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LETTER TO THE EDITOR

The power law as behavioral illusion: reappraising the reappraisals

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Abstract

Marken and Shaffer (Exp Brain Res 235:1835–1842, 2017) have argued that the power law of movement, which is generally thought to reflect the mechanisms that produce movement, is actually an example of what Powers (Psychol Rev 85:417–435, 1978) dubbed a behavioral illusion, where an observed relationship between variables is seen as revealing something about the mechanisms that produce a behavior when, in fact, it does not. Zago et al. (Exp Brain Res. https://doi.org/10.1007/s00221-017-5108-z, 2017) and Taylor (Exp Brain Res, https://doi.org/10.1007/s00221-018-5192-8, 2018) have "reappraised" this argument, claiming that it is based on logical, mathematical, statistical and theoretical errors. In the present paper we answer these claims and show that the power law of movement is, indeed, an example of a behavioral illusion. However, we also explain how this apparently negative finding can point the study of movement in a new and more productive direction, with research aimed at understanding movement in terms of its purposes rather than its causes.

Keywords Power law of movement · Cause–effect model · Control model · Behavioral illusion · Controlled variables

It is generally assumed that the power law of movement—the approximately 1/3 or 2/3 power relationship between curvature and velocity that is consistently found for voluntarily produced movements—reveals something about the mechanisms that underlie movement production (e.g., Zago et al. 2016). In a recent paper (Marken and Shaffer 2017), we showed that this assumption is likely to be based on what Powers (1978) called a behavioral illusion, where an observed relationship between variables is seen as revealing something about the mechanisms that produce the observed behavior when, in fact, it does not. Not surprisingly, the appearance of our paper led to considerable consternation inside (and outside) the power law research community, resulting in two published "reappraisals"; one by Zago et al. (2017, in the following abbreviated as Z/M) and the other by Taylor (2018; in the following abbreviated as Taylor). Both "reappraisals" rejected our conclusion, claiming that it is based on logical, mathematical, statistical and theoretical errors. In this paper, we answer each of these claims and show how our apparently negative findings can point to more useful directions for research on the mechanisms underlying the production of movement.

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The facts of logic

We began our paper by pointing out that voluntary movements are controlled rather than caused results of the mechanisms—specifically, the muscle forces—that produce them. We made this point by noting that "… different muscle forces are required to produce the same movement trajectory on different occasions due to variations in the circumstances that exist each time the movement is produced" (Marken and Shaffer 2017, p. 1836), which is the definition of movement as a controlled result of action (Marken 1988). Based on this observation, we concluded that the power law could not possibly reveal the mechanisms that produce movements because there is no direct causal path from mechanism (muscle forces) to movement.

Z/M agree that "muscle forces will not be consistently related to the curvature and velocity of the movement" (p. 13). But they assert that this fact does not imply that the power law cannot reveal the mechanisms of movement and they conclude that our "inference" that it does "…is logically a non sequitur, given the complex relationship between

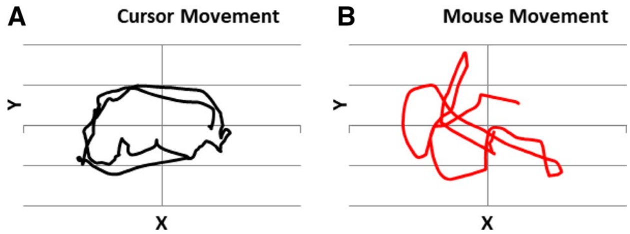

Fig. 1 Movement trajectories of the cursor (**a**) and mouse (**b**) made during a typical trial of a voluntary movement task where the subject was asked to move the cursor in an elliptical trajectory

muscle forces and movement kinematics" (p. 13). Z/M do not explain how the "complex relationships" to which they refer make our inference a logical non sequitur, but we can explain the logic behind our inference. We will do this using data from a simple voluntary movement task where a person is asked to use a mouse to make elliptical cursor movements on a computer display screen. At each instant during a trial the position of the cursor is determined by the sum of two variables: the position of the mouse and the value of a low-pass filtered noise disturbance generated by the computer.

