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Symplectic resolutions for nilpotent orbits

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Abstract. In this paper, firstly we calculate Picard groups of a nilpotent orbit $\mathcal{O}$ in a classical complex simple Lie algebra and discuss the properties of being $\mathbb{Q}$-factorial and factorial for the normalization $\widetilde{\mathcal{O}}$ of the closure of $\mathcal{O}$. Then we consider the problem of symplectic resolutions for $\widetilde{\mathcal{O}}$. Our main theorem says that for any nilpotent orbit $\mathcal{O}$ in a semi-simple complex Lie algebra, equipped with the Kostant-Kirillov symplectic form ω, if for a resolution $\pi : Z \to \widetilde{\mathcal{O}}$, the 2-form $\pi^*(\omega)$ defined on $\pi^{-1}(\mathcal{O})$ extends to a symplectic 2-form on Z, then Z is isomorphic to the cotangent bundle $T^*(G/P)$ of a projective homogeneous space, and π is the collapsing of the zero section. It proves a conjecture of Cho-Miyaoka-Shepherd-Barron in this special case. Using this theorem, we determine all varieties $\widetilde{\mathcal{O}}$ which admit such a resolution.

0. Introduction

Since A. Beauville's pioneering paper [Be2], symplectic singularities have received a particular attention by many mathematicians. Recall that a holomorphic 2-form on a smooth variety is *symplectic* if it is closed and non-degenerate at every point. A normal algebraic variety V (always over $k = \mathbb{C}$) is said to have *symplectic singularities* (or to be a *symplectic variety*) if there exists a holomorphic symplectic 2-form ω on V_{reg} such that for any resolution of singularities $\pi : \widetilde{V} \to V$, the 2-form $\pi^*\omega$ defined a priori on $\pi^{-1}(V_{reg})$ can be extended to a regular 2-form on $\widetilde{V}$. If furthermore the 2-form $\pi^*\omega$ extends to a symplectic 2-form on $\widetilde{V}$ for some resolution of V, then we say that V admits a *symplectic resolution*.

A resolution of singularities $\pi : \widetilde{V} \to V$ for a symplectic variety V is called *crepant* if the canonical bundle of $\widetilde{V}$ is trivial. As we will show later, a resolution is crepant if and only if it is a symplectic resolution. In particular, we see that the existence of a symplectic resolution is independent of the special symplectic form on V_{reg}.

Examples of symplectic singularities are quotients of symplectic singularities by finite groups of automorphisms, preserving a symplectic 2-form on the regular locus. A particular case is the quotient of a complex vector space $\mathbb{C}^{2n}$ by a finite group G of symplectic automorphisms. The problem of symplectic resolutions for such quotient singularities has been studied by D. Kaledin in [Kal] and M. Verbitsky in [Ver]. We will come back to this problem in a subsequent paper [Fu]. The motivation is to generalize the McKay correspondence to higher dimensions. Strikingly, if $\mathbb{C}^{2n}/G$ admits a symplectic resolution, then the general problem of McKay correspondence has been solved by D. Kaledin [Ka2].

In general, it is difficult to determine whether a symplectic variety admits a symplectic resolution or not. A general conjecture on symplectic resolutions says (see [CMSB]) that any birational contraction of a smooth symplectic variety can be modelled locally upon the collapsing of the cotangent bundle $T^*(G/P)$ of a homogenous space. In the case of isolated symplectic singularities, this has been proved in [CMSB] with some extra hypothesis.

Another important class of symplectic singularities is the normalization $\widetilde{\mathcal{O}}$ of the closure of a nilpotent orbit $\mathcal{O}$ in a semi-simple complex Lie algebra $\mathfrak{g}$. As is well-known there exists a canonical symplectic 2-form ω on any adjoint orbit $\mathcal{O}$ (identified with a co-adjoint orbit via the Killing form), called the Kostant-Kirillov form. It was first proved by D. Panyushev [Pan] that this symplectic form extends to any resolution, so $\widetilde{\mathcal{O}}$ is a variety with symplectic singularities. The main purpose of this paper is to give an affirmative answer to the above conjecture for these singularities. More precisely, we prove the following

Theorem 0.1 **(Main theorem)** *For any symplectic resolution $\pi : (Z, \Omega) \to (\widetilde{\mathcal{O}}, \omega)$, there exists a parabolic subgroup P of G, such that (Z, Ω) is isomorphic to $(T^*(G/P), \Omega_{can})$, where Ω_{can} is the canonical symplectic form on $T^*(G/P)$. Furthermore, under this isomorphism, the map π corresponds to*

$$T^*(G/P) \simeq G \times^P \mathfrak{u} \to \mathfrak{g}, \qquad (g, X) \mapsto Ad(g)X,$$

where $\mathfrak{u}$ is the nilradical of $\mathfrak{p} = Lie(P)$.

Recall that an element $X \in \mathfrak{u}$ is a *Richardson element* if $dim(G \cdot X) = 2dim(G/P)$, or equivalently $P \cdot X$ is dense in $\mathfrak{u}$ and P contains the identity component of the centralizer $Z_G(X)$ (see [Hes]). The orbit $G \cdot X$ is called a *Richardson orbit*, which plays an important role in representation theory. It was Richardson who has shown that every parabolic subgroup in G has Richardson elements (see also our Proposition 3.10). As a direct corollary, our theorem implies that if the closure of a nilpotent orbit $\mathcal{O}$ admits a symplectic resolution, then it is a Richardson orbit.

The key point of the proof is to study the $\mathbb{C}^*$-action on $\overline{\mathcal{O}}$. Using the same idea as in [Kal] and [Ver], we show that this action can be lifted to any symplectic resolution Z of $\widetilde{\mathcal{O}}$. Then we use some standard analysis for this action on Z, as done in [Kal] and [Nak].

Let us give a brief outline of the contents of the paper.

• Sect. 1 recalls some basic definitions. We prove some easy propositions which reveal the relationship between crepant resolutions, symplectic resolutions and resolutions with small exceptional set.

• In Sect. 2 we calculate Picard groups of nilpotent orbits of classical type. In particular, we prove that for a nilpotent orbit $\mathcal{O}$ in a simple complex Lie algebra of B-C-D type, $Pic(\mathcal{O}) = Hom(\pi_1(\mathcal{O}), \mathbb{C}^*)$. Explicite formulas are given. Then we use these results to determine when the normalization $\widetilde{\mathcal{O}}$ is $\mathbb{Q}$-factorial or factorial.

• In Sect. 3, firstly we prove the main theorem. Then we use it and some results of W. Hesselink [Hes] to determine, in terms of the partition corresponding to the orbit $\mathcal{O}$, all normal varieties $\widetilde{\mathcal{O}}$ which admit a symplectic resolution.

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1. Preliminaries

Let V be an irreducible complex algebraic variety. A morphism $\pi : \widetilde{V} \to V$ is called a *resolution* (or *desingularization*) if π is projective and π induces an isomorphism outside the singular locus $Sing(V)$ of V. Since we are working over $\mathbb{C}$, Hironaka's big theorem says that every V admits a desingularization. In this paper we will consider a particular class of resolutions (called symplectic resolutions). Suppose that the canonical divisor K_V of V is a Cartier divisor. In this case, there is the following notion.

Definition 1.1 *A resolution $\pi : \widetilde{V} \to V$ is called crepant if $\pi^*(K_V) = K_{\widetilde{V}}$, i.e. π preserves the canonical class.*

Recall that a holomorphic 2-form on a smooth variety is *symplectic* if it is closed and non-degenerate at every point.

Definition 1.2 *A normal algebraic variety V is said to have symplectic singularities (or to be a symplectic variety) if there exists a symplectic 2-form ω on V_{reg} such that for any resolution $\pi : \widetilde{V} \to V$, the 2-form $\pi^*\omega$ defined a priori on $\pi^{-1}(V_{reg})$ extends to a holomorphic 2-form Ω on $\widetilde{V}$.*

Basic examples of varieties with symplectic singularities are quotients of symplectic singularities by finite groups of automorphisms, preserving a symplectic 2-form on the regular locus (see [Be2]). Some classification theorems for isolated symplectic singularities have been proved since the original paper [Be2].

Definition 1.3 *A resolution* $\pi : \widetilde{V} \to V$ *for a symplectic variety* (V, ω) *is called symplectic if the 2-form* Ω *extending* $\pi^*\omega$ *is a symplectic 2-form on* $\widetilde{V}$.

Note that for a symplectic variety V, its canonical sheaf is trivial, thus a symplectic resolution is a crepant resolution. The converse is also true, as shown by the following proposition (see also [Kal] and [Ver]).

