On the Inverse Scattering Problem for Jacobi Matrices with the Spectrum on an Interval, a Finite System of Intervals or a Cantor Set of Positive Length

A. Volberg, P. Yuditskii

Dept. of Mathematics, Michigan State University, East Lansing, MI 48824, USA.
E-mail: volberg@math.msu.edu; yuditski@math.msu.edu

Abstract: When solving the inverse scattering problem for a discrete Sturm–Liouville operator with a rapidly decreasing potential, one gets reflection coefficients $s_\pm$ and invertible operators $I + \mathcal{H}_{s_\pm}$, where $\mathcal{H}_{s_\pm}$ is the Hankel operator related to the symbol $s_\pm$. The Marchenko–Faddeev theorem [8] (in the continuous case, for the discrete case see [4, 6]), guarantees the uniqueness of the solution of the inverse scattering problem. In this article we ask the following natural question – can one find a precise condition guaranteeing that the inverse scattering problem is uniquely solvable and that operators $I + \mathcal{H}_{s_\pm}$ are invertible? Can one claim that uniqueness implies invertibility or vise versa?

Moreover, we are interested here not only in the case of decreasing potential but also in the case of asymptotically almost periodic potentials. So we merge here two mostly developed cases of the inverse problem for Sturm–Liouville operators: the inverse problem with (almost) periodic potential and the inverse problem with the fast decreasing potential.

Main Results

The asymptotics of polynomials orthogonal on a homogeneous set, which we described earlier [10], indicated strongly that there should be a scattering theory for Jacobi matrices with an almost periodic background as it exists in the classical case of a constant background. Note that in this case left and right asymptotics are not necessarily the same almost periodic coefficient sequences, but they are of the same spectral class. In this work, we present all principal ingredients of such a theory: reflection/transmission coefficients, Gelfand–Levitan–Marchenko transformation operators, a Riemann–Hilbert problem related to the inverse scattering problem. Now we can say finally that the reflectionless Jacobi matrices with homogeneous spectrum are those whose reflection coefficient is zero.

Moreover, we extend the theory in depth and show that a reflection coefficient determines uniquely a Jacobi matrix of the Szegő class, and both transformation operators are invertible if and only if the spectral density satisfies the matrix A_2 condition [13].

Concerning the A_2 condition in the inverse scattering, we have to mention, at least as indirect references, [9, Chapter 2, Sect. 4] and [2]. Generally references to stationary scattering and inverse scattering problems in connection with spatial asymptotics can be found in [5], where explicit expressions of reflection and transmission coefficients in terms of Weyl functions and phases, asymptotic wave functions were given. Reference [12] gives a complete introduction to Jacobi operators, their spectral and perturbation theories.

Let J be a Jacobi matrix defining a bounded self–adjoint operator on $l^2(\mathbb{Z})$:

$$Je_n = p_n e_{n-1} + q_n e_n + p_{n+1} e_{n+1}, \quad n \in \mathbb{Z}, \tag{0.1}$$

where $\{e_n\}$ is the standard basis in $l^2(\mathbb{Z})$, $p_n > 0$. The resolvent matrix-function is defined by the relation

$$R(z) = R(z, J) = \mathcal{E}^*(J - z)^{-1}\mathcal{E}, \tag{0.2}$$

where $\mathcal{E} : \mathbb{C}^2 \to l^2(\mathbb{Z})$ in such a way that

$$\mathcal{E} \begin{bmatrix} c_{-1} \\ c_0 \end{bmatrix} = e_{-1}c_{-1} + e_0 c_0.$$

This matrix-function possesses an integral representation

$$R(z) = \int \frac{d\sigma}{x - z} \tag{0.3}$$

with a 2×2 matrix-measure having compact support on $\mathbb{R}$. J is unitary equivalent to the operator multiplication by an independent variable on

$$L^2_{d\sigma} = \left\{ f = \begin{bmatrix} f_{-1}(x) \\ f_0(x) \end{bmatrix} : \int f^* d\sigma \, f < \infty \right\}.$$

The spectrum of J is called absolutely continuous if the measure $d\sigma$ is absolutely continuous with respect to the Lebesgue measure on the real axis,

$$d\sigma(x) = \rho(x) \, dx. \tag{0.4}$$

Let J_0 be a Jacobi matrix with constant coefficients, $p_n = 1$, $q_n = 0$ (the so-called Chebyshev matrix). It has the following functional representation, besides the general one mentioned above. Note that the resolvent set of J_0 is the domain $\bar{\mathbb{C}} \setminus [-2, 2]$. Let $z(\zeta) : \mathbb{D} \to \bar{\mathbb{C}} \setminus [-2, 2]$ be a uniformization of this domain, $z(\zeta) = 1/\zeta + \zeta$. With respect to the standard basis $\{t^n\}_{n \in \mathbb{Z}}$ in

$$L^2 = \left\{ f(t) : \int_{\mathbb{T}} |f|^2 \, dm \right\},$$

the matrix of the operator of multiplication by $z(t)$, $t \in \mathbb{T}$, is the Jacobi matrix J_0, since $z(t)t^n = t^{n-1} + t^{n+1}$.

The famous Bernstein–Szegő theorem implies the following proposition (for a matrix modification of the Szegő condition, see [1]).

Proposition 0.1. *Let J be a Jacobi matrix whose spectrum is an interval $[-2, 2]$. Assume that the spectrum is absolutely continuous and the density of the spectral measure satisfies the condition*

$$\log \det \rho(z(t)) \in L^1. \tag{0.5}$$

Then

$$p_n \to 1, \quad q_n \to 0, \quad n \to \pm\infty. \tag{0.6}$$

Moreover, there exist generalized eigenvectors

$$p_n e^+(n-1, t) + q_n e^+(n, t) + p_{n+1} e^+(n+1, t) = z(t) e^+(n, t),$$
$$p_n e^-(-n, t) + q_n e^-(-n-1, t) + p_{n+1} e^-(-n-2, t) = z(t) e^-(-n-1, t), \tag{0.7}$$

such that the following asymptotics hold true:

$$s(t) e^{\pm}(n, t) = s(t) t^n + o(1), \quad n \to +\infty,$$
$$s(t) e^{\pm}(n, t) = t^n + s_{\mp}(t) t^{-n-1} + o(1), \quad n \to -\infty \tag{0.8}$$

in L^2.

To clarify the meaning of the words "generalized eigenvectors", we need some definitions and notation.

The matrix

$$S(t) = \begin{bmatrix} s_- & s \\ s & s_+ \end{bmatrix} (t) \tag{0.9}$$

is called the scattering matrix-function. It is a unitary-valued matrix-function with the following symmetry property:

$$S^*(\bar{t}) = S(t), \tag{0.10}$$

and analytic property:

$$s(t) \text{ is boundary values of an outer function.} \tag{0.11}$$

We still denote by $s(\zeta)$, $\zeta \in \mathbb{D}$, the values of the function inside the disk, and subsequently, we assume that s meets the normalization condition $s(0) > 0$.

In fact, this means that each of the entries $s_\pm$ (the so-called reflection coefficient) determines the matrix $S(t)$ in a unique way. Indeed, since

$$|s(t)|^2 + |s_\pm(t)|^2 = 1, \tag{0.12}$$

using (0.11), we have

$$s(\zeta) = e^{\frac{1}{2} \int_{\mathbb{T}} \frac{t+\zeta}{t-\zeta} \log\{1 - |s_\pm(t)|^2\} \, dm}.$$

Then, we can solve for $s_\mp$ the relation

$$\bar{s}_+ s + \bar{s} s_- = 0. \tag{0.13}$$

With the function $s_\pm$ we associate the metric

$$||f||^2_{s_\pm} = \frac{1}{2}\left\langle \begin{bmatrix} 1 & \overline{s_\pm(t)} \\ s_\pm(t) & 1 \end{bmatrix} \begin{bmatrix} f(t) \\ \bar{t}f(\bar{t}) \end{bmatrix}, \begin{bmatrix} f(t) \\ \bar{t}f(\bar{t}) \end{bmatrix} \right\rangle$$

$$= \langle f(t) + \bar{t}(s_\pm f)(\bar{t}),\, f(t)\rangle, \quad f \in L^2.$$

Note that the conditions (0.11), (0.12) guarantee that $||f||_{s_\pm} = 0$ implies $f = 0$. We denote by $L^2_{dm,s_\pm}$ or $L^2_{s_\pm}$ (for shortness) the closure of L^2 with respect to this new metric.

The following relation sets a unitary map from $L^2_{s_+}$ to $L^2_{s_-}$:

$$s(t)f^-(t) = \bar{t}f^+(\bar{t}) + s_+(t)f^+(t),$$

moreover, in this case,

$$||f^+||^2_{s_+} = ||f^-||^2_{s_-} = \frac{1}{2}\{||sf^+||^2 + ||sf^-||^2\},$$

and the inverse map is of the form

$$s(t)f^+(t) = \bar{t}f^-(\bar{t}) + s_-(t)f^-(t).$$

We say that a Jacobi matrix J with the spectrum $[-2, 2]$ is of Szegö class if its spectral measure $d\sigma$ satisfies (0.4), (0.5).

Theorem 0.1. *Let J be a Jacobi matrix of Szegö class with the spectrum $E = [-2, 2]$. Then J possesses the scattering representation, i.e.: there exists a unique unitary-valued matrix-function $S(t)$ of the form (0.9) with the properties: (0.10), (0.11), and a unique pair of Fourier transforms*

$$\mathcal{F}^\pm : l^2(\mathbb{Z}) \to L^2_{s_\pm}, \quad (\mathcal{F}^\pm Jf)(t) = z(t)(\mathcal{F}^\pm f)(t), \tag{0.14}$$

determining each other by the relations

$$s(t)(\mathcal{F}^\pm f)(t) = \bar{t}(\mathcal{F}^\mp f)(\bar{t}) + s_\mp(t)(\mathcal{F}^\mp f)(t), \tag{0.15}$$

and having the following analytic properties

$$s\mathcal{F}^\pm(l^2(\mathbb{Z}_\pm)) \subset H^2, \tag{0.16}$$

and asymptotic properties

$$e^\pm(n, t) = t^n + o(1) \quad in\ L^2_{s_\pm}, \quad n \to +\infty, \tag{0.17}$$

where

$$e^+(n, t) = (\mathcal{F}^+ e_n)(t), \quad e^-(n, t) = (\mathcal{F}^- e_{-n-1})(t).$$

(*As before, $\{e_n\}$ is the standard basis in $l^2(\mathbb{Z})$*).

Remark 0.1. Show that (0.17) is equivalent to (0.8). Due to

$$\begin{bmatrix} 1 & \bar{s}_\pm \\ s_\pm & 1 \end{bmatrix} = \begin{bmatrix} |s|^2 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \bar{s}_\pm \\ 1 \end{bmatrix} \begin{bmatrix} s_\pm & 1 \end{bmatrix}, \tag{0.18}$$

(0.17) is equivalent to $(n \to +\infty)$

$$s(t)e^\pm(n, t) = s(t)t^n + o(1) \quad \text{in } L^2,$$
$$s_\pm(t)e^\pm(n, t) + \bar{t}e^\pm(n, \bar{t}) = s_\pm(t)t^n + \bar{t}^{n+1} + o(1) \quad \text{in } L^2.$$

Using (0.15), we rewrite the second relation into the form

$$s(t)e^\mp(-n - 1, t) = t^{-n-1} + s_\pm(t)t^n + o(1) \quad \text{in } L^2.$$

Substituting $n := -n - 1$, we get the second relation of (0.8).

A fundamental question is how to recover the Jacobi matrix from the scattering matrix, in fact, from the reflection coefficient s_+ (or s_-)? When can this be done? Do we have a uniqueness theorem?

