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Renormalized Squares of White Noise and Other Non-Gaussian Noises as Lévy Processes on Real Lie Algebras

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Abstract: It is shown how the relations of the renormalized squared white noise defined by Accardi, Lu, and Volovich [ALV99] can be realized as factorizable current representations or Lévy processes on the real Lie algebra $\mathfrak{sl}_2$. This allows to obtain its Itô table, which turns out to be infinite-dimensional. The linear white noise without or with number operator is shown to be a Lévy process on the Heisenberg–Weyl Lie algebra or the oscillator Lie algebra. Furthermore, a joint realization of the linear and quadratic white noise relations is constructed, but it is proved that no such realizations exist with a vacuum that is an eigenvector of the central element and the annihilator. Classical Lévy processes are shown to arise as components of Lévy processes on real Lie algebras and their distributions are characterized. In particular the square of white noise analogue of the quantum Poisson process is shown to have a χ^2 probability density and the analogue of the field operators to have a density proportional to $|\Gamma(\frac{m_0+ix}{2})|^2$, where Γ is the usual Γ-function and m_0 a real parameter.

1. Introduction

The stochastic limit of quantum theory [ALV00b] shows that stochastic equations (both classical and quantum) are equivalent to white noise Hamiltonian equations. This suggests a natural extension of stochastic calculus to higher powers of white noise. The program to develop such an extension was formulated in [ALV95] where it was also shown that it requires some kind of renormalization. As a first step towards the realization of this program a new type of renormalization was introduced in [ALV99] which led to a closed set of algebraic relations for the renormalized square of white noise (SWN) and to the construction of a Hilbert space representation for these relations. This construction was extended by Śniady [Śni00] to a family of processes including non Boson noises and simplified in [AS00a] who also showed that the interacting Fock space

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constructed in [ALV99] was in fact canonically isomorphic to the Boson Fock space of the finite difference algebra, introduced by Feinsilver [Fei89] and Boukas [Bou88, Bou91]. Commenting upon this result U. Franz, and independently a few months later K. R. Parthasarathy, (private communications) pointed out that the commutation relations of the SWN define a Lévy process on the Lie algebra of $SL(2,\mathbb{R})$ or, equivalently, a representation of a current algebra over this Lie algebra, and suggested that the theory of representations of current algebras, developed in the early seventies by Araki, Streater, Parthasarathy, Schmidt, Guichardet, ... (see [PS72, Gui72] and the references therein) might be used to produce a more direct construction of the Fock representation of the SWN as well as different ones. In the present paper we prove that this is indeed the case. As a by-product we reduce the stochastic integration with respect to the SWN to the usual stochastic integration in the sense of Hudson and Parthasarathy [Par92] and this also allows to write down their corresponding Itô tables (see Eq. (2.2)).

After the renormalization procedure (which we shall not discuss here, simply taking its output as our starting point) the algebraic relations, defining the SWN are:

$$b_\phi b^+_\psi - b^+_\psi b_\phi = \gamma\langle\phi,\psi\rangle + n_{\overline{\phi}\psi}, \tag{1.1a}$$

$$n_\phi b_\psi - b_\psi n_\phi = -2b_{\overline{\phi}\psi}, \tag{1.1b}$$

$$n_\phi b^+_\psi - b^+_\psi n_\phi = 2b^+_{\phi\psi}, \tag{1.1c}$$

$$(b_\phi)^* = b^+_\phi, \qquad (n_\phi)^* = n_{\overline{\phi}}, \tag{1.1d}$$

where γ is a fixed strictly positive real parameter (coming from the renormalization) and

$$\phi,\psi \in \Sigma(\mathbb{R}_+) = \{\phi = \sum_{i=1}^n \phi_i \mathbf{1}_{[s_i,t_i[};\ \phi_i \in \mathbb{C}, s_i < t_i \in \mathbb{R}_+, n \in \mathbb{N}\}$$

the algebra of step functions on $\mathbb{R}_+$ with bounded support and finitely many values. Furthermore b^+ and n are linear and b is anti-linear in the test functions.

We want to find a Hilbert space representation of these relations, i.e. we want to construct an Hilbert space $\mathcal{H}$, a dense subspace $D \subseteq \mathcal{H}$ and three maps b, b^+, n from $\Sigma(\mathbb{R}_+)$ to $\mathcal{L}(D)$, the algebra of adjointable linear operators on D, such that the above relations are satisfied.

The simple current algebra $\mathfrak{g}^{\mathbb{T}}$ of a real Lie algebra $\mathfrak{g}$ over a measure space $(\mathbb{T},\mathcal{T},\mu)$ is defined as the space of simple functions on $\mathbb{T}$ with values in $\mathfrak{g}$,

$$\mathfrak{g}^{\mathbb{T}} = \left\{ X = \sum_{i=1}^n X_i \mathbf{1}_{M_i};\ X_i \in \mathfrak{g}, M_i \in \mathcal{T}, n \in \mathbb{N} \right\}.$$

This is a real Lie algebra with the Lie bracket and the involution defined pointwise. The SWN relations (1.1) imply that any realization of SWN on a pre-Hilbert space D defines a representation π of the current algebra $\mathfrak{sl}_2^{\mathbb{R}_+}$ of the real Lie algebra $\mathfrak{sl}_2$ over $\mathbb{R}_+$ (with the Borel σ-algebra and the Lebesgue measure) on D by

$$B^-\mathbf{1}_{[s,t[} \mapsto b_{1_{[s,t[}}, \quad B^+\mathbf{1}_{[s,t[} \mapsto b^+_{1_{[s,t[}}, \quad M\mathbf{1}_{[s,t[} \mapsto \gamma(t-s) + n_{1_{[s,t[}},$$

where $\mathfrak{sl}_2$ is the three-dimensional real Lie algebra spanned by $\{B^+, B^-, M\}$, with the commutation relations

$$[B^-, B^+] = M, \quad [M, B^\pm] = \pm 2B^\pm,$$

and the involution $(B^-)^* = B^+$, $M^* = M$. The converse is obviously also true, every representation of the current algebra $\mathfrak{sl}_2^{\mathbb{R}_+}$ defines a realization of the SWN relations (1.1). Looking only at indicator functions of intervals we get a family of $*$-representations $(j_{st})_{0\le s\le t}$ on D of the Lie algebra $\mathfrak{sl}_2$,

$$j_{st}(X) = \pi(X\mathbf{1}_{[s,t[}), \qquad \text{for all } X \in \mathfrak{sl}_2.$$

By the universal property these $*$-representations extend to $*$-representations of the universal enveloping algebra $\mathcal{U}(\mathfrak{sl}_2)$ of $\mathfrak{sl}_2$. If there exists a vector Ω in $\mathcal{L}(D)$ such that the representations corresponding to disjoint intervals are independent (in the sense of Definition 2.1, Condition 2), i.e. if they commute and their expectations in the state $\Phi(\cdot) = \langle \Omega, \cdot\,\Omega\rangle$ factorize, then $(j_{st})_{0\le s\le t}$ is a Lévy process on $\mathfrak{sl}_2$ (in the sense of Definition 2.1). This condition is satisfied in the constructions in [ALV99, AS00a, Śni00]. They are of 'Fock type' and have a fixed special vector, the so-called vacuum, and the corresponding vector state has the desired factorization property.

On the other hand, given a Lévy process on $\mathfrak{sl}_2$ on a pre-Hilbert space D, we can construct a realization of the SWN relations (1.1) on D. Simply set

$$b_\phi = \sum_{i=1}^n \overline{\phi_i}\, j_{s_i,t_i}(B^-),\ b_\phi^+ = \sum_{i=1}^n \phi_i\, j_{s_i,t_i}(B^+),\ n_\phi = \sum_{i=1}^n \phi_i \left(j_{s_i,t_i}(M) - \gamma(t_i - s_i)\mathrm{id}_D\right),$$

for $\phi = \sum_{i=1}^n \phi_i \mathbf{1}_{[s_i,t_i[} \in \Sigma(\mathbb{R}_+)$.

We see that in order to construct realizations of the SWN relations we can construct Lévy processes on $\mathfrak{sl}_2$. Furthermore, all realizations that have a vacuum vector in which the expectations factorize, will arise in this way.

In this paper we show how to classify the Lévy processes on $\mathfrak{sl}_2$ and how to construct realizations of these Lévy processes acting on (a dense subspace of) the symmetric Fock space over $L^2(\mathbb{R}_+, H)$ for some Hilbert space H. Given the *generator* L of a Lévy process, we immediately can write down a realization of the process; see Eq. (2.1). The theory of Lévy processes has been developed for arbitrary involutive bialgebras, cf. [ASW88, Sch93], but here it will be sufficient to consider enveloping algebras of Lie algebras. This allows some simplification, in particular we do not need to make explicit use of the coproduct. The construction of this sub-class of Lévy process is based on the theory of "factorizable unitary representation of current algebras" and the abelian subprocesses of these processes are the stationary independent increment processes of classical probability (cf. Sect. 4 below).

As already specified, the SWN naturally leads to the real Lie algebra $\mathfrak{sl}_2$, but we shall also consider several other real Lie algebras, including the Heisenberg–Weyl Lie algebra $\mathfrak{hw}$, the oscillator Lie algebra $\mathfrak{osc}$, and the finite-difference Lie algebra $\mathfrak{fd}$.

This paper is organized as follows. In Sect. 2, we recall the definitition of Lévy processes on real Lie algebras and present their fundamental properties. We also outline how the Lévy processes on a given real Lie algebra can be characterized and constructed as a linear combination of the four basic processes of Hudson–Parthasarathy quantum stochastic calculus: number, creation, annihilation and time.

In Sect. 3, we list all Gaussian Lévy processes or Lévy processes associated to integrable unitary irreducible representations for several real Lie algebras in terms of their generators or Schürmann triples (see Definition 2.2). We also give explicit realizations on a boson Fock space for several examples. These examples include the processes on the finite-difference Lie algebra defined by Boukas [Bou88, Bou91] and by Parthasarathy and Sinha [PS91] as well as a process on $\mathfrak{sl}_2$ that has been considered previously by Feinsilver and Schott [FS93, Sect. 5.IV]. See also [VGG73] for factorizable current representations of current groups over $SL(2,\mathbb{R})$.

Finally, in Sect. 4, we show that the restriction of a Lévy process to one single hermitian element of the real Lie algebra always gives rise to a classical Lévy process. We give a characterization of this process in terms of its Fourier transform. For several examples we also explicitly compute its Lévy measure or its marginal distribution. It turns out that the densities of self-adjoint linear combinations of the SWN operators $b_{\mathbf{1}_{[s,t[}}$, $b^+_{\mathbf{1}_{[s,t[}}$, $n_{\mathbf{1}_{[s,t[}}$ in the realization considered in [ALV99, AS00a, Śni00] are the measures of orthogonality of the Laguerre, Meixner, and Meixner–Pollaczek polynomials.

2. Lévy Processes on Real Lie Algebras

In this section we give the basic definitions and properties of Lévy processes on real Lie algebras. This is a special case of the theory of Lévy processes on involutive bialgebras, for more detailed accounts on these processes see [Sch93],[Mey95, Chapter VII],[FS99]. For a list of references on factorizable representations of current groups and algebras and a historical survey, we refer to [Str00, Sect. 5].

By a real Lie algebra we will mean a pair $\mathfrak{g}_\mathbb{R} = (\mathfrak{g}, *)$ consisting of a Lie algebra $\mathfrak{g}$ over the field of complex numbers $\mathbb{C}$ and an involution $* : \mathfrak{g} \to \mathfrak{g}$. These pairs are in one-to-one correspondence with the Lie algebras over the field of real numbers $\mathbb{R}$. To recover a Lie algebra $\mathfrak{g}_0$ over $\mathbb{R}$ from a pair $(\mathfrak{g}, *)$, simply take the anti-hermitian elements, i.e. set $\mathfrak{g}_0 = \{X \in \mathfrak{g} | X^* = -X\}$. Note that it is not possible to take the hermitian elements, because the commutator of two hermitian elements in not again hermitian. Given a Lie algebra $\mathfrak{g}_0$ over $\mathbb{R}$, the involution on its complexification $\mathfrak{g} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$ is defined by $(X + iY)^* = -X + iY$ for $X, Y \in \mathfrak{g}_0$.

We denote by $\mathcal{U}(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$ and by $\mathcal{U}_0(\mathfrak{g})$ the non-unital subalgebra of $\mathcal{U}$ generated by $\mathfrak{g}$. If $X_1, \ldots, X_d$ is a basis of $\mathfrak{g}$, then

$$\{X_1^{n_1} \cdots X_d^{n_d} | n_1, \ldots, n_d \in \mathbb{N}, n_1 + \cdots + n_d \geq 1\}$$

is a basis of $\mathcal{U}_0(\mathfrak{g})$. Furthermore, we extend the involution on $\mathfrak{g}$ as an anti-linear anti-homomorphism to $\mathcal{U}(\mathfrak{g})$ and $\mathcal{U}_0(\mathfrak{g})$.

Definition 2.1. *Let D be a pre-Hilbert space and $\Omega \in D$ a unit vector. We call a family $\big(j_{st} : \mathcal{U}(\mathfrak{g}) \to \mathcal{L}(D)\big)_{0\leq s\leq t}$ of unital $*$-representations of $\mathcal{U}(\mathfrak{g})$ a* **Lévy process** *on $\mathfrak{g}_\mathbb{R}$ over D (with respect to Ω), if the following conditions are satisfied.*

1. *(Increment property) We have*

$$j_{st}(X) + j_{tu}(X) = j_{su}(X)$$

   for all $0 \leq s \leq t \leq u$ and all $X \in \mathfrak{g}$.
2. *(Boson independence) We have $[j_{st}(X), j_{s't'}(Y)] = 0$ for all $X, Y \in \mathfrak{g}$, $0 \leq s \leq t \leq s' \leq t'$ and*

$$\langle \Omega, j_{s_1t_1}(u_1) \cdots j_{s_nt_n}(u_n)\Omega\rangle = \langle \Omega, j_{s_1t_1}(u_1)\Omega\rangle \cdots \langle \Omega, j_{s_nt_n}(u_n)\Omega\rangle$$

   for all $n \in \mathbb{N}$, $0 \leq s_1 \leq t_1 \leq s_2 \leq \cdots \leq t_n$, $u_1, \ldots, u_n \in \mathcal{U}(\mathfrak{g})$.