The results of a typical 30-s test trial are shown in Fig. 1. Figure 1a shows that the cursor is moved in an approximately elliptical trajectory. The coefficients of the power relationship between radius of curvature (R) and tangential velocity (V) and between curvature (C) and angular velocity (A) for this trajectory are 0.3 and 0.7, respectively. These values are very close to the 1/3 and 2/3 values that are typically found in power law studies. The fit of the power relationship to the cursor trajectory is also close to what is found in power law studies; the R^2 value for the power relationships between R and V and between C and A are 0.56 and 0.87, respectively.

Figure 1b shows the mouse movements that were used to produce the elliptical cursor movements shown in Fig. 1a. Due to the effect of the disturbance on the position of the cursor, the trajectory of the mouse movements was very different than the resulting elliptical trajectory of the cursor movements. The coefficients of the power relationship between R and V and between C and A for this mouse movement trajectory were 0.05 and 0.98, respectively. These values are not close to the 1/3 and 2/3 values that are typically found in power law studies. Moreover, the fit of the power relationship to the mouse trajectory is quite poor; the R^2 value for the power relationships between R and V and between C and A are 0.01 and 0.68, respectively.

In this study, mouse movements represent "…the different muscle forces that are required to produce the same movement trajectory on different occasions" (Marken and Shaffer 2017, p. 1836); the time-varying disturbance represents the "…variations in the circumstances that exist each time a movement is produced" (p. 1836); and the cursor movements are the voluntary movements that follow the power law. The results of this study show that, contrary to Z/M, the mechanisms that produce voluntary movements are not revealed in the power law that characterizes the movement trajectory. This can be seen in the fact that the power relationship between curvature and velocity for the mechanism (mouse movements) that produces the movement trajectory (cursor movements) differs considerably from that for the voluntarily produced movement trajectory itself. It can also be seen in the fact that the correlation between mouse and cursor movements is only 0.53, meaning that only about 28% of the variance in the voluntary cursor movement trajectory is accounted for by the mechanism (mouse movement) that produces it.

Z/M claim that we are making a logical error by saying that the power law cannot reveal the mechanisms of movement. We believe that this study shows that Z/M are making a factual error by saying that it can.

A simple twist of math

Both Z/M and Taylor claim that we made a mathematical error by showing that the measures of movement velocity and curvature used in power law studies are mathematically related. Specifically, we noted that:

$$V = R^{1/3} \cdot D^{1/3} \tag{1}$$

and

$$A = C^{2/3} \cdot D^{1/3}, \tag{2}$$

where D is what we called the "cross-product" variable and is equal to $|\dot{X} \cdot \ddot{Y} - \ddot{X} \cdot \dot{Y}|$, the cross-product of the first and second derivatives of movement in the X and Y dimensions.[1] We derived Eqs. 1 and 2 because we thought that a mathematical relationship between measures of the curvature and speed of movement might explain the consistent finding of the approximately 1/3 or 2/3 power law relationship between these variables despite the lack of a consistent causal connection to the mechanisms that produce this relationship.

According to Z/M our mathematical error was a failure to take into account the "…simultaneous dependence of D on both speed and curvature" (p. 7). Based on this they conclude that the equation relating speed (measured as A), curvature (measured as C) and D, shown in Eq. 2 above, "…represents a simple mathematical identity and does not imply that A depends on two independent variables, D and C" (p. 7). Taylor, using V and R as the measures of speed and curvature, also says that our error was a failure to properly

[1] We now understand D to be a measure of affine velocity (Maoz et al. 2006; Pollick and Sapiro 1997), but we will continue to refer to D as the "cross-product" variable when we are discussing the analysis in Marken and Shaffer (2017).

analyze the role of the cross-product variable, D, in our analysis. Taylor says that this happened because we misunderstood the equations for computing V and R that are given in Gribble and Ostry (1996).