Proposition 1.1 *Let* (V, ω) *be a symplectic variety of dimension* $2n$ *and* $\pi : \widetilde{V} \to V$ *a resolution, then* π *is a crepant resolution if and only if* π *is a symplectic resolution.*

Proof. We have seen that a symplectic resolution is a crepant resolution. Now suppose that π is crepant. Since ω^n has no zeros on V_{reg}, it extends to a global section of K_V. That π is crepant implies that $\pi^*(\omega^n)$ extends to a global section on $\widetilde{V}$ without zeroes. By our assumption, V has symplectic singularities, so $\pi^*\omega$ extends to a 2-form Ω on $\widetilde{V}$, furthermore we have $\Omega^n = \pi^*(\omega^n)$, thus Ω^n has no zeroes on $\widetilde{V}$, which implies that Ω is non-degenerate everywhere. Now on $\pi^{-1}(V_{reg})$, we have $d\Omega = d\pi^*(\omega) = \pi^*(d\omega) = 0$. Since $\pi^{-1}(V_{reg})$ is open-dense, $d\Omega = 0$ on $\widetilde{V}$, i.e. the 2-form Ω is closed, which gives that Ω is a symplectic 2-form, i.e. π is a symplectic resolution. □

In particular, we see that the existence of a symplectic resolution is independent of the special symplectic form on V_{reg}. This easy proposition gives the following criterion for a resolution being symplectic in the case of $Sing(V)$ having higher codimension.

Proposition 1.2 *Let* V *be a symplectic variety with* $codim(Sing(V)) \geq 3$. *Then* V *has a symplectic resolution if and only if there exists a resolution* $\pi : \widetilde{V} \to V$ *such that* $codim(\pi^{-1}(Sing(V))) \geq 2$.

Proof. Let $\pi : \widetilde{V} \to V$ be a symplecitc resolution. Suppose that the exceptional set $E = \pi^{-1}(Sing(V))$ has a component of codimension 1, then by Corollary 1.5 of [Nam] (where the proof works without the assumption "projective") the image of this component by π in V should be of codimension 2. This is impossible since π is a resolution of singularities and $codim(Sing(V)) \geq 3$, so we have $codim(E) \geq 2$.

Conversely, if π is resolution such that $codim(\pi^{-1}(Sing(V))) \geq 2$, then $\widetilde{V} - \pi^{-1}(V_{reg})$ has codimension ≥ 2, so $K_{\widetilde{V}} = K_{\pi^{-1}(V_{reg})} = \pi^*(K_{V_{reg}})$, which shows that $K_{\widetilde{V}}$ is trivial. Thus the resolution π is crepant, which is also symplectic by the above proposition. □

Recall that a variety is $\mathbb{Q}$-*factorial* if any Weil divisor has a multiple that is a Cartier divisor.

Corollary 1.3 *A locally* $\mathbb{Q}$-*factorial normal variety* V *whose singular locus* $Sing(V)$ *is not of pure co-dimension 2 does not admit any symplectic resolution.*

Proof. By an argument of [Deb] (p. 28), V being normal and locally $\mathbb{Q}$-factorial implies that for any birational map $\pi : Z \to V$, the exceptional set $E \subset Z$ of π is of pure codimension 1 in Z. Now the corollary follows in the same way as the precedent proposition. □

Here we recall some basic notions concerning nilpotent orbits. A detailed and excellent discussion can be found in [C-M]. Let $\mathfrak{g}$ be a semi-simple complex Lie algebra and let G_{ad} (or G for short) be the identity component of its automorphism group, which is called the adjoint group of $\mathfrak{g}$. Recall that each adjoint orbit $\mathcal{O}$ (identified with a co-adjoint orbit in $\mathfrak{g}^*$ via the Killing form) carries a canonical G-invariant symplectic 2-form ω, the Kostant-Kirillov form. An adjoint orbit is closed if and only if it is a semi-simple orbit. For a nilpotent orbit $\mathcal{O}$, its closure $\overline{\mathcal{O}}$ in $\mathfrak{g}$ is not necessarily normal. It is shown in [Pan] (see also [Be1]) that the normalization $\widetilde{\mathcal{O}}$ of $\mathcal{O}$ is a symplectic variety.

Another feature of nilpotent orbits is the existence of a $\mathbb{C}^*$-action. This can be seen as follows. Take a nilpotent element $X \in \mathfrak{g}$, we want to show that λX is conjugate to X for any $\lambda \in \mathbb{C}^*$. By the Jacobson-Morozov theorem, there exists a standard triple (H, X, Y) for X, i.e. we have

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H.$$

So we have an isomorphism $\phi : \mathfrak{sl}_2 \to \mathbb{C}\langle H, X, Y\rangle$. Now it is clear that $\lambda X = gXg^{-1}$ where $g = exp(cH)$ with $c \in \mathbb{C}$ satisfying $exp(2c) = \lambda$. For this action, we have

Lemma 1.4 *For $\lambda \in \mathbb{C}^*$, we have $\lambda^*\omega = \lambda\omega$.*

Proof. For an element $X \in \mathfrak{g}$, let ξ^X be the vector field

$$\xi^X(Z) = \frac{d}{dt}|_{t=0}\, exp(tX) \cdot Z = [X, Z].$$

Then for any $\lambda \in \mathbb{C}^*$, we have $\lambda_*\xi^X(Z) = \frac{d}{dt}|_{t=0}\lambda exp(tX) \cdot Z = [X, \lambda Z] = \xi^X(\lambda Z)$. Let $\kappa(\cdot, \cdot)$ be the Killing form on $\mathfrak{g}$, then by definition

$$\omega_Z\big(\xi^X(Z), \xi^Y(Z)\big) = \kappa(Z, [X, Y]).$$

Now

$$\begin{aligned}\lambda^*\omega_Z(\xi^X(Z), \xi^Y(Z)) &= \omega_{\lambda Z}(\lambda_*(\xi^X(Z)), \lambda_*(\xi^Y(Z)))\\ &= \omega_{\lambda Z}(\xi^X(\lambda Z), \xi^Y(\lambda Z)) = \kappa(\lambda Z, [X, Y]) = \lambda\omega_Z(\xi^X(Z), \xi^Y(Z)),\end{aligned}$$

so we have $\lambda^*\omega = \lambda\omega$. □

Now $\mathfrak{g}$ is an $\mathfrak{sl}_2$-module via the above isomorphism ϕ, so $\mathfrak{g}$ is decomposed as $\mathfrak{g} = \oplus_{i\in\mathbb{Z}}\mathfrak{g}_i$, where $\mathfrak{g}_i = \{Z \in \mathfrak{g} | [H, Z] = iZ\}$. The nilpotent orbit $\mathcal{O}$ is called *even* if $\mathfrak{g}_1 = 0$, or equivalently if $\mathfrak{g}_{2k+1} = 0$ for all $k \in \mathbb{Z}$ (see Lemma 3.8.7 [C-M]). Let $\mathfrak{p} = \oplus_{i\geq 0}\mathfrak{g}_i$ and P a connected subgroup in G with Lie algebra $\mathfrak{p}$. Set $\mathfrak{n} = \oplus_{i\geq 2}\mathfrak{g}_i$. Then for an even orbit $\mathcal{O}$, there is an isomorphism between $(\mathfrak{g}/\mathfrak{p})^*$ and $\mathfrak{n}$.

Proposition 1.5 **(Springer's resolution)** *Let $\mathcal{O}$ be an even nilpotent orbit in $\mathfrak{g}$, then there exists a G-equivariant resolution of singularities*

$$\pi : T^*(G/P) \simeq G \times^P \mathfrak{n} \to \overline{\mathcal{O}}, \quad (g, X) \mapsto Ad(g)X.$$

Now the morphism π factorizes through $\widetilde{\mathcal{O}}$, so for an even orbit $\mathcal{O}$, the variety $\widetilde{\mathcal{O}}$ admits a symplectic resolution. The motivation of this paper is to find out all varieties $\widetilde{\mathcal{O}}$ which admit a symplectic resolution. This is achieved at the end of Sect. 3.

2. Picard groups and $\mathbb{Q}$-factority

2.1. Picard groups of nilpotent orbits

Let $\mathfrak{g}$ be a complex simple Lie algebra, and G the adjoint group of $\mathfrak{g}$. For a nilpotent element $X \in \mathfrak{g}$, we denote by $\mathcal{O}_X$ the nilpotent orbit $G \cdot X$. It is isomorphic to G/G^X, where G^X is the stabilizer of X in G. The purpose of this section is to calculate Picard groups for these nilpotent orbits in case of $\mathfrak{g}$ being of classical type.

Instead of working with G, we will work with the universal covering G_{sc} of G. In this case we have $\mathcal{O}_X = G_{sc}/G_{sc}^X$, where G_{sc}^X is the stabilizer of X in G_{sc}.

