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Crossed Products of C^*-Algebras and Spectral Analysis of Quantum Hamiltonians

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Abstract: We study spectral properties of a hamiltonian by analyzing the structure of certain C^*-algebras to which it is affiliated. The main tool we use for the construction of these algebras is the crossed product of abelian C^*-algebras (generated by the classical potentials) by actions of groups. We show how to compute the quotient of such a crossed product with respect to the ideal of compact operators and how to use the resulting information in order to get spectral properties of the hamiltonians. This scheme provides a unified approach to the study of hamiltonians of anisotropic and many-body systems (including quantum fields).

1. Introduction

1.1. Algebras of hamiltonians. Throughout this paper X will be a locally compact not compact abelian group (with the operation denoted additively) equipped with a Haar measure $\mathrm{d}x$. For example, X could be a finite dimensional vector space over a p-adic field (or rather the underlying additive group). We shall call an *algebra of (classical) interactions on X* any C^*-algebra $\mathscr{A}$ of functions such that[1]

$$C_\infty(X) \subset \mathscr{A} \subset C_{\mathrm{b}}^{\mathrm{u}}(X) \text{ and } \mathscr{A} \text{ is stable under translations.} \tag{1.1}$$

Let X^* be the group dual to X and for $h : X^* \to \mathbb{C}$ Borel let $h(P)$ be the operator on $L^2(X)$ defined by $h(P) = \mathcal{F}^* M_h \mathcal{F}$, where $\mathcal{F}$ is the Fourier transformation and M_h is the operator of multiplication by h in $L^2(X^*)$. A self-adjoint operator H on $L^2(X)$ of the form $h(P) + v(Q)$, where h is a real continuous function on X^* such that $\lim_{k\to\infty} |h(k)| = \infty$ and $v(Q)$ is the operator of multiplication by a function $v \in \mathscr{A}$,

[1] $C_{\mathrm{b}}^{\mathrm{u}}(X)$ is the C^*-algebra of complex uniformly continuous bounded functions on X; then $C_\infty(X)$, $C_0(X)$ and $C_{\mathrm{c}}(X)$ are the subalgebras of functions which have a limit at infinity, are convergent to zero at infinity, or have compact support, respectively. If $\mathscr{H}$ is a Hilbert space then $B(\mathscr{H})$ is the space of bounded linear operators on $\mathscr{H}$ and $K(\mathscr{H})$ the subspace of compact operators.

will be called an *elementary hamiltonian of type* $\mathscr{A}$ (if X^* is compact, there is, of course, no condition at infinity on h). We use the symbols Q and P in order to keep notations close to those of quantum mechanics, where Q and P are the position and momentum observables respectively; here we do not attach any meaning to them.

We recall that the C^*-algebra generated by a family of self-adjoint operators is the smallest C^*-algebra which contains their resolvents.

Definition 1.1. *Denote by* $\mathscr{A} \rtimes X$ *the* C^**-algebra of operators on* $L^2(X)$ *generated by the elementary hamiltonians of type* $\mathscr{A}$*. A self-adjoint operator* H *on* $L^2(X)$ *such that* $(H+i)^{-1} \in \mathscr{A} \rtimes X$ *is called* ***hamiltonian of type*** $\mathscr{A}$*.*

We shall often refer to $\mathscr{A} \rtimes X$ as the C^**-algebra of hamiltonians (or energy observables) of type* $\mathscr{A}$. The notation $\mathscr{A} \rtimes X$ is standard in the theory of C^*-algebras, meaning crossed product of $\mathscr{A}$ by an action of the group X. Theorem 1.1 below justifies its use in the present context. Note that we shall also use the terminology "C^*-algebra of hamiltonians" for algebras which are not crossed products (see (1.5) and Sects. 1.7 and 1.8).

In this paper we shall use operator algebra techniques in order to study the spectral properties of type $\mathscr{A}$ hamiltonians. More precisely, we show that the algebra $\mathscr{A} \rtimes X$ has a rather remarkable structure which allows us to propose a general method of computation of the essential spectrum of these hamiltonians. The same ideas allow one to prove the Mourre estimate for certain choices of conjugate operators, but this question will be treated very briefly in this paper, in connection with a quantum field model, see Sect. 1.7. In [12,13] the method is applied to a general class of dispersive N-body hamiltonians, and in [20] to quantum field models with a particle number cut-off.

We emphasize that the class of type $\mathscr{A}$ hamiltonians is much larger than the elementary ones. Although in concrete examples we take $H = h(P) + V$, in general V is not a function but only a symmetric operator satisfying certain conditions (it could be a pseudo-differential operator). Theorem 2.1 gives a perturbative method of constructing such hamiltonians (see Theorem 6.1 and [13] for applications). On the other hand, the C^*-algebras $\mathscr{A} \rtimes X$ can often be described in direct terms, as in Theorems 1.2 and 1.3, and this allows one to get very general classes of hamiltonians of type $\mathscr{A}$ which have no natural decompositions into a sum of a kinetic and a potential part as above (see, for example, the comments after Theorem 1.4 from Sect. 1.6). We mention that some algebras are such that it is always possible to define the kinetic part of a hamiltonian affiliated to them. For example, in the context of Sect. 5.2 the kinetic component of H is its quotient with respect to the ideal $\mathscr{C}_{L,0}(X)$. One can similarly define the kinetic part of observables affiliated to the graded C^*-algebra associated to N-body systems, see Sect. 1.4.

A slight modification of the algebra $\mathscr{A}$, obtained by taking the tensor product with an algebra of compact operators, allows one to greatly improve the applications of the theory. More precisely, let $\mathbf{E}$ be a complex Hilbert space. Then the C^*-algebra $\mathscr{A}^{\mathbf{E}} \equiv \mathscr{A} \otimes K(\mathbf{E})$ has a natural structure of X-algebra (see Sect. 3.1) and so the crossed product $\mathscr{A}^{\mathbf{E}} \rtimes X$ is well defined. Theorem 4.1 extends in an obvious way to the present case, the space $L^2(X)$ being replaced by $L^2(X; \mathbf{E}) \equiv L^2(X) \otimes \mathbf{E}$. Corollary 4.1 holds in the form (see Sect. 1.3 for the notations)

$$C_0(X)^{\mathbf{E}} \rtimes X \cong [\![C_0(X)^{\mathbf{E}} \cdot C_0(X^*)]\!] = K(L^2(X; \mathbf{E})) = \mathscr{K}(X) \otimes K(\mathbf{E}).$$

Most of what we do in the rest of the paper can be extended with no difficulty to this setting, so we shall not stress this point. This trivial mathematical extension is, however, quite useful in applications to the spectral theory of differential operators with operator valued coefficients (e.g. Dirac operators) and also in other contexts (see Sect. 1.4). In order to treat couplings between two systems it is important to consider tensor products with C^*-algebras more general than $K(\mathbf{E})$; this less trivial problem is studied in Sect. 1.8.

We give now a short description of the content of the paper. The rest of this introduction is devoted to a detailed presentation of the method we use and of several classes of examples, including ones which do not belong to the crossed product setting introduced above. We hope that this will clarify the scope and power of this algebraic approach. In Sect. 2 we discuss several questions concerning the self-adjoint operators affiliated to C^*-algebras, their essential spectra and the connection with the problem of computing quotients of C^*-algebras with respect to some ideals. Theorem 2.2 will be particularly useful later on. Sections 3 and 4 are devoted to a short presentation of the theory of crossed products of C^*-algebras by actions of abelian groups with emphasis on some results that we need and which we have not been able to find in the literature (at least in a sufficiently explicit form). Especially important for us are Theorems 3.1 and 4.1. Many examples are given in this introduction, but we devote the whole Sect. 5 to a detailed study of one of the most interesting of these algebras: that suggested by the work of Klaus [27] on potentials with infinitely many "bumps". In Sect. 6 we point out a large class of hamiltonians affiliated to it. The appendix is devoted to the rather long proof of Theorem 1.2.

We have to mention that we decided to change the title of the preprint version [22] of this paper because there are substantial modifications in the presentation of the results and in the subjects we treat: besides a quite different introduction (the examples in the second part of Sect. 1.6 and the Sects. 1.7, 1.8 are new), we have eliminated topics and examples which are either not so important or will be developed elsewhere. On the other hand, several proofs are more detailed in the preprint, which could be useful especially to a novice in C^*-algebras. In [26] one can find a preliminary description of our results. We also note that recently the preprints [30, 31] containing various applications of our ideas have appeared.

There is a very large literature on the applications of the abstract theory of C^*-algebras to the study of spectral properties of various classes of operators. One can find references on this question in [14] and [32]. The work of H. O. Cordes [9] is partly relevant in our context. C^*-algebras also appear in many other branches of mathematical physics: statistical mechanics and quantum field theory (algebras of local observables, crossed products), scattering theory (algebras of asymptotic observables). But our purposes and techniques are quite different.

Let us notice that C^*-algebras generated by the energy observable appear already in the work of J. Bellissard in relation with solid state physics. In [3, 4] he pointed out a remarkable connection between K-theory, cyclic cohomology of C^*-algebras, and quantum Hall effect, opening thus the way to many other applications of algebraic methods in the study of models in condensed matter physics (cf. [3–5] and also [6, 8] and references therein). In particular, he considers the C^*-algebra generated by the translates of the hamiltonian of a physical system and shows that under certain conditions it is a crossed product. Although this should be considered as an "algebra of hamiltonians" in our terminology, it is quite different from those we consider here, being rather tightly related to *one* hamiltonian H (see Sect. 1.3 for more comments on this question). In the models considered by Bellissard this is an advantage and allows him and his co-workers to get much more precise spectral properties of H which lead, for example, to a beautiful mathematical description of the quantum Hall effect. The algebras which appear in our work are much larger and, in a certain sense, simpler. So we can treat a large class of models but we are not able to study finer spectral properties of the hamiltonians. On the other hand, we are mainly concerned with the quotient of the C^*-algebra of hamiltonians of a system with respect to the ideal of compact operators. In the situations studied by Bellissard, this does not seem to be of interest, since the C^*-algebras that appear in his most important applications do not contain compact operators. The techniques we use do not give anything interesting concerning almost periodic or random operators. In fact, we think that the main future development of the ideas presented in this paper will concern many-body systems (consisting of a large and variable number of particles) and quantum fields.

1.2. C^*-algebra techniques. We shall explain the main ideas of our approach in a more general setting. Assume that we are asked to compute the essential spectrum $\boldsymbol{\sigma}_{\text{ess}}(H)$ of a self-adjoint operator H acting on a Hilbert space $\mathscr{H}$. If H_0 is a second self-adjoint operator on $\mathscr{H}$ such that $(H+i)^{-1} - (H_0+i)^{-1} \in K(\mathscr{H})$, then $\boldsymbol{\sigma}_{\text{ess}}(H) = \boldsymbol{\sigma}_{\text{ess}}(H_0)$ (Weyl's theorem). Thus an idea would be: try to make H simpler by adding to it a compact operator (for unbounded H the addition being interpreted in a generalized sense: the difference of the resolvents should be compact). The problem is that in many physically important situations it is impossible to get a simpler H_0 by this procedure: think about the 3-body problem or the most elementary anisotropic hamiltonian $H = P^2 + V(Q)$ on $L^2(\mathbb{R})$, where the function V has distinct limits at $\pm\infty$.

C^*-algebras offer a straightforward solution to this difficulty: there is always an H_0 simpler than H that can be used in the preceding argument, there is even an optimal one. Of course, a new kind of problem appears: this operator does not act (in a natural way) in the initial Hilbert space $\mathscr{H}$. The main purpose of our paper is to show how to solve such problems, first in a general framework and then in a concrete but rather remarkable situation.

Consider the quotient C^*-algebra $C(\mathscr{H}) = B(\mathscr{H})/K(\mathscr{H})$ (this is called the *Calkin algebra*). If $H \in B(\mathscr{H})$, let $\widehat{H}$ be its image in $C(\mathscr{H})$. Then Weyl's theorem can be stated as $\boldsymbol{\sigma}_{\text{ess}}(H) = \boldsymbol{\sigma}(\widehat{H})$. The "abstract" operator $\widehat{H}$ will be the optimal choice we talked about before ($\widehat{H}$ is abstract in the sense that it does not act on a Hilbert space).

In order to use this for unbounded H, we have to define the notion of self-adjoint operator in a purely algebraic setting. It is convenient[2], and physically motivated, to define an *observable affiliated to a C^*-algebra* $\mathscr{C}$ as a morphism $H : C_0(\mathbb{R}) \to \mathscr{C}$ (see Sect. 2.1 for basic C^*-algebra terminology). In order to keep close to standard notations, we denote $\varphi(H)$ (not $H(\varphi)$) the image of $\varphi \in C_0(\mathbb{R})$ through this morphism. The *spectrum* of the observable H is defined by:

$$\boldsymbol{\sigma}(H) = \{\lambda \in \mathbb{R} \mid \varphi \in C_0(\mathbb{R}) \text{ and } \varphi(\lambda) \neq 0 \Longrightarrow \varphi(H) \neq 0\}. \tag{1.2}$$

If $\mathcal{P} : \mathscr{C} \to \mathscr{C}'$ is a morphism into a second C^*-algebra $\mathscr{C}'$ we can define the image $H' = \mathcal{P}[H]$ of H through $\mathcal{P}$ as the observable affiliated to $\mathscr{C}'$ given by $\varphi(H') = \mathcal{P}[\varphi(H)]$. In particular, if $\mathscr{J}$ is an ideal in $\mathscr{C}$ and $\widehat{\mathscr{C}} = \mathscr{C}/\mathscr{J}$ is the quotient algebra, we may define $\widehat{H}$ as the observable affiliated to $\widehat{\mathscr{C}}$ given by $\widehat{H} = \pi[H]$, where π is the canonical morphism $\mathscr{C} \to \widehat{\mathscr{C}}$.

[2] The notion of observable is related to that of a self-adjoint operator affiliated with a C^*-algebra in the sense of Woronowicz. Following a suggestion of the referee, we discuss this question in Sect. 2.1.

Assume that the C^*-algebra $\mathscr{C}$ is realized on a Hilbert space $\mathscr{H}$, i.e. $\mathscr{C}$ is a C^*-subalgebra of $B(\mathscr{H})$, and let us take above $\mathscr{J} = \mathscr{C} \cap K(\mathscr{H})$. A self-adjoint operator H on $\mathscr{H}$ is called *affiliated to* $\mathscr{C}$ if $(H - z)^{-1} \in \mathscr{C}$ for some $z \in \mathbb{C} \setminus \boldsymbol{\sigma}(H)$. This implies $\varphi(H) \in \mathscr{C}$ for all $\varphi \in C_0(\mathbb{R})$, so each self-adjoint operator on $\mathscr{H}$ affiliated to $\mathscr{C}$ defines an observable affiliated to $\mathscr{C}$. Then $\widehat{H}$ is well defined as an observable affiliated to the (abstract) C^*-algebra $\widehat{\mathscr{C}}$. We recall that a real number λ does not belong to the essential spectrum of a self-adjoint operator H if and only if $\varphi(H) \in K(\mathscr{H})$ for some $\varphi \in C_0(\mathbb{R})$ such that $\varphi(\lambda) \neq 0$. Hence from (1.2) we get $\boldsymbol{\sigma}_{\text{ess}}(H) = \boldsymbol{\sigma}(\widehat{H})$ (see also (2.1)).

We stress the fact that the operation which associates $\widehat{H}$ to H has no meaning from a strictly hilbertian point of view. Indeed, in most cases $\widehat{H}$ is not an operator on $\mathscr{H}$, because $\widehat{\mathscr{C}}$ has no natural realization on $\mathscr{H}$. But we shall see in Sect. 2.1 that if H is "strictly" affiliated to $\mathscr{C}$, then one can realize $\widehat{H}$ as a self-adjoint operator in each nondegenerate representation of $\widehat{\mathscr{C}}$.

After these preliminary definitions, we go back to our problem: we would like to study the spectral properties of a given self-adjoint operator H on a Hilbert space $\mathscr{H}$. We shall consider only spectral properties which are stable under compact perturbations, for example the determination of the essential spectrum or the validity of the Mourre estimate (which indeed is, in a sense which can be made precise, stable under such perturbations). As we explained before, H is also an observable affiliated to $B(\mathscr{H})$, so $\widehat{H}$ is well defined as an observable affiliated to the Calkin algebra $C(\mathscr{H})$. Clearly, what we really have to do, is to study the spectral properties of $\widehat{H}$, and for this we have first to compute it! This cannot be a trivial task since the Calkin algebra is a rather complex object, e.g. it cannot be faithfully represented on a separable Hilbert space. Now we come to the main point of our approach. Assume that H is affiliated to a C^*-subalgebra $\mathscr{C} \subset B(\mathscr{H})$. Then $\widehat{H}$ is affiliated to the C^*-subalgebra $\widehat{\mathscr{C}} = \mathscr{C} / [\mathscr{C} \cap K(\mathscr{H})]$ of $C(\mathscr{H})$ and $\widehat{\mathscr{C}}$ could be much simpler than $C(\mathscr{H})$. If we can determine $\widehat{\mathscr{C}}$ rather explicitly, we have good chances to obtain an explicit expression of $\widehat{H}$, and so to say something interesting about H. Our purpose is to show that this strategy works and allows one to treat in a systematic way hamiltonians of physical systems with a complicated structure.

The preceding formulation could give the wrong impression that the main object is H and that $\mathscr{C}$ is an auxiliary construction needed only at an intermediate step of the computation. But in the most interesting situations this is not the case: $\mathscr{C}$ is necessary not only to isolate the natural and general class of hamiltonians (H is *defined* by its affiliation to $\mathscr{C}$), but also for the formulation of the results (again, in the convenient degree of generality). In this respect, it is instructive to compare the statements of Theorems 5.3, 5.6 and 6.1. Thus, we simply forget the hamiltonian and state the problem we have to solve as follows: a C^*-subalgebra $\mathscr{C} \subset B(\mathscr{H})$ being given, describe $\widehat{\mathscr{C}} \subset C(\mathscr{H})$.

We have to point out situations in which this approach is useless. First, we cannot expect to get an interesting result if $\mathscr{C} \cap K(\mathscr{H}) = \{0\}$, because then $\mathscr{C} \cong \widehat{\mathscr{C}}$ (where $\cong$ means "canonically isomorphic"). For example, this is the case if the algebra of classical interactions consists of almost periodic functions. On the other hand, assume that $\mathscr{C}_0$ is a C^*-algebra of operators on $\mathscr{H}$ such that $\mathscr{C}_0 \cap K(\mathscr{H}) = \{0\}$ (this means that $\mathscr{C}_0$ has no nonzero finite rank projections) and let $\mathscr{C} = \mathscr{C}_0 + K(\mathscr{H})$. Then $\mathscr{C}$ is a C^*-algebra and the projection $\mathscr{C} \to \mathscr{C}_0$ associated with the linear direct sum decomposition which defines $\mathscr{C}$ is a morphism which gives a canonical identification $\widehat{\mathscr{C}} \cong \mathscr{C}_0$. So in this case $\widehat{\mathscr{C}}$ is naturally realized on the same Hilbert space as $\mathscr{C}$. Our approach is not really useful in such a simple situation. Indeed, a self-adjoint operator H on $\mathscr{H}$ is affiliated to $\mathscr{C}$ if and only if there is a self-adjoint operator H_0 affiliated to $\mathscr{C}_0$

such that $(H+i)^{-1}-(H_0+i)^{-1}$ is compact, and then $\widehat{H}=H_0$. Now the fact that $\boldsymbol{\sigma}_{\text{ess}}(H)=\boldsymbol{\sigma}(H_0)$ is a standard fact of Hilbert space nature and the algebraic approach we propose is, at this level, irrelevant. The quantum mechanical two-body problem is a particular case of this example, see (1.4).

We note that the operator H_0 is *not* densely defined in general: strictly speaking, it is only an observable affiliated to $\mathscr{C}_0$. For example, if H has purely discrete spectrum, H_0 is the observable ∞ defined by $\varphi(\infty)=0$ for all $\varphi\in C_0(\mathbb{R})$. This trivial observable appears quite often in practical computations (see Sect. 1.6). Equivalent characterizations of ∞, which is affiliated to any C^*-algebra $\mathscr{C}$, is $\boldsymbol{\sigma}(\infty)=\emptyset$, or $D(\infty)=\{0\}$ in each representation of $\mathscr{C}$.

1.3. Relevance of crossed products. The preceding strategy has been applied in several situations in [22] and [23]. A treatment of the dispersive N-body problem and of some quantum field models (including the proof of the Mourre estimate) along these lines can be found in [12,13] and [20]. In this paper the main emphasis is on the technique of crossed products (see Sect. 3), which allows one to do the computations at an abelian level. The next result (whose proof can be found at the end of Sect. 4) explains the relevance of these objects in our context.

We shall use the following notation: if $\mathscr{H}$ is a Hilbert space and $\mathscr{S}$, $\mathscr{T}$ are subalgebras of $B(\mathscr{H})$ then $\mathscr{S}\cdot\mathscr{T}$ is the set of sums of the form $S_1T_1+\cdots+S_nT_n$ with $S_i\in\mathscr{S}$, $T_i\in\mathscr{T}$, and $[\![\mathscr{S}\cdot\mathscr{T}]\!]$ its norm closure. We also identify $C_0(X^*)$ with an algebra of operators on $L^2(X)$ with the help of the map $\psi\mapsto\psi(P)$. $\mathscr{A}$ is as in (1.1) and is identified with the corresponding algebra of multiplication operators on $L^2(X)$. The group operation in X^* will be denoted additively.

Theorem 1.1. *Let $h:X^*\to\mathbb{R}$ be a continuous non-constant function such that $\lim_{k\to\infty}|h(k)|=\infty$. Then the C^*-algebra generated by the self-adjoint operators of the form $h(P+k)+v(Q)$, with $k\in X^*$ and $v\in\mathscr{A}$ real, is equal to $[\![\mathscr{A}\cdot C_0(X^*)]\!]$. Moreover, this space is canonically isomorphic to the crossed product of the C^*-algebra $\mathscr{A}$ by the action τ of X defined by $(\tau_x\varphi)(y)=\varphi(y-x)$.*

The result can be restated as follows: *$\mathscr{A}\rtimes X$ is the smallest C^*-algebra of operators on $L^2(X)$ which contains $(h(P)+v(Q)+i)^{-1}$ for all $v\in\mathscr{A}$ real and which is stable under all the automorphisms*[3] $S\mapsto V_kSV_k^*$.

Note that the theorem is stronger than expected, the function h being fixed. Then it is easily seen that it implies the following in the case $X=\mathbb{R}^n$. Let h be a real elliptic polynomial of order m and let $\mathscr{A}^\infty$ be the set of $\varphi\in\mathscr{A}$ which are of class C^∞ and such that all their derivatives belong to $\mathscr{A}$ too. Then $\mathscr{A}\rtimes X$ is the C^*-algebra generated by the self-adjoint operators of the form $h(P)+V$, where V runs over the set of symmetric differential operators of order $<m$ with coefficients in $\mathscr{A}^\infty$.

From this one can also see the main difference between the C^*-algebras considered by J. Bellissard in [3,4] and those we work with here: Bellissard fixes h *and* v and takes the algebra generated by the operators $h(P)+v(Q+x)$, $x\in X$, while we fix h and let v vary on a quite large set of symmetric operators modeled by $\mathscr{A}$. In our case crossed products appear essentially by definition, while in the cases studied in [3,4] this is a rather subtle feature. Note also that, although we start with a given kinetic energy h, the

[3] Here and later on we denote by V_k and U_x the unitary operators on $L^2(X)$ defined by the relations $(V_kf)(x)=k(x)f(x)$ and $(U_xf)(y)=f(x+y)$.

final algebra $\mathscr{A} \rtimes X$ is independent of it. It is not clear for us whether this property holds in the situations studied in [3–6].

Now we explain the procedure of reduction to an abelian situation. This is based on certain properties (described in Sect. 3) of the correspondence $\mathscr{A} \mapsto \mathscr{A} \rtimes X$. We set $\mathscr{K}(X) = K(L^2(X))$ and $\mathscr{B}(X) = B(L^2(X))$ (these C^*-algebras depend only on X, not on the choice of the Haar measure). One has $\mathscr{K}(X) \cong C_0(X) \rtimes X$ (see Corollary 4.1). Thus from (1.1) we see that $\mathscr{K}(X)$ is an ideal in $\mathscr{A} \rtimes X$. The quotient C^*-algebra $\mathscr{A} \rtimes X/\mathscr{K}(X)$ is well defined and, by Theorem 3.2,

$$\mathscr{A} \rtimes X \Big/_{\mathscr{K}(X)} \cong [\mathscr{A}/C_0(X)] \rtimes X. \tag{1.3}$$

This relation reduces the problem of the computation of the quotient of the two noncommutative algebras from the left-hand side to an easier abelian problem: that of giving a convenient description of $\mathscr{A}/C_0(X)$.