We take these criticisms to mean that both Z/M and Taylor believe that Eqs. 1 and 2 are either misleading or incorrect. If this were true it would certainly undermine our analysis, which is based on the observation that the linear equations relating curvature to speed of movement are:

$$\log(V) = 1/3 \cdot \log(R) + 1/3 \cdot \log(D) \quad (3)$$

and

$$\log(A) = 2/3 \cdot \log(C) + 1/3 \cdot \log(D), \quad (4)$$

which imply that velocity is an exact mathematical function of both curvature and the variable D for any curved movement trajectory. Moreover, Eqs. 3 and 4 imply that a regression analysis that omitted the variable D from the analysis (as is typically done in power law research) would result in a biased estimate of the true coefficient of the power relationship between speed and curvature, the size of the bias dependent on the correlation between measures of curvature and the variable D. So if our derivation of Eqs. 3 and 4 is correct, power law researchers have discovered a "law" that is "forced" by their method of analysis. If, however, our derivation is wrong, then our analysis of the power law in Marken and Shaffer (2017) is either misleading (per Z/M) or incorrect (per Taylor). However, we will show that our mathematical derivation of Eqs. 3 and 4 is correct and, thus, the analysis in our paper is neither misleading nor wrong.

First, we note that the analysis in Marken and Shaffer (2017) is not misleading because, contrary to the criticisms of Z/M, regression analysis does not assume that the criterion variable (speed) "depends on two independent variables". The regression analysis takes into account any correlation that might exist between the predictor variables when solving for the coefficients that give the best fit to the speed measurements. So the fact that the cross-product variable is functionally related to curvature does not give a misleading impression of the meaning of Eqs. 1 and 2 or of the linear equations (Eqs. 3 and 4) derived from them.

And second, our analysis is not wrong because, contrary to Taylor, we did not misunderstand the equations for computing V and R that are given in Gribble and Ostry (1996). Our purported misunderstanding was taking $\dot{X}$ and $\dot{Y}$ to be the same variables in the Gribble and Ostry equations for both V and R. According to Taylor, it should have been obvious to us that $\dot{X}$ and $\dot{Y}$ are actually different in these two equations (even though they were notated the same) because, in the equation for V, $\dot{X}$ and $\dot{Y}$ "…are values observed in an experiment and are used to compute the corresponding velocity" (p. 5), whereas in the equation for R, $\dot{X}$ and $\dot{Y}$ "…are arbitrary parameters, corresponding to any velocity " (p. 5). In fact, $\dot{X}$ and $\dot{Y}$ are not parameters in the equation for V or R but, rather, are values calculated from the data to get the measures of both V and R.

Taylor also argues that our analysis is wrong because it should have been obvious that $\dot{X}$ and $\dot{Y}$ are derivatives with respect to time in the expression for V, whereas they are derivatives with respect to space in the expression for R (p. 5). In fact, the Gribble and Ostry equations are derived from those given in Viviani and Stucchi (1992), where the denominators of the equations for both V and R are expressed in terms of derivatives of a spatial variable, φ, (Viviani and Stucchi 1992, Appendix Eqs. A5 and A6, respectively). Gribble and Ostry transformed the Viviani and Stucchi equations so that all derivatives that had been with respect to φ became derivatives with respect to time. So we were correct to treat $\dot{X}$ and $\dot{Y}$ as being the same in the Gribble and Ostry equations for both V and R.

Further evidence that our mathematical derivation of Eqs. 3 and 4 is correct is the fact that the same derivation has been used by other power law researchers, specifically Maoz et al. (2006) and Pollick and Sapiro (1997), whose work was referenced favorably by Z/M in their "reappraisal" of our analysis. In particular, we find in Maoz et al. the following equation:

$$V = \kappa^{-1/3} \cdot \alpha^{1/3}, \quad (5)$$

which is equivalent to our Eq. 1 but with the curvature variable, κ, equal to 1/R, so that the exponent of κ is − 1/3 rather than 1/3. The variable α is the same as our variable D; it is called α because what we had called the "cross-product variable" is actually a measure of affine velocity. Maoz et al. go on to note that Eq. 5 can be linearized as

$$\log(V) = -1/3 \cdot \log(\kappa) + 1/3 \cdot \log(\alpha), \quad (6)$$

which is equivalent to our Eq. 3. We take the fact that Z/M found no fault with the Maoz et al. analysis that led to Eq. 6 as an implicit recognition of the correctness of our Eqs. 3 and 4 and of Eqs. 1 and 2 from which they are derived.