Let $\mathfrak{g}_i = \{Z \in \mathfrak{g} | \, [H, Z] = iZ\}$, then $\mathfrak{g}$ is decomposed as $\mathfrak{g} = \oplus_{i \in \mathbb{Z}} \mathfrak{g}_i$. Let $\mathfrak{g}_i^X = \mathfrak{g}^X \cap \mathfrak{g}_i$ and $\mathfrak{u}^X = \oplus_{i>0} \mathfrak{g}_i^X$. If we define $\mathfrak{g}^\phi = \{Z \in \mathfrak{g} | \, [Z, X] = [Z, Y] = [Z, H] = 0\}$, then $\mathfrak{g}^X = \mathfrak{u}^X \oplus \mathfrak{g}^\phi$. Let U^X be the connected subgroup of G_{sc}^X with Lie algebra $\mathfrak{u}^X$, and let G_{sc}^ϕ be the centralizer of $Im(\phi)$ in G_{sc}, then we have

Proposition 2.1 **(Barbasch-Vogan, Kostant)** *There is a semi-direct product decomposition $G_{sc}^X = U^X \cdot G_{sc}^\phi$.*

Recall that the character group $\mathcal{X}(G)$ of an algebraic group G is defined to be the abelian group of algebraic group morphisms between G and $\mathbb{C}^*$, i.e. $\mathcal{X}(G) = Hom(G, \mathbb{C}^*)$.

Lemma 2.2 *We have $\mathcal{X}(G_{sc}^X) = \mathcal{X}(G_{sc}^\phi)$.*

Proof. Note that the algebra $\mathfrak{u}^X$ is nilpotent, so the group U^X is unipotent. As is well-known, every unipotent group has trivial character group. Now our lemma follows from the above proposition. □

So to calculate $\mathcal{X}(G_{sc}^X)$, we need to calculate the character group $\mathcal{X}(G_{sc}^\phi)$. In the classical cases, we can describe explicitly the subgroup G_{sc}^ϕ. Before giving the description, we need some notations. For any group H, let H_Δ^m denote the diagonal copy of H in H^m. If $H_1, \cdots, H_m$ are matrix groups, let $S(\prod_i H_i)$ be the subgroup of $\prod_i H_i$ consisting of m-tuples of matrices whose determinants have product 1.

Recall that a *partition* **d** of n is a tuple $[d_1, \cdots, d_N]$ such that $d_1 \geq d_2 \geq \cdots \geq d_N > 0$ and $\sum_{j=1}^N d_j = n$. In classical cases, a nilpotent orbit can be encoded by some partition **d**. To illustrate the idea, let us consider the case of $\mathfrak{sl}_n$. Every nilpotent element $X \in \mathfrak{sl}_n$ is conjugate to an element of the form $diag(J_{d_1}, \cdots, J_{d_N})$, where J_{d_j} is a Jordan block of type $d_j \times d_j$ and $\mathbf{d} = [d_1, \cdots, d_N]$ is a partition of n. It is clear that this partition is invariant under conjugation, thus to the nilpotent orbit $\mathcal{O}_X$ we associate the partition **d**, which establishes a bijection between nilpotent orbits in $\mathfrak{sl}_n$ and partitions of n. A similar bijection exists for nilpotent orbits in $\mathfrak{sp}_{2n}$ and $\mathfrak{so}_m$ (see Sect. 5.1 of [C-M]). It is easy to see that the orbit $\mathcal{O}_X$ is even if and only if all the parts d_i have the same parity. For a partition $\mathbf{d} = [d_1, \cdots, d_N]$, we put $r_i = \#\{j | d_j = i\}$ and $s_i = \#\{j | d_j \geq i\}$. Then we have (see [C-M] Theorem 6.1.3):

Proposition 2.3 (Springer-Steinberg)

$$G_{sc}^{\phi} = \begin{cases} S(\prod_i (GL_{r_i})^i_{\Delta}) & \mathfrak{g} = \mathfrak{sl}_n; \\ \prod_{i\, odd} (Sp_{r_i})^i_{\Delta} \times \prod_{i\, even} (O_{r_i})^i_{\Delta} & \mathfrak{g} = \mathfrak{sp}_{2n}; \\ double\ cover\ of\ S(\prod_{i\, even} (Sp_{r_i})^i_{\Delta} \times \prod_{i\, odd} (O_{r_i})^i_{\Delta}) & \mathfrak{g} = \mathfrak{so}_m. \end{cases}$$

Lemma 2.4 *If $\mathfrak{g} = \mathfrak{sl}_n$, then $\mathcal{X}(G_{sc}^X) = \mathbb{Z}^{k-1}$, where $k = \#\{i | r_i \neq 0\}$ is the number of distinct d_i in* **d**.

Proof. Recall that $\mathcal{X}(GL_r) \cong \mathbb{Z}$ for any $r > 0$, where the isomorphism is given by $\mathbb{Z} \ni l \mapsto (A \mapsto det(A)^l)$. So $\mathcal{X}(\prod_i (GL_{r_i})^i_{\Delta}) \cong \mathbb{Z}^k$, where $k = \#\{i | r_i \neq 0\}$ is the number of distinct d_i in the partition **d**. When we take the operation "S", the determinant of the first factor GL_{r_1} is determined by the others, so $\mathcal{X}(G_{sc}^X) = \mathbb{Z}^{k-1}$. □

Proposition 2.5 *Let $\mathcal{O}_X$ be a nilpotent orbit in $\mathfrak{sl}_n$ corresponding to the partition* **d**, *and k the number of distinct d_i in* **d**. *Then we have $Pic(\mathcal{O}_X) = \mathbb{Z}^{k-1}$.*

Proof. By a result of V. Popov ([Pop]), the Picard group of a connected semisimple Lie group is isomorphic to its fundamental group. So the Picard group $Pic(G_{sc})$ of the simply connected semi-simple group G_{sc} is trivial. Now the following exact sequence ([KKV]):

$$0 \to \mathcal{X}\big(G_{sc}^X\big) \to Pic(\mathcal{O}_X) \to Pic(G_{sc}) = 0,$$

gives that $Pic(\mathcal{O}_X) = \mathcal{X}(G_{sc}^X)$, which is equal to $\mathbb{Z}^{k-1}$ by the above lemma. □

Now we consider the case of a simple Lie algebra of B-C-D type.

Theorem 2.6 *Let $\mathfrak{g}$ be a simple complex Lie algebra of B-C-D type, and $\mathcal{O}_X$ a nilpotent orbit in $\mathfrak{g}$, then we have $Pic(\mathcal{O}_X) = \mathcal{X}(\pi_1(\mathcal{O}_X))$, where $\pi_1(\mathcal{O}_X)$ is the fundamental group of $\mathcal{O}_X$.*

Proof. By our Lemma 2.2, we have $\mathcal{X}((G_{sc}^X)^\circ) = \mathcal{X}((G_{sc}^\phi)^\circ)$. By the above theorem of Springer-Steinberg, we see that $(G_{sc}^\phi)^\circ$ is product of copies of semi-simple Lie groups, thus $\mathcal{X}((G_{sc}^\phi)^\circ)$ is trivial. So we have

$$\mathcal{X}\big(G_{sc}^X\big) = \mathcal{X}\big(G_{sc}^X/\big(G_{sc}^X\big)^\circ\big).$$

By Lemma 6.1.1 of [C-M], the component group $G_{sc}^X/(G_{sc}^X)^\circ$ is isomorphic to the fundamental group $\pi_1(\mathcal{O}_X)$ of $\mathcal{O}_X$, thus we have $\mathcal{X}(G_{sc}^X) = \mathcal{X}(\pi_1(\mathcal{O}_X))$. Now a similar argument as we did in the above proof gives that $Pic(\mathcal{O}_X) = \mathcal{X}(G_{sc}^X) = \mathcal{X}(\pi_1(\mathcal{O}_X))$. □

Combining with explicit formulas for $\pi_1(\mathcal{O}_X)$ given in [C-M] (Cor. 6.1.6), we can give the following formulas for $Pic(\mathcal{O}_X)$. Recall that we say a partition **d** is *rather odd* if all of its odd parts have multiplicity one.

