structures but whose Euler characteristic is non-zero. Therefore they do not admit $\mathcal{F}$-structures of positive rank. For instance the Euler characteristic of any simply connected closed 4-manifold is strictly positive, but for any m, n, k, l, the manifold $n\mathbb{C}P^2 \# k\overline{\mathbb{C}P}^2 \# mK3 \# l(S^2 \times S^2)$ admits a $\mathcal{T}$-structure of rank zero. This will follow form the results in Sect. 5. Hence, general $\mathcal{F}$-structures are abundant in comparison with polarized ones.

In Sect. 6 we show:

Theorem A. *If the closed manifold M admits an $\mathcal{F}$-structure then the minimal entropy of M is 0.*

The theorem and (1) yield the following corollary, which generalizes the result of K. Yano [45] that closed manifolds which admit non-trivial S^1-actions have simplicial volume 0 (there is also a proof of the latter result in [20]).

Corollary. *Let M be a closed manifold. If M admits an $\mathcal{F}$-structure then the simplicial volume of M is 0.*

Hence the existence of an $\mathcal{F}$-structure, possibly of rank zero, also imposes constraints on the topology of the manifold.

The method employed in the proof of Theorem A is general enough that allows us to apply it to the study of other types of collapsing. We will say that *M collapses with curvature bounded from below* if there exists a sequence of metrics g_j for which the sectional curvature is uniformly bounded from below, but their volumes approach zero as j goes to infinity. Similarly we will say that *M* collapses with Ricci (respectively, scalar) curvature bounded from below if there exists a sequence of metrics g_j for which the Ricci (respectively, scalar) curvature is uniformly bounded from below, but their volumes approach zero as j goes to infinity. In Sect. 7 we prove:

Theorem B. *If the closed manifold M admits an $\mathcal{F}$-structure then M collapses with curvature bounded from below.*

Clearly if *M* collapses with curvature bounded form below then it also collapses with Ricci and scalar curvatures bounded from below. As we explain in Sect. 7 if *M* has dimension ≥ 3, then it collapses with scalar curvature bounded form below if and only if it collapses with *bounded scalar curvature*. This is in turn equivalent to having non-negative *Yamabe invariant*.

It is interesting to remark that for instance the manifold $T^4 \# \overline{\mathbb{C}P}^2$ admits an $\mathcal{F}$-structure but it does *not* collapse with bounded Ricci curvature (see Sect. 7). Therefore our Theorem B can be regarded as an optimal extension of the results of Cheeger and Gromov in the sense that there is no stronger

collapsing phenomena for general $\mathcal{F}$-structures other than the one claimed in the theorem.

C. LeBrun proved in [26] that the Yamabe invariant of any compact complex surface of general type is strictly negative. It follows from Theorem B that these surfaces do not admit $\mathcal{F}$-structures. Among these surfaces of general type there are simply connected ones which are homeomorphic (but not diffeomorphic) to connected sums of $\mathbb{C}P^2$'s and $\overline{\mathbb{C}P}^2$'s. Hence in dimension 4 there are simply connected closed manifolds which do not admit $\mathcal{F}$-structures and they are homeomorphic to manifolds that do admit them. We do not know if this a phenomenon specific of dimension 4. In dimension ≥ 5 the second author showed in [37] that any simply connected manifold has non-negative Yamabe invariant. This opens the possibility that *any* closed simply connected manifold of dimension ≥ 5 admits an $\mathcal{F}$-structure. Morever it is possible for this structure to be polarized in odd dimensions. In fact we show in Sect. 8:

Theorem C. *Every simply connected closed smooth 5-manifold M admits a $\mathcal{T}$-structure. Moreover, suppose that either:*

1. *M is spin;*
2. *M is the non-trivial S^3-bundle over S^2 or the Wu-manifold $SU(3)/SO(3)$;*
3. *M is a connected sum of manifolds of types 1 or 2.*

Then M admits a polarized $\mathcal{T}$-structure.

We do not know if *every* closed simply connected non-spin 5-manifold admits a polarized $\mathcal{T}$-structure, even though it appears to be the case.

These results can be used to give fairly complete solutions to the minimal entropy problem for simply connected manifolds of dimensions 4 and 5.

Theorem D. *Let M be a closed manifold obtained by taking connected sums of copies of S^4, $\mathbb{C}P^2$, $\overline{\mathbb{C}P}^2$, $S^2 \times S^2$ and the K3 surface. Then $\mathrm{h}(M) = 0$ and the minimal entropy problem can be solved for M if and only if M is diffeomorphic to S^4, $\mathbb{C}P^2$, $S^2 \times S^2$, $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ or $\mathbb{C}P^2 \# \mathbb{C}P^2$.*

A manifold M like in Theorem D realizes many intersection forms of simply connected 4-manifolds. In fact the 11/8-conjecture (see [11,12]) states that any smooth simply connected 4-manifold is homeomorphic to a manifold as in Theorem D. Hence, if one assumes the 11/8-conjecture, Theorem D is saying that any smooth simply connected 4-manifold is homeomorphic to one whose minimal entropy is zero and for which we know the answer to the minimal entropy problem.

The proof of Theorem D is partially based on the fact that the $K3$ surface admits a $\mathcal{T}$-structure. In fact we show that any elliptic compact complex surface admits a $\mathcal{T}$-structure. We also show that $\mathcal{T}$-structures behave relatively well with respect to the usual operations of connected sums and surgeries on manifolds.

For simply connected 5-manifolds, we have a complete answer to the minimal entropy problem :

Theorem E. *Let M be a closed simply connected 5-manifold. Then $h(M) = 0$ and the minimal entropy problem can be solved for M if and only if M is diffeomorphic to S^5, $S^3 \times S^2$, the nontrivial S^3-bundle over S^2 or the Wu-manifold $SU(3)/SO(3)$.*

The common feature of the nine manifolds listed in Theorems D and E is that they are *elliptic*. This means that their loop space homology grows polynomially for every coefficient field (cf. Sect. 3, [15,16,21] and references therein). In fact, as we will see in Sect. 3, these are the only elliptic manifolds in dimensions 4 and 5. Hence Theorems D and E characterize this very much studied class of manifolds as that for which the minimal entropy problem can be solved or, equivalently, as that for which there exists a smooth metric g with $h_{\text{top}}(g) = 0$. It is tempting to speculate that perhaps the same phenomenon occurs in any dimension.

We would like to close this introduction by illustrating some of the ideas with specific examples. A 5-dimensional Brieskorn variety of type (a_1, a_2, a_3, a_4) is given by the intersection of the 7-sphere in $\mathbb{C}^4$ with the zero set of:

$$f(z_1, z_2, z_3, z_4) = z_1^{a_1} + z_2^{a_2} + z_3^{a_3} + z_4^{a_4}.$$

This gives a large class of simply connected 5-manifolds. In fact, they are all spin and their second homology group can be computed using the algorithm described in [33,38]. The Brieskorn varieties admit very simple polarized $\mathcal{T}$-structures: they have a canonically defined action of S^1 which is locally free (but not free in general). If we let

$$q_i = \text{lcm}(a_1, a_2, a_3, a_4)/a_i$$

then the action is given by:

$$e^{i\theta}(z_1, z_2, z_3, z_4) = \left(e^{q_1 i\theta} z_1, e^{q_2 i\theta} z_2, e^{q_3 i\theta} z_3, e^{q_4 i\theta} z_4 \right).$$

For example, the Brieskorn variety M_2 defined by $(2, 3, 3, 3)$ coincides with the spin manifold whose second homology group is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. It has the property that its loop space homology grows exponentially with $\mathbb{Z}_2$ coefficients and hence for any C^∞ Riemannian metric g, $h_{\text{top}}(g) > 0$ (see Theorem 8.3 in Sect. 8). Since $\text{MinVol}(M_2) = h(M_2) = 0$ it follows that the minimal entropy problem for M_2 cannot be solved. It is interesting to note that M_2 has the rational cohomology ring of the 5-sphere and hence its loop space homology with rational coefficients is actually bounded, i.e., M_2 is rationally elliptic.

2. Preliminaries on simplicial volume, minimal volume and topological entropy

The purpose of this section is to present some of the basic material and definitions that we will need later on.

2.1. Simplicial volume.

Let M be a closed manifold. Denote by C_* the real chain complex of M: a chain $c \in C_*$ is a finite linear combination $\sum_i r_i \sigma_i$ of singular simplices σ_i in M with real coefficients r_i. Define the *simplicial l^1-norm* in C_* by setting $|c| = \sum_i |r_i|$. This norm gives rise to a pseudo-norm on the homology $H_*(M, \mathbb{R})$ by setting

$$\|[\alpha]\| = \inf\{|z| \; : \; z \in C_* \text{ and } [z] = [\alpha]\}.$$

When M is orientable, define the *simplicial volume* of M, denoted $\|M\|$, to be the simplicial norm of the fundamental class. The simplicial volume is also called *Gromov's invariant*, since it was first introduced by Gromov in [20].

2.2. Minimal volume and collapsing.

The *minimal volume* $\mathrm{MinVol}(M)$ of a manifold M is defined to be the infimum of $\mathrm{Vol}(M, g)$ over all metrics g such that the sectional curvature K_g of g satisfies $|K_g| \leq 1$. This differential invariant was introduced by M. Gromov in [20].

As we mentioned in the introduction we have [9]:

Proposition 2.1. *If M admits a polarized $\mathcal{F}$-structure, then* $\mathrm{MinVol}(M) = 0$.

2.3. Topological entropy and curvature.

We recall in this subsection the definition of the topological entropy of the geodesic flow of a Riemannian metric g on a closed manifold M. The geodesic flow of g is a flow ϕ_t that acts on SM, the unit sphere bundle of M, which is a closed hypersurface of the tangent bundle of M. In general the topological entropy is defined for an arbitrary continuous flow (or map) on a compact metric space.

Let (X, d) be a compact metric space and let $\phi_t : X \to X$ be a continuous flow. For each $T > 0$ we define a new distance function

$$d_T(x, y) := \max_{0 \leq t \leq T} d(\phi_t(x), \phi_t(y)).$$

Since X is compact, we can consider the minimal number of balls of radius $\varepsilon > 0$ in the metric d_T that are necessary to cover X. Let us denote this number by $N(\varepsilon, T)$. We define

$$h(\phi, \varepsilon) := \limsup_{T \to \infty} \frac{1}{T} \log N(\varepsilon, T).$$

Observe now that the function $\varepsilon \mapsto h(\phi, \varepsilon)$ is monotone decreasing and therefore the following limit exists:

$$h_{\text{top}}(\phi) := \lim_{\varepsilon \to 0} h(\phi, \varepsilon).$$

The number $h_{\text{top}}(\phi)$ thus defined is called the *topological entropy* of the flow ϕ_t. Intuitively, this number measures the orbit complexity of the flow. The positivity of $h_{\text{top}}(\phi)$ indicates complexity or "chaos" of some kind in the dynamics of ϕ_t. The topological entropy $h_{\text{top}}(\phi)$ may also be defined as $h_{\text{top}}(\phi_1)$ using the entropy of the time one-map or it may be defined in either of the following ways. All the definitions give the same number $h_{\text{top}}(\phi)$ which is independent of the choice of metric [22, 44].

A set $Y \subset X$ is called a (T, ε)-separated set if given different points $y, y' \in Y$ we have $d_T(y, y') \geq \varepsilon$. Let $S(T, \varepsilon)$ denote the maximal cardinality of a (T, ε)-separated set. Then

$$h_{\text{top}}(\phi) = \lim_{\varepsilon \to 0} \limsup_{T \to \infty} \frac{1}{T} \log S(T, \varepsilon).$$

A set $Z \subset X$ is called a (T, ε)-spanning set if for all $x \in X$ there exists $z \in Z$ such that $d_T(x, z) \leq \varepsilon$. Let $M(T, \varepsilon)$ denote the minimal cardinality of a (T, ε)-spanning set. Then

$$h_{\text{top}}(\phi) = \lim_{\varepsilon \to 0} \limsup_{T \to \infty} \frac{1}{T} \log M(T, \varepsilon).$$

Given a compact subset $K \subset X$ (not necessarily invariant) we can define the topological entropy of the flow with respect to the set K, $h_{\text{top}}(\phi, K)$, simply by considering separated (spanning) sets of K.

