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Singular Spectrum of Lebesgue Measure Zero for One-Dimensional Quasicrystals

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Abstract: The spectrum of one-dimensional discrete Schrödinger operators associated to strictly ergodic dynamical systems is shown to coincide with the set of zeros of the Lyapunov exponent if and only if the Lyapunov exponent exists uniformly. This is used to obtain the Cantor spectrum of zero Lebesgue measure for all aperiodic subshifts with uniform positive weights. This covers, in particular, all aperiodic subshifts arising from primitive substitutions including new examples such as e.g. the Rudin–Shapiro substitution.

Our investigation is not based on trace maps. Instead it relies on an Oseledec type theorem due to A. Furman and a uniform ergodic theorem due to the author.

1. Introduction

This article is concerned with discrete random Schrödinger operators associated to minimal topological dynamical systems. This means we consider a family $(H_\omega)_{\omega\in\Omega}$ of operators acting on $\ell^2(\mathbb{Z})$ by

$$(H_\omega u)(n) \equiv u(n+1) + u(n-1) + f(T^n\omega)u(n), \tag{1}$$

where Ω is a compact metric space, $T : \Omega \longrightarrow \Omega$ is a homeomorphism and $f : \Omega \longrightarrow \mathbb{R}$ is continuous. The dynamical system (Ω, T) is called minimal if every orbit is dense. For minimal (Ω, T), there exists a set $\Sigma \subset \mathbb{R}$ s.t.

$$\sigma(H_\omega) = \Sigma, \quad \text{for all} \quad \omega \in \Omega, \tag{2}$$

where we denote the spectrum of the operator H by $\sigma(H)$ (cf. [6,36]).

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We will be particularly interested in the case that (Ω, T) is a subshift over a finite alphabet $A \subset \mathbb{R}$. In this case Ω is a closed subset of $A^{\mathbb{Z}}$, invariant under the shift operator $T : A^{\mathbb{Z}} \longrightarrow A^{\mathbb{Z}}$ given by $(Ta)(n) \equiv a(n+1)$ and f is given by $f : \Omega \longrightarrow A \subset \mathbb{R}$, $f(\omega) \equiv \omega(0)$. Here, A carries the discrete topology and $A^{\mathbb{Z}}$ is given the product topology.

Operators associated to subshifts arise in the quantum mechanical treatment of quasicrystals (cf. [3,40] for background on quasicrystals). Various examples of such operators have been studied in recent years. The main examples can be divided in two classes. These classes are given by primitive substitution operators (cf. e.g. [4,5,7,11,41,42]) and Sturmian operators respectively more generally circle map operators (cf. e.g. [6,12, 15,16,26,27,30]). A recent survey can be found in [14].

For these classes and in fact for arbitrary operators associated to subshifts satisfying suitable ergodicity and aperiodicity conditions, one expects the following features:
$(\mathcal{S})$ Purely singular spectrum; $(\mathcal{A})$ absence of eigenvalues; $(\mathcal{Z})$ Cantor spectrum of Lebesgue measure zero.

Note that $(\mathcal{S})$ combined with $(\mathcal{A})$ implies purely singular continuous spectrum and note also that $(\mathcal{S})$ is a consequence of $(\mathcal{Z})$. Let us mention that $(\mathcal{S})$ is by now completely established for all relevant subshifts due to recent results of Last–Simon [34] in combination with earlier results of Kotani [32]. For discussion of $(\mathcal{A})$ and further details we refer the reader to the cited literature.

The aim of this article is to investigate $(\mathcal{Z})$ and to relate it to ergodic properties of the underlying subshifts.

The property $(\mathcal{Z})$ has been investigated for several models by a number of authors: Following work by Bellissard–Bovier–Ghez [5], the most general result for primitive substitutions so far has been obtained by Bovier/Ghez [7]. They can treat a large class of substitutions which is given by an algorithmically accessible condition. The Rudin–Shapiro substitution does not belong to this class. For arbitrary Sturmian operators, Bellissard–Iochum–Scoppola–Testard established $(\mathcal{Z})$ [6], thereby extending the work of Sütő in the golden mean case [41,42]. A different approach, which recovers some of these results, is given in [13,19].