3. *(Stationarity) The functional* $\varphi_{st} : \mathcal{U}(\mathfrak{g}) \to \mathbb{C}$ *defined by*

$$\varphi_{st}(u) = \langle \Omega, j_{st}(u)\Omega \rangle, \qquad u \in \mathcal{U}(\mathfrak{g}),$$

depends only on the difference $t - s$.

4. *(Weak continuity) We have* $\lim_{t \searrow s} \langle \Omega, j_{st}(u)\Omega \rangle = 0$ *for all* $u \in \mathcal{U}_0(\mathfrak{g})$.

If $(j_{st})_{0 \le s \le t}$ is a Lévy process on $\mathfrak{g}_{\mathbb{R}}$, then the functionals $\varphi_t = \langle \Omega, j_{0t}(\cdot)\Omega \rangle : \mathcal{U}(\mathfrak{g}) \to \mathbb{C}$ are actually states. Furthermore, they are differentiable w.r.t. t and

$$L(u) = \lim_{t \searrow 0} \frac{1}{t} \varphi_t(u), \qquad u \in \mathcal{U}_0(\mathfrak{g}),$$

defines a positive hermitian linear functional on $\mathcal{U}_0(\mathfrak{g})$. In fact one can prove that the family (φ_t) is a convolution semigroup on $\mathfrak{g}_{\mathbb{R}}$ whose generator is L. The functional L is also called the *generator* of the process.

Let $\big(j^{(1)}_{st} : \mathcal{U}(\mathfrak{g}) \to \mathcal{L}(D^{(1)})\big)_{0 \le s \le t}$ and $\big(j^{(2)} : \mathcal{U}(\mathfrak{g}) \to \mathcal{L}(D^{(2)})\big)_{0 \le s \le t}$ be two Lévy processes on $\mathfrak{g}_{\mathbb{R}}$ with respect to the state vectors $\Omega^{(1)}$ and $\Omega^{(2)}$, resp. We call them *equivalent*, if all their moments agree, i.e. if

$$\langle \Omega^{(1)}, j^{(1)}_{s_1 t_1}(u_1) \cdots j^{(1)}_{s_n t_n}(u_n)\Omega^{(1)} \rangle = \langle \Omega^{(2)}, j^{(2)}_{s_1 t_1}(u_1) \cdots j^{(2)}_{s_n t_n}(u_n)\Omega^{(2)} \rangle,$$

for all $n \in \mathbb{N}$, $0 \le s_1 \le t_1 \le s_2 \le \cdots \le t_n$, $u_1, \ldots, u_n \in \mathcal{U}(\mathfrak{g})$.

By a GNS-type construction, one can associate to every generator a Schürmann triple.

Definition 2.2. *A* **Schürmann triple** *on* $\mathfrak{g}_{\mathbb{R}}$ *is a triple* (ρ, η, L), *where* ρ *is a* $*$*-representation of* $\mathcal{U}_0(\mathfrak{g})$ *on some pre-Hilbert space* D, $\eta : \mathcal{U}_0(\mathfrak{g}) \to D$ *is a surjective* ρ*-1-cocycle, i.e. it satisfies*

$$\eta(uv) = \rho(u)\eta(v),$$

for all $u, v \in \mathcal{U}_0(\mathfrak{g})$, *and* $L : \mathcal{U}_0(\mathfrak{g}) \to \mathbb{C}$ *is a hermitian linear functional such that the bilinear map* $(u, v) \mapsto -\langle \eta(u^*), \eta(v) \rangle$ *is the 2-coboundary of* L *(w.r.t. the trivial representation), i.e.*

$$L(uv) = \langle \eta(u^*), \eta(v) \rangle$$

for all $u, v \in \mathcal{U}_0(\mathfrak{g})$.

Let (ρ, η, L) be a Schürmann triple on $\mathfrak{g}_{\mathbb{R}}$, acting on a pre-Hilbert space D. We can define a Lévy process on the symmetric Fock space $\Gamma\big(L^2(\mathbb{R}_+, D)\big) = \bigoplus_{n=0}^{\infty} L^2(\mathbb{R}_+, D)^{\odot n}$ by setting

$$j_{st}(X) = \Lambda_{st}\big(\rho(X)\big) + A^*_{st}\big(\eta(X)\big) + A_{st}\big(\eta(X^*)\big) + L(X)(t - s)\mathrm{id}, \tag{2.1}$$

for $X \in \mathfrak{g}$, where Λ_{st}, A^*_{st}, A_{st} denote the conservation, creation, and annihilation processes on $\Gamma\big(L^2(\mathbb{R}_+, D)\big)$, cf. [Par92,Mey95]. It is straightforward to check that we have

$$\big[j_{st}(X), j_{st}(Y)\big] = j_{st}\big([X, Y]\big), \qquad \text{and} \qquad j_{st}(X)^* = j_{st}(X^*)$$

for all $0 \le s \le t$, $X, Y \in \mathfrak{g}$. By the universal property, the family

$$\Big(j_{st} : \mathfrak{g} \to \mathcal{L}\Big(\Gamma\big(L^2(\mathbb{R}_+, D)\big)\Big)\Big)_{0 \le s \le t}$$

extends to a unique family $(j_{st})_{0\le s\le t}$ of unital $*$-representations of $\mathcal{U}(\mathfrak{g})$, and it is not difficult to verify that this family is a Lévy process with generator L on $\mathfrak{g}_{\mathbb{R}}$ over $\Gamma\big(L^2(\mathbb{R}_+, D)\big)$ with respect to the Fock vacuum Ω.

The following theorem shows that the correspondence between (equivalence classes of) Lévy processes and Schürmann triples is one-to-one and that the representation (2.1) is universal.

Theorem 2.1. *[Sch93] Two Lévy processes on $\mathfrak{g}_{\mathbb{R}}$ are equivalent if and only if their Schürmann triples are unitarily equivalent. A Lévy process $(k_{st})_{0\le s\le t}$ with generator L and Schürmann triple (ρ, η, L) is equivalent to the Lévy process $(j_{st})_{0\le s\le t}$ associated to (ρ, η, L) defined in Eq. (2.1).*

Remark 2.1. Since we know the Itô table for the four H-P integrators,

$\bullet$	$\mathrm{d}A^*(u)$	$\mathrm{d}\Lambda(F)$	$\mathrm{d}A(u)$	$\mathrm{d}t$
$\mathrm{d}A^*(v)$	0	0	0	0
$\mathrm{d}\Lambda(G)$	$\mathrm{d}A^*(Gu)$	$\mathrm{d}\Lambda(GF)$	0	0
$\mathrm{d}A(v)$	$\langle v, u\rangle\mathrm{d}t$	$\mathrm{d}A(F^*v)$	0	0
$\mathrm{d}t$	0	0	0	0

for all $F, G \in \mathcal{L}(D)$, $u, v \in D$, we can deduce the Itô tables for the Lévy processes on $\mathfrak{g}_{\mathbb{R}}$. The map d_L associating elements u of the universal enveloping algebra to the corresponding quantum stochastic differentials $\mathrm{d}_L u$ defined by

$$\mathrm{d}_L u = \mathrm{d}\Lambda\big(\rho(u)\big) + \mathrm{d}A^*\big(\eta(u)\big) + \mathrm{d}A\big(\eta(u^*)\big) + L(u)\mathrm{d}t, \tag{2.2}$$

is a $*$-homomorphism from $\mathcal{U}_0(\mathfrak{g})$ to the Itô algebra over D, see [FS99, Proposition 4.4.2]. It follows that the dimension of the Itô algebra generated by $\{\mathrm{d}_L X;\, X \in \mathfrak{g}\}$ is at least the dimension of D (since η is supposed surjective) and not bigger than $(\dim D + 1)^2$. If D is infinite-dimensional, then its dimension is also infinite. Note that it depends on the choice of the Lévy process.

Due to Theorem 2.1, the problem of characterizing and constructing all Lévy processes on a given real Lie algebra can be decomposed into the following steps. First, classify all $*$-representations of $\mathcal{U}(\mathfrak{g})$ (modulo unitary equivalence), this will give the possible choices for the representation ρ in the Schürmann triple. Next determine all surjective ρ-1-cocycles. We distinguish between trivial cocycles, i.e. cocycles which are of the form

$$\eta(u) = \rho(u)\omega, \qquad u \in \mathcal{U}_0(\mathfrak{g})$$

for some vector $\omega \in D$ in the representation space of ρ, and non-trivial cocycles, i.e. cocycles, which can not be written in this form. We will denote the space of all cocycles of a given $*$-representation ρ on some pre-Hilbert space D by $Z^1(\mathcal{U}_0(\mathfrak{g}), \rho, D)$, that of trivial ones by $B^1(\mathcal{U}_0(\mathfrak{g}), \rho, D)$. The quotient $H^1(\mathcal{U}_0(\mathfrak{g}), \rho, D) = Z^1(\mathcal{U}_0(\mathfrak{g}), \rho, D)/B^1(\mathcal{U}_0(\mathfrak{g}), \rho, D)$ is called the first cohomology group of ρ. In the last step we determine all generators L that turn a pair (ρ, η) into a Schürmann triple (ρ, η, L). This can again also be viewed as a cohomological problem. If η is a ρ-1-cocycle, then the bilinear map $(u, v) \mapsto -\langle \eta(u^*), \eta(v)\rangle$ is a 2-cocycle for the trivial representation, i.e. it satisfies $-\big\langle \eta\big((uv)^*\big), \eta(w)\big\rangle + \big\langle \eta(u^*), \eta(vw)\big\rangle = 0$ for all $u, v, w \in \mathcal{U}_0(\mathfrak{g})$. For L we can take any hermitian functional that has the map $(u, v) \mapsto -\langle \eta(u^*), \eta(v)\rangle$ as coboundary, i.e.

L has to satisfy $L(u^*) = \overline{L(u)}$ and $L(uv) = \langle \eta(u^*), \eta(v) \rangle$ for all $u, v \in \mathcal{U}_0(\mathfrak{g})$. If η is trivial, then such a functional always exists, we can take $L(u) = \langle \omega, \rho(u)\omega \rangle$. For a given pair (ρ, η), L is determined only up to a hermitian 0-1-cocycle, i.e. a hermitian functional ℓ that satisfies $\ell(uv) = 0$ for all $u, v \in \mathcal{U}_0(\mathfrak{g})$.

Remark 2.2. A linear $*$-map $\pi : \mathfrak{g} \to \mathcal{L}(D)$ is called a projective $*$-representation of $\mathfrak{g}$, if there exists a bilinear map $\alpha : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$, such that

$$\big[\pi(X), \pi(Y)\big] = \pi\big([X, Y]\big) + \alpha(X, Y)\mathrm{id},$$

for all $X, Y \in \mathfrak{g}$. Every projective $*$-representation defines a $*$-representation of a central extension $\tilde{\mathfrak{g}}$ of $\mathfrak{g}$. As a vector space $\tilde{\mathfrak{g}}$ is defined as $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{C}$. The Lie bracket and the involution are defined by

$$\big[(X, \lambda), (Y, \mu)\big] = \big([X, Y], \alpha(X, Y)\big), \qquad (X, \lambda)^* = (X^*, \overline{\lambda})$$

for $(X, \lambda), (Y, \mu) \in \tilde{\mathfrak{g}}$. It is not hard to check that

$$\tilde{\pi}\big((X, \lambda)\big) = \pi(X) + \lambda\,\mathrm{id}$$

defines a $*$-representation of $\tilde{\mathfrak{g}}$. If the cocycle α is trivial, i.e. if there exists a (hermitian) linear functional β such that $\alpha(X, Y) = \beta([X, Y])$ for all $X, Y \in \mathfrak{g}$, then the central extension is trivial, i.e. $\tilde{\mathfrak{g}}$ is isomorphic to the direct sum of $\mathfrak{g}$ with the (abelian) one-dimensional Lie algebra $\mathbb{C}$. Such an isomorphism is given by $\mathfrak{g} \oplus \mathbb{C} \ni (X, \mu) \mapsto (X, \beta(X) + \mu) \in \tilde{\mathfrak{g}}$. This implies that in this case

$$\pi_\beta(X) = \tilde{\pi}\big((X, \beta(X))\big) = \pi(X) + \beta(X)\mathrm{id}$$

defines a $*$-representation of $\mathfrak{g}$.

For a pair (ρ, η) consisting of a $*$-representation ρ and a ρ-1-cocycle η we can always define a family of projective $*$-representations $(k_{st})_{0 \le s \le t}$ of $\mathfrak{g}$ by setting

$$k_{st}(X) = \Lambda_{st}\big(\rho(X)\big) + A^*_{st}\big(\eta(X)\big) + A_{st}\big(\eta(X^*)\big),$$

for $X \in \mathfrak{g}$, $0 \le s \le t$. Using the commutation relations of the creation, annihilation, and conservation operators, one finds that the 2-cocycle α is given by $(X, Y) \mapsto \alpha(X, Y) = \langle \eta(X^*), \eta(Y) \rangle - \langle \eta(Y^*), \eta(X) \rangle$. If it is trivial, then $(k_{st})_{0 \le s \le t}$ can be used to define a Lévy process on $\mathfrak{g}$. More precisely, if there exists a hermitian functional ψ on $\mathcal{U}_0(\mathfrak{g})$ such that $\psi(uv) = \langle \eta(u^*), \eta(v) \rangle$ holds for all $u, v \in \mathcal{U}_0(\mathfrak{g})$, then (ρ, η, ψ) is a Schürmann triple on $\mathfrak{g}$ and therefore defines a Lévy process on $\mathfrak{g}$. But even if such a hermitian functional ψ does not exist, we can define a Lévy process on $\tilde{\mathfrak{g}}$ by setting

$$\tilde{k}_{st}\big((X, \lambda)\big) = \Lambda_{st}\big(\rho(X)\big) + A^*\big(\eta(X)\big) + A\big(\eta(X^*)\big) + (t - s)\lambda\,\mathrm{id},$$

for $(X, \lambda) \in \tilde{\mathfrak{g}}$, $0 \le s \le t$.