We mention here an important point: in general, there are many (equivalent) descriptions of $\mathscr{A}/C_0(X)$, each of them having its own merits, so that we end up with quite different descriptions of $\mathscr{A} \rtimes X/\mathscr{K}(X)$, hence of the essential spectrum of the hamiltonian. For example, Theorems 5.2 and 5.5 give rather different characterizations of the same quotient, and this furnishes the quite different descriptions of $\boldsymbol{\sigma}_{\text{ess}}(H)$ from Theorems 5.3 and 5.6 (see also Sect. 1.4).

1.4. Examples. It is worthwhile to begin with the simplest situation: $\mathscr{A} = C_\infty(X) = \mathbb{C} + C_0(X)$. The corresponding algebra of $\mathscr{A}$ type hamiltonians will be denoted $\mathscr{T}(X)$ and will be called the *two-body algebra* (the hamiltonians of a particle in external fields vanishing at infinity generate such an algebra). By Corollary 3.1, and since X is not compact, we have (with linear direct sums)

$$\mathscr{T}(X) = C_\infty(X) \rtimes X = (\mathbb{C} + C_0(X)) \rtimes X = C_0(X^*) + \mathscr{K}(X) \subset \mathscr{B}(X). \tag{1.4}$$

We get $\mathscr{T}(X)/\mathscr{K}(X) \cong C_0(X^*)$ either by using (1.3) and $C_\infty(X)/C_0(X) \cong \mathbb{C}$, or directly because $C_0(X^*) \cap \mathscr{K}(X) = \{0\}$. The canonical surjection $\mathscr{T}(X) \to C_0(X^*)$ is given by $\widehat{S} = \text{s-lim}_{x\to\infty} U_x S U_x^*$ (note that $\text{w-lim}_{x\to\infty} U_x = 0$).

We consider next the simplest anisotropic behavior: one dimensional physical systems with different asymptotics at plus and minus infinity in configuration space. Let $X = \mathbb{R}$ and $\mathscr{A} = C(\overline{\mathbb{R}})$ be the algebra of continuous bounded complex functions on $\mathbb{R}$ which have limits at $\pm\infty$. Observe that if $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ is the two-point compactification of $\mathbb{R}$ with the natural topology, then $C(\overline{\mathbb{R}})$ is the set of functions in $C_{\text{b}}^{\text{u}}(\mathbb{R})$ which extend continuously to $\overline{\mathbb{R}}$. It is very easy to describe the quotient $C(\overline{\mathbb{R}})/C_0(\mathbb{R})$: we have two morphisms $\varphi \mapsto \lim_{x\to\pm\infty} \varphi(x)$ from $C(\overline{\mathbb{R}})$ onto $\mathbb{C}$ and the intersection of their kernels is $C_0(\mathbb{R})$, so we get an identification $C(\overline{\mathbb{R}})/C_0(\mathbb{R}) \cong \mathbb{C} \oplus \mathbb{C}$. Taking the cross product by the action of $\mathbb{R}$ is also easy. If $\mathscr{C}(\overline{\mathbb{R}}) = C(\overline{\mathbb{R}}) \rtimes \mathbb{R}$ then we consider the morphisms $\mathcal{P}_\pm : \mathscr{C}(\overline{\mathbb{R}}) \to C_0(\mathbb{R}^*)$ given by $\mathcal{P}_\pm[T] = \text{s-lim}_{x\to\pm\infty} U_x T U_x^*$. The map $T \mapsto (\mathcal{P}_-[T], \mathcal{P}_+[T])$ is a surjective morphism of $\mathscr{C}(\overline{\mathbb{R}})$ onto $C_0(\mathbb{R}^*) \oplus C_0(\mathbb{R}^*)$ and its kernel equals $\mathscr{K}(\mathbb{R})$. Thus we get

$$\mathscr{C}(\overline{\mathbb{R}})/\mathscr{K}(\mathbb{R}) \cong C_0(\mathbb{R}^*) \oplus C_0(\mathbb{R}^*).$$

Physically interesting self-adjoint operators affiliated to $\mathscr{C}(\overline{\mathbb{R}})$ include Schrödinger operators with locally singular potentials and different asymptotics at $\pm\infty$.

In order to improve the applications of the theory we use the idea presented at the end of Sect. 1.1. Let $C^{\mathbf{E}}(\overline{\mathbb{R}}) = C(\overline{\mathbb{R}}) \otimes K(\mathbf{E})$ be the C^*-algebra of norm continuous functions $\mathbb{R} \to K(\mathbf{E})$ which have limits at $\pm\infty$ (in the norm topology). Then

$$\mathscr{C}(\overline{\mathbb{R}})^{\mathbf{E}} = C^{\mathbf{E}}(\overline{\mathbb{R}}) \rtimes \mathbb{R} = \mathscr{C}(\overline{\mathbb{R}}) \otimes K(\mathbf{E}) = [\![C^{\mathbf{E}}(\overline{\mathbb{R}}) \cdot C_0(\mathbb{R}^*)]\!]$$

(we used Proposition 3.2) and exactly as above we get

$$\mathscr{C}(\overline{\mathbb{R}})^{\mathbf{E}}/K(L^2(\mathbb{R};\mathbf{E})) \cong C_0^{\mathbf{E}}(\mathbb{R}^*) \oplus C_0^{\mathbf{E}}(\mathbb{R}^*),$$

the isomorphism being induced by the same $\mathcal{P}_\pm$. Less trivial is the proof of the Mourre estimate for operators affiliated to $\mathscr{C}(\overline{\mathbb{R}})^{\mathbf{E}}$; our results in this direction will be published elsewhere. The applications cover the spectral theory of elliptic operators on asymptotically cylindric (star shaped) domains in $\mathbb{R}^n$ (with different asymptotics at various ends) and on some manifolds with cylindrical ends. Our work on these questions has been especially motivated by [7, 10, 16] and references therein. By taking $\mathbf{E} = \mathbb{C}^2$ we cover one dimensional Dirac operators with different asymptotics at $\pm\infty$.

The preceding example has a simple and interesting extension which goes beyond the crossed product framework. Let

$$\mathscr{C}^{\mathbf{E}} = [\![C^{\mathbf{E}}(\overline{\mathbb{R}}) \cdot C^{\mathbf{E}}(\overline{\mathbb{R}^*})]\!] = [\![C^{\mathbf{E}}(\overline{\mathbb{R}}) \cdot C(\overline{\mathbb{R}^*})]\!]. \tag{1.5}$$

This is a C^*-algebra (but not a crossed product) which admits an intrinsic description of the same nature as that of Theorem 1.2, and $\mathscr{C}(\overline{\mathbb{R}})^{\mathbf{E}} \subset \mathscr{C}^{\mathbf{E}}$. We clearly have $\mathscr{C}^{\mathbf{E}} = \mathscr{C} \otimes K(\mathbf{E})$, where $\mathscr{C}$ is the algebra corresponding to the choice $\mathbf{E} = \mathbb{C}$. If $T \in \mathscr{C}^{\mathbf{E}}$ then $T_\pm = \text{s-lim}_{x\to\pm\infty} U_x T U_x^*$ and $T^\pm = \text{s-lim}_{k\to\pm\infty} V_k^* T V_k$ exist, and the map $T \mapsto (T_-, T_+, T^-, T^+)$ is a morphism of $\mathscr{C}^{\mathbf{E}}$ into

$$C^{\mathbf{E}}(\overline{\mathbb{R}^*}) \oplus C^{\mathbf{E}}(\overline{\mathbb{R}^*}) \oplus C^{\mathbf{E}}(\overline{\mathbb{R}}) \oplus C^{\mathbf{E}}(\overline{\mathbb{R}})$$

with kernel equal to $K(L^2(\mathbb{R};\mathbf{E}))$ and range given by the compatibility relations $(T_-)^\pm = (T^\pm)_-$ and $(T_+)^\pm = (T^\pm)_+$. Thus we get

$$\mathscr{C}^{\mathbf{E}}/K(L^2(\mathbb{R};\mathbf{E})) \hookrightarrow C^{\mathbf{E}}(\overline{\mathbb{R}^*}) \oplus C^{\mathbf{E}}(\overline{\mathbb{R}^*}) \oplus C^{\mathbf{E}}(\overline{\mathbb{R}}) \oplus C^{\mathbf{E}}(\overline{\mathbb{R}}).$$

Hence if H is an observable affiliated to $\mathscr{C}^{\mathbf{E}}$ then one can associate to it four asymptotic observables $H_\pm$, $H^\pm$ (the first two $H_\pm$ correspond to $Q \to \pm\infty$ while the other ones $H^\pm$ correspond to $P \to \pm\infty$) and spectral properties like essential spectrum, thresholds, Mourre estimate, of H can be expressed in terms of these observables. For example

$$\boldsymbol{\sigma}_{\text{ess}}(H) = \boldsymbol{\sigma}(H_-) \cup \boldsymbol{\sigma}(H_+) \cup \boldsymbol{\sigma}(H^-) \cup \boldsymbol{\sigma}(H^-).$$

This is an elementary but physically interesting situation when the essential spectrum of H is not determined by its localizations at infinity if the infinity is interpreted only in the Q sense (compare with Theorem 1.4).

We now go back to an arbitrary X and consider the problem of constructing algebras of classical interactions. These are in bijective correspondence with a certain class of compactifications of X: the spectrum $X(\mathscr{A})$ of $\mathscr{A}$ is a compact space and X is homeomorphically and densely embedded in $X(\mathscr{A})$; then $\mathscr{A}$ is the set of functions in $C_{\mathrm{b}}^{\mathrm{u}}(X)$ which have continuous extensions to $X(\mathscr{A})$, i.e. $\mathscr{A} = C(X(\mathscr{A}))$. For example, $C_\infty(X)$ is related to the simplest compactification of X, the Alexandroff compactification, and $C(\overline{\mathbb{R}})$ involves the two-point compactification of $\mathbb{R}$.

The compactifications of X are quotients of the Stone-Čech compactification βX. However, instead of considering explicitly βX, one can construct algebras of interactions by using limits at infinity along certain filters. Let $\mathcal{F}$ be a filter on X and assume that $\mathcal{F}$ is translation invariant (if $A \in \mathcal{F}$, $x \in X$, then $x + A \in \mathcal{F}$) and finer than the Fréchet filter (which is the family of subsets with relatively compact complements). We consider the space of $\varphi \in C_{\mathrm{b}}^{\mathrm{u}}(X)$ which have a limit at infinity along the filter $\mathcal{F}$:

$$C_{\mathcal{F}}(X) = \{\varphi \in C_{\mathrm{b}}^{\mathrm{u}}(X) \mid \lim_{\mathcal{F}} \varphi \text{ exists}\}. \tag{1.6}$$

It is clear that $C_{\mathcal{F}}(X)$ is an algebra of classical interactions. If $\mathcal{F}$ is the Fréchet filter then $C_{\mathcal{F}}(X) = C_\infty(X)$. There is a largest non-trivial ideal in $C_{\mathcal{F}}(X)$, namely $C_{\mathcal{F},0}(X) = \{\varphi \in C_{\mathrm{b}}^{\mathrm{u}}(X) \mid \lim_{\mathcal{F}} \varphi = 0\}$, and $C_{\mathcal{F}}(X) = \mathbb{C} + C_{\mathcal{F},0}(X)$.

We consider a particular class of algebras of the preceding form. Fix a closed set L such that $L + \Lambda \neq X$ if Λ is compact (this means that there are points as far as we wish from L). Then the complements of the sets $L + \Lambda$ form the base of a filter $\mathcal{F}_L$ which is translation invariant and finer than the Fréchet filter. The algebra $C_L(X) \equiv C_{\mathcal{F}_L}(X)$ consists of functions $\varphi \in C_{\mathrm{b}}^{\mathrm{u}}(X)$ which tend to a constant when we are far away from L. The difficulty of the problem of describing the quotient algebra $C_L(X)/C_0(X)$ appears already in the seemingly elementary case when L is a straight line in $X = \mathbb{R}^2$.

In Sect. 5 we shall study in detail the preceding problem for a class of sets that we call sparse. More precisely, the set L is called *sparse* if it is locally finite and for each compact Λ of X there is a finite set $F \subset L$ such that if $l \in M = L \setminus F$ and $l' \in L \setminus \{l\}$ then $(l + \Lambda) \cap (l' + \Lambda) = \emptyset$. The corresponding class of hamiltonians generalizes that of the Schrödinger operators with "widely separated bumps" introduced by M. Klaus in [27] (see also [11,25]). This suggests calling $\mathscr{C}_L(X) \equiv C_L(X) \rtimes X$ the *bumps algebra*. As we shall see, this example fits very nicely in our framework, the quotient algebra having an especially interesting structure (see Sect. 2.3 for the notations):

$$\mathscr{C}_L(X) \big/ \mathscr{K}(X) \hookrightarrow \mathscr{T}(X)^{[L]} \big/ \mathscr{K}(X)^{(L)}. \tag{1.7}$$

This implies the following about the essential spectrum of an arbitrary observable H affiliated to $\mathscr{C}_L(X)$. First, note that there is a (highly not unique) family $(H_l)_{l \in L}$ of observables affiliated to the two-body algebra $\mathscr{T}(X)$ such that the quotient of $\prod_{l \in L} H_l$ with respect to the ideal $\mathscr{K}(X)^{(L)}$ is equal to the image of $\widehat{H}$ through the embedding (1.7). Then

$$\boldsymbol{\sigma}_{\mathrm{ess}}(H) = \bigcap_{\substack{F \subset L \\ F \text{ finite}}} \overline{\bigcup_{l \in L \setminus F} \boldsymbol{\sigma}(H_l)}. \tag{1.8}$$

In Sect. 5.4 we give a second description of the quotient algebra hence a new formula for the essential spectrum: for each ultrafilter $\varkappa$ finer than the Fréchet filter on L the limit $\operatorname{s-lim}_l U_l H U_l^* \equiv H_\varkappa$ when $l \to \infty$ along $\varkappa$ exists in the strong resolvent sense and

$$\boldsymbol{\sigma}_{\mathrm{ess}}(H) = \bigcup_{\varkappa} \boldsymbol{\sigma}(H_\varkappa). \tag{1.9}$$

In Sect. 6 we shall point out an explicit and quite general class of self-adjoint operators H affiliated to $\mathscr{C}_L(\mathbb{R}^n)$ and compute corresponding families $(H_l)_{l\in L}$. This makes the connection with the class of interactions studied by Klaus in [27]. We also mention a connection between (1.9) and previous work of Bellissard [3,4]. Indeed, the set $\widetilde{L}$ of ultrafilters finer than the Fréchet filter on L has a natural topology such that $l^\infty(L)/c_0(L) \cong C(\widetilde{L})$. But this is related to the spectrum of the quotient algebra $\mathscr{C}_L(X)/\mathscr{K}(X)$, as it can be deduced from (1.7). We do not insist on this aspect because the use of ultrafilters seems to us more convenient in computations (see the last part of Sect. 1.6).

The set of algebras of interactions is obviously a complete lattice for the inclusion order relation. Indeed, an arbitrary intersection of algebras $\mathscr{A}$ satisfying (1.1) satisfies it too. The existence of the upper bound for arbitrary families is a consequence, because $C_{\mathrm{b}}^{\mathrm{u}}(X)$ is the largest algebra of interactions. These operations allow one to construct new algebras from the existing ones.

Algebras of N-body type are constructed using a slight modification of this idea. Note first that if $Y \subset X$ is a closed subgroup, then we can embed $C_0(X/Y) \subset C_{\mathrm{b}}^{\mathrm{u}}(X)$ with the help of the map $\varphi \mapsto \varphi \circ \pi_Y$, where π_Y is the canonical surjection of X onto the quotient group X/Y. $C_0(X/Y)$ does not contain $C_\infty(X)$ but is translation invariant, so we may construct the crossed product $\mathscr{C}(Y) = C_0(X/Y) \rtimes X$. Now assume that X is (the underlying additive group of) a real finite dimensional vector space. Then the norm closure $\mathscr{A}$ of $\sum_Y C_0(X/Y)$, where Y runs over the set of all vector subspaces of X, is an algebra of interactions and $\mathscr{C} = \mathscr{A} \rtimes X$ is a generalized version of the algebras of hamiltonians appearing in the N-body problem. We refer to [13] for a detailed study of $\mathscr{C}$ and of the hamiltonians affiliated to it (including the Mourre estimate).

We mention however that $\mathscr{C}$ has a quite remarkable structure: *it is graded by the lattice of all subspaces Y of X*, and so belongs to a general class of algebras whose quotient with respect to the compacts can be computed explicitly. More precisely, $\mathscr{C}$ is the norm closure in $\mathscr{B}(X)$ of $\sum_Y \mathscr{C}(Y)$, one has $\mathscr{C}(Y)\cdot\mathscr{C}(Z) \subset \mathscr{C}(Y\cap Z)$, and the sum $\sum_{Y\in\mathcal{L}} \mathscr{C}(Y)$ is direct (linearly and topologically) if $\mathcal{L}$ is finite. For each subspace Y we define $\mathscr{C}_Y$ as the closure of the sum $\sum_{Z\supset Y} \mathscr{C}(Z)$. Then there is a canonical linear projection $\mathcal{P}_Y$ of $\mathscr{C}$ onto $\mathscr{C}_Y$ which is also a morphism. The map $T \mapsto (\mathcal{P}_Y[T])_{Y\in\mathcal{H}}$, where $\mathcal{H}$ is the set of hyperplanes of X, induces an embedding

$$\mathscr{C}/\mathscr{K}(X) \subset \textstyle\prod_{Y\in\mathcal{H}} \mathscr{C}_Y$$

whose range consists of families $(T_Y)_{Y\in\mathcal{H}}$ such that $\{T_Y \mid Y \in \mathcal{H}\}$ is a compact set in $\mathscr{C}$. As a corollary, if H is an observable affiliated to $\mathscr{C}$, we have

$$\boldsymbol{\sigma}_{\mathrm{ess}}(H) = \bigcup_{Y\in\mathcal{H}} \boldsymbol{\sigma}(H_Y),$$

where $H_Y = \mathcal{P}_Y[H]$. We note that H_Y can also be expressed, as before, as a strong limit of translated observables $U_x H U_x^*$.

Graded C^*-algebras which are not crossed products are also useful. For example, to each symplectic space one can associate an algebra of this type, to which hamiltonians of N-body systems in constant external magnetic fields are affiliated (see [23]). The simplest case is $C_0(X) + C_0(X^*) + \mathscr{K}(X)$.

In this paper we consider only crossed products of algebras of interactions $\mathscr{A} \subset C_{\mathrm{b}}^{\mathrm{u}}(X)$ on which X acts by translations. It is quite interesting, however, to replace X by non commutative groups: this gives the possibility to treat particle systems in magnetic fields which do not vanish at infinity. Indeed, we propose in [23] to consider groups G which are extensions of X by abelian groups. Then $\mathscr{A} \rtimes G$ is the C^*-algebra of energy observables of systems having X as configuration space, subject to internal interactions of type $\mathscr{A}$, and whose "momentum observable" derives from the symmetry group G which is determined by the external magnetic field.

1.5. Intrinsic description of algebras of hamiltonians. In the case of sparse sets it is possible to define the algebra $\mathscr{C}_L(X)$ in a rather simple way without mentioning crossed products. This is the content of the next theorem, where we characterize the elements of $\mathscr{C}_L(X)$ in geometric terms (involving the phase space $X \oplus X^*$). Its rather long proof will be given in the Appendix Sect. 7. We denote by $\chi_{L_\Lambda^c}(Q)$ the operator of multiplication by the characteristic function of the set $X \setminus (L+\Lambda)$. Below (and later on), when a symbol such as $T^{(*)}$ appears in a relation involving an operator T, we mean that the relation is satisfied (or has to be satisfied) both by T and by T^*.

Theorem 1.2. *A bounded operator T on $L^2(X)$ belongs to $\mathscr{C}_L(X)$ if and only if*

(i) $\lim_{x\to 0} \|(U_x - 1)T^{(*)}\| = 0,$
(ii) $\lim_{k\to 0} \|V_k T V_k^* - T\| = 0,$
(iii) *there exists $\widetilde{T} \in C_0(X^*)$ such that for each $\varepsilon > 0$ there is a compact set $\Lambda \subset X$ such that $\|\chi_{L_\Lambda^c}(Q)(T - \widetilde{T})^{(*)}\| < \varepsilon$.*

One has $T \in \mathscr{C}_{L,0}(X)$ if and only if one can take $\widetilde{T} = 0$.

It is interesting to note that this statement is of the same nature as the Riesz–Kolmogorov characterization of the compact operators in $L^2(X)$ (see [21]; we set $S^\perp = 1 - S$): *An operator $T \in \mathscr{B}(X)$ is compact if and only if it satisfies one of the following equivalent conditions:*

(i) $\lim_{x\to 0} \|(U_x - 1)T\| = 0$ and $\lim_{k\to 0} \|(V_k - 1)T\| = 0$;
(ii) $\forall \varepsilon > 0\ \exists \varphi \in C_c(X)\ \exists \psi \in C_c(X^*)$ such that $\|\varphi(Q)^\perp T\| + \|\psi(P)^\perp T\| < \varepsilon$.

There are characterizations similar to that of Theorem 1.2 in many of the concrete examples of C^*-algebras of hamiltonians. The case of the (generalized) N-body algebra is treated in [13]. The graded C^*-algebra associated with a symplectic space admits a similar description, see [23]. The following very simple description of the algebra $C_b^u(\mathbb{R}^n) \rtimes \mathbb{R}^n$ has been obtained in [13] by the methods of the Appendix Sect. 7 (recall the results of R. Beals [2], although they belong to the rather different setting of smooth pseudo-differential operators):

Theorem 1.3. *A bounded operator T on $L^2(\mathbb{R}^n)$ belongs to $C_b^u(\mathbb{R}^n) \rtimes \mathbb{R}^n$ if and only if* $\lim_{x\to 0} \|(U_x - 1)T^{(*)}\| \to 0$ *and* $\lim_{k\to 0} \|V_k^* T V_k - T\| \to 0$.

1.6. Localizations at infinity. We consider here the largest algebra of interactions $\mathscr{A} = C_b^u(X)$. A rather detailed sketch of the proof of Theorem 1.4 can be found in [22]; complete proofs and applications will be published elsewhere. We must first give a "convenient" description of $C_b^u(X)/C_0(X)$. As we mentioned above, there are many such descriptions; our choice is motivated by the desire to obtain an algorithm efficient in practical computations.

Let $\varphi \in C_b^u(X)$ and let $\varkappa$ be an ultrafilter on X finer than the Fréchet filter; we denote by γX the set of all such ultrafilters. The *localization at infinity* of φ at the point $\varkappa$ is the function $\varphi_\varkappa \in C_b^u(X)$ given by $\varphi_\varkappa(x) = \lim_y \varphi(x+y)$, where $y \to \infty$ along the filter $\varkappa$ and the limit exists locally uniformly in x. For example, it is easy to check that all the localizations at infinity of φ are constant functions if and only if $\lim_{y\to\infty}[\varphi(x+y) - \varphi(y)] = 0$ for each $x \in X$. It can be shown that the map $\varphi \mapsto \{\varphi_\varkappa\}_{\varkappa\in\gamma X}$ is a morphism $C_b^u(X) \to C_b^u(X)^{[\gamma X]}$ (see (2.5)) with $C_0(X)$ as kernel. Thus we get a canonical embedding $C_b^u(X)/C_0(X) \subset C_b^u(X)^{[\gamma X]}$. From (1.3) and (3.7) we then deduce an embedding $C_b^u(X) \rtimes X/\mathscr{K}(X) \subset \left[C_b^u(X) \rtimes X\right]^{[\gamma X]}$ which, in turn, allows one to prove the following:

Theorem 1.4. *Let H be an observable affiliated to the algebra $C_b^u(X) \rtimes X$. Then for each $\varkappa \in \gamma X$ the strong limit* $\text{s-lim}_x U_x H U_x^* = H_\varkappa$ *exists when $x \to \infty$ along $\varkappa$ and*

$$\boldsymbol{\sigma}_{\text{ess}}(H) = \bigcup\nolimits_\varkappa \boldsymbol{\sigma}(H_\varkappa). \tag{1.10}$$

By strong convergence we mean $\text{s-lim}_x \theta(U_x H U_x^*) = \theta(H_\varkappa)$ for each $\theta \in C_0(X)$. The observables $H_\varkappa$ are affiliated to $C_b^u(X) \rtimes X$ and will be called *localizations at infinity of H*. The proof of the theorem and a better insight of the objects involved require the Stone–Čech compactification of X. We give some applications of Theorem 1.4 with $X = \mathbb{R}^n$ in order to make the connection with [24].