A canonical starting point in the investigation of $(\mathcal{Z})$ for subshifts is the fundamental result of Kotani [32] that the set $\{E \in \mathbb{R} : \gamma(E) = 0\}$ has Lebesgue measure zero if (Ω, T) is an aperiodic subshift. Here, γ denotes the Lyapunov exponent (precise definition given below). This reduces the problem $(\mathcal{Z})$ to establishing the equality

$$\Sigma = \{E \in \mathbb{R} : \gamma(E) = 0\}. \tag{3}$$

As do all other investigations of $(\mathcal{Z})$ so far, our approach starts from (3). Unlike the earlier treatments mentioned above our approach does not rely on the so called trace maps. Instead, we present a new method, the cornerstones of which are the following: (1) A strong type of Oseledec theorem by A. Furman [21]. (2) A uniform ergodic theorem for a large class of subshifts by the author [37]. This new setting allows us

$(*)$ to characterize validity of (3) for arbitrary strictly ergodic dynamical systems by an essentially ergodic property viz by uniform existence of the Lyapunov exponent (Theorem 1),

$(**)$ to present a large class of subshifts satisfying this property (Theorem 2).

Here, $(*)$ gives the new conceptual point of view of our treatment and $(**)$ gives a large class of examples. Put together $(*)$ and $(**)$ provide a soft argument for $(\mathcal{Z})$ for a large class of examples which contains, among other examples, all primitive substitutions.

The paper is organized as follows. In Sect. 2 we present the subshifts we will be interested in, introduce some notation and state our results. In Sect. 3, we recall results of Furman [21] and of the author [37] and adopt them to our setting. Section 4 is devoted to a proof of our results. Finally, in Sect. 5 we provide some further comments and discuss a variant of our main result.

2. Notation and Results

In this section we discuss basic material concerning topological dynamical systems and the associated operators and state our results.

As usual a dynamical system is said to be strictly ergodic if it is uniquely ergodic (i.e. there exists only one invariant probability measure) and minimal. A minimal dynamical system is called aperiodic if there does not exist an $n \in \mathbb{Z}$, $n \neq 0$, and $\omega \in \Omega$ with $T^n\omega = \omega$.

As mentioned already, our main focus will be the case that (Ω, T) is a subshift over the finite alphabet $A \subset \mathbb{R}$. We will then consider the elements of (Ω, T) as double sided infinite words and use notation and concepts from the theory of words. In particular, we then associate to Ω the set $\mathcal{W}$ of words associated to Ω consisting of all finite subwords of elements of Ω. The length $|x|$ of a word $x \equiv x_1 \ldots x_n$ with $x_j \in A$, $j = 1, \ldots, n$, is defined by $|x| \equiv n$. The number of occurrences of $v \in \mathcal{W}$ in $x \in \mathcal{W}$ is denoted by $\sharp_v(x)$.

We can now introduce the class of subshifts we will be dealing with. They are those satisfying uniform positivity of weights (PW) given as follows:

(PW) There exists a $C > 0$ with $\liminf_{|x|\to\infty} \frac{\sharp_v(x)}{|x|}|v| \geq C$ for every $v \in \mathcal{W}$.

One might think of (PW) as a strong type of minimality condition. Indeed, minimality can easily be seen to be equivalent to $\liminf_{|x|\to\infty} |x|^{-1}\sharp_v(x)|v| > 0$ for every $v \in \mathcal{W}$ [39]. The condition (PW) implies strict ergodicity [37]. The class of subshifts satisfying (PW) is rather large. By [37], it contains all linearly repetitive subshifts (see [20,33] for definition and thorough study of linearly repetitive systems). Thus, it contains, in particular, all subshifts arising from primitive substitutions as well as all those Sturmian dynamical systems whose rotation number has bounded continued fraction expansion [20,33,38].

In our setting the class of subshifts satisfying (PW) appears naturally as it is exactly the class of subshifts admitting a strong form of the uniform ergodic theorem [37]. Such a theorem in turn is needed to apply Furmans results (s. below for details).

After this discussion of background from dynamical systems we are now heading towards introducing key tools in spectral theoretic considerations viz transfer matrices and Lyapunov exponents.

The operator norm $\|\cdot\|$ on the set of 2×2-matrices induces a topology on $GL(2,\mathbb{R})$ and $SL(2,\mathbb{R})$. For a continuous function $A : \Omega \longrightarrow GL(2,\mathbb{R})$, $\omega \in \Omega$, and $n \in \mathbb{Z}$, we define the cocycle $A(n,\omega)$ by

$$A(n,\omega) \equiv \begin{cases} A(T^{n-1}\omega)\cdots A(\omega) & : \quad n > 0 \\ Id & : \quad n = 0 \\ A^{-1}(T^n\omega)\cdots A^{-1}(T^{-1}\omega) & : \quad n < 0. \end{cases}$$

By Kingman's subadditive ergodic theorem (cf. e.g. [31]), there exists $\Lambda(A) \in \mathbb{R}$ with

$$\Lambda(A) = \lim_{n\to\infty} \frac{1}{n} \log \|A(n,\omega)\|$$

for μ a. e. $\omega \in \Omega$ if (Ω, T) is uniquely ergodic with invariant probability measure μ. Following [21], we introduce the following definition.