Corollary 2.7 *Let $\mathcal{O}_X$ be a nilpotent orbit in $\mathfrak{g}$ corresponding to the partition* **d**. *Let a be the number of distinct odd d_i and let b be the number of distinct even d_i. Then we have*

(1) For $\mathfrak{g} = \mathfrak{sp}_{2n}$, we have $Pic(\mathcal{O}_X) = (\mathbb{Z}/2\mathbb{Z})^b$;
(2) For $\mathfrak{g} = \mathfrak{so}_{2n+1}$, if **d** *is rather odd, then $Pic(\mathcal{O}_X)$ is an extension of $\mathbb{Z}/2\mathbb{Z}$ by $(\mathbb{Z}/2\mathbb{Z})^{a-1}$, otherwise $Pic(\mathcal{O}_X) = (\mathbb{Z}/2\mathbb{Z})^{a-1}$;*
(3) For $\mathfrak{g} = \mathfrak{so}_{2n}$, if **d** *is rather odd, then $Pic(\mathcal{O}_X)$ is an extension of $\mathbb{Z}/2\mathbb{Z}$ by $(\mathbb{Z}/2\mathbb{Z})^{max\{0,a-1\}}$, otherwise $Pic(\mathcal{O}_X) = (\mathbb{Z}/2\mathbb{Z})^{max\{0,a-1\}}$.*

Corollary 2.8 *For any nilpotent orbit $\mathcal{O}_X$ in a simple Lie algebra of B-C-D type, the Picard group $Pic(\mathcal{O}_X)$ is finite.*

2.2. $\mathbb{Q}$-factority and symplectic resolutions

Let $\mathfrak{g}$ be a simple Lie algebra of classical type, and $\mathcal{O}_X$ be a nilpotent orbit. The closure $\overline{\mathcal{O}}_X$ is not always normal (for precise results see [KP]). We denote by $\widetilde{\mathcal{O}}_X$ its normalization. In this section, we will give some results on the properties of being $\mathbb{Q}$-factorial and factorial for the normal variety $\widetilde{\mathcal{O}}_X$.

For an irreducible algebraic variety V, which is smooth in codimension 1, we denote by $Cl(V)$ its divisor class group, i.e. the Weil divisors modulo linear equivalences ([Har]). The following lemma is easily proved.

Lemma 2.9 *Let V be an irreducible algebraic variety, which is smooth in codimension 1. Let $\pi : \widetilde{V} \to V$ be the normalization. Then the induced map $\pi^* : Cl(V) \to Cl(\widetilde{V})$ is a group isomorphism.*

Proposition 2.10 *Let $\mathfrak{g}$ be a simple Lie algebra of B-C-D type, and let $\mathcal{O}_X$ be a nilpotent orbit in $\mathfrak{g}$, then the normal variety $\widetilde{\mathcal{O}}_X$ is $\mathbb{Q}$-factorial.*

Proof. Since $\overline{\mathcal{O}}_X$ is smooth in codimension 1, by the precedent lemma, we have $Cl(\widetilde{\mathcal{O}}_X) \cong Cl(\overline{\mathcal{O}}_X)$. Now by Proposition II.6.5 of [Har], we have $Cl(\overline{\mathcal{O}}_X) \cong Cl(\mathcal{O}_X)$. That $\mathcal{O}_X$ is smooth gives $Cl(\mathcal{O}_X) = Pic(\mathcal{O}_X)$, thus $Cl(\widetilde{\mathcal{O}}_X) \cong Pic(\mathcal{O}_X)$. Now by our calculations in the precedent section, we know that $Pic(\mathcal{O}_X)$ is a finite group, so $Cl(\widetilde{\mathcal{O}}_X) \otimes_{\mathbb{Z}} \mathbb{Q} \cong Pic(\mathcal{O}_X) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$. In particular, the map

$$Pic(\widetilde{\mathcal{O}}_X) \otimes_{\mathbb{Z}} \mathbb{Q} \to Cl(\widetilde{\mathcal{O}}_X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is surjective, $\widetilde{\mathcal{O}}_X$ is $\mathbb{Q}$-factorial. □

Proposition 2.11 *Let $\mathfrak{g}$ be a simple Lie algebra, and $\mathcal{O}_X$ be a nilpotent orbit in $\mathfrak{g}$ corresponding to the partition $\mathbf{d} = [d_1, \cdots, d_N]$. Then we have:*

(1) For $\mathfrak{g} = \mathfrak{sl}_n$, the normal variety $\widetilde{\mathcal{O}}_X$ is factorial if and only if $d_1 = d_2 = \cdots = d_N$, i.e. there is only one distinct d_i;
(2) For $\mathfrak{g} = \mathfrak{sp}_{2n}$, the normal variety $\widetilde{\mathcal{O}}_X$ is factorial if and only if every d_i is odd;
(3) For $\mathfrak{g} = \mathfrak{so}_{2n}$, the normal variety $\widetilde{\mathcal{O}}_X$ is factorial if and only if there exists exact one distinct odd d_i;
(4) For $\mathfrak{g} = \mathfrak{so}_{2n+1}$, the normal variety $\widetilde{\mathcal{O}}_X$ is factorial if and only if there exists just one distinct odd d_i with multiplicity at least 3.

Proof. From Proposition II.6.2 of [Har], the affine normal variety $\widetilde{\mathcal{O}}_X$ is factorial if and only if $Cl(\widetilde{\mathcal{O}}_X) = 0$. By the proof of the above proposition, this is equivalent to $Pic(\mathcal{O}_X) = 0$. Now we just do a case-by-case check, based on our Proposition 2.5 and Corollary 2.7. For example, when $\mathfrak{g} = \mathfrak{so}_{2n+1}$, $Pic(\mathcal{O}_X) = 0$ if and only if $\mathbf{d}$ is not rather odd and $a = 1$, i.e. $\mathbf{d}$ has only one distinct odd d_i, with multiplicity at least 2. But the sum $\sum d_i = 2n + 1$ is odd, so the multiplicity should be at least 3. □

The above two propositions give a rather clear description of the property of being $\mathbb{Q}$-factorial or factorial of the normal variety $\widetilde{\mathcal{O}}_X$, with one exception $\mathfrak{g} = \mathfrak{sl}_n$. This will be discussed further in Sect. 3. Applying these results to symplectic resolutions, we have

Proposition 2.12 *Let $\mathfrak{g}$ be a simple Lie algebra of type B-C-D, and $\mathcal{O}_X$ be a nilpotent orbit in $\mathfrak{g}$. If $\overline{\mathcal{O}}_X - \mathcal{O}_X$ is not of pure co-dimension 2 in $\overline{\mathcal{O}}_X$, then the normal variety $\widetilde{\mathcal{O}}_X$ does not admit any symplectic resolution.*

Proof. This comes directly from the above Proposition 2.10 and Corollary 1.3. □

Examples: In the case of $\mathfrak{g} = \mathfrak{sp}_6$, this proposition shows that $\widetilde{\mathcal{O}}_{[2,2,1,1]}$ and the closure of the minimal orbit do not have any symplectic resolution. In the case of $\mathfrak{g} = \mathfrak{so}_8$, we find 2 orbits which do not admit any symplectic resolution, corresponding to the partitions $[3, 2^2, 1]$ and $[2^2, 1^4]$, while the other nilpotent orbits are all even, thus having a symplectic resolution by Springer's resolution.

Corollary 2.13 *Let $\mathfrak{g}$ be a simple Lie algebra of type B-C-D. The closure of the minimal orbit $\overline{\mathcal{O}}_{min}$ in $\mathfrak{g}$ does not admit any symplectic resolution.*

Remark 1 This is also true for minimal orbits in exceptional Lie algebras by a similar argument. We just need to note that the Picard groups for these cases are trivial. This follows also from our discussions in Sect. 3.4.

As we will see later, the closure of every nilpotent orbit in $\mathfrak{sl}_n$ admits a symplectic resolution. This is more or less known to some experts.

3. Symplectic resolutions for nilpotent orbits

3.1. Lifting the action of $\mathbb{C}^*$ and G

Let $\mathfrak{g}$ be a semi-simple complex Lie algebra and G its adjoint group. Let $\mathcal{O}$ be a nilpotent orbit in $\mathfrak{g}$ and $\widetilde{\mathcal{O}}$ the normalization of $\overline{\mathcal{O}}$. Recall that there exists an action of G (resp. $\mathbb{C}^*$) on $\overline{\mathcal{O}}$. The purpose of this section is to prove that this action can be lifted to any symplectic resolution of $\widetilde{\mathcal{O}}$. Let $\pi : Z \to \widetilde{\mathcal{O}}$ be a symplectic resolution for $\widetilde{\mathcal{O}}$, and Ω the symplectic 2-form on Z extending $\pi^*\omega$, where ω is the Kostant-Kirillov symplectic form on $\mathcal{O}$. We denote also by π the map $Z \to \overline{\mathcal{O}}$, which should not cause any confusion.

Proposition 3.1 *The action of G (resp. $\mathbb{C}^*$) on $\overline{\mathcal{O}}$ lifts to Z, in such a way that π is G-equivariant (resp. $\mathbb{C}^*$-equivariant).*

Proof. The proof is inspired from the proof of Theorem 1.3 in [Kal] and Theorem 2.5 in [Ver]. Firstly we will show that the action of $\mathfrak{g}$ on $\overline{\mathcal{O}}$ can be lifted to an action of $\mathfrak{g}$ on Z. To this end, let $X \in \mathfrak{g}$. Consider the vector field ξ^X on $\mathcal{O}$ defined by $\xi^X(Y) = \frac{d}{dt}|_{t=0}\, exp(tX) \cdot Y = [X, Y]$, where $\cdot$ denotes the adjoint action of G on $\mathfrak{g}$. Now the symplectic form ω on $\mathcal{O}$ gives an isomorphism $\Omega^1(\mathcal{O}) \simeq \mathcal{T}(\mathcal{O})$ between 1-forms and vector fields on $\mathcal{O}$. Let us denote by α_X the 1-form on $\mathcal{O}$ corresponding to the vector field ξ^X. The key point is the following claim.