We show that for an arbitrary function $s_+(t)$ satisfying

$$\overline{s_+(\bar{t})} = s_+(t) \quad \text{and} \quad \log\{1 - |s_+(t)|^2\} \in L^1, \tag{0.19}$$

there exists a Jacobi matrix J of Szegö class such that $s_+(t)$ is its reflection coefficient in the scattering representation. However we can construct a matrix with this property, at least, in two different ways.

First, consider the space

$$H^2_{s_+} = \operatorname{clos}_{L^2_{s_+}} H^2,$$

and introduce the Hankel operator $\mathcal{H}_{s_+} : H^2 \to H^2$,

$$\mathcal{H}_{s_+} f = P_+ \bar{t}(s_+ f)(\bar{t}), \quad f \in H^2,$$

where P_+ is the Riesz projection from L^2 onto H^2. This operator determines the metric in $H^2_{s_+}$:

$$\|f\|^2_{s_+} = \langle f(t) + \bar{t}(s_+ f)(\bar{t}), f(t) \rangle$$
$$= \langle (I + \mathcal{H}_{s_+})f, f \rangle, \quad \forall f \in H^2.$$

Lemma 0.1. *Under the assumptions (0.19), the space $H^2_{s_+}$ is a space of holomorphic functions with a reproducing kernel. Moreover, $sf \in H^2$ for any $f \in H^2_{s_+}$, and the reproducing vector k_{s_+}:*

$$\langle f, k_{s_+} \rangle = f(0), \quad \forall f \in H^2_{s_+},$$

is of the form

$$k_{s_+} = (I + \mathcal{H}_{s_+})^{[-1]}\mathbf{1} := \lim_{\epsilon \to 0^+} (\epsilon + I + \mathcal{H}_{s_+})^{-1}\mathbf{1} \quad \text{in } L^2_{s_+}. \tag{0.20}$$

Put $K_{s_+}(t) = k_{s_+}(t)/\sqrt{k_{s_+}(0)}$.

Theorem 0.2. *Let $s_+(t)$ satisfy (0.19). Then the system of functions $\{t^n K_{s_+ t^{2n}}(t)\}_{n\in\mathbb{Z}}$ forms an orthonormal basis in $L^2_{s_+}$. With respect to this basis, operator multiplication by $z(t)$ is a Jacobi matrix J of Szegö class. Moreover, the initial function $s_+(t)$ is the reflection coefficient of the scattering matrix-function $S(t)$, associated to J by Theorem 0.1, and*

$$e^+(n, t) = t^n K_{s_+ t^{2n}}(t).$$

On the other hand, the system of functions $\{t^n K_{s_- t^{2n}}(t)\}_{n\in\mathbb{Z}}$ forms an orthonormal basis in $L^2_{s_-}$, and we are able to define a Jacobi matrix $\tilde{J}$ by the relation

$$z(t)\tilde{e}^+(n, t) = \tilde{p}_n \tilde{e}^+(n-1, t) + \tilde{q}_n \tilde{e}^+(n, t) + \tilde{p}_{n+1}\tilde{e}^+(n+1, t),$$

where $\{\tilde{e}^+(n, t)\}$ is the dual system to the system $\{t^n K_{s_- t^{2n}}(t)\}$ (see (0.15)), i.e.:

$$s(t)\tilde{e}^+(-n-1, t) = \bar{t}^{n+1} K_{s_- t^{2n}}(\bar{t}) + s_-(t)t^n K_{s_- t^{2n}}(t).$$

Even the invertibility condition for the operators $(I + \mathcal{H}_{s_\pm})$ does not guarantee that operators J and $\tilde{J}$ are the same (see the *Example* at the end of Sect. 2). But if $J = \tilde{J}$, then the uniqueness theorem takes place.

Theorem 0.3. *Let s_+ satisfy (0.19). Then the reflection coefficient s_+ determines a Jacobi matrix J of Szegö class in a unique way if and only if the following relations take place*

$$s(0)K_{s_\pm}(0)K_{s_\mp t^{-2}}(0) = 1. \tag{0.21}$$

Corollary 0.1. *Let J be a Jacobi matrix of Szegö class with the spectrum $[-2, 2]$ and let ρ be the density of its spectral measure. If*

$$\int_{-2}^{2} \rho^{-1}(x)\, dx < \infty,$$

then there is no other Jacobi matrix of Szegö class with the same scattering matrix– function $S(t)$.

It is important to know when the operators $(I + \mathcal{H}_{s_\pm})$, playing a central role in the inverse scattering problem, are invertible in the proper sense of the word.

Theorem 0.4. *Let J be a Jacobi matrix of Szegö class with the spectrum $[-2, 2]$. Let ρ be the density of its spectral measure and let s_+ be the reflection coefficient of its scattering matrix-function. Then the following statements are equivalent.*

1. *The spectral density ρ satisfies condition A_2.*
2. *The reflection coefficient s_+ determines a Jacobi matrix of Szegö class uniquely and both operators $(I + \mathcal{H}_{s_\pm})$ are invertible.*

To extend these results to the case when a spectrum E is a finite system of intervals or a standard Cantor set of positive measure [15], see also [3], we need only to introduce a counterpart of the Hardy space.

Let $z(\zeta) : \mathbb{D} \to \Omega$ be a uniformization of the domain $\Omega = \bar{\mathbb{C}} \setminus E$. Thus there exists a discrete subgroup Γ of the group $SU(1, 1)$ consisting of elements of the form

$$\gamma = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}, \quad \gamma_{11} = \overline{\gamma_{22}}, \ \gamma_{12} = \overline{\gamma_{21}}, \ \det\gamma = 1,$$

such that $z(\zeta)$ is automorphic with respect to Γ, i.e., $z(\gamma(\zeta)) = z(\zeta)$, $\forall \gamma \in \Gamma$, and any two preimages of $z_0 \in \Omega$ are Γ–equivalent, i.e.,

$$z(\zeta_1) = z(\zeta_2) \implies \exists \gamma \in \Gamma : \zeta_1 = \gamma(\zeta_2).$$

We normalize $z(\zeta)$ by the conditions $z(0) = \infty$, $(\zeta z)(0) > 0$.

A character of Γ is a complex–valued function $\alpha : \Gamma \to \mathbb{T}$, satisfying

$$\alpha(\gamma_1 \gamma_2) = \alpha(\gamma_1)\alpha(\gamma_2), \quad \gamma_1, \gamma_2 \in \Gamma.$$

The characters form an Abelian compact group denoted by Γ^*.

For a given character $\alpha \in \Gamma^*$, as usual let us define

$$H^\infty(\Gamma, \alpha) = \{f \in H^\infty : f(\gamma(\zeta)) = \alpha(\gamma)f(\zeta), \ \forall \gamma \in \Gamma\}.$$

Generally, a group Γ is said to be of *Widom type* if for any $\alpha \in \Gamma^*$ the space $H^\infty(\Gamma, \alpha)$ is not trivial (contains a non-constant function).

A group of Widom type acts dissipatively on $\mathbb{T}$ with respect to dm, that is there exists a measurable (fundamental) set $\mathbb{E}$, which does not contain any two Γ-equivalent points, and the union $\cup_{\gamma \in \Gamma}\gamma(\mathbb{E})$ is a set of full measure. We can choose $\mathbb{E}$ possessing the symmetry property: $t \in \mathbb{E} \Rightarrow \bar{t} \in \mathbb{E}$. For the space of square summable functions on $\mathbb{E}$ (with respect to the Lebesgue measure), we use the notation $L^2_{dm|\mathbb{E}}$.

Let f be an analytic function in $\mathbb{D}$, $\gamma \in \Gamma$ and $k \in \mathbb{N}$. Then we put

$$f|[\gamma]_k = \frac{f(\gamma(\zeta))}{(\gamma_{21}\zeta + \gamma_{22})^k}.$$

Notice that $f|[\gamma]_2 = f \ \forall \gamma \in \Gamma$ means that the form $f(\zeta)d\zeta$ is invariant with respect to the substitutions $\zeta \to \gamma(\zeta)$ ($f(\zeta)d\zeta$ is an Abelian integral on $\mathbb{D}/\Gamma$). Analogically, $f|[\gamma] = \alpha(\gamma)f \ \forall \gamma \in \Gamma$ means that the form $|f(\zeta)|^2 |d\zeta|$ is invariant with respect to these substitutions.

We recall that a function $f(\zeta)$ is of Smirnov class, if it can be represented as a ratio of two functions from H^∞ with an outer denominator.

Definition. *Let Γ be a group of Widom type. The space $A_1^2(\Gamma, \alpha)$ $(A_2^1(\Gamma, \alpha))$ is formed by functions f, which are analytic on $\mathbb{D}$ and satisfy the following three conditions*

    1)   *f is of Smirnov class,*

    2) $f|[\gamma] = \alpha(\gamma)f$ $(f|[\gamma]_2 = \alpha(\gamma)f)$ $\forall \gamma \in \Gamma$,

    3) $\displaystyle\int_{\mathbb{E}} |f|^2 \, dm < \infty$ $\left(\displaystyle\int_{\mathbb{E}} |f| \, dm < \infty\right)$.

$A_1^2(\Gamma, \alpha)$ is a Hilbert space with the reproducing kernel $k^\alpha(\zeta, \zeta_0)$, moreover

$$0 < \inf_{\alpha \in \Gamma^*} k^\alpha(\zeta_0, \zeta_0) \le \sup_{\alpha \in \Gamma^*} k^\alpha(\zeta_0, \zeta_0) < \infty. \tag{W}$$

Put

$$k^\alpha(\zeta) = k^\alpha(\zeta, 0) \quad \text{and} \quad K^\alpha(\zeta) = \frac{k^\alpha(\zeta)}{\sqrt{k^\alpha(0)}}.$$

We need one more special function. The Blaschke product

$$b(\zeta) = \zeta \prod_{\gamma \in \Gamma, \gamma \neq 1_2} \frac{\gamma(0) - \zeta}{1 - \overline{\gamma(0)}\zeta} \frac{|\gamma(0)|}{\gamma(0)}$$

is called the *Green's function* of Γ with respect to the origin. It is a character-automorphic function, i.e., there exists $\mu \in \Gamma^*$ such that $b \in H^\infty(\Gamma, \mu)$. Note, if $G(z) = G(z, \infty)$ denotes the Green's function of the domain Ω, then

$$G(z(\zeta)) = -\log|b(\zeta)|.$$

Theorem ([7]). *Let Γ be a group of Widom type. The following statements are equivalent:*

(1) *The function $K^\alpha(0)$ is continuous on Γ^*.*
(2) $\sup\{|f(0)| : f \in H^\infty(\Gamma, \alpha), \|f\| \leq 1\} \to 1, \quad \alpha \to 1_{\Gamma^*}.$
(3) *The Direct Cauchy Theorem holds:*

$$\int_{\mathbb{E}} \frac{f}{b}(t) \frac{dt}{2\pi i} = \frac{f}{b'}(0), \quad \forall f \in A_2^1(\Gamma, \mu). \tag{DCT}$$

(4) *Let $\overline{t A_1^2(\Gamma, \alpha^{-1})} = \{g = \overline{t f} : f \in A_1^2(\Gamma, \alpha^{-1})\}$. Then*

$$L_{dm|\mathbb{E}}^2 = \overline{t A_1^2(\Gamma, \alpha^{-1})} \oplus A_1^2(\Gamma, \alpha) \quad \forall \alpha \in \Gamma^*.$$

(5) *Every invariant subspace $M \subset A_1^2(\Gamma, \alpha)$ (i.e. $\phi M \subset M \ \forall \phi \in H^\infty(\Gamma)$) is of the form*

$$M = \Delta A_1^2(\Gamma, \beta^{-1}\alpha)$$

for some character-automorphic inner function $\Delta \in H^\infty(\beta)$.