Definition 1. *Let (Ω, T) be strictly ergodic. The continuous function $A : (\Omega, T) \longrightarrow GL(2,\mathbb{R})$ is called uniform if the limit $\Lambda(A) = \lim_{n\to\infty} \frac{1}{n} \log \|A(n,\omega)\|$ exists for all $\omega \in \Omega$ and the convergence is uniform on Ω.*

Remark 1. It is possible to show that uniform existence of the limit in the definition already implies uniform convergence. The author learned this from Furstenberg and Weiss [22]. They actually have a more general result. Namely, they consider a continuous subadditive cocycle $(f_n)_{n\in\mathbb{N}}$ on a minimal (Ω, T) (i.e. f_n are continuous real-valued functions on Ω with $f_{n+m}(\omega) \le f_n(\omega) + f_m(T^n\omega)$ for all $n, m \in \mathbb{N}$ and $\omega \in \Omega$). Their result then gives that existence of $\phi(\omega) = \lim_{n\to\infty} n^{-1} f_n(\omega)$ for all $\omega \in \Omega$ implies constancy of ϕ as well as uniform convergence.

For spectral theoretic investigations a special type of $SL(2,\mathbb{R})$-valued function is relevant. Namely, for $E \in \mathbb{R}$, we define the continuous function $M^E : \Omega \longrightarrow SL(2,\mathbb{R})$ by

$$M^E(\omega) \equiv \begin{pmatrix} E - f(T\omega) & -1 \\ 1 & 0 \end{pmatrix}. \tag{4}$$

It is easy to see that a sequence u is a solution of the difference equation

$$u(n+1) + u(n-1) + (f(T^n\omega) - E)u(n) = 0 \tag{5}$$

if and only if

$$\begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix} = M^E(n,\omega) \begin{pmatrix} u(1) \\ u(0) \end{pmatrix}, \; n \in \mathbb{Z}. \tag{6}$$

By the above considerations, M^E gives rise to the average $\gamma(E) \equiv \Lambda(M^E)$. This average is called the Lyapunov exponent for the energy E. It measures the rate of exponential growth of solutions of (5). Our main result now reads as follows.

Theorem 1. *Let (Ω, T) be strictly ergodic. Then the following are equivalent:*

(i) *The function M^E is uniform for every $E \in \mathbb{R}$.*

(ii) $\Sigma = \{E \in \mathbb{R} : \gamma(E) = 0\}$.

In this case the Lyapunov exponent $\gamma : \mathbb{R} \longrightarrow [0,\infty)$ is continuous.

Remark 2. (a) As will be seen later on, M^E is always uniform for E with $\gamma(E) = 0$ and for $E \in \mathbb{R} \setminus \Sigma$. From this point of view, the theorem essentially states that M^E can not be uniform for $E \in \Sigma$ with $\gamma(E) > 0$.

(b) Continuity of the Lyapunov exponent can easily be inferred from (ii) (though this does not seem to be in the literature). More precisely, continuity of γ on $\{E \in \mathbb{R} : \gamma(E) = 0\}$ is a consequence of subharmonicity. Continuity of γ on $\mathbb{R} \setminus \Sigma$ follows from the Thouless formula (see e. g. [10] for discussion of subharmonicity and the Thouless formula). Below, we will show that continuity of γ follows from (i) and this will be crucial in our proof of (i) $\Longrightarrow$ (ii).

Having studied $(*)$ of the introduction in the above theorem, we will now state our result on $(**)$.

Theorem 2. *If (Ω, T) is a subshift satisfying (PW), then the function M^E is uniform for each $E \in \mathbb{R}$.*

Remark 3. (a) Uniformity of M^E is rather unusual. This is, of course, clear from Theorem 1. Alternatively, it is not hard to see directly that it already fails for discrete almost periodic operators. More precisely, the Almost–Mathieu-Operator with coupling bigger than 2 has uniform positive Lyapunov exponent [24]. By a deterministic version of the theorem of Oseledec (cf. Theorem 8.1 of [34] for example), this would force pure point spectrum for all these operators, if M^E were uniform on the spectrum. However, there are examples of such Almost–Mathieu Operators without point spectrum [2,29].
(b) The above theorem generalizes [18,35], which in turn unified the work of Hof [25] on primitive substitutions and of Damanik and the author [17] on certain Sturmian subshifts.
(c) The theorem is a rather direct consequence of the subadditive theorem of [37].

The two theorems yield some interesting conclusions. We start with the following consequence of Theorem 1 concerning $(\mathcal{Z})$. A proof is given in Sect. 4.