The following proposition gives an idea of the dynamical significance of the topological entropy (for proofs see [22, 44]).

Proposition 2.2. *The topological entropy satisfies the following properties:*

1. *For any two closed subsets Y_1, Y_2 in X,*

$$h_{\text{top}}(\phi, Y_1 \cup Y_2) = \max_{i=1,2} h_{\text{top}}(\phi, Y_i);$$

2. *If $Y_1 \subset Y_2$ then $h_{\text{top}}(\phi, Y_1) \leq h_{\text{top}}(\phi, Y_2)$;*
3. *Let $\phi_t^i : X_i \to X_i$ for $i = 1, 2$ be two flows and let $\pi : X_1 \to X_2$ be a continuous map commuting with ϕ_t^i i.e. $\phi_t^2 \circ \pi = \pi \circ \phi_t^1$. If π is onto, then $h_{\text{top}}(\phi^1) \geq h_{\text{top}}(\phi^2)$ and if π is finite-to-one, then $h_{\text{top}}(\phi^1) \leq h_{\text{top}}(\phi^2)$.*
4. *Let $\phi_t^i : X_i \to X_i$ for $i = 1, 2$ be two flows and let $\psi_t := \phi_t^1 \times \phi_t^2$ be the product flow on $X_1 \times X_2$. Then $h_{\text{top}}(\psi) = h_{\text{top}}(\phi^1) + h_{\text{top}}(\phi^2)$.*
5. *Given $c \in \mathbb{R}$, let $c\phi_t$ be the flow given by $c\phi_t := \phi_{ct}$. Then $h_{\text{top}}(c\phi) = |c|h_{\text{top}}(\phi)$.*

Next we shall state a useful result of R. Bowen that we will need later.

Proposition 2.3. [Corollary 18 in [7]] *Let (X, d) and (Y, e) be compact metric spaces and $\phi_t : X \to X$ a flow. Suppose $\pi : X \to Y$ is a continuous map such that $\pi \circ \phi_t = \pi$. Then*

$$h_{top}(\phi) = \sup_{y \in Y} h_{top}(\phi, \pi^{-1}(y)).$$

Given a Riemannian metric g, let d be any distance function compatible with the topology of SM. Since the geodesic flow is a smooth flow on SM we can attach to it its topological entropy that we denote by $h_{top}(g)$ to stress its dependence on the Riemannian metric g. There is a formula, known as Mañé's formula, that gives a nice alternative way of thinking about $h_{top}(g)$. Given p and q in M and $T > 0$, define $n_T(p, q)$ as the number of geodesic arcs joining p and q with length $\leq T$. R. Mañé showed in [30] that

$$h_{top}(g) = \lim_{T \to \infty} \frac{1}{T} \log \int_{M \times M} n_T(p, q) \, dp \, dq.$$

Using property 5 in Proposition 2.2 it is easy to check how entropy behaves under scaling: if c is any positive constant, then $h_{top}(cg) = \frac{h_{top}(g)}{\sqrt{c}}$.

We now describe a basic relationship between entropy and curvature.

Let (M^n, g) be a closed Riemannian manifold and let K_{max} be a positive upper bound for the sectional curvature. It was proved in [36] that

$$h_{top}(g) \leq \frac{n-1}{2} \sqrt{K_{max}} - \frac{\min_{v \in SM} r(v)}{2\sqrt{K_{max}}},$$

where SM is the unit sphere bundle of M and $r(v)$ is the Ricci curvature in the direction of $v \in SM$.

Let k be a positive number such that $|K(P)| \leq k$ for all 2-planes P. Then, clearly $r \geq -(n-1)k \, g$ and hence the previous inequality gives

(2) $$h_{top}(g) \leq \frac{n-1}{2} \sqrt{k} + \frac{n-1}{2} \sqrt{k} = (n-1)\sqrt{k}.$$

The latter inequality was first proved by A. Manning in [29].

2.4. **An important chain of inequalities.** Let (M, g) be a closed Riemannian manifold and let $\widetilde{M}$ be its universal covering endowed with the induced metric. Given $x \in \widetilde{M}$, let $V(x, r)$ be the volume of the ball with center x and radius r. Set

$$\lambda(g) := \lim_{r \to +\infty} \frac{1}{r} \log V(x, r).$$

Manning [28] showed that the limit exists and it is independent of x.

Set

$$\lambda(M) := \inf\{\lambda(g) \mid g \text{ is a smooth metric on } M \text{ with } \mathrm{Vol}(M, g) = 1\}.$$

It is well known [31] that $\lambda(g)$ is positive if and only if $\pi_1(M)$ has exponential growth. Manning's inequality [28] asserts that for any metric g,

$$\lambda(g) \leq h_{\text{top}}(g). \tag{3}$$

In particular, it follows that if $\pi_1(M)$ has exponential growth then $h_{\text{top}}(g)$ is positive for any metric g. This fact was first observed by E.I. Dinaburg in [10]. Gromov showed in [20] that if $\text{Vol}(M, g) = 1$, then there is a positive constant $c(n)$ such that

$$c(n)\|M\| \leq [\lambda(g)]^n. \tag{4}$$

Finally it was observed in [34] that using (2) it is easy to show that

$$[h(M)]^n \leq (n-1)^n \text{MinVol}(M). \tag{5}$$

Hence if we combine (3), (4) and (5), we obtain the following chain of inequalities:

$$c(n)\|M\| \leq [\lambda(M)]^n \leq [h(M)]^n \leq (n-1)^n \text{MinVol}(M). \tag{6}$$

The only known manifolds with $h(M) > 0$ are manifolds with $\|M\| \neq 0$. For these manifolds $\pi_1(M)$ has exponential growth.

2.5. **Entropy of products and submersions.**

Lemma 2.4. 1. *Let (M_1, g_1) and (M_2, g_2) be two compact Riemannian manifolds. Endow $M_1 \times M_2$ with the product metric $g_1 \times g_2$. Then*

$$h_{\text{top}}(g_1 \times g_2) = \sqrt{[h_{\text{top}}(g_1)]^2 + [h_{\text{top}}(g_2)]^2}.$$

2. *Let $(M, g_M) \mapsto (N, g_N)$ be a Riemannian submersion where M and N are compact manifolds. Then $h_{\text{top}}(g_M) \geq h_{\text{top}}(g_N)$.*

Proof. Let us prove the first item. Let $f : S(M_1 \times M_2) \to S^1$ be the function given by

$$f(x_1, v_1, x_2, v_2) = (|v_1|_{x_1}, |v_2|_{x_2}).$$

Since the geodesics in $M_1 \times M_2$ are products of geodesics in M_1 and M_2, the function f is constant along the orbits of the geodesic flow of $M_1 \times M_2$. It follows from Proposition 2.3 that

$$h_{\text{top}}(g_1 \times g_2) = \sup_{c \in S^1} h_{\text{top}}(f^{-1}(c)).$$

If we write $c = (l, m)$, it is easy to check using Proposition 2.2 that

$$h_{\text{top}}(f^{-1}(c)) = l\, h_{\text{top}}(g_1) + m\, h_{\text{top}}(g_2)$$

from which we obtain right away the first equality in the lemma.

To prove the second item, let $H \subset SM$ be the set of all horizontal unit vectors. Clearly the geodesic flow of (M, g_M) leaves H invariant. Let $\tau : H \to SN$ be the restriction to H of the differential of the submersion map. Since horizontal geodesics project to geodesics, τ is a surjective map that intertwines the geodesic flow of (M, g_M) restricted to H with the geodesic flow of (N, g_N). It follows from Proposition 2.2 that $h_{\text{top}}(g_M) \geq h_{\text{top}}(g_N)$. □

3. Elliptic manifolds in dimensions 4 and 5

Let M be a closed simply connected manifold and let ΩM be the space of based loops. Let k_p be the prime field of characteristic p, p prime or zero. Following Y. Félix, S. Halperin and J.C. Thomas we say that M is *elliptic* if for each p, the homology of the loop space:

$$\sum_{i=0}^{n} \dim H_i(\Omega M, k_p),$$

grows polynomially with n (cf. [15, 16, 21] and references therein).

Elliptic manifolds are rare. However a number of geometrically interesting spaces are elliptic:

1. homogeneous spaces;
2. manifolds M admitting a fibration $F \to M \to B$ with F and B elliptic;
3. manifolds M for which the algebra $H^*(M, k_p)$ is generated by two elements for all p;
4. manifolds M admitting a smooth action by a compact Lie group with a simply connected codimension one orbit;
5. connected sums $M \# N$ with the algebras $H^*(M, \mathbb{Z})$ and $H^*(N, \mathbb{Z})$ each generated by a single class.

The manifold M is said to be *rationally elliptic* if the total rational homotopy $\pi_*(M) \otimes \mathbb{Q}$ is finite dimensional, i.e. there exists a positive integer i_0 such that for all $i \geq i_0$, $\pi_i(M) \otimes \mathbb{Q} = 0$. This property is known to be equivalent to the polynomial growth of $\sum_{i=0}^{n} \dim H_i(\Omega M, \mathbb{Q})$. Obviously an elliptic manifold is rationally elliptic. We will see that for smooth 4-manifolds ellipticity and rational ellipticity are equivalent. This is no longer the case for 5-manifolds as we will see below.

Lemma 3.1. *Suppose that M is 4-dimensional and let b_2 be the second Betti number of M. If M is rationally elliptic then $b_2 \leq 2$.*

Proof. It is shown in [18, Corollary 1.3] (cf. also [14]) that if M^n is rationally elliptic then,

$$(7) \qquad \sum_{k \geq 1} 2k \dim (\pi_{2k}(M) \otimes \mathbb{Q}) \leq n.$$

Since M is simply connected the Hurewicz isomorphism theorem implies that

$$b_2 = \dim\ H_2(M, \mathbb{Q}) = \dim\ (\pi_2(M) \otimes \mathbb{Q}).$$

Since $n = 4$, using (7) we obtain $2\,b_2 \leq 4$. $\square$

Lemma 3.2. *Let M be a closed smooth simply connected 4-manifold. The following are equivalent:*

1. *M is elliptic;*
2. *M is rationally elliptic;*
3. *M is homeomorphic to S^4, $\mathbb{C}P^2$, $S^2 \times S^2$, $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ or $\mathbb{C}P^2 \# \mathbb{C}P^2$.*

Moreover, if M is not elliptic then $\sum_{i=0}^{n} \dim H_i(\Omega M, \mathbb{Q})$ grows exponentially.

Proof. Obviously 1 implies 2. Let us prove that 2 implies 3. Suppose that M is rationally elliptic. By Lemma 3.1, $b_2 \leq 2$. Since M is smooth, the Kirby-Siebenmann obstruction vanishes. Therefore by M. Freedman's theory [17], the homeomorphism type of M is completely determined by the intersection form of M. It follows that if $b_2 = 0$, M is homeomorphic to S^4 and if $b_2 = 1$, M is homeomorphic to $\mathbb{C}P^2$. When $b_2 = 2$, the possible intersection forms are

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

These forms correspond to $S^2 \times S^2$, $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ and $\mathbb{C}P^2 \# \mathbb{C}P^2$ respectively.

On the other hand S^4, $\mathbb{C}P^2$ and $S^2 \times S^2$ are homogeneous spaces and hence they are elliptic (see property 1 above). By property 5 above, $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ and $\mathbb{C}P^2 \# \mathbb{C}P^2$ are elliptic.

Finally, it is well known that the homology of the loop space with rational coefficients can either grow polynomially or exponentially. $\square$

Remark 3.3. For an arbitrary simply connected manifold M it is known that if $\sum_{i=0}^{n} \dim H_i(\Omega M, k_p)$ does not grow polynomially then it must grow at least like $\lambda^{\sqrt{n}}$ for some $\lambda > 1$ [16]. There is a conjecture that says that the growth should in fact be exponential, but this is only known for rational coefficients (as we mentioned at the end of the proof of the last lemma) and for primes p strictly bigger that the dimension of M.