Tangled up in statistics

The results of our mathematical analysis led us to infer that the consistent finding of a power law might be the result of the failure to include what we called the cross-product variable (D) in the regression analysis that is typically used to determine whether a movement trajectory follows the power law. The typical approach to determining whether the relationship between curvature and speed is fit by a power law is to do a regression analysis with the log of the measures of curvature as the predictor variable and the log of the measures of speed as the criterion variable. For example, the regression equation used to evaluate the fit of a 2/3 power function to the relationship between curvature measured as C and velocity measured as A is:

$$\log(A) = k + \beta \cdot \log(C). \tag{7}$$

However, our analysis suggests that the appropriate regression equation for evaluating the relationship between C and A should include the cross-product variable D:

$$\log(A) = k + \beta_1 \cdot \log(C) + \beta_2 \cdot \log(D). \tag{8}$$

We pointed out that if C alone is used as a predictor variable then β will be a biased estimate of the true value of the coefficient, β_1, of log (C) in Eq. 8; according to Eq. 4 the true value of β_1 is 2/3. The bias results from omitting log (D) from the regression analysis and the amount of bias is given by a statistical technique called omitted variable bias (OVB) analysis (Wooldridge 2009).

Both Z/M and Taylor are very critical of our use of OVB analysis, but their criticisms reflect a lack of understanding of some fundamentals of both OVB and multiple regression analysis. For example, Taylor says that we used OVB analysis "…to "correct" the power relation between V and R by adding a value ∂ found using the statistical analysis" (p. 7). Actually, we used OVB analysis to predict the degree to which the power law coefficient, β, found by regressing log R on log V deviates from the "true" (mathematically derived) value. The "true" value of β is 1/3 if Eq. 1 describes the correct mathematical relationship between R and V. The variable Taylor calls ∂ is the one we called δ and it is not a "correction" factor but, rather, a measure of how much the observed value of β is expected to deviate from the true value, 1/3, in a regression analysis that omits the variable D. The value of δ is calculated from the trajectory data—specifically, what is calculated is the covariance between the curvature and the cross-product measures—and it was found to predict the deviation of observed from the "true" value of β exactly (Marken and Shaffer, 2017, p. 1841).

Z/M claim that our application of OVB analysis is "…ill-grounded, since OVB applies to linear regressions of a dependent variable on one or more independent variables, and here C is an independent variable but D is not."(p. 12) But, as we mentioned earlier, multiple regression analysis does not depend on the predictor variables being "independent", either in the sense of their being uncorrelated with each other or in the sense of their being the variables manipulated in an experiment.

Z/M go on to claim that our OVB analysis is useless for learning about the physiological underpinnings of the relationship between speed and curvature because it is based on equations "…that are a mathematical identity that must always be satisfied" (p. 12). Taylor seems to agree when he states that "the "cross-product" correction [in the OVB analysis] is exactly enough to remove the observed effect of R on V, leaving only the tautology log (V) = log (V)" (p. 7).

Despite the alleged uselessness of OVB analysis, Z/M allow that such an analysis would be useful in the study of noise effects on the power law and they point to the work of Maoz et al. (2006) as an example. Maoz et al. used OVB analysis to estimate the average amount by which the regression-based estimate of the power coefficient, β, relating velocity (measured as V) to curvature (measured as $\kappa = 1/R$) would deviate from the "true" value of − 1/3 (per Eq. 6 above) when the variable α (affine velocity, which corresponds to our cross-product variable, D) is omitted from the analysis. They did this analysis on randomly generated movement trajectories and found that the average deviation of the regression-based estimate of β from − 1/3 was quite small, only 0.05. Since, per OVB analysis, the size of the deviation of β from − 1/3 depends on the size of the correlation between log(κ) and log(α), the small average deviation from − 1/3 means that this correlation is typically close to zero for arbitrarily produced movement trajectories.