Definition ([3]). *A measurable set E is homogeneous if there is an $\eta > 0$ such that*

$$|(x - \delta, x + \delta) \cap E| \geq \eta\delta \quad \text{for all } 0 < \delta < 1 \quad \text{and all } x \in E. \tag{C}$$

A standard Cantor set of positive length is an example of a homogeneous set [3], see also [10]. Let E be a homogeneous set, then the domain $\Omega = \bar{C} \setminus E$ (respectively the group Γ) is of Widom type and the Direct Cauchy Theorem holds.

Recall that a sequence of real numbers $\{p_n\} \in l^\infty(\mathbb{Z})$ is called uniformly almost periodic if the set of sequences $\{\{p_{n+l}\}, l \in \mathbb{Z}\}$ is a precompact in $l^\infty(\mathbb{Z})$. The general way to produce a sequence of this type looks as follows: let $\mathcal{G}$ be a compact Abelian group, and let $f(g)$ be a continuous function on $\mathcal{G}$, then

$$p_n := f(g_0 + ng_1), \quad g_0, g_1 \in \mathcal{G},$$

is an almost periodic sequence. A Jacobi matrix is almost periodic if the coefficient sequences are almost periodic. We denote by $J(E)$ the class of almost periodic Jacobi matrices with absolutely continuous homogeneous spectrum E. In fact, if $E = [-2, 2]$ then $J(E) = \{J_0\}$. In what follows the class $J(E)$ will substitute the Chebyshev matrix in the case when the spectrum E is not an interval but a general homogeneous set. First of all this class can be described as follows.

Theorem ([11]). *Let E be a homogeneous set. Let $z : \mathbb{D} \to \bar{\mathbb{C}} \setminus E$ be a uniformizing mapping. Then the systems of functions $\{b^n K^{\alpha \mu^{-n}}\}_{n \in \mathbb{Z}_+}$ and $\{b^n K^{\alpha \mu^{-n}}\}_{n \in \mathbb{Z}}$ form an orthonormal basis in $A_1^2(\Gamma, \alpha)$ and in $L_{dm|\mathbb{E}}^2$, respectively, for any $\alpha \in \Gamma^*$. With respect to this basis, the operator multiplication by $z(t)$ is a three–diagonal almost periodic Jacobi matrix $J(\alpha)$. Moreover,*

$$J(E) = \{J(\alpha) : \; \alpha \in \Gamma^*\},$$

and $J(\alpha)$ is a continuous function on Γ^*.

We say that a Jacobi matrix J with the spectrum E is of Szegö class if its spectral measure is absolutely continuous, $d\sigma(x) = \rho(x) \, dx$, and $\rho(z(t))$ satisfies (0.5).

Theorem 0.5. *Let J be a Jacobi matrix of Szegö class with a homogeneous spectrum E. Then J possesses the scattering representation, i.e.: there exists a unique unitary-valued matrix-function $S(t)$ of the form (0.9) with the properties (0.10), (0.11), and a unique pair of Fourier transforms*

$$\mathcal{F}^{\pm} : l^2(\mathbb{Z}) \to L_{dm|\mathbb{E}, s_{\pm}}^2, \quad (\mathcal{F}^{\pm} J f)(t) = z(t)(\mathcal{F}^{\pm} f)(t), \tag{0.22}$$

determining each other by the relations

$$s(t)(\mathcal{F}^{\pm} f)(t) = \bar{t}(\mathcal{F}^{\mp} f)(\bar{t}) + s_{\mp}(t)(\mathcal{F}^{\mp} f)(t), \tag{0.23}$$

and having the following analytic properties:

$$s\mathcal{F}^{\pm}(l^2(\mathbb{Z}_{\pm})) \subset A_1^2(\Gamma, \alpha_{\mp}^{-1}), \tag{0.24}$$

and asymptotic properties

$$e^{\pm}(n, t) = b^n(t) K^{\alpha_{\pm} \mu^{-n}}(t) + o(1) \quad in \; L_{dm|\mathbb{E}, s_{\pm}}^2, \quad n \to +\infty, \tag{0.25}$$

where

$$e^+(n, t) = (\mathcal{F}^+ e_n)(t), \quad e^-(n, t) = (\mathcal{F}^- e_{-n-1})(t),$$

and $L_{dm|\mathbb{E}, s_{\pm}}^2$ is the closure of the functions from $L_{dm|\mathbb{E}}^2$ with respect to the metric

$$||f||_{s_{\pm}}^2 = \frac{1}{2} \left\langle \begin{bmatrix} 1 & \overline{s_{\pm}(t)} \\ s_{\pm}(t) & 1 \end{bmatrix} \begin{bmatrix} f(t) \\ \bar{t} f(\bar{t}) \end{bmatrix}, \begin{bmatrix} f(t) \\ \bar{t} f(\bar{t}) \end{bmatrix} \right\rangle, \quad f \in L_{dm|\mathbb{E}}^2.$$

Theorems 0.2–0.4 also have their closely parallel counterparts in the case when the spectrum is a homogeneous set, see Theorems 1.1, 2.1 and 3.1 in combination with Theorem 4.1.

We finish this paper with a remark on a connection between this new type inverse scattering problem and a Riemann–Hilbert problem.

1. In the Model Space

Let E be a homogeneous set. Let $z(\zeta) : \mathbb{D}/\Gamma \sim \bar{\mathbb{C}} \setminus E$ be a uniformization and $b(\zeta)$ be the Green's function. Throughout the paper we assume that $(bz)(0) = 1$. Let $\mathbb{E} \subset \mathbb{T}$ be a symmetric fundamental set ($t \in \mathbb{E} \Rightarrow \bar{t} \in \mathbb{E}$).

With a function $s_+(t) \in L^\infty_{dm|\mathbb{E}}$ such that

$$\overline{s_+(\bar{t})} = s_+(t) \quad \text{and} \quad 1 - |s_+(t)|^2 > 0 \text{ a.e. on } \mathbb{E}, \tag{1.1}$$

we associate the metric

$$\|f\|^2_{s_+} = \frac{1}{2}\left\langle \begin{bmatrix} 1 & \overline{s_+(t)} \\ s_+(t) & 1 \end{bmatrix} \begin{bmatrix} f(t) \\ \bar{t}f(\bar{t}) \end{bmatrix}, \begin{bmatrix} f(t) \\ \bar{t}f(\bar{t}) \end{bmatrix} \right\rangle$$

$$= \langle f(t) + \bar{t}(s_+f)(\bar{t}), f(t) \rangle, \quad f \in L^2_{dm|\mathbb{E}}.$$

Condition (1.1) guarantee that $\|f\|_{s_+} = 0$ implies $f = 0$. We denote by $L^2_{dm|\mathbb{E},s_+}$ or $L^2_{s_+}$ (for shortness) the closure of $L^2_{dm|\mathbb{E}}$ with respect to this metric.

Lemma 1.1. *The operator multiplication by $z(t)$ in $L^2_{s_+}$ is unitary equivalent to the operator multiplication by $z(t)$ in $L^2_{dm|\mathbb{E}}$.*

Proof. Let us put

$$\begin{bmatrix} g(t) \\ \bar{t}g(\bar{t}) \end{bmatrix} = \begin{bmatrix} 1 & \overline{s_+(t)} \\ s_+(t) & 1 \end{bmatrix}^{1/2} \begin{bmatrix} f(t) \\ \bar{t}f(\bar{t}) \end{bmatrix}, \quad f \in L^2_{dm|\mathbb{E}}.$$

In this case $\|f\|_{s_+} = \|g\|$. The system of identities

$$\begin{bmatrix} 1 & \overline{s_+(t)} \\ s_+(t) & 1 \end{bmatrix}^{1/2} \begin{bmatrix} (zf)(t) \\ \bar{t}(zf)(\bar{t}) \end{bmatrix} = \begin{bmatrix} 1 & \overline{s_+(t)} \\ s_+(t) & 1 \end{bmatrix}^{1/2} z(t) \begin{bmatrix} f(t) \\ \bar{t}f(\bar{t}) \end{bmatrix}$$

$$= z(t) \begin{bmatrix} g(t) \\ \bar{t}g(\bar{t}) \end{bmatrix} = \begin{bmatrix} (zg)(t) \\ \bar{t}(zg)(\bar{t}) \end{bmatrix}$$

finishes the proof. ☐

Let $\alpha_+ \in \Gamma^*$. Further, we assume that $s_+ \in L^\infty(\Gamma, \alpha_+^{-2})$ and

$$\log(1 - |s_+(t)|^2) \in L^1. \tag{1.2}$$

We define an outer function s, $s(0) > 0$, by the relation

$$|s(t)|^2 = 1 - |s_+(t)|^2, \quad t \in \mathbb{T}.$$

It is a character-automorphic function such that $\overline{s(\bar{t})} = s(t)$. It is convenient to denote its character by $\alpha_+^{-1}\alpha_-^{-1}$, i.e., $s \in H^\infty(\Gamma, \alpha_+^{-1}\alpha_-^{-1})$.

Let us discuss some properties of the space

$$H^2_{s_+}(\alpha_+) := \text{clos}_{L^2_{s_+}} A^2_1(\Gamma, \alpha_+).$$

First of all, we define "a Hankel operator" $\mathcal{H}_{s_+} : A_1^2(\Gamma, \alpha_+) \to A_1^2(\Gamma, \alpha_+)$,

$$\mathcal{H}_{s_+} f = P_{A_1^2(\Gamma, \alpha_+)} \bar{t}(s_+ f)(\bar{t}).$$

Note that this operator, indeed, does not depend on "an analytical part" of its symbol, more precisely,

$$\mathcal{H}_{(s_+ + \epsilon)} = \mathcal{H}_{s_+}, \quad \forall \epsilon \in H^\infty(\Gamma, \alpha_+^{-2}).$$

Besides, in the classical case $E = [-2, 2]$, $\Gamma = \{1_2\}$, $\mathbb{E} = \mathbb{T}$, with a function

$$s_+(t) = \sum_{n \in \mathbb{Z}} a_n t^n$$

is associated the operator $\mathcal{H}_{s_+} : H^2 \to H^2$ having the representation

$$\mathcal{H}_{s_+} = \begin{bmatrix} a_{-1} \ a_{-2} \ a_{-3} \ \cdots \\ a_{-2} \ a_{-3} \ \cdots \\ a_{-3} \ \cdots \\ \cdots \end{bmatrix}$$

with respect to the standard basis $\{t^n\}_{n \in \mathbb{Z}_+}$ in H^2.