Theorem 3.4 (Following a suggestion of S. Halperin). *Let M be a closed $(2s - 1)$-connected manifold of dimension $4s + 1$ with $s \geq 1$. Then M is elliptic if and only if $H_{2s}(M, \mathbb{Z})$ is 0, $\mathbb{Z}$ or $\mathbb{Z}_2$. Moreover, if M is not elliptic the homology of the loop space of M grows exponentially for some field of coefficients k_p.*

Proof. It follows from a theorem of S. Eilenberg and J.C. Moore [13, Theorem 12.1] that the homology of the loop space can be computed as

$$H_*(\Omega M, k_p) \cong \operatorname{Tor}^{C^*(M)}(k_p, k_p),$$

where $C^*(M)$ is the differential graded algebra given by the normalized singular cochains with coefficients in k_p. (In fact, Eilenberg and Moore mention in his paper that this special case of their theorem has to be attributed to J.F. Adams [1].)

It can be seen that for a manifold M satisfying the hypotheses of the theorem there exists a *quism* between $C^*(M)$ and $(H^*(M, k_p), 0)$. This means a morphism of differential graded algebras with the property that induces isomorphisms in homology. Since a quism preserves Tor it follows that

$$H_*(\Omega M, k_p) \cong \operatorname{Tor}^{H^*(M, k_p)}(k_p, k_p).$$

We now make use of the following lemma whose proof will be given after completing the proof of the theorem.

Lemma 3.5. *The sum of the dimensions of* $\operatorname{Tor}^{H^*(M, k_p)}(k_p, k_p)$ *grows exponentially unless* $\dim H_{2s}(M, k_p) \leq 1$. *Conversely if* $\dim H_{2s}(M, k_p) \leq 1$ *then the sum of the dimensions of* $\operatorname{Tor}^{H^*(M, k_p)}(k_p, k_p)$ *grow polynomially.*

A result of C.T.C. Wall [43] (see also the corollary before Lemma F in [4]) using the linking form ensures that the torsion part of $H_{2s}(M, \mathbb{Z})$ always has the form $B + B$ or $B + B + \mathbb{Z}_2$ for some finite abelian group B. Hence if M is elliptic, the lemma implies that B must be zero and when the $\mathbb{Z}_2$ factor appears the rank of $H_{2s}(M, \mathbb{Z})$ should be zero.   □

Proof of the lemma. Let us set for brevity $k := k_p$. Observe that $R := H^*(M, k)$ is a (graded) commutative local ring with residue field k that satisfies Poincaré duality. We note that it suffices to prove the lemma ignoring the grading of R because $\operatorname{Tor}^R_{p,q}(k, k) = 0$ for $q > p(4s+1)$ (the first integer indicates the resolution degree and the second the internal grading).

Let $a := \dim H_{2s}(M, k) = \dim H_{2s+1}(M, k)$ and let $\mathfrak{m} := H_{2s}(M, k) \oplus H_{2s+1}(M, k) \oplus H_{4s+1}(M, k)$ be the maximal ideal of R. Given a finitely generated R-module U, let $U_0 := U/\mathfrak{m}U$. U_0 is a finite dimensional vector space over k. Below we will use the following form of Nakayama's lemma: if $\varphi : U \to V$ is a morphism of R-modules such that the induced morphism $\varphi^0 : U_0 \to V_0$ is surjective, then φ is also surjective.

To compute $\operatorname{Tor}^R(k, k)$ we need to take a projective resolution of k regarded as a R-module in the obvious way. Since R is local a R-module is projective if and only if is free. Hence, we will construct a resolution of the form:

$$\cdots \to R^{b_i} \xrightarrow{\partial_i} \cdots \to R^{b_1} \xrightarrow{\partial_1} R \xrightarrow{\partial_0} k \to 0.$$

The first map ∂_0 is given simply by

$$\partial_0(x, y, z, t) = x,$$

where $(x, y, z, t) \in R = H_0(M, k) \oplus H_{2s}(M, k) \oplus H_{2s+1}(M, k) \oplus H_{4s+1}(M, k)$
and we identify $H_0(M, k)$ with k. Clearly Ker $\partial_0 = \mathfrak{m}$.

We will now define a surjective morphism $\partial_1 : R^{2a} \to \mathfrak{m}$. Let $\mathbf{1} = (1, 0, 0, 0) \in R$. Clearly $\mathbf{1}$ generates R and hence given any free module R^b, the elements $\mathbf{e}_i = (0, \dots, \mathbf{1}_i, \dots, 0)$ for $1 \leq i \leq b$ generate R^b. Hence, to define ∂_1 it suffices to indicate the images of the $\mathbf{e}_i$'s. Pick a basis of $H_{2s}(M, k) \oplus H_{2s+1}(M, k)$ (which has dimension $2a$) and let ∂_1 be determined by a bijection between the generators of R^{2a} and this basis.

Note that $\mathfrak{m}_0 = \mathfrak{m}/\mathfrak{m}^2 \cong H_{2s}(M, k) \oplus H_{2s+1}(M, k)$. Hence ∂_1^0 is an isomorphism and by Nakayama's lemma ∂_1 is surjective.

Let $Q \subset R$ be the ideal given by those elements of the form $(0, 0, 0, t)$. Note that

1. $\mathfrak{m}\text{Ker}\,\partial_1 = Q^{2a}$;
2. Ker $\partial_1/\mathfrak{m}\text{Ker}\,\partial_1$ has dimension $4a^2 - 1$.

To define ∂_2, we take R^{4a^2-1} and we map the canonical $4a^2 - 1$ generators of R^{4a^2-1} onto a basis of Ker $\partial_1/\mathfrak{m}\text{Ker}\,\partial_1$. This gives a surjective morphism as before.

By continuing in this fashion we find that at the i-th step of the construction of the resolution we have:

1. $\mathfrak{m}\text{Ker}\,\partial_{i-1} = Q^{b_{i-1}}$;
2. Ker $\partial_{i-1}/\mathfrak{m}\text{Ker}\,\partial_{i-1}$ has dimension $2ab_{i-1} - b_{i-2}$.

Therefore $b_i = 2ab_{i-1} - b_{i-2}$. This implies that the growth of sequence b_i is exponential if $a > 1$ (with exponent $a + \sqrt{a^2 - 1}$) and at most linear if $a \leq 1$.

Now observe that we have the isomorphism $R^{b_i} \otimes_R k \cong k^{b_i}$ and under this isomorphism the map $\partial_i \otimes 1$ is zero. Thus the differential of the complex $R^{b_i} \otimes_R k$ is zero, so the dimensions of $\text{Tor}^R(k, k)$ over k grow exactly as the b_i's. $\qquad\square$

Closed simply connected smooth 5-manifolds have been classified by S. Smale in the spin case [41] and by D. Barden [4] in the general case. We will now briefly describe the classification.

The oriented (5-dimensional) cobordism group has order 2. The non-trivial cobordism class is formed by the manifolds for which the Stiefel-Whitney number $w^2 \cup w^3 \neq 0$. Let M be a closed simply connected smooth 5-manifold. If M bounds, then the torsion part of $H_2(M, \mathbb{Z})$ is isomorphic to $G \oplus G$ for some finite Abelian group G. If M belongs to the non-trivial cobordism class then the torsion part of its second homology group is of the form $\mathbb{Z}_2 \oplus G \oplus G$, where G is again a finite Abelian group.

The second Stiefel-Whitney class of a simply connected closed manifold is given by a homomorphism $w^2 : H_2(M, \mathbb{Z}) \to \mathbb{Z}_2$. There exists a basis of

the Abelian group $H_2(M, \mathbb{Z})$ such that it has the maximal possible number of elements (for a basis of the Abelian group) and such that w^2 does not vanish in at most one of the elements of the basis. If the order of this element is 2^i then i depends only on M.

This invariant $i(M)$ together with $H_2(M, \mathbb{Z})$ is a complete set of invariants for simply connected closed 5-manifolds.

Let $X_{-1} = SU(3)/SO(3)$ be the Wu-manifold, which is characterized by $i(X_{-1}) = 1$ and $H_2(X_{-1}, \mathbb{Z}) = \mathbb{Z}_2$. Let $X_0 = S^5$, $M_\infty = S^3 \times S^2$ and $X_\infty = \eta_3$ (the only non-trivial S^3-bundle over S^2).

For $1 \leq j < \infty$ let X_j be a closed simply connected non-spin 5-manifold such that $H_2(X_j, \mathbb{Z}) = \mathbb{Z}_{2^j} \oplus \mathbb{Z}_{2^j}$. Then $i(X_j) = j$. Also let M_j be a spin manifold with $H_2(M_j, \mathbb{Z}) = \mathbb{Z}_j \oplus \mathbb{Z}_j$. Of course, $i(M_j) = 0$.

Then Barden proves that any simply connected closed 5-manifold M is diffeomorphic to a connected sum of some of these manifolds. More precisely, $M = X_j \# M_{k_1} \# \ldots \# M_{k_l}$ where $-1 \leq j \leq \infty$, $k_1 > 1$ and k_i divides k_{i+1} for all i. Note that then $i(M) = j$ and $H_2(M, \mathbb{Z}) = \mathbb{Z}_{2^j} \oplus \mathbb{Z}_{2^j} \oplus \mathbb{Z}_{k_1} \oplus \mathbb{Z}_{k_1} \oplus \ldots \oplus \mathbb{Z}_{k_s} \oplus \mathbb{Z}_{k_s}$, unless $j = -1$ in which case the first two factors should be replaced by one copy of $\mathbb{Z}_2$. As a consequence of Theorem 3.4 and the classification of simply connected 5-manifolds we obtain:

Corollary 3.6. *Let M be a closed simply connected 5-manifold. Then M is elliptic if and only if M is diffeomorphic to:*

1. *S^5;*
2. *$S^3 \times S^2$ whose second homology group is $\mathbb{Z}$ and is a spin manifold;*
3. *η_3, the nontrivial S^3-bundle over S^2, whose second homology group is $\mathbb{Z}$ and is not spin;*
4. *the Wu-manifold $X_{-1} = SU(3)/SO(3)$ whose second homology group is $\mathbb{Z}_2$ and is not spin.*

Moreover if M is not elliptic, the homology of the loop space of M grows exponentially for some field of coefficients k_p.

Table 1. The elliptic list in dimensions 4 and 5

dim 4	dim 5
S^4	S^5
$\mathbb{C}P^2$	$S^3 \times S^2$
$S^2 \times S^2$	$X_{-1} = SU(3)/SO(3)$
$\eta_2 = \mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$	η_3
$\mathbb{C}P^2 \# \mathbb{C}P^2$	

4. Existence of a metric with zero entropy on each manifold in the elliptic list

4.1. **Dimension 4.** The standard symmetric metrics on S^4 and $\mathbb{C}P^2$ have all the geodesics closed and with the same period, and hence their geodesic flows have zero topological entropy. On $S^2 \times S^2$ consider the product metric of the round metric on S^2; it follows from part (1) in Lemma 2.4 that the geodesic flow of the product metric has zero entropy.

The manifold $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ is η_2, the non-trivial S^2-bundle over S^2, and it is known to be diffeomorphic to the space that we now describe. Represent $S^3 \subset \mathbb{C}^2$ as pairs of complex numbers (z_1, z_2) with $|z_1|^2 + |z_2|^2 = 1$. Let S^1 act on S^3 by

$$(w, (z_1, z_2)) \mapsto (wz_1, wz_2),$$

where $w \in S^1$ is a complex number with modulus one. Let S^1 also act on S^2 by rotations. Consider the space $M = S^3 \times_{S^1} S^2$ obtained by taking the quotient of $S^3 \times S^2$ by the diagonal action of S^1. The manifold M is diffeomorphic to $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$. Endow S^3 and S^2 with the canonical metrics of curvature one. By part (1) of Lemma 2.4 the product metric on $S^3 \times S^2$ has zero entropy. By part (2) in Lemma 2.4 the submersion metric on $M = S^3 \times_{S^1} S^2$ will also have a geodesic flow with zero entropy.