Claim The 1-form $\pi^*(\alpha_X)$ extends to the whole of Z.

Proof of the claim. Take an Hermitian metric h on Z. Set $U = \pi^{-1}(\mathcal{O})$. Then $\pi^*(\alpha_X)$ can be extended to the whole of Z unless it has singularities on the complement $Z - U$. So we need to show that for any compact set $K \subset Z$ and for any $z \in U \cap K$, the Hermitian norm of $\pi^*(\alpha_X)$ is bounded by some constant depending on K.

Since π is analytic, thus Lipschitz on compact subsets $K \subset Z$. By rescaling the metric h, we can suppose that $\pi|_K$ decreases distance. Here the metric on $\mathcal{O}$ is the one induced from the metric on $\mathfrak{g}$. We need to show that on $\pi(K) \cap \mathcal{O}$, the 1-form α_X has bounded norm. This follows from the fact that the vector field ξ^X has bounded norm on $\pi(K) \cap \mathcal{O}$ and ω is G-invariant, so it also has bounded norm on $\pi(K) \cap U$. □

Let us denote by $\tilde{\alpha}$ the extended 1-form on Z. The symplectic form Ω on Z gives a 1-1 correspondence between 1-forms and vector fields on Z, so we get a vector field ζ^X on Z. If we denote by $\phi_X(z, t)$ the flow of this vector field, then $\mathfrak{g}$ acts on Z via $X \cdot z = \phi_X(z, 1)$. This gives an action of G_{sc} on Z. The center of G_{sc} acts trivially on the open-dense set $\pi^{-1}(\mathcal{O})$, so it acts trivially on the whole of Z. Since G is the quotient of G_{sc} by its center, we get an action of G on Z.

A similar argument shows that we can also lift the $\mathbb{C}^*$-action to Z. It is clear from the construction that π is G-equivariant (resp. $\mathbb{C}^*$-equivariant). □

Corollary 3.2 *The symplectic form Ω is G-invariant, and for the $\mathbb{C}^*$-action, we have $\lambda^*\Omega = \lambda\Omega$ for any $\lambda \in \mathbb{C}^*$.*

Proof. This comes from the above proposition and the corresponding properties of ω, which is G-invariant and satisfies $\lambda^*\omega = \lambda\omega$ (Lemma 1.4). □

3.2. Main theorem

In this section we will prove the following theorem, which gives an affirmative answer in the case of nilpotent orbits to the conjecture of Cho-Miyaoka-Shepherd-Barron [CMSB].

Theorem 3.3 **(Main theorem)** *Let $\mathfrak{g}$ be a semi-simple complex Lie algebra, and G its adjoint group. Consider a nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}$, equipped with the Kostant-Kirillov form ω. Then for any symplectic resolution $\pi : (Z, \Omega) \to (\widetilde{\mathcal{O}}, \omega)$, there exists a parabolic subgroup P of G, such that (Z, Ω) is isomorphic to $(T^*(G/P), \Omega_{can})$, where Ω_{can} is the canonical symplectic form on $T^*(G/P)$. Furthermore, under this isomorphism, the map π becomes*

$$T^*(G/P) \simeq G \times^P \mathfrak{u} \to \mathfrak{g}, \qquad (g, X) \mapsto Ad(g)X,$$

where $\mathfrak{u}$ is the nilradical of $\mathfrak{p} = Lie(P)$.

The idea of the proof is to study the $\mathbb{C}^*$-action on Z, as done in the papers of D. Kaledin [Kal] and H. Nakajima [Nak]. A detailed account of the general theory can be found in Sect. 6 of [Kal]. One may also consult the excellent book of H. Nakajima [Na2].

In the following we also denote by π the map $Z \to \overline{\mathcal{O}}$. We will study this map intead of $Z \to \widetilde{\mathcal{O}}$. By our Proposition 3.1, the $\mathbb{C}^*$-action on $\overline{\mathcal{O}}$ can be lifted on Z. Let $Z^{\mathbb{C}^*}$ be the fixed points subvariety in Z under this $\mathbb{C}^*$-action. Put $2n = dim(Z)$. As we will see later, one difficulty of the proof is to show that $\pi^{-1}(0) = Z^{\mathbb{C}^*}$.

Lemma 3.4 *There exists a G-equivariant attraction $p : Z \to Z^{\mathbb{C}^*}$.*

Proof. For any point $x \in Z$, define $\phi_x : \mathbb{C}^* \to Z$ to be $\phi_x(\lambda) = \lambda \cdot x$. Let $\psi_x : \mathbb{C} \to \overline{\mathcal{O}}$ be $\psi_x(\lambda) = \lambda\pi(x)$. Since π is $\mathbb{C}^*$-equivariant, we have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{C}^* & \xrightarrow{\phi_x} & Z \\ \downarrow & & \downarrow \pi \\ \mathbb{C} & \xrightarrow{\psi_x} & \overline{\mathcal{O}} \end{array}$$

Applying the valuative criterion of properness for the projective map π, we get a unique morphism $\widetilde{\phi}_x : \mathbb{C} \to Z$ extending ϕ_x. It is clear that $\widetilde{\phi}_x(0) \in Z^{\mathbb{C}^*}$. Now we define $p : Z \to Z^{\mathbb{C}^*}$ by $p(x) = \widetilde{\phi}_x(0)$, which is the attraction map for the $\mathbb{C}^*$-action. That the G-action commutes with the $\mathbb{C}^*$-action on Z implies that p is G-equivariant. □

Now take a fixed point $z \in Z^{\mathbb{C}^*}$, the action of $\mathbb{C}^*$ on Z induces a weight decomposition

$$T_z Z = \oplus_{p \in \mathbb{Z}} T_z^p Z,$$

where $T_z^p Z = \{v \in T_z Z | \lambda_* v = \lambda^p v\}$.

Definition 3.1 *The $\mathbb{C}^*$-action is called* definite *at the point z if $T_z^p Z = 0$ for all $p < 0$.*

Lemma 3.5 *There exists an irreducible smooth component Z_0 of $Z^{\mathbb{C}^*}$ on which the $\mathbb{C}^*$-action is definite.*

Proof. Consider the open-dense set $\pi^{-1}(\mathcal{O})$, which is isomorphic to $\mathcal{O}$, so G acts on it transitively. By our precedent lemma, p is G-equivariant, so the G-action on $p(\pi^{-1}(\mathcal{O}))$ is also transitive. As a corollary, $p(\pi^{-1}(\mathcal{O}))$ is connected, then it is contained in a connected component, say Z_0, of $Z^{\mathbb{C}^*}$. This means that the attraction subvariety of Z_0 contains the open-dense set $\pi^{-1}(\mathcal{O})$ of Z, so the $\mathbb{C}^*$-action is definite on Z_0 (see Lemma 6.1 [Kal]), i.e. $T_z^q Z = 0$ for $q < 0$. Since Z is smooth, it is well-known that the fixed points variety $Z^{\mathbb{C}^*}$ is a union of smooth connected components, thus Z_0 is smooth and irreducible. □

Lemma 3.6 *The closed subvariety Z_0 is projective, n-dimensional and Lagrangian w.r.t. Ω.*

Proof. Since π is projective, the variety $\pi^{-1}(0)$ is projective. That π is $\mathbb{C}^*$-equivariant implies that Z_0 is a closed subvariety in $\pi^{-1}(0)$, so Z_0 is projective.

The tangent space of Z_0 at the point z equals to $T_z Z_0 = T_z^0 Z$. Now take two vectors $v_1 \in T_z^p Z$ and $v_2 \in T_z^q Z$, then the equation $\lambda^* \Omega = \lambda \Omega$ implies

$$\lambda \Omega_z(v_1, v_2) = (\lambda^* \Omega)_z(v_1, v_2) = \Omega_{\lambda \cdot z}(\lambda_*(v_1), \lambda_*(v_2)) = \lambda^{p+q} \Omega_z(v_1, v_2).$$

So $\Omega_z(v_1, v_2) = 0$ if $p + q \neq 1$. Now that $T_z^q Z = 0$ for $q < 0$ implies that for $p \geq 2$, the space $T_z^p Z$ is orthogonal to $T_z Z$ w.r.t. Ω, thus $T_z^p Z = 0$, so $T_z Z = T_z^0 Z \oplus T_z^1 Z$. Furthermore Ω gives a duality between $T_z^0 Z$ and $T_z^1 Z$, so $dim(T_z^0 Z) = n$, i.e. Z_0 is of dimension n. This also gives that Z_0 is Lagrangian with respect to the symplectic form Ω. □

Since the decomposition by the attraction subvarieties is locally closed, the variety $V = p^{-1}(Z_0)$ containing $\pi^{-1}(\mathcal{O})$ is open-dense in Z. As easily seen, it is also G-invariant. From now on, we will only consider $p : V \to Z_0$ instead of $p : Z \to Z^{\mathbb{C}^*}$. At the end of this section, we will prove that $V = Z$.