The operator $\mathcal{H}_{s_+}$ determines the metric in $H_{s_+}^2(\alpha_+)$:

$$\begin{aligned} ||f||_{s_+}^2 &= \langle f(t) + \bar{t}(s_+ f)(\bar{t}), \ f(t) \rangle \\ &= \langle (I + \mathcal{H}_{s_+}) f, f \rangle, \quad f \in A_1^2(\Gamma, \alpha_+). \end{aligned}$$

Lemma 1.2. *Under the assumptions (1.2), the space $H_{s_+}^2(\alpha_+)$ is a space of holomorphic functions with a reproducing kernel. Moreover, $sf \in A_1^2(\Gamma, \alpha_-^{-1})$ for any $f \in H_{s_+}^2(\alpha_+)$, and the reproducing vector $k_{s_+}^{\alpha_+}$:*

$$\langle f, k_{s_+}^{\alpha_+} \rangle = f(0), \quad \forall f \in H_{s_+}^2(\alpha_+),$$

is of the form

$$k_{s_+}^{\alpha_+} = (I + \mathcal{H}_{s_+})^{[-1]} k^{\alpha_+} := \lim_{\epsilon \to 0^+} (\epsilon + I + \mathcal{H}_{s_+})^{-1} k^{\alpha_+} \quad in \ L_{s_+}^2. \quad (1.3)$$

Proof. From the inequality

$$\begin{bmatrix} 1 & \overline{s_+(t)} \\ s_+(t) & 1 \end{bmatrix} - \begin{bmatrix} |s(t)|^2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} |s_+(t)|^2 & \overline{s_+(t)} \\ s_+(t) & 1 \end{bmatrix} \geq 0, \quad (1.4)$$

it follows that

$$||sf||^2 \leq 2||f||_{s_+}^2 \quad \forall f \in L_{s_+}^2.$$

Thus, if a sequence $\{f_n\}$, $f_n \in A_1^2(\Gamma, \alpha_+)$, converges in $H_{s_+}^2(\alpha_+)$, then the sequence $\{sf_n\}$ converges in $A_1^2(\Gamma, \alpha_-^{-1})$. In the same way we have boundedness of the functional $f \to f(0)$,

$$|f(0)|^2 \leq \frac{1}{|s(0)|^2} |(sf)(0)|^2 \leq \frac{2}{|s(0)|^2} ||f||_{s_+}^2 k^{\alpha_-^{-1}}(0).$$

Let us prove (1.3). Let $\epsilon > 0$, then for the norm of the difference we have an estimate

$$\|k_{s_+}^{\alpha_+} - (\epsilon + I + \mathcal{H}_{s_+})^{-1}k^{\alpha_+}\|_{s_+}^2$$
$$= k_{s_+}^{\alpha_+}(0) - 2\{(\epsilon + I + \mathcal{H}_{s_+})^{-1}k^{\alpha_+}\}(0)$$
$$\quad + \langle(I + \mathcal{H}_{s_+})(\epsilon + I + \mathcal{H}_{s_+})^{-1}k^{\alpha_+}, (\epsilon + I + \mathcal{H}_{s_+})^{-1}k^{\alpha_+}\rangle$$
$$\leq k_{s_+}^{\alpha_+}(0) - \{(\epsilon + I + \mathcal{H}_{s_+})^{-1}k^{\alpha_+}\}(0). \tag{1.5}$$

Therefore,

$$\{(\epsilon + I + \mathcal{H}_{s_+})^{-1}k^{\alpha_+}\}(0) \leq k_{s_+}^{\alpha_+}(0). \tag{1.6}$$

Besides, (1.5) implies that (1.3) follows from the relation

$$\lim_{\epsilon \to 0}\{(\epsilon + I + \mathcal{H}_{s_+})^{-1}k^{\alpha_+}\}(0) = k_{s_+}^{\alpha_+}(0). \tag{1.7}$$

Let us prove (1.7). Since the function

$$\{(\epsilon + I + \mathcal{H}_{s_+})^{-1}k^{\alpha_+}\}(0) = \langle(\epsilon + I + \mathcal{H}_{s_+})^{-1}k^{\alpha_+}, k^{\alpha_+}\rangle$$

decreases with ϵ and it is bounded by (1.6), there exists a limit

$$\lim_{\epsilon \to 0}\{(\epsilon + I + \mathcal{H}_{s_+})^{-1}k^{\alpha_+}\}(0) \leq k_{s_+}^{\alpha_+}(0). \tag{1.8}$$

On the other hand, for any $f \in A_1^2(\Gamma, \alpha_+)$ and $\epsilon > 0$ the following inequalities hold:

$$|f(0)|^2 \leq \langle(\epsilon + I + \mathcal{H}_{s_+})^{-1}k^{\alpha_+}, k^{\alpha_+}\rangle\langle(\epsilon + I + \mathcal{H}_{s_+})f, f\rangle$$
$$\leq \{\lim_{\epsilon \to 0}\langle(\epsilon + I + \mathcal{H}_{s_+})^{-1}k^{\alpha_+}, k^{\alpha_+}\rangle\}\langle(\epsilon + I + \mathcal{H}_{s_+})f, f\rangle,$$

that is

$$|f(0)|^2 \leq \{\lim_{\epsilon \to 0}\langle(\epsilon + I + \mathcal{H}_{s_+})^{-1}k^{\alpha_+}, k^{\alpha_+}\rangle\}\|f\|_{s_+}^2.$$

Putting $f = k_{s_+}^{\alpha_+}$, we have

$$k_{s_+}^{\alpha_+}(0) \leq \lim_{\epsilon \to 0}\langle(\epsilon + I + \mathcal{H}_{s_+})^{-1}k^{\alpha_+}, k^{\alpha_+}\rangle.$$

Comparing this inequality with (1.8), we get (1.7), thus (1.3) is proved. □

We define $s_- \in L^\infty(\Gamma, \alpha_-^{-2})$ by

$$s_-(t) = -\overline{s_+(t)}s(t)/\overline{s(t)}.$$

In this case

$$S(t) = \begin{bmatrix} s_- & s \\ s & s_+ \end{bmatrix}(t)$$

is a unitary-valued matrix function possessing properties (0.10), (0.11).

Lemma 1.3. *The following relation sets a unitary map from $L^2_{s_+}$ to $L^2_{s_-}$:*

$$s(t)f^-(t) = \bar{t}f^+(\bar{t}) + s_+(t)f^+(t).$$

In this case,

$$\|f^+\|^2_{s_+} = \|f^-\|^2_{s_-} = \frac{1}{2}\{\|sf^+\|^2 + \|sf^-\|^2\},$$

and the inverse map is of the form

$$s(t)f^+(t) = \bar{t}f^-(\bar{t}) + s_-(t)f^-(t).$$

Moreover, this unitary map intertwines the operator multiplication by $z(t)$ in $L^2_{s_\pm}$.

Proof. The first statement follows from the identities

$$\begin{bmatrix} 1 & \bar{s}_+ \\ s_+ & 1 \end{bmatrix} = \begin{bmatrix} \bar{s}_+ & 1 \\ 1 & s_+ \end{bmatrix} \begin{bmatrix} 1/\bar{s} & 0 \\ 0 & 1/s \end{bmatrix} \begin{bmatrix} 1 & \bar{s}_- \\ s_- & 1 \end{bmatrix} \begin{bmatrix} 1/s & 0 \\ 0 & 1/\bar{s} \end{bmatrix} \begin{bmatrix} s_+ & 1 \\ 1 & \bar{s}_+ \end{bmatrix}$$

and (0.18).

Since $z(t) = z(\bar{t})$, $t \in \mathbb{D}$, the last statement is evident.   □

Lemma 1.4. *Let $K^{\alpha_+}_{s_+}(t) = k^{\alpha_+}_{s_+}(t)/\sqrt{k^{\alpha_+}_{s_+}(0)}$. The system of functions $\{b^n(t)K^{\alpha_+\mu^{-n}}_{s_+b^{2n}}(t)\}$ forms an orthonormal basis in $H^2_{s_+}(\alpha_+)$ when $\{n \in \mathbb{Z}_+\}$ and in $L^2_{s_+}$ when $\{n \in \mathbb{Z}\}$. With respect to this basis the operator multiplication by $z(t)$ is a Jacobi matrix.*

Proof. First, we note that

$$\{f : f \in H^2_{s_+}(\alpha_+), \ f(0) = 0\} = \{f = b\tilde{f} : \tilde{f} \in H^2_{s_+b^2}(\alpha_+\mu^{-1})\}.$$

Therefore,

$$H^2_{s_+}(\alpha_+) = \{K^{\alpha_+}_{s_+}(t)\} \oplus bH^2_{s_+b^2}(\alpha_+\mu^{-1}).$$

Iterating this relation, we get that $\{b^n(t)K^{\alpha_+\mu^{-n}}_{s_+b^{2n}}(t)\}_{n\in\mathbb{Z}_+}$ is an orthonormal basis in $H^2_{s_+}(\alpha_+)$, since $\cap_{n\in\mathbb{Z}_+}b^n H^2_{s_+b^{2n}}(\alpha_+\mu^{-n}) = \{0\}$.

Then, we note that an arbitrary function $f \in L^2_{s_+}$ can be approximated with the given accuracy by a function f_1 from $L^2_{dm|\mathbb{E}}$. This function, in its turn, can be approximated by a function $f_2 \in b^n A^2_1(\Gamma, \alpha_+\mu^{-n})$ with a suitable n. Therefore, linear combinations of functions from $\{b^n(t)K^{\alpha_+\mu^{-n}}_{s_+b^{2n}}(t)\}$ are dense in $L^2_{s_+}$. Since this system of functions is orthonormal, it forms a basis in $L^2_{s_+}$.

Since $bz \in H^\infty(\Gamma, \mu)$, we have

$$z : b^n H^2_{s_+b^{2n}}(\alpha_+\mu^{-n}) \to b^{n-1}H^2_{s_+b^{2n-2}}(\alpha_+\mu^{-n+1}).$$

For this reason, in the basis $\{b^n(t)K^{\alpha_+\mu^{-n}}_{s_+b^{2n}}(t)\}_{n\in\mathbb{Z}}$, the matrix of the operator multiplication by $z(t)$ has only one non-zero entry over diagonal in each column. But the operator is self-adjoint, therefore, the matrix is a three-diagonal Jacobi matrix.   □

Lemma 1.5. *Let* $e^+(n, t) = b^n(t) K_{s_+ b^{2n}}^{\alpha+\mu^{-n}}(t)$, $n \in \mathbb{Z}$. *Define*

$$s(t)e^-(n, t) = \bar{t} e^+(-n - 1, \bar{t}) + s_+(t)e^+(-n - 1, t).$$

Then $\{e^-(n, t)\}$ *is an orthonormal basis in* $L_{s_-}^2$,

$$s(t)e^-(n, t) \in A_1^2(\Gamma, \alpha_+^{-1}), \quad n \in \mathbb{Z}_+, \tag{1.9}$$

and

$$e^-(0, 0)(be^+)(-1, 0) = \frac{b'(0)}{s(0)}. \tag{1.10}$$

Proof. Lemma 1.3 and Lemma 1.4 imply immediately that $\{e^-(n, t)\}$ is an orthonormal basis in $L_{s_-}^2$. Moreover, $s(t)e^-(n, t) \in L_{dm|\mathbb{E}}^2$. To prove (1.9) consider a scalar product $(f \in A_1^2(\Gamma, \alpha_+))$

$$\langle \bar{t} f(\bar{t}), s(t)e^-(n, t) \rangle = \frac{1}{2} \left\langle \begin{bmatrix} f(t) \\ \bar{t} f(\bar{t}) \end{bmatrix}, \begin{bmatrix} 1 & \overline{s_+(t)} \\ s_+(t) & 1 \end{bmatrix} \begin{bmatrix} e^+(n, t) \\ \bar{t} e^+(n, \bar{t}) \end{bmatrix} \right\rangle$$

$$= \frac{1}{2} \left\langle \begin{bmatrix} f(t) \\ \bar{t} f(\bar{t}) \end{bmatrix}, \begin{bmatrix} 1 & \overline{s_+(t)} \\ s_+(t) & 1 \end{bmatrix} \begin{bmatrix} (b^{-n-1} K_{s_+ b^{-2n-2}}^{\alpha+\mu^{n+1}})(t) \\ \bar{t}(b^{-n-1} K_{s_+ b^{-2n-2}}^{\alpha+\mu^{n+1}})(\bar{t}) \end{bmatrix} \right\rangle$$

$$= \langle b^{n+1} f, K_{s_+ b^{-2n-2}}^{\alpha+\mu^{n+1}} \rangle_{s_+ b^{-2n-2}} = 0, \quad \forall n \geq 0.$$

To prove (1.10), we write

$$s(0)e^-(0, 0) = \langle s(t)e^-(0, t), k^{\alpha_+^{-1}}(t) \rangle. \tag{1.11}$$

Due to the Direct Cauchy Theorem, the reproducing kernel k^α possesses the following property:

$$\bar{t} k^{\alpha_+^{-1}}(\bar{t}) = \frac{b'(0)}{k^{\alpha+\mu}(0)} \frac{k^{\alpha+\mu}(t)}{b(t)}. \tag{1.12}$$