Z/M say that the message of the Maoz et al. (2006) study is the opposite of ours, their message being that empirical speed–curvature power laws are not mathematical/statistical artifacts but, rather "…real and require a critical investigation of the properties of D to account for compliance or deviation of empirical β values relative to the prototypical 2/3 value found in elliptic drawings" (p. 12). But this is the message of Moaz et al. only if one assumes that the "true" value of the power coefficient (− 1/3 for Moaz et al.) is the one that results from the physiological processes that produce the movement and that the variance in the affine velocity variable, α (or, equivalently, D), that results in compliance or deviation from that value is the result of "noise" factors. This amounts to assuming that one's theory of the cause of the power law is correct and any deviation of data from the theory is the fault of the data. In fact, when the results of the Moaz et al. study are interpreted correctly we see the message of their and other similar studies (e.g., Pollick, and Sapiro 1997) as being perfectly consistent with ours, which is that the power law coefficient that is found using a regression analysis that omits the cross-product (or affine velocity) variable depends on characteristics of the trajectory itself and says nothing about the mechanisms that produced those trajectories.

An error, in theory

At the heart of the criticisms of our paper by Z/M and Taylor is the assumption that the power law is a result of a direct causal connection between curvature and speed of movement or between these variables and the physiological mechanisms that produce them. That is, it is assumed that the power law is a reflection of what Powers (1978) called the "general cause–effect model of behavior" (p. 423). This assumption is made explicit in the statement by Z/M that the power law involves a "…causal relationship between curvature and speed..." (p. 13). Since the main conclusion of our paper was that these apparent causal relationships are misleading—indeed, an illusion—it is not surprising that the main thrust of the responses to our paper have been to show that this is not the case. This can be seen in Taylor's efforts to show that measures of curvature and speed are "independent" in the sense that measures of speed (*V*) cannot be correctly represented as a mathematical function of measures of curvature (*C*). Presumably this would show that curvature and speed can be causally related. While it is not clear to us how the mathematical independence of these variables would justify this conclusion, the question is moot since, as we have shown above (Eqs. 5 and 6), there is a mathematical dependence between measures of speed and curvature and this fact is well known to power law researchers.

Z/M take several different approaches to showing that the power law is a result of causal processes. They start by noting that "Empirical speed-curvature power laws for human drawing have different exponents" (p. 3). Z/M replicate the Huh and Sejnowski (2015) finding that different movement trajectories can result in quite different power law exponents and conclude that this reveals differences in the physiological mechanisms that cause these different movements. However, the difference in exponents for different trajectories can also be explained by the difference in the mathematical properties of the trajectories themselves. Indeed, this has already been pointed out by Moaz et al. (2006) and Pollick and Sapiro (1997), who note that the observed exponent of the power law for a particular movement trajectory will deviate from the "ideal" or "prototype" value (1/3 or 2/3) depending on the degree to which measures of affine velocity (the variable D in Eqs. 3 and 4) are correlated with measures of curvature. That is, variations in the exponent of the power law for different movement trajectories depend on the nature of the trajectory itself and not on how it was produced.

Z/M go on to argue that "Empirical power laws do not depend on how curvature is computed" (p. 5). The goal was to show that a power law relationship between curvature (*C*) and speed (*A*) is not a consequence of the mathematical relationship between *C* and *A* shown in Eq. 4. We derived that relationship under the assumption that C is computed as:

$$\mathrm{C} = \left|\dot{X}\cdot\ddot{Y} - \ddot{X}\cdot\dot{Y}\right| / (\dot{X}^2 + \dot{Y}^2)^{3/2}. \tag{9}$$

If a power law relationship between *C* and *A* is found using a different computational formula for *C* it would presumably rule out the possibility that the power law is a statistical artifact of the mathematical relationship between *C* and *A* that is based on computing *C* per Eq. 9.

Z/M computed *C* using two formulas other than Eq. 9 and found that the estimates of the power law coefficient, β, using these values were the same as when *C* was computed using Eq. 9. We have confirmed this result with one of their alternative measures of *C*:

$$C' = \left|\frac{\mathrm{d}}{\mathrm{d}t}\arctan(\dot{Y}/\dot{X})/(\dot{X}^2 + \dot{Y}^2)^{1/2}\right|. \tag{10}$$

This result does seem to show that the power law is not a consequence of the mathematical relationship between curvature and velocity shown in Eq. 4. However, we found that Eq. 4 still holds when *C* is computed according to Eq. 10. That is, when we do a log—log regression analysis predicting speed (*A*) as a function of curvature measured as C' and the cross-product variable *D*, we found that the coefficient of C' (β_1) is exactly 2/3 and the coefficient of the cross-product variable (β_2) is exactly 1/3, as predicted by Eq. 4; and the R^2 value for the regression is 1.0. We take this to mean that a log–log regression of an appropriate measure of curvature (*C*) and affine velocity (the cross-product variable *D*) as the predictor variables will account for all the variance in the speed (*A*) of movement at each instant in the trajectory.