We close this section with two useful lemmata on cohomology groups.

Schürmann triples (ρ, η, L), where the $*$-representation ρ is equal to the trivial representation defined by $0 : \mathcal{U}_0(\mathfrak{g}) \ni u \mapsto 0 \in \mathcal{L}(D)$ are called *Gaussian*, as well as the corresponding processes, cocycles, and generators (cf. Corollary 4.1 for a justification of this definition). The following lemma completely classifies all Gaussian cocycles of a given Lie algebra.

Lemma 2.1. *Let D be an arbitrary complex vector space, and* 0 *the trivial representation of* $\mathfrak{g}$ *on* D*. We have*

$$Z^1(\mathcal{U}_0(\mathfrak{g}), 0, D) \cong \big(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]\big)^*, \qquad B^1(\mathcal{U}_0(\mathfrak{g}), 0, D) = \{0\},$$

and therefore $\dim H^1(\mathcal{U}_0(\mathfrak{g}), 0, D) = \dim \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$.

Proof. Let ϕ be a linear functional on $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$, then we can extend it to a unique 0-1-cocycle on the algebra $\mathcal{U}_0(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])$ (this is the free abelian algebra over $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$), which we denote by $\tilde{\phi}$. Denote by π the canonical projection from $\mathfrak{g}$ to $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$, by the universal property of the enveloping algebra it has a unique extension $\tilde{\pi} : \mathcal{U}_0(\mathfrak{g}) \to \mathcal{U}_0(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])$. We can define a cocycle η_ϕ on $\mathcal{U}_0(\mathfrak{g})$ by $\eta_\phi = \tilde{\phi} \circ \tilde{\pi}$. Furthermore, since any 0-1-cocycle on $\mathcal{U}_0(\mathfrak{g})$ has to vanish on $[\mathfrak{g},\mathfrak{g}]$ (because $Y = [X_1, X_2]$ implies $\eta(Y) = 0\eta(X_2) - 0\eta(X_1) = 0$), the map $\phi \mapsto \eta_\phi$ is bijective. □

The following lemma shows that a representation of $\mathcal{U}(\mathfrak{g})$ can only have non-trivial cocycles, if the center of $\mathcal{U}_0(\mathfrak{g})$ acts trivially.

Lemma 2.2. *Let ρ be a representation of* $\mathfrak{g}$ *on some vector space D and let $C \in \mathcal{U}_0(\mathfrak{g})$ be central. If $\rho(C)$ is invertible, then*

$$H^1(\mathcal{U}_0(\mathfrak{g}), \rho, D) = \{0\}.$$

Proof. Let η be a ρ-cocycle on $\mathcal{U}_0(\mathfrak{g})$ and $C \in \mathcal{U}_0(\mathfrak{g})$ such that $\rho(C)$ is invertible. Then we get

$$\rho(C)\eta(u) = \eta(Cu) = \eta(uC) = \rho(u)\eta(C)$$

and therefore $\eta(u) = \rho(u)\rho(C)^{-1}\eta(C)$ for all $u \in \mathcal{U}_0(\mathfrak{g})$, i.e. $\eta(u) = \rho(u)\omega$, where $\omega = \rho(C)^{-1}\eta(C)$. This shows that all ρ-cocycles are trivial. □

3. Examples

In this section we completely classify the Gaussian generators for several real Lie algebras and determine the non-trivial cocycles for some or all of their integrable unitary irreducible representations, i.e. those representations that arise by differentiating unitary irreducible representations of the corresponding Lie group. These are $*$-representations of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ on some pre-Hilbert space D for which the Lie algebra elements are mapped to essentially self-adjoint operators. For some of the processes we give explicit realizations on the boson Fock space.

3.1. White noise or Lévy processes on $\mathfrak{hw}$ *and* $\mathfrak{osc}$*.* The Heisenberg–Weyl Lie algebra $\mathfrak{hw}$ is the three-dimensional Lie algebra with basis $\{A^+, A^-, E\}$, commutation relations

$$[A^-, A^+] = E, \quad [A^\pm, E] = 0,$$

and involution $(A^-)^* = A^+$, $E^* = E$. Adding a hermitian element N with commutation relations

$$[N, A^\pm] = \pm A^\pm, \quad [N, E] = 0,$$

we obtain the four-dimensional oscillator Lie algebra $\mathfrak{osc}$.

We begin with the classification of all Gaussian generators on these two Lie algebras.

Proposition 3.1. (a) *Let* $v_1, v_2 \in \mathbb{C}^2$ *be two vectors and* $z \in \mathbb{C}$ *an arbitrary complex number. Then*

$$\rho(A^+) = \rho(A^-) = \rho(E) = 0,$$
$$\eta(A^+) = v_1, \quad \eta(A^-) = v_2, \quad \eta(E) = 0,$$
$$L(A^+) = z, \quad L(A^-) = \overline{z}, \quad L(E) = ||v_1||^2 - ||v_2||^2,$$

defines the Schürmann triple on $D = \operatorname{span}\{v_1, v_2\}$ *of a Gaussian generator on* $\mathcal{U}_0(\mathfrak{hw})$. *Furthermore, all Gaussian generators on* $\mathcal{U}_0(\mathfrak{hw})$ *arise in this way.*
(b) *The Schürmann triples of Gaussian generators on* $\mathcal{U}_0(\mathfrak{osc})$ *are all of the form*

$$\rho(N) = \rho(A^+) = \rho(A^-) = \rho(E) = 0,$$
$$\eta(N) = v, \quad \eta(A^+) = \eta(A^-) = \eta(E) = 0,$$
$$L(N) = b, \quad L(A^+) = L(A^-) = L(E) = 0,$$

with $v \in \mathbb{C}$, $b \in \mathbb{R}$.

Proof. The form of the Gaussian cocycles on $\mathcal{U}_0(\mathfrak{hw})$ and $\mathcal{U}_0(\mathfrak{osc})$ follows from Lemma 2.1. Then one checks that for all these cocycles there do indeed exist generators and computes their general form. □

Therefore from (2.2) we get, for an arbitrary Gaussian Lévy process on $\mathfrak{hw}$:

$$\begin{aligned} \mathrm{d}_L A^+ &= \mathrm{d}A^*(v_1) + \mathrm{d}A(v_2) + z\mathrm{d}t, \\ \mathrm{d}_L A^- &= \mathrm{d}A^*(v_2) + \mathrm{d}A(v_1) + \overline{z}\mathrm{d}t, \\ \mathrm{d}_L E &= \big(||v_1||^2 - ||v_2||^2\big)\mathrm{d}t, \end{aligned}$$

and the Itô table

$\bullet$	$\mathrm{d}_L A^+$	$\mathrm{d}_L A^-$	$\mathrm{d}_L E$
$\mathrm{d}_L A^+$	$\langle v_2, v_1\rangle \mathrm{d}t$	$\langle v_2, v_2\rangle \mathrm{d}t$	0
$\mathrm{d}_L A^-$	$\langle v_1, v_1\rangle \mathrm{d}t$	$\langle v_1, v_2\rangle \mathrm{d}t$	0
$\mathrm{d}_L E$	0	0	0

For $||v_1||^2 = 1$ and $v_2 = 0$, this is the usual Itô table for the creation and annihilation process in Hudson-Parthasarathy calculus.

Any integrable unitary irreducible representation of $\mathfrak{hw}$ is equivalent either to one of the one-dimensional representations defined by

$$\pi_z(A^+) = z, \quad \pi_z(A^-) = \overline{z}, \quad \pi_z(E) = 0,$$

for some $z \in \mathbb{C}$, or to one of the infinite-dimensional representations defined by

$$\rho_h(A^+)e_n = \sqrt{(n+1)h}\, e_{n+1}, \quad \rho_h(A^-)e_n = \sqrt{nh}\, e_{n-1}, \quad \rho_h(E)e_n = he_n, \tag{3.1}$$

and

$$\rho_{-h}(A^-)e_n = \sqrt{(n+1)h}\, e_{n+1}, \quad \rho_{-h}(A^+)e_n = \sqrt{nh}\, e_{n-1}, \quad \rho_{-h}(E)e_n = -he_n,$$

where $h > 0$, and $\{e_0, e_1, \dots\}$ is a orthonormal basis of ℓ^2. By Lemma 2.2, the representations ρ_h have no non-trivial cocycles. But by a simple computation using the defining relations of $\mathfrak{hw}$ we see that, for $z \neq 0$, the representations of the form $\pi_z \,\mathrm{id}_D$ also have only one trivial cocycle. From $A^+E = EA^+$ we get

$$z\eta(E) = \eta(A^+E) = \eta(EA^+) = \pi_z(E)\eta(A^+) = 0,$$

and therefore $\eta(E) = 0$. But $E = A^-A^+ - A^+A^-$ implies

$$0 = \eta(E) = \pi_z(A^-)\eta(A^+) - \pi_z(A^+)\eta(A^-) = \bar{z}\eta(A^+) - z\eta(A^-),$$

and we see that η is the coboundary of $\omega = z^{-1}\eta(A^+)$. Thus the integrable unitary irreducible representations (except the trivial one) of $\mathfrak{hw}$ have no non-trivial cocycles.

Let us now consider the oscillator Lie algebra $\mathfrak{osc}$. The elements E and $NE - A^+A^-$ generate the center of $\mathcal{U}_0(\mathfrak{osc})$. If we want an irreducible representation of $\mathcal{U}(\mathfrak{osc})$, which has non-trivial cocycles, they have to be represented by zero. But this implies that we have also $\rho(A^+) = \rho(A^-) = 0$ (since we are only interested in $*$-representations). Thus we are lead to study the representations ρ_ν defined by

$$\rho_\nu(N) = \nu\,\mathrm{id}_\mathrm{D}, \quad \rho_\nu(A^+) = \rho_\nu(A^-) = \rho_\nu(E) = 0,$$

with $\nu \in \mathbb{R}\backslash\{0\}$. It is straightforward to determine all their cocycles and generators.

Proposition 3.2. *For $\nu \in \mathbb{R}$, $\nu \notin \{-1, 0, 1\}$, all cocycles of ρ_ν are of the form*

$$\eta(N) = v, \quad \eta(A^+) = \eta(A^-) = \eta(E) = 0,$$

for some $v \in D$ and thus trivial (coboundaries of $\omega = \nu^{-1}v$).

For $\nu = 1$ they are of the form

$$\eta(N) = v_1, \quad \eta(A^+) = v_2, \quad \eta(A^-) = \eta(E) = 0,$$

and for $\nu = -1$ of the form

$$\eta(N) = v_1, \quad \eta(A^-) = v_2, \quad \eta(A^+) = \eta(E) = 0,$$

with some vectors $v_1, v_2 \in D$. Therefore we get

$$\dim H^1(\mathcal{U}_0(\mathfrak{osc}), \rho_{\pm 1}, D) = 1, \quad \dim B^1(\mathcal{U}_0(\mathfrak{osc}), \rho_{\pm 1}, D) = 1$$

and

$$\dim H^1(\mathcal{U}_0(\mathfrak{osc}), \rho_\nu, D) = 0, \quad \dim B^1(\mathcal{U}_0(\mathfrak{osc}), \rho_\nu, D) = 1$$

for $\nu \in \mathbb{R}\backslash\{-1, 0, 1\}$.

Let now $\nu = 1$, the case $\nu = -1$ is similar, since ρ_1 and ρ_{-1} are related by the automorphism $N \mapsto -N$, $A^+ \mapsto A^-$, $A^- \mapsto A^+$, $E \mapsto -E$. It turns out that for all the cocycles given in the preceding proposition there exists a generator, and we obtain the following result.

Proposition 3.3. *Let $v_1, v_2 \in \mathbb{C}^2$ and $b \in \mathbb{R}$. Then $\rho = \rho_1$,*

$$\eta(N) = v_1, \quad \eta(A^+) = v_2, \quad \eta(A^-) = \eta(E) = 0,$$
$$L(N) = b, \quad L(E) = ||v_2||^2, \quad L(A^+) = \overline{L(A^-)} = \langle v_1, v_2\rangle,$$

defines a Schürmann triple on $\mathfrak{osc}$ acting on $D = \operatorname{span}\{v_1, v_2\}$. The corresponding quantum stochastic differentials are

$$\begin{aligned} \mathrm{d}_L N &= \mathrm{d}\Lambda(\mathrm{id}) + \mathrm{d}A^*(v_1) + \mathrm{d}A(v_1) + b\mathrm{d}t, \\ \mathrm{d}_L A^+ &= \mathrm{d}A^*(v_2) + \langle v_1, v_2\rangle \mathrm{d}t, \\ \mathrm{d}_L A^- &= \mathrm{d}A(v_2) + \langle v_2, v_1\rangle \mathrm{d}t, \\ \mathrm{d}_L E &= ||v_2||^2 \mathrm{d}t, \end{aligned}$$

and they satisfy the following Itô table

$\bullet$	$\mathrm{d}_L A^+$	$\mathrm{d}_L N$	$\mathrm{d}_L A^-$	$\mathrm{d}_L E$
$\mathrm{d}_L A^+$	0	0	0	0
$\mathrm{d}_L N$	$\mathrm{d}_L A^+$	$\mathrm{d}_L N + \big(\|\|v_1\|\|^2 - b\big)\mathrm{d}t$	0	0
$\mathrm{d}_L A^-$	$\mathrm{d}_L E$	$\mathrm{d}_L A^-$	0	0
$\mathrm{d}_L E$	0	0	0	0

Note that for $||v_1||^2 = b$, this is the usual Itô table of the four fundamental noises of Hudson–Parthasarathy calculus.