Theorem 1.3 allows us to get many hamiltonians affiliated to $C_b^u(\mathbb{R}^n) \rtimes \mathbb{R}^n$ which cannot be obtained with the help of Theorem 2.1 (because the perturbation V will not be comparable with H_0). Assume that H is a self-adjoint operator on $L^2(\mathbb{R}^n)$ such that $\mathscr{G} = D(|H|^{1/2}) \subset D(\theta(P))$ with $\theta(p) \to \infty$ if $p \to \infty$. Then $T = (H+i)^{-1}$ satisfies the first condition of Theorem 1.3 (and conversely). To ensure the second condition, we ask $V_k\mathscr{G} \subset \mathscr{G}$ for all $k \in \mathbb{R}^n$ and $\lim_{k\to 0} \|V_k^* H V_k - H\|_{\mathscr{G}\to\mathscr{G}^*} = 0$. Then H is affiliated to $C_b^u(\mathbb{R}^n) \rtimes \mathbb{R}^n$.

For example, consider a generalized elementary hamiltonian $H = h(P) + V(Q)$, where h, V are real functions on $\mathbb{R}^n$. Assume h continuous, polynomially bounded, $h(p) \to \infty$ if $p \to \infty$, and $\lim_{k\to 0} \sup_p |h(p+k) - h(p)|(1+|h(p)|)^{-1} = 0$. Let V be locally integrable and assume that its negative part is form bounded with respect to $h(P)$ with relative bound < 1. Then H is a well defined self-adjoint operator (sum in the sense of forms) with $\mathscr{G} = D(|h(P)|^{1/2}) \cap D(V_+(Q)^{1/2})$ and the preceding conditions are satisfied. We have $U_x H U_x^* = h(P) + V(x+Q)$, so the localizations at infinity of H are determined by the (suitably defined) localizations at infinity of the function V. Thus, in order to compute $\boldsymbol{\sigma}_{\text{ess}}(H)$, we are once again reduced to an abelian situation. The “elementary” case, when $V \in C_b^u(\mathbb{R}^n)$, is very easy: we have $H_\varkappa = h(P) + V_\varkappa(Q)$, where the localizations at infinity $V_\varkappa$ are as defined before.

More interesting is the case of unbounded potentials. For simplicity we consider functions V bounded from below and of class C^{m+1} for some $m \in \mathbb{N}$, such that $V^{(\alpha)}(x) \to 0$ if $|\alpha| = m+1$ and $x \to \infty$. Then, if $\varkappa \in \gamma\mathbb{R}^n$, there are only two possibilities (the limits are taken along $\varkappa$): either $\lim_y V(x+y) = +\infty$ for almost all $x \in X$, or $\lim_y V(x+y) =: V_\varkappa(x)$ exists (and is finite) locally uniformly in $x \in X$. In the second case $V_\varkappa$ is a polynomial (bounded from below) of degree $\leq m$ and these polynomials will be called localizations at infinity of V. Strictly speaking, V has one more localization, the function equal to $+\infty$ almost everywhere; but the corresponding $H_\varkappa$ is the observable ∞ and $\boldsymbol{\sigma}(\infty) = \emptyset$, so it does not contribute to the union from (1.10). The next result covers those from [24] when the magnetic field is absent.

Theorem 1.5. *Under the preceding conditions*

$$\boldsymbol{\sigma}_{\text{ess}}(h(P)+V)=\bigcup_v \boldsymbol{\sigma}(h(P)+v),$$

where the union is performed over all the localizations at infinity v of V.

We shall give an explicit example in the case $n=1$. Note that if $\varkappa \in \gamma\mathbb{R}$ then either $[0,\infty)\in\varkappa$ or $(-\infty,0)\in\varkappa$. Thus there are two contributions $\boldsymbol{\sigma}^{\pm}_{\text{ess}}(H)$ to the union from (1.10) and $\boldsymbol{\sigma}_{\text{ess}}(H)=\boldsymbol{\sigma}^{+}_{\text{ess}}(H)\cup\boldsymbol{\sigma}^{-}_{\text{ess}}(H)$. We take $H=h(P)+V(Q)$ on $L^2(\mathbb{R})$, where h is as before and $V:\mathbb{R}\to\mathbb{R}$ is continuous and bounded from below. Then H is affiliated to $C^{\mathrm{u}}_{\mathrm{b}}(\mathbb{R})\rtimes\mathbb{R}$ and $\boldsymbol{\sigma}^{\pm}_{\text{ess}}(H)$ is determined by the behavior of V at $\pm\infty$. Assume now that for large positive x we have $V(x)=x^a\omega(x^\theta)$ with $a\geq 0$, $0<\theta<1$ and ω a positive continuous periodic function with period 1. Moreover, we assume that ω vanishes only at the points of $\mathbb{Z}$ and that there are real numbers $\lambda,\mu>0$ such that $\omega(t)\sim\lambda|t|^\mu$ when $t\to 0$ (different asymptotics from left and right can be treated). Then there are three possibilities:

(1) If $a<\mu(1-\theta)$ the localizations at $+\infty$ of V are all the non-negative constant functions, thus $\boldsymbol{\sigma}^{+}_{\text{ess}}(H)=[\inf h,+\infty)$.
(2) If $a=\mu(1-\theta)$ the localizations at $+\infty$ of V are the functions $v(x)=\lambda|\theta x+c|^\mu$ with $c\in\mathbb{R}$. Thus $\boldsymbol{\sigma}^{+}_{\text{ess}}(H)=\boldsymbol{\sigma}(h(P)+\lambda|\theta Q|^\mu)$, hence it is a discrete not empty set.
(3) If $a>\mu(1-\theta)$ the only localization at $+\infty$ of V is $+\infty$, so $\boldsymbol{\sigma}^{+}_{\text{ess}}(H)=\emptyset$.

1.7. Quantum fields. We shall discuss here the C^*-algebra of hamiltonian operators of a quantum field, extending thus the results from [20], where only models with a particle number cut-off are considered. Our main purpose is to explain how one can derive a Mourre estimate from a knowledge of this algebra, so we shall restrict ourselves to the case when the one-particle Hilbert space is $\mathfrak{H}=L^2(\mathbb{R}^s)$, although most of the next considerations are valid in an abstract and general setting, like in [20]. We refer to [15] for a proof of the Mourre estimate for the $P(\phi)_2$ model and for the second quantization formalism that we use without further explanation. We recall only that the field operator is $\phi(u)=(a(u)+a^*(u))/\sqrt{2}$ if $u\in\mathfrak{H}$.

The Hilbert space generated by the states of the field is the symmetric Fock space $\Gamma(\mathfrak{H})$. We take $C_0(\mathbb{R}^{s*})$ as a C^*-algebra of one-particle kinetic energies and our purpose is to study models for which the "elementary" hamiltonians (compare with Sect. 1.1) are of the form $d\Gamma(\omega)+W$, where ω is affiliated to $C_0(\mathbb{R}^{s*})$ with $\inf\omega\equiv m>0$ and W is a polynomial in the field operators with a particle number cut-off (we stress that one of the main points of our approach is to start with a small class of elementary hamiltonians which, however, should generate a C^*-algebra to which the physically realistic hamiltonians are affiliated). Let $\mathfrak{C}=C_\infty(\mathbb{R}^{s*})$. An argument similar to that of the proof of Theorem 1.1 justifies the following definition: the *algebra of energy observables* of the quantum field is the C^*-algebra $\mathscr{C}$ generated by the operators $\phi(u)\Gamma(S)$, where $u\in\mathfrak{H}$ and $S\in\mathfrak{C}$ with $\|S\|<1$. If we denote $\mathscr{K}(\mathfrak{H})=K(\Gamma(\mathfrak{H}))$, the main result is:

Theorem 1.6. *There is a unique morphism $\mathcal{P}:\mathscr{C}\to\mathfrak{C}\otimes\mathscr{C}$ such that $\mathcal{P}[\phi(u)\Gamma(S)]=S\otimes[\phi(u)\Gamma(S)]$. The kernel of this morphism is $\mathscr{K}(\mathfrak{H})$ (which is a subset of $\mathscr{C}$). Thus*

$$\widehat{\mathscr{C}}\equiv\mathscr{C}/\mathscr{K}(\mathfrak{H})\hookrightarrow\mathfrak{C}\otimes\mathscr{C}. \tag{1.11}$$

It is interesting to note that one can proceed as in Sect. 1.1 and define $\mathscr{C}$ as a kind of crossed product: if the *algebra of interactions* is the C^*-algebra $\mathscr{A}$ obtained by taking $\mathfrak{C} = \mathbb{C}$ above and the *algebra of kinetic energies* is the C^*-algebra $\mathscr{B}$ generated by the operators $\Gamma(S)$ with $S \in \mathfrak{C}$, $\|S\| \leq 1$, then $\mathscr{C} = [\![\mathscr{A} \cdot \mathscr{B}]\!]$ (compare with Theorem 4.1). We have $\mathscr{K}(\mathfrak{H}) \subset \mathscr{A}$ and there is a unique morphism $\mathcal{P}_0 : \mathscr{A} \to \mathscr{A}$ such that $\mathcal{P}_0[\phi(u)\Gamma(\lambda)] = \lambda\phi(u)\Gamma(\lambda)$ if $\lambda \in \mathbb{C}$ and $|\lambda| \leq 1$. $\mathcal{P}_0$ is surjective and has $\mathscr{K}(\mathfrak{H})$ as kernel, so we get a canonical identification $\widehat{\mathscr{A}} \cong \mathscr{A}$. An easy and interesting consequence is that *all the operators in $\mathscr{A}$ have a countable spectrum* (note that $\|\mathcal{P}_0^k[T]\| \to 0$ if $k \to \infty$). $\mathscr{A}$ is also the algebra generated by $\phi(u)\varphi(\mathrm{N})$ with $u \in \mathfrak{H}$ and $\varphi \in C_{\mathrm{c}}(\mathbb{R})$, where N is the particle number operator, and $\mathcal{P}_0$ is uniquely determined by the relation $\mathcal{P}_0[\phi(u)\varphi(\mathrm{N})] = \phi(u)\varphi(\mathrm{N}+1)$.

In the present situation the most convenient affiliation criterion is the following: if H is a self-adjoint bounded from below operator on $\Gamma(\mathfrak{H})$, and if $\mathrm{e}^{-H} \in \mathscr{C}$, then H is affiliated to $\mathscr{C}$. For example, if ω is as above and the symmetric operator W is a (generalized) polynomial in the field operators, and if $W_n = \chi_n(\mathrm{N})W\chi_n(\mathrm{N})$ (where $n \in \mathbb{N}$ and χ is the characteristic function of $[0, n]$), then it is easy to see that $\mathrm{e}^{-W_n}\Gamma(\mathrm{e}^{-\omega}) \in \mathscr{C}$ and $\mathcal{P}\left[\mathrm{e}^{-W_n}\Gamma(\mathrm{e}^{-\omega})\right] = \mathrm{e}^{-\omega} \otimes \left[\mathrm{e}^{-W_{n-1}}\Gamma(\mathrm{e}^{-\omega})\right]$. Then the "norm convergence" version of the Trotter–Kato formula shows that $H(n) = d\Gamma(\omega) + W_n$ is affiliated to $\mathscr{C}$ and $\mathcal{P}\left[\mathrm{e}^{-H(n)}\right] = \mathrm{e}^{-\omega} \otimes \mathrm{e}^{-H(n-1)}$. If there is a self-adjoint operator H such that $\mathrm{e}^{-H(n)} \to \mathrm{e}^{-H}$ in norm as $n \to \infty$, we get that H is affiliated to $\mathscr{C}$ and $\widehat{H} = \omega \otimes 1 + 1 \otimes H$. These ideas must be used in conjunction with the fact that affiliation is preserved by convergence in the norm resolvent sense of sequences of self-adjoint operators. In this way one can prove, for example, that the hamiltonian of the $P(\phi)_2$ model ($s = 1$) with a spatial cut-off is affiliated to $\mathscr{C}$.

We come now to the question of the Mourre estimate for a hamiltonian H of the preceding type. We refer to [20] for a résumé of the Mourre method adapted to the present case. Here we consider only conjugate operators of the form $A = d\Gamma(\mathfrak{a})$, where $\mathfrak{a} = F(P)Q + QF(P)$ and F is a vector field of class C_{c}^∞; such an A will be called *standard*. A self-adjoint operator on $\Gamma(\mathfrak{H})$ which is of class[4] $C_{\mathrm{u}}^1(A)$ or $C^{1,1}(A)$ for each standard A will be called of class C_{u}^1 or $C^{1,1}$, respectively.

Theorem 1.7. *Let H be a bounded from below hamiltonian strictly affiliated to $\mathscr{C}$ and such that $\widehat{H} = \omega(P) \otimes 1 + 1 \otimes H$, where $\omega : \mathbb{R}^s \to \mathbb{R}$ (the one-particle kinetic energy) is a function of class C^1, $\inf \omega \equiv m > 0$, and $\omega(p) \to \infty$ if $p \to \infty$. Then $\boldsymbol{\sigma}_{\mathrm{ess}}(H) = [m + \inf H, \infty)$. Assume that H is of class C_{u}^1. Denote $\boldsymbol{\kappa}(\omega)$ the set of critical values of the function ω, let $\boldsymbol{\kappa}_n(\omega) = \boldsymbol{\kappa}(\omega) + \cdots + \boldsymbol{\kappa}(\omega)$ (n terms), and define the threshold set of H by*

$$\boldsymbol{\tau}(H) = \textstyle\bigcup_{n=1}^\infty \left[\boldsymbol{\kappa}_n(\omega) + \boldsymbol{\sigma}_{\mathrm{p}}(H)\right] \tag{1.12}$$

where $\boldsymbol{\sigma}_{\mathrm{p}}(H)$ is the set of eigenvalues of H. Then $\boldsymbol{\tau}(H)$ is a closed set and H admits a standard local conjugate operator at each point not in $\boldsymbol{\tau}(H)$. In particular, the eigenvalues of H which do not belong to $\boldsymbol{\tau}(H)$ are of finite multiplicity and their accumulation points belong to $\boldsymbol{\tau}(H)$. If H is of class $C^{1,1}$, then it has no singular continuous spectrum outside $\boldsymbol{\tau}(H)$. If we also assume that $\boldsymbol{\kappa}(\omega)$ is countable, then $\boldsymbol{\tau}(H)$ is countable too, so H has no singular continuous spectrum.

[4] H is of class $C_{\mathrm{u}}^1(A)$ if the map $t \mapsto \mathrm{e}^{itA}(H+i)^{-1}\mathrm{e}^{-itA}$ is of class C^1 in norm. The $C^{1,1}(A)$ class is defined by requiring that this map be of Besov class $B_\infty^{1,1}$, a slightly stronger regularity condition.

The preceding result is a rather straightforward consequence of Theorem 1.6, as explained in [20]. We use standard operators A associated with C_c^∞ vector fields satisfying $F(p)\cdot\nabla\omega(p)\geq 0$. We define the *A-threshold set* $\boldsymbol{\tau}_A(H)$ of H as the set of real numbers λ such that: if $\varphi(H)^*[H,iA]\varphi(H)\geq a|\varphi(H)|^2+K$ with a real, $\varphi\in C_c^\infty$, $\varphi(\lambda)\neq 0$, and K a compact operator, then $a\leq 0$. Obviously $\boldsymbol{\tau}_A(H)\subset\boldsymbol{\sigma}_{\mathrm{ess}}(H)$. The *A-critical set* $\boldsymbol{\kappa}_A(H)$ of H is defined in the same way but with $K=0$. Let $\boldsymbol{\sigma}_{\mathrm{p}}(H)$ be the set of eigenvalues of H. Then $\boldsymbol{\kappa}_A(H)=\boldsymbol{\tau}_A(H)\cup\boldsymbol{\sigma}_{\mathrm{p}}(H)$ and $\boldsymbol{\kappa}_A(H)\setminus\boldsymbol{\tau}_A(H)$ consists of eigenvalues of finite multiplicity which can accumulate only toward $\boldsymbol{\tau}_A(H)$. The expression for $\widehat{H}$ given in the theorem implies $\boldsymbol{\tau}_A(H)=\boldsymbol{\kappa}_{\mathfrak{a}}(\omega(P))+\boldsymbol{\kappa}_A(H)$ (see [20]). This suggests to consider the set $\boldsymbol{\tau}(H)$ satisfying the relation

$$\boldsymbol{\tau}(H)=\boldsymbol{\kappa}(\omega)+\left[\boldsymbol{\tau}(H)\cup\boldsymbol{\sigma}_{\mathrm{p}}(H)\right]=[\boldsymbol{\kappa}(\omega)+\boldsymbol{\tau}(H)]\cup\left[\boldsymbol{\kappa}(\omega)+\boldsymbol{\sigma}_{\mathrm{p}}(H)\right].$$

The unique solution is given by (1.12).

Observe that the strict positivity condition $m>0$ plays an important role above. This is no longer necessary if we consider hamiltonians with a particle number cut-off, as in [20]. Indeed, if H is given by a formal expression $H=d\Gamma(\omega)+W$, the restrictions $H_n=\chi_n(\mathrm{N})H\chi_n(\mathrm{N})$ are often well defined self-adjoint operators and they satisfy $\widehat{H_n}=\omega\otimes 1+1\otimes H_{n-1}$. Then the threshold set of H_n is defined by the relation (with $\boldsymbol{\sigma}_{\mathrm{p}}(H_0)=\{0\}$):

$$\boldsymbol{\tau}(H_n)=\textstyle\bigcup_{i=1}^{n}\left[\boldsymbol{\kappa}_i(\omega)+\boldsymbol{\sigma}_{\mathrm{p}}(H_{n-i})\right]. \tag{1.13}$$

1.8. Coupling of two systems. We have mentioned in Sect. 1.1 that in the applications it is often useful to consider C^*-algebras of hamiltonians of the form $\mathscr{C}\otimes K(\mathrm{E})$. Physically speaking, this means that we couple the system having $\mathscr{C}$ as C^*-algebra of energy observables with a confined system having $K(\mathrm{E})$ as C^*-algebra of hamiltonians (the observables affiliated to $K(\mathrm{E})$ have purely discrete spectrum). We shall consider now the coupling of two arbitrary systems. Assume that $\mathscr{C}_1,\mathscr{C}_2$ are C^*-algebras of operators on the Hilbert spaces $\mathscr{H}_1,\mathscr{H}_2$ respectively such that $K(\mathscr{H}_i)\subset\mathscr{C}_i$. We think of $\mathscr{C}_i$ as the algebra of hamiltonians of the system i which has $\mathscr{H}_i$ as state space. Then we take $\mathscr{H}_1\otimes\mathscr{H}_2$ as the state space of the coupled system and $\mathscr{C}_1\otimes\mathscr{C}_2$ as its algebra of energy observables. Since $K(\mathscr{H}_1)\otimes K(\mathscr{H}_2)=K(\mathscr{H}_1\otimes\mathscr{H}_2)$ we are in a situation similar to the preceding ones: $K(\mathscr{H}_1\otimes\mathscr{H}_2)\subset\mathscr{C}_1\otimes\mathscr{C}_2\subset B(\mathscr{H}_1\otimes\mathscr{H}_2)$. We shall prove in Sect. 2.4 that *if the C^*-algebras $\mathscr{C}_1,\mathscr{C}_2$ are nuclear, then there is a canonical embedding*

$$\widehat{\mathscr{C}_1\otimes\mathscr{C}_2}\subset\left[\widehat{\mathscr{C}_1}\otimes\mathscr{C}_2\right]\oplus\left[\mathscr{C}_1\otimes\widehat{\mathscr{C}_2}\right], \tag{1.14}$$

where a hat means a quotient with respect to the ideal of compact operators. In particular, let $\mathcal{P}_i$ be the natural morphism $\mathscr{C}_i\to\widehat{\mathscr{C}_i}$. If H is an observable affiliated to $\mathscr{C}_1\otimes\mathscr{C}_2$, let $H_1=(\mathcal{P}_1\otimes 1)[H]$ and $H_2=(1\otimes\mathcal{P}_2)[H]$. Then we get $\boldsymbol{\sigma}_{\mathrm{ess}}(H)=\boldsymbol{\sigma}(H_1)\cup\boldsymbol{\sigma}(H_2)$. For example, these results allow one to study quantum fields interacting with N-body systems.

2. Observables and Their Essential Spectra

2.1. Observables and self-adjoint operators. We recall several notations and conventions which are usual in the theory of C^*-algebras. A $*$-homomorphism between two C^*-algebras will be called *morphism*. $\mathscr{A}\cong\mathscr{B}$ means that the C^*-algebras $\mathscr{A}$ and $\mathscr{B}$

are canonically isomorphic; in such a situation the canonical morphism is either obvious from the context or we give it explicitly. By *ideal* we mean a closed bilateral (hence self-adjoint) ideal. We make the same conventions for the more general case of Banach $*$-algebras.

We say that an observable H affiliated to a C^*-algebra $\mathscr{C}$ is *strictly affiliated* to $\mathscr{C}$ if the linear subspace generated by $\{\varphi(H)S \mid \varphi \in C_0(\mathbb{R}),\ S \in \mathscr{C}\}$ is dense in $\mathscr{C}$. Now consider the case where the C^*-algebra $\mathscr{C}$ is realized on a Hilbert space $\mathscr{H}$. The affiliation of a self-adjoint operator H on $\mathscr{H}$ to $\mathscr{C}$ has been defined in Sect. 1.2 and the strict affiliation is defined in an obvious way. We mention that there are no observables strictly affiliated to $B(\mathscr{H})$ (if $\dim \mathscr{H} = \infty$) and that the operator of multiplication by the function $h(x) = x + x^{-1}$ in $L^2(\mathbb{R})$ is affiliated to $C_0(\mathbb{R})$ but not strictly.

The observables affiliated to $\mathscr{C}$ can always be realized as operators on $\mathscr{H}$, but these operators are not densely defined in general. On the other hand, if $\mathscr{C}$ is *nondegenerate on* $\mathscr{H}$ (i.e. if the elements Sf, with $S \in \mathscr{C}$ and $f \in \mathscr{H}$, generate a dense linear subspace), then the correspondence between self-adjoint operators on $\mathscr{H}$ strictly affiliated to $\mathscr{C}$ and observables strictly affiliated to $\mathscr{C}$ defined above is bijective (see [13]).

We stress once again the fact that if $\mathscr{J}$ is an ideal in $\mathscr{C}$ and H is a self-adjoint operator affiliated to $\mathscr{C}$, then the quotient $\widehat{H}$ is a well defined observable affiliated to $\widehat{\mathscr{C}} = \mathscr{C}/\mathscr{J}$. But this operation is meaningless in a pure Hilbert space setting: in most cases $\widehat{H}$ has no meaning as an operator on $\mathscr{H}$. However, by the preceding remarks, if H is strictly affiliated to $\mathscr{C}$ then one can realize $\widehat{H}$ as a self-adjoint operator in each nondegenerate representation of $\widehat{\mathscr{C}}$.

There is a close connection between the notion of observable *strictly* affiliated to $\mathscr{C}$ and that of a self-adjoint operator affiliated with $\mathscr{C}$ as it was defined by Woronowicz in [36] (according to [37], this notion first appeared in [1]; see also Chapter 9 in [29]). More precisely, if H is such an observable, let T_H be the closure of the operator defined on the dense subset of $\mathscr{C}$ consisting of elements of the form $\varphi(H)S$ with $\varphi \in C_c(\mathbb{R})$ by $T_H\varphi(H)S = \varphi_1(H)S$, where $\varphi_1(\lambda) = \lambda\varphi(\lambda)$. Then $H \mapsto T_H$ is a bijection between the set of observables strictly affiliated to $\mathscr{C}$ and the set of self-adjoint operators affiliated with $\mathscr{C}$ in the sense of Woronowicz. Let us note that the observables affiliated to $C_0(X)$ are the continuous functions on open subsets of X, whereas the self-adjoint operators affiliated with the same algebra in the sense of Woronowicz are the functions from $C(X)$.

The point of view of Woronowicz is convenient in two respects: (1) it is easy to consider operators more general than self-adjoint, and (2) there is an obvious candidate for the sum of two such operators. On the other hand, our definition makes the operation of taking the image through a morphism (hence of taking the quotient with respect to an ideal) very natural and easy to define, and this is the operation of main interest in our approach. Moreover, we emphasize that observables not strictly affiliated to $\mathscr{C}$ play an important role here: for example, most of the localizations at infinity which appear in Theorem 1.4 are of such type. Besides the trivial observable $H = \infty$ (defined by $\varphi(H) = 0$ for all $\varphi \in C_0(\mathbb{R})$) and those of the examples given above, we mention that the hamiltonian of an N-body system with hard-core interactions is affiliated but not strictly to the N-body algebra (see Sect. 1.4).

Let H_0 be a self-adjoint bounded from below operator on $\mathscr{H}$. Let V be a continuous symmetric sesquilinear form on $\mathscr{G} = D(|H_0|^{\frac{1}{2}})$ such that $V \geq -\mu H_0 - \delta$ as forms on $\mathscr{G}$, for some numbers $\mu \in [0, 1)$ and $\delta \in \mathbb{R}$. Then the form sum $H = H_0 + V$ is a self-adjoint operator on $\mathscr{H}$ with the same form domain as H_0. We are interested in conditions which ensure the affiliation of H to $\mathscr{C}$ if H_0 is affiliated to $\mathscr{C}$. The following result is from [13]. Let λ be any real number such that $H_0 + \lambda \geq c > 0$.