Corollary 2.1. *Let (Ω, T) be an aperiodic strictly ergodic subshift. If M^E is uniform for every $E \in \mathbb{R}$, then the spectrum Σ is a Cantor set of Lebesgue measure zero.*

As $\Sigma = \{E : \gamma(E) = 0\}$ holds for arbitrary Sturmian dynamical subshifts [6,41] (cf. [19] as well), Theorem 1 immediately implies the following corollary.

Corollary 2.2. *Let $(\Omega(\alpha), T)$ be a Sturmian dynamical system with rotation number α. Then M^E is uniform for every $E \in \mathbb{R}$.*

Remark 4. So far uniformity of M^E for Sturmian systems could only be established for rotation numbers with bounded continued fraction expansion [17]. Moreover, the corollary is remarkable as a general type of uniform ergodic theorem actually fails as soon as the continued fraction expansion of α is unbounded [37,38].

Theorem 1, Theorem 2 and Corollary 2.1 directly yield the following corollary.

Corollary 2.3. *Let (Ω, T) be a subshift sastisfying (PW). Then $\Sigma = \{E \in \mathbb{R} : \gamma(E) = 0\}$. If (Ω, T) is furthermore aperiodic, then Σ is a Cantor set of Lebesgue measure zero.*

Remark 5. For aperiodic (Ω, T) satisfying (PW), this gives an alternative proof of $(\mathcal{S})$.

As discussed above primitive substitutions satisfy (PW). As validity of $(\mathcal{Z})$ for primitive substitutions has been a special focus of earlier investigations (cf. the discussion in Sect. 1 and Sect. 5), we explicitly state the following consequence of the foregoing corollary.

Corollary 2.4. *Let (Ω, T) be aperiodic and associated to a primitive substitution, then Σ is a Cantor set of Lebesgue measure zero.*

3. Key Results

In this section, we present (consequences of) results of Furman [21] and of the author [37].

We start with some simple facts concerning uniquely ergodic systems. Define for a continuous $b : \Omega \longrightarrow \mathbb{R}$ and $n \in \mathbb{Z}$ the averaged function $A_n(b) : \Omega \longrightarrow \mathbb{R}$ by

$$A_n(b)(\omega) \equiv \begin{cases} n^{-1}\sum_{k=0}^{n-1} b(T^k\omega) & : \quad n > 0 \\ 0 & : \quad n = 0 \\ |n|^{-1}\sum_{k=1}^{|n|} b(T^{-k}\omega) & : \quad n < 0. \end{cases} \tag{7}$$

Moreover, for a continuous b as above and a finite measure μ on Ω we set $\mu(b) \equiv \int_\Omega b(\omega)\, d\mu(\omega)$. The following proposition is well known, see e.g. [43].

Proposition 3.1. *Let (Ω, T) be uniquely ergodic with invariant probability measure μ. Let b be a continuous function on Ω. Then the averaged functions $A_n(b)$ converge uniformly towards the constant function with value $\mu(b)$ for $|n|$ tending to infinity.*

The following consequence of a result by A. Furman is crucial to our approach.

Lemma 3.2. *Let (Ω, T) be strictly ergodic with invariant probability measure μ. Let $B : \Omega \longrightarrow SL(2,\mathbb{R})$ be uniform with $\Lambda(B) > 0$. Then, for arbitrary $U \in \mathbb{C}^2 \setminus \{0\}$ and $\omega \in \Omega$, there exist constants $D, \kappa > 0$ such that $\|B(n,\omega)U\| \geq D\exp(\kappa|n|)$ holds for all $n \geq 0$ or for all $n \leq 0$. Here, $\|\cdot\|$ denotes the standard norm on $\mathbb{C}^2$.*

Proof. Theorem 4 of [21] states that uniformity of B implies that (in the notation of [21]) either $\Lambda(B) = 0$ or B is continuously diagonalizable. As we have $\Lambda(B) > 0$, we infer that B is continuously diagonalizable. This means that there exist continuous functions $C : \Omega \longrightarrow GL(2,\mathbb{R})$ and $a, d : \Omega \longrightarrow \mathbb{R}$ with

$$B(1,\omega) = C(T\omega)^{-1}\begin{pmatrix} \exp(a(\omega)) & 0 \\ 0 & \exp(d(\omega)) \end{pmatrix} C(\omega).$$

By multiplication and inversion, this immediately gives

$$B(n,\omega) = C(T^n\omega)^{-1}\begin{pmatrix} \exp(nA_n(a)(\omega)) & 0 \\ 0 & \exp(nA_n(d)(\omega)) \end{pmatrix} C(\omega), \quad n \in \mathbb{Z}. \tag{8}$$

As $C : \Omega \longrightarrow GL(2,\mathbb{R})$ is continuous on the compact space Ω, there exists a constant $\rho > 0$ with

$$0 < \rho \leq \|C(\omega)\|, |\det C(\omega)|, \|C^{-1}(\omega)\|, |\det C^{-1}(\omega)| \leq \frac{1}{\rho} < \infty, \quad \text{for all } \omega \in \Omega. \tag{9}$$

In view of (8) and (9), exponential growth of terms as $\|B(n,\omega)U\|$ will follow from suitable upper and lower bounds on $A_n(a)(\omega)$ and $A_n(d)(\omega)$ for large $|n|$. To obtain these bounds we proceed as follows.