The above results are explained by the fact that when both the alternative measure of curvature and the cross-product variable are included in a regression, the analysis "partials out" the component of the variance in the cross-product variable that is correlated with both speed and curvature and "reallocates" it to curvature, resulting in the "true" estimate of the coefficient of the curvature variable, 2/3. Because of this, OVB analysis exactly predicts how much the regression-based estimate of the power coefficient of curvature, measured as C' (per Eq. 10), will differ from the "true value (2/3, per Eq. 4), when the cross-product variable, *D*, is left out of the regression analysis. So the use of alternative measures of curvature to determine the power law does not nullify our conclusion that a regression analysis that omits measures of affine velocity (the variable *D*) from the analysis forces a biased estimate of the true (mathematical) coefficient (1/3 or 2/3) of the power function that relates curvature to the velocity of a movement trajectory.

Z/M then argue that "The power law is not obligatory in physical systems." They note, for example, that, unlike the movement trajectories produced by biological systems, the movement trajectories of some physical systems, such as the orbits of the planets, do not follow the 1/3 or 2/3 power law, a fact that we confirmed for ourselves. However, we found that, while the power law is not obligatory in physical systems, it is found for many of the trajectories that are produced by such systems. For example, we found a good fit of the power law to the trajectories produced by the toy helicopters in the experiment conducted by Shaffer et al. (2013) and analyzed in Marken and Shaffer (2017). We also found a good fit of the power law to the movement trajectories of Frisbees thrown to dogs (Shaffer et al. 2004) and of footballs thrown to oneself (Shaffer et al. 2015).

We also found that the power law is no more obligatory for biological systems than it is for physical systems, as can be seen in Fig. 1b. Here the trajectory of the mouse movements that were produced by a biological system—the person making the cursor movements in Fig. 1a—was not obligated to follow the power law. As noted above, the power coefficients for the mouse movement trajectory were 0.05 and 0.98, rather than the "prototypical" values of 0.33 and 0.67, respectively. Moreover, we have shown (Marken and Shaffer 2017, p. 1836) that whether a trajectory is produced by a biological or physical system, a regression analysis based on Eqs. 3 or 4 always finds that the "true" power coefficient for the relationship between the speed and curvature of the movement trajectory is 1/3 (for R versus V) or 2/3 (for C versus A), with an R^2 value of 1.0 in both cases. Again, this result shows that the value of the power coefficient relating speed to curvature depends on the nature of the trajectory itself and not on how it was produced.

Finally, Z/M argue that "Different biological constraints may give rise to the speed curvature power law" (p. 9). The point here is that the power law is "non-trivial" (in the sense that it is not a statistical artifact) because it reflects the physiological constraints that underlie the production of these movements. Z/M make this point by comparing the movements made by biological systems to those made by orthogonal harmonic oscillators. They note that such oscillators will produce elliptical movement trajectories that precisely follow the power law even when they are "… generated by coupled angular motions at the limb joints" (p. 9). They point to research that suggests that these non-orthogonal limb movements can still produce these elliptical trajectories because the limb segments are moved "…with appropriate inter-segmental phase shifts" (p. 9). Z/M imply that these "appropriate phase shafts" are the result of "physiological constraints" that are revealed by analyzing the relationship between the speed and curvature of the resulting movement.

We had some difficulty imagining the kind of physiological constraints that could be clever enough to know what phase shifts would be "appropriate" to compensate for the varying non-orthogonal effects of limb coupling at every instant during the production of a movement. Therefore, we suggest that it is more likely that these phase shifts are the result of muscle forces that are part of a negative-feedback control loop, as in the control model of arm movement developed by Powers (2008, pp. 134–144). And, like the mouse movements shown in Fig. 1, the muscle forces that compensate for the varying non-orthogonal effects of limb coupling are not revealed by analyzing the movements that these forces produce.