Theorem 2.1. *If H_0 is strictly affiliated to $\mathscr{C}$ and $(H_0+\lambda)^{-\alpha}V(H_0+\lambda)^{-1/2}$ belongs to $\mathscr{C}$ for some $\alpha \geq 1/2$, then H is strictly affiliated to $\mathscr{C}$.*

2.2. A formula for $\boldsymbol{\sigma}_{\text{ess}}(H)$. Let H be an observable affiliated to a C^*-algebra $\mathscr{C}$, $\mathscr{J}$ an ideal in $\mathscr{C}$, and $\widehat{H}$ the quotient of H with respect to $\mathscr{J}$. Clearly

$$\boldsymbol{\sigma}(\widehat{H}) = \{\lambda \in \mathbb{R} \mid \varphi \in C_0(\mathbb{R}) \text{ and } \varphi(\lambda) \neq 0 \Longrightarrow \varphi(H) \notin \mathscr{J}\}. \tag{2.1}$$

In this subsection we give a description of $\boldsymbol{\sigma}(\widehat{H})$ in a situation important for us.

Let $\{\mathscr{C}_i\}_{i\in I}$ be an arbitrary family of C^*-algebras. We recall the definition of their direct product $\prod_{i\in I}\mathscr{C}_i$ and their direct sum $\bigoplus_{i\in I}\mathscr{C}_i$:

$$\prod_{i\in I}\mathscr{C}_i = \{S = (S_i)_{i\in I} \mid S_i \in \mathscr{C}_i \text{ and } \|S\| := \sup_{i\in I}\|S_i\| < \infty\},$$

$$\bigoplus_{i\in I}\mathscr{C}_i = \{S = (S_i)_{i\in I} \mid S_i \in \mathscr{C}_i \text{ and } \|S_i\| \to 0 \text{ as } i \to \infty\}.$$

These are C^*-algebras for the usual operations and $\bigoplus_{i\in I}\mathscr{C}_i$ is an ideal in $\prod_{i\in I}\mathscr{C}_i$. We denote by $\prod_{i\in I} S_i$ and $\bigoplus_{i\in I} S_i$ an element of $\prod_{i\in I}\mathscr{C}_i$ and $\bigoplus_{i\in I}\mathscr{C}_i$ respectively.

If for each $i \in I$ an observable H_i affiliated to $\mathscr{C}_i$ is given, we may associate to it an observable $H = \prod_{i\in I} H_i$ affiliated to $\mathscr{C} = \prod_{i\in I}\mathscr{C}_i$ by setting $\varphi(H) = \prod_{i\in I}\varphi(H_i)$ for each $\varphi \in C_0(\mathbb{R})$. It is easily shown that H is affiliated to the subalgebra $\bigoplus_{i\in I}\mathscr{C}_i$ if and only if $H_i \to \infty$ as $i \to \infty$ in I in the following sense: for each compact real set K there is a finite subset $F \subset I$ such that $\boldsymbol{\sigma}(H_i) \cap K = \emptyset$ if $i \in I \setminus F$. One has

$$\boldsymbol{\sigma}(H) = \overline{\bigcup_{i\in I}\boldsymbol{\sigma}(H_i)}, \tag{2.2}$$

and if H is affiliated to $\bigoplus_{i\in I}\mathscr{C}_i$ then the union is already closed. We will need the following generalization of this relation.

Theorem 2.2. *For each $i \in I$ let $\mathscr{J}_i$ be an ideal in $\mathscr{C}_i$ and let $\mathscr{J} = \bigoplus_{i\in I}\mathscr{J}_i$, so that $\mathscr{J}$ is an ideal in $\mathscr{C} = \prod_{i\in I}\mathscr{C}_i$. Denote by $\widehat{H_i}$ the quotient of H_i in $\mathscr{C}_i/\mathscr{J}_i$ and let $\widehat{H}$ be the quotient of H in $\mathscr{C}/\mathscr{J}$. Then*

$$\boldsymbol{\sigma}(\widehat{H}) = \bigcap_{\substack{F\subset I\\ F \text{ finite}}} \left\{ \left(\bigcup_{i\in F}\boldsymbol{\sigma}(\widehat{H_i})\right) \cup \left(\overline{\bigcup_{j\in I\setminus F}\boldsymbol{\sigma}(H_j)}\right)\right\}. \tag{2.3}$$

Proof. Let $\lambda \notin \boldsymbol{\sigma}(\widehat{H})$. By (2.1) there exists $\varphi \in C_0(\mathbb{R})$ such that $\varphi(\mu) = 1$ on a neighborhood J of λ and $\varphi(H) \in \mathscr{J}$. Thus for all $i \in I$ one has $\varphi(H_i) \in \mathscr{J}_i$ and $\|\varphi(H_i)\| \to 0$ as $i \to \infty$ in I. The first assertion shows (again by (2.1)) that $\lambda \notin \bigcup_{i\in I}\boldsymbol{\sigma}(\widehat{H_i})$ and the second one ensures the existence of a finite set $F \subset I$ such that $\|\varphi(H_i)\| < 1$ if $i \notin F$. But $\sup\{|\varphi(x)| \mid x \in \boldsymbol{\sigma}(H_i)\} = \|\varphi(H_i)\| < 1$, so $J \cap \boldsymbol{\sigma}(H_i) = \emptyset$ for all $i \in I \setminus F$, hence $\lambda \notin \overline{\bigcup_{i\in I\setminus F}\boldsymbol{\sigma}(H_i)}$. Since $\boldsymbol{\sigma}(\widehat{H_i}) \subset \boldsymbol{\sigma}(H_i)$ for all i, we get $\lambda \notin \bigcup_{i\in I}\boldsymbol{\sigma}(\widehat{H_i}) \cup \overline{\bigcup_{i\in I\setminus F}\boldsymbol{\sigma}(H_i)} = \bigcup_{i\in F}\boldsymbol{\sigma}(\widehat{H_i}) \cup \overline{\bigcup_{i\in I\setminus F}\boldsymbol{\sigma}(H_i)}$ for some finite F.

Conversely, if λ does not belong to the r.h.s. of (2.3) (which is a closed set of the form $\bigcap_{F\subset I}\Sigma_F$) there is a compact neighborhood J of λ disjoint from it. Since the upper directed family of open sets $\mathbb{R}\setminus\Sigma_F$ covers J, there is F_0 (a finite subset of I) such that $J \subset \mathbb{R}\setminus\Sigma_{F_0}$. Thus $J \cap \bigcup_{i\in F_0}\boldsymbol{\sigma}(\widehat{H_i})$ and $J \cap \bigcup_{i\in I\setminus F_0}\boldsymbol{\sigma}(H_i)$ are empty sets. This means that there is a $\varphi \in C_c(J)$, with $\varphi = 1$ on a neighborhood of λ, such that $\varphi(H_i) \in \mathscr{J}_i$ for all $i \in F_0$ and $\varphi(H_i) = 0$ for all $i \notin F_0$. In particular $\varphi(H_i) \in \mathscr{J}_i$ for all $i \in I$ and $\|\varphi(H_i)\| \to 0$ as $i \to \infty$ in I. Thus $\varphi(\widehat{H}) \in \mathscr{J}$, i.e. $\lambda \notin \boldsymbol{\sigma}(\widehat{H})$. □

It is interesting to remark on the similarity between (2.1) and one of the characterizations of the usual notion of essential spectrum in a Hilbert space setting (see Sect. 1.2). It is thus natural to call this set the *essential spectrum of H with respect to the ideal $\mathscr{J}$* and to denote it $\mathscr{J}\text{-}\boldsymbol{\sigma}_{\text{ess}}(H)$. Then (2.3) may be written as:

$$\mathscr{J}\text{-}\boldsymbol{\sigma}_{\text{ess}}(H) = \bigcap_{\substack{F\subset I \\ F\,\text{finite}}} \left\{ \left(\bigcup_{i\in F} \mathscr{J}_i\text{-}\boldsymbol{\sigma}_{\text{ess}}(H_i)\right) \cup \left(\overline{\bigcup_{j\in I\setminus F} \boldsymbol{\sigma}(H_j)}\right)\right\}.$$

Assume, more specifically, that each $\mathscr{C}_i$ is realized on a Hilbert space $\mathscr{H}_i$ and that $\mathscr{J}_i = K(\mathscr{H}_i)$. Let $\mathscr{H} = \bigoplus_{i\in I} \mathscr{H}_i$ and let us realize $\mathscr{C}$ on $\mathscr{H}$ in the usual way. It is easy to show that $\mathscr{C} \cap K(\mathscr{H}) = \mathscr{J}$. We get

$$\boldsymbol{\sigma}_{\text{ess}}(H) = \bigcap_{\substack{F\subset I \\ F\,\text{finite}}} \left\{ \left[\bigcup_{i\in F} \boldsymbol{\sigma}_{\text{ess}}(H_i)\right] \cup \left[\overline{\bigcup_{i\in I\setminus F} \boldsymbol{\sigma}(H_i)}\right]\right\}. \tag{2.4}$$

2.3. Restricted products of C^*-algebras. We consider here the case when $\mathscr{C}_i \equiv \mathscr{A}$ is an algebra independent of $i \in I$. Then we denote by $\mathscr{A}^{(I)}$ the C^*-algebra $\bigoplus_{i\in I} \mathscr{A}$. Besides the direct product and the direct sum one can introduce now a third C^*-algebra:

$$\mathscr{A}^{[I]} = \{(S_i)_{i\in I} \in \textstyle\prod_{i\in I} \mathscr{A} \mid \{S_i \mid i \in I\} \text{ is relatively compact in } \mathscr{A}\}. \tag{2.5}$$

This is a C^*-subalgebra of $\prod_{i\in I} \mathscr{A}$ and $\mathscr{A}^{(I)}$ is an ideal in $\mathscr{A}^{[I]}$. We denote by $c_0(I;\mathscr{A})$ and $l^\infty(I;\mathscr{A})$ the C^*-algebras consisting of $\mathscr{A}$ valued maps on I which converge to zero at infinity (I being equipped with the discrete topology) or are bounded, respectively. Then $\mathscr{A}^{(I)} = c_0(I;\mathscr{A})$ and $\prod_{i\in I} \mathscr{A} = l^\infty(I;\mathscr{A})$. Moreover,

$$\mathscr{A}^{[I]} = l^{\text{rc}}(I;\mathscr{A}) := \{S : I \to \mathscr{A} \mid S \text{ has relatively compact range in } \mathscr{A}\}.$$

Lemma 2.1. *If $\mathscr{A}$ is a C^*-algebra, then*

$$\mathscr{A}^{(I)} \cong c_0(I) \otimes \mathscr{A} \quad\text{and}\quad \mathscr{A}^{[I]} \cong l^\infty(I) \otimes \mathscr{A}. \tag{2.6}$$

Proof. The first relation is obvious. To prove the second one, assume that $\mathscr{A} \subset B(\mathscr{H})$ and realize $l^\infty(I)$ as a C^*-algebra on $l^2(I)$ in the standard way. Then $l^\infty(I) \otimes_{\text{alg}} \mathscr{A}$ is realized on $l^2(I) \otimes \mathscr{H} = l^2(I;\mathscr{H})$ as the set $l^\infty_{\text{fin}}(I;\mathscr{A})$ of operators of multiplication by functions $F : I \to \mathscr{A}$ such that the range of F is included in a finite dimensional subspace of $\mathscr{A}$. Finally, the fact that the closure of $l^\infty_{\text{fin}}(I;\mathscr{A})$ in $l^\infty(I;\mathscr{A})$ is equal to $l^{\text{rc}}(I;\mathscr{A})$ is easy to prove. □

One more object will appear naturally in our later investigations: the *I-asymptotic algebra* of $\mathscr{A}$. This is the quotient algebra:

$$\mathscr{A}^{\langle I\rangle} := \mathscr{A}^{[I]} \big/ \mathscr{A}^{(I)}. \tag{2.7}$$

The following description of $\mathscr{A}^{\langle I\rangle}$ explains the name we gave it. Let $\widetilde{I} \equiv \gamma I$ be the set of ultrafilters $\varkappa$ on I finer than the Fréchet filter, equipped with its natural topology of compact space ($\widetilde{I}$ is the boundary of I in its Stone-Čech compactification). Then for each $S = (S_i)_{i \in I} \in \mathscr{A}^{[I]}$ and each $\varkappa \in \widetilde{I}$ the limit $\lim_i S_i := S_\varkappa$ as $i \to \infty$ along the filter $\varkappa$ exists (because the range of the map $S : I \to \mathscr{A}$ is included in a compact subset of $\mathscr{A}$). It can be shown that the morphism $S \mapsto (S_\varkappa)_{\varkappa \in \widetilde{I}}$ has $\mathscr{A}^{(I)}$ as kernel and induces an isomorphism (see [22] for details)

$$\mathscr{A}^{\langle I \rangle} \cong C(\widetilde{I}\,; \mathscr{A}). \tag{2.8}$$

We now describe a certain class of C^*-subalgebras of $\mathscr{A}^{[I]}$ containing the ideal $\mathscr{A}^{(I)}$. Let $\mathcal{I}$ be a finite partition of I consisting of infinite sets and

$$\mathscr{A}^{\mathcal{I}} = \{(S_i)_{i \in I} \in \mathscr{A}^{[I]} \mid \lim_{i \to \infty, i \in J} S_i \equiv S_J \text{ exists in } \mathscr{A},\ \forall J \in \mathcal{I}\}. \tag{2.9}$$

Clearly, this is the set of $S \in \mathscr{A}^{[I]}$ such that, for each $J \in \mathcal{I}$ and for each ultrafilter $\varkappa \in \widetilde{I}$ with $J \in \varkappa$, the limit $S_\varkappa$ depends only on J. Note that for each $\varkappa \in \widetilde{I}$ there is a unique $J \in \mathcal{I}$ such that $J \in \varkappa$ so $\mathcal{I}$ defines also a partition of $\widetilde{I}$ (consisting of subsets which are open and closed, as it can be easily shown). The following fact is a consequence of the definition (2.9):

$$\mathscr{A}^{\mathcal{I}} \Big/ \mathscr{A}^{(I)} \cong \bigoplus\nolimits_{J \in \mathcal{I}} \mathscr{A}. \tag{2.10}$$

2.4. Tensor products. We prove here a result implying (1.14). Let $\mathscr{C}_1, \mathscr{C}_2$ be *nuclear* C^*-algebras equipped with ideals $\mathscr{J}_1, \mathscr{J}_2$. For each i let $\mathcal{P}_i : \mathscr{C}_i \to \mathscr{C}_i / \mathscr{J}_i$ be the canonical surjection and let us consider the tensor products of these morphisms with the identity map. We get morphisms $\mathcal{P}_1' = \mathcal{P}_1 \otimes 1$ and $\mathcal{P}_2' = 1 \otimes \mathcal{P}_2$ of $\mathscr{C}_1 \otimes \mathscr{C}_2$ into $\widehat{\mathscr{C}_1} \otimes \mathscr{C}_2$ and $\mathscr{C}_1 \otimes \widehat{\mathscr{C}_2}$ respectively.

Theorem 2.3. *The kernel of the morphism*

$$\mathcal{P}_1' \oplus \mathcal{P}_2' : \mathscr{C}_1 \otimes \mathscr{C}_2 \to \big[\widehat{\mathscr{C}_1} \otimes \mathscr{C}_2\big] \oplus \big[\mathscr{C}_1 \otimes \widehat{\mathscr{C}_2}\big]$$

is equal to $\mathscr{J}_1 \otimes \mathscr{J}_2$.

Proof. The nuclearity of $\mathscr{C}_2$ implies that the kernel of $\mathcal{P}_1'$ is equal to $\mathscr{J}_1 \otimes \mathscr{C}_2$ (see Theorem 6.5.2 in [32]). For the same reason we get $\ker \mathcal{P}_2' = \mathscr{C}_1 \otimes \mathscr{J}_2$. It remains to prove that

$$\big[\mathscr{J}_1 \otimes \mathscr{C}_2\big] \bigcap \big[\mathscr{C}_1 \otimes \mathscr{J}_2\big] = \mathscr{J}_1 \otimes \mathscr{J}_2. \tag{2.11}$$

Only the inclusion $\subset$ is not trivial, so assume that S belongs to the left hand side of (2.11). For each $\varepsilon > 0$ we can find $K_1, \dots, K_n \in \mathscr{J}_1$ and $T_1, \dots, T_n \in \mathscr{C}_2$ such that the operator $S' = \sum K_i \otimes T_i$ satisfies $\|S - S'\| \leq \varepsilon$. Since $\mathscr{J}_1$ has an approximate identity, we can find $K' \in \mathscr{J}_1$ with $\|K'\| \leq 1$ and $\|K'K_i - K_i\| \leq \varepsilon/(n\|T_i\|)$ for each i. Then $\|K' \otimes 1 \cdot S' - S'\| \leq \varepsilon$, hence $\|S - K' \otimes 1 \cdot S\| \leq 3\varepsilon$. Similarly we find $K'' \in \mathscr{J}_2$ with $\|K''\| \leq 1$ and $\|S - S \cdot 1 \otimes K''\| \leq 3\varepsilon$. Thus $\|S - K' \otimes 1 \cdot S \cdot 1 \otimes K''\| \leq 6\varepsilon$. Finally $\|S - K' \otimes 1 \cdot S' \cdot 1 \otimes K''\| \leq 7\varepsilon$. Since $K' \otimes 1 \cdot S' \cdot 1 \otimes K'' \in \mathscr{J}_1 \otimes \mathscr{J}_2$ and ε is arbitrary, we get $S \in \mathscr{J}_1 \otimes \mathscr{J}_2$. ◻

3. Crossed Products

3.1. Definition of crossed products. In this section we first recall the definition of crossed products in the particular case of abelian groups and then we discuss several results which we have not been able to locate in the literature in a form convenient to us. We fix a locally compact abelian group X and a Haar measure $\mathrm{d}x$ on it. But note that the crossed products $\mathscr{A} \rtimes X$ defined below are independent of the choice of $\mathrm{d}x$.

We shall say that a C^*-algebra $\mathscr{A}$ is an *X-algebra* if a homomorphism $\alpha : x \mapsto \alpha_x$ of X into the group of automorphisms of $\mathscr{A}$ is given, such that for each $a \in \mathscr{A}$ the map $x \mapsto \alpha_x(a)$ is continuous. A subalgebra of $\mathscr{A}$ is called *stable* if it is left invariant by all the automorphisms α_x. If $(\mathscr{A}, \alpha)$ and $(\mathscr{B}, \beta)$ are two X-algebras, a morphism $\phi : \mathscr{A} \to \mathscr{B}$ is called an *X-morphism* (or *covariant* morphism) if $\phi[\alpha_x(a)] = \beta_x[\phi(a)]$ for all $x \in X$ and $a \in \mathscr{A}$.

Let $\mathscr{A}$ be an X-algebra and let $L^1(X; \mathscr{A})$ be the Banach $*$-algebra constructed as follows. As a Banach space it is just the space of (Bochner) integrable (equivalence classes of) functions $S : X \to \mathscr{A}$. The product and the involution are defined by:

$$(S \cdot T)(x) = \int_X S(y)\, \alpha_y[T(x-y)]\, \mathrm{d}y, \tag{3.1}$$

$$S^*(x) = \alpha_x[S(-x)^*]. \tag{3.2}$$

Assume, furthermore, that $\mathscr{A}$ is realized on a Hilbert space $\mathscr{H}$ and let $\mathscr{H}_X = L^2(X; \mathscr{H})$. Then we get a faithful representation of $L^1(X; \mathscr{A})$ on $\mathscr{H}_X$, the so-called *left regular representation*, by defining the action of $S \in L^1(X; \mathscr{A})$ onto $\xi \in \mathscr{H}_X$ by

$$(S \bullet \xi)(x) = \int_X \alpha_{-x}[S(x-y)]\, \xi(y)\, \mathrm{d}y. \tag{3.3}$$

Definition 3.1. *If $\mathscr{A}$ is an X-algebra, then the* **crossed product** *$\mathscr{A} \rtimes X$ of $\mathscr{A}$ by the action α of X, is the enveloping C^*-algebra of $L^1(X; \mathscr{A})$.*

Thus $\mathscr{A} \rtimes X$ is the completion of $L^1(X; \mathscr{A})$ under the largest C^*-norm on it, and each representation of $L^1(X; \mathscr{A})$ extends to a representation of $\mathscr{A} \rtimes X$ (for the notion of enveloping C^*-algebra see Sect. 2.7 in [17]). Due to the fact that X is abelian, hence amenable, the crossed product defined above coincides with the "reduced crossed product" (Theorems 7.7.5 and 7.7.7 in [34]): the left regular representation of $L^1(X; \mathscr{A})$ extends to a faithful representation of $\mathscr{A} \rtimes X$. In particular, $\mathscr{A} \rtimes X$ is canonically isomorphic to the closure in $B(\mathscr{H}_X)$ of the $*$-algebra of operators of the form (3.3).

Heuristically, one should think of $\mathscr{A} \rtimes X$ as a kind of twisted tensor product of the algebras $\mathscr{A}$ and $C_0(X^*)$, where X^* is the group dual to X. In fact, if the action of X on $\mathscr{A}$ is trivial, then $\mathscr{A} \rtimes X = \mathscr{A} \otimes C_0(X^*)$.

3.2. Functorial properties. The correspondence $\mathscr{A} \mapsto \mathscr{A} \rtimes X$ extends to a covariant functor from the category of X-algebras (with X-morphisms as morphisms) into the category of C^*-algebras. Indeed, if $\phi : \mathscr{A} \to \mathscr{B}$ is an X-morphism, then it clearly induces a morphism $\phi_0 : L^1(X; \mathscr{A}) \to L^1(X; \mathscr{B})$ by the formula $(\phi_0 S)(x) := \phi[S(x)]$. Hence we may define the morphism $\phi_* : \mathscr{A} \rtimes X \to \mathscr{B} \rtimes X$ as the canonical extension of ϕ_0 to the enveloping algebras. A very useful fact is described in the next theorem (see [22] for a detailed proof).

Theorem 3.1. *Let $\mathscr{J}, \mathscr{A}, \mathscr{B}$ be X-algebras and let*

$$0 \longrightarrow \mathscr{J} \xrightarrow{\ \phi\ } \mathscr{A} \xrightarrow{\ \psi\ } \mathscr{B} \longrightarrow 0$$

be an exact sequence of X-morphisms. Then

$$0 \longrightarrow \mathscr{J} \rtimes X \xrightarrow{\ \phi_*\ } \mathscr{A} \rtimes X \xrightarrow{\ \psi_*\ } \mathscr{B} \rtimes X \longrightarrow 0$$

is an exact sequence.

Let $\mathscr{J}$ be a stable ideal of an X-algebra $\mathscr{A}$. By Theorem 3.1, if $j : \mathscr{J} \to \mathscr{A}$ is the inclusion map, then $j_* : \mathscr{J} \rtimes X \to \mathscr{A} \rtimes X$ is an isometric morphism of $\mathscr{J} \rtimes X$ onto an ideal of $\mathscr{A} \rtimes X$. From now on *we shall identify $\mathscr{J} \rtimes X$ with its image under j_**. So, $\mathscr{J} \rtimes X$ is just the closure in $\mathscr{A} \rtimes X$ of the ideal $L^1(X; \mathscr{J})$ of $L^1(X; \mathscr{A})$.

Now the quotient C^*-algebra $\mathscr{B} = \mathscr{A}/\mathscr{J}$ has a natural structure of X-algebra such that the canonical morphism $\mathscr{A} \to \mathscr{A}/\mathscr{J}$ is a X-morphism. Theorem 3.1 says also that the morphism $\mathscr{A} \rtimes X \to \left[\mathscr{A}/\mathscr{J}\right] \rtimes X^2$ associated with it has $\mathscr{J} \rtimes X$ as kernel. We thus get the following reformulation of Theorem 3.1:

Theorem 3.2. *If $\mathscr{J}$ is a stable ideal of a X-algebra $\mathscr{A}$ then*

$$\mathscr{A} \rtimes X \Big/ \mathscr{J} \rtimes X \cong \left[\mathscr{A}/\mathscr{J}\right] \rtimes X. \tag{3.4}$$

The simplest case of the preceding situation is that when the exact sequence splits, so that $\mathscr{A}/\mathscr{J}$ can be realized as a stable C^*-subalgebra of $\mathscr{A}$. Then we have:

Corollary 3.1. *Let $\mathscr{J}$ be a stable ideal and $\mathscr{B}$ a stable C^*-subalgebra of $\mathscr{A}$ such that $\mathscr{A} = \mathscr{B} + \mathscr{J}$ direct linear sum. Then $\mathscr{J} \rtimes X$ is an ideal in $\mathscr{A} \rtimes X$, $\mathscr{B} \rtimes X$ is a C^*-subalgebra of $\mathscr{A} \rtimes X$, and $\mathscr{A} \rtimes X = \mathscr{B} \rtimes X + \mathscr{J} \rtimes X$ is direct linear sum.*

Corollary 3.2. *Let $\mathscr{A}, \mathscr{B}$ be X-algebras and let $\mathscr{A} \oplus \mathscr{B}$ be equipped with the natural X-algebra structure. Then*

$$(\mathscr{A} \oplus \mathscr{B}) \rtimes X \cong (\mathscr{A} \rtimes X) \oplus (\mathscr{B} \rtimes X). \tag{3.5}$$

Proposition 3.1. *If $\phi : \mathscr{A} \to \mathscr{B}$ is an injective or surjective X-morphism then $\phi_* : \mathscr{A} \rtimes X \to \mathscr{B} \rtimes X$ is injective or surjective respectively. In particular, if $\mathscr{A}$ is a stable C^*-subalgebra of the X-algebra $\mathscr{B}$, then $\mathscr{A} \rtimes X$ can be identified with a C^*-subalgebra of $\mathscr{B} \rtimes X$.*

The assertion is obvious in the surjective case. For the injective case, see Proposition 7.7.9 in [34]. So what we proved above for ideals is valid for subalgebras too.