Substituting (1.12) in (1.11), we obtain

$$s(0)e^-(0, 0) = \frac{b'(0)}{2k^{\alpha+\mu}(0)} \left\langle \begin{bmatrix} 1 & \overline{s_+(t)} \\ s_+(t) & 1 \end{bmatrix} \begin{bmatrix} e^+(-1, t) \\ \bar{t} e^+(-1, \bar{t}) \end{bmatrix}, \begin{bmatrix} (b^{-1} k^{\alpha+\mu})(t) \\ \bar{t}(b^{-1} k^{\alpha+\mu})(\bar{t}) \end{bmatrix} \right\rangle$$

$$= \frac{b'(0)}{k^{\alpha+\mu}(0)} \langle K_{s_+ b^{-2}}^{\alpha+\mu}(t), k^{\alpha+\mu}(t) \rangle_{s_+ b^{-2}}.$$

Using (1.3), we have

$$s(0)e^-(0, 0) = \frac{b'(0)}{k^{\alpha+\mu}(0) K_{s_+ b^{-2}}^{\alpha+\mu}(0)} \lim_{\epsilon \to 0} \langle (\epsilon + I + \mathcal{H}_{s_+ b^{-2}})^{-1} k^{\alpha+\mu}, k^{\alpha+\mu} \rangle_{s_+ b^{-2}}$$

$$= \frac{b'(0)}{k^{\alpha+\mu}(0) K_{s_+ b^{-2}}^{\alpha+\mu}(0)} \lim_{\epsilon \to 0} \langle (I + \mathcal{H}_{s_+ b^{-2}})(\epsilon + I + \mathcal{H}_{s_+ b^{-2}})^{-1} k^{\alpha+\mu}, k^{\alpha+\mu} \rangle$$

$$= \frac{b'(0)}{k^{\alpha+\mu}(0) K_{s_+ b^{-2}}^{\alpha+\mu}(0)} \{k^{\alpha+\mu}(0) - \lim_{\epsilon \to 0} \epsilon \langle (\epsilon + I + \mathcal{H}_{s_+ b^{-2}})^{-1} k^{\alpha+\mu}, k^{\alpha+\mu} \rangle\}.$$

Since the limit (1.7) exists, finally, we get

$$s(0)e^-(0,0) = \frac{b'(0)}{K^{\alpha+\mu}_{s_+b^{-2}}(0)} = \frac{b'(0)}{(be^+)(-1,0)}.$$

The lemma is proved.   □

Lemma 1.6. *Let* $||s_+|| < 1$. *Then*

$$K^{\alpha\pm}_{s\pm}(0)K^{\alpha\mp\mu}_{s\mp b^{-2}}(0) = \frac{b'(0)}{s(0)}.$$

Proof. Note that operators $(I + \mathcal{H}_{s\pm b^n})$ are invertible.

We use the notation of Lemma 1.5. As we know, $s(t)e^-(0,t) \in A^2_1(\Gamma, \alpha^{-1}_+)$. But, in the case under consideration, $1/s \in H^\infty(\Gamma, \alpha_+\alpha_-)$. Hence, the function $e^-(0,t)$ itself belongs to $A^2_1(\Gamma, \alpha_-)$. Therefore, we can project each term onto $A^2_1(\Gamma, \alpha_-)$ in the relation

$$\bar{t}(se^+)(-1,\bar{t}) = e^-(0,t) + \bar{t}(s_-e^-)(0,\bar{t}).$$

On the right-hand side we get

$$P_{A^2_1(\Gamma,\alpha_-)}\{e^-(0,t) + \bar{t}(s_-e^-)(0,\bar{t})\} = (I + \mathcal{H}_{s_-})e^-(0,t).$$

To evaluate the left-hand side, using (1.10), we write

$$s(t)e^+(-1,t) = s(0)(be^+)(-1,0)\frac{k^{\alpha^{-1}_-\mu}(t)}{b(t)k^{\alpha^{-1}_-\mu}(0)} + g(t)$$

$$= \frac{b'(0)}{e^-(0,0)}\frac{k^{\alpha^{-1}_-\mu}(t)}{b(t)k^{\alpha^{-1}_-\mu}(0)} + g(t), \quad g \in A^2_1(\Gamma, \alpha^{-1}_-).$$

Using (1.12), we get

$$P_{A^2_1(\Gamma,\alpha_-)}\{\bar{t}(se^+)(-1,\bar{t})\} = \frac{k^{\alpha_-}(t)}{e^-(0,0)} = (I + \mathcal{H}_{s_-})e^-(0,t).$$

Thus,

$$e^-(0,t)e^-(0,0) = (I + \mathcal{H}_{s_-})^{-1}k^{\alpha_-}.$$

In particular, $e^-(0,0) = K^{\alpha_-}_{s_-}(0)$, and (1.10) becomes the statement of the lemma.   □

Lemma 1.7. *Assume that for some Jacobi matrix* J *there exists a pair of unitary transforms*

$$\mathcal{F}^\pm : l^2(\mathbb{Z}) \to L^2_{s\pm}, \quad (\mathcal{F}^\pm Jf)(t) = z(t)(\mathcal{F}^\pm f)(t),$$

determining each other by the relations

$$s(t)(\mathcal{F}^\pm f)(t) = \bar{t}(\mathcal{F}^\mp f)(\bar{t}) + s_\mp(t)(\mathcal{F}^\mp f)(t),$$

such that

$$s\mathcal{F}^\pm(l^2(\mathbb{Z}_\pm)) \subset A^2_1(\Gamma, \alpha^{-1}_\mp). \tag{1.13}$$

As before, we put

$$e^+(n, t) = (\mathcal{F}^+ e_n)(t), \quad e^-(n, t) = (\mathcal{F}^- e_{-n-1})(t). \tag{1.14}$$

Then $e^\pm(n, t)$ has at the origin zero (poles) of multiplicity n, $n > 0$ ($-n$, $n < 0$). Furthermore, $\mathcal{F}^\pm(l^2(\mathbb{Z}_\pm)) \supset H^2_{s_\pm}(\alpha_\pm)$, and, hence,

$$e^\pm(0, 0) \geq K^{\alpha_\pm}_{s_\pm}(0). \tag{1.15}$$

The equality in (1.15) takes place if and only if $e^\pm(0, t) = K^{\alpha_\pm}_{s_\pm}(t)$.

Proof. Let us show that the annihilator of the linear space $A_1^2(\Gamma, \alpha^+) \subset L^2_{s_+}$ contains $\mathcal{F}^+\{l^2(\mathbb{Z}_-)\}$. For $f \in A_1^2(\Gamma, \alpha^+)$ and $e^+(-n-1, t)$, $n \geq 0$, we have

$$
\begin{aligned}
\langle f(t), e^+(-n-1, t) \rangle_{s_+} &= \frac{1}{2} \left\langle \begin{bmatrix} f(t) \\ \bar{t} f(\bar{t}) \end{bmatrix}, \begin{bmatrix} 1 & \overline{s_+(t)} \\ s_+(t) & 1 \end{bmatrix} \begin{bmatrix} e^+(-n-1, t) \\ \bar{t} e^+(-n-1, \bar{t}) \end{bmatrix} \right\rangle \\
&= \langle f(t), e^+(-n-1, t) + \overline{t s_+(t)} e^+(-n-1, \bar{t}) \rangle \\
&= \langle f(t), \bar{t}(se^-)(n, \bar{t}) \rangle.
\end{aligned}
$$

By (1.13) and (DCT), the last scalar product equals zero. Therefore,

$$H^2_{s_+}(\alpha_+) = \text{clos}_{L^2_{s_+}} A_1^2(\Gamma, \alpha^+) \subset \{\mathcal{F}^+(l^2(\mathbb{Z}_-))\}^\perp = \mathcal{F}^+(l^2(\mathbb{Z}_+)).$$

Now, from the three-term recurrent relation

$$z(t)s(t)e^+(n, t) = p_n s(t)e^+(n-1, t) + q_n s(t)e^+(n, t) + p_{n+1} s(t)e^+(n+1, t), \tag{1.16}$$

and (1.13) it follows that $e^+(n, t)$, $n > 0$, has in the origin zero, at least of multiplicity n.

Since $K^{\alpha_+}_{s_+}(t) \in \mathcal{F}^+(l^2(\mathbb{Z}_+))$, it possesses the decomposition

$$K^{\alpha_+}_{s_+}(t) = \sum_{n \in \mathbb{Z}_+} a_n e^+(n, t).$$

Since $e^+(n, 0) = 0$, $n > 0$,

$$a_0 = \frac{K^{\alpha_+}_{s_+}(0)}{e^+(0, 0)}$$

in this decomposition. But,

$$|a_0|^2 \leq \sum |a_n|^2 = \|K^{\alpha_+}_{s_+}(t)\|^2_{s_+} = 1.$$

Thus, (1.15) and the lemma are proved. □

Lemma 1.8 ([10]). *Let $f \in L^\infty(\alpha^{-2})$. Then*

$$P_{A_1^2(\Gamma, \alpha)} \left\{ \bar{t}(f b^n K^{\alpha \mu^{-n}})(\bar{t}) \right\} \to 0, \quad n \to +\infty,$$

where $P_{A_1^2(\Gamma, \alpha)}$ is the orthogonal projection from $L^2_{dm|\mathbb{E}}$ onto $A_1^2(\Gamma, \alpha)$.

Proof. Let us denote by $\Delta^\beta(t)$ an extremal function of the problem

$$\Delta^\beta(0) = \sup\{\phi(0) : \phi \in H^\infty(\Gamma, \beta), \ ||\phi|| \le 1\}.$$

Using properties (1), (2) of a group of Widom type with (DCT), Theorem [7], and compactness of Γ^*, for any $\epsilon > 0$, we can find a finite covering of Γ^*,

$$\Gamma^* = \bigcup_{j=1}^{l(\epsilon)} \{\beta : \ \mathrm{dist}(\beta, \beta_j) \le \eta(\epsilon)\}$$

such that

$$2 \left| 1 - \Delta^{\beta_j^{-1}\beta}(0) \frac{K^{\beta_j}(0)}{K^\beta(0)} \right| \le \epsilon^2, \quad \mathrm{dist}(\beta, \beta_j) \le \eta(\epsilon).$$

It means that

$$\|(\Delta^{\beta_j^{-1}\beta} K^{\beta_j}) - K^\beta\|^2 \le 1 + 1 - 2\Delta^{\beta_j^{-1}\beta}(0) \frac{K^{\beta_j}(0)}{K^\beta(0)} \le \epsilon^2, \quad \mathrm{dist}(\beta, \beta_j) \le \eta(\epsilon).$$

For fixed β one can find n_0 such that

$$\|P_{b^n A_1^2(\Gamma, \alpha^2 \beta^{-1} \mu^{-n})} \bar{t}(f K^\beta)(\bar{t})\| \le \epsilon, \ \forall n > n_0.$$

Therefore, there exists n_0 such that

$$\|P_{b^n A_1^2(\Gamma, \alpha^2 \beta_j^{-1} \mu^{-n})} \bar{t}(f K^{\beta_j})(\bar{t})\| \le \epsilon, \ \forall n > n_0, \ 1 \le j \le l(\epsilon).$$

Now, let $n > n_0 = n_0(\epsilon)$ and let $\beta_j : \mathrm{dist}(\beta_j, \alpha \mu^{-n}) \le \eta(\epsilon)$. For $h \in A_1^2(\Gamma, \alpha)$, we write

$$\langle \bar{t}(f b^n K^{\alpha \mu^{-n}})(\bar{t}), h \rangle$$
$$= \langle \bar{t}(b^n f[K^{\alpha \mu^{-n}} - \Delta^{\alpha \mu^{-n} \beta_j^{-1}} K^{\beta_j}])(\bar{t}), h \rangle + \langle \bar{t}(b^n \Delta^{\alpha \mu^{-n} \beta_j^{-1}} f K^{\beta_j})(\bar{t}), h \rangle.$$