3.2. SWN or Lévy processes on $\mathfrak{sl}_2$. The Lie algebra $\mathfrak{sl}_2$ is the three-dimensional simple Lie algebra with basis $\{B^+, B^-, M\}$, commutation relations

$$[B^-, B^+] = M, \quad [M, B^\pm] = \pm 2B^\pm,$$

and involution $(B^-)^* = B^+$, $M^* = M$. Its center is generated by the Casimir element

$$C = M(M-2) - 4B^+B^- = M(M+2) - 4B^-B^+.$$

We have $[\mathfrak{sl}_2, \mathfrak{sl}_2] = \mathfrak{sl}_2$, and so $\mathcal{U}_0(\mathfrak{sl}_2)$ has no Gaussian cocycles, cf. Lemma 2.1, and therefore no Gaussian generators either. Let us now determine all the non-trivial cocycles for the integrable unitary irreducible representations of $\mathfrak{sl}_2$.

It is known that, beyond the trivial representation ρ_0 there are three families of equivalence classes of integrable unitary irreducible representation of $\mathfrak{sl}_2$ (given in Eqs. (3.3), (3.4), (3.5) below), see, e.g., [GLL90] and the references therein. We will consider them separately. We begin to consider the lowest and highest weight representations. These families of representations are parametrized by a real number m_0 and are induced by $\rho(M)\Omega = m_0\Omega$, $\rho(B^-)\Omega = 0$, and $\rho(M)\Omega = -m_0\Omega$, $\rho(B^+)\Omega = 0$, respectively. The lowest weight representations are spanned by the vectors $v_n = \rho(B^+)^n\Omega$, with $n \in \mathbb{N}$. We get

$$\begin{aligned} \rho(B^+)v_n &= v_{n+1}, \\ \rho(B^-)v_n &= \rho\left(B^-(B^+)^n\right)\Omega = \rho\left(\frac{1}{4}\big(M(M+2) - C\big)(B^+)^{n-1}\right)\Omega \\ &= n(n+m_0-1)\rho(B^+)^{n-1}\Omega = n(n+m_0-1)v_{n-1}, \\ \rho(M)v_n &= (2n+m_0)v_n. \end{aligned}$$

If we want to define an inner product on span $\{v_n; n \in \mathbb{N}\}$ such that $\rho(M)^* = \rho(M)$ and $\rho(B^-)^* = \rho(B^+)$, then the v_n have to be orthogonal and their norms have to satisfy the recurrence relation

$$||v_{n+1}||^2 = \langle \rho(B^+)v_n, v_{n+1}\rangle = \langle v_n, \rho(B^-)v_{n+1}\rangle = (n+1)(n+m_0)||v_n||^2. \quad (3.2)$$

It follows there exists an inner product on span $\{v_n; n \in \mathbb{N}\}$ such that the lowest weight representation with $\rho(M)\Omega = m_0\Omega$, $\rho(B^-)\Omega = 0$ is a $*$-representation, if and only if the coefficients $(n+1)(n+m_0)$ in Eq. (3.2) are non-negative for all $n = 0, 1, \ldots$, i.e. if and only if $m_0 \geq 0$. For $m_0 = 0$ we get the trivial one-dimensional representation $\rho_0(B^+)\Omega = \rho_0(B^-)\Omega = \rho_0(M)\Omega = 0$ (since $||v_1||^2 = 0$), for $m_0 > 0$ we get

$$\rho^+_{m_0}(B^+)e_n = \sqrt{(n+1)(n+m_0)}\, e_{n+1}, \quad (3.3a)$$

$$\rho^+_{m_0}(M)e_n = (2n+m_0)e_n, \quad (3.3b)$$

$$\rho^+_{m_0}(B^-)e_n = \sqrt{n(n+m_0-1)}\, e_{n-1}, \quad (3.3c)$$

where $\{e_0, e_1, \ldots\}$ is an orthonormal basis of ℓ^2. Note that the Casimir element acts as $\rho^+_{m_0}(C)e_n = m_0(m_0-2)e_n$. Similarly we see that there exists a $*$-representation ρ containing a vector Ω such that $\rho(B^+)\Omega = 0$, $\rho(M)\Omega = -m_0\Omega$, if and only if $m_0 \geq 0$. For $m_0 = 0$ this is the trivial representation, for $m_0 > 0$ it is of the form

$$\rho^-_{m_0}(B^-)e_n = \sqrt{(n+1)(n+m_0)}\, e_{n+1}, \quad (3.4a)$$

$$\rho^-_{m_0}(M)e_n = -(2n+m_0)e_n, \quad (3.4b)$$

$$\rho^-_{m_0}(B^+)e_n = \sqrt{n(n+m_0-1)}\, e_{n-1}, \quad (3.4c)$$

and $\rho^-_{m_0}(C)e_n = m_0(m_0-2)e_n$. The integrable unitary irreducible representations of $\mathfrak{sl}_2$, belonging to the third class, have no highest or lowest weight vector. They are parametrized by two real numbers m_0, c and are induced by $\rho(M)\Omega = m_0\Omega$, $\rho(C)\Omega = c\Omega$. Note that since C is central, the second relation implies actually $\rho(C) = c\,\mathrm{id}$. The vectors $\{v_{\pm n} = \rho(B^\pm)^n\Omega; n \in \mathbb{N}\}$ form a basis for the induced representation,

$$\begin{aligned}
\rho(M)v_n &= (2n+m_0)v_n,\\
\rho(B^+)v_n &= \begin{cases} v_{n+1} & \text{if } n \geq 0,\\ \frac{(m_0+2n+2)(m_0+2n)-c}{4} v_{n+1} & \text{if } n < 0,\end{cases}\\
\rho(B^-)v_n &= \begin{cases} \frac{(m_0+2n-2)(m_0+2n)-c}{4} v_{n-1} & \text{if } n > 0,\\ v_{n-1} & \text{if } n \leq 0.\end{cases}
\end{aligned}$$

We look again for an inner product that turns this representation into a $*$-representation. The v_n have to be orthogonal for such an inner product and their norms have to satisfy the recurrence relations

$$\begin{aligned}
||v_{n+1}||^2 &= \frac{(m_0+2n+2)(m_0+2n)-c}{4}||v_n||^2, \quad \text{for } n \geq 0,\\
||v_{n-1}||^2 &= \frac{(m_0+2n-2)(m_0+2n)-c}{4}||v_n||^2, \quad \text{for } n \leq 0.
\end{aligned}$$

Therefore we can define a positive definite inner product on span $\{v_n; n \in \mathbb{Z}\}$, if and only if $\lambda(\lambda+2) > c$ for all $\lambda \in m_0 + 2\mathbb{Z}$. We can restrict ourselves to $m_0 \in [0, 2[$, because the representations induced by (c, m_0) and $(c, m_0 + 2k)$, $k \in \mathbb{Z}$ turn out to be unitarily equivalent. We get the following family of integrable unitary irreducible representations of $\mathcal{U}(\mathfrak{sl}_2)$:

$$\rho_{cm_0}(B^+)e_n = \frac{1}{2}\sqrt{(m_0+2n+2)(m_0+2n)-c}\; e_{n+1}, \tag{3.5a}$$

$$\rho_{cm_0}(M)e_n = (2n+m_0)e_n, \tag{3.5b}$$

$$\rho_{cm_0}(B^-)e_n = \frac{1}{2}\sqrt{(m_0+2n-2)(m_0+2n)-c}\; e_{n-1}, \tag{3.5c}$$

where $\{e_n; n \in \mathbb{Z}\}$ is an orthonormal basis of $\ell^2(\mathbb{Z})$, $m_0 \in [0, 2[$, $c < m_0(m_0-2)$.

Due to Lemma 2.2, we are interested in representations in which C is mapped to zero. There are, up to unitary equivalence, only three such representations, the trivial or zero representation (which has no non-zero cocycles at all, by Lemma 2.1), and the two representations $\rho^\pm = \rho_2^\pm$ on ℓ^2 defined by

$$\begin{aligned}
\rho^\pm(M)e_n &= \pm(2n+2)e_n,\\
\rho^+(B^+)e_n &= \sqrt{(n+1)(n+2)}\; e_{n+1},\\
\rho^+(B^-)e_n &= \sqrt{n(n+1)}\; e_{n-1},\\
\rho^-(B^+)e_n &= \sqrt{n(n+1)}\; e_{n-1},\\
\rho^-(B^-)e_n &= \sqrt{(n+1)(n+2)}\; e_{n+1},
\end{aligned}$$

for $n \in \mathbb{N}$, where $\{e_0, e_1, \dots\}$ is an orthonormal basis of ℓ^2. The representations ρ^+ and ρ^- are not unitarily equivalent, but they are related by the automorphism $M \mapsto -M$, $B^+ \mapsto B^-$, $B^- \mapsto B^+$. Therefore it is sufficient to study ρ^+. Let η be a ρ^+-1-cocycle. Since $\rho^+(B^+)$ is injective, we see that η is already uniquely determined by $\eta(B^+)$, since the relations $[M, B^+] = 2B^+$ and $[B^-, B^+] = M$ imply

$$\begin{aligned}
\eta(M) &= \rho^+(B^+)^{-1}\big(\rho^+(M)-2\big)\eta(B^+),\\
\eta(B^-) &= \rho^+(B^+)^{-1}\big(\rho^+(B^-)\eta(B^+)-\eta(M)\big).
\end{aligned}$$

In fact, we can choose any vector for $\eta(B^+)$, the definitions above and the formula $\eta(uv) = \rho^+(u)\eta(v)$ for $u, v \in \mathcal{U}_0(\mathfrak{sl}_2)$ will extend it to a unique ρ^+-1-cocycle. This cocycle is a coboundary, if and only if the coefficient v_0 in the expansion $\eta(B^+) = \sum_{n=0}^\infty v_n e_n$ of $\eta(B^+)$ vanishes, and an arbitrary ρ^+-1-cocycle is a linear combination of the non-trivial cocyle η_1 defined by

$$\eta_1\left((B^+)^n M^m (B^-)^r\right) = \begin{cases} 0 & \text{if } n = 0,\\ \delta_{r,0}\delta_{m,0}\rho(B^+)^{n-1}e_0 & \text{if } n \geq 1,\end{cases} \tag{3.6}$$

and a coboundary. In particular, for η with $\eta(B^+) = \sum_{n=0}^\infty v_n e_n$, we get $\eta = v_0\eta_1 + \partial\omega$ with $\omega = \sum_{n=0}^\infty \frac{v_{n+1}}{\sqrt{(n+1)(n+2)}} e_n$. Thus we have shown the following.

Proposition 3.4. *We have*

$$\dim H^1(\mathcal{U}_0(\mathfrak{sl}_2), \rho^{\pm}, \ell^2) = 1$$

and $\dim H^1(\mathcal{U}_0(\mathfrak{sl}_2), \rho, \ell^2) = 0$ *for all other integrable unitary irreducible representations of* $\mathfrak{sl}_2$.

Since $[\mathfrak{sl}_2, \mathfrak{sl}_2] = \mathfrak{sl}_2$, all elements of $\mathcal{U}_0(\mathfrak{sl}_2)$ can be expressed as linear combinations of products of elements of $\mathcal{U}_0(\mathfrak{sl}_2)$. Furthermore one checks that

$$L(u) = \langle \eta(u_1^*), \eta(u_2) \rangle, \qquad \text{for } u = u_1 u_2, \quad u_1, u_2 \in \mathcal{U}_0(\mathfrak{sl}_2)$$

is independent of the decomposition of u into a product and defines a hermitian linear functional. Thus there exists a unique generator for every cocycle on $\mathfrak{sl}_2$.

Example 3.1. We will now construct the Lévy process for the cocycle η_1 defined in Eq. (3.6) and the corresponding generator. We get

$$\begin{aligned} L(M) &= \langle \eta_1(B^+), \eta_1(B^+) \rangle - \langle \eta_1(B^-), \eta_1(B^-) \rangle = 1, \\ L(B^+) &= L(B^-) = 0, \end{aligned}$$

and therefore

$$\begin{aligned} \mathrm{d}_L M &= \mathrm{d}\Lambda\big(\rho^+(M)\big) + \mathrm{d}t, \\ \mathrm{d}_L B^+ &= \mathrm{d}\Lambda\big(\rho^+(B^+)\big) + \mathrm{d}A^*(e_0), \\ \mathrm{d}_L B^- &= \mathrm{d}\Lambda\big(\rho^+(B^-)\big) + \mathrm{d}A(e_0). \end{aligned} \tag{3.7}$$

The Itô table is infinite-dimensional. This is the process that leads to the realization of SWN that was constructed in the previous works [ALV99,AS00a,Śni00].

For the Casimir element we get

$$\mathrm{d}_L C = -2\mathrm{d}t.$$

For this process we have $j_{st}(B^-)\Omega = 0$ and $j_{st}(M)\Omega = (t-s)\Omega$ for all $0 \le s \le t$. From our previous considerations about the lowest weight representation of $\mathfrak{sl}_2$ we can now deduce that for fixed s and t the representation j_{st} of $\mathfrak{sl}_2$ restricted to the subspace $j_{st}\big(\mathcal{U}(\mathfrak{sl}_2)\big)\Omega$ is equivalent to the representation ρ^+_{t-s} defined in Eq. (3.3).