Assume without loss of generality $\mu(a) \geq \mu(d)$. By (9), (8) and Proposition 3.1, we then have

$$0 < \Lambda(B) = \Lambda(C(T\cdot)^{-1}BC) = \Lambda(\begin{pmatrix} \exp(a(\cdot)) & 0 \\ 0 & \exp(d(\cdot)) \end{pmatrix} = \mu(a). \tag{10}$$

Moreover, $\det B(\omega) = 1$ implies $\det B(n,\omega) = 1$ for all $n \in \mathbb{Z}$. Thus, taking determinants, logarithms and averaging with $\frac{1}{n}$ in (8), we infer

$$0 = A_n(a)(\omega) + A_n(d)(\omega) + \frac{1}{n}\log|\det(C(T^n\omega)^{-1}C(\omega))|.$$

Taking the limit $n \to \infty$ in this equation and invoking (9) as well as Proposition 3.1, we obtain $\mu(a) = -\mu(d)$. As $\mu(a) > 0$ by (10), Proposition 3.1 then shows that there exists $\kappa > 0$, e.g. $\kappa = \frac{1}{2}\mu(a)$, s.t. for large $|n|$, we have

$$A_n(a)(\omega) > \kappa, \quad \text{and} \quad A_n(d)(\omega) < -\kappa \quad \text{for all } \omega \in \Omega.$$

Now, the statement of the lemma is a direct consequence of (8) and (9). □

Lemma 3.3. *Let* (Ω, T) *be strictly ergodic. Let* $A : \Omega \longrightarrow SL(2,\mathbb{R})$ *be uniform. Let* (A_n) *be a sequence of continuous* $SL(2,\mathbb{R})$*-valued functions converging to* A *in the sense that* $d(A_n, A) \equiv \sup_{\omega\in\Omega}\{\|A_n(\omega) - A(\omega)\|\} \longrightarrow 0, n \longrightarrow \infty$. *Then,* $\Lambda(A_n) \longrightarrow \Lambda(A)$, $n \longrightarrow \infty$.

Proof. This is essentially a result of [21]. More precisely, Theorem 5 of [21] shows that $\Lambda(A_n)$ converges to $\Lambda(A)$ whenever the following holds: A is a uniform $GL(2,\mathbb{R})$-valued function and $d(A_n, A) \longrightarrow 0$ and $d(A_n^{-1}, A^{-1}) \longrightarrow 0$, $n \longrightarrow \infty$. Now, for functions A_n, A with values in $SL(2,\mathbb{R})$, it is easy to see that $d(A_n^{-1}, A^{-1}) \longrightarrow 0$, $n \longrightarrow \infty$ if $d(A_n, A) \longrightarrow 0$, $n \longrightarrow \infty$. The proof of the lemma is finished. □

Lemma 3.4. *Let* (Ω, T) *be uniquely ergodic. Let* $A : \Omega \longrightarrow GL(2,\mathbb{R})$ *be continuous. Then, the inequality* $\limsup_{n\to\infty} n^{-1}\log\|A(n,\omega)\| \leq \Lambda(A)$ *holds uniformly on* Ω.

Proof. This follows from Corollary 2 of [21] (cf. Theorem 1 of [21] as well). □

Finally, we need the following lemma providing a large supply of uniform functions if (Ω, T) is a subshift satisfying (PW).

Lemma 3.5. *Let* (Ω, T) *be a subshift satisfying (PW). Let* $F : \mathcal{W} \longrightarrow \mathbb{R}$ *satisfy* $F(xy) \leq F(x) + F(y)$ *(i.e.* F *is subadditive). Then, the limit* $\lim_{|x|\to\infty}\frac{F(x)}{|x|}$ *exists.*

Proof. This is just Proposition 4.2 of [37]. □

4. Proofs of the Main Results

In this section, we use the results of the foregoing section to prove the theorems stated in Sect. 2.

We start with some lemmas needed for the proof of Theorem 1.