We are left with the case of $M = \mathbb{C}P^2 \# \mathbb{C}P^2$ which is in fact the only tricky case. The manifold M can be obtained from two copies of $S^3 \times_{S^1} D^2$ where D^2 is the 2-disk and S^1 acts diagonally, glued along their boundary $S^3 \times_{S^1} S^1 = S^3$ by an orientation reversing map. In [35] the first author proved that the metrics considered by J. Cheeger in [8] have zero topological entropy.

4.2. **Dimension 5.** The round metric on S^5 and the product metric on $S^3 \times S^2$ clearly have zero entropy.

For the Wu manifold X_{-1} we proceed as follows. Let us consider a biinvariant metric on $SU(3)$. Since every geodesic is the orbit of a 1-parameter subgroup and since $SU(3)$ is compact it follows easily that all the Jacobi fields grow at most linearly. Therefore all the Liapunov exponents of *every* geodesic in $SU(3)$ are zero. It follows from Ruelle's inequality [39] that all measure entropies are zero. Hence, by the variational principle, the topological entropy of the geodesic flow of $SU(3)$ must be zero. Endow X_{-1} with the submersion metric. It follows from part (2) in Lemma 2.4 that this metric has zero topological entropy.

We are left with η_3. This is handled in a similar way with the help of the next lemma which gives a convenient way of expressing η_3 using group actions.

Lemma 4.1. *Consider on $S^3 \times S^3 \subset \mathbb{C}^2 \times C^2$ the action of S^1 given by*

$$(w, (z_1, z_2, z_3, z_4)) \mapsto (wz_1, wz_2, wz_3, z_4),$$

*where $w \in S^1$ is a complex number with modulus one. This action is fixed
point free and the quotient of $S^3 \times S^3$ by this action is η_3.*

Proof. Let M be the quotient of $S^3 \times S^3$ by the circle action. A simple argument with the long exact sequence of the fibration shows that M is simply connected and $\pi_2(M) = \mathbb{Z}$. By the Hurewicz theorem $H_2(M, \mathbb{Z}) = \mathbb{Z}$. Note that M contains a copy of $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ given by the projection to M of the subset of $S^3 \times S^3$ given by {imaginary part of $z_4 = 0$} and hence M is not spin. It follows from the Barden-Smale classification that the only closed simply connected non spin 5-manifold with $H_2(M, \mathbb{Z}) = \mathbb{Z}$ is η_3. $\square$

Using the lemma it is easy to construct a metric on η_3 with zero entropy. Consider on $S^3 \times S^3$ the product metric and on the quotient the submersion metric. By Lemma 2.4 the metric thus constructed on η_3 has zero entropy.

5. $\mathcal{F}$-structures and minimal entropy

We begin by considering the case of a non-trivial S^1-action. This preliminary result will not be used in the proof for the case of a general $\mathcal{F}$-structure. But we think that its much simpler proof gives a nice picture of the ideas behind the general case.

Theorem 5.1. *Suppose that the closed connected smooth manifold M admits a non-trivial S^1-action. Then the minimal entropy of M is 0.*

Proof. First consider a metric g on M which is invariant under the S^1-action. This is obtained as usual by averaging any given Riemannian metric over the orbits.

Now consider the manifold $\bar{M} = M \times S^1$ and for any $\delta > 0$ the Riemannian metric $\bar{g}_\delta = g + \delta dt^2$ (where dt^2 is the Euclidean metric on S^1) on $\bar{M}$.

Define a (free) S^1-action on $\bar{M}$ by

$$\lambda.(x, \theta) = (\lambda._M x, \lambda\theta)$$

The quotient of $\bar{M}$ by this action is diffeomorphic to M and the metric $\bar{g}_\delta$ is invariant through the action; therefore it induces a metric g_δ on M. The projection

$$\pi : (\bar{M}, \bar{g}_\delta) \to (M, g_\delta)$$

is a Riemannian submersion, and therefore the entropy of g_δ is bounded above by the entropy of $\bar{g}_\delta$ (see Sect. 2.5) which is actually equal to the entropy of g (see Sect. 2.5). Therefore to prove the theorem it is enough to show that the volume of (M, g_δ) approaches 0 as δ approaches 0. We will prove this now.

First we identify the quotient (of $\bar{M}$ by the S^1-action) with M via the diffeomorphism which sends $x \in M$ to the class of $(x, 1)$. Let v_x be the

vector tangent to the $._M$-action at x and let ω be the tangent to the canonical S^1-action on S^1 (which gives the usual trivialization of the tangent space of S^1). Let $\varepsilon_x = g(v_x, v_x)$.

The tangent vector to the action on $\bar{M}$ is (v_x, ω). If $\varepsilon_x \neq 0$, the $\bar{g}_\delta$-orthogonal subspace to this vector is spanned by $(v_x, -\frac{\varepsilon_x}{\delta}\omega)$ and the subspace of vectors of the form $(v, 0)$ where $v \in V_x \subset T_x M$, the subspace of vectors g-orthogonal to v_x. It is clear that g_δ and g coincide on V_x. Moreover, $g_\delta(v_x, v) = 0$ for all $v \in V_x$.

Since

$$(v_x, 0) = \frac{\varepsilon_x/\delta}{1 + (\varepsilon_x/\delta)} (v_x, \omega) + \frac{1}{1 + (\varepsilon_x/\delta)} (v_x, -(\varepsilon_x/\delta)\omega),$$

we have that

$$g_\delta(v_x, v_x) = \left(\frac{1}{1 + (\varepsilon_x/\delta)}\right)^2 \bar{g}_\delta ((v_x, -(\varepsilon_x/\delta)\omega), (v_x, -(\varepsilon_x/\delta)\omega))$$

$$= \frac{\varepsilon_x + \varepsilon_x^2/\delta}{(1 + (\varepsilon_x/\delta))^2} = \frac{\delta}{\delta + \varepsilon_x} g(v_x, v_x).$$

This implies the following equation for the volume elements of the two metrics:

$$\mathrm{dvol}(g_\delta) = \sqrt{\delta}\frac{1}{\sqrt{\delta + \varepsilon}}\mathrm{dvol}(g).$$

This formula will be enough to show that the volume of (M, g_δ) approaches 0 with δ. Note first that the formula shows that the volume of any region computed with g_δ is always at most the volume of the same region computed with g (independently of δ). Given any $\varrho > 0$, we can find an open neighborhood of the fixed point set of the S^1-action on M which has g-volume less than $\varrho/2$. Then the g_δ-volume of this neighborhood will also be less than $\varrho/2$ for any δ. Away from the neighborhood, ε has a positive lower bound, and the volume formula clearly shows that the g_δ-volume of the complement of the neighborhood is of the order of $\sqrt{\delta}$ for δ small. Therefore, for δ small enough the volume of the complement will also be less than $\varrho/2$. This completes the proof of the theorem.   □

This result should be compared to the collapsing with bounded sectional curvature of Cheeger and Gromov [9, 20]. If the manifold M admits a locally free S^1-action then picking a Riemannian metric g on M invariant through the action and then shrinking along the orbit produces a sequence of metrics with uniformly bounded curvature and volume and injectivity radius converging to zero. This is not true if the action has fixed points. To get a geometrical picture of our theorem one can consider the canonical S^1-action on S^2 which has fixed points in the poles. The metrics produced in the proof of the theorem will shrink the horizontal circles by a non-constant factor, which approaches 1 near the poles. At the poles the curvature will

blow-up and the injectivity radius will stay uniformly bounded from below. But the volume will collapse and the entropy will stay bounded.

Cheeger and Gromov introduced in [9,20] the concept of $\mathcal{F}$-structures and generalize the previous result to manifolds admitting $\mathcal{F}$-structures with certain special properties: polarized $\mathcal{F}$-structures of positive rank. There exist plenty of examples of closed manifolds which admit $\mathcal{F}$-structures but which cannot be collapsed with bounded sectional curvature; manifolds whose minimal volume is non-zero. Therefore they do not admit polarized $\mathcal{F}$-structures of positive rank. We will show that the minimal entropy does vanish in the presence of general $\mathcal{F}$-structures. We will follow the notation of [9] as closely as possible, and the reader should check that reference for any detail about the definition and many constructions related to $\mathcal{F}$-structures we will use. We consider first the definition of an $\mathcal{F}$-structure.

A sheaf of tori $\mathcal{S}$ over the smooth manifold M is said to *act* on M if for each open subset U of M there is a local action of the group of sections $\mathcal{S}(U)$ on U, with the obvious compatibility between restriction homomorphisms of the sheaf and restrictions of the local actions (a local action of a group G is an action defined only on a neighborhood of $\{e\} \times U \subset G \times U$). The action divides M into *orbits* and a subset of M is called *saturated* if it is a union of orbits.

Definition 5.2. *An $\mathcal{F}$-structure on a smooth closed manifold M is given by an action on M of a sheaf $\mathcal{S}$ of tori together with a finite cover of M by saturated open subsets $\{U_1, \ldots, U_N\}$ such that:*

(a) *On each U_i there is a locally constant subsheaf $\mathcal{S}_i$ of $\mathcal{S}$ and a finite normal covering $\pi_i : \tilde{U}_i \to U_i$ such that the structure homomorphisms of $\pi_i^*(\mathcal{S}_i)$ give isomorphisms between the global sections and the stalks.*

(b) *The local action of the sections defines a smooth, effective torus action*

$$._i : T^{k_i} \times \tilde{U}_i \to \tilde{U}_i,$$

(c) *The stalk of the sheaf at any $x \in M$ is spanned by the stalks of the subsheaves corresponding to the U_i's which contain x and non-empty intersections of the U_i's also have a finite covering such that the pull back of the sheaf spanned by the corresponding $\mathcal{S}_i$'s gives rise to a global torus action as before.*

Definition 5.3. *An $\mathcal{F}$-structure is called a $\mathcal{T}$-structure if all the coverings $\pi_i : \tilde{U}_i \to U_i$ are trivial.*

Remark 5.4. The dimension of the orbit through x is called the *rank* of $\mathcal{F}$ at x. The minimum of the dimensions of the orbits is called the *rank* of the $\mathcal{F}$-structure. The $\mathcal{F}$-structure is called *polarized* if the torus actions defined on the finite coverings are locally-free .

Remark 5.5. Our definition of $\mathcal{F}$-structure is essentially the same as the one in [9]. More precisely, one can see that given any $\mathcal{F}$-structure as defined by Cheeger and Gromov there exists an *atlas* with the properties in our definition (see p. 317 in [9]).

Remark 5.6. A $\mathcal{T}$-structure is given by a covering by open subsets and a torus action on each subset such that any intersection of the open subsets is invariant through the corresponding actions and these commute. The stalk over any point x of the sheaf appearing in Definition 5.2 is the maximal torus which is acting on x. The definition is of course the same as the original one given by Gromov in [20], except that it is only asked that the torus actions are effective (but not necessarily locally free).

Example 5.7. Any non-trivial S^1-action on M is of course a $\mathcal{T}$-structure on M. Hence, for instance, S^4 and $\mathbb{C}P^2$ admit $\mathcal{T}$-structures although they cannot admit any polarized $\mathcal{F}$-structure.

Example 5.8. The compact complex surface $K3$ admits a $\mathcal{T}$-structure, even though it does not admit any non-trivial S^1-action [3]. Actually every elliptic compact complex surface admits a $\mathcal{T}$-structure as we will show below.

We will see now that $\mathcal{T}$-structures behave relatively well with respect to the usual operations of connected sums and surgeries on manifolds. T. Soma proved in [42] that the family of 3-manifolds which admit polarized $\mathcal{T}$-structures is closed under connected sums. As pointed out by Gromov in [20], this result generalizes to any odd dimension. We will see now that the result also holds for the family of manifolds which admit general $\mathcal{T}$-structures and for any dimension greater than 2.