There exists a canonical symplectic 2-form Ω_{can} on the cotangent bundle $T^* Z_0$, which comes from the Liouville form. There is also a natural $\mathbb{C}^*$-action on $T^* Z_0$, considered as a vector bundle over Z_0.

Lemma 3.7 *There exists a $\mathbb{C}^*$-equivariant isomorphism $i : V \to T^* Z_0$, which identifies also the two symplectic structures.*

Proof. Since the $\mathbb{C}^*$-action is definite on Z_0 and the $\mathbb{C}^*$-action on $T_z Z = T_z^0 Z \oplus T_z^1 Z$ is the same for any $z \in Z_0$, a classical work of Bialynicki-Birula (see [BB]) implies that the attraction $p : V \to Z_0$ makes V a vector bundle of rank n over Z_0, and the $\mathbb{C}^*$-action on this vector bundle is the natural one. Let us identity Z_0 with the zero section of this bundle. Since for any $z \in Z_0$, $T_z Z = T_z Z_0 \oplus T_z^1 Z$, the induced $\mathbb{C}^*$-action on the normal bundle N of Z_0 in V is the natural one when we consider N as a vector bundle over Z_0. This gives a $\mathbb{C}^*$-equivariant isomorphism between V and the total space of the normal bundle N. Let us also denote by Ω the symplectic form on N, which comes from the symplectic form on V.

Now we establish an isomorphism between (N, Ω) and $(T^* Z_0, \Omega_{can})$ as follows. Take a point $z \in Z_0$, and a vector $v \in N_z$. Since Z_0 is Lagrangian in the two symplectic spaces, there exists a unique vector $w \in T_z^* Z_0$ such that $\Omega_z(v, u) = \Omega_{can,z}(w, u)$ for any $u \in T_z Z_0$. We define the map $i : N \to T^* Z_0$ to be $i(v) = w$. Now it is easy to see that i is a symplectic isomorphism. □

From now on, we will denote by Ω (instead of Ω_{can}) the canonical symplectic 2-form on $T^* Z_0$. Now we will study the action of G on Z. Since the G-action commutes with the $\mathbb{C}^*$-action, the fixed points $Z^{\mathbb{C}^*}$ is G-invariant. Since Z_0 is a connected component of $Z^{\mathbb{C}^*}$ and G is connected, Z_0 is also G-invariant, thus we get a G-action on Z_0. This action induces naturally an action of G on the total space of TZ_0, which is given by $g \cdot (z, v) = (g \cdot z, dg(v))$. By taking the dual, we have a G-action on $T^* Z_0$.

Lemma 3.8 *The G-action on $T^* Z_0$ is isomorphic to the one described above.*

Proof. For an element $g \in G$, we write ϕ_g the action of g on $T^* Z_0$. The action of G on Z_0 will be simply denoted by $\cdot$. For an element $z \in Z_0 \subset T^* Z_0$, we have a natural decomposition $T_z(T^* Z_0) = T_z^* Z_0 \oplus T_z Z_0$, which is also isotropic w.r.t. Ω. The G-action on $T_z Z_0$ is the natural differential one. Since the G-action commutes with the $\mathbb{C}^*$-action, for a vector $v_2 \in T_z^* Z_0$, considered as a tangent vector in $T_z(T^* Z_0)$, we have $d\phi_g(v_2) = \frac{d}{d\lambda}|_{\lambda=0}\phi_g(\lambda v_2) = \frac{d}{d\lambda}|_{\lambda=0}\lambda\phi_g(v_2) = \phi_g(v_2)$.

For any vector $v_1 \in T_z Z_0$, since Ω is G-invariant, we have

$$\begin{aligned}\Omega_z(v_1, v_2) &= \phi_g^* \Omega_z(v_1, v_2) \\ &= \Omega_{g\cdot z}(d\phi_g(v_1), d\phi_g(v_2)) = \Omega_{g\cdot z}(dg(v_1), \phi_g(v_2)).\end{aligned}$$

That Ω is the natural symplectic structure on $T^* Z_0$ implies that $\phi_g(v_2) = (dg)^*(v_2)$, i.e. the action of G on $T^* Z_0$ is the natural one induced from the action of G on Z_0. □

Lemma 3.9 *The variety Z_0 contains an open-dense G-orbit.*

Proof. The argument in the proof of Lemma 3.5 shows that G acts transitively on the set $p(\pi^{-1}(\mathcal{O})) \subset Z_0$. Here $p : T^* Z_0 \to Z_0$ is the canonical projection. Since $\pi^{-1}(\mathcal{O})$ is open, $p(\pi^{-1}(\mathcal{O}))$ is an open-dense G-orbit. □

The author wants to thank F. Knop and M. Brion for having provided the proof of the following proposition, which is implicit in [Kno].

Proposition 3.10 *Let $\mathfrak{g}$ be a semi-simple complex Lie algebra and G its adjoint group. Let P be a closed subgroup of G and $\mathfrak{p}$ its Lie algebra. Then P has a dense orbit (via the co-adjoint action) in $(\mathfrak{g}/\mathfrak{p})^*$ if and only if P is parabolic.*

Proof. Suppose that P has a dense orbit in $(\mathfrak{g}/\mathfrak{p})^*$, then its image in $\mathfrak{g}^*//G$ is a point. This means that the space G/P has rank 0 (see Satz 5.4 [Kno]). Now Satz 9.1 of *loc. cit.* implies that all isotropy groups of G/P are parabolic, thus P is parabolic.

The converse is a theorem of Richardson [Ric]. □

Corollary 3.11 *The G-action on Z_0 is transitive, and Z_0 is isomorphic to G/P for some parabolic subgroup P of G.*

Proof. By our Lemma 3.9, Z_0 has an open orbit, say $U = G/P$, where P is a subgroup of G. Now $T^*(G/P) \simeq G \times^P (\mathfrak{g}/\mathfrak{p})^*$. Note that G has an open orbit in $T^*(G/P)$, which is isomorphic to $\pi^{-1}(\mathcal{O})$, so P has an open dense orbit in $(\mathfrak{g}/\mathfrak{p})^*$, where the action is the co-adjoint one by our Lemma 3.8. Now the above proposition gives that P is parabolic. As a corollary, the variety G/P is projective which is also dense in Z_0, so $Z_0 = G/P$. □

By our above lemmas, V is isomorphic to $T^*(G/P) = G \times^P \mathfrak{u}$, where $\mathfrak{u}$ is the nilradical of $\mathfrak{p} = Lie(P)$ which is identified with $(\mathfrak{g}/\mathfrak{p})^*$ via the Killing form. Under this isomorphism, the G-action on $G \times^P \mathfrak{u}$ is the natural one $g \cdot (h, X) = (gh, X)$.

Lemma 3.12 *Under these isomorphisms, the map* $\pi : V \simeq G \times^P \mathfrak{u} \to \overline{\mathcal{O}}$ *becomes* $(g, X) \mapsto Ad(g)X$.

Proof. Take a point $(1, X) \in 1 \times \mathfrak{u} \cap \pi^{-1}(\mathcal{O})$, possibly by using a translation on G/P, we can suppose $\pi(1, X) = X$. Since π is G-equivariant, for any $p \in P$, we have $\pi(1, Ad(p)X) = \pi(1, p \cdot X) = p \cdot \pi(1, X) = Ad(p)X$. Now that $P \cdot X$ is dense in $\mathfrak{u}$ implies that $\pi(1, Y) = Y$ for any $Y \in \mathfrak{u}$, which gives $\pi(g, Y) = \pi(g \cdot (1, Y)) = g \cdot \pi(1, Y) = Ad(g)Y$ for any $g \in G$. □

Corollary 3.13 *We have* $Z_0 = \pi^{-1}(0) \cap V$, *i.e.* Z_0 *is a connected component of* $\pi^{-1}(0)$.

Proof. Note that for any $(g, X) \in G \times^P \mathfrak{u}$, $\pi(g, X) = Ad(g)X$ is 0 if and only if $X = 0$, i.e. $Z_0 = \pi^{-1}(0) \cap V$. This implies that for any $z \in \pi^{-1}(0)$ such that $\lim_{\lambda \to 0} \lambda \cdot z \in Z_0$, then $z \in Z_0$, so Z_0 is a connected component of $\pi^{-1}(0)$. □

Now to finish the proof of our theorem, we need to show that $V = Z$, which is equivalent to that $\pi^{-1}(0)$ is connected by the above corollary. Suppose that there existed another connected component W of $\pi^{-1}(0)$. Let $Z_1, \cdots, Z_l$ be the connected components of $Z^{\mathbb{C}^*} \cap W$.