Then

$$|\langle \bar{t}(b^n f[K^{\alpha \mu^{-n}} - \Delta^{\alpha \mu^{-n} \beta_j^{-1}} K^{\beta_j}])(\bar{t}), h \rangle|$$
$$\le \|f\| \|h\| \|K^{\alpha \mu^{-n}} - \Delta^{\alpha \mu^{-n} \beta_j^{-1}} K^{\beta_j}\| \le \epsilon ||f|| \, ||h||,$$

and

$$|\langle \bar{t}(b^n \Delta^{\alpha \mu^{-n} \beta_j^{-1}} f K^{\beta_j})(\bar{t}), h \rangle| = |\langle \bar{t}(f K^{\beta_j})(\bar{t}), b^n \overline{\Delta^{\alpha \mu^{-n} \beta_j^{-1}}}(\bar{t})h \rangle|$$
$$\le \|P_{b^n A_1^2(\Gamma, \alpha^2 \beta_j^{-1} \mu^{-n})} \bar{t}(f K^{\beta_j})(\bar{t})\| \, ||h|| \le \epsilon ||h||.$$

Therefore,

$$|\langle P_{A_1^2(\Gamma, \alpha)} \left\{ \bar{t}(f b^n K^{\alpha \mu^{-n}})(\bar{t}) \right\}, h \rangle| \le \epsilon(1 + ||f||) ||h||.$$

Putting $h = P_{A_1^2(\Gamma, \alpha)} \left\{ \bar{t}(f b^n K^{\alpha \mu^{-n}})(\bar{t}) \right\}$, we get

$$||P_{A_1^2(\Gamma, \alpha)} \left\{ \bar{t}(f b^n K^{\alpha \mu^{-n}})(\bar{t}) \right\} || \le \epsilon(1 + ||f||).$$

The lemma is proved. □

Proposition 1.1. *Assume that for some Jacobi matrix J there exists a pair of unitary transforms*

$$\mathcal{F}^{\pm} : l^2(\mathbb{Z}) \to L^2_{s_{\pm}}, \quad (\mathcal{F}^{\pm} Jf)(t) = z(t)(\mathcal{F}^{\pm} f)(t),$$

determining each other by the relations

$$s(t)(\mathcal{F}^{\pm} f)(t) = \bar{t}(\mathcal{F}^{\mp} f)(\bar{t}) + s_{\mp}(t)(\mathcal{F}^{\mp} f)(t), \tag{1.17}$$

such that (1.13) holds. Then the following relations are equivalent:

$$e^+(n, t) = b^n(t) K^{\alpha + \mu^{-n}} + o(1) \quad in \ L^2_{s_+}, \tag{1.18}$$

$$\bar{t} p_n \{ e^+(n, t) e^+(n - 1, \bar{t}) - e^+(n - 1, t) e^+(n, \bar{t}) \} = z'(t), \tag{1.19}$$

$$s(0) e^+(0, 0)(be^-)(-1, 0) = b'(0), \tag{1.20}$$

where $\{e^{\pm}(n, t)\}$ is defined by (1.14).

Proof. $(1.18) \Rightarrow (1.19)$. It follows from two remarks. First, the form on the left in (1.19) does not depend on n (it is the Wronskian of the recurrence relation (0.7)). Second, the identity

$$\bar{t} \frac{K^{\alpha}(0)}{K^{\alpha\mu}(0)} \{ K^{\alpha}(t)(K^{\alpha\mu}/b)(\bar{t}) - (K^{\alpha\mu}/b)(t) K^{\alpha}(\bar{t}) \} = z'(t)$$

holds for any $\alpha \in \Gamma^*$.

$(1.19) \Rightarrow (1.20)$. Let us introduce the matrix

$$\Phi(t) = \begin{bmatrix} e^-(-1, t) & -e^-(0, t) \\ -e^+(0, t) & e^+(-1, t) \end{bmatrix}. \tag{1.21}$$

Then (1.17) implies

$$\bar{t} \Phi(\bar{t}) = -S(t) \Phi(t).$$

In particular, with the help of (1.19), we get

$$s(t) = -\bar{t} \frac{e^+(0, t) e^+(-1, \bar{t}) - e^+(-1, t) e^+(0, \bar{t})}{e^-(-1, t) e^+(-1, t) - e^-(0, t) e^+(0, t)}$$

$$= \frac{-z'(t)}{p_0 \{ e^-(-1, t) e^+(-1, t) - e^-(0, t) e^+(0, t) \}}. \tag{1.22}$$

Since $b(t) e^{\pm}(-1, t)$ are holomorphic functions (in fact, of Smirnov class)

$$s(0) = \frac{b'(0)}{p_0 (be^-)(-1, 0)(be^+)(-1, 0)}.$$

Now, we only have to mention that $p_0 (be^{\pm})(-1, 0) = e^{\pm}(0, 0)$.

$(1.20) \Rightarrow (1.18)$. This is non–trivial part of the proposition. The main step is to prove that

$$\lim_{n \to +\infty} \frac{(b^{-n} e^+)(n, 0)}{K^{\alpha + \mu^{-n}}(0)} = 1. \tag{1.23}$$

By Lemma (1.7) we have an estimate from below,

$$(b^{-n}e^+)(n, 0) \geq K_{s_+ b^{2n}}^{\alpha + \mu^{-n}}(0) \geq \{(\epsilon + I + \mathcal{H}_{s_+ b^{2n}})^{-1}k^{\alpha + \mu^{-n}}\}^{1/2}(0)$$

$$= \frac{1}{\sqrt{1 + \epsilon}} K_{\frac{s_+}{1+\epsilon}b^{2n}}^{\alpha + \mu^{-n}}(0). \tag{1.24}$$

To get an estimate from above we use (1.20).

Let us note that due to the recurrence relation, the form

$$p_n\{e^+(n - 1, t)e^-(-n - 1, t) - e^+(n, t)e^-(-n, t)\}$$

also does not depend on n. Thus, a relation like (1.20) holds for all n:

$$(b^{-n}e^+)(n, 0)(b^{n+1}e^-)(-n - 1, 0) = p_n(b^{-n+1}e^+)(n - 1, 0)(b^{n+1}e^-)(-n - 1, 0)$$

$$= e^+(0, 0)(be^-)(-1, 0) = b'(0)/s(0).$$

Therefore,

$$(b^{-n}e^+)(n, 0) = \frac{b'(0)}{s(0)} \frac{1}{(b^{n+1}e^-)(-n - 1, 0)}$$

$$\leq \frac{b'(0)}{s(0)} \frac{1}{K_{s_- b^{-2n-2}}^{\alpha - \mu^{n+1}}(0)}$$

$$\leq \frac{b'(0)}{s(0)} \frac{1}{\{(\epsilon + I + \mathcal{H}_{s_- b^{-2n-2}})^{-1}k^{\alpha - \mu^{n+1}}\}^{1/2}(0)}$$

$$= \frac{b'(0)}{s(0)} \frac{\sqrt{1 + \epsilon}}{K_{s_{\epsilon,-}b^{-2n-2}}^{\alpha - \mu^{n+1}}(0)}, \tag{1.25}$$

where $s_{\epsilon,-} := s_-/(1 + \epsilon)$.

With the function $s_{\epsilon,-}$, let us associate the functions s_ϵ, $s_{\epsilon,+}$ and the character $\alpha_{\epsilon,+}$ (note that $s_{\epsilon,+}$ is not $\frac{1}{1+\epsilon}s_+$, but $s_{\epsilon,+} = -\bar{s}_{\epsilon,-}(s_\epsilon/\bar{s}_\epsilon)$). It is important that $s_\epsilon(0)$ and $\alpha_{\epsilon,+}$ depend continuously on ϵ.

By Lemma 1.6,

$$\frac{b'(0)}{K_{s_{\epsilon,-}b^{-2n-2}}^{\alpha - \mu^{n+1}}(0)} = s_\epsilon(0) K_{s_{\epsilon,+}b^{2n}}^{\alpha_{\epsilon,+}\mu^{-n}}(0). \tag{1.26}$$

Substituting (1.26) in (1.25), and combining the result with (1.24), we obtain

$$\frac{1}{\sqrt{1 + \epsilon}} K_{\frac{s_+}{1+\epsilon}b^{2n}}^{\alpha + \mu^{-n}}(0) \leq (b^{-n}e^+)(n, 0) \leq \sqrt{1 + \epsilon} \frac{s_\epsilon(0)}{s(0)} K_{s_{\epsilon,+}b^{2n}}^{\alpha_{\epsilon,+}\mu^{-n}}(0). \tag{1.27}$$

Lemma 1.8 implies that for any $f \in L^\infty(\Gamma, \alpha_+^{-2})$ with $||f|| < 1$ we have

$$\lim_{n \to +\infty} \frac{K_{fb^{2n}}^{\alpha + \mu^{-n}}(0)}{K^{\alpha + \mu^{-n}}(0)} \to 1.$$

Indeed,

$$|k_{fb^{2n}}^{\alpha+\mu^{-n}}(0) - k^{\alpha+\mu^{-n}}(0)| = |\langle \mathcal{H}_{fb^{2n}} k^{\alpha+\mu^{-n}}, (I + \mathcal{H}_{fb^{2n}})^{-1} k^{\alpha+\mu^{-n}}\rangle|$$

$$= |\langle \bar{t}(fb^n k^{\alpha+\mu^{-n}})(\bar{t}), b^n(I + \mathcal{H}_{fb^{2n}})^{-1} k^{\alpha+\mu^{-n}}\rangle|$$

$$\leq ||P_{A_1^2(\Gamma,\alpha_+)}\{\bar{t}(fb^n k^{\alpha+\mu^{-n}})(\bar{t})\}||\,||b^n(I+\mathcal{H}_{fb^{2n}})^{-1} k^{\alpha+\mu^{-n}}||$$

$$\leq \frac{1}{1-||f||}||P_{A_1^2(\Gamma,\alpha_+)}\{\bar{t}(fb^n k^{\alpha+\mu^{-n}})(\bar{t})\}||\,||k^{\alpha+\mu^{-n}}|| \to 0,$$

as $n \to +\infty$.

Also, since $\alpha_{\epsilon,+}$ depends continuously on ϵ and $K^{\alpha_+}(0)$ is continuous on a compact group Γ^*, for any $\delta > 0$ we can choose ϵ so small that

$$\frac{K^{\alpha_{\epsilon,+}\mu^{-n}}(0)}{K^{\alpha_+\mu^{-n}}(0)} \leq 1 + \delta, \quad \forall n.$$

Thus, returning to (1.27), we obtain

$$\frac{1}{\sqrt{1+\epsilon}} \leq \liminf_{n\to\infty} \frac{(b^{-n}e^+)(n,0)}{K^{\alpha_+\mu^{-n}}(0)} \leq \limsup_{n\to\infty} \frac{(b^{-n}e^+)(n,0)}{K^{\alpha_+\mu^{-n}}(0)} \leq \sqrt{1+\epsilon}\frac{s_\epsilon(0)}{s(0)}(1+\delta).$$

Since ϵ and δ are arbitrary small, (1.23) is proved.