Power law as behavioral illusion

As noted above, our fundamental theoretical error, according to Z/M and Taylor, was the failure to see that movement production is a cause–effect process. This can be seen in the fact that their criticisms of our paper are all aimed at showing that the speed of a movement depends on (is caused by) its curvature and/or that the relationship between the speed and curvature of a movement depends on a third variable, such as "physiological constraints. But we believe it is Z/M and Taylor who are making the theoretical error—the error of seeing voluntary movement as a cause–effect rather than a control process. As a result, we see evidence that in their criticisms of our paper Z/M and Taylor are succumbing to the "behavioral illusion", the very phenomenon that is demonstrated by our analysis of the power law in the paper that they criticize.

As noted above, the behavioral illusion occurs when an observed relationship between variables is seen as revealing something about the mechanisms that produce a behavior when, in fact, it does not. For example, the behavioral illusion occurs when "reinforcement" is seen as "selecting" the behavior that produced it (Marken and Powers 1989; Yin 2013, pp. 342–343) or when a tap on the patellar tendon is seen as the cause of the knee-jerk response (Marken 2014b, p. 123). The illusion occurs when the behavior under study is assumed to be that of an open-loop, cause–effect system when it is actually that of a closed-loop control system (Powers 1978).

Our analysis shows that the power law of movement is just such an illusion. The power law is seen as revealing something about the mechanisms that produce movement trajectories when these trajectories are seen as the outputs of an open-loop, cause effect system. We have argued, however, that these trajectories are variables controlled by a closed-loop control system, a fact demonstrated by the results of the movement trajectory experiment shown in Fig. 1. The power law relationship between the speed and curvature of these trajectories, like the one in Fig. 1a, is a consequence of the mathematical relationship between these variables (Eqs. 1 and 2 above) and reveals nothing about the mechanisms, like the mouse movements in Fig. 1b, that produce it.

Taylor expresses some doubt about whether or not we have shown the power law to be an example of a behavioral illusion. But he asserts that "Whether it is or not, and whether their analysis is correct or not, the research questions would have been unaffected" (p. 12). Presumably one of the main research questions alluded to is "…the issue that has been the object of so much research: why does the observed velocity of movement of a living organism along a curve so often approximate a power function of the local radius of curvature, and under what conditions does the power vary over such a wide range?" (p. 6). This statement actually contains two questions, both of which are "affected"—indeed, answered—if the power law is an example of a behavioral illusion.

The answer to the first question is that velocity always approximates a 1/3 or 2/3 power function of curvature because measures of velocity are mathematically a power function of measures of curvature, per Eqs. 1 and 2. The closeness of the approximation depends on the covariance between measures of curvature and affine velocity. The answer to the second question is that the range of variation in the power coefficient for movements made by living organisms is not actually that wide (e.g., Maoz et al. 2006). But to the extent that there is variation in the power coefficient, the reason for it is the difference in the relationship between affine velocity and curvature in different trajectories. This answer is based on our OVB analysis of the power law which shows that a power coefficient that is determined by a regression analysis that omits measures of affine velocity as a predictor variable from the analysis will deviate from the "prototype" value (1/3 or 2/3) by an amount that is proportional to the covariance between the curvature and affine velocity of the observed trajectory.

Doing research on purpose

Our aim in Marken and Shaffer (2017) was not to bury power law research but to point it in a new and more productive direction; one based on an understanding of behavior as a control rather than a cause–effect process. This new approach to research is aimed at finding the purposes rather than the causes of behavior (Marken 2014b). When behavior is viewed as a control process, its purpose can be seen as maintaining perceptual inputs in goal or reference states that are specified by the behaving system itself. The perceptual inputs that the behaving system is maintaining in reference states are called controlled variables and the main goal of research when behavior is viewed as a control process is to determine what perceptual input variables the system is controlling. This general approach to the study of the behavior of living control systems is called the *test for the controlled variable* (Powers 1978, p. 432) or simply the test.