Following [Wlo], we say that Z_i is an *immediate predecessor* of Z_j iff there exists a non-fixed point $x \in Z$ such that

$$\lim_{\lambda \to 0} \lambda \cdot x \in Z_i, \textit{ and } \lim_{\lambda \to \infty} \lambda \cdot x \in Z_j.$$

Since Z_i, Z_j are in $\pi^{-1}(0)$, such a point x is necessarily in $\pi^{-1}(0)$, and in fact in the connected component W of $\pi^{-1}(0)$. We say that Z_i *precedes* Z_j and write $Z_i \prec Z_j$ if there exists a sequence $Z_{k_0} = Z_i, Z_{k_1}, \cdots, Z_{k_s} = Z_j$ such that $Z_{k_{m-1}}$ is an immediate predecessor of Z_{k_m} for all m. Since Z is quasi-projective and smooth, the following lemma is a restatement of Lemma 1 in [Wlo], which is though elmentary but essential to the work of [Wlo].

Lemma 3.14 *The relation* $\prec$ *is an order, i.e. there does not exist any component* Z_j *such that* $Z_j \prec Z_j$.

The following corollary is clear from the above lemma.

Corollary 3.15 *There exists a component* Z_i *which is minimal with respect to the relation* $\prec$, *i.e. there does not exist any* Z_j *with* $Z_j \prec Z_i$.

Now to finish the proof of the main theorem, we show that the $\mathbb{C}^*$-action is also definite on Z_i. The key point here is to note that for any $z \in Z$, $\lim_{\lambda \to 0} \pi(\lambda \cdot z) = \lim_{\lambda \to 0} \lambda \pi(z) = 0$, so $\lambda \cdot z$ lies in a neighborhood of the compact variety $\pi^{-1}(0)$. This implies that for a point $x \in Z^{\mathbb{C}^*}$ the negatively weighted vectors $v \in T_x^p Z$, with $p < 0$ could only possibly come from the $\mathbb{C}^*$-action on $\pi^{-1}(0)$. Now Z_i is minimal with respect to the relation $\prec$, then there does not exist any $z \in Z$ such that $\lim_{\lambda \to \infty} \lambda \cdot z \in Z_i$, so there is no negatively weighted tangent vector on Z_i, i.e. the $\mathbb{C}^*$-action is definite on Z_i. As a corollary, the attraction subvariety of Z_i should contain an open set in Z, which is impossible, since the attraction subvariety of Z_0 is already an open-dense set. So $\pi^{-1}(0)$ is connected and equals to Z_0, which gives $V = Z$. The proof of our main theorem is thus completed. □

Remark 2 Here $\overline{\mathcal{O}}$ is not necessarily normal, so we could not use Zariski's main theorem to deduce that $\pi^{-1}(0)$ is connected.

3.3. Symplectic resolutions: classical cases

Let $\mathfrak{g}$ be a simple Lie algebra and G its adjoint group. Recall that a nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}$ is called *polarizable* if there exists a parabolic subalgebra $\mathfrak{p} \subseteq \mathfrak{g}$ such that $\mathfrak{u} \cap \mathcal{O}$ is dense in $\mathfrak{u}$, where $\mathfrak{u}$ is the nilradical of $\mathfrak{p}$. The Lie group P corresponding to $\mathfrak{p}$ is called a *polarization* of $\mathcal{O}$. In the literature, these orbits are also called *Richardson orbits*. For such an orbit, we have $dim(\mathcal{O}) = 2\, dim(G/P)$. As shown by Richardson, every parabolic subalgebra corresponds to a polarizable orbit. In the case of $\mathfrak{g} = \mathfrak{sl}_n$, every nilpotent orbit is polarizable. As a direct corollary of our main theorem, we have

Proposition 3.16 *If the normal variety $\widetilde{\mathcal{O}}$ admits a symplectic resolution, then $\mathcal{O}$ is a Richardson orbit.*

So in the following we will only consider Richardson orbits. For an element $X \in \mathcal{O}$, let G^X be the stabilizer of X in G, and $(G^X)^\circ$ the identity component. The component group $G^X/(G^X)^\circ$ is denoted by $A(\mathcal{O})$. Suppose that $\mathcal{O}$ is polarizable and let P be a polarization. Let P^X be the stabilizer of X in P. Then we have $(G^X)^\circ \subseteq P^X \subseteq G^X$. Put $A_P(\mathcal{O}) = P^X/(G^X)^\circ$, which is also the stabilizer of P in $A(\mathcal{O})$. Note that $A_P(\mathcal{O})$ is a subgroup of the abelian group $A(\mathcal{O})$. We denote by $N(P) = [A(\mathcal{O}) : A_P(\mathcal{O})]$ the index of $A_P(\mathcal{O})$ in $A(\mathcal{O})$. It turns out that this number is useful to our problem, as shown by the following:

Proposition 3.17 *Let $\mathcal{O}$ be a nilpotent orbit in a simple Lie algebra $\mathfrak{g}$. Then the symplectic variety $\widetilde{\mathcal{O}}$ admits a symplectic resolution if and only if there exists a polarization P of $\mathcal{O}$ such that $N(P) = 1$.*

Proof. By our main theorem, any symplectic resolution for $\overline{\mathcal{O}}$ is of the form

$$\pi : T^*(G/P) \simeq G \times^P \mathfrak{u} \to \overline{\mathcal{O}}, \quad \pi(g, X) = Ad(g)X,$$

for some parobolic subgroup P in G. In particular, we see that P gives a polarization of $\mathcal{O}$. As shown in [BK](§7), the map π is projective and of degree $N(P) = [G^X : P^X]$. So it is birational if and only if $N(P) = 1$, in which case it gives an isomorphism from $\pi^{-1}(\mathcal{O})$ to $\mathcal{O}$, thus we get a symplectic resolution for $\overline{\mathcal{O}}$, and also for $\widetilde{\mathcal{O}}$. □

Corollary 3.18 *The closure of any nilpotent orbit in $\mathfrak{sl}_n$ admits a symplectic resolution.*

This corollary comes directly from the proposition and the fact that $A(\mathcal{O}) = 1$ for any nilpotent orbit in $\mathfrak{sl}_n$ (see for example Corollary 6.1.6 of [C-M]). The following corollary follows from Corollary 1.3.

Corollary 3.19 *Consider a nilpotent orbit $\mathcal{O}$ in $\mathfrak{sl}_n$. If $\overline{\mathcal{O}} - \mathcal{O}$ is not of pure codimension 2, then $\overline{\mathcal{O}}$ is not locally $\mathbb{Q}$-factorial.*

In the following we will consider the classical cases $\mathfrak{g} = \mathfrak{so}_m$ and $\mathfrak{g} = \mathfrak{sp}_m$. The first question is to decide when a nilpotent orbit $\mathcal{O}$ is polarizable, and the second question is to decide when there exists a polarization P such that $N(P) = 1$. We will use notations and results of [Hes] to settle these two questions.

First, some notations. Here all congruences are modulo 2. Let $\varepsilon = 0$ for $\mathfrak{so}_m$ and $\varepsilon = 1$ for $\mathfrak{sp}_m$. A natural number $q \geq 0$ is called *admissible* if $q \equiv m$ and $q \neq 2$ if $\varepsilon = 0$. Set

$$Pai(m, q) = \{partitions\ \pi\ of\ m \mid \pi_j \equiv 1\ \ if\ \ j \leq q;\ \pi_j \equiv 0\ \ if\ \ j > q\},$$

which parametrizes some Levi types of parabolic subgroups in G. Let $\mathcal{P}_\varepsilon(m)$ be the partitional parameterizations of nilpotent orbits in $\mathfrak{g}$, i.e. the partitions $\mathbf{d}$ such that for any $i \equiv \varepsilon$, the number $\#\{j | d_j = i\}$ is even (see Sect. 5.1 in [C-M]). Then there exists an injective Spaltensein mapping

$$S_q : Pai(m, q) \to \mathcal{P}_\varepsilon(m).$$

As shown by Theorem 7.1.(a) in [Hes], a nilpotent orbit $\mathcal{O}$ is polarizable if and only if there exists some admissible q such that the partition of the orbit is in the image of S_q. If we denote by

$$J(d) = \{j | d_j \equiv \varepsilon\} \cup \{j, j+1 | j \equiv m, d_j = d_{j+1}\}$$
$$j_1(d) = sup\{j \in J(d) | d_j \equiv 1\}\ (-\infty\ \ if\ empty)$$
$$j_0(d) = min\{j \in J(d) | d_j \equiv 0\}\ (+\infty\ \ if\ empty),$$

then a partition $\mathbf{d}$ is in the image of S_q if and only if (see Prop. 6.5 [Hes]):

$$j_1(d) \leq q < j_0(d)\ \text{ and }\ d_j \equiv d_{j+1}\ \text{ if }\ j \equiv m+1.$$

Let $B(d) = \{j \in \mathbb{N} | d_j > d_{j+1}; d_j \equiv \varepsilon + 1\}$ and $u = \frac{1}{2}(-1)^\varepsilon(\#\{j | d_j \equiv 1\} - q)$, then we have

Proposition 3.20 (Theorem 7.1 [Hes]) *Suppose* $\mathbf{d}$ *is in the image of S_q. Let P be an associated parabolic subgroup. Then*

$$N(P) = \begin{cases} 2^u & if\ \ q + \varepsilon \geq 1\ \text{ or }\ B(d) = \emptyset; \\ 2^{u-1} & if\ \ q = \varepsilon = 0\ \text{ and }\ B(d) \neq \emptyset. \end{cases}$$

Now we will do a case-by-case check.