Example 3.2. Let now ρ be one of the lowest weight representations defined in (3.3) with $m_0 > 0$, and let η be the trivial cocycle defined by

$$\eta(u) = \rho^+_{m_0}(u) e_0,$$

for $u \in \mathcal{U}_0(\mathfrak{sl}_2)$. There exists a unique generator for this cocycle, and the corresponding Lévy process is defined by

$$\begin{aligned} \mathrm{d}_L M &= \mathrm{d}\Lambda\big(\rho^+_{m_0}(M)\big) + m_0 \mathrm{d}A^*(e_0) + m_0 \mathrm{d}A(e_0) + m_0 \mathrm{d}t, \\ \mathrm{d}_L B^+ &= \mathrm{d}\Lambda\big(\rho^+_{m_0}(B^+)\big) + \sqrt{m_0}\mathrm{d}A^*(e_1), \\ \mathrm{d}_L B^- &= \mathrm{d}\Lambda\big(\rho^+_{m_0}(B^-)\big) + \sqrt{m_0}\mathrm{d}A(e_1). \end{aligned} \tag{3.8}$$

For the Casimir element we get

$$\mathrm{d}_L C = m_0(m_0 - 2)\big(\mathrm{d}\Lambda(\mathrm{id}) + \mathrm{d}A^*(e_0) + \mathrm{d}A(e_0) + \mathrm{d}t\big).$$

3.3. White noise and its square or Lévy processes on $\mathfrak{sl}_2 \oplus_\alpha \mathfrak{hw}$. We can define an action α of the Lie algebra $\mathfrak{sl}_2$ on $\mathfrak{hw}$ by

$$\alpha(M): \begin{array}{ll} A^+ \mapsto A^+, \\ E \ \ \mapsto 0, \\ A^- \mapsto -A^-, \end{array} \qquad \alpha(B^+): \begin{array}{ll} A^+ \mapsto 0, \\ E \ \ \mapsto 0, \\ A^- \mapsto -A^+, \end{array} \qquad \alpha(B^-): \begin{array}{ll} A^+ \mapsto A^-, \\ E \ \ \mapsto 0, \\ A^- \mapsto 0. \end{array}$$

The $\alpha(X)$ are derivations and satisfy $\big(\alpha(X)Y\big)^* = -\alpha(X^*)Y^*$ for all $X \in \mathfrak{sl}_2$, $Y \in \mathfrak{hw}$. Therefore we can define a new Lie algebra $\mathfrak{sl}_2 \oplus_\alpha \mathfrak{hw}$ as the semi-direct sum of $\mathfrak{sl}_2$ and $\mathfrak{hw}$, it has the commutation relations $\big[(X_1, Y_1), (X_2, Y_2)\big] = \big([X_1, X_2], [Y_1, Y_2] + \alpha(X_1)Y_2 - \alpha(X_2)Y_1\big)$ and the involution $(X, Y)^* = (X^*, Y^*)$. In terms of the basis $\{B^\pm, M, A^\pm, E\}$ the commutation relations are

$$[B^-, B^+] = M \quad [M, B^\pm] = \pm 2B^\pm,$$
$$[A^-, A^+] = E, \quad [E, A^\pm] = 0,$$
$$[B^\pm, A^\mp] = \mp A^\pm, \quad [B^\pm, A^\pm] = 0,$$
$$[M, A^\pm] = \pm A^\pm, \quad [E, B^\pm] = 0, \quad [M, E] = 0.$$

The action α has been chosen in order to obtain these relations, which also follow from the renormalization rule introduced in [ALV00b].

In the following we identify $\mathcal{U}(\mathfrak{hw})$ and $\mathcal{U}(\mathfrak{sl}_2)$ with the corresponding subalgebras in $\mathcal{U}(\mathfrak{sl}_2 \oplus_\alpha \mathfrak{hw})$.

Note that for any $c \in \mathbb{R}$, span$\{N = M + cE, A^+, A^-, E\}$ forms a Lie subalgebra of $\mathfrak{sl}_2 \oplus_\alpha \mathfrak{hw}$ that is isomorphic to $\mathfrak{osc}$.

There exist no Gaussian Lévy processes on $\mathfrak{sl}_2 \oplus_\alpha \mathfrak{hw}$, since $[\mathfrak{sl}_2 \oplus_\alpha \mathfrak{hw}, \mathfrak{sl}_2 \oplus_\alpha \mathfrak{hw}] = \mathfrak{sl}_2 \oplus_\alpha \mathfrak{hw}$. But, like for every real Lie algebra, there exist non-trivial $*$-representations of $\mathfrak{sl}_2 \oplus_\alpha \mathfrak{hw}$, and thus also Lévy processes, it is sufficient to take, e.g., a trivial cocycle.

The following result shows that the usual creation and annihilation calculus cannot be extended to a joint calculus of creation and annihilation and their squares.

Proposition 3.5. *Let (ρ, η, L) be the Schürmann triple on $\mathfrak{hw}$ defined in Proposition 3.1 a), and denote the corresponding Lévy process by $(j_{st})_{0\le s\le t}$. There exists no Lévy process $(\tilde{j}_{st})_{0\le s\le t}$ on $\mathfrak{sl}_2 \oplus_\alpha \mathfrak{hw}$ such that*

$$\big(\tilde{j}_{st}|_{\mathcal{U}(\mathfrak{hw})}\big) \cong (j_{st}),$$

unless $(j_{st})_{0\le s\le t}$ is trivial, i.e. $j_{st}(u) = 0$ for all $u \in \mathcal{U}_0(\mathfrak{hw})$.

Proof. We will assume that $(\tilde{j}_{st})$ exists and show that this implies $||v_1||^2 = ||v_2||^2 = |z|^2 = 0$, i.e. $L = 0$.

Let $(\tilde{\rho}, \tilde{\eta}, \tilde{L})$ be the Schürmann triple of $(\tilde{j}_{st})$. If $\big(\tilde{j}_{st}|_{\mathcal{U}(\mathfrak{hw})}\big) \cong (j_{st})$, then we have $\tilde{L}|_{\mathcal{U}_0(\mathfrak{hw})} = L$, and therefore the triple on $\mathfrak{hw}$ obtained by restriction of $(\tilde{\rho}, \tilde{\eta}, \tilde{L})$ is equivalent to (ρ, η, L) and there exists an isometry from $D = \eta\big(\mathcal{U}_0(\mathfrak{hw})\big)$ into $\tilde{D} = \tilde{\eta}\big(\mathcal{U}_0(\mathfrak{sl}_2 \oplus_\alpha \mathfrak{hw})\big)$, such that we have

$$\tilde{\rho}|_{\mathcal{U}(\mathfrak{hw})\times D} = \rho, \qquad \text{and } \tilde{\eta}|_{\mathcal{U}_0(\mathfrak{hw})} = \eta,$$

if we identify D with its image in $\tilde{D}$.

From $[B^+, A^-] = -A^+$ and $[B^-, A^+] = A^-$, we get

$$-\tilde{\eta}(A^+) = \tilde{\rho}(B^+)\eta(A^-) - \tilde{\rho}(A^-)\tilde{\eta}(B^+),$$
$$\tilde{\eta}(A^-) = \tilde{\rho}(B^-)\eta(A^+) - \tilde{\rho}(A^+)\tilde{\eta}(B^-).$$

Taking the inner product with $\tilde{\eta}(A^+) = \eta(A^+) = v_1$ and $\tilde{\eta}(A^-) = \eta(A^-) = v_2$, resp., we get

$$\begin{aligned} -||v_1||^2 &= \langle v_1, \rho(B^+)v_2\rangle - \langle v_1, \tilde{\rho}(A^-)\tilde{\eta}(B^+)\rangle \\ &= \langle v_1, \rho(B^+)v_2\rangle - \langle \rho(A^+)v_1, \tilde{\eta}(B^+)\rangle = \langle v_1, \rho(B^+)v_2\rangle, \\ ||v_2||^2 &= \langle v_2, \rho(B^-)v_1\rangle, \end{aligned}$$

since $\tilde{\rho}(A^\pm)|_D = \rho(A^\pm)$. Therefore

$$-||v_1||^2 = \langle v_1, \rho(B^+)v_2\rangle = \overline{\langle v_2, \rho(B^-)v_1\rangle} = ||v_2||^2,$$

and thus $||v_1||^2 = ||v_2||^2 = 0$. But $A^+ = -[B^+, A^-]$ and

$$\begin{aligned} L(A^+) = \tilde{L}(A^+) &= \langle \tilde{\eta}(A^+), \tilde{\eta}(B^+)\rangle - \langle \tilde{\eta}(B^-), \tilde{\eta}(A^-)\rangle \\ &= \langle v_1, \tilde{\eta}(B^+)\rangle - \langle \tilde{\eta}(B^-), v_2\rangle \end{aligned}$$

which now implies that $z = L(A^+) = 0$. □

Śniady [Śni00] has posed the question, if it is possible to define a joint calculus for the linear white noise and the square of white noise. Formulated in our context, his answer to this question is that there exists no Lévy process on $\mathfrak{sl}_2 \oplus_\alpha \mathfrak{hw}$ such that

$$j_{st}(E) = (t-s)\mathrm{id}, \quad \text{and} \quad j_{st}(A^-)\Omega = j_{st}(B^-)\Omega = 0,$$

for all $0 \le s \le t$. We are now able to show the same under apparently much weaker hypotheses.

Corollary 3.1. *Every Lévy process on $\mathfrak{sl}_2 \oplus_\alpha \mathfrak{hw}$ such that the state vector Ω is an eigenvector for $j_{st}(E)$ and $j_{st}(A^-)$ for some pair s and t with $0 \le s < t$ is trivial on $\mathfrak{hw}$, i.e. it has to satisfy $j_{st}|_{\mathcal{U}_0(\mathfrak{hw})} = 0$ for all $0 \le s \le t$.*

Proof. Assume that such a Lévy process exists. Then it would be equivalent to its realization on a boson Fock space defined by Eq. (2.1). Therefore we see that the state vector is an eigenvector of $j_{st}(E)$ and $j_{st}(A^-)$, if and only if the Schürmann triple of $(j_{st})_{0\le s\le t}$ satisfies $\eta(E) = \eta(A^-) = 0$. If we show that the only Schürmann triples on $\mathfrak{hw}$ satisfying this condition are the Gaussian Schürmann triples, then our result follows from Proposition 3.5.

Let (ρ, η, L) be a Schürmann triple on $\mathfrak{hw}$ such that $\eta(E) = \eta(A^-) = 0$. Then the vector $\eta(A^+)$ has to be cyclic for ρ. We get

$$\rho(E)\eta(A^+) = \rho(A^+)\eta(E) = 0,$$

since E and A^+ commute. From $[A^-, A^+] = E$, we get

$$\rho(A^-)\eta(A^+) = \rho(A^+)\eta(A^-) + \eta(E) = 0.$$

But

$$\begin{aligned}||\rho(A^+)\eta(A^+)||^2 &= \langle\eta(A^+), \rho(A^-)\rho(A^+)\eta(A^+)\rangle \\ &= \langle\eta(A^-), \rho(A^+)\rho(A^-)\eta(A^+)\rangle + \langle\eta(A^+), \rho(E)\eta(A^+)\rangle \\ &= 0\end{aligned}$$

shows that $\rho(A^+)$ also acts trivially on $\eta(A^+)$ and therefore the restriction of the triple (ρ, η, L) to $\mathcal{U}(\mathfrak{hw})$ is Gaussian. □

The SWN calculus defined in Example 3.1 can only be extended in the trivial way, i.e. by setting it equal to zero on $\mathfrak{hw}$, $\tilde{j}_{st}|_{\mathfrak{hw}} = 0$.

Proposition 3.6. *Let $(j_{st})_{0\leq s\leq t}$ be the Lévy process on $\mathfrak{sl}_2$ defined in (3.7). The only Lévy process $(\tilde{j}_{st})_{0\leq s\leq t}$ on $\mathfrak{sl}_2 \oplus_\alpha \mathfrak{hw}$ such that*

$$(\tilde{j}_{st}|_{\mathcal{U}(\mathfrak{sl}_2)}) \cong (j_{st})$$

is the process defined by $\tilde{j}_{st} = j_{st} \circ \pi$ for $0 \leq s \leq t$, where π is the canonical homomorphism $\pi : \mathcal{U}(\mathfrak{sl}_2 \oplus_\alpha \mathfrak{hw}) \to \mathcal{U}\big((\mathfrak{sl}_2 \oplus_\alpha \mathfrak{hw})/\mathfrak{hw}\big) \cong \mathcal{U}(\mathfrak{sl}_2)$.

Proof. We proceed as in the proof of Proposition 3.5, we assume that $(\tilde{j}_{st})_{0\leq s\leq t}$ is such an extension, and then we show that this necessarily implies $\tilde{\rho}|_{\mathcal{U}_0(\mathfrak{hw})} = 0$, $\tilde{\eta}|_{\mathcal{U}_0(\mathfrak{hw})} = 0$, and $\tilde{L}|_{\mathcal{U}_0(\mathfrak{hw})} = 0$ for its Schürmann triple $(\tilde{\rho}, \tilde{\eta}, \tilde{L})$. We know that the restriction of the Schürmann triple $(\tilde{\rho}, \tilde{\eta}, \tilde{L})$ to the subalgebra $\mathfrak{sl}_2$ and the representation space $D = \tilde{\eta}\big(\mathcal{U}_0(\mathfrak{sl}_2)\big)$ has to be equivalent to the Schürmann triple (ρ, η, L) defined in Example 3.1.

Our main tool are the following two facts, which can be deduced from our construction of the irreducible $*$-representations of $\mathfrak{sl}_2$ in Subsect. 3.2. Let π be an arbitrary $*$-representation of $\mathfrak{sl}_2$. Then $\pi(B^-)v = 0$ and $\pi(M)v = \lambda v$, with $\lambda < 0$ implies $v = 0$. And if we have a vector $v \neq 0$ that satisfies $\pi(B^-)v = 0$ and $\pi(M)v = \lambda v$ with $\lambda \geq 0$, then π restricted to $\pi\big(\mathcal{U}(\mathfrak{sl}_2)\big)v$ is equivalent to the lowest weight representation $\rho^+_{m_0}$ with $m_0 = \lambda$.

First, we show in several steps that $\eta(B^+)$ is cyclic for $\tilde{\rho}$ and exhibit several vectors in $\tilde{D} = \tilde{\eta}\big(\mathcal{U}_0(\mathfrak{sl}_2 \oplus_\alpha \mathfrak{hw})\big)$ which are lowest weight vectors for $\mathfrak{sl}_2$. Using this information we can then prove that $\tilde{\rho}$, $\tilde{\eta}$, and $\tilde{L}$ vanish on $\mathfrak{hw}$ (and therefore also on $\mathcal{U}_0(\mathfrak{hw})$).