There are various ways to do the test, one of which involves inserting into computer models different hypotheses about the perceptual input variables that are under control when organisms are performing various behaviors to see which of these hypotheses results in model behavior that is most like that of the system under study (Marken 2014a). This was the approach we took in our study of the variables a person controls when intercepting a toy helicopter (Shaffer et al. 2013). We were able to show that one of the variables controlled when a person moves to intercept the helicopter is the velocity rather than the commonly assumed acceleration of the vertical motion of the image of the helicopter. Once we had identified the inputs controlled when intercepting toy helicopters we were able to use the model to account for the object interception behavior that is seen under many different circumstances (e.g., Shaffer et al. 2015).

In our paper (Marken and Shaffer 2017), we reported that we had found that the model-generated pursuit movements from the Shaffer et al. (2013) study "…followed a power law with an exponent equivalent to that found in other studies of similarly curved movement trajectories" (p. 1843). We noted that the model produced this result "without any attempt to produce trajectories that followed a power law" (p. 1843). Nevertheless, Taylor finds that this model "…contains and implies nothing that would explain why the movements of either people or toy helicopters conform to the power law" (p. 8). It seems to us that it does explain why these movements conform to a power law; it is because the movements that are made in order to intercept objects follow trajectories where the correlation between curvature and the affine component of velocity happens to be quite low. Thus the regression used to determine the fit of the data to a power law will find a power coefficient close to the "prototype" value of 1/3 or 2/3.

However, Taylor's complaint may refer to the fact that our model of object interception did not explain how the movement trajectories themselves were produced; it did not explain the "mechanism" that produces a movement trajectory that follows the power law. So in order to show that a control model of the sort we described in Marken and Shaffer (2017) can explain why movements conform to a power law, we developed a simple control model that produces cursor movement trajectories like that in Fig. 1a—movements that conform to the power law—by the mechanism of varying mouse movements as shown in Fig. 1b.

The model explains the mechanism by which the power law-conforming cursor trajectories are produced to the extent that the model-produced mouse movements match the actual mouse movements that were used to produce these trajectories. And the model-produced mouse movements match the actual mouse movement quite well; the correlation between model and actual mouse movements (Fig. 1b) was 0.998. On average, the correlation between the model and actual mouse movements that produced power law-conforming cursor trajectories was 0.995.

We concluded our paper on the power law by recommending that future research on movement production be focused on understanding movement behavior by "…determining the nature of the perceptual variables that are being controlled (being maintained in specified reference states) by the behaving system" (p. 1544). Given the very general nature of this plea, we have to admit that Taylor is correct when he says: "It is quite possible, even likely, that something about the processes involved in the control of certain perceptions accounts for the power law and the variations in the observed power, but Marken and Shaffer do not pursue this line of enquiry." So we will take this opportunity to briefly describe what research aimed at discovering the perceptions controlled in movement production would look like.

We will use as an example of such research a study by Viviani and Stucchi (1992). In that study the authors presented subjects with spots of light moving in ellipses and "scribbles" on a computer screen and had them press the ">" or "<" key to make the motion of the spot appear to be of uniform velocity. The key press increased or decreased the β exponent of a power function equation that was used to generate the motion of the light spot. So the subjects are asked to control a perception of uniform motion and they are asked to do it by pressing the appropriate keys to compensate for disturbances to the velocity of motion produced by changes in the power coefficient β. The researchers found that over many different conditions the subjects did this by keeping β fairly close to the "prototype" power law coefficient, 0.33. This hints at the possibility that the perception being controlled is affine velocity and that it is being controlled at a constant reference level. This possibility is suggested by our OVB analysis and the work of Pollick and Sapiro (1997), both of which suggest that keeping affine velocity constant (and, thus, uncorrelated with curvature) will result in motion that conforms to the power law.

Further research using the test for the controlled variable would be needed to determine exactly what perceptual variable is being controlled when an organism produces movements with uniform speed. The variable may be affine velocity but it may be some other, closely related variable, such as some function of the ratio of movement velocity to curvature. But once the controlled perceptual input variable is identified it should be possible to build a simple control system model that explains the movement behavior produced by living systems as it occurs under many different circumstances, as was the case with our object interception model noted above. To paraphrase Powers' conclusion to his 1978 *Psychological Review* paper (p. 434): for a thousand unconnected empirical generalizations about movement behavior that are based on superficial similarities between features of movement trajectories, we here substitute one general underlying principle: control of input.

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