Now we are in a position to prove (1.18). Consider the norm of the difference

$$||e^+(n,t) - b^n K^{\alpha+\mu^{-n}}||_{s_+}^2 = 1 + ||b^n K^{\alpha+\mu^{-n}}||_{s_+}^2 - 2\langle e^+(n,t), b^n K^{\alpha+\mu^{-n}}\rangle_{s_+}.$$

Since

$$||b^n K^{\alpha+\mu^{-n}}||_{s_+}^2 = 1 + \langle b^n K^{\alpha+\mu^{-n}}, \bar{t}(s_+ b^n K^{\alpha+\mu^{-n}})(\bar{t})\rangle,$$

using Lemma 1.8, we conclude that

$$||b^n K^{\alpha+\mu^{-n}}||_{s_+}^2 \to 1, \ n \to +\infty.$$

Let us evaluate the scalar product

$$\langle e^+(n,t), b^n K^{\alpha+\mu^{-n}}\rangle_{s_+} = \langle se^-(-n-1,t), \bar{t}(b^n K^{\alpha+\mu^{-n}})(\bar{t})\rangle$$

$$= \langle se^-(-n-1,t), b^{-n}b^{-1} K^{\alpha_+^{-1}\mu^{n+1}}\rangle$$

$$= \frac{s(0)(b^{n+1}e^-)(-n-1,0)}{K^{\alpha_+^{-1}\mu^{n+1}}(0)}$$

$$= \frac{K^{\alpha+\mu^{-n}}(0)}{(b^{-n}e^+)(n,0)} \to 1, \ n \to +\infty.$$

The proposition is proved. □

The following theorem shows that an arbitrary function s_+, possessing (1.1), (1.2), is the reflection coefficient of a Jacobi matrix of Szegö class.

Theorem 1.1. *Let a function $s_+ \in L^\infty(\Gamma, \alpha_+^{-2})$, $||s_+|| \leq 1$, $\overline{s_+(\bar{t})} = s_+(t)$, be such that that $\log(1 - |s_+|^2) \in L^1$. Let an outer function s, $s(0) > 0$, and s_- be associated to s_+ by the relations*

$$|s|^2 = 1 - |s_+|^2, \quad s_- = -\bar{s}_+ s/\bar{s}.$$

Then the system of functions

$$e^+(n, t) = b^n K_{s_+ b^{2n}}^{\alpha + \mu^{-n}}$$

forms an orthonormal basis in $L_{s_+}^2$. The dual system, defined by

$$s(t)e^-(n, t) = \bar{t}e^+(-n - 1, \bar{t}) + s_+(t)e^+(-n - 1, t),$$

forms an orthonormal basis in $L_{s_-}^2$. The subspaces of $L_{s_\pm}^2$, that formed by functions with vanishing negative Fourier coefficients with respect to these bases, are spaces of holomorphic character-automorphic forms; moreover,

$$sf^\pm \in A_1^2(\Gamma, \alpha_\mp^{-1}) \text{ if } f^\pm \in \text{clos}_{L_{s_\pm}^2} \text{span}\{e^\pm(n, t) : n \geq 0\}.$$

Further,

$$e^\pm(n, t) = b^n K^{\alpha_\pm \mu^{-n}} + o(1) \text{ in } L_{s_\pm}^2,$$

and with respect to these bases the operator multiplication by $z(t)$ is a Jacobi matrix J of Szegö class.

Proof. All statements, besides the last one, only summarize results of Lemmas 1.4, 1.5 and Proposition 1.1. To prove that J is of Szegö class we evaluate its spectral density $\rho(x)$.

Using the definition of the resolvent matrix–function, we get

$$R(z) = \begin{bmatrix} \langle (z(t) - z)^{-1}e^+(-1, t), e^+(-1, t)\rangle_{s_+} & \langle (z(t) - z)^{-1}e^+(0, t), e^+(-1, t)\rangle_{s_+} \\ \langle (z(t) - z)^{-1}e^+(-1, t), e^+(0, t)\rangle_{s_+} & \langle (z(t) - z)^{-1}e^+(0, t), e^+(0, t)\rangle_{s_+} \end{bmatrix}.$$

Note that if $f^\pm \in L_{s_\pm}^2$ are related by $s(t)f^-(t) = \bar{t}f^+(\bar{t}) + s_+(t)f^+(t)$ then

$$\left(\frac{f^+(t)}{z(t) - z}\right)^- = \frac{f^-(t)}{z(t) - z}.$$

Therefore, using Lemma 1.3, we have

$$R(z) = \frac{1}{2} \int_{\mathbb{E}} \begin{bmatrix} e^+(-1, t) & e^+(0, t) \\ e^-(0, t) & e^-(-1, t) \end{bmatrix}^* \begin{bmatrix} e^+(-1, t) & e^+(0, t) \\ e^-(0, t) & e^-(-1, t) \end{bmatrix} \frac{|s(t)|^2 \, dm}{z(t) - z},$$

and, substituting $s(t)$ from (1.22), we obtain

$$R(z) = \frac{1}{2} \int_{\mathbb{E}} \frac{\begin{bmatrix} e^+(-1, t) & e^+(0, t) \\ e^-(0, t) & e^-(-1, t) \end{bmatrix}^* \begin{bmatrix} e^+(-1, t) & e^+(0, t) \\ e^-(0, t) & e^-(-1, t) \end{bmatrix}}{p_0^2|e^-(-1, t)e^+(-1, t) - e^-(0, t)e^+(0, t)|^2} \frac{|z'(t)|^2 \, dm}{z(t) - z}$$

$$= \frac{1}{2} \int_{\mathbb{E}} \frac{\tilde{\Phi}^{-1*}(t)\tilde{\Phi}^{-1}(t)}{z(t) - z} \frac{|z'(t)|^2 \, |dt|}{2\pi p_0^2},$$

where

$$\tilde{\Phi}(t) = \begin{bmatrix} e^-(-1, t) & -e^+(0, t) \\ -e^-(0, t) & e^+(-1, t) \end{bmatrix}. \tag{1.28}$$

Thus,

$$2\pi p_0^2 \rho(z(t)) = \tilde{\Phi}^{-1*}(t)\tilde{\Phi}^{-1}(t)|z'(t)|, \tag{1.29}$$

and

$$\det\{2\pi p_0 \rho(z(t))\} = \frac{|z'(t)|^2}{p_0^2 |\det \tilde{\Phi}(t)|^2} = |s(t)|^2. $$

The theorem is proved. □

Let us note, by the way, that $\Phi(t)$ (see (1.21)) and $\tilde{\Phi}(t)$ are related by $\tilde{\Phi}(\bar{t}) = \Phi^*(t)$ and, besides (1.28),

$$2\pi p_0^2 \rho(z(t)) = \Phi^{-1}(t)\Phi^{-1*}(t)|z'(t)|. \tag{1.30}$$

2. Existence and Uniqueness

We start this section with a remark that the spectral measure $d\sigma$ determines a Jacobi matrix uniquely, but it is not an arbitrary 2×2 matrix–measure, or, say, a real-valued (all entries are real) 2×2 matrix–measure.

Indeed, one can represent J as a two dimensional perturbation of an orthogonal sum of a pair of one–sided Jacobi matrices, i.e.:

$$J = \begin{bmatrix} J_- & 0 \\ 0 & J_+ \end{bmatrix} + p_0\langle \, , e_{-1}\rangle e_0 + p_0\langle \, , e_0\rangle e_{-1},$$

where $J_\pm = P_{l^2(\mathbb{Z}_\pm)} J | l^2(\mathbb{Z}_\pm)$. This formula implies that

$$R(z) = \begin{bmatrix} r_-^{-1}(z) & p_0 \\ p_0 & r_+^{-1}(z) \end{bmatrix}^{-1}, \tag{2.1}$$

where

$$r_-(z) = r(z, J_-) = \langle (J_- - z)^{-1} e_{-1}, e_{-1}\rangle = \int \frac{d\sigma_-(x)}{x - z},$$

$$r_+(z) = r(z, J_+) = \langle (J_+ - z)^{-1} e_0, e_0\rangle = \int \frac{d\sigma_+(x)}{x - z}.$$

Thus, the real-valued matrix-measure $d\sigma$ is determined by two scalar measures $d\sigma_\pm$ (with the normalization $\int d\sigma_\pm = 1$) and a constant p_0.

In what follows $\hat{f}(x) \in L^2_{d\sigma}$ denotes the image of $f \in l^2(\mathbb{Z})$ in the spectral representation. Recall that

$$\hat{e}_{-1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \hat{e}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and

$$(\widehat{Jf})(x) = x\hat{f}(x).$$

Let $\{P_n^\pm(z)\}$ be the orthonormal polynomials with respect to the (scalar) measure $d\sigma_\pm$ and

$$Q_n^\pm(z) := \int \frac{P_n^\pm(x) - P_n^\pm(z)}{x - z} d\sigma_\pm(x)$$

(so-called polynomials of the second kind). In these terms

$$\hat{e}_n(x) = \begin{bmatrix} -p_0 Q_n^+(x) \\ P_n^+(x) \end{bmatrix}, \quad n \geq 0,$$

$$\hat{e}_{-n-1}(x) = \begin{bmatrix} P_n^-(x) \\ -p_0 Q_n^-(x) \end{bmatrix}, \quad n \geq 0. \tag{2.2}$$

Now, we prove Theorem 0.5.

Proof of Theorem 0.5, the uniqueness part. The function $e^\pm(0,\zeta)/e^\pm(-1,\zeta)$ is Γ-automorphic, thus it defines a meromorphic function in $\bar{\mathbb{C}} \setminus E$,

$$\tilde{r}_\pm(z(\zeta)) := -\frac{e^\pm(0,\zeta)}{p_0 e^\pm(-1,\zeta)}.$$

The recurrence relations imply that $\tilde{r}_\pm(z)$ possesses the same decomposition into a continued fraction as $r_\pm(z)$. Therefore,

$$r_\pm(z(\zeta)) = -\frac{e^\pm(0,\zeta)}{p_0 e^\pm(-1,\zeta)}. \tag{2.3}$$

By Proposition 1.1 the asymptotic (1.18) implies the identity (1.19). Using this identity, we get ($t \in \mathbb{T}$)

$$r_\pm(z(t)) - \overline{r_\pm(z(t))} = -p_0 \frac{e^\pm(0,t)\overline{e^\pm(-1,t)} - e^\pm(-1,t)\overline{e^\pm(0,t)}}{|p_0 e^\pm(-1,t)|^2}$$

$$= \frac{-tz'(t)}{|p_0 e^\pm(-1,t)|^2}.$$

This means that an outer part of the function $e^\pm(-1,\zeta)$ is determined uniquely. But then (2.3) means that an outer part of $e^\pm(0,\zeta)$ is determined uniquely, and since $b(\zeta)e^\pm(-1,\zeta)$ and $e^\pm(0,\zeta)$ are of Smirnov class, these functions are determined up to a common inner factor $\Delta_\pm(\zeta)$, i.e.,

$$e^\pm(0,\zeta) = \Delta_\pm(\zeta)\tilde{e}^\pm(0,\zeta) \text{ and } e^\pm(-1,\zeta) = \Delta_\pm(\zeta)\tilde{e}^\pm(-1,\zeta), \tag{2.4}$$

where the inner parts of $\tilde{e}^\pm(0,\zeta)$, $\tilde{e}^\pm(-1,\zeta)$ are relatively prime.

To show that $\Delta_\pm(\zeta) = 1$, we use (0.23), (0.24). Since

$$s(t)e^\mp(0,t) = \bar{t}e^\pm(-1,\bar{t}) + s_\pm(t)e^\pm(-1,t),$$

$$s(t)e^\mp(-1,t) = \bar{t}e^\pm(0,\bar{t}) + s_\pm(t)e^\pm(0,t), \tag{2.5}$$

we have

$$s(t)\{e^\mp(-1,t)e^\pm(-1,t) - e^\mp(0,t)e^\pm(0,t)\}$$
$$= \bar{t}\{e^\pm(0,\bar{t})e^\pm(-1,t) - e^\pm(-1,\bar{t})e^\pm(0,t)\}.$$

Substituting (2.4) and using the symmetry

$$\overline{\tilde{e}^{\pm}(0,\bar{t})} = \tilde{e}^{\pm}(0,t), \quad \overline{\tilde{e}^{\pm}(-1,\bar{t})} = \tilde{e}^{\pm}(-1,t),$$

we obtain

$$s(t)b^2(t)\{e^{\mp}(-1,t)\tilde{e}^{\pm}(-1,t) - e^{\mp}(0,t)\tilde{e}^{\pm}(0,t)\}$$
$$= \bar{t}\Delta_{\pm}(\bar{t})b^2(t)\{\overline{\tilde{e}^{\pm}(0,t)}\tilde{e}^{\pm}(-1,t) - \overline{\tilde{e}^{\pm}(-1,t)}\tilde{e}^{\pm}(0,t)\}$$
$$= -b^2(t)z'(t)\{p_0\overline{\Delta_{\pm}(\bar{t})}\}^{-1}.$$

Since the first expression here is a function of Smirnov class and $b^2 z'$ is an outer function, we conclude that $\Delta_{\pm}(t)$ is a constant.