Step 1: $\tilde{\eta}(A^-) = 0$.
The relations $[B^-, A^-] = 0$ and $[M, A^-] = -A^-$ imply $\tilde{\rho}(B^-)\tilde{\eta}(A^-) = \tilde{\rho}(A^-)\eta(B^-) = 0$ and $-\tilde{\eta}(A^-) = \tilde{\rho}(M)\tilde{\eta}(A^-) - \tilde{\rho}(A^-)\eta(M) = \tilde{\rho}(M)\tilde{\eta}(A^-)$.

Step 2: If $u_0 = \tilde{\rho}(A^-)\eta(B^+) = \tilde{\eta}(A^+) \neq 0$, then it generates an $\mathfrak{sl}_2$-representation that is equivalent to ρ^+_1.
Since $\tilde{\eta}(A^-) = 0$, the relation $[A^-, B^+] = A^+$ implies $\tilde{\eta}(A^+) = \tilde{\rho}(A^-)\eta(B^+) - \tilde{\rho}(B^+)\tilde{\eta}(A^-) = \tilde{\rho}(A^-)\eta(B^+)$. Furthermore $[B^-, A^+] = A^-$ and $[M, A^+] = A^+$ yield $\tilde{\rho}(B^-)\tilde{\eta}(A^+) = \tilde{\rho}(A^+)\eta(B^-) + \tilde{\eta}(A^-) = 0$ and $\tilde{\rho}(M)\tilde{\eta}(A^+) = \tilde{\rho}(A^+)\eta(M) + \tilde{\eta}(A^+) = \tilde{\eta}(A^+)$.

Step 3: The $\mathfrak{sl}_2$-representation generated from $v_0 = \tilde{\rho}(A^-)\tilde{\eta}(A^+) = \tilde{\eta}(E)$ is equivalent to the trivial one, i.e. $\tilde{\rho}(B^-)\tilde{\eta}(E) = \tilde{\rho}(M)\tilde{\eta}(E) = \tilde{\rho}(B^+)\tilde{\eta}(E) = 0$.
We get $\tilde{\rho}(B^-)\tilde{\eta}(E) = \tilde{\rho}(M)\tilde{\eta}(E) = 0$ from the relations $[M, E] = 0$ and $[B^-, E] = 0$, and $\tilde{\rho}(B^+)\tilde{\eta}(E) = 0$ follows from our basic facts on $\mathfrak{sl}_2$-representations.

Step 4: $\tilde{\eta}(E) = 0$ and $w_0 = \tilde{\rho}(A^+)\tilde{\eta}(A^+)$ is the lowest weight vector of an $\mathfrak{sl}_2$-representation equivalent to ρ_2^+ (unless $w_0 = 0$).
Applying twice the relation $[B^-, A^+] = A^-$ and once $[A^-, A^+] = E$, we get

$$\begin{aligned}\tilde{\rho}(B^-)\tilde{\rho}(A^+)\tilde{\eta}(A^+) &= \tilde{\rho}(A^+)\tilde{\rho}(B^-)\tilde{\eta}(A^+) + \tilde{\rho}(A^-)\tilde{\eta}(A^+)\\ &= \tilde{\rho}(A^+)\tilde{\rho}(A^+)\eta(B^-) + \tilde{\rho}(A^+)\tilde{\eta}(A^-)\\ &\quad + \tilde{\rho}(A^+)\tilde{\eta}(A^-) + \tilde{\eta}(E)\\ &= \tilde{\eta}(E).\end{aligned}$$

We can use this relation to compute the norm of $\tilde{\eta}(E)$,

$$||\tilde{\eta}(E)||^2 = \langle\tilde{\eta}(E), \tilde{\rho}(B^-)\tilde{\rho}(A^+)\tilde{\eta}(A^+)\rangle = \langle\tilde{\rho}(B^+)\tilde{\eta}(E), \tilde{\rho}(A^+)\tilde{\eta}(A^+)\rangle = 0,$$

since $\tilde{\rho}(B^+)\tilde{\eta}(E) = 0$.
Using twice the relation $[M, A^+] = A^+$, one also obtains $\tilde{\rho}(M)w_0 = 2w_0$.

Step 5: $\tilde{\rho}(E) = 0$.
The results of Steps 1, 2, and 4 and the surjectivity of $\tilde{\eta}$ imply that $\eta(B^+)$ is cyclic for $\tilde{\rho}$, i.e. any vector $v \in D$ can be written in the form $v = \tilde{\rho}(u)\eta(B^+)$ for some $u \in \mathcal{U}(\mathfrak{sl}_2 \oplus_\alpha \mathfrak{hw})$. Since E is central, we get

$$\tilde{\rho}(E)v = \tilde{\rho}(E)\tilde{\rho}(u)\eta(B^+) = \tilde{\rho}(uB^+)\tilde{\eta}(E) = 0$$

for all $v \in D$.

Step 6: $w_0 = 0$.
We can compute the norm of $\tilde{\rho}(B^+)w_0 = \tilde{\rho}(B^+)\tilde{\rho}(A^+)\tilde{\eta}(A^+)$ in two different ways. Since A^+ and B^+ commute, we get

$$\begin{aligned}||\tilde{\rho}(B^+)w_0||^2 &= ||\tilde{\rho}(A^+)^2\eta(B^+)||^2 = \langle\eta(B^+), \tilde{\rho}(A^-)^2\tilde{\rho}(A^+)^2\eta(B^+)\rangle\\ &= \langle\tilde{\rho}(A^-)^2\eta(B^+), \tilde{\rho}(A^-)^2\eta(B^+)\rangle\\ &= ||\tilde{\rho}(A^-)\tilde{\eta}(A^+)||^2 = ||\tilde{\eta}(E)||^2 = 0,\end{aligned}$$

where we also used $\tilde{\rho}(E) = 0$.
If $w_0 \neq 0$, then $\tilde{\rho}$ restricted to $\tilde{\rho}(\mathcal{U}(\mathfrak{sl}_2)w_0$ is equivalent to ρ_2^+, so in particular the vectors $w_n = \tilde{\rho}(B^+)^n, n \geq 0$, must be an orthogonal family of non-zero vectors with $||w_1||^2 = 6||w_0||^2$ by Eq. (3.2). But we have just shown $||w_1||^2 = 0$.

Step 7: $u_0 = 0$ and $\tilde{\rho}|_{\mathfrak{hw}} = 0$.
We get

$$\begin{aligned}||u_0||^2 &= \langle\tilde{\rho}(A^-)\eta(B^+), \tilde{\rho}(A^-)\eta(B^+)\rangle = \langle\eta(B^+), \tilde{\rho}(A^+)\tilde{\rho}(A^-)\eta(B^+)\rangle\\ &= \langle\eta(B^+), \tilde{\rho}(A^+)\tilde{\eta}(A^+)\rangle = \langle\eta(B^+), w_0\rangle = 0.\end{aligned}$$

Therefore we have $\tilde{\eta}|_{\mathfrak{hw}} = 0$ and $\tilde{D} = D = \text{span}\left\{\eta\big((B^+)^k\big)|k = 1, 2, \dots\right\}$. From this we can deduce $\tilde{\rho}(A^+)\eta\big((B^+)^k\big) = \tilde{\rho}((B^+)^k\big)\tilde{\eta}(A^+) = 0$, i.e. $\tilde{\rho}(A^+) = 0$ and therefore also $\tilde{\rho}(A^-) = \tilde{\rho}(A^+)^* = 0$.

Step 8: $\tilde{L}|_{\mathfrak{hw}} = 0$.
Finally, using, e.g., the relations $[M, A^\pm] = \pm A^\pm$ and $E = [A^-, A^+]$, one can show that the generator $\tilde{L}$ also vanishes on $\mathfrak{hw}$,

$$\begin{aligned}\pm\tilde{L}(A^\pm) &= \langle\eta(M), \tilde{\eta}(A^\pm)\rangle - \langle\tilde{\eta}(A^\mp), \eta(M)\rangle = 0,\\ \tilde{L}(E) &= ||\tilde{\eta}(A^+)||^2 - ||\tilde{\eta}(A^-)||^2 = 0. \qquad \square\end{aligned}$$

But there do exist non-trivial Lévy processes such that $j_{st}(A^-)\Omega = j_{st}(B^-)\Omega = 0$ for all $0 \le s \le t$, as the following example shows:

Example 3.3. Let $h > 0$ and let ρ_h be the Fock representation of $\mathcal{U}(\mathfrak{hw})$ defined in (3.1). This extends to a representation of $\mathcal{U}(\mathfrak{sl}_2 \oplus_\alpha \mathfrak{hw})$, if we set

$$\rho_h(B^+) = \frac{\rho_h(A^+)^2}{2h}, \quad \rho_h(M) = \frac{\rho_h(A^+A^- + A^-A^+)}{2h}, \quad \rho_h(B^+) = \frac{\rho_h(A^-)^2}{2h}.$$

The restriction of this representation to $\mathfrak{sl}_2$ is a direct sum of the two lowest weight representations $\rho^+_{1/2}$ and $\rho^+_{3/2}$, the respective lowest weight vectors are e_0 and e_1. For the cocycle we take the coboundary of the "lowest weight vector" $e_0 \in \ell^2$, i.e. we set

$$\eta(u) = \rho_h(u)e_0$$

for $u \in \mathcal{U}_0(\mathfrak{sl}_2 \oplus_\alpha \mathfrak{hw})$, and for the generator

$$L(u) = \langle e_0, \rho_h(u)e_0\rangle$$

for $u \in \mathcal{U}_0(\mathfrak{sl}_2 \oplus_\alpha \mathfrak{hw})$. This defines a Schürmann triple on $\mathfrak{sl}_2 \oplus_\alpha \mathfrak{hw}$ over ℓ^2 and therefore

$$\begin{aligned}
\mathrm{d}_L B^+ &= \frac{1}{2h}\mathrm{d}\Lambda\left(\rho_h(A^+)^2\right) + \frac{1}{\sqrt{2}}\mathrm{d}A^*(e_2),\\
\mathrm{d}_L A^+ &= \mathrm{d}\Lambda\big(\rho_h(A^+)\big) + \sqrt{h}\mathrm{d}A^*(e_1),\\
\mathrm{d}_L M &= \frac{1}{2h}\mathrm{d}\Lambda\big(\rho_h(A^+A^- + A^-A^+)\big) + \frac{1}{2}\mathrm{d}A^*(e_0) + \frac{1}{2}\mathrm{d}A(e_0) + \frac{1}{2}\mathrm{d}t,\\
\mathrm{d}_L E &= h\mathrm{d}\Lambda(\mathrm{id}) + h\mathrm{d}A^*(e_0) + h\mathrm{d}(e_0) + h\mathrm{d}t,\\
\mathrm{d}_L A^- &= \mathrm{d}\Lambda\big(\rho_h(A^-)\big) + \sqrt{h}\mathrm{d}A(e_1),\\
\mathrm{d}_L B^- &= \frac{1}{2h}\mathrm{d}\Lambda\left(\rho_h(A^-)^2\right) + \frac{1}{\sqrt{2}}\mathrm{d}A(e_2),
\end{aligned}$$

defines a Lévy process $\mathfrak{sl}_2 \oplus_\alpha \mathfrak{hw}$, acting on the Fock space over $L^2(\mathbb{R}_+, \ell^2)$. The Itô table of this process is infinite-dimensional. The restriction of this process to $\mathfrak{sl}_2$ is equivalent to the process defined in Example 3.2 with $m_0 = \frac{1}{2}$.

One can easily verify that $j_{st}(A^-)$ and $j_{st}(B^-)$ annihilate the vacuum vector of $\Gamma\big(L^2(\mathbb{R}_+, \ell^2)\big)$.

We have $\rho_h(C) = -\frac{3}{4}\mathrm{id}$, and therefore

$$\mathrm{d}_L C = -\frac{3}{4}\big(\mathrm{d}\Lambda(\mathrm{id}) + \mathrm{d}A^*(e_0) + \mathrm{d}A(e_0) + \mathrm{d}t\big).$$

3.4. Higher order noises. Let us now consider the infinite-dimensional real Lie algebra $\mathfrak{wn}$ that is spanned by $\{B_{n,m}; n, m \in \mathbb{N}\}$ with the commutation relations obtained by the natural extension, to higher powers of the white noise, of the renormalization rule introduced in [ALV99], i.e.:

$$[B_{n_1,m_1}, B_{n_2,m_2}] = \sum_{k=1}^{n_2\wedge m_1} \frac{m_1!n_2!}{(m_1-k)!(n_2-k)!k!} c^k B_{n_1+n_2-k,m_1+m_2-k} - \sum_{k=1}^{n_1\wedge m_2} \frac{m_2!n_1!}{(m_2-k)!(n_1-k)!k!} c^k B_{n_1+n_2-k,m_1+m_2-k}$$

for $n_1, n_2, m_1, m_2 \in \mathbb{N}$, and involution $\left(B_{n,m}\right)^* = B_{m,n}$, where $c \geq 0$ is some fixed positive parameter. These relations can be obtained by taking the quotient of the universal enveloping algebra $\mathcal{U}(\mathfrak{hw})$ of $\mathfrak{hw}$ with respect to the ideal generated by $E = c\mathbf{1}$. The basis elements $B_{n,m}$ are the images of $(A^+)^n(A^-)^m$.

We can embed $\mathfrak{hw}$ and $\mathfrak{sl}_2 \oplus_\alpha \mathfrak{hw}$ into $\mathfrak{wn}$ by

$$A^+ \mapsto \frac{B_{1,0}}{\sqrt{c}}, \quad A^- \mapsto \frac{B_{0,1}}{\sqrt{c}}, \quad E \mapsto B_{0,0},$$
$$B^+ \mapsto \frac{1}{2c}B_{2,0}, \quad B^- \mapsto \frac{1}{2c}B_{0,2}, \quad M \mapsto \frac{1}{c}B_{1,1} + \frac{1}{2}B_{0,0}.$$

There exist no Gaussian Lévy processes on $\mathfrak{wn}$, since $[\mathfrak{wn}, \mathfrak{wn}] = \mathfrak{wn}$.