Proposition 3.2. *Let $\mathscr{A}$ be an X-algebra and let $\mathscr{B}$ be a nuclear (e.g. abelian) C^*-algebra. Equip $\mathscr{A} \otimes \mathscr{B}$ with the X-algebra structure defined by $\alpha_x(a \otimes b) = \alpha_x(a) \otimes b$. Then*

$$(\mathscr{A} \otimes \mathscr{B}) \rtimes X \cong (\mathscr{A} \rtimes X) \otimes \mathscr{B}. \tag{3.6}$$

Proposition 2.4 in [35] asserts more than this (in [22] one can find an elementary proof of the last proposition).

3.3. Direct products. We discuss now the behavior of the crossed product under infinite direct products and sums. Let $\{\mathscr{A}_i\}_{i\in I}$ be an arbitrary family of C^*-algebras. Assume that each $\mathscr{A}_i$ is an X-algebra, the corresponding group of automorphisms being α^i. Then one may define $\alpha : X \to \mathrm{Aut}(\prod_{i\in I} \mathscr{A}_i)$ by $\alpha_x[(a_i)_{i\in I}] = (\alpha^i_x[a_i])_{i\in I}$. In this way we do not (in general) get an X-algebra structure on $\prod_{i\in I} \mathscr{A}_i$ because the continuity condition is not satisfied. However, we may define an "equicontinuous product" algebra as the largest subalgebra on which α acts continuously:

$$\prod_{i\in I}^{X} \mathscr{A}_i = \{(a_i)_{i\in I} \in \prod_{i\in I} \mathscr{A}_i \mid \lim_{x\to 0} \sup_{i\in I} \|\alpha^i_x[a_i] - a_i\| = 0\}.$$

This is naturally an X-algebra which contains $\bigoplus_{i\in I} \mathscr{A}_i$ as a stable subalgebra, thus $\bigoplus_{i\in I} \mathscr{A}_i$ becomes an X-algebra too.

Proposition 3.3. $\left(\bigoplus_{i\in I} \mathscr{A}_i\right) \rtimes X \cong \bigoplus_{i\in I} (\mathscr{A}_i \rtimes X)$.

Proof. Denote $\mathscr{A} = \bigoplus_{i\in I} \mathscr{A}_i$. Since each $\mathscr{A}_i$ is an ideal in $\mathscr{A}$, we have canonical embeddings of $\mathcal{A}_i \equiv \mathscr{A}_i \rtimes X$ as ideals in $\mathcal{A} \equiv \mathscr{A} \rtimes X$. Now it suffices to show two things: 1) $\mathcal{A}_i \cdot \mathcal{A}_j = 0$ if $i \neq j$ and 2) the linear subspace $\sum_{i\in I} \mathcal{A}_i$ generated by $\bigcup_{i\in I} \mathcal{A}_i$ is dense in $\mathcal{A}$. Both assertions follow easily from the fact that $\mathcal{A}_i$ is the closure of $L^1(X; \mathscr{A}_i)$ in $\mathcal{A}$. $\square$

We shall go beyond direct sums only in the particular case we need. Assume that $\mathscr{A}$ is an X-algebra and I is a set. The algebras $\mathscr{A}^{(I)}$ and $\mathscr{A}^{[I]}$ have been introduced in Sect. 2.3. Then the C^*-algebra $\mathscr{A}^{[I]}$ is an X-algebra, and $\mathscr{A}^{(I)}$ is a stable ideal in it, if we set $\alpha_x[(a_i)_{i\in I}] = (\alpha_x[a_i])_{i\in I}$. Indeed, for each $\varepsilon > 0$ there is a finite set $K \subset \mathscr{A}$ such that $\mathrm{dist}\,(a_i, K) < \varepsilon$ for all $i \in I$. Then

$$\|\alpha_x[(a_i)_{i\in I}] - (a_i)_{i\in I}\| = \sup_{i\in I} \|\alpha_x[a_i] - a_i\| \leq 2\varepsilon + \sup_{b\in K} \|\alpha_x[b] - b\|$$

and the last term is $< \varepsilon$ if x is in a suitable neighborhood of zero in X. Note that we could also consider the X-algebra $\prod_{i\in I}^{X} \mathscr{A}$ which depends on the action of X on $\mathscr{A}$ and which contains $\mathscr{A}^{[I]}$ as a stable ideal.

Proposition 3.4. *If $\mathscr{A}$ is an X-algebra and I a set, then*

$$\mathscr{A}^{(I)} \rtimes X \cong (\mathscr{A} \rtimes X)^{(I)} \ \text{ and } \ \mathscr{A}^{[I]} \rtimes X \cong (\mathscr{A} \rtimes X)^{[I]}. \tag{3.7}$$

Moreover, the I-asymptotic algebra $\mathscr{A}^{\langle I\rangle}$ has a canonical structure of X-algebra and one has

$$(\mathscr{A} \rtimes X)^{[I]} \Big/ (\mathscr{A} \rtimes X)^{(I)} \equiv (\mathscr{A} \rtimes X)^{\langle I\rangle} \cong C(\widetilde{I}; \mathscr{A} \rtimes X). \tag{3.8}$$

The first identification of (3.7) is a particular case of Proposition 3.3. The second one is a consequence of Proposition 3.2 and of Lemma 2.1. The last part of the proposition follows from Theorem 3.2 and the representation (2.8).

4. Pseudo-Differential Operators

In this section we show that certain crossed products can be faithfully represented as algebras of pseudo-differential operators on $L^2(X)$. We first recall some facts concerning the harmonic analysis on X (see [18]).

Let X^* be the locally compact abelian group dual to X. The Fourier transform of $u \in L^1(X)$ is the function $\mathcal{F}u \equiv \widehat{u} : X^* \to \mathbb{C}$ given by $\widehat{u}(k) = \int_X \overline{k(x)}u(x)\,\mathrm{d}x$. Then $\mathcal{F}$ is a linear map $L^1(X) \to C_0(X^*)$ and we shall equip X^* with the unique Haar measure $\mathrm{d}k$ such that $\mathcal{F}$ induces a unitary map $\mathcal{F} : L^2(X, \mathrm{d}x) \to L^2(X^*, \mathrm{d}k)$. >From $\mathcal{F}^{-1} = \mathcal{F}^*$ we get $(\mathcal{F}^{-1}v)(x) = \int_{X^*} k(x)v(k)\,\mathrm{d}k$ for $v \in L^2(X^*)$. The dual group $(X^*)^*$ of X^* is identified with X, each $x \in X$ being seen as a character of X^* through the formula $x(k) = k(x)$. Then the Fourier transform of $\psi \in L^1(X^*)$ is given by $\widehat{\psi}(x) = (\mathcal{F}^*\psi)(-x)$. For each $\psi \in C_0(X^*)$ we define the operator $\psi(P) \in \mathscr{B}(X)$ by $\psi(P) = \mathcal{F}^*M_\psi\mathcal{F}$, where M_ψ is the operator of multiplication by ψ in $L^2(X^*)$ (P is the X^* valued momentum observable). The injective morphism $\psi \mapsto \psi(P)$ gives us an embedding $C_0(X^*) \subset \mathscr{B}(X)$.

We recall the embedding of the C^*-algebra $C_\mathrm{b}^\mathrm{u}(X)$ in $\mathscr{B}(X)$ obtained by associating to $\varphi \in C_\mathrm{b}^\mathrm{u}(X)$ the operator of multiplication by the function φ. In order to avoid ambiguities we often denote this operator by $\varphi(Q)$ (Q is the X valued position observable).

We have two strongly continuous unitary representations $\{U_x\}_{x\in X}$ and $\{V_k\}_{k\in X^*}$ of X and X^* in $L^2(X)$ defined by $(U_x f)(y) = f(x + y)$ and $(V_k f)(y) = k(y)f(y)$ respectively. The group C^*-algebra of X is the C^*-subalgebra of $\mathscr{B}(X)$ generated by the convolution operators $\int_X u(z)U_z\,\mathrm{d}z$ with $u \in L^1(X)$, and is canonically isomorphic to $C_0(X^*)$. The isomorphism is determined by the formula $\psi(P) = \int_X \widehat{\psi}(z)U_z\,\mathrm{d}z$ for $\psi \in C_0(X^*)$ such that $\widehat{\psi} \in L^1(X)$.

The group X acts in a natural way on $C_\mathrm{b}^\mathrm{u}(X)$: if $x \in X$ and if we denote by $\tau_x\varphi$ the function $y \mapsto \varphi(y - x)$ then for $\varphi \in C_\mathrm{b}^\mathrm{u}(X)$ we have $\tau_x\varphi \in C_\mathrm{b}^\mathrm{u}(X)$ and $x \mapsto \tau_x\varphi \in C_\mathrm{b}^\mathrm{u}(X)$ is norm continuous. We consider a C^*-subalgebra $\mathscr{A}$ of $C_\mathrm{b}^\mathrm{u}(X)$ stable under translations: $\tau_x\varphi \in \mathscr{A}$ if $x \in X$ and $\varphi \in \mathscr{A}$. Then $\mathscr{A}$ is an X-algebra and we are interested in the crossed product $\mathscr{C} = \mathscr{A} \rtimes X$ of $\mathscr{A}$ by the action $\alpha_x := \tau_{-x}$ of X. In such a situation the crossed product $\mathscr{A} \rtimes X$ has an especially useful faithful representation that we shall describe below.

Let us use the embedding $C_\mathrm{b}^\mathrm{u}(X) \subset \mathscr{B}(X)$ and observe that $(\tau_x\varphi)(Q) \equiv \tau_x[\varphi(Q)] = U_x^*\varphi(Q)U_x$. In particular $\mathscr{A} \subset \mathscr{B}(X)$ and our purpose is to show that $\mathscr{A} \rtimes X$ can also be realized as a C^*-algebra of operators on the Hilbert space $L^2(X)$.

Theorem 4.1. *Let $\mathscr{A}$ be a C^*-subalgebra of $C_\mathrm{b}^\mathrm{u}(X)$ stable under translations. Then the linear subspace $[\![\mathscr{A} \cdot C_0(X^*)]\!]$ is a C^*-algebra on the Hilbert space $L^2(X)$ and*

$$[\![\mathscr{A} \cdot C_0(X^*)]\!] \cong \mathscr{A} \rtimes X \tag{4.1}$$

in the sense that there is a unique isomorphism $\Phi : [\![\mathscr{A} \cdot C_0(X^*)]\!] \to \mathscr{A} \rtimes X$ such that $\Phi[\varphi(Q)\psi(P)] = S_{\varphi,\psi}$ for all $\varphi \in \mathscr{A}$ and $\psi \in C_0(X^*)$ with $\widehat{\psi} \in L^1(X)$. Here $S_{\varphi,\psi}$ is the element $y \mapsto S_{\varphi,\psi}(\cdot, y) \in \mathscr{A}$ of $L^1(X; \mathscr{A})$ defined by the function $S_{\varphi,\psi}(x, y) = \varphi(x)\widehat{\psi}(y)$.

Proof. The fact that $[\![\mathscr{A} \cdot C_0(X^*)]\!]$ is a C^*-algebra can easily be proved directly, but it is also a consequence of the next arguments. By the comments which follow Definition 3.1 we have the following description of $\mathscr{A} \rtimes X$. Let $\mathscr{H} = L^2(X)$ and

$$\mathscr{H}_X = L^2(X; \mathscr{H}) \cong \mathscr{H} \otimes L^2(X) \cong L^2(X \times X).$$

To each integrable function $S : X \to \mathscr{A}$ we associate an operator $S\bullet$ acting on $\mathscr{H}_X$ in the following manner: if $\xi : X \to \mathscr{H}$ is L^2, then

$$(S \bullet \xi)(y) = \int_X \tau_y[S(y-z)]\,\xi(z)\,\mathrm{d}z = \int_X U_y^* S(z) U_y\,\xi(y-z)\,\mathrm{d}z. \tag{4.2}$$

The map $S \mapsto S\bullet$ of $L^1(X;\mathscr{A})$ into $B(\mathscr{H}_X)$ is linear and injective. Equip $L^1(X;\mathscr{A})$ with a structure of $*$-algebra by asking that $S \mapsto S\bullet$ be a $*$-morphism; then we set $\|S\| := \|S\bullet\|_{B(\mathscr{H}_X)}$. The completion of $L^1(X;\mathscr{A})$ under this norm will then be identified with a C^*-subalgebra of $B(\mathscr{H}_X)$ and this C^*-algebra is (canonically isomorphic to) the crossed product $\mathscr{A} \rtimes X$.

This representation, however, is not convenient for our purposes. We thus construct a new one with the help of the unitary operator $W : \mathscr{H}_X \to \mathscr{H}_X$ defined as $(W\xi)(x,y) := \xi(x-y,x)$. Note that its adjoint is given by $(W^*\xi)(x,y) = \xi(y,y-x)$.

If $S \in L^1(X;\mathscr{A})$ then S may also be viewed as a function $S : X \times X \to \mathbb{C}$ with the convention $S(y) = S(\cdot,y) \in \mathscr{A}$. Similarly, an element $\xi : X \to \mathscr{H}$ of $\mathscr{H}_X$ is interpreted as a function $\xi : X \times X \to \mathbb{C}$ by setting $\xi(y) = \xi(\cdot,y)$. Then (4.2) may be written as:

$$(S \bullet \xi)(x,y) = \int_X S(x-y,z)\,\xi(x,y-z)\,\mathrm{d}z,$$

which allows us to compute:

$$\begin{aligned}
(W^* S \bullet\ W\xi)(x,y) &= (S \bullet\ W\xi)(y,y-x) \\
&= \int_X S(x,z)\,(W\xi)(y,y-x-z)\,\mathrm{d}z = \int_X S(x,z)\,\xi(x+z,y)\,\mathrm{d}z \\
&= \int_X S(x,z)\left[(U_z \otimes 1)\xi\right](x,y)\,\mathrm{d}z = \int_X \left\{\left[S(Q,z)\,U_z \otimes 1\right]\xi\right\}(x,y)\,\mathrm{d}z.
\end{aligned}$$

In other terms,

$$W^*\ [S\bullet]\ W = \left[\int_X S(Q,z)\,U_z\,\mathrm{d}z\right] \otimes 1.$$

Consider the particular case when $S(x,y) = S_{\varphi,\psi}(x,y) = \varphi(x)\widehat{\psi}(y)$ as in the statement of the theorem. Then the above integral is equal to $\varphi(Q)\psi(P)$. So we have

$$W^*\left[S_{\varphi,\psi}\bullet\right] W = [\varphi(Q)\psi(P)] \otimes 1.$$

Since the subspace generated by the elements of the form $S_{\varphi,\psi}$ is dense in $L^1(X;\mathscr{A})$, the assertions of the theorem follow easily. □

Corollary 4.1. $\mathscr{K}(X) = [\![C_0(X) \cdot C_0(X^*)]\!] \cong C_0(X) \rtimes X$.

The first equality is easy to prove. Then the canonical isomorphism with $C_0(X) \rtimes X$ follows from Theorem 4.1 (for another proof of the isomorphism of $\mathscr{K}(X)$ with the crossed product $C_0(X) \rtimes X$ see Proposition 3.3 in [35]).

Theorem 1.1 is a consequence of Theorem 4.1 and of the next proposition.

Proposition 4.1. *Let $\mathscr{A}$ be a C^*-subalgebra of $C_{\mathrm{b}}^{\mathrm{u}}(X)$ which contains the constants and is stable under translations. Let $h : X^* \to \mathbb{R}$ be a continuous non-constant function such that* $\lim_{k\to\infty} |h(k)| = \infty$. *Then $\mathscr{A} \rtimes X$ is the C^*-algebra generated by the self-adjoint operators of the form $h(P+k) + V(Q)$, with $k \in X^*$ and $V \in \mathscr{A}$ real.*

Proof. Let $\mathscr{C}$ be the C^*-algebra generated by the operators $H = h(P+k) + V(Q) \equiv H_0 + V(Q)$, with $k \in X^*$ and $V \in \mathscr{A}$ real. By making a norm convergent series expansion for large z,

$$(z - H)^{-1} = \sum_{n \geq 0} (z - H_0)^{-1}[V(Q)(z - H_0)^{-1}]^n,$$

we get $\mathscr{C} \subset \mathscr{A} \rtimes X$. It remains to prove the opposite inclusion. For each $\mu \in \mathbb{R}$ the operator $H_\mu = h(P+k) + \mu V(Q)$ is affiliated to $\mathscr{C}$ and $(H_\mu - i)^{-1}$ is norm derivable at $\mu = 0$ with derivative $-(H_0 - i)^{-1}V(Q)(H_0 - i)^{-1}$. We thus have $(H_0 - i)^{-1}V(Q)(H_0 - i)^{-1} \in \mathscr{C}$. Let $\theta \in C_{\mathrm{c}}(\mathbb{R})$ with $\theta(0) = 1$ and $\varepsilon > 0$. Since H_0 is affiliated to $\mathscr{C}$, we get $\theta(\varepsilon H_0)(H_0 - i) \in \mathscr{C}$, and so $\theta(\varepsilon H_0)V(Q)(H_0 - i)^{-1} \in \mathscr{C}$. From the uniform continuity of V, and since $(h(p+k) - i)^{-1} \to 0$ when $p \to \infty$ in X^*, we get $\|(U_x - 1)V(Q)(H_0 - i)^{-1}\| \to 0$ if $x \to 0$ in X. This implies $\lim_{\varepsilon \to 0} \theta(\varepsilon H_0)V(Q)(H_0 - i)^{-1} = V(Q)(H_0 - i)^{-1}$ in norm in $\mathscr{B}(X)$ (indeed, for $T \in \mathscr{B}(X)$ we have $\lim_{x \to 0} \|(U_x - 1)T\| = 0$ if and only if for each $\delta > 0$ there is $\eta \in C_{\mathrm{c}}(X^*)$ such that $\|\eta(P)^\perp T\| < \delta$). Hence $V(Q)(h(P+k) - i)^{-1} \in \mathscr{C}$ for each $k \in X^*$ and each $V \in \mathscr{A}$ real. But H_0 is affiliated to $\mathscr{C}$, so this implies $\varphi(Q)\psi(P) \in \mathscr{C}$ for all $\varphi \in \mathscr{A}$ (not necessarily real) and all ψ in the $*$-subalgebra $\mathscr{B} \subset C_0(X^*)$ generated by functions of the form $p \mapsto \xi(h(p+k))$ with $\xi \in C_c(\mathbb{R})$ and $k \in X^*$. By the Stone–Weierstrass theorem, $\mathscr{B}$ is dense in $C_0(X^*)$. Hence, since the set of $\psi \in C_0(X^*)$ such that $\varphi(Q)\psi(P) \in \mathscr{C}$ is norm closed and contains $\mathscr{B}$, we finally obtain $\varphi(Q)\psi(P) \in \mathscr{C}$ for all $\varphi \in \mathscr{A}$, $\psi \in C_0(X^*)$. $\square$

5. Bumps Algebras

5.1. The algebra of classical interactions. In this section we will consider algebras of interactions determined by sets $L \subset X$ by the following rule: the interaction tends to a constant when the distance to L tends to infinity.

For $L \subset X$ closed and $\Lambda \subset X$ compact, set $L_\Lambda \equiv L + \Lambda = \{x + y \mid x \in L,\ y \in \Lambda\}$ and $L^c_\Lambda = X \setminus L_\Lambda$. For example, if X is equipped with an invariant metric and Λ is the closed ball of radius r, then L^c_Λ is the set of points at distance $> r$ from L. Note that $\Lambda \subset \Lambda' \Rightarrow L_\Lambda \subset L_{\Lambda'}$, in particular $L^c_{\Lambda_1 \cup \Lambda_2} \subset L^c_{\Lambda_1} \cap L^c_{\Lambda_2}$. Moreover, for each $x \in X$ we have $x + L^c_\Lambda = L^c_{x+\Lambda}$.

If L has the property $L_\Lambda \neq X$ if Λ is compact, then the family of open sets $\{L^c_\Lambda \mid \Lambda \subset X \text{compact}\}$ is the base of a filter $\mathcal{F}_L$ which, by the preceding remarks, is translation invariant and finer than the Fréchet filter. Thus we are in the general framework described in Sect. 1.4 and we can introduce the algebra $C_{\mathcal{F}_L}(X)$. We recall it, with notations adapted to the present situation.

We denote by L-$\lim \varphi$ the limit along the filter $\mathcal{F}_L$. Thus, if $\varphi : X \to \mathbb{C}$ then L-$\lim \varphi$ exists if and only if there is a complex number $c \equiv L$-$\lim \varphi$ with the property: for each $\varepsilon > 0$ one can find a compact set $\Lambda \subset X$ such that $|\varphi(x) - c| < \varepsilon$ if $x \notin L_\Lambda$.

Let $C_L(X)$ be the translation invariant C^*-subalgebra of $C^{\mathrm{u}}_{\mathrm{b}}(X)$ defined by

$$C_L(X) = \{\varphi \in C^{\mathrm{u}}_{\mathrm{b}}(X) \mid L\text{-}\lim \varphi \text{ exists}\},$$

and let us point out the following subalgebras:

$$C_{L,0}(X) = \{\varphi \in C^{\mathrm{u}}_{\mathrm{b}}(X) \mid L\text{-}\lim \varphi = 0\},$$
$$C_{L,\mathrm{c}}(X) = \{\varphi \in C^{\mathrm{u}}_{\mathrm{b}}(X) \mid \exists \Lambda \subset X \text{ compact such that } \operatorname{supp} \varphi \subset L_\Lambda\}.$$

Clearly $C_{L,0}$ is an ideal of C_L and one has $C_L = \mathbb{C} + C_{L,0}$, so C_L is the unital algebra associated with $C_{L,0}$. Since $C_0 \subset C_{L,0}$, we have

$$C_L \big/ C_0 \;\cong\; \mathbb{C} + C_{L,0} \big/ C_0\,, \tag{5.1}$$

i.e. $C_L \,/\, C_0$ is the unital algebra associated with $C_{L,0} \,/\, C_0$. We want to apply the general theory from Sects. 3 and 4 to the algebra $\mathscr{A} = \mathscr{C}_L(X) = C_L(X) \rtimes X$, hence we have to give an explicit description of the quotient algebra $C_L(X)/C_0(X)$. By the preceding remarks, we are reduced to the problem of computing $C_{L,0}(X)/C_0(X)$. However, this is not an easy task if no further conditions are put on L.

For this reason *we shall assume from now on that L is sparse,* in the following sense: L is locally finite and for each compact Λ of X there is a finite set $F \subset L$ such that if $l \in M = L \setminus F$ and $l' \in L \setminus \{l\}$ then $(l + \Lambda) \cap (l' + \Lambda) = \emptyset$.

If the topology of X is given by an invariant metric d, we can restate the definition of sparsity as follows. Let $\delta : L \to \mathbb{R}$ be defined by $\delta(l) = \inf_{l' \in L\setminus\{l\}} d(l, l')$. Then L is sparse if and only if $\delta(l) > 0$ for all l and $\delta(l) \to \infty$ when $l \to \infty$. Such a set is much more rarefied than the uniformly discrete sets usually considered in the theory of quasicrystals. Note also that the Delone sets considered in [28] have the property $L + \Lambda = X$ if Λ is a sufficiently large compact, hence are quite different from the kind of sets L studied here.

We begin by describing some properties of the space $C_{L,\mathrm{c}}$. We recall the Ascoli theorem for the case of a locally compact space X: *a bounded subset $\mathcal{K}$ of $C_0(X)$ is relatively compact in $C_0(X)$ if and only if $\mathcal{K}$ is an equicontinuous family of functions and for each $\varepsilon > 0$ there is a compact set $\Lambda \subset X$ such that $|\varphi(x)| < \varepsilon$ for all $\varphi \in \mathcal{K}$ and $x \in X \setminus \Lambda$.*

If Λ is a compact subset of X we shall denote by $C_\mathrm{c}(\Lambda)$ the set of continuous functions on X with support included in Λ.