Theorem 5.9. *Suppose X and Y are n-dimensional manifolds, $n > 2$, which admit a $\mathcal{T}$-structure. Then $X\#Y$ also admits a $\mathcal{T}$-structure.*

Proof. Pick a point $x \in X$ so that x lies in only one of the open subsets of the $\mathcal{T}$-structure (for this one might need to do some harmless changes in the $\mathcal{T}$-structure, like eliminating any open subset which is contained in the union of the others). We can also assume that the torus acting on the open subset containing x is of dimension one and that x lies on a regular orbit.

Now pick a small $(n-1)$-ball D_x centered at x and transverse to the S^1-action. The union of the orbits through D_x form an embedded solid torus $S^1 \times D_x$. Repeat the same procedure to obtain an embedded solid torus $S^1 \times D_y$ in Y containing a point $y \in Y$. We will perform the connected sum inside $S^1 \times D_x$ and $S^1 \times D_y$.

First divide D_x into an inner ball and an outer annulus: $D_x = D_{\varepsilon_1} \cup (S^{n-2} \times [\varepsilon_1, \varepsilon_2])$. We can identify $S^1 \times D_x \# S^1 \times D_y$ with $S^1 \times D_x - S^{n-2} \times D^2$, where D^2 is a small 2-dimensional ball centered at a point in the middle of $S^{n-2} \times [\varepsilon_1, \varepsilon_2]$ and transverse to S^{n-2} in $S^1 \times S^{n-2} \times [\varepsilon_1, \varepsilon_2]$. The component of the boundary corresponding to the boundary of the deleted $S^{n-2} \times D^2$ is identified with the boundary of $S^1 \times D_y$.

We can now describe the $\mathcal{T}$-structure on $X\#Y$. On $(X - S^1 \times D_x) \cup S^1 \times D_{\varepsilon_1}$ leave the initial $\mathcal{T}$-structure. On $S^1 \times S^{n-2} \times [\varepsilon_1, \varepsilon_2] - (S^{n-2} \times D^2)$ consider any non-trivial S^1-action on the S^{n-2}-factor (here is where we need the hypothesis $n > 2$). The action induced on each component of the

boundary glues to the canonical action on the S^1-factor to create a T^2-action (in case n is even it will have orbits of dimension 1). Finally on $Y - (S^1 \times D_y)$ leave the initial $\mathcal{T}$-structure. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Theorem 5.10. *Every compact complex elliptic surface admits a $\mathcal{T}$-structure.*

Proof. For the proof we will need smooth descriptions of the surfaces: see [19, 26] for details. Every elliptic surface of Euler characteristic 0 is obtained by performing logarithmic transforms on a *basic* elliptic surface. Every elliptic surface is obtained by taking the *fiber sum* of an elliptic surface of Euler characteristic 0 and some rational elliptic surfaces, and then blowing up some points.

Basic surfaces are fiber bundles with fibers T^2 and structure group in $SL(2, \mathbb{Z})$. Hence they admit a polarized $\mathcal{T}$-structure whose orbits are the fibers.

Now let $B \times T^2$ be a neighborhood of a fiber on a basic surface M, where B is identified with the unit ball in $\mathbb{C} = \mathbb{R}^2$. Fix a positive integer m and integers a, b such that (a, b) has order m in $\mathbb{Z}_m \oplus \mathbb{Z}_m$. Let $F : B \times T^2 \to B \times T^2$ be given by $F(z, t) = (e^{2\pi i/m}z, t)$. F generates a group G_1 of diffeomorphisms of $B \times T^2$ of order m. The quotient of $B \times T^2$ by this group is again diffeomorphic to $B \times T^2$. Consider also the map $L : B \times T^2 \to B \times T^2$ given by

$$L(z, t) = \left(e^{2\pi i/m}z, \left(t_1 e^{2a\pi i/m}, t_2 e^{2b\pi i/m}\right)\right).$$

L generates a group G_2 of diffeomorphism of $B \times T^2$ of order m which acts freely on $B \times T^2$. The map $P : S^1 \times T^2/G_1 \to S^1 \times T^2/G_2$,

$$P(z, t) = \left(z, \left(z^a t_1, z^b t_2\right)\right)$$

is a diffeomorphism. The logarithmic transform (of order m) at the fiber over $(0, 0)$ in M is the elliptic surface $\widetilde{M}$ obtained by gluing $M - B \times T^2$ and $B \times T^2/G_2$ via this diffeomorphism. Clearly the obvious S^1-action on B (which fixes $(0, 0)$) induces an S^1-action on $B \times T^2/G_2$ which commutes with the action on the fibers. Hence $\widetilde{M}$ admits a $\mathcal{T}$-structure (with orbits of dimension 0,1,2 and 3).

Rational elliptic surfaces are diffeomorphic to $S = \mathbb{C}P^2 \# 9\overline{\mathbb{C}P}^2$ and therefore admit $\mathcal{T}$-structures by the previous theorem. Nevertheless we will need to perform fiber sums and so we will give another $\mathcal{T}$-structure on it, compatible with the elliptic fibration. To do this we need first to give a description of the surface as an elliptic surface (see [26]). Let $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ and consider the involution $I(z) = -z$ of T^2. Let $H : S^2 \to S^2$ be rotation of $180°$ around the z-axis. The diffeomorphism

$$J = (I, H) : T^2 \times S^2 \to T^2 \times S^2$$

has 8 fixed points. Identify a neighborhood of each of these points with a ball B in $\mathbb{C}^2$. Consider $U = \{(z, l) \in B \times \mathbb{C}P^1 : z \in l\}$. The canonical

projection $\pi_1|_U : U \to B$ induces an isomorphism away from the preimage of 0. Construct a surface $\widetilde{S}$ by replacing the eight copies of B with U in $T^2 \times S^2$. The involution J extends to an involution $\widetilde{J}$ on $\widetilde{S}$ which has 8 spheres as the set of fixed points. Then $S = \widetilde{S}/\widetilde{J}$. Let $\pi : S^2 \to S^2 = S^2/H$ be the projection. Then $\pi \circ \pi_2 : T^2 \times S^2 \to S^2$ induces a map $p : \widetilde{S} \to S^2$ which commutes with $\widetilde{J}$ and so induces a map $S \to S^2$ whose generic fiber is T^2; this map expresses S as an elliptic surface.

Note that the S^1-action on B given by $\lambda(w_1, w_2) = (w_1, \lambda w_2)$ commutes with J and induces an S^1-action on U. We can extend this action to an S^1-action defined on a neighborhood of the fibers of p over the north and south poles. This action commutes with $\widetilde{J}$ and so induces an action on a neighborhood of the fibers of $S \to S^2$ over the poles.

Away from the fibers over the poles S actually is the total space of a fiber bundle with structure group $\{Id, I\}$. There is then a polarized $\mathcal{T}$-structure defined on this piece, whose orbits are the fibers. On the boundary of the neighborhoods around the fibers over the poles the two actions commute. This defines a $\mathcal{T}$-structure on S.

The fiber sum of two elliptic surfaces is done as follows: pick regular fibers on each surface identifying neighborhoods of them with $D \times T^2$ (D is a small 2-ball). Delete the corresponding regular fiber from each surface and then glue both surfaces along $(D \# D) \times T^2$. The diffeomorphism class of the resulting surface will depend only on the isotopy class of the diffeomorphism chosen to identify the fibers with T^2. We can therefore take the diffeomorphism to be in $SL(2, \mathbb{Z})$ and we can see that the $\mathcal{T}$-structures we defined on the surfaces of Euler characteristic 0 and the rational elliptic surfaces glue well along the fiber sum.

Finally blowing up points means, in terms of diffeomorphisms, to take connected sums with $\overline{\mathbb{C}P}^2$'s. Such a connected sum admits a $\mathcal{T}$-structure by the previous theorem.   □

We can now also see that inside the family of manifolds with $\mathcal{T}$-structures one can perform surgery on spheres which are "well positioned" with respect to the $\mathcal{T}$-structure.

Definition 5.11. *Let M be a manifold with a fixed $\mathcal{T}$-structure. An embedded k-sphere $\mathbf{S}^k$ is said to be completely transversal with respect to the $\mathcal{T}$-structure if:*

1) $\mathbf{S}^k$ *intersects only one of the open subsets of the $\mathcal{T}$-structure.*
2) *The torus acting on the open subset of (1) has dimension 1 and the orbits passing through $\mathbf{S}^k$ form an embedded $\mathbf{S}^k \times S^1$ with trivial normal bundle.*

Remark 5.12. Note in particular that the normal bundle of a completely transversal sphere is trivial.

Example 5.13. If X admits a $\mathcal{T}$-structure and Y is any other manifold then $X \times Y$ admits an obvious $\mathcal{T}$-structure. Any homotopy class in Y which can

be represented by an embedded sphere with trivial normal bundle (in Y) can be represented by a completely transversal sphere (in $X \times Y$).

Theorem 5.14. *Let M^n be a manifold with a $\mathcal{T}$-structure. Let $\mathbf{S}^k$ be a completely transversal sphere (with respect to the given $\mathcal{T}$-structure). The manifold $\hat{M}$, obtained by performing surgery on $\mathbf{S}$, also admits a $\mathcal{T}$-structure. Moreover, if n and $n-k$ are odd and the structure on M^n is polarized, then $\hat{M}$ also admits a polarized $\mathcal{T}$-structure.*

Proof. Let $S^k \times S^1 \times D^{n-k-1}$ be a tubular neighborhood of the union of the orbits through $\mathbf{S}$. Consider the unit n-sphere $S^n \subset \mathbb{R}^{n+1}$. Pick a nontrivial S^1-action on S^n, for instance complex multiplication in the first 2 coordinates. In case n is odd we can pick a free S^1-action. Choose a regular orbit of the action and a disc D^{n-1} transverse to the orbit. Pick a canonical embedded k-sphere $S_0^k \subset D^{n-1}$ and a tubular neighborhood $S_0^k \times S^1 \times D^{n-k-1}$ of the union of the orbits through S_0^k. The manifold $\hat{M}$ is obtained by gluing M and S^n along $\mathbf{S}$ and S_0^k. But gluing two copies of $S^k \times S^1 \times D^{n-k-1}$ along the k-spheres is the same as taking the product of a k-sphere with the connected sum of two copies of $S^1 \times D^{n-k-1}$. Hence in this glued part we can consider the $\mathcal{T}$-structure we defined in the proof of Theorem 5.9, which on each component of the boundary coincides with the structure of M and S^n, respectively. This clearly defines a $\mathcal{T}$-structure on $\hat{M}$. This structure is polarized if n and $n-k$ are odd. $\qquad\square$

6. Collapsing with bounded entropy: Proof of Theorem A

In this section we will prove that the minimal entropy of a closed manifold which admits an $\mathcal{F}$-structure vanishes. The general idea of the proof is quite simple. Given an $\mathcal{F}$-structure on M we define a polarized $\mathcal{F}$-structure on $M \times T^k$ for some k and consider a Riemannian metric on the product which is invariant through all the torus actions. Then we collapse the metric along the orbits of the $\mathcal{F}$-structure on $M \times T^k$. The procedure constructs metrics which are invariant by the canonical T^k-action on $M \times T^k$. Taking the quotient by this action gives a Riemannian submersion over a metric on M. Now, for the polarized structure on $M \times T^k$, Cheeger and Gromov [9] proved that the sectional curvatures of the collapsed metrics are uniformly bounded. Therefore the entropy of the metrics are also uniformly bounded (see Sect. 2.3) and since entropy is non-increasing under Riemannian submersions (see Sect. 2.5), the collapsed metrics on M also have uniformly bounded entropy. The theorem therefore reduces to the proof that the volumes of the metrics on M collapse. Note that the only properties about entropy we will use in the proof are its bound in terms of curvature and its behaviour under Riemannian submersions. Since Riemannian submersions do not decrease sectional curvatures, the same proof works for any quantity that depends only on lower bounds for the sectional curvature. We will

use this remark in the next section to study certain curvature invariants for manifolds admitting $\mathcal{F}$-structures.