Let ρ_c be the Fock representation defined in Eq. (3.1). Setting

$$\rho(B_{n,m}) = \rho_c\left((A^+)^n(A^-)^m\right), \qquad n, m \in \mathbb{N},$$

we get a $*$-representation of $\mathcal{U}(\mathfrak{wn})$. If we set $\eta(u) = \rho(u)e_0$ and $L(u) = \langle e_0, \rho(u)e_0\rangle$ for $u \in \mathcal{U}_0(\mathfrak{wn})$, then we obtain a Schürmann triple on $\mathfrak{wn}$. For this triple we get

$$\mathrm{d}_L B_{n,m} = \mathrm{d}\Lambda\left(\rho_c(A^+)^n\rho_c(A^-)^m\right) + \delta_{m0}\sqrt{c^n n!}\,\mathrm{d}A^*(e_n) + \delta_{n0}\sqrt{c^m m!}\,\mathrm{d}A(e_m) + \delta_{n0}\delta_{m0}\mathrm{d}t,$$

for the differentials. Note that we have $j_{st}(B_{nm})\Omega = 0$ for all $m \geq 1$ and $0 \leq s \leq t$ for the associated Lévy process.

3.5. Other examples: Lévy processes on $\mathfrak{fd}$ *and* $\mathfrak{gl}_2$. The goal of this subsection is to explain the relation of the present paper to previous works by Boukas [Bou88, Bou91] and Parthasarathy and Sinha [PS91].

We introduce the two real Lie algebras $\mathfrak{fd}$ and $\mathfrak{gl}_2$. The finite-difference Lie algebra $\mathfrak{fd}$ is the three-dimensional solvable real Lie algebra with basis $\{P, Q, T\}$, commutation relations

$$[P, Q] = [T, Q] = [P, T] = T,$$

and involution $P^* = Q$, $T^* = T$, cf. [Fei87]. This Lie algebra is actually the direct sum of the unique non-abelian two-dimensional real Lie algebra and the one-dimensional abelian Lie algebra, its center is spanned by $T - P - Q$.

The Lie algebra $\mathfrak{gl}_2$ of the general linear group $GL(2;\mathbb{R})$ is the direct sum of $\mathfrak{sl}_2$ with the one-dimensional abelian Lie algebra. As a basis of $\mathfrak{gl}_2$ we will choose $\{B^+, B^-, M, I\}$, where B^+, B^-, and M are a basis of the Lie subalgebra $\mathfrak{sl}_2$, and I is hermitian and central. Note that $T \mapsto M + B^+ + B^-$, $P \mapsto (M - I)/2 + B^-$,

$Q \mapsto (M - I)/2 + B^+$ defines an injective Lie algebra homomorphism from $\mathfrak{fd}$ into $\mathfrak{gl}_2$, i.e. we can regard $\mathfrak{fd}$ as a Lie subalgebra of $\mathfrak{gl}_2$.

Following ideas by Feinsilver [Fei89], Boukas [Bou88, Bou91] constructed a calculus for $\mathfrak{fd}$, i.e. he constructed a Lévy process on it and defined stochastic integrals with respect to it. He also derived the Itô formula for these processes and showed that their Itô table is infinite-dimensional. His realization is not defined on the boson Fock space, but on the so-called finite-difference Fock space especially constructed for his $\mathfrak{fd}$ calculus. Parthasarathy and Sinha constructed another Lévy process on $\mathfrak{fd}$, acting on a boson Fock space, in [PS91]. They gave an explicit decomposition of the operators into conservation, creation, annihilation, and time, thereby reducing its calculus to Hudson–Parthasarathy calculus.

Accardi and Skeide [AS00a, AS00b] noted that they were able to recover Boukas' $\mathfrak{fd}$ calculus from their SWN calculus. In fact, since $\mathfrak{gl}_2$ is a direct sum of $\mathfrak{sl}_2$ and the one-dimensional abelian Lie algebra, any Lévy process $(j_{st})_{0\le s\le t}$ on $\mathfrak{sl}_2$ can be extended (in many different ways) to a Lévy process $(\tilde{j}_{st})_{0\le s\le t}$ on $\mathfrak{gl}_2$. We will only consider the extensions defined by

$$\tilde{j}_{st}|_{\mathfrak{sl}_2} = j_{st}, \quad \text{and} \quad \tilde{j}_{st}(I) = \lambda(t-s)\mathrm{id}, \qquad \text{for } 0 \le s \le t,$$

for $\lambda \in \mathbb{R}$. Since $\mathfrak{fd}$ is a Lie subalgebra of $\mathfrak{gl}_2$, we also get a Lévy process on $\mathfrak{fd}$ by restricting $(\tilde{j}_{st})_{0\le s\le t}$ to $\mathcal{U}(\mathfrak{fd})$.

If we take the Lévy process on $\mathfrak{sl}_2$ defined in Example 3.1 and $\lambda = 1$, then we get

$$\begin{aligned}
\mathrm{d}_L P &= \mathrm{d}\Lambda\big(\rho^+(M/2 + B^-)\big) + \mathrm{d}A(e_0),\\
\mathrm{d}_L Q &= \mathrm{d}\Lambda\big(\rho^+(M/2 + B^+)\big) + \mathrm{d}A^*(e_0),\\
\mathrm{d}_L T &= \mathrm{d}\Lambda\big(\rho^+(M + B^+ + B^-)\big) + \mathrm{d}A^*(e_0) + \mathrm{d}A(e_0) + \mathrm{d}t.
\end{aligned}$$

It can be checked that this Lévy process is equivalent to the one defined by Boukas.

If we take instead the Lévy process on $\mathfrak{sl}_2$ defined in Example 3.2, then we get

$$\begin{aligned}
\mathrm{d}_L P &= \mathrm{d}\Lambda\big(\rho^+_{m_0}(M/2 + B^-)\big) + \mathrm{d}A^*\left(\frac{m_0}{2}e_0\right) + \mathrm{d}A\left(\frac{m_0}{2}e_0 + \sqrt{m_0}e_1\right) + \frac{m_0-\lambda}{2}\mathrm{d}t,\\
\mathrm{d}_L Q &= \mathrm{d}\Lambda\big(\rho^+_{m_0}(M/2 + B^+)\big) + \mathrm{d}A^*\left(\frac{m_0}{2}e_0 + \sqrt{m_0}e_1\right) + \mathrm{d}A\left(\frac{m_0}{2}e_0\right) + \frac{m_0-\lambda}{2}\mathrm{d}t,\\
\mathrm{d}_L T &= \mathrm{d}\Lambda\big(\rho^+_{m_0}(M + B^+ + B^-)\big) + \mathrm{d}A^*(m_0 e_0 + \sqrt{m_0}e_1)\\
&\quad + \mathrm{d}A(m_0 e_0 + \sqrt{m_0}e_1) + m_0\mathrm{d}t\\
&= \mathrm{d}_L P + \mathrm{d}_L Q + \lambda\mathrm{d}t.
\end{aligned}$$

For $m_0 = \lambda = 2$, this is exactly the Lévy process defined in [PS91]. Note that in that case the representation $\rho^+_2 = \rho^+$ and the Fock space agree with those of Boukas' process, but the cocycle and the generator are different. Therefore the construction of [PS91] leads to the same algebra as Boukas', but not to the same quantum process – a fact that had already been noticed by Accardi and Boukas [AB00].

4. Classical Processes

Let $(j_{st})_{0\le s\le t}$ be a Lévy process on a real Lie algebra $\mathfrak{g}_{\mathbb{R}}$ over $\Gamma = \Gamma\big(L^2(\mathbb{R}_+, D)\big)$, fix a hermitian element Y, $Y^* = Y$, of $\mathfrak{g}_{\mathbb{R}}$, and define a map $y : \Sigma(\mathbb{R}_+) \to \mathcal{L}(\Gamma)$ by

$$y_\phi = \sum_{k=1}^{n} \phi_k j_{s_k t_k}(Y), \qquad \text{for } \phi = \sum_{k=1}^{n} \phi_k \mathbf{1}_{[s_k, t_k[} \in \Sigma(\mathbb{R}_+).$$

It is clear that the operators $\{y_\phi; \phi \in \Sigma(\mathbb{R}_+)\}$ commute, since y is the restriction of $\pi : \mathfrak{g}^{\mathbb{R}_+} \ni \psi = \sum_{k=1}^n \psi_k \mathbf{1}_{[s_k,t_k[} \mapsto \sum_{k=1}^n j_{s_k t_k}(\psi_k) \in \mathcal{L}(\Gamma)$ to the abelian current algebra $\mathbb{C}Y^{\mathbb{R}_+}$ over $\mathbb{C}Y$. Furthermore, if ϕ is real-valued, then y_ϕ is hermitian, since Y is hermitian. Therefore there exists a classical stochastic process $(\tilde{Y}_t)_{t\geq 0}$ whose moments are given by

$$\mathbb{E}(\tilde{Y}_{t_1} \cdots \tilde{Y}_{t_n}) = \langle \Omega, y_{\mathbf{1}_{[0,t_1[}} \cdots y_{\mathbf{1}_{[0,t_n[}} \Omega \rangle, \qquad \text{for all } t_1, \ldots, t_n \in \mathbb{R}_+.$$

Since the expectations of $(j_{st})_{0\leq s\leq t}$ factorize, we can choose $(\tilde{Y}_t)_{t\geq 0}$ to be a Lévy process. If $j_{st}(Y)$ is even essentially self-adjoint, then the marginal distribution of $(\tilde{Y}_t)_{t\geq 0}$ is uniquely determined.

We will now give a characterization of $(\tilde{Y}_t)_{t\geq 0}$. First, we need two lemmas.

Lemma 4.1. *Let $X \in \mathcal{L}(D)$, $u, v \in D$, and suppose furthermore that the series $\sum_{n=0}^\infty \frac{(tX)^n}{n!} w$ and $\sum_{n=0}^\infty \frac{(tX^*)^n}{n!} w$ converge in D for all $w \in D$. Then we have*

$$e^{\Lambda(X)} A(v) = A\left(e^{-X^*} v\right) e^{\Lambda(X)},$$
$$e^{A^*(u)} A(v) = \big(A(v) - \langle v, u\rangle\big) e^{A^*(u)},$$
$$e^{A^*(u)} \Lambda(X) = \big(\Lambda(X) - A^*(Xu)\big) e^{A^*(u)}$$

on the algebraic boson Fock space over D.

Proof. This can be deduced from the formula for the adjoint actions, $\operatorname{Ad} e^X Y = e^X Y e^{-X} = Y + [X, Y] + \frac{1}{2}\big[X, [X, Y]\big] + \cdots = e^{\operatorname{ad} X} Y$. □

The following formula gives the normally ordered form of the generalized Weyl operators and is a key tool to calculate the characteristic functions of classical subprocesses of Lévy processes on real Lie algebras.

Lemma 4.2. *Let $X \in \mathcal{L}(D)$ and $u, v \in D$ and suppose furthermore that the series $\sum_{n=0}^\infty \frac{(tX)^n}{n!} w$ and $\sum_{n=0}^\infty \frac{(tX^*)^n}{n!} w$ converge in D for all $w \in D$. Then we have*

$$\exp\big(\Lambda(X) + A^*(u) + A(v) + \alpha\big) = \exp\big(A^*(\tilde{u})\big) \exp\big(\Lambda(X)\big) \exp\big(A(\tilde{v})\big) \exp(\tilde{\alpha})$$

on the algebraic boson Fock space over D, where

$$\tilde{u} = \sum_{n=1}^\infty \frac{X^{n-1}}{n!} u, \quad \tilde{v} = \sum_{n=1}^\infty \frac{(X^*)^{n-1}}{n!} v, \quad \tilde{\alpha} = \alpha + \sum_{n=2}^\infty \langle v, \frac{X^{n-2}}{n!} u\rangle.$$

Proof. Let $\omega \in D$ and set $\omega_1(t) = \exp t\big(\Lambda(X) + A^*(u) + A(v) + \alpha\big)\omega$ and

$$\omega_2(t) = \exp\Big(A^*\big(\tilde{u}(t)\big)\Big) \exp\big(t\Lambda(X)\big) \exp\Big(A\big(\tilde{v}(t)\big)\Big) \exp\big(\tilde{\alpha}(t)\big)\omega$$

for $t \in [0, 1]$, where

$$\tilde{u}(t) = \sum_{n=1}^\infty \frac{t^n X^{n-1}}{n!} u, \ \tilde{v}(t) = \sum_{n=1}^\infty \frac{t^n (X^*)^{n-1}}{n!} v, \ \tilde{\alpha}(t) = t\alpha + \sum_{n=2}^\infty \langle v, \frac{t^n X^{n-2}}{n!} u\rangle.$$

Then we have $\omega_1(0) = \omega = \omega_2(0)$. Using Lemma 4.1, we can also check that

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega_1(t) = \big(\Lambda(X) + A^*(u) + A(v) + \alpha\big)\omega \exp t\big(\Lambda(X) + A^*(u) + A(v) + \alpha\big)\omega$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\omega_2(t) = {} & A^*\left(\frac{\mathrm{d}\tilde{u}}{\mathrm{d}t}(t)\right)\exp\Big(A^*\big(\tilde{u}(t)\big)\Big)\exp\big(t\Lambda(X)\big)\exp\Big(A\big(\tilde{v}(t)\big)\Big)\exp\big(\tilde{\alpha}(t)\big)\omega \\ & + \exp\Big(A^*\big(\tilde{u}(t)\big)\Big)\Lambda(X)\exp\big(t\Lambda(X)\big)\exp\Big(A\big(\tilde{v}(t)\big)\Big)\exp\big(\tilde{\alpha}(t)\big)\omega \\ & + \exp\Big(A^*\big(\tilde{u}(t)\big)\Big)\exp\big(t\Lambda(X)\big)A\left(\frac{\mathrm{d}\tilde{v}}{\mathrm{d}t}(t)\right)\exp\Big(A\big(\tilde{v}(t)\big)\Big)\exp\big(\tilde{\alpha}(t)\big)\omega \\ & + \exp\Big(A^*\big(\tilde{u}(t)\big)\Big)\exp\big(t\Lambda(X)\big)\exp\Big(A\big(\tilde{v}(t)\big)\Big)\frac{\mathrm{d}\tilde{\alpha}}{\mathrm{d}t}(t)\exp\big(\tilde{\alpha}(t)\big)\omega\end{aligned}$$

coincide for all $t \in [0, 1]$. Therefore we have $\omega_1(1) = \omega_2(1)$. □

Theorem 4.1. *Let $(j_{st})_{0\le s\le t}$ be a Lévy process on a real Lie algebra $\mathfrak{g}_\mathbb{R}$ with Schürmann triple (ρ, η, L). Then for any hermitian element Y of $\mathfrak{g}_\mathbb{R}$ such that $\eta(Y)$ is analytic for $\rho(Y)$, the associated classical Lévy process $(\tilde{Y}_t)_{t\ge 0}$ has characteristic exponent*

$$\Psi(\lambda) = i\lambda L(Y) + \sum_{n=2}^{\infty}\frac{(i\lambda)^n}{n!}\langle\eta(Y^*), \rho(Y)^{n-2}\eta(Y)\rangle,$$

($\rho(Y)^0 = \mathrm{id}$) for λ in some neighborhood of zero.