Lemma 4.1. *Let* (Ω, T) *be strictly ergodic. If* M^E *is uniform for every* $E \in \mathbb{R}$ *then* $\Sigma = \{E \in \mathbb{R} : \gamma(E) = 0\}$ *and the Lyapunov exponent* $\gamma : \mathbb{R} \longrightarrow [0,\infty)$ *is continuous.*

Proof. We start by showing continuity of the Lyapunov exponent. Consider a sequence (E_n) in $\mathbb{R}$ converging to $E \in \mathbb{R}$. As the function M^E is uniform by assumption, by Lemma 3.3, it suffices to show that $d(M^{E_n}, M^E) \to 0, n \to \infty$. This is clear from the definition of M^E in (4).

Now, set $\Gamma \equiv \{E \in \mathbb{R} : \gamma(E) = 0\}$. The inclusion $\Gamma \subset \Sigma$ follows from general principles (cf. e.g. [10]). Thus, it suffices to show the opposite inclusion $\Sigma \subset \Gamma$. By (2), it suffices to show $\sigma(H_\omega) \subset \Gamma$ for a fixed $\omega \in \Omega$.

Assume the contrary. Then there exists spectrum of H_ω in the complement $\Gamma^c \equiv \mathbb{R} \setminus \Gamma$ of Γ in $\mathbb{R}$. As γ is continuous, the set Γ^c is open. Thus, the spectrum of H_ω can only exist in Γ^c, if spectral measures of H_ω actually give weight to Γ^c. By standard results on the generalized eigenfunction expansion [8], there exists then an $E \in \Gamma^c$ admitting a polynomially bounded solution $u \neq 0$ of (5). By (6), this solution satisfies $(u(n+1), u(n))^t = M^E(n,\omega)(u(1), u(0))^t$, $n \in \mathbb{Z}$, where v^t denotes the transpose of v. By $E \in \Gamma^c$, we have $\Lambda(M^E) \equiv \gamma(E) > 0$. As M^E is uniform by assumption, we can thus apply Lemma 3.2 to M^E to obtain that $\|(u(n+1), u(n))^t\|$ is, at least, exponentially growing for large values of n or large values of $-n$. This contradicts the fact that u is polynomially bounded and the proof is finished. □

Lemma 4.2. *If* (Ω, T) *is uniquely ergodic,* M^E *is uniform for each* $E \in \mathbb{R}$ *with* $\gamma(E) = 0$.

Proof. By $\det M^E(\omega) = 1$, we have $1 \le \|M^E(n,\omega)\|$ and therefore $0 \le \liminf_{n\to\infty} n^{-1} \log \|M^E(n,\omega)\| \le \limsup_{n\to\infty} n^{-1} \log \|M^E(n,\omega)\|$. Now, the statement follows from Lemma 3.4. □

The following lemma is probably well known. However, as we could not find it in the literature, we include a proof.

Lemma 4.3. *If* (Ω, T) *is strictly ergodic,* M^E *is uniform with* $\gamma(E) > 0$ *for each* $E \in \mathbb{R} \setminus \Sigma$.

Proof. Let $E \in \mathbb{R} \setminus \Sigma$ be given. The proof will be split in four steps. Recall that Σ is the spectrum of H_ω for every $\omega \in \Omega$ by (2) and thus E belongs to the resolvent of H_ω for all $\omega \in \Omega$.

Step 1. For every $\omega \in \Omega$, there exist unique (up to a sign) normalized $U(\omega), V(\omega) \in \mathbb{R}^2$ such that $\|M^E(n,\omega)U(\omega)\|$ is exponentially decaying for $n \longrightarrow \infty$ and $\|M^E(n,\omega)V(\omega)\|$ is exponentially decaying for $n \longrightarrow -\infty$. The vectors $U(\omega), V(\omega)$ are linearly independent. For fixed $\omega \in \Omega$ they can be chosen to be continuous in a neighborhood of ω.

Step 2. Define the matrix $C(\omega)$ by $C(\omega) \equiv (U(\omega), V(\omega))$. Then $C(\omega)$ is invertible and there exist functions $a, b : \Omega \longrightarrow \mathbb{R} \setminus \{0\}$ such that

$$C(T\omega)^{-1} M^E(\omega) C(\omega) = \begin{pmatrix} a(\omega) & 0 \\ 0 & b(\omega) \end{pmatrix}. \tag{11}$$

Step 3. The functions $|a|, |b|, \|C\|, \|C^{-1}\| : \Omega \longrightarrow \mathbb{R}$ are continuous.

Step 4. M^E is uniform with $\gamma(E) > 0$.