In the proof of the theorem we will need the following elementary lemma from linear algebra:

Lemma 6.1. *Let (V_1, h_1) and (V_2, h_2) be two real vector spaces of dimension l with inner products. Let F be a subspace of $V_1 \oplus V_2$ of dimension l which intersects trivially with both V_1 and V_2 such that for any $(v, w) \in F$, $h_1(v, v) \leq h_2(w, w)$. Then:*

a. *Consider F as the graph of a map $\widetilde{F} : V_2 \to V_1$ and let $I : V_1 \to V_1$ be given by $I(v) = \pi_1 \circ \pi_F(v, 0)$ ($\pi_F : V_1 \oplus V_2 \to F$ is the orthogonal projection). Then $(\det I)^2 \geq 4^{-l}(\det \widetilde{F})^{4l}$.*

b. *Given any λ, $0 < \lambda \leq 1$ consider the inner product $\bar{h}_\lambda$ on $V_1 \oplus V_2$ defined by $\lambda(h_1 + h_2)|_F + (h_1 + h_2)|_{F^\perp}$. Let h_λ be the inner product on V_1 obtained as the quotient of $\bar{h}_\lambda$ (by π_1). Then $dvol(h_\lambda) \leq dvol(h_1)$.*

Proof. a) Consider an orthonormal basis $\{v_1, \ldots, v_l\}$ of (V_1, h_1). If $I(v_j) = \sum_{i=1}^{l} a_{ij} v_i$ and we let $A = (a_{ij})$ then $(\det I)^2 = \det(A^t A)$. But $A^t A$ is a positive definite symmetric matrix, and therefore it has l positive eigenvalues $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_l$ and $(\det I)^2 = \mu_1 \ldots \mu_l \geq \mu_1{}^l$.

Now consider an orthonormal basis $\{w_1, \ldots, w_l\}$ of (V_2, h_2) and let $\widetilde{F}(w_j) = \sum_{i=1}^{l} b_{ij} v_i$. Let $\gamma = (\det \widetilde{F})^2$. If $B = (b_{ij})$ then $\gamma = \det(B^t B)$. Again, $B^t B$ is a positive definite symmetric matrix. Moreover, since $h_1(v, v) \leq h_2(w, w)$ for any $(v, w) \in F$ we have that no eigenvalue of $B^t B$ is greater than 1. Therefore the smallest eigenvalue is at least γ. This means that for all $(v, w) \in F$, $h_1(v, v) \geq \gamma h_2(w, w)$.

Now if $(v^\perp, w^\perp) \in F^\perp$ we can find the unique vector $(v^\perp, w^*) \in F$ whose first coordinate is $v^\perp$. Then

$$h_1(v^\perp, v^\perp) = -h_2(w^\perp, w^*) \leq \|w^\perp\| \|w^*\| \leq \frac{1}{\sqrt{\gamma}} \|w^\perp\| \|v^\perp\|$$

and so $h_2(w^\perp, w^\perp) \geq \gamma h_1(v^\perp, v^\perp)$

Therefore, if $h_1(v, v) = 1$ and $(v, 0) = f + f^\perp$ with $f = (I(v), w) \in F$ and $f^\perp = (v^*, -w) \in F^\perp$, we have that

$$h_1(I(v), I(v)) \geq \gamma h_2(w, w) \geq \gamma^2 h_1(v^*, v^*).$$

But $I(v) + v^* = v$, and therefore either $h_1(v^*, v^*) \geq 1/4$ or $h_1(I(v), I(v)) \geq 1/4$. In any case, $h_1(I(v), I(v)) \geq (1/4)\gamma^2$.

This means that $\mu_1 \geq (1/4)\gamma^2$ and so $(\det I)^2 \geq 4^{-l}\gamma^{2l}$, proving (a).

b) Given any $v \in V_1$ write $(v, 0) = f_v + f_v^\perp$ (in $F \oplus F^\perp$). The map $L : V_1 \to V_1 \oplus V_2$, $L(v) = f_v + \lambda f_v^\perp$, is a monomorphism. Moreover, the image of L is included in the $\bar{h}_\lambda$-orthogonal complement of V_2, $V_2^{\perp_\lambda}$, and therefore L gives an isomorphism between V_1 and $V_2^{\perp_\lambda}$.

Pick any $v \in V_1$. The map $\pi_1 \circ L : V_1 \to V_1$ is an isomorphism. Therefore there exists a unique $a \in V_1$ such that

$$v = \pi_1(f_a + \lambda f_a^\perp) = \pi_1((f_{a1}, f_{a2}) + \lambda(f_{a1}^\perp, f_{a2}^\perp)) = f_{a1} + \lambda f_{a1}^\perp.$$

Since $(h_1 + h_2)(f_a, f_a^\perp) = 0$, we have that

$$h_1(f_{a1}, f_{a1}^\perp) + h_2(f_{a2}, f_{a2}^\perp) = 0.$$

But since $f_a + f_a^\perp = (a, 0)$, we have that $f_{a2} = -f_{a2}^\perp$. Therefore

$$h_1(f_{a1}, f_{a1}^\perp) = h_2(f_{a2}, f_{a2}) \geq 0.$$

There is a unique $b \in V_1$ such that $(b, f_{a2}^\perp) \in F$. Then since $h_1(b, b) \leq h_2(f_{a2}^\perp, f_{a2}^\perp)$ and $h_1(b, f_{a1}^\perp) + h_2(f_{a2}^\perp, f_{a2}^\perp) = 0$, we have that $h_2(f_{a2}^\perp, f_{a2}^\perp) \leq h_1(f_{a1}^\perp, f_{a1}^\perp)$.

Therefore

$$\begin{aligned}
h_\lambda(v, v) &= \bar{h}_\lambda(f_a + \lambda f_a^\perp, f_a + \lambda f_a^\perp) \\
&= \lambda(h_1 + h_2)(f_a, f_a) + \lambda^2(h_1 + h_2)(f_a^\perp, f_a^\perp) = \\
&= \lambda h_1(f_{a1}, f_{a1}) + \lambda h_2(f_{a2}, f_{a2}) + \lambda^2 h_1(f_{a1}^\perp, f_{a1}^\perp) + \lambda^2 h_2(f_{a2}^\perp, f_{a2}^\perp) \\
&\leq h_1(f_{a1}, f_{a1}) + \lambda h_1(f_{a1}, f_{a1}^\perp) + \lambda^2 h_1(f_{a1}^\perp, f_{a1}^\perp) + \lambda h_1(f_{a1}, f_{a1}^\perp) = \\
&= h_1(f_{a1} + \lambda f_{a1}^\perp, f_{a1} + \lambda f_{a1}^\perp) = h_1(v, v).
\end{aligned}$$

Hence for any $v \in V_1$, $h_\lambda(v, v) \leq h_1(v, v)$ and (b) follows. $\qquad\square$

We are now ready to prove our theorem.

Theorem A. *If the closed manifold M admits an $\mathcal{F}$-structure then the minimal entropy of M is 0.*

Proof. Let $U_1, \ldots, U_N$ be the open covering corresponding to an $\mathcal{F}$-structure on M; with corresponding actions $._1, \ldots, ._N$ by tori $T^{k_1}, \ldots, T^{k_N}$ on the coverings $\widetilde{U}_i$'s.

We can construct a *regular atlas* for the structure as in [9], Lemma 1.2. Namely we construct a new open cover $W_1, \ldots, W_J$ of M obtained by considering all non-empty intersections of the U_i's and then removing from each set the "unnecessary" parts. Each W_i has a finite cover $\widetilde{W}_i$ where there is defined an effective torus action. For instance, if one had $U_1 \cap U_2 \neq \emptyset$ then one would consider $W_1 = U_1 \cap U_2$, $W_2 \subset U_1$, $W_3 \subset U_2$ so that $W_2 \cap W_3 = \emptyset$ and both are invariant through the corresponding action (note that on $\widetilde{W}_1$ one has defined a $T^{k_1+k_2}$ action).

A Riemannian metric g on M is called invariant if on each of the open subsets W_i of the $\mathcal{F}$-structure the corresponding sheaf of torus acts by isometries. An invariant metric always exists, at least after replacing the open subsets W_i by slightly smaller ones. Such a metric is constructed in [9], Lemma 1.3.

Let us then fix a Riemannian metric g on M invariant through the $\mathcal{F}$-structure. Each W_i is, essentially, the intersection of certain number of U_i's. Assume that $W_1, \ldots, W_J$ are ordered in a non-increasing way with respect to the number of the U_i's intersecting. Therefore, if $i > j$ and $W_i \cap W_j \neq \emptyset$ the torus action on W_i, restricted to $W_i \cap W_j$, is embedded in the action on W_j. Consider smooth functions $f_i : M \to [0, 1]$, supported in W_i which are constant along the orbits and such that $\{f_i = 1\}_{i=1,\ldots,J}$ covers M.

Let $K = \sum_{i=1}^{N} k_i$ and let $\bar{M} = M \times T^K$ and $\bar{g} = g + dx^2$ (where dx^2 is the standard Euclidean metric on the K-torus). For each open subset U_i consider the following (free) T^{k_i}-action on $\tilde{U}_i \times T^K$:

$$\cdot_{\bar{i}} : T^{k_i} \times \tilde{U}_i \times T^K \to \tilde{U}_i \times T^K$$

$$(\lambda, (x, t_1, \ldots, t_i, \ldots, t_N)) \mapsto (\lambda \cdot_i x, (t_1, \ldots, \lambda t_i, \ldots, t_N))$$

where $t_j \in T^{k_j}$.

These formulas clearly define an $\mathcal{F}$-structure on $\bar{M}$. But what is more important to us is that it is actually a polarized $\mathcal{F}$-structure of positive rank. Note that on the W_i's all the torus actions corresponding to the U_i's which are intersecting glue together to get a free torus action on $\tilde{W}_i \times T^K$ (where the dimension of the torus acting is the sum of the corresponding k_i's).

Pull back the functions f_i to obtain smooth functions $\bar{f}_i$ on $\bar{M}$. Note that the functions $\bar{f}_i$ are invariant through both the torus action (on $W_i \times T^K$) coming from M and the canonical T^K-action on the T^K-factor of $\bar{M}$. The same is true for the metric $\bar{g}$.

Now we proceed to collapse the metric $\bar{g}$ along the orbits of the $\mathcal{F}$-structure on $\bar{M}$. This is done in [9], Theorem 3.1. We will describe the procedure, since we need to make some computations on it. Fix a small $\delta > 0$. For technical reasons it is convenient to first replace $\bar{g}$ by $\bar{g}_0 = (\log^2 \delta) \bar{g}$. We construct a metric $\bar{g}_1$ on $\bar{M}$ by multiplying the metric $\bar{g}_0$ by $\delta^{\bar{f}_1}$ in the directions tangent to the orbits of the torus action on $W_1 \times T^K$ (and leaving the same metric in the directions orthogonal to the orbits). Note that the T^K-action on $\bar{M}$ given by the canonical action on the T^K-factor is isometric with respect to $\bar{g}_1$. Repeating this procedure J-times we get a metric $\bar{g}_\delta = \bar{g}_J$ which is invariant under the T^K-action just mentioned.

Let g_δ be the metric induced on $M = \bar{M}/T^K$. The projection $(\bar{M}, \bar{g}_\delta) \to (M, g_\delta)$ is a Riemannian submersion. Therefore the entropy of g_δ is bounded above by the entropy of $\bar{g}_\delta$ (see Sect. 2.5). The entropy of $\bar{g}_\delta$ on the other hand is bounded above by $(n-1)\sqrt{K_0}$, where K_0 is an upper bound for the absolute value of the sectional curvature of $\bar{g}_\delta$ (see Sect. 2.3). But it is proved in [9], Theorem 3.1, that the sectional curvature of $\bar{g}_\delta$ is bounded independently of δ.

Therefore we got that:
$$h_{\text{top}}(g_\delta) \leq c_1$$
where c_1 is some constant independent of δ.

We will now estimate the volume of (M, g_δ). We will do this by comparing the volume element of g_δ with that of g.