Proof. The characteristic exponent $\Psi(\lambda)$, $\lambda \in \mathbb{R}$, is defined by $\mathbb{E}(e^{i\lambda\tilde{Y}_t}) = e^{t\Psi(\lambda)}$, so we have to compute

$$\mathbb{E}\left(e^{i\lambda\tilde{Y}_t}\right) = \langle\Omega, e^{i\lambda j_{0t}(Y)}\Omega\rangle$$

for $j_{0t}(Y) = \Lambda_{0t}\big(\rho(Y)\big) + A^*_{0t}\big(\eta(Y)\big) + A_{0t}\big(\eta(Y)\big) + tL(Y)$. Using Lemma 4.2, we get

$$\mathbb{E}\left(e^{i\lambda\tilde{Y}_t}\right) = \exp\left(it\lambda L(Y) + t\sum_{n=2}^{\infty}\left\langle\eta(Y^*), \frac{(i\lambda)^n\rho(Y)^{n-2}}{n!}\eta(Y)\right\rangle\right). \qquad □$$

Remark 4.1. Note that $\Psi(\lambda)$ is nothing else than $\sum_{n=1}^{\infty}\frac{(i\lambda)^n}{n!}L(Y^n)$. It is also possible to give a more direct proof of the theorem, using the convolution of functionals on $\mathcal{U}(\mathfrak{g})$ instead of the boson Fock space realization of $(j_{st})_{0\le s\le t}$.

We give two corollaries of this result, the first justifies our definition of Gaussian generators.

Corollary 4.1. *Let L be a Gaussian generator on $\mathfrak{g}_\mathbb{R}$ with corresponding Lévy process $(j_{st})_{0\le s\le t}$. Then for any hermitian element Y the associated classical Lévy process $(\tilde{Y}_t)_{t\ge 0}$ is Gaussian with mean and variance*

$$\mathbb{E}(\tilde{Y}_t) = tL(Y), \qquad \mathbb{E}(\tilde{Y}_t^2) = ||\eta(Y)||^2 t, \qquad \textit{for } t \ge 0.$$

We see that in this case we can take $\big(||\eta(Y)||B_t + L(Y)t\big)_{t\geq 0}$ for $(\tilde{Y}_t)_{t\geq 0}$, where $(B_t)_{t\geq 0}$ is a standard Brownian motion.

The next corollary deals with the case where L is the restriction to $\mathcal{U}_0(\mathfrak{g})$ of a positive functional on $\mathcal{U}(\mathfrak{g})$.

Corollary 4.2. *Let (ρ, η, L) be a Schürmann triple on $\mathfrak{g}_{\mathbb{R}}$ whose cocycle is trivial, i.e. there exists a vector $\omega \in D$ such that $\eta(u) = \rho(u)\omega$ for all $u \in \mathcal{U}_0(\mathfrak{g})$, and whose generator is of the form $L(u) = \langle\omega, \rho(u)\omega\rangle$, for all $u \in \mathcal{U}_0(\mathfrak{g})$. Suppose furthermore that the vector ω is analytical for $\rho(Y)$, i.e. that $e^{u\rho(Y)}\omega := \sum_{n=1}^{\infty} \frac{u^n\rho(Y)^n}{n!}\omega$ converges for sufficiently small u. Then the classical stochastic process $(\tilde{Y}_t)_{t\geq 0}$ associated to $(j_{st})_{0\leq s\leq t}$ and Y is a compound Poisson process with characteristic exponent*

$$\Psi(u) = \left\langle \omega, \left(e^{iu\rho(Y)} - 1\right)\omega\right\rangle.$$

Remark 4.2. If the operator $\rho(Y)$ is even (essentially) self-adjoint, then we get the Lévy measure of $(\tilde{Y}_t)_{t\geq 0}$ by evaluating its spectral measure in the state vector ω,

$$\mu(\mathrm{d}\lambda) = \langle\omega, \mathrm{d}\mathbb{P}_\lambda \omega\rangle,$$

where $\rho(Y) = \int \lambda \mathrm{d}\mathbb{P}_\lambda$ is the spectral resolution of (the closure of) $\rho(Y)$.

Corollary 4.2 suggests to call a Lévy process on $\mathfrak{g}$ with trivial cocycle $\eta(u) = \rho(u)\omega$ and generator $L(u) = \langle\omega, \rho(u)\omega\rangle$ for $u \in \mathcal{U}_0(\mathfrak{g})$ a *Poisson process* on $\mathfrak{g}$.

Example 4.1. Let $(j_{st})_{0\leq s\leq t}$ be the Lévy process on $\mathfrak{sl}_2$ defined in Example 3.2 and let $Y = B^+ + B^- + \beta M$ with $\beta \in \mathbb{R}$. The operator $X = \rho^+_{m_0}(Y)$ is essentially self-adjoint. We now want to characterize the classical Lévy process $(\tilde{Y}_t)_{t\geq 0}$ associated to Y and $(j_{st})_{0\leq s\leq t}$ in the manner described above. Corollary 4.2 tells us that $(\tilde{Y}_t)_{t\geq 0}$ is a compound Poisson process with characteristic exponent

$$\Psi(u) = \left\langle e_0, \left(e^{iuX} - 1\right) e_0\right\rangle.$$

We want to determine the Lévy measure of $(\tilde{Y}_t)_{t\geq 0}$, i.e. we want to determine the measure μ on $\mathbb{R}$, for which

$$\Psi(u) = \int \left(e^{iux} - 1\right)\mu(\mathrm{d}x).$$

This is the spectral measure of X evaluated in the state $\langle e_0, \cdot\, e_0\rangle$. Note that the polynomials $p_n \in \mathbb{R}[x]$ defined by the condition

$$e_n = p_n(X)e_0,$$

$n = 0, 1, \ldots$, are orthogonal w.r.t. μ, since

$$\int p_n(x)p_m(x)\mu(\mathrm{d}x) = \langle e_0, p_n(X)p_m(X)e_0\rangle = \langle p_n(X)e_0, p_m(X)e_0\rangle = \delta_{nm},$$

for $n, m \in \mathbb{N}$. Looking at the definition of X, we can easily identify the three-term-recurrence relation satisfied by the p_n. We get

$$Xe_n = \sqrt{(n+1)(n+m_0)}e_{n+1} + \beta(2n+m_0)e_n + \sqrt{n(n+m_0-1)}e_{n-1},$$

for $n \in \mathbb{N}$, and therefore

$$(n+1)P_{n+1} + (2\beta n + \beta m_0 - x)P_n + (n + m_0 - 1)P_{n-1} = 0,$$

with initial condition $P_{-1} = 0$, $P_0 = 1$, for the rescaled polynomials

$$P_n = \prod_{k=1}^{n} \sqrt{\frac{n}{n+m_0}}\, p_n.$$

According to the value of β we have to distinguish three cases.

1. $|\beta| = 1$: In this case we have, up to rescaling, Laguerre polynomials, i.e.

$$P_n(x) = (-\beta)^n L_n^{(m_0-1)}(\beta x),$$

   where the Laguerre polynomials $L_n^{(\alpha)}$ are defined as in [KS94, Eq. (1.11.1)]. The measure μ can be obtained by normalizing the measure of orthogonality of the Laguerre polynomials; it is equal to

$$\mu(\mathrm{d}x) = \frac{|x|^{m_0-1}}{\Gamma(m_0)} e^{-\beta x} \mathbf{1}_{\beta\mathbb{R}_+} \mathrm{d}x.$$

   If $\beta = +1$, then this measure is, up to a normalization parameter, the usual χ^2-distribution (with parameter m_0) of probability theory. The operator X is then positive and therefore $(\tilde{Y}_t)_{t\geq 0}$ is a subordinator, i.e. a Lévy process with values in $\mathbb{R}_+$, or, equivalently, a Lévy process with non-decreasing sample paths.
2. $|\beta| < 1$: In this case we find the Meixner–Pollaczek polynomials after rescaling,

$$P_n(x) = P_n^{(m_0/2)}\left(\frac{x}{2\sqrt{1-\beta^2}}; \pi - \arccos\beta\right).$$

   For the definition of these polynomials see, e.g., [KS94, Eq. (1.7.1)]. For the measure μ we get

$$\mu(\mathrm{d}x) = C \exp\left(\frac{(\pi - 2\arccos\beta)x}{2\sqrt{1-\beta^2}}\right) \left|\Gamma\left(\frac{m_0}{2} + \frac{ix}{2\sqrt{1-\beta^2}}\right)\right|^2 \mathrm{d}x,$$

   where C has to be chosen such that μ is a probability measure.
3. $|\beta| > 1$: In this case we get Meixner polynomials after rescaling,

$$P_n(x) = \begin{cases} (-1)^n \prod_{k=1}^n \frac{k+m_0-1}{k} M_n\left(\frac{x}{c-1/c} - \frac{m_0}{2}; m_0; c^2\right) & \text{if } \beta > +1, \\ \prod_{k=1}^n \frac{k+m_0-1}{k} M_n\left(-\frac{x}{c-1/c} + \frac{m_0}{2}; m_0; c^2\right) & \text{if } \beta < -1, \end{cases}$$

   where

$$c = \begin{cases} \beta - \sqrt{\beta^2 - 1} & \text{if } \beta > +1, \\ -\beta - \sqrt{\beta^2 - 1} & \text{if } \beta < -1. \end{cases}$$

The definition of these polynomials can be found, e.g., in [KS94, Eq. (1.9.1)]. The density μ is again the measure of orthogonality of the polynomials P_n (normalized to a probability measure). We therefore get

$$\mu = C \sum_{n=0}^{\infty} \frac{c^{2n}(m_0)_n}{n!} \delta_{\mathrm{sgn}\beta((c-1/c)(n+m_0/2))},$$

where $C^{-1} = \sum_{n=0}^{\infty} \frac{c^{2n}(m_0)_n}{n!} = (1-c^2)^{-m_0}$. Here $(m_0)_n$ denotes the Pochhammer symbol, $(m_0)_n = m_0(m_0+1)\cdots(m_0+n-1)$.

Example 4.2. Let now $(j_{st})_{0\le s\le t}$ be the Lévy process on $\mathfrak{sl}_2$ defined in Example 3.1 and let again $Y = B^+ + B^- + \beta M$ with $\beta \in \mathbb{R}$. We already noted in Example 3.1 that j_{st} is equivalent to ρ^+_{t-s} for fixed s and t. Therefore the marginal distributions of the classical Lévy process $(\tilde{Y}_t)_{t\ge 0}$ are exactly the distributions of the operator X that we computed in the previous example (with $m_0 = t$).

For $\beta = 1$, we recover [Bou91, Theorem 2.2]. The classical Lévy process associated to $T = B^+ + B^- + M$ is an exponential or Gamma process with Fourier transform

$$\mathbb{E}\left(e^{iu\tilde{Y}_t}\right) = (1-iu)^{-t}$$

and marginal distribution $\nu_t(\mathrm{d}x) = \frac{x^{t-1}}{\Gamma(t)} e^{-x} \mathbf{1}_{\mathbb{R}_+} \mathrm{d}x$. This is a subordinator with Lévy measure $x^{-1} e^{-x} \mathbf{1}_{\mathbb{R}_+} \mathrm{d}x$, see, e.g., [Ber96].

For $\beta > 1$, we can write the Fourier transform of the marginal distributions ν_t as

$$\mathbb{E}(e^{iu\tilde{Y}_t}) = \exp t \left(\frac{iu(c-1/c)}{2} + \sum_{n=1}^{\infty} \frac{c^{2n}}{n} \left(e^{iun(c-1/c)} - 1 \right) \right).$$

This shows that we can define $(\tilde{Y}_t)_{t\ge 0}$ as a sum of Poisson processes with a drift, i.e. if $\left((N_t^{(n)})_{t\ge 0} \right)_{n\ge 1}$ are independent Poisson processes (with intensity and jump size equal to one), then we can take

$$\tilde{Y}_t = (c-1/c) \left(\sum_{n=1}^{\infty} n N^{(n)}_{c^{2n}t/n} + \frac{t}{2} \right), \qquad \text{for } t \ge 0.$$

The marginal distributions of these processes for the different values of β and their relation to orthogonal polynomials are also discussed in [FS93, Chapter 5].

5. Conclusion

We have shown that the theories of factorizable current representations of Lie algebras and Lévy processes on $*$-bialgebras provide an elegant and efficient formalism for defining and studying quantum stochastic calculi with respect to additive operator processes satisfying Lie algebraic relations. The theory of Lévy processes on $*$-bialgebras can also handle processes whose increments are not simply additive, but are composed by more complicated formulas, the main restriction is that they are independent (in the tensor sense). This allows to answer questions that could not be handled by direct computational methods, such as the computation of the SWN Itô table, the simultaneous realization of linear and squared white noise on the same Hilbert space, or the characterization of the associated classical processes.

After the completion of the present article, Accardi, Hida, and Kuo [AHK01] have shown that using white noise calculus it is possible to obtain a closed Itô table for the quadratic covariations of the three basic square of white noise operators. But the coefficients in their Itô table contain functions of the Hida derivative and its adjoint.

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