Lemma 5.1. (i) *$C_{L,\mathrm{c}}$ is a dense self-adjoint ideal in $C_{L,0}$.*
(ii) *A function φ belongs to $C_{L,\mathrm{c}}$ if and only if there is a compact set $\Lambda \subset X$ and an equicontinuous bounded family $\{\varphi_l\}_{l\in L}$ of elements of $C_\mathrm{c}(\Lambda)$ such that $\varphi = \sum_{l\in L} \tau_l \varphi_l$.*
(iii) *The linear subspace generated by functions of the form $\sum_{l\in M} \tau_l \varphi$, where M is a subset of L and $\varphi \in C_\mathrm{c}(X)$, is dense in $C_{L,0}$.*

Proof. (i) Clearly $C_{L,\mathrm{c}}$ is a self-adjoint (non-closed) ideal in C_L. We shall prove its density in $C_{L,0}$. Let Λ be a compact neighborhood of 0 in X and let $\theta \in C_\mathrm{c}(\Lambda)$ such that $0 \le \theta \le 1$ and $\theta = 1$ on a neighborhood Λ_0 of zero. Denote n the maximal number of sets $l + \Lambda$, with $l \in L$, which have a non-empty intersection. Since L is a sparse subset of X the number n is finite. Then $\Theta(x) = \sum_{l\in L} \theta(x - l)$ is well defined, $0 \le \Theta(x) \le n$, and

$$|\Theta(x) - \Theta(y)| \le n \sup_{l\in L} |\theta(x - l) - \theta(y - l)| \le n\|\tau_{y-x}\theta - \theta\|,$$

where $\|\cdot\|$ is the sup norm. So Θ is uniformly continuous and $\Theta \in C_{L,\mathrm{c}}$.

Let $\varphi \in C_{L,0}$ and $\varepsilon > 0$. Then there is a compact neighborhood Λ of zero in X such that $|\varphi(x)| < \varepsilon$ if $x \notin L + \Lambda$. Choose $F \subset L$ finite such that $(l + \Lambda) \cap (l' + \Lambda) = \emptyset$ if $l \in M \equiv L \setminus F$ and $l' \in L,\ l' \neq l$, and denote K the compact set $\bigcup_{l\in F}[l + \Lambda]$. Observe that if $x \notin K$ and θ is as above then $\Theta(x) = \sum_{l\in M} \theta(x - l)$ and the supports of the functions $\tau_l\theta$ are disjoint if $l \in M$; in particular $0 \le \Theta(x) \le 1$. Let η be a continuous function such that $0 \le \eta \le 1$, $\eta = 0$ on a neighborhood of K and $\eta = 1$ on a neighborhood of infinity. If we denote $\eta\varphi = \psi$ we have

$$\varphi - (1-\eta)\varphi - \Theta\psi = \psi - \Theta\psi.$$

If $x \in K$ then the r.h.s. above takes the value zero in x. If $x \notin K$ and $x \in L + \Lambda_0$ then there is a unique $l \in M$ such that $x \in l + \Lambda_0$, hence $\Theta(x) = \theta(x - l) = 1$ and $\psi(x) - \Theta(x)\psi(x) = 0$. If $x \notin K$ and $x \notin L + \Lambda_0$ then $|\psi(x)| < \varepsilon$ and $0 \le \Theta(x) \le 1$, so $|\psi(x) - \Theta(x)\psi(x)| < \varepsilon$. Thus we have

$$\|\varphi - (1-\eta)\varphi - \Theta\psi\| \le \varepsilon.$$

Since $(1-\eta)\varphi \in C_{\mathrm{c}}(X)$ and $\Theta\psi \in C_{L,\mathrm{c}}$, we proved that for each $\varepsilon > 0$ there is $\varphi_\varepsilon \in C_{L,\mathrm{c}}$ such that $\|\varphi - \varphi_\varepsilon\| < \varepsilon$. Hence $C_{L,\mathrm{c}}$ is dense in $C_{L,0}$.
(ii) If $\varphi \in C_{L,\mathrm{c}}$ then there is a compact $\Lambda \subset X$ such that $\operatorname{supp}\varphi \subset L + \Lambda$. Since L is sparse one can write $L + \Lambda = K \cup \bigcup_{l \in M}(l + \Lambda)$, where the compact set $K \subset X$ and M are chosen such that the sets which appear in the preceding union are pairwise disjoint. For $l \in M$ we define φ_l by $\varphi_l(x) := \varphi(x + l)$ if $x \in \Lambda$ and $\varphi_l = 0$ otherwise. If $l \in L \setminus M$ the definition of φ_l is to a large extent arbitrary, e.g. we may take $\varphi_l = 0$ for all but one $l \equiv l_0$ and choose conveniently φ_{l_0}; this is possible if K is large enough. Conversely, it suffices to notice that the equicontinuity of the family $\{\varphi_l\}_{l \in L}$ implies the uniform continuity of φ.
(iii) Because of the first part of the lemma it suffices to prove that for each φ as in (ii) and for each $\varepsilon > 0$ there is a partition $\{L_1, \dots, L_k\}$ of L and there are functions $\phi_1, \dots, \phi_k \in C_{\mathrm{c}}(\Lambda)$ such that $\|\varphi - \sum_{i=1}^k \sum_{l \in L_i} \tau_l \phi_i\| < \varepsilon$. By the Ascoli theorem, $\{\varphi_l \mid l \in L\}$ is a relatively compact subset of $C_{\mathrm{c}}(\Lambda)$, hence there is a finite number of functions $\phi_1 := \varphi_{l_1}, \dots, \phi_k := \varphi_{l_k}$ and there is a partition $\{L_1, \dots, L_k\}$ of L such that $\|\varphi_l - \phi_i\| < \varepsilon/n$ for $l \in L_i$, where n is the maximal number of sets of the form $l + \Lambda$ which have non-empty intersection. Then, for each x,

$$\begin{aligned} |\varphi(x) - \sum_{i=1}^{k} \sum_{l \in L_i} \tau_l \phi_i(x)| &= \left| \sum_{i=1}^{k} \sum_{l \in L_i} \tau_l (\varphi_l - \phi_i)(x) \right| \\ &\le n \sup_{i=1,\dots,k} \sup_{l \in L_i} \|\tau_l(\varphi_l - \phi_i)\| \le \varepsilon. \qquad \square \end{aligned}$$

We are now ready to compute the quotient $C_{L,0}/C_0$. We recall the notation $C_0(X)^{\langle L \rangle} \equiv C_0(X)^{[L]}/C_0(X)^{(L)}$ (see Sect. 2.3) and denote by π the canonical morphism $C_0(X)^{[L]} \to C_0(X)^{\langle L \rangle}$.

Theorem 5.1. *There is a unique morphism $\mathcal{J} : C_{L,0}(X) \to C_0(X)^{\langle L \rangle}$ such that $\mathcal{J}(\varphi) = \pi[(\varphi_l)_{l \in L}]$ if $\varphi = \sum_{l \in L} \tau_l \varphi_l$, with $\{\varphi_l\}_{l \in L}$ an equicontinuous bounded family in $C_0(\Lambda)$ for some compact $\Lambda \subset X$. The morphism $\mathcal{J}$ is surjective and* $\ker \mathcal{J} = C_0(X)$. *In particular, $\mathcal{J}$ induces a canonical isomorphism:*

$$C_{L,0}(X) \Big/ C_0(X) \;\cong\; C_0(X)^{[L]} \Big/ C_0(X)^{(L)} \;\equiv\; C_0(X)^{\langle L \rangle}. \tag{5.2}$$

Remark. As a consequence of Lemma 5.1 and of the identifications (2.6), $\mathcal{J}$ is the unique morphism $C_{L,0} \to C_0^{\langle L \rangle}$ such that for each subset M of L and each function $\varphi \in C_{\mathrm{c}}(X)$ one has $\mathcal{J}(\sum_{l \in M} \tau_l \varphi) = \pi(\chi_M \otimes \varphi)$, where χ_M is the characteristic function of the set M. There is another description of $\mathcal{J}$ based on the identification (2.8), but we shall make it explicit only in the case of the algebra $C_{L,0} \rtimes X$.

Proof. The uniqueness of $\mathcal{J}$ is a consequence of (i) from Lemma 5.1. The surjectivity of $\mathcal{J}$ is also easy to prove: since the range of a morphism is closed, it suffices to show that the set of elements of the form $(\varphi_l)_{l\in L}$, where the family $\{\varphi_l\}_{l\in L}$ is as in the statement of the theorem, is dense in $C_0^{[L]}$. But this is a straightforward consequence of the Ascoli theorem, because $C_0^{[L]}$ consists of relatively compact families of elements of C_0 (see Definition (2.5)).

We prove the existence of $\mathcal{J}$. Note first that for a given $\varphi \in C_{L,\mathrm{c}}$ a family $\{\varphi_l\}_{l\in L}$ which verifies $\varphi = \sum_{l\in L} \tau_l \varphi_l$ is not unique. However, L being a sparse set, the functions φ_l which correspond to large enough l are uniquely defined. In particular, the image of $\{\varphi_l\}_{l\in L}$ in the quotient $C_0^{[L]}/C_0^{(L)}$ depends only on φ. Thus $\mathcal{J}$ is well defined on $C_{L,\mathrm{c}}$ and it is clearly a morphism with $\|\mathcal{J}\| \le 1$. This allows us to extend it by continuity to all $C_{L,0}$.

It remains to prove that the kernel of $\mathcal{J}$ is C_0. We first make a preliminary general remark. *Let $\mathscr{A}$ be a C^*-algebra and let $S = (S_l)_{l\in L} \in \mathscr{A}^{[L]}$ such that $\|\widehat{S}\| < \varepsilon$ for some $\varepsilon > 0$, where $\widehat{S}$ is the image of S in $\mathscr{A}^{\langle L\rangle} \equiv \mathscr{A}^{[L]}/\mathscr{A}^{(L)}$. Then there is a finite set $F \subset L$ such that $\|S_l\| < 2\varepsilon$ if $l \notin F$.* Indeed,

$$\|\widehat{S}\| = \inf\{\|(S_l - T_l)_{l\in L}\| \mid T \equiv (T_l)_{l\in L} \in \mathscr{A}^{(L)}\}$$

so there is $T \in \mathscr{A}^{(L)}$ such that $\|(S_l - T_l)_{l\in L}\| = \sup_{l\in L} \|S_l - T_l\| < 3\varepsilon/2$. Then $\|S_l\| < \|T_l\| + 3\varepsilon/2$ and $\|T_l\| \to 0$ as $l \to \infty$. So there is a finite set $F \subset L$ such that $\|T_l\| < \varepsilon/2$ if $l \notin F$, which proves the remark.

Let $\varphi \in C_{L,0}$ be such that $\mathcal{J}(\varphi) = 0$ and let $\varepsilon > 0$. Then there is $\psi \in C_{L,\mathrm{c}}$ such that $\|\varphi - \psi\| < \varepsilon$ and $\|\mathcal{J}(\psi)\| < \varepsilon$. Choose a compact $\Lambda \subset X$ and a bounded equicontinuous family $\{\psi_l\}_{l\in L}$ in $C_0(\Lambda)$ such that $\psi = \sum_{l\in L} \tau_l \psi_l$. Then $(\psi_l)_{l\in L} \in C_0(\Lambda)^{[L]}$ and $\|\pi[(\psi_l)_{l\in L}]\| = \|\mathcal{J}(\psi)\| < \varepsilon$ hence, by the preceding remark, there is a finite set $F \subset L$ such that $\|\psi_l\| < 2\varepsilon$ if $l \notin F$. But $\psi(x) = \sum_{l\in L} \psi_l(x - l)$ and if x is outside some compact then at most one term in the sum is non-zero, so $|\psi(x)| < \varepsilon$ for x in some neighborhood of infinity. Then $|\varphi(x)| \le |\varphi(x) - \psi(x)| + |\psi(x)| < 3\varepsilon$ for such x. Since ε is arbitrary, this shows $\varphi \in C_0$. □

Corollary 5.1. *The quotient algebra $C_L(X)/C_0(X)$ is canonically isomorphic to the unital C^*-algebra associated with $C_0(X)^{\langle L\rangle}$. In particular, there is a natural embedding:*

$$C_L(X)\big/C_0(X) \hookrightarrow C_\infty(X)^{[L]}\big/C_0(X)^{(L)}. \tag{5.3}$$

Proof. The first assertion follows from (5.1). To get (5.3) we use the canonical embedding $\mathbb{C} + C_0(X)^{[L]} \hookrightarrow C_\infty(X)^{[L]}$, which associates to $\lambda + (\varphi_l)_{l\in L}$ the element $(\lambda + \varphi_l)_{l\in L}$. □

Note that we have a simple description of the range of the embedding (5.3): this is the quotient of the space of the elements of the form $(\lambda + \varphi_l)_{l\in L}$ with $\lambda \in \mathbb{C}$ and $(\varphi_l)_{l\in L} \in C_0(X)^{[L]}$.

Remark. We have considered a generalization of the class of sparse sets. We do not give the details because it does not involve essentially new ideas; we shall, however, describe it here succinctly. Let L be the union of a family $\mathcal{B}$ of pairwise disjoint compact sets such that for each compact Λ of X there is a finite set $\mathcal{B}_\Lambda \subset \mathcal{B}$ such that $(B+\Lambda)\cap(B'+\Lambda) = \emptyset$ if $B \notin \mathcal{B}_\Lambda$ and $B' \neq B$. Then we say that L is a *dispersed* set. If there is a compact set K such that each $B \in \mathcal{B}$ is a subset of a translate of K (i.e. L is "uniformly" dispersed), then L is equivalent to a sparse set, in the sense that $\mathcal{F}_L$ coincides to $\mathcal{F}_{L^0}$ for some sparse set L^0. Indeed, it suffices to replace each compact set $B \in \mathcal{B}$ by a point sitting inside it. In the case of a sparse set the main role in the computation of the quotient is played by the algebra $C_0(X)^{[L]}$ consisting of relatively compact families $\{\varphi_l\}_{l\in L}$ of elements of $C_0(X)$. For a general dispersed set this has to be replaced with the algebra $\prod_{l\in L}^X C_0(X)$ (see Sect. 3.3 for the notations) consisting of equicontinuous such families (compare with the statement of the Ascoli theorem in Sect. 5.1).

5.2. Hamiltonians of type $C_L(X)$. We are now ready to introduce the C^*-algebra of energy observables corresponding to quantum systems with interactions having sparse supports. For this we take the crossed product of $C_L(X)$ by the action of translations on the locally compact group X. Note that the second equality below is a consequence of Theorem 4.1.

Definition 5.1. $\mathscr{C}_L(X) := C_L(X) \rtimes X = [\![C_L(X) \cdot C_0(X^*)]\!]$.

In the same manner we may define the smaller C^*-algebra,

$$\mathscr{C}_{L,0}(X) := C_{L,0}(X) \rtimes X = [\![C_{L,0}(X) \cdot C_0(X^*)]\!]. \tag{5.4}$$

Then, by (3.5) we can write $\mathscr{C}_L(X)$ as a linear direct sum,

$$\mathscr{C}_L(X) = C_0(X^*) + \mathscr{C}_{L,0}(X). \tag{5.5}$$

The algebra $\mathscr{C}_{L,0}$ is an ideal of $\mathscr{C}_L$ and $\mathscr{C}_L \to C_0(X^*)$ is a surjective morphism which gives the pure kinetic energy part, and $\mathscr{C}_L/\mathscr{C}_{L,0} \cong C_0(X^*)$. On the other hand, $C_0(X)$ being a stable ideal of C_L, the crossed product subalgebra $C_0(X) \rtimes X$ is an ideal of $\mathscr{C}_L(X)$. We recall that $C_0(X) \rtimes X = \mathscr{K}(X)$.

The general theory exposed in Sect. 3 allows us to give a complete characterization both of the quotient algebras $\mathscr{C}_{L,0}(X)/\mathscr{K}(X)$ and $\mathscr{C}_L(X)/\mathscr{K}(X)$ in terms of much simpler objects involving only the compact operator algebra $\mathscr{K}(X)$ and the two-body algebra:

Theorem 5.2. *The quotient algebra $\mathscr{C}_{L,0}(X)/\mathscr{K}(X)$ is canonically isomorphic to the L-asymptotic algebra $\mathscr{K}(X)^{\langle L\rangle}$. One has a natural embedding:*

$$\mathscr{C}_L(X)\Big/\mathscr{K}(X) \hookrightarrow \mathscr{T}(X)^{[L]}\Big/\mathscr{K}(X)^{(L)}. \tag{5.6}$$

Proof. The first assertion follows from Proposition 3.4 because of (5.2). In order to prove the second assertion we start with the embedding (5.3) and use (3.4), Proposition 3.1 and (3.7) to get

$$\begin{aligned}
\mathscr{C}_L\Big/\mathscr{K}(X) &\equiv (C_L \rtimes X)\Big/(C_0(X) \rtimes X) = C_L\Big/C_0(X) \rtimes X \\
&\hookrightarrow C_\infty(X)^{[L]}\Big/C_0(X)^{(L)} \rtimes X = (C_\infty(X)^{[L]} \rtimes X)\Big/(C_0(X)^{(L)} \rtimes X) \\
&= (C_\infty(X) \rtimes X)^{[L]}\Big/(C_0(X) \rtimes X)^{(L)} = \mathscr{T}(X)^{[L]}\Big/\mathscr{K}(X)^{(L)}.
\end{aligned}$$

□

Remarks. (i) As in the abelian case, we have a precise description of the range of the embedding (5.6): it is the quotient with respect to $\mathscr{K}(X)^{(L)}$ of the subspace of $\mathscr{T}(X)^{[L]}$ consisting of sequences of the form $(\psi(P)+K_l)_{l\in L}$ for some $\psi \in C_0(X^*)$ and $(K_l)_{l\in L} \in \mathscr{K}(X)^{[L]}$.
(ii) The algebra $\mathscr{T}(X)^{[L]}$ has an obvious faithful representation on the Hilbert space $\mathscr{H} = \bigoplus_L L^2(X)$. In this representation we have $\mathscr{K}(X)^{(L)} = \mathscr{T}(X)^{[L]} \bigcap K(\mathscr{H})$.

Theorem 5.2 is the main result of this section: it allows, via Theorem 2.2, to compute the essential spectrum of a hamiltonian affiliated to the algebra of energy observables $\mathscr{C}_L$ in terms of spectra of hamiltonians affiliated to the two-body algebra. The details are as follows.

If H is an observable affiliated to $\mathscr{C}_L$ and if $\widehat{H}$ is its image through the canonical morphism $\mathscr{C}_L \to \mathscr{C}_L/\mathscr{K}(X)$, then there is a family $(H_l)_{l\in L}$ of observables affiliated to the two-body algebra $\mathscr{T}(X)$ such that the quotient of $\prod_{l\in L} H_l$ with respect to the ideal $\mathscr{K}(X)^{(L)}$ is equal to the image of $\widehat{H}$ through the embedding (5.6). Such a family $(H_l)_{l\in L}$ will be called a *representative* of H. By the discussion above we have $\prod_{l\in L}(H_l - z)^{-1} \in \mathscr{T}(X)^{[L]}$ and the component of $(H_l - z)^{-1}$ in $C_0(X^*)$ is independent of $l \in L$, so $\boldsymbol{\sigma}_{\text{ess}}(H_l)$ is independent of l. Thus the next result is a consequence of Theorem 2.2.

Theorem 5.3. *If H is an observable affiliated to $\mathscr{C}_L(X)$ and $\{H_l\}_{l\in L}$ is a representative of H, then*

$$\boldsymbol{\sigma}_{\text{ess}}(H) = \bigcap_{\substack{F\subset L\\ F\,\text{finite}}} \overline{\bigcup_{l\in L\setminus F} \boldsymbol{\sigma}(H_l)}.$$

It is quite easy to give examples of a self-adjoint operator affiliated to $\mathscr{C}_L$ with a nontrivial essential spectrum. Let $h : X^* \to \mathbb{R}$ be a continuous divergent function (by “divergent” we mean $\lim_{\kappa\to\infty} h(\kappa) = \infty$). Then $H_0 = h(P)$ is a self-adjoint operator strictly affiliated to $\mathscr{C}_L$, hence if V is a self-adjoint operator in the multiplier algebra of $\mathscr{C}_L$ then $H = H_0 + V$ is also strictly affiliated to $\mathscr{C}_L$ and we may apply to it the Theorems 5.3 and 5.6. More explicitly, we may take $V = \sum_{l\in L} \tau_l \varphi_l(Q)$, where $\{\varphi_l\}$ is as in (ii) of Lemma 5.1, in which case the operators H_l of Theorem 5.3 are given by $H_l = h(P) + \varphi_l(Q)$. Much more singular perturbations are, however, allowed, as we shall show later on.

We close this paragraph by pointing out the interesting particular case when there is only a finite number of *types of bumps*. Let $\mathcal{L}$ be a finite partition of L consisting of infinite sets M and let $\mathscr{C}_L(X)^{\mathcal{L}}$ be the space of $S \in \mathscr{C}_L(X)$ such that the limit $\text{s-lim}_{l\in M, l\to\infty} U_l S U_l^* := S_M$ exists for each $M \in \mathcal{L}$. This is clearly a C^*-subalgebra of $\mathscr{C}_L(X)$ which contains $\mathscr{K}(X)$. $\mathscr{C}_L(X)^{\mathcal{L}}$ is the set of $S \in \mathscr{C}_L(X)$ such that for each $M \in \mathcal{L}$ and each ultrafilter $\varkappa \in \widetilde{L}$ with $M \in \varkappa$ the limit $S_\varkappa$ is independent of $\varkappa$. By using the remarks made in the last part of Sect. 2.3 one can prove that

$$\mathscr{C}_L(X)^{\mathcal{L}} \big/ \mathscr{K}(X) \hookrightarrow \bigoplus_{M\in\mathcal{L}} \mathscr{T}(X). \tag{5.7}$$

If H is an observable affiliated to $\mathscr{C}_L(X)^{\mathcal{L}}$ then $\text{s-lim}_{l\in M, l\to\infty} U_l H U_l^* := H_M$ exists in the strong resolvent sense for each $M \in \mathcal{L}$ and

$$\boldsymbol{\sigma}_{\text{ess}}(H) = \bigcup_{M\in\mathcal{L}} \boldsymbol{\sigma}(H_M). \tag{5.8}$$

5.3. Dense subalgebras. We shall describe here a class of elements of the algebra $\mathscr{C}_{L,0}$. This will give us a version of Theorem 5.2 independent of the constructions from the abelian case.

Proposition 5.1. *Let $\{K_l\}_{l\in L}$ be a relatively compact family of compact operators on $\mathscr{H}$. Assume that there is a compact set $\Lambda \subset X$ such that $K_l = \chi_\Lambda(Q)\, K_l\, \chi_\Lambda(Q)$ for all $l \in L$. Then the series $\sum_{l\in L} U_l^*\, K_l\, U_l$ converges in the strong operator topology and its sum belongs to $\mathscr{C}_{L,0}$. The set of operators of the preceding form is dense in $\mathscr{C}_{L,0}$. More precisely, the linear subspace generated by the operators $\sum_{l\in M} U_l^* K U_l$, where M is a subset of L and $K \in \mathscr{K}(X)$ has the property $K = \chi_\Lambda(Q)\, K\, \chi_\Lambda(Q)$ for some compact set $\Lambda \subset X$, is dense in $\mathscr{C}_{L,0}$.*

For the proof we need the following noncommutative version of the Ascoli theorem (which follows from the Riesz–Kolmogorov compacity criterion, see [21]): *a bounded subset $\mathscr{K} \subset \mathscr{B}(X)$ is a relatively compact set of compact operators if and only if it satisfies the following equivalent conditions:*

(i) $\lim_{x\to 0} \sup_{T\in\mathscr{K}} \|(U_x - 1)T^{(*)}\| = 0$ *and* $\lim_{k\to 0} \sup_{T\in\mathscr{K}} \|(V_k - 1)T^{(*)}\| = 0$.