Proposition 3.21 *For $\mathfrak{g} = \mathfrak{sp}_{2n}$ (resp. $\mathfrak{g} = \mathfrak{so}_{2n+1}$), let $\mathcal{O}$ be a nilpotent orbit in $\mathfrak{g}$ corresponding to the partition $\mathbf{d} = [d_1, \cdots, d_N]$. Then the following three conditions are equivalent:*

(i) $\mathcal{O}$ is polarizable and there exists a polarization P such that $N(P) = 1$;
(ii) there exists an even (resp. odd) number $q \geq 0$ such that the first q parts $d_1, \cdots, d_q$ are odd and the other parts are even;
(iii) the normal variety $\widetilde{\mathcal{O}}$ has a symplectic resolution.

Proof. The equivalence between (i) and (iii) has been established earlier. In the following, we will prove the equivalence between conditions (i) and (ii).

For the case $\mathfrak{g} = \mathfrak{sp}_{2n}$, we have $\varepsilon = 1$, so $N(P) = 2^u$, which gives that $N(P) = 1$ if and only if $u = 0$, i.e. if and only if $q = \#\{j|d_j \equiv 1\}$. Now the condition (i) is equivalent to saying that the partition $\mathbf{d}$ is in the image of $S_q : Pai(2n, q) \to \mathcal{P}_1(2n)$. This condition is equivalent to having $j_1(d) \leq q < j_0(d)$ and $d_j \equiv d_{j+1}$ if $j \equiv 1$. Note that here the set $J(d)$ contains the set $\{j|d_j \equiv 1\}$, thus $j_1(d) = sup\{j|d_j \equiv 1\}$. Now $j_1(d) \leq q = \#\{j|d_j \equiv 1\}$ implies that the first q parts of $\mathbf{d}$ are odd, and the others are even. In this case, the conditions $q < j_0(d)$ and $d_j \equiv d_{j+1}$ if $j \equiv 1$ are satisfied automatically. So we have the equivalence of the conditions (i) and (ii).

In the case of $\mathfrak{g} = \mathfrak{so}_{2n+1}$, since q is admissible, $q \equiv 1$, thus $q \geq 1$, which gives $N(P) = 2^u$. So $N(P) = 1$ if and only if $u = 0$, i.e. $q = \#\{j|d_j \equiv 1\}$. Now the set $\{j|d_j \equiv 0\}$ is contained in $J(d)$, thus $j_0(d) = min\{j|d_j \equiv 0\}$. The condition $j_0(d) > q$ gives that the first q parts should be odd, and the others are even. If this is satisfied, then the orbit is polarizable by some P with $N(P) = 1$. So we have the equivalence between (i) and (ii). □

Examples: Consider $\mathfrak{so}_7$, the variety $\widetilde{\mathcal{O}}_{[3,2,2]}$ has a symplectic resolution. So in the case of $\mathfrak{so}_7$, only $\overline{\mathcal{O}}_{min}$ does not admit any symplectic resolution. For $\mathfrak{sp}_6$, we see that the variety $\widetilde{\mathcal{O}}_{[4,1,1]}$ does not admit any symplectic resolution.

Proposition 3.22 *For $\mathfrak{g} = \mathfrak{so}_{2n}$, let $\mathcal{O}$ be a nilpotent orbit corresponding to the partition $\mathbf{d} = [d_1, \cdots, d_N]$. Then the following three conditions are equivalent:*

(i) $\mathcal{O}$ is polarizable and there exists a polarization P such that $N(P) = 1$;
(ii) either there exists some even number $q \neq 2$ such that the first q parts of $\mathbf{d}$ are odd and the others are even or there exists exactly 2 odd parts which are at the positions $2k - 1$ and $2k$ in the partition $\mathbf{d}$ for some k;
(iii) the normal variety $\widetilde{\mathcal{O}}$ has a symplectic resolution.

Proof. We need to establish the equivalence between (i) and (ii). There are three cases:

Case (a): $q = \varepsilon = 0$ and $B(d) \neq \emptyset$.
In this case we should have $u = 1$, then $\#\{j | d_j \equiv 1\} = 2$, i.e. there are exactly 2 odd parts. The polarizable condition $d_j \equiv d_{j+1}$ if $j \equiv 1$ gives that the two parts are at the positions $2k - 1$ and $2k$ in **d** for some k. In this case, the other conditions as $B(d) \neq \emptyset$ and $j_1(d) \leq 0 < j_0(d)$ are satisfied.

Case (b): $B(d) = \emptyset$
In this case we find that there is no odd part in **d**.

Case (c): $q \geq 2$ and $q \equiv 0$.
Since q should be admissible, thus $q \neq 2$, i.e. $q \geq 4$ even. Now $N(P) = 1$ gives $u = 0$, i.e. $q = \#\{j | d_j \equiv 1\}$. Now the set $\{j | d_j \equiv 0\}$ is contained in $J(d)$, so $j_0(d) > q$ gives that the first q parts of the partition **d** should be odd, and thus the other parts should be even. If this is satisfied, then the other conditions are also satisfied.

The above analysis for the three cases gives the equivalence between the conditions (i) and (ii). □

Remark 3 By Theorem 7.1.(d) of [Hes], it is also possible to determine the number of non-conjugate polarizations P for a Richardson orbit $\mathcal{O}$ such that $N(P) = 1$, i.e. we could give the number of different symplectic resolutions for a Richardson orbit. Interested readers are encouraged to provide such a list.

3.4. Symplectic resolutions: exceptional cases

Firstly we have the following

Proposition 3.23 *Let $\mathfrak{g}$ be one of the following exceptional simple complex Lie algebras: G_2, F_4, E_6, and $\mathcal{O}$ a nilpotent orbit in $\mathfrak{g}$. Then the normal variety $\widetilde{\mathcal{O}}$ admits a symplectic resolution if and only if $\mathcal{O}$ is a Richardson orbit.*

Proof. Note that an even orbit is always a Richardson orbit, and it admits a symplectic resolution by Springer's resolution. Among the non-even orbits, a list of Richardson orbits in exceptional simple Lie algebra $\mathfrak{g}$ can be found in [Hir]. Here we use notations of [C-M] for nilpotent orbits in exceptional algebras. When $\mathfrak{g} = G_2$, every Richardson orbit is even. When $\mathfrak{g} = F_4$, there is only one non-even Richardson orbit: C_3. In the case of $\mathfrak{g} = E_6$, there are 5 non-even Richardson orbits: $2A_1$, $A_2{+}2A_1$, A_3, $A_4{+}A_1$, $D_5(a_1)$. Now the table given in [C-M] (Chap. 8) shows that all these orbits are simply connected, thus $A(\mathcal{O}) = 1$, so for any polarization P for $\mathcal{O}$, the collapsing $T^*(G/P) \to \overline{\mathcal{O}}$ gives a symplectic resolution. □

In the case of $\mathfrak{g} = E_7$, there are 5 non-even Richardson orbits, two of which have trivial component groups (thus admit a symplectic resolution):

$D_5 + A_1$ and $D_6(a_1)$; while for the other three ($D_4(a_1) + A_1$, $A_4 + A_1$ and $D_5(a_1)$) we do not know, whose component groups are S_2.

In the case of $\mathfrak{g} = E_8$, there are 7 non-even Richardson orbits, three of which have trivial component groups (thus admit a symplectic resolution): $A_4 + A_2 + A_1$, $A_6 + A_1$ and $E_7(a_1)$; while for the other four ($D_6(a_1)$, $D_7(a_2)$, $E_6(a_1)+A_1$ and $E_7(a_3)$), whose component groups are S_2, we do not know the answer.

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