Let $(\phi_1, \ldots, \phi_n)$ be a g-orthonormal basis of $T_x M$. Then

$$\mathrm{dvol}(g_\delta) = \sqrt{\det(g_\delta(\phi_i, \phi_j))}\,\mathrm{dvol}(g).$$

Since the volume element at a point depends only on the value of the metric at the point, it is the same to work on W_i or on the corresponding finite covering. Therefore from now on we will think that we are working with a $\mathcal{T}$-structure to simplify the notation. Fix any point $x \in M$ and any point $(x, t) \in \bar{M}$ which projects to x. We have to check how the volume element changes at each step in the construction of $\bar{g}_\delta$. Of course there is no change in the step i if x does not belong to W_i. So let us assume for instance that $x \in W_1$. Moreover, assume that x is not a fixed point for the torus action (the set of fixed points has volume 0 with respect to any Riemannian metric). We want to compare the volume elements at x of g_1 and g (g_i is of course the quotient of $\bar{g}_i$ under the T^K-action on $\bar{M}$).

Assume that the orbit through x of the torus action has dimension l. There is then an orthonormal set of vectors $w_1, \ldots, w_l \in T_t(T^K)$ and some linearly independent vectors $v_1, \ldots, v_l \in T_x M$ so that the vectors $(v_1, w_1), \ldots, (v_l, w_l)$ are tangent to the orbit on $\bar{M}$, and the directions orthogonal to the w_i's act trivially on M at x. Let H be the subspace of $T_{(x,t)}(M \times T^K)$ spanned by this l vectors (the tangent space to the orbit on $\bar{M}$), let $V = < v_1, \ldots, v_l > \subset T_x M$ be the tangent space to the orbit in M and let $W = < w_1, \ldots, w_l > \subset T_t(T^K)$.

Let $v_{l+1}, \ldots, v_n$ be a g-orthonormal basis of the space g-orthogonal to the orbit (in M). Note that $v_{l+1}, \ldots, v_n$ are also g_1-orthogonal to the orbit and $g_1(v_{l+j}, v_{l+k}) = \delta_j^k (\log \delta)^2$. Therefore

$$\det\left((g_1(v_i, v_j))_{1 \le i, j \le n}\right) = (\log \delta)^{2(n-l)} \det\left((g_1(v_i, v_j))_{1 \le i, j \le l}\right).$$

Recall that the metric $\bar{g}_1$ is obtained by multiplying by $\delta^{\bar{f}_1}$ the values of $\bar{g}_0$ on H.

From now on we restrict our attention to $V \oplus W$, since its orthogonal complement plays no real role in the construction of g_1.

We can assume that for any unitary tangent vector to any of the tori (acting on any of the W_i), the derivative of the action in that direction has g-norm at most one.

Therefore we are under the hypothesis of our Linear Algebra Lemma 6.1.

Consider now a g-orthonormal basis of V; call them $v_1^0, \ldots, v_l^0$. For each i write

$$(v_i^0, 0) = h_i + h_i^\perp,$$

where $h_i \in H$ and $h_i^\perp \in H^\perp$ (the $\bar{g}$-orthogonal complement of H in $V \oplus W$). Let $I(v_i^0) = \pi_1(h_i) \in V$.

Now, for each $i = 1, \ldots, l$, consider the vector

$$w_i^0 = h_i + \delta^{\bar{f}_1} h_i^\perp.$$

The vector w_i^0 is $\bar{g}_1$-orthogonal to the tangent space to the torus factor. Its first coordinate is, of course, $I(v_i^0) + \delta^{\bar{f}_1} \pi_1(h_i^\perp)$.

Assume that $\bar{f}_1(x) = 1$. Then

$$\det\left(g\big(\pi_1(w_i^0), \pi_1(w_j^0)\big)\right) = \det\left(g\big(I(v_i^0), I(v_j^0)\big)\right) + o(\delta).$$

Note also that:

$$g_1\big(\pi_1(w_i^0), \pi_1(w_j^0)\big) = \bar{g}_1(w_i^0, w_j^0) = \log^2\delta\left(\delta^{\bar{f}_1}\bar{g}(h_i, h_j) + \delta^{2\bar{f}_1}\bar{g}(h_i^\perp, h_j^\perp)\right).$$

Therefore,

$$\frac{\det\left(g_1\big(\pi_1(w_i^0), \pi_1(w_j^0)\big)\right)}{\det\left(g\big(\pi_1(w_i^0), \pi_1(w_j^0)\big)\right)} = \frac{o\big(\log^{2l}(\delta)\delta^l\big)}{\det\left(g\big(I(v_i^0), I(v_j^0)\big)\right) + o(\delta)}.$$

Now, in the region where $\det\big(g(v_i, v_j)\big) > \delta^{1/(4l)}$, we have from part (a) of Lemma 6.1 that

$$\det\left(g\big(I(v_i^0), I(v_j^0)\big)\right) = (\det I)^2 \geq \frac{1}{4^l}\delta^{1/2}$$

and, therefore,

$$\mathrm{dvol}(g_1) = (\log\delta)^{n-l}\sqrt{\frac{\det\left(g_1\big(\pi_1(w_i^0), \pi_1(w_j^0)\big)\right)}{\det\left(g\big(\pi_1(w_i^0), \pi_1(w_j^0)\big)\right)}}\,\mathrm{dvol}(g)$$

$$= o(\delta^{1/4}\log^n\delta)\mathrm{dvol}(g).$$

Therefore the g_1-volume of the region where $\det\big(g(v_i, v_j)\big) > \delta^{1/(4l)}$ and $f_1(x) = 1$ approaches 0 as δ does.

The g-volume of the region $\det\big(g(v_i, v_j)\big) < \varrho^2$ is of the order of ϱ. Therefore the g_1-volume of the region where $\det\big(g(v_i, v_j)\big) < \delta^{1/(4l)}$ is of the order of $\log^n(\delta)\delta^{1/(8l)}$ (using part (b) of Lemma 6.1) and therefore it also approaches 0 with δ.

This of course implies that $\mathrm{Vol}(\{f_1 = 1\}, g_\delta)$ approaches 0 with δ.

Finally note that in the passage from $\bar{g}$ to $\bar{g}_1$ there are two steps: first we multiply by $\log^2\delta$ to obtain $\bar{g}_0$ and then we collapse along the orbits multiplying by $\delta^{\bar{f}_1}$. Lemma 6.1, part (b), tells us that the second of these steps does not increase volumes (on M with the quotient metric). To go from $\bar{g}_1$ to $\bar{g}_2$ only the second step is performed. Therefore when passing from

g_1 to g_2 the volume of the region $f_1 = 1$ will remain small, while by taking δ small we can make the volume of the region $f_2 = 1$ small. Hence for δ small enough the g_δ-volume of the whole M will be as small as desired.

Since the entropy of g_δ is bounded above independently of δ, the theorem is proved. □

7. Collapsing $\mathcal{F}$-structures and curvature invariants: Proof of Theorem B

There are many natural invariants of a smooth manifold which measure the possible size of the curvature of a Riemannian metric of some fixed volume. In this section we will recall some of them and study what can be said about them for manifolds which admit $\mathcal{F}$-structures. In every case we restrict attention to the metrics verifying certain bounds on its curvature and search for the infimum of the volumes.

Given a fixed closed smooth manifold M we consider the following subsets of the family $\mathcal{M}$ of all Riemannian metrics on M:

$$\mathcal{M}_{|K|} = \{g : |K| \leq 1\}$$
$$\mathcal{M}_K = \{g : K \geq -1\}$$
$$\mathcal{M}_{|r|} = \{g : |r| \leq n - 1\}$$
$$\mathcal{M}_r = \{g : r \geq -(n - 1)\}$$
$$\mathcal{M}_{|s|} = \{g : |s| \leq n(n - 1)\}$$
$$\mathcal{M}_s = \{g : s \geq -n(n - 1)\}$$

where K, r and s denote as usual the sectional, Ricci and scalar curvature, respectively. Now define (see [20,25])

$$\mathrm{MinVol}(M) = \inf_{g \in \mathcal{M}_{|K|}} \mathrm{Vol}(M, g)$$
$$\mathrm{Vol}_K(M) = \inf_{g \in \mathcal{M}_K} \mathrm{Vol}(M, g)$$
$$\mathrm{Vol}_{|r|}(M) = \inf_{g \in \mathcal{M}_{|r|}} \mathrm{Vol}(M, g)$$
$$\mathrm{Vol}_r(M) = \inf_{g \in \mathcal{M}_r} \mathrm{Vol}(M, g)$$
$$\mathrm{Vol}_{|s|}(M) = \inf_{g \in \mathcal{M}_{|s|}} \mathrm{Vol}(M, g)$$
$$\mathrm{Vol}_s(M) = \inf_{g \in \mathcal{M}_s} \mathrm{Vol}(M, g)$$

Cheeger and Gromov proved that if M is a closed manifold which admits a polarized $\mathcal{F}$-structure of positive rank then $\mathrm{MinVol}(M) = 0$.

It is easy to check the same proof of the theorem in the previous section proves that if the closed manifold M admits an $\mathcal{F}$-structure, then $\mathrm{Vol}_K(M) = 0$. Of course, this implies that $\mathrm{Vol}_r(M) = \mathrm{Vol}_s(M) = 0$.

More can be said about the scalar curvature. Let us first recall some facts about the Yamabe invariant (or *sigma constant* in [40]). More details and references can be found for instance in [26, 37, 40].

Given a conformal class of metrics $\mathcal{C}$ on M the Yamabe constant of $\mathcal{C}$, denoted by $Y(M, \mathcal{C})$, is the infimum of the integral of the scalar curvature over all metrics in $\mathcal{C}$ of volume 1 (integrating with respect to the volume element of the same metric). The infimum is actually realized: this is a very deep result obtained in several steps by Yamabe, Trudinger, Aubin and Schoen. Metrics realizing the infimum have constant scalar curvature and are usually called Yamabe metrics. The Yamabe invariant of M, $Y(M)$, is then defined as

$$Y(M) = \sup_{\mathcal{C}} Y(M, \mathcal{C}).$$

M admits a metric of strictly positive scalar curvature if and only if $Y(M) > 0$. In this case, if the dimension of M is at least 3, M also admits scalar flat metrics and so $\mathrm{Vol}_{|s|}(M) = \mathrm{Vol}_s(M) = 0$.

Now assume that $Y(M) \leq 0$. Let $g \in \mathcal{M}_s(M)$. Then there exists a Riemannian metric $\hat{g} = e^f g$ conformal to g with constant scalar curvature and with the same volume as g. The scalar curvature of $\hat{g}$ can be written in terms of f and g. From there it is easy to see that $s_{\hat{g}} \geq -n(n-1)$ (see for instance [24]). But since $Y(M, \mathcal{C}_g) \leq 0$, $s_{\hat{g}} \leq 0$. Therefore $\bar{g} \in \mathcal{M}_{|s|}(M)$ and $\mathrm{Vol}(M, \bar{g}) = \mathrm{Vol}(M, g)$. Hence:

Proposition 7.1. *For any closed smooth manifold M of dimension greater than 2, $\mathrm{Vol}_{|s|}(M) = \mathrm{Vol}_s(M)$.*

Summarizing, we have proved the following:

Theorem 7.2. *If M admits an $\mathcal{F}$-structure, $\dim M > 2$, then*

$$\mathrm{Vol}_K(M) = \mathrm{Vol}_r(M) = \mathrm{Vol}_{|s|}(M) = \mathrm{Vol}_s(M) = 0.$$

The last equality is equivalent to $Y(M) \geq 0$.

Clearly this theorem implies Theorem B in the introduction.

As we mentioned before there are plenty of examples of closed manifolds M which admit $\mathcal{F}$-structures and verify $\mathrm{MinVol}(M) > 0$. Also C. LeBrun proved (see [26, 27]) that, for instance, an elliptic compact complex surface collapses with bounded Ricci curvature (i.e $\mathrm{Vol}_{|r|} = 0$) if and only if it is minimal. Therefore we have that, for instance, $\mathrm{Vol}_{|r|}(T^4 \# \overline{\mathbb{C}P}^2) > 0$, while $T^4 \# \overline{\mathbb{C}P}^2$ does admit an $\mathcal{F}$-structure.