(ii) *For each $\varepsilon > 0$ there are $\varphi \in C_c(X)$ and $\psi \in C_c(X^*)$ such that*

$$\|\varphi(Q)^\perp T^{(*)}\| + \|\psi(P)^\perp T^{(*)}\| < \varepsilon \ \ \textit{for all } T \in \mathscr{K}.$$

Thus, a family $\{K_l\}_{l\in L}$ satisfying $K_l = \chi_\Lambda(Q)\, K_l\, \chi_\Lambda(Q)$ is relatively compact if and only if

$$\lim_{x\to 0} \sup_{l\in L} \|(U_x - 1)K_l^{(*)}\| = 0. \tag{5.9}$$

Proof. Notice first that finite sums $\sum_{l\in F} U_l^*\, K_l\, U_l$ are compact operators, so belong to $\mathscr{C}_{L,0}$. Hence we just have to prove the first part of the theorem under the assumption $(l+\Lambda)\cap(l'+\Lambda) = \emptyset$ if $K_l \neq 0,\ K_{l'} \neq 0$. The series $T = \sum_{l\in L} U_l^*\, K_l\, U_l$ converges strongly because the operators $U_l^*\, K_l\, U_l$ are pairwise orthogonal for large l. The family $\{K_l\}_{l\in L}$ being relatively compact, for each $\varepsilon > 0$ there is a finite subset I of L such that L decomposes into a disjoint union $\bigsqcup_{i\in I} L_i$ and for each $i \in I$ we have $\|K_l - K_i\| < \varepsilon$ for all $l \in L_i$. Let then $T_\varepsilon \equiv \sum_{i\in I}\sum_{l\in L_i} U_l^*\, K_i\, U_l$ and estimate for each $f \in \mathscr{H}$:

$$\begin{aligned}\|(T - T_\varepsilon)f\|^2 &= \|\textstyle\sum_{i\in I}\sum_{l\in L_i} \chi_\Lambda(Q-l) U_l^*\,(K_l - K_i)\, U_l \chi_\Lambda(Q-l) f\,\|^2\\ &= \textstyle\sum_{i\in I}\sum_{l\in L_i} \|\chi_\Lambda(Q-l)\, U_l^*\,(K_l - K_i)\, U_l \chi_\Lambda(Q-l) f\|^2\\ &\le \varepsilon^2 \textstyle\sum_{l\in L} \|\chi_\Lambda(Q-l)\, f\|^2 = \varepsilon^2 \|\sum_{l\in L} \chi_{\Lambda+l}(Q)\, f\,\|^2 \le \varepsilon^2\|f\|^2.\end{aligned}$$

Thus it suffices to show that $T_\varepsilon \in \mathscr{C}_{L,0}$ which actually means that it suffices to prove the proposition for the case when $K_l \equiv K$ is independent of l. Note that in the arguments below one can substitute to L any subset of it.

So let K be a compact operator and Λ a compact subset of X such that $K = \chi_\Lambda(Q) K \chi_\Lambda(Q)$. We shall prove that $\widetilde{K} \equiv \sum_{l\in L} U_l^*\, K\, U_l \in \mathscr{C}_{L,0}$ (the series being strongly convergent by the same argument as above). The set of $\psi \in C_0(X^*)$ such that $\psi = \widehat{\eta}$ for some $\eta \in C_c(X)$ is dense in $C_0(X^*)$ (see (4.13) in [18]). By using also Corollary 4.1, we see that for each $\varepsilon > 0$ there are functions $\varphi_1, \dots, \varphi_n, \eta_1, \dots, \eta_n$ in $C_c(X)$ such that $\|K - S\| < \varepsilon$, where $S = \sum_{i=1}^n \varphi_i(Q)\widehat{\eta}_i(P)$. Since $(\widehat{\eta}(P) f)(x) =$

$\int_X \eta(x-y) f(y)\,\mathrm{d}y$ we have $(Sf)(x) = \int_X \left(\sum_i \varphi_i(x)\eta_i(x-y)\right) f(y)\,\mathrm{d}y$. Let Γ be a compact set such that the supports of the functions φ_i, η_i are included in Γ and let Ω be the compact set $\Lambda \cup \Gamma \cup (\Gamma - \Gamma)$. If $\chi_\Omega \equiv \chi_\Omega(Q)$ then $K = \chi_\Omega K \chi_\Omega,\ S = \chi_\Omega S \chi_\Omega$. This shows in particular that the series $\sum_{l\in L} U_l^* S U_l$ is strongly convergent (L being sparse) and its sum $\widetilde{S}$ can be computed:

$$\widetilde{S} = \textstyle\sum_i \sum_l U_l^* \varphi_i(Q) \widetilde{\eta}_i(P) = \sum_i [\sum_l \varphi_i(Q-l)] \widetilde{\eta}_i(P) \equiv \sum_i \phi_i(Q) \widehat{\eta}_i(P).$$

The functions ϕ_i belong to $C_{L,0}$ by Lemma 5.1, so $\widetilde{S} \in \mathscr{C}_{L,0}$. We have $\widetilde{K} - \widetilde{S} = \sum_l \chi_{l+\Omega} U_l^* (K-S) U_l \chi_{l+\Omega}$ and there is a finite set $F \subset L$ such that, if $l, l' \in M \equiv L \setminus F,\ l \neq l'$, then $(l+\Omega) \cap (l'+\Omega) = \emptyset$. Then

$$\begin{aligned} \big\| \widetilde{K} f - \widetilde{S} f - \textstyle\sum_{l\in F} U_l^*(K-S) U_l f \big\|^2 &= \big\| \textstyle\sum_{l\in M} \chi_{l+\Omega} U_l^* (K-S) U_l \chi_{l+\Omega} f \big\|^2 \\ &= \textstyle\sum_{l\in M} \big\| \chi_{l+\Omega} U_l^*(K-S) U_l \chi_{l+\Omega} f \big\|^2 \le \varepsilon^2 \sum_{l\in M} \|\chi_{l+\Omega} f\|^2 \le \varepsilon^2 \|f\|^2. \end{aligned}$$

Thus for each $\varepsilon > 0$ there is an operator $T = \widetilde{S} - \sum_{l\in F} U_l^*(K-S)U_l \in \mathscr{C}_{L,0}$ such that $\|\widetilde{k} - T\| \le \varepsilon$. Hence $\widetilde{K} \in \mathscr{C}_{L,0}$.

The fact that the linear subspace generated by the operator sums $\sum_{l\in M} U_l^* \varphi(Q) \widehat{\eta}(P) U_l$, with $M \subset L$ and $\varphi, \eta \in C_{\mathrm{c}}(X)$, is dense in $\mathscr{C}_{L,0}$ follows from Theorem 4.1, Lemma 5.1 and the preceding arguments (where L can be replaced by M). □

The next result, a more explicit version of Theorem 5.2, is a straightforward consequence of Proposition 5.1 and Theorem 5.2 (see also (2.6)).

Theorem 5.4. *There is a unique morphism $\mathscr{C}_L \to \mathscr{T}(X)^{[L]}/\mathscr{K}(X)^{(L)}$ such that the image of an element of the form $\psi(P) + \sum_{l\in M} U_l^* K U_l$, where $\psi \in C_0(X^*)$, $M \subset L$, and $K \in \mathscr{K}(X)$ is such that $K = \chi_\Lambda(Q)\, K\, \chi_\Lambda(Q)$ for some compact set $\Lambda \subset X$, is the quotient of the element $\chi_M \otimes (\psi(P) + K) \in \mathscr{T}(X)^{[L]}$ with respect to the ideal $\mathscr{K}(X)^{(L)}$. The kernel of this morphism is $\mathscr{K}(X)$ and its restriction to $\mathscr{C}_{L,0}$ induces the canonical isomorphism of $\mathscr{C}_{L,0}/\mathscr{K}(X)$ with the L-asymptotic algebra of compact operators $\mathscr{K}(X)^{\langle L\rangle}$.*

5.4. Another description of the quotient. Let us give now a second description of the quotient algebra $\mathscr{C}_L(X)/\mathscr{K}(X)$, based on the formalism exposed in Sect. 2.3. According to the notations introduced there, we shall denote by $\widetilde{L}$ the set of ultrafilters on L finer than the Fréchet filter; $\widetilde{L}$ is a compact topological space. We denote $\lim_{l,\varkappa}$ the limit over l along a filter $\varkappa$.

Theorem 5.5. *If $S \in \mathscr{C}_L$ and $\varkappa \in \widetilde{L}$ the limit* $\operatorname{s-lim}_{l,\varkappa} U_l S U_l^* = S_\varkappa$ *exists in the strong operator topology and belongs to $\mathscr{T}(X)$. The component of $S_\varkappa$ in $C_0(X^*)$ is equal to that of S in $C_0(X^*)$. The map $S \mapsto (S_\varkappa)_{\varkappa\in\widetilde{L}}$ is a morphism $\mathscr{C}_L \to C(\widetilde{L}\,; \mathscr{T}(X))$ with kernel $\mathscr{K}(X)$ and range equal to the set of $(S_\varkappa)_{\varkappa\in\widetilde{L}}$ such that the component of $S_\varkappa$ in $C_0(X^*)$ is independent of $\varkappa$.*

Proof. One has a unique decomposition of S into a sum $T + S'$ with $T \in C_0(X^*)$ and $S' \in \mathscr{C}_{L,0}$. Since $U_l T U_l^* = T$, it suffices to consider $T = 0$. Then by (iii) of Lemma 5.1 and (5.4) it suffices to take $S = \varphi(Q)\psi(P)$ with $\varphi = \sum_{m\in M} \tau_m \varphi_0$ for some subset $M \subset L$ and some $\varphi_0 \in C_0(\Lambda)$, $\Lambda \subset X$ compact, and with $\psi \in C_0(X^*)$.

Since $U_l S U_l^* = (\tau_{-l}\varphi)(Q)\psi(P)$, it suffices to show that $\lim_{l,\varkappa} \tau_{-l}\varphi$ exists uniformly on compacts on X. There are only two possibilities: either $M \in \varkappa$, or $L \setminus M \in \varkappa$; in the first case we shall prove that $\lim_{l,\varkappa} \tau_{-l}\varphi = \varphi_0$ and in the second one that $\lim_{l,\varkappa} \tau_{-l}\varphi = 0$. Indeed, let $K \subset X$ be a compact set and let $x \in K$. Then $\tau_{-l}\varphi(x) = \sum_{m\in M} \varphi_0(x + l - m)$. If $\varphi_0(x + l - m) \neq 0$ then $l \in m + (\Lambda - K)$ and $\Lambda - K$ is a compact set. So if l is large enough then $\varphi_0(x + l - m) \neq 0$ only if $l = m$ (L being sparse). So for large l one has either $\tau_{-l}\varphi(x) = \varphi_0(x)$ or $\tau_{-l}\varphi(x) = 0$ (independently of $x \in K$).

We have thus shown that the limit $\text{s-}\lim_{l,\varkappa} U_l S U_l^* := S_\varkappa$ exists for each $\varkappa \in \widetilde{L}$. The argument also gives the explicit form of the limit for a class of operators S which is dense in $\mathscr{C}_L$. Namely, assume that S is of the form $\psi_0(P) + \sum_{i=1}^n \varphi_i(Q)\psi_i(P) \equiv T + S'$, with $\psi_1, \ldots, \psi_n \in C_0(X^*)$ and $\varphi_i = \sum_{j=1}^{k_i} \sum_{l\in L_{ij}} \tau_l \varphi_{ij}$, where, for each i, $\{L_{i1}, \ldots, L_{ik_i}\}$ is a partition of L and $\varphi_{ij} \in C_{\mathrm{c}}(X)$. For each i there is a unique $j(i) \in \{1, \ldots, k_i\}$ such that $L_{ij(i)} \in \varkappa$. Then

$$S_\varkappa = \psi_0(P) + \sum_{i=1}^n \varphi_{ij(i)}(Q)\psi_i(P). \tag{5.10}$$

Thus $S_\varkappa \in \mathscr{T}(X)$ and its projection on $C_0(X^*)$ is $\psi_0(P)$, which is the component of S in $C_0(X^*)$. This remains valid for all S by continuity and density.

Finally, consider the image $\widehat{S'}$ of S' in $\mathscr{K}(X)^{\langle L\rangle}$ given by Theorem 5.2 and identify $\mathscr{K}(X)^{\langle L\rangle} \cong C(\widetilde{L}\,; \mathscr{K}(X))$, cf. (2.8). Then $\widehat{S'}$ will be the family of operators $S_\varkappa$ defined by (5.10), so the theorem is proved. □

The following result is a straightforward consequence of Theorem 5.5.

Theorem 5.6. *Let H be an observable affiliated to $\mathscr{C}_L(X)$. Then* $\text{s-}\lim_{l,\varkappa} U_l H U_l^* \equiv H_\varkappa$ *exists in the strong resolvent sense for each $\varkappa \in \widetilde{L}$ and*

$$\boldsymbol{\sigma}_{\text{ess}}(H) = \bigcup_{\varkappa\in\widetilde{L}} \boldsymbol{\sigma}(H_\varkappa).$$

If $\{H_l\}_{l\in L}$ is a representative of H, then for $\varkappa \in \widetilde{L}$ one also has $H_\varkappa = \text{u-}\lim_{l,\varkappa} H_l$ (limit in the norm resolvent sense).

6. An Explicit Class of Hamiltonians

We shall construct here a large class of hamiltonians affiliated to the algebra $\mathscr{C}_L(X)$. We consider explicitly only the case $X = \mathbb{R}^n$ in order to be able to use the standard theory of Sobolev spaces. However, our arguments easily extend to other groups.

We begin with a remark concerning the definition of the hamiltonians. In the sequel we use the abbreviation $\langle\cdot\rangle = (1 + |\cdot|^2)^{1/2}$. Let $\mathscr{H}^s = \mathscr{H}^s(\mathbb{R}^n)$ be the scale of Sobolev spaces; here $s \in \mathbb{R}$, $\mathscr{H}^0 = L^2(\mathbb{R}^n) \equiv \mathscr{H}$. Let s, t be real numbers such that $0 \le t < s$. Let H_0 be a self-adjoint operator in $\mathscr{H}$ with $D(|H_0|^{1/2}) \subset \mathscr{H}^s$ and let $V : \mathscr{H}^t \to \mathscr{H}^{-t}$ be a symmetric operator. Then $\langle P\rangle^{2s} \le C(|H_0|+1)$ for some constant C and for each $\varepsilon > 0$ there is a constant $c < \infty$ such that $\pm V \le \varepsilon\langle P\rangle^{2s} + c$. Thus the form sum $H_0 + V$ defines a self-adjoint operator H in $\mathscr{H}$ with form domain equal to that of H_0. The self-adjoint operators from the next theorem should be interpreted in this sense.

Theorem 6.1. *Let $h : \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that*

$$C^{-1}|x|^{2s} \leq |h(x)| \leq C|x|^{2s} \text{ for } |x| > R, \tag{6.1}$$

for some constants $s > 0$, $C > 0$ and $R < \infty$; denote $H_0 = h(P)$. Let $t \in [0, s)$ real, let L be a sparse subset of $\mathbb{R}^n$, and let $\{W_l\}_{l\in L}$ be a family of symmetric operators in $B(\mathscr{H}^t, \mathscr{H}^{-t})$ with the following property:t here is a number $a > 2n$ such that

$$\sup_{l\in L} \|\langle Q\rangle^a W_l\|_{B(\mathscr{H}^t,\mathscr{H}^{-t})} < \infty. \tag{6.2}$$

Then the series $\sum_{l\in L} U_l^* W_l U_l$ converges in the strong topology of $B(\mathscr{H}^t, \mathscr{H}^{-t})$ and its sum is a symmetric operator $W : \mathscr{H}^t \to \mathscr{H}^{-t}$. Let $H = H_0 + W$, $H_l = H_0 + W_l$ be the self-adjoint operators in $\mathscr{H}$ defined as form sums. Then H is strictly affiliated to $\mathscr{C}_L$, H_l is strictly affiliated to $\mathscr{T}(X)$, and the family $\{H_l\}_{l\in L}$ is a representative of H. In particular:

$$\boldsymbol{\sigma}_{\text{ess}}(H) = \bigcap_{\substack{F\subset L\\ F \text{finite}}} \overline{\bigcup_{l\in L\setminus F} \boldsymbol{\sigma}(H_l)}. \tag{6.3}$$

If $\varkappa$ is an ultrafilter on L finer than the Fréchet filter, then $\text{u-lim}_{l\in\varkappa} H_l := H_\varkappa$ exists in the norm resolvent sense, one has $H_\varkappa = \text{s-lim}_{l,\varkappa} U_l H U_l^*$ in the strong resolvent sense, and

$$\boldsymbol{\sigma}_{\text{ess}}(H) = \bigcup_{\varkappa\in\widetilde{L}} \boldsymbol{\sigma}(H_\varkappa). \tag{6.4}$$

Remarks. (i) If $s \leq n/2$ and $W_l : \mathbb{R}^n \to \mathbb{R}$ are Borel functions satisfying the condition $\int_{|y-x|<1} |W_l(y)| \cdot |y-x|^{-n+2s-\lambda} \, dy \leq c\langle x\rangle^{-a}$ $\forall x \in \mathbb{R}^n$ for some constants $c, \lambda > 0$, then the operators W_l of multiplication by the functions W_l satisfy (6.2) for some $t < s$. If $s > n/2$ then the simpler condition $\int_{|y-x|<1} |W_l(y)| \, dy \leq c\langle x\rangle^{-a}$ suffices. If s is an integer and W_l is a differential operator of order less than $2s$, similar explicit conditions on its coefficients can be stated.

(ii) The condition $a > 2n$ is not natural and should be improved to $a > n$, but the version of the Cotlar-Stein lemma that we use in the proof does not allow us to get such a result. However, the assumption on a may be relaxed in terms of the "degree of rarefaction" of L: the greatest lower bound for a is actually inversely proportional to it, see Lemma 6.2.

(iii) As explained in Sect. 1.1, we can replace the Hilbert space $L^2(X)$ of physical states by $L^2(X; \mathrm{E})$ where E is a finite dimensional Hilbert space. This allows us to treat, for example, Dirac hamiltonians H_0 perturbed by the same class of potentials W. The condition (6.1) is satisfied with $s = 1/2$, $|h(x)|$ being interpreted as $[h(x)^2]^{1/2}$ ($h(x)$ is a self-adjoint operator in E). Note that H_0 is not semibounded in this case. Theorems 5.3 and 5.6 remain valid without any change in this context.

(iv) The assumption $t < s$ is not essential and can be improved to $t = s$, which allows one to treat perturbations of the same order as H_0. But then one must add other conditions in order to give a sense to the sums $H_0 + W_l$ and $H_0 + W$ as self-adjoint operators. This question is of some importance if one wants to treat Dirac operators with Coulomb potentials or second order perturbations of the Laplace operator, but is outside the main scope of this paper.

(v) Assume that h is bounded from below, so that its range is of the form $J = [\mu, \infty)$ for some real μ. Then the spectrum of the operator H_l is of the form $\boldsymbol{\sigma}(H_l) = J \cup D_l$, where D_l is a discrete subset of $(-\infty, \mu)$. From (6.3) it follows that the *essential* spectrum of H is of the form $J \cup D$, where $D \subset (-\infty, \mu)$ could have a quite complicated structure. The spectrum inside J is probably also of a rather complex nature: singularly continuous, absolutely continuous, and pure point spectrum could coexist. The methods used in this paper do not allow us to study such fine properties of H (we do not expect that H admits conjugate operators locally inside J). However, in the preprint version [22] of this article we proved that the wave operators corresponding to the elastic channel exist in rather general situations (the difficulty appears when the W_l are of the same order of magnitude, e.g. do not depend on l). Our results extend those from [25] and are valid in any dimension $n \geq 2$. In particular, the absolutely continuous spectrum of H is often equal to J. On the other hand, if $n = 1$, taking into account the results from [33], we are tempted to think that the wave operators do not exist and there is no absolutely continuous spectrum if W_l is independent of l.

We begin now the proof of the theorem and first recall the Cotlar-Stein lemma:

Lemma 6.1. *Let $\{B_l\}_{l\in L}$ be a family of operators in $B(\mathscr{H}_1, \mathscr{H}_2)$ for some Hilbert spaces $\mathscr{H}_1, \mathscr{H}_2$. Assume that*

$$\sup_{l\in L} \sum_{m\in L} \max\left\{ \|B_l B_m^*\|_{B(\mathscr{H}_2)}^{1/2}\,,\ \|B_l^* B_m\|_{B(\mathscr{H}_1)}^{1/2} \right\} = b < \infty.$$

Then $\sum_{l\in L} B_l \equiv B$ exists in the strong operator topology and $\|B\|_{B(\mathscr{H}_1,\mathscr{H}_2)} \leq b$.

Theorem 6.1 will be a consequence of the next lemma. We denote by $\|\cdot\|_{u,v}$ the norm in $B(\mathscr{H}^u, \mathscr{H}^v)$.

Lemma 6.2. *Let $L \subset \mathbb{R}^n$ such that $|l - m| \geq$ const. > 0 if l, m are distinct points of L, and let $a > 2n$. Then there is $C > 0$ such that, for each family of operators $W_l \in B(\mathscr{H}^u, \mathscr{H}^v)$ with $W_l = 0$ for all but a finite number of l, the following estimate holds:*

$$\left\| \sum_{l\in L} U_l^* W_l U_l \right\|_{u,v} \leq C \sup_{l\in L} \max\left\{ \|\langle Q\rangle^a W_l\|_{u,v}\,,\ \|W_l \langle Q\rangle^a\|_{u,v} \right\}. \tag{6.5}$$

Proof. Let us denote $B_l \equiv U_l^* \langle P\rangle^v W_l \langle P\rangle^{-u} U_l$ and check that the hypotheses of the Cotlar-Stein lemma are satisfied. For each couple l, m of points of L we estimate:

$$\begin{aligned} \|B_l B_m^*\| &= \|\langle P\rangle^v W_l \langle P\rangle^{-2u} U_{l-m} W_m^* \langle P\rangle^v\| \\ &\leq \|W_l \langle Q\rangle^a\|_{u,v} \cdot \|\langle P\rangle^u \langle Q\rangle^{-a} \langle P\rangle^{-2u} U_{l-m} \langle Q\rangle^{-a} \langle P\rangle^u\| \cdot \|\langle Q\rangle^a W_m^*\|_{-v,-u}. \end{aligned}$$

We have $\|W_m \langle Q\rangle^a\|_{u,v} = \|\langle Q\rangle^a W_m^*\|_{-v,-u}$. By standard commutator estimates, there is a bounded operator S such that $\langle P\rangle^u \langle Q\rangle^{-a} \langle P\rangle^{-2u} = S \langle P\rangle^{-u} \langle Q\rangle^{-a}$, thus the middle norm in the last term of the above inequality may be majorated by $\|S\|$ times the quantity:

$$\|\langle P\rangle^{-u} \langle Q\rangle^{-a} U_{l-m} \langle Q\rangle^{-a} U_{l-m}^* \langle P\rangle^u\| = \|\langle Q\rangle^{-a} \langle Q - (m-l)\rangle^{-a}\|_{u,u}.$$

Let us denote by C a generic positive finite constant. By interpolation between 0 and an integer $N > |u|$ the above quantity is dominated by

$$C\, \|\langle Q\rangle^{-a} \langle Q - (m-l)\rangle^{-a}\|_{N,N} \leq C \sup_{x\in\mathbb{R}^n} \sup_{|\alpha|\leq N} \left| \varphi_{lm}^{(\alpha)}(x) \right|,$$

where $\varphi_{lm}(x) \equiv \langle x\rangle^{-a}\langle x-(m-l)\rangle^{-a}$. Furthermore, for each α there is a constant c_α such that

$$\left|\varphi_{lm}^{(\alpha)}(x)\right| \le c_\alpha\ |\varphi_{lm}(x)| \le C\,\langle l-m\rangle^{-a}.$$

Hence $\|B_l B_m^*\| \le C\,\langle l-m\rangle^{-a}\sup_{l\in L}\|W_l\langle Q\rangle^a\|_{u,v}$. Similarly we obtain the estimate $\|B_l^* B_m\| \le C\,\langle l-m\rangle^{-a}\sup_{l\in L}\|\langle Q\rangle^a W_l\|_{u,v}$ which finally yields

$$\max\left\{\|B_l B_m^*\|\,,\ \|B_l^* B_m\|\right\} \le C\,\langle l-m\rangle^{-a}\max\left\{\|\langle Q\rangle^a W_l\|_{u,v}\,,\ \|W_l\langle Q\rangle^a\|_{u,v}\right\}.$$

The hypothesis $a > 2n$ is then sufficient to ensure $\sum_{m\in L}\langle l-m\rangle^{-a/2} \le \text{const.} < \infty$ independently of $l \in L$, hence the hypotheses of Lemma 6.1 are verified. □

If H_0 is as in Theorem 6.1 then it is obviously affiliated to $C_0(X^*) \subset \mathscr{C}_L$. From Theorem 4.1 it follows easily that H_0 is strictly affiliated to $\mathscr{C}_L$. Note that we can assume $H_0 \ge 1$. Then we write

$$H_0^{-1/2} W H_0^{-1/2} = H_0^{-1/2}\langle P\rangle^s \cdot \langle P\rangle^{-s} W \langle P\rangle^{-s} \cdot \langle P\rangle^s H_0^{-1/2}. \tag{6.6}$$

Below we shall prove that

$$\langle P\rangle^{-s} W \langle P\rangle^{-s} \in \mathscr{C}_L. \tag{6.7}$$

From Theorem 4.1 it follows that the elements of the form $\theta_1(P)T\theta_2(P)$ with $\theta_k \in C_c(X^*)$ are dense in $\mathscr{C}_L$. Then the relations (6.6) and (6.7) imply $H_0^{-1/2} W H_0^{-1/2} \in \mathscr{C}_L$. Finally, Theorem 6.1 is a consequence of the affiliation criterion Theorem 2.1.