Ad Step 1. This can be seen by standard arguments. Here is a sketch of the construction. Fix $\omega \in \Omega$ and set $u_0(n) \equiv (H_\omega - E)^{-1}\delta_0(n)$ and $u_{-1}(n) \equiv (H_\omega - E)^{-1}\delta_{-1}(n)$, where δ_k , $k \in \mathbb{Z}$, is given by $\delta_k(k) = 1$ and $\delta_k(n) = 0$, $k \neq n$. By Combes–Thomas arguments, see e.g. [10], the initial conditions $(u_0(0), u_0(1))$ and $(u_{-1}(0), u_{-1}(1))$ give rise to solutions of (5) which decay exponentially for $n \to \infty$. It is easy to see that not both of these solutions can vanish identically. Thus, after normalizing, we find a vector $U(\omega)$ with the desired properties. The continuity statement follows easily from continuity of $\omega \mapsto (H_\omega - E)^{-1}x$, for $x \in \ell^2(\mathbb{Z})$. The construction for $V(\omega)$ is similar. Uniqueness follows by standard arguments from constancy of the Wronskian. Linear independence is clear as E is not an eigenvalue of H_ω.

Ad Step 2. The matrix C is invertible by linear independence of U and V. The uniqueness statements of Step 1, show that there exist functions $a, b : \Omega \longrightarrow \mathbb{R}$ with $M^E(\omega)U(\omega) = a(\omega)U(T\omega)$ and $M^E(\omega)V(\omega) = b(\omega)V(T\omega)$. This easily yields (11). As the left hand side of this equation is invertible, the right hand side is invertible as well. This shows that a and b do not vanish anywhere.

Ad Step 3. Direct calculations show that the functions in question do not change if $U(\omega)$ or $V(\omega)$ or both are replaced by $-U(\omega)$ resp. $-V(\omega)$. By Step 1, such a replacement can be used to provide a version of V and U continuous around an arbitrary $\omega \in \Omega$. This gives the desired continuity.

Ad Step 4. As $\|C\|$ and $\|C^{-1}\|$ are continuous by Step 3 and Ω is compact, there exists a constant $\kappa > 0$ with $\kappa \leq \|C(\omega)\|, \|C^{-1}(T\omega)\| \leq \kappa^{-1}$ for every $\omega \in \Omega$. Thus, uniformity of M^E will follow from uniformity of $\omega \mapsto C^{-1}(T\omega)M^E(\omega)C(\omega)$, which in turn will follow by Step 2 from uniformity of

$$\omega \mapsto D(\omega) \equiv \begin{pmatrix} |a|(\omega) & 0 \\ 0 & |b|(\omega) \end{pmatrix}.$$

As $|a|$ and $|b|$ are continuous by Step 3 and do not vanish by Step 2, the functions $\ln|a|, \ln|b| : \Omega \longrightarrow \mathbb{R}$ are continuous. The desired uniformity of D follows now by Proposition 3.1 (see proof of Lemma 3.2 for similar reasoning). Positivity of $\gamma(E)$ is immediate from Step 1. □

A simple but crucial step in the proof of Theorem 2 is to relate the transfer matrices to subadditive functions. This will allow us to use Lemma 3.5 to show that the uniformity assumption of Lemma 3.2 and Lemma 3.3 holds for subshifts satisfying (PW). We proceed as follows. Let (Ω, T) be a strictly ergodic subshift and let $E \in \mathbb{R}$ be given. To the matrix valued function M^E we associate the function $F^E : \mathcal{W} \longrightarrow \mathbb{R}$ by setting

$$F^E(x) \equiv \log\|M^E(|x|, \omega)\|,$$

where $\omega \in \Omega$ is arbitrary with $\omega(1)\cdots\omega(|x|) = x$. It is not hard to see that this is well defined. Moreover, by submultiplicativity of the norm $\|\cdot\|$, we infer that F^E satisfies $F^E(xy) \leq F^E(x) + F^E(y)$.

Proposition 4.4. *M^E is uniform if and only if the limit* $\lim_{|x|\to\infty} \frac{F^E(x)}{|x|}$ *exists.*

Proof. This is straightforward. □

Now, we can prove the results stated in Sect. 2.

Proof of Theorem 1. The implication (i)$\Longrightarrow$(ii) is an immediate consequence of Lemma 4.1. This lemma also shows continuity of the Lyapunov exponent. The implication (ii)$\Longrightarrow$(i) follows from Lemma 4.2 and Lemma 4.3. □

Proof of Corollary 2.1. As Σ is closed and has no discrete points by general principles on random operators, the Cantor property will follow if Σ has measure zero. But this follows from the assumption and Theorem 1, as the set $\{E \in \mathbb{R} : \gamma(E) = 0\}$ has measure zero by the results of Kotani theory discussed in the introduction. □

Proof of Theorem 2. This is immediate from Lemma 3.5 and Proposition 4.4. □

5. Further Discussion

In this section we will present some comments on the results proven in the previous sections.