8. Minimal entropy in dimensions 4 and 5: Proofs of Theorems C, D and E

We will study in this section the minimal entropy of simply connected manifolds of dimensions four and five. The aim is to give an idea of up to what point the previous results can be used to compute minimal entropies.

Let us begin with dimension four. Homeomorphism types have been classified by Freedman. But the main feature in dimension four is the comparison between homeomorphism classes and diffeomorphism classes. Freedman's results say that the homeomorphism type of a smooth closed simply connected 4-manifold is determined by the intersection form. Not every possible intersection form can be realized by a smooth manifold and the number of diffeotypes corresponding to each homeotype is essentially unknown. With regards to the question of which intersection forms are realized by a smooth manifold it all comes down to the well-known 11/8-conjecture. Namely, the basic examples of (homeomorphism types of) simply connected smooth four-manifolds are S^4, $S^2 \times S^2$, $\mathbb{C}P^2$ and $K3$. By taking connected sums of them (with different orientations) one can realize many intersection forms. Namely, connected sums of $\mathbb{C}P^2$'s realize all positive definite intersection forms (by the well-known result of Donaldson) and varying the orientations of some of the factors one gets all odd forms. Finally, the complicated part of the analysis comes from the indefinite even intersection forms. Let H be the intersection form of $S^2 \times S^2$:

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and let

$$E_8 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix}.$$

Every even indefinite bilinear form is equivalent to $kE_8 + lH$ for some integers k and $l \geq 0$. Rohlin's theorem says that for a smooth closed spin 4-manifold the signature is divisible by 16. For simply connected 4-manifolds the spin condition means exactly that the intersection form is even. Therefore we have that for the intersection forms of smooth simply connected 4-manifolds, k is even. The intersection form of the $K3$ surface is $-2E_8 + 3H$. By taking connected sums of K_3's and $S^2 \times S^2$ we see that any such bilinear form can be realized as the intersection form of a smooth 4-manifold if $l \geq (3/2)|k|$. The 11/8-conjecture says precisely that these are exactly all the bilinear forms which come from smooth simply connected 4-manifolds. Therefore in the previous sections we have shown that:

Theorem 8.1. *Assuming the $11/8$-conjecture, every closed simply connected smooth 4-manifold is homeomorphic to one whose minimal entropy is 0.*

There are simply connected compact complex surfaces of general type which are homeomorphic to connected sums of $\mathbb{C}P^2$'s (with different orientations). Nevertheless, they do not collapse with bounded scalar curvature (see [26]) and so they cannot admit $\mathcal{F}$-structures from the results of the previous section. The following question therefore seems very intriguing:

Question: Is the minimal entropy of a simply connected compact complex surface of general type positive?

Let us now consider 5-dimensional manifolds. As we explained in Sect. 3, closed simply connected smooth 5-manifolds have been classified by S. Smale [41] and D. Barden [4]. We will use this classification to prove the next theorem which clearly implies Theorem C in the introduction.

Theorem 8.2. *Every simply connected closed smooth 5-manifold M admits a $\mathcal{T}$-structure and hence $\mathrm{h}(M) = 0$ and $\mathrm{Vol}_K(M) = 0$. Moreover, M admits a polarized $\mathcal{T}$-structure and hence $\mathrm{MinVol}(M) = 0$ unless M is cobordant to 0 and non-spin with $1 < i(M) < \infty$.*

Proof. We will prove that M admits a $\mathcal{T}$-structure and then apply Theorems A and B.

By Theorem 5.4 it is enough to show that each of the building blocks of the classification (see Sect. 3) admits a $\mathcal{T}$-structure.

Consider a smoothly embedded 2-sphere **S** representing j-times a generator of $H_2(S^2 \times S^3, \mathbb{Z})$ $(1 < j < \infty)$. M_j is obtained by performing surgery on **S**. In the same way X_j is obtained by performing surgery on a sphere representing 2^j-times the generator of $H_2(X_\infty, \mathbb{Z})$ (note that even multiples of the generator have trivial normal bundles). This is easy to check since these manifolds are characterized by their homology groups and whether they are spin or not.

Of course, $X_0 = S^5$ and $M_\infty = S^2 \times S^3$ admit free S^1-actions. X_∞ also admits a free S^1-action since the Hopf action on S^3 commutes with the structure group of the bundle. If we consider X_∞ as the quotient of $S^3 \times S^3 \subset \mathbb{C}^2 \times \mathbb{C}^2$ by the S^1-action

$$(w, (z_1, z_2, z_3, z_4)) \mapsto (wz_1, wz_2, wz_3, z_4),$$

then the Hopf action is given by complex multiplication in the last two coordinates. But to construct $\mathcal{T}$-structures on all the X_j's consider the S^1-action on X_∞ given by complex multiplication on the last coordinate. This action has fixed points, of course. Call this action A_2, and A_1 the free "Hopf"-action. The second homology of X_∞ is generated by the image (under the projection) of $\{z_3 = 0, z_4 = 1\}$. Call this 2-sphere **S₀**. Now, given any small ε consider the 2-sphere

$$\mathbf{S}_\varepsilon = \left\{ (z_1, z_2, \varepsilon z_2, (1 - \varepsilon^2\|z_2\|^2)^{1/2}) \right\} / S^1 \subset X_\infty.$$

$\mathbf{S}_\varepsilon$ is homologous to $\mathbf{S_0}$ and they intersect only at $N = (1, 0, 0, 1)$. If we set the imaginary part of z_4 to be 0, we get the non-trivial S^2-bundle over S^2,

$S^3 \times S^2/S^1 \subset S^3 \times S^3/S^1$, which is diffeomorphic to $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$. We can modify $\mathbf{S_0} \cup \mathbf{S_e}$ inside $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ to obtain a smoothly embedded sphere $\mathbf{S}$ representing twice the generator of $H_2(X_\infty, \mathbb{Z})$. The orbits of the A_2-action passing through $\mathbf{S}$ form an embedded $S^2 \times S^1 \subset X_\infty$. Its normal bundle is $\mathbf{D} \times S^1$, where $\mathbf{D}$ is the D^2-bundle over S^2 with Euler characteristic 4. $\mathbf{D}$ can be represented as the quotient of $S^3 \times D^2$ under the S^1-action

$$\lambda(z_1, z_2, z_3) = (\lambda z_1, \lambda z_2, \lambda^4 z_3).$$

There is then a canonical S^1-action A_3 on $\mathbf{D}$ given by complex multiplication in the last coordinate. Define a $\mathcal{T}$-structure on X_∞ by leaving the A_2 action on $X_\infty - \mathbf{D} \times S^1$ and giving to $\mathbf{D} \times S^1$ the A_3 action. The zero section of $\mathbf{D}$ is the embedded $S^2 \times S^1$ and is exactly the fixed point set of A_3. For any j, $1 < j < \infty$, consider a 2-sphere embedded in $S^2 \times S^1$ representing 2^{j-1}-times the generator of the second homology group. This sphere represents 2^j-times the generator of $H_2(X_\infty, \mathbb{Z})$. Its normal bundle is $D^j \times \mathbb{R}$, which is isomorphic to the trivial bundle $S^2 \times D^3$. D^j is usually presented as the union of two copies of $D^2 \times D^2$ glued along $S^1 \times D^2$ by the map

$$(\lambda, z) \mapsto (\lambda, \lambda^{2^{j+1}} z).$$

Namely, $\lambda^{2^{j+1}}$ is considered as a map $\gamma : S^1 \to SO(2)$ and then we identify (λ, z) with $(\lambda, \gamma(\lambda)(z))$. The identification of $D^j \times \mathbb{R}$ with $S^2 \times D^3$ is obtained by an homotopy of the loop $(\gamma, 1)$ in $SO(3)$ with the constant loop 1. The action A_3 can then be viewed in $S^2 \times D^3$ as:

$$\lambda.(x, y) = (\varphi(x)^{-1}, \lambda_\times(\varphi(x)(y))),$$

for a map $\varphi : S^2 \to SO(3)$. Here $\lambda_\times$ means complex multiplication in the first two (real) coordinates. Since $\pi_2(SO(3)) = 1$ the map φ is null-homotopic. Therefore we can define an S^1-action on $S^2 \times (D_3 - \{0\})$ which is equal to A_3 in an exterior annulus and to

$$\lambda.(x, y) = (x, \lambda_\times y)$$

in an inner annulus.

X_j is obtained by deleting $S^2 \times D^3$ of X_∞ and gluing $D^3 \times S^2$ along the boundaries. Giving any S^1-action to the D^3-factor of the glued $D^3 \times S^2$ clearly defines a $\mathcal{T}$-structure on X_j. These $\mathcal{T}$-structures are not polarized.

The Wu-manifold $X_{-1} = SU(3)/SO(3)$ admits a locally-free S^1-action: simply embed S^1 in $SU(3)$ by sending $\lambda \in S^1$ to the diagonal matrix with λ, λ, λ^{-2} as the diagonal coefficients and then follow by matrix multiplication.

Finally, for any j, $1 < j < \infty$, M_j is obtained by performing surgery on a sphere $\mathbf{S}$ representing j-times the generator of $H_2(S^2 \times S^3, \mathbb{Z})$, which can be represented by a completely transversal sphere for the Hopf action on the S^3-factor (in the sense of Sect. 5). One then obtains by Theorem 5.14 a polarized $\mathcal{T}$-structure on M_j.

This finishes the first part of the theorem. The last statement follows because the fact that M is either non-cobordant to zero or it is cobordant to 0 but either $i(M) = 0, 1$ or ∞, means that in the factorization of M as connected sum of building blocks only appear M_j's, X_{-1}, X_1 and X_∞ and we have put polarized $\mathcal{T}$-structures on these manifolds ($X_1 = X_{-1} \# X_{-1}$).

<div align="right">□</div>

Proof of Theorem D. We shall make use of the following remarkable fact which is a consequence of results M. Gromov, Y. Yomdin and the Morse theory of the loop space. A proof can be found in [34].

Theorem 8.3. *Let M be a closed simply connected smooth manifold. Suppose that the loop space homology of M*

$$\sum_{i=0}^{n} \dim H_i(\Omega M, k_p)$$

grows exponentially with n for some field of coefficients k_p. Then, any C^∞ Riemannian metric has positive topological entropy.

Let M be a closed manifold obtained by taking connected sums of copies of S^4, $\mathbb{C}P^2$, $\overline{\mathbb{C}P}^2$, $S^2 \times S^2$ and the $K3$ surface. Since S^4, $\mathbb{C}P^2$, $\overline{\mathbb{C}P}^2$ and $S^2 \times S^2$ admit a circle action and the $K3$ surface admits a $\mathcal{T}$-structure by Theorem 5.10, it follows from Theorem 5.9 that M admits a $\mathcal{T}$-structure. By Theorem A, the minimal entropy of M vanishes.

Suppose now that M is diffeomorphic to one of the five manifolds listed in Theorem D. By the results in Sect. 4 each of these manifolds admits a smooth metric g with $\mathrm{h_{top}}(g) = 0$ and hence the minimal entropy problem can be solved for M.

On the other hand, suppose that the minimal entropy problem can be solved for M. Since $h(M) = 0$ it follows that M admits a smooth metric with zero topological entropy. Theorem 8.3 and Lemma 3.2 imply that M must be diffeomorphic to one of the five manifolds listed in Theorem D.

<div align="right">□</div>

Proof of Theorem E. Let M be a closed simply connected 5-manifold. Theorems C and A imply that the minimal entropy of M is zero.

Suppose now that M is diffeomorphic to one of the four manifolds listed in Theorem E. By the results in Sect. 4 each of these manifolds admit a smooth metric g with $\mathrm{h_{top}}(g) = 0$ and hence the minimal entropy problem can be solved for M.

On the other hand, suppose that the minimal entropy problem can be solved for M. Since $h(M) = 0$ it follows that M admits a smooth metric with zero topological entropy. Theorem 8.3 and Corollary 3.6 imply that M must be diffeomorphic to one of the four manifolds listed in Theorem E.

<div align="right">□</div>

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