We shall prove (6.7) by constructing a family of symmetric operators $\{W_\varepsilon\}$ in $\mathscr{C}_L$ which approximates W in the norm of $B(\mathscr{H}^s, \mathscr{H}^{-s})$. Choose $\theta \in C_c^\infty(\mathbb{R}^n)$ such that $0 \le \theta \le 1$ and $\theta(0) = 1$, and set $\Theta_\varepsilon = \theta(\varepsilon Q)\theta(\varepsilon P)$. Let $W_\varepsilon = \sum_{l\in L} U_l^* W_{l,\varepsilon} U_l$, where $W_{l,\varepsilon} = \Theta_\varepsilon W_l \Theta_\varepsilon^*$. Observe first that, for each $\varepsilon \in (0,1]$, $\{W_{l,\varepsilon}\}_{l\in L}$ is a relatively compact family of compact symmetric operators on $\mathscr{H}$. Hence, by Proposition 5.1, W_ε belongs to $\mathscr{C}_{L,0}$ for each $\varepsilon > 0$. It remains thus only to show the convergence $\|W_\varepsilon - W\|_{s,-s} \to 0$ as $\varepsilon \to 0$. Let $2n < \alpha < a$, where a is as in Theorem 6.1. We shall use Lemma 6.2 with W replaced by $W_\varepsilon - W$, a by α, and $u = s,\ v = -s$. Then the first norm in the r.h.s. of the corresponding inequality of type (6.5) is estimated as follows:

$$\begin{aligned}\|\langle Q\rangle^\alpha (W_{l,\varepsilon} - W_l)\|_{s,-s} &\le \|\langle Q\rangle^\alpha(\Theta_\varepsilon - 1)W_l\Theta_\varepsilon^*\|_{s,-s} + \|\langle Q\rangle^\alpha W_l(\Theta_\varepsilon^* - 1)\|_{s,-s}\\ &\le \|\langle Q\rangle^\alpha(\Theta_\varepsilon - 1)\langle Q\rangle^{-a}\|_{-t,-s}\cdot\|\langle Q\rangle^a W_l\|_{t,-t}\cdot\|\Theta_\varepsilon^*\|_{s,t}\\ &\le \|\langle Q\rangle^\alpha W_l\langle Q\rangle^{a-\alpha}\|_{t,-s}\cdot\|\langle Q\rangle^{\alpha-a}(\Theta_\varepsilon^* - 1)\|_{s,t}.\end{aligned}$$

We shall use the scale of spaces $\mathscr{H}_v^u$ defined by the norms $\|\langle P\rangle^u\langle Q\rangle^v\cdot\|$. By hypothesis, the family of operators W_l is bounded in $B(\mathscr{H}^t, \mathscr{H}_a^{-t})$. By interpolation, and since $t < s$, we get that it is also bounded in $B(\mathscr{H}_{\alpha-a}^t, \mathscr{H}_\alpha^{-s})$. So in order to show that $\|\langle Q\rangle^\alpha(W_{l,\varepsilon} - W_l)\|_{s,-s} \to 0$ if $\varepsilon \to 0$ it suffices to prove the next two relations:

$$\begin{aligned}&\|\langle P\rangle^{-s}\langle Q\rangle^\alpha(\Theta_\varepsilon - 1)\langle Q\rangle^{-a}\langle P\rangle^t\|\\ &\qquad = \|\langle P\rangle^{-s}\langle Q\rangle^\alpha(\Theta_\varepsilon - 1)\langle Q\rangle^{-\alpha}\langle P\rangle^s\cdot\langle P\rangle^{-s}\langle Q\rangle^{\alpha-a}\langle P\rangle^t\| \to 0,\\ &\|\langle P\rangle^{-s}(\Theta_\varepsilon - 1)\langle Q\rangle^{\alpha-a}\langle P\rangle^t\|\\ &\qquad = \|\langle P\rangle^{-s}(\Theta_\varepsilon - 1)\langle P\rangle^s\cdot\langle P\rangle^{-s}\langle Q\rangle^{\alpha-a}\langle P\rangle^t\| \to 0.\end{aligned}$$

The operator $\langle P\rangle^{-s}\langle Q\rangle^{\alpha-a}\langle P\rangle^t$ is compact, so it suffices to show that $\Theta_\varepsilon \to 1$ strongly in $B(\mathscr{H}_v^u)$ when $\varepsilon \to 0$, for each $u, v \in \mathbb{R}$. But this is an easy consequence of the next more precise lemma and its analog with the roles of Q and P interchanged. Theorem 6.1 is proved. □

Lemma 6.3. *Let θ be in the Schwartz space $\mathscr{S}(\mathbb{R}^n)$ and let $u, v \in \mathbb{R}$. Then*

$$\lim_{\varepsilon\to 0} \|[\langle Q\rangle^v, \theta(\varepsilon P)]\langle Q\rangle^{-v}\|_{B(\mathscr{H}^u)} = 0.$$

Proof. We use the Fourier representation $\theta(\varepsilon P) = \int U_{\varepsilon x}\widehat{\theta}(x)\,\mathrm{d}x$ in order to compute

$$\begin{aligned}[\langle Q\rangle^v, \theta(\varepsilon P)]\langle Q\rangle^{-v} &= \int U_{\varepsilon x}(U^*_{\varepsilon x}\langle Q\rangle^v U_{\varepsilon x}\langle Q\rangle^{-v} - 1)\,\widehat{\theta}(x)\,\mathrm{d}x \\ &= \int U_{\varepsilon x}(\langle Q-\varepsilon x\rangle^v\langle Q\rangle^{-v} - 1)\,\widehat{\theta}(x)\,\mathrm{d}x.\end{aligned}$$

It is clear that $\|\langle Q-x\rangle^v\langle Q\rangle^{-v}\|_{B(\mathscr{H}^u)} \le C\langle x\rangle^r$ for some positive numbers C and r, so we shall have $\|[\langle Q\rangle^v, \theta(\varepsilon P)]\langle Q\rangle^{-v}\|_{B(\mathscr{H}^u)} \le$ const. Then, by an easy interpolation argument, it suffices to prove the lemma in the case $u = 0$. Now the dominated convergence theorem shows that it is enough to prove $\|\langle Q-x\rangle^v\langle Q\rangle^{-v} - 1\| \to 0$ when $x \to 0$. But this is a consequence of $(1+|x|)^{-1} \le \langle y-x\rangle\langle y\rangle^{-1} \le (1+|x|)$. □

7. Appendix

In this appendix we shall prove Theorem 1.2. It clearly suffices to consider only the case of $T \in \mathscr{C}_{L,0}$. We denote by $\mathscr{A}$ the set of operators verifying the conditions (i)-(iii) of the theorem with $\widetilde{T} = 0$. Clearly $\mathscr{A}$ is a C^*-algebra.

We first show the easy inclusion $\mathscr{C}_{L,0} \subset \mathscr{A}$. By (5.4) it suffices to show that operators of the form $\varphi(Q)\psi(P)$ with $\varphi \in C_{L,\mathrm{c}}(X)$ and $\psi \in C_0(X^*)$ belong to $\mathscr{A}$. Since $U_x\varphi(Q)U_x^* = \varphi(Q+x)$, we have

$$\|(U_x-1)\varphi(Q)\psi(P)\| \le \|\varphi(Q+x)-\varphi(Q)\|\,\|\psi(P)\| + \|\varphi(Q)\|\,\|(U_x-1)\psi(P)\|.$$

The function φ is uniformly continuous so the first term in the r.h.s. above tends to zero as $x \to 0$. Since

$$\|(U_x-1)\psi(P)\| = \sup_{k\in X^*}|k(x)-1|\,|\psi(k)|$$

and $\psi(k) \to 0$ as $k \to \infty$ in X^* we see that this norm also tends to zero as $x \to 0$. Thus (i) is satisfied. Further, (ii) is an immediate consequence of the uniform continuity of ψ because

$$V_k\varphi(Q)\psi(P)V_k^* - \varphi(Q)\psi(P) = \varphi(Q)\,[\psi(P+k)-\psi(P)].$$

Also, $\chi_{L^c_\Lambda}(Q)T = \chi_{L^c_\Lambda}(Q)\varphi(Q)\psi(P)$ is zero if Λ is large enough, by the properties of $\varphi \in C_{L,\mathrm{c}}(X)$. In order to treat the term $T\chi_{L^c_\Lambda}(Q)$ of (iii) we recall that $C_0(X^*)\cdot C_{L,\mathrm{c}}(X)$ is dense in $\mathscr{C}_{L,0}$, so for each $\varepsilon > 0$ one may find $\psi_1, \dots, \psi_n \in C_0(X^*)$ and $\varphi_1, \dots, \varphi_n \in C_{L,\mathrm{c}}(X)$ such that $\|T - \sum_1^n \psi_i(P)\varphi_i(Q)\| < \varepsilon$. For Λ large enough we have $\varphi_i(Q)\chi_{L^c_\Lambda}(Q) = 0$ for all i, hence $\|T\chi_{L^c_\Lambda}(Q)\| < \varepsilon$. This finishes the proof of $\mathscr{C}_{L,0} \subset \mathscr{A}$. The reciprocal assertion is less elementary and we devote the rest of the Appendix to its proof.

In what follows we shall need three groups of automorphisms of $B(\mathscr{H})$, namely $\{\mathcal{U}_x\}_{x\in X}$, $\{\mathcal{V}_k\}_{k\in X^*}$ and $\{\mathcal{W}_\xi\}_{\xi\in X\times X^*}$, defined on every $T \in B(\mathscr{H})$ by $\mathcal{U}_x[T] := U_xTU_x^*$, $\mathcal{V}_k[T] := V_kTV_k^*$ and $\mathcal{W}_{(x,k)} := \mathcal{U}_x\mathcal{V}_k$ respectively. Notice that $[\mathcal{U}_x, \mathcal{V}_k] = 0$ for each couple $(x,k) \in X \times X^*$. Hence $\mathcal{W}_\xi$ is a representation on $B(\mathscr{H})$ of the locally compact group $\Xi := X \times X^*$ equipped with the Haar measure $\mathrm{d}\xi = \mathrm{d}x \otimes \mathrm{d}k$. This representation is continuous if we equip $B(\mathscr{H})$ with the strong operator topology but it is not norm continuous. It is clear that

$$C^0_{\mathrm{u}}(\Xi) := \{T \in B(\mathscr{H}) \mid \Xi \ni \xi \mapsto \mathcal{W}_\xi[T] \in B(\mathscr{H}) \text{ is norm continuous}\} \tag{7.1}$$

is a C^*-subalgebra of $B(\mathscr{H})$.

If $T \in B(\mathscr{H})$ and $u \in L^1(X)$, $v \in L^1(X^*)$, we denote

$$T_{u,v} := \int_\Xi \mathcal{W}_\xi[T]\,(u \otimes v)(\xi)\;\mathrm{d}\xi. \tag{7.2}$$

This is related to the Wigner transform and Husimi quantization, see [19]. In the next three lemmas we give properties of this object which show that $T_{u,v}$ is a “regularization” of T in a similar manner in which the convolution of a function is a smoothing of this function. We mention that the regularization in x, realized by u, is not needed for the proof of Theorem 1.2, but is useful in other contexts (see [23]).

Lemma 7.1. *For each $T \in B(\mathscr{H})$ the following statements are equivalent:*

(i) *For each $\varepsilon > 0$ there are $u \in L^1(X)$ and $v \in L^1(X^*)$ such that $\|T_{u,v} - T\| < \varepsilon$.*
(ii) $T \in C^0_{\mathrm{u}}(\Xi)$.
(iii) $\lim_{x\to 0} \|(\mathcal{U}_x - 1)T\| = 0$ *and* $\lim_{k\to 0} \|(\mathcal{V}_k - 1)T\| = 0$.

Moreover, if one of these conditions is satisfied, the functions u and v from (i) *may be chosen such that their Fourier transforms $\widehat{u}$, $\widehat{v}$ have compact support.*

Proof. First, (ii) is equivalent to (iii), as a consequence of

$$\|\mathcal{W}_{(x,k)}[T] - T\| = \|\mathcal{U}_x(\mathcal{V}_k[T] - T) + (\mathcal{U}_x[T] - T)\| \le \|\mathcal{V}_k[T] - T\| + \|\mathcal{U}_x[T] - T)\|.$$

We prove now the equivalence between (i) and (ii). For each couple $(y, p) \in X \times X^*$ we have

$$\mathcal{W}_{(y,p)}[T_{u,v}] \equiv (\mathcal{U}_y\mathcal{V}_p)[T_{u,v}] = \int_X \int_{X^*} (\mathcal{U}_x\mathcal{V}_k)[T]\,u(x-y)\,v(k-p)\,\mathrm{d}x\,\mathrm{d}k.$$

Since the translations act continuously on L^1 we see that the map $(y, p) \mapsto \mathcal{W}_{(y,p)}[T_{u,v}]$ is norm continuous. So $T_{u,v} \in C^0_{\mathrm{u}}(\Xi)$ for each $T \in B(\mathscr{H})$. Then if (i) holds we get (ii) because $C^0_{\mathrm{u}}(\Xi)$ is a norm closed subspace of $B(\mathscr{H})$. Conversely, assume that (iii) holds. It can be shown that for every open set $\Lambda \neq \emptyset$ of X and for every $\varepsilon > 0$ there is $u \in L^1(X)$ such that $u \ge 0$, $\int_X u = 1$, $\int_{X\setminus\Lambda} u \le \varepsilon$ and $\widehat{u} \in C_{\mathrm{c}}(X^*)$ (put $u = \widehat{\psi}$ in Lemma 2.1 from [21]). Similarly, for each neighborhood Γ of 0 in X^* there is $v \in L^1(X^*)$ with $v \ge 0$, $\int_{X^*} v = 1$, $\int_{X^*\setminus\Gamma} v \le \varepsilon$ and $\widehat{v} \in C_{\mathrm{c}}(X)$. Then we have

$$\begin{aligned}
\|T_{u,v} - T\| &= \left\| \int_{X\times X^*} (\mathcal{W}_{(x,k)}[T] - T)\,u(x)\,v(k)\,\mathrm{d}x\,\mathrm{d}k \right\| \\
&\le \int_X \|(\mathcal{U}_x - 1)T\|\,u(x)\,\mathrm{d}x + \int_{X^*} \|(\mathcal{V}_k - 1)T\|\,v(k)\,\mathrm{d}k \\
&\le \sup_{x\in\Lambda} \|(\mathcal{U}_x - 1)T\| + \sup_{k\in\Gamma} \|(\mathcal{V}_k - 1)T\| + 4\varepsilon\|T\|.
\end{aligned}$$

By choosing Λ, Γ such that the first two terms in the last member above are small, we see that (i) holds. Moreover, we also proved the last assertion of the lemma. □

Lemma 7.2. *$\mathscr{A} \subset C^0_{\mathrm{u}}(\Xi)$ and if $T \in \mathscr{A}$ and $u \in L^1(X)$, $v \in L^1(X^*)$ then $T_{u,v} \in \mathscr{A}$.*

Proof. The condition (i) of Theorem 1.2 is stronger than $\lim_{x\to 0} \|(\mathcal{U}_x - 1)T\| = 0$ from (iii) of Lemma 7.1, hence each $T \in \mathscr{A}$ verifies the statements (i)–(iii) of this lemma. It remains to prove that $T_{u,v} \in \mathscr{A}$, which obviously follows by dominated convergence if $\mathcal{W}_\xi[T] \in \mathscr{A}$ for any $\xi \in X \times X^*$.

Let, more generally, ψ be the Fourier transform of an integrable measure on X and $T \in \mathscr{A}$. We shall prove that $\psi(P)T \in \mathscr{A}$. Since $U_x = x(P)$ the operator $\psi(P)T$ clearly satisfies condition (i) of Theorem 1.2. Then

$$[V_k, \psi(P)T] = \{V_k \psi(P) V_k^* - \psi(P)\} V_k T + \psi(P)[V_k, T].$$

Since $V_k \psi(P) V_k^* = \psi(P + k)$ and $\psi \in C^{\mathrm{u}}_{\mathrm{b}}(X^*)$, condition (ii) of Theorem 1.2 is also satisfied. Finally, to check (iii) we use the representation $\psi(P) = \int_X U_x \widehat{\psi}\,(\mathrm{d}x)$. If Λ is a compact subset of X then

$$\|\chi_{L^c_\Lambda}(Q)\psi(P)T\| \le \int_X \|\chi_{L^c_\Lambda}(Q) U_x T\|\, |\widehat{\psi}|(dx).$$

Let $\varepsilon > 0$. Since $\widehat{\psi}$ is an integrable measure, there is a compact subset K of X such that $\int_{X\setminus K} |\widehat{\psi}|(dx) < \varepsilon/(2\|T\|)$, hence

$$\|\chi_{L^c_\Lambda}(Q)\psi(P)T\| \le \frac{\varepsilon}{2} + \int_K \|\chi_{L^c_\Lambda}(Q - x)T\|\, |\widehat{\psi}|(dx).$$

Since $T \in \mathscr{A}$, there is a compact set M in X such that $\|\chi_{L^c_M}(Q)T\| \le \varepsilon[2|\widehat{\psi}|(K)]^{-1}$. If we take $\Lambda = M - K$ then Λ is also a compact in X and for each $x \in K$ we have $M \subset x + \Lambda$, so $L_M \subset L_\Lambda + x$. But $\chi_{L^c_\Lambda}(Q - x) = \chi_{(L_\Lambda + x)^c}(Q)$ so $\|\chi_{L^c_\Lambda}(Q - x)T\| \le \|\chi_{L^c_M}(Q)T\|$. With this choice of Λ we shall thus have $\|\chi_{L^c_\Lambda}(Q)\psi(P)T\| \le \varepsilon$. □

Lemma 7.3. *Let $T_{u,v}$ be given by (7.2) with $\widehat{v}$ of compact support. Then there is a compact set Λ in X such that for $\theta_1,\ \theta_2 \in C_{\mathrm{c}}(X)$ with*

$$\operatorname{supp}\theta_1 \bigcap (\operatorname{supp}\theta_2 + \Lambda) = \emptyset$$

one has $\theta_1(Q)\, T_{u,v}\, \theta_2(Q) = 0$.

Proof. We have to prove

$$\theta_1(Q) \int_X \int_{X^*} V_k U_x T U_x^* V_k^*\, u(x)\, v(k)\ \mathrm{d}x\, \mathrm{d}k\ \theta_2(Q) = 0$$

for $\operatorname{supp}\theta_1$ and $\operatorname{supp}\theta_2$ sufficiently far away one from another and for all $T \in B(\mathscr{H})$. By the weak density of the finite rank operators, it suffices to assume T of rank one and, by the polarization identity, we may take T of the form $|g\rangle\langle g|$ for some $g \in \mathscr{H}$. Then for any $f_1,\ f_2 \in \mathscr{H}$ one has

$$\begin{aligned}
&\langle f_1, \theta_1(Q)T_{u,v}\theta_2(Q)f_2\rangle \\
&\quad=\iint_{X\times X^*} \langle f_1, \theta_1(Q)V_kU_xg\rangle\, \langle V_kU_xg, \theta_2(Q)f_2\rangle\, u(x)\, v(k)\ \mathrm{d}x\,\mathrm{d}k \\
&\quad=\iint_{X\times X^*} \mathrm{d}x\,\mathrm{d}k\, u(x)\, v(k)\, \times \\
&\qquad\times \iint_{X\times X} \overline{f_1}(x_1)\, \theta_1(x_1)\, g(x_1+x)\, k(x_1)\overline{k}(x_2)\, \overline{g}(x_2+x)\, \theta_2(x_2)\, f_2(x_2)\ \mathrm{d}x_1\,\mathrm{d}x_2 \\
&\quad=\iiint_{X\times X\times X} \mathrm{d}x\,\mathrm{d}x_1\,\mathrm{d}x_2\, u(x)\, \overline{f_1}(x_1)\, \theta_1(x_1)\, g(x_1+x)\, \overline{g}(x_2+x)\, \theta_2(x_2)\, f_2(x_2)\times \\
&\qquad\times \int_{X^*} k(x_1-x_2)\, v(k)\,\mathrm{d}k.
\end{aligned}$$

The last integral over X^* equals $(\mathcal{F}^{-1}v)(x_1-x_2)$, thus the triple integral above will be non zero only if x_1-x_2 belongs to the compact set $\operatorname{supp}\mathcal{F}^{-1}v$. Then $x_1 = x_2+(x_1-x_2) \in (\operatorname{supp}\theta_2 + \operatorname{supp}\mathcal{F}^{-1}v) \bigcap \operatorname{supp}\theta_1$ shows that it suffices to choose $\Lambda = \operatorname{supp}\mathcal{F}^{-1}v$. □

As a consequence of the previous results, for each $T \in \mathscr{A}$ and each $\varepsilon > 0$ there are $u \in L^1(X)$ and $v \in L^1(X^*)$ with $\widehat{v}$ compactly supported such that $T_{u,v} \in \mathscr{A}$, $\|T_{u,v} - T\| < \varepsilon$ and such that the conclusion of Lemma 7.3 be satisfied. Hence there is (and we may fix it) a compact set $\Lambda \subset X$ such that Lemma 7.3 is valid and for which we have

$$\|\chi_{L^c_\Lambda}(Q)T_{u,v}\| + \|T_{u,v}\chi_{L^c_\Lambda}(Q)\| < \varepsilon. \tag{7.3}$$

Let now $\theta \in C_{\mathrm{c}}(X)$ with $0 \le \theta \le 1$ and $\theta = 1$ on Λ. For each $l \in L$ we denote simply by θ^l both the map $x \mapsto \theta(x-l)$ and the operator of multiplication by this function (in what follows we shall freely use the same condensed notation for other functions on X too). Since the set L is sparse, we may find a subset $M \subset L$ with a finite complementary such that

$$\operatorname{supp}\theta^l \bigcap (\operatorname{supp}\theta^{l'} + \Lambda) = \emptyset \ \text{ if }\ l,\ l' \in M \text{ and } l \neq l'.$$

Lemma 7.3 gives then $\theta^l T_{u,v}\theta^{l'} = 0$ for all the pairs l, l' as above. Hence, setting $\phi = \sum_{l\in M}\theta^l$ one obtains

$$\phi\, T_{u,v}\, \phi = \sum_{l,l'\in M} \theta^l T_{u,v}\theta^{l'} = \sum_{l\in M} \theta^l T_{u,v}\theta^l.$$

On the other hand, since $0 \le \phi \le 1$ there is a bounded, compactly supported function φ such that $1-\phi = \varphi + (1-\phi)\chi_{L^c_\Lambda}$. This gives the following decomposition of $T_{u,v}$:

$$\begin{aligned}
T_{u,v} &= \phi\, T_{u,v}\, \phi + (1-\phi)T_{u,v}\,\phi + T_{u,v}(1-\phi) \\
&= \sum_{l\in M} \theta^l T_{u,v}\theta^l + (\varphi\, T_{u,v}\,\phi + T_{u,v}\,\varphi) + ((1-\phi)\chi_{L^c_\Lambda} T_{u,v}\,\phi + T_{u,v}\chi_{L^c_\Lambda}(1-\phi)).
\end{aligned}$$

Let us observe that for each $S \in \mathscr{A}$ and each bounded function with compact support φ on X the operators $\varphi(Q)S$ and $S\varphi(Q)$ are compact. Indeed, choose $\phi \in C_{\mathrm{c}}(X)$ such that $\varphi\phi = \varphi$. It suffices thus to show that $\phi(Q)S$ is compact. The second member of the estimate

$$\|(U_x - 1)\phi(Q)S\| \leq \|\phi(Q)\| \, \|(U_x - 1)S\| + \|U_x\phi(Q)U_x^* - \phi(Q)\| \, \|S\|,$$

tends to zero because $S \in \mathscr{A}$ and ϕ is uniformly continuous. This shows that $\phi(Q)S$ satisfies the hypothesis of the Riesz-Kolmogorov compacity criterion.

So $\varphi \, T_{u,v} \, \phi + T_{u,v} \, \varphi$ is a compact operator K. Thus we may use (7.3) to get

$$\|T_{u,v} - \sum_{l \in M} \theta^l T_{u,v} \theta^l - K\| < \varepsilon.$$

In this manner, we are reduced to the proof of the assertion $\sum_{l \in M} \theta^l T_{u,v} \theta^l \in \mathscr{C}_{L,0}$. This may be reformulated as $\sum_{l \in M} U_l^* \, K_l \, U_l \in \mathscr{C}_{L,0}$ if we take into account that $\theta^l \equiv \theta(Q - l) = U_l^* \theta(Q) U_l$ and if we denote by K_l the compact operator $\theta(Q)U_l T_{u,v} U_l^* \theta(Q)$. It is straightforward to check that the family of these compacts verifies the hypotheses of Proposition 5.1 (see (5.9)). This finishes the proof of the inclusion $\mathscr{A} \subset \mathscr{C}_{L,0}$, hence that of Theorem 1.2.

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