As shown in the introduction and the proof of Theorem 1, the problem $(\mathcal{Z})$ for subshifts can essentially be reduced to establishing the inclusion $\Sigma \subset \{E \in \mathbb{R} : \gamma(E) = 0\}$. This has been investigated for various models by various authors [5–7, 13, 19, 42]. All these proofs rely on the same tool viz trace maps (see [1, 9] for study of trace maps as well). Trace maps are very powerful as they capture the underlying hierarchical structures. Besides being applicable in the investigation of $(\mathcal{Z})$, trace maps are extremely useful because

- trace map bounds are an important tool to prove absence of eigenvalues.

Actually, most of the cited literature studies both $(\mathcal{A})$ and $(\mathcal{Z})$. In fact, $(\mathcal{Z})$ can even be shown to follow from a strong version of $(\mathcal{A})$ [19] (cf. [13] as well). While this makes the trace map approach to $(\mathcal{Z})$ very attractive, it has two drawbacks:

- The analysis of the actual trace maps may be quite hard or even impossible.
- The trace map formalism only applies to substitution-like subshifts.

Thus, trace map methods can not be expected to establish zero-measure spectrum in a generality comparable to the validity of the underlying Kotani result.

Let us now compare this with the method presented above. Essentially, our method has a complementary profile: It does not seem to give information concerning absence of eigenvalues. But on the other hand it only requires a weak ergodic type condition. This condition is met by subshifts satisfying (PW) and this class of subshifts contains all primitive substitutions. In particular, it gives information on the Rudin-Shapiro substitution which so far had been unattainable. Moreover, quite likely, the condition (PW) will be satisfied for certain circle maps, where $(\mathcal{Z})$ could not be proven by other means.

All the same, it seems worthwhile pointing out that (PW) does not contain the class of Sturmian systems whose rotation number has an unbounded continued fraction expansion. This is in fact the only class known to satisfy $(\mathcal{Z})$ (and much more [6, 12, 15–17, 27, 28, 41]) not covered by (PW). For this class, one can use the implication (ii) $\Longrightarrow$ (i) of Theorem 1, to conclude uniform existence of the Lyapunov exponent as done in Corollary 2.2. Still it seems desirable to give a direct proof of uniform existence of the Lyapunov exponent for these systems.

Finally, let us give the following strengthening of (the proof of) Theorem 1. It may be of interest whenever the strictly ergodic system is not a subshift.

Theorem 3. *Let (Ω, T) be strictly ergodic. Then,*

$$\Sigma = \{E \in \mathbb{R} : \gamma(E) = 0\} \cup \{E \in \mathbb{R} : \ M^E \textit{ is not uniform}\},$$

where the union is disjoint.

Proof. The union is disjoint by Lemma 4.2. The inclusion "$\supset$" follows from Lemma 4.3.

To prove the inclusion "$\subset$", let $E \in \mathbb{R}$ with M^E uniform and $\gamma(E) > 0$ be given. By Lemma 3.3, we infer positivity of the Lyapunov exponent for all $F \in \mathbb{R}$ close to E. Moreover, by Theorem 4 of [21], for $F \in \mathbb{R}$ with $\gamma(F) > 0$, uniformity of M^F is equivalent to existence of an $n \in \mathbb{N}$ and a continuous $C : \Omega \longrightarrow GL(2, \mathbb{R})$ such that all entries of $C(T^n\omega)^{-1}M^F(n,\omega)C(\omega)$ are positive for all $\omega \in \Omega$. By uniformity of M^E this latter condition holds for M^E. By continuity of $(F,\omega) \mapsto C(T^n\omega)^{-1}M^F(n,\omega)C(\omega)$ and compactness of Ω, it must then hold for M^F as well whenever F is sufficiently close to E.

These considerations prove existence of an open interval $I \subset \mathbb{R}$ containing E on which uniformity of the transfer matrices and positivity of the Lyapunov exponent hold (cf. top of p. 811 of [21] for related arguments). Now, replacing Γ^c with I, one can easily adopt the proof of Lemma 4.1 to obtain the desired inclusion. □

Note added. After this work was completed, we learned about the very recent preprint "Measure Zero Spectrum of a Class of Schrödinger Operators" by Liu–Tan–Wen–Wu (mp-arc 01-189). They present a detailed and thorough analysis of trace maps for primitive substitutions. Based on this analysis, they establish $(\mathcal{Z})$ for all primitive substitutions thereby extending the approach developed in [5,7,9,41].

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