Since

$$\bar{t}\Phi(\bar{t}) = -S(t)\Phi(t) \tag{2.6}$$

with $\Phi(t)$ defined by (1.21), $S(t)$ is also determined in a unique way. At last, by the recurrence relations we get the same conclusion with respect to all functions $\{e^{\pm}(n,\zeta)\}$, not only for $n = -1, 0$. $\quad\square$

Proof of Theorem 0.5, the existence part. The key instrument is the following theorem [11]: if $r(z)$ is a meromorphic function in $\bar{\mathbb{C}} \setminus E$ such that $\operatorname{Im}r(z)/\operatorname{Im}z \geq 0$ and poles of $r(z(\zeta))$ satisfy the Blaschke condition, then $r(z(\zeta))$ is a function of bounded characteristic in $\mathbb{D}$ without a singular component in the multiplicative representation.

Let us show that poles of $r_{\pm}(z(\zeta))$ satisfy the Blaschke condition. Diagonal entries $R_{-1,-1}(z)$ and $R_{0,0}(z)$ of the resolvent matrix–function $R(z)$ are holomorphic in $\bar{\mathbb{C}} \setminus E$. By the theorem mentioned above they are functions of bounded characteristic. Using the force of (2.1),

$$-1/R_{-1,-1}(z) = -1/r_{-}(z) + p_0^2 r_{+}(z),$$
$$-1/R_{0,0}(z) = -1/r_{+}(z) + p_0^2 r_{-}(z).$$

This means that poles of $r_{\pm}$ are subsets of poles of $1/R_{-1,-1}$ and $1/R_{0,0}$. Thus $r_{\pm}(z(\zeta))$ are functions of bounded characteristic.

Now, let us use the Szegö condition $\log \det \operatorname{Im} R(z(t)) \in L^1$. Since

$$\det \operatorname{Im} R^{-1}(z(t)) = |\det R^{-1}(z(t))|^2 \det \operatorname{Im} R(z(t)),$$

using again (2.1), we have

$$\log \operatorname{Im} r_{-}^{-1}(z(t)) + \log \operatorname{Im} r_{+}^{-1}(z(t)) = \log \det \operatorname{Im} R^{-1}(z(t)) \in L^1.$$

Therefore, each of the functions $\log \operatorname{Im} r_{\pm}(z(t))$ belongs to L^1. Thus we can represent $r_{\pm}(z)$ (uniquely) in the form

$$r_{\pm}(z(\zeta)) = -\frac{e^{\pm}(0,\zeta)}{p_0 e^{\pm}(-1,\zeta)},$$

where $e^{\pm}(0,\zeta)$ and $b(\zeta)e^{\pm}(-1,\zeta)$ are functions of Smirnov class with coprime inner parts (in fact, they are Blaschke products) such that

$$\bar{t}p_0\{e^{\pm}(0,t)e^{\pm}(-1,\bar{t}) - e^{\pm}(-1,t)e^{\pm}(0,\bar{t})\} = z'(t), \tag{2.7}$$

and $e^{\pm}(0, 0) > 0$, $(be^{\pm})(-1, 0) > 0$. Note that

$$p_0 = \frac{e^{\pm}(0, 0)}{(be^{\pm})(-1, 0)}.$$

As soon as the functions $e^{\pm}(0, \zeta)$ and $e^{\pm}(-1, \zeta)$ have been constructed we are able to introduce $S(t)$ and $\mathcal{F}^{\pm}$ in their terms.

First, let us write down an expression for the resolvent matrix-function:

$$R(z(\zeta)) = \begin{bmatrix} -p_0 \frac{e^-(-1,\zeta)}{e^-(0,\zeta)} & p_0 \\ p_0 & -p_0 \frac{e^+(-1,\zeta)}{e^+(0,\zeta)} \end{bmatrix}^{-1}$$

$$= -(p_0 \Phi)^{-1}\Psi = -\tilde{\Psi}(p_0\tilde{\Phi})^{-1}, \tag{2.8}$$

where Φ and $\tilde{\Phi}$ are as in (1.21) and (1.28) respectively, and

$$\Psi(\zeta) = \tilde{\Psi}(\zeta) = \begin{bmatrix} e^-(0, \zeta) & 0 \\ 0 & e^+(0, \zeta) \end{bmatrix}.$$

Therefore,

$$p_0^2\{R(z(t)) - R^*(z(t))\} = -tz'(t)\Phi^{-1}(t)\Phi^{-1*}(t) = -tz'(t)\tilde{\Phi}^{-1*}(t)\tilde{\Phi}^{-1}(t), \tag{2.9}$$

since (see (2.7))

$$p_0\{\Psi\Phi^* - \Phi\Psi^*\} = p_0\{\tilde{\Phi}^*\tilde{\Psi} - \tilde{\Psi}^*\tilde{\Phi}\} = tz'.$$

From (2.9) and $\tilde{\Phi}^*(t) = \Phi(\bar{t})$ we get immediately that the matrix–function $S(t)$ defined by (2.6) is unitary–valued. Let us show that its element $s(\zeta)$ is an outer function. In fact, we have to show that the function $b^2(\zeta) \det \Phi(\zeta)$ is an outer function (see (1.22)). To this end let us use the representation for the diagonal entries of $R(z)$ (see (2.8))

$$R_{-1,-1}(z(\zeta)) = -\frac{e^+(-1, \zeta)e^-(0, \zeta)}{p_0 \det \Phi(\zeta)},$$

$$R_{0,0}(z(\zeta)) = -\frac{e^-(-1, \zeta)e^+(0, \zeta)}{p_0 \det \Phi(\zeta)}.$$

Let Δ be an inner part of $b^2(\zeta) \det \Phi(\zeta)$. Since $R_{0,0}(z(\zeta))$ is of Smirnov class, Δ is a divisor of $e^-(-1, \zeta)e^+(0, \zeta)$. If Δ is not trivial, then it has a non–trivial divisor Δ_1 that is a divisor of one of these functions, say, $e^-(-1, \zeta)$. Since $e^-(-1, \zeta)$ and $e^-(0, \zeta)$ are coprime (and Δ_1 is a divisor of $b^2(\zeta) \det \Phi(\zeta)$), the Δ_1 is a divisor of $e^+(0, \zeta)$, and, therefore, it is not a divisor of $e^+(-1, \zeta)$. Thus, Δ_1 is not a divisor of the product $e^+(-1, \zeta)e^-(0, \zeta)$. But this means that $R_{-1,-1}(z(\zeta))$ is not of Smirnov class. We arrive at a contradiction, hence Δ is a constant.

We define $\mathcal{F}^{\pm}$ by the formulas

$$(\mathcal{F}^+ f)(t) = \left[e^+(-1, t) \; e^+(0, t)\right] \hat{f}(z(t)),$$
$$(\mathcal{F}^- f)(t) = \left[e^-(0, t) \; e^-(-1, t)\right] \hat{f}(z(t)). \tag{2.10}$$

Evidently, $(\mathcal{F}^{\pm} J f)(t) = z(t)(\mathcal{F}^{\pm} f)(t)$ and by (2.6), (0.23) are fulfilled. Using the formula for the spectral density $\rho(x) = \frac{1}{\pi} \mathrm{Im}\, R(x)$ and (2.9), we have

$$\int_E \hat{f}^*(x)\, \rho(x)dx\, \hat{f}(x) = \frac{1}{2} \int_{\mathbb{E}} \hat{f}^*(z(t))(p_0\tilde{\Phi})^{*-1}(p_0\tilde{\Phi})^{-1} \hat{f}(z(t))\, |z'(t)|^2 dm(t).$$

Since

$$\tilde{\Phi}^{-1}(t) = \frac{1}{\det \tilde{\Phi}(t)} \begin{bmatrix} e^+(-1,t) & e^+(0,t) \\ e^-(0,t) & e^-(-1,t) \end{bmatrix},$$

we obtain

$$\|f\|^2 = \|\hat{f}\|_{L^2_{d\sigma}}^2 = \frac{1}{2}\{\|s\mathcal{F}^+ f\|^2 + \|s\mathcal{F}^- f\|^2\}$$
$$= \|\mathcal{F}^+ f\|_{s_+}^2 = \|\mathcal{F}^- f\|_{s_-}^2.$$

Thus $\mathcal{F}^+$ is an isometry, and since this map is invertible,

$$\begin{bmatrix} \hat{f}_{-1}(z(t)) \\ \hat{f}_0(z(t)) \end{bmatrix} = -\frac{p_0}{z'(t)} \begin{bmatrix} \bar{t}e^+(0,\bar{t}) & -e^+(0,t) \\ -\bar{t}e^+(-1,\bar{t}) & e^+(-1,t) \end{bmatrix} \begin{bmatrix} g(t) \\ \bar{t}g(\bar{t}) \end{bmatrix},$$

where $g = \mathcal{F}^+ f$, it is a unitary map.

Further, using (2.2), for $n \geq 0$ we have

$$e^+(n,\zeta) = \begin{bmatrix} e^+(-1,\zeta) & e^+(0,\zeta) \end{bmatrix} \begin{bmatrix} -p_0 Q_n^+(z(\zeta)) \\ P_n^+(z(\zeta)) \end{bmatrix}.$$

Due to the well known properties of orthogonal polynomials these functions have no singularity at the origin and hence they are functions of Smirnov class. This easily implies (0.24).

At last, our maps possess properties (1.19) (or (1.20)), in force of Proposition 1.1, (0.25) holds. The theorem is proved. $\square$

Theorem 2.1. *Let* $s_+ \in L^\infty(\Gamma, \alpha_+^{-2})$, $\|s_+\| \leq 1$, $\overline{s_+(\bar{t})} = s_+(t)$, *satisfy* $\log(1 - |s_+|^2) \in L^1$. *Then the reflection coefficient* s_+ *determines a Jacobi matrix of Szegö class in a unique way if and only if*

$$s(0)K_{s_\pm}^{\alpha_\pm}(0)K_{s_\mp b^{-2}}^{\alpha_\mp \mu}(0) = b'(0). \tag{2.11}$$

Proof. Assume on the contrary that

$$s(0)K_{s_+}^{\alpha_+}(0)K_{s_- b^{-2}}^{\alpha_-\mu}(0) \neq b'(0). \tag{2.12}$$

We construct two Jacobi matrices. First, we consider the basis

$$e^+(n,t) = b^n(t)K_{s_+ b^{2n}}^{\alpha_+ \mu^{-n}}(t), \tag{2.13}$$

and by J we denote the operator multiplication by $z(t)$ in $L^2_{s_+}$ with respect to this basis (Lemma 1.4). Then, starting with the basis $\{b^n(t)K_{s_- b^{2n}}^{\alpha_- \mu^{-n}}(t)\}$ in $L^2_{s_-}$, we introduce the basis

$$s(t)\tilde{e}^+(-n-1,t) = \bar{t}(b^n K_{s_- b^{2n}}^{\alpha_- \mu^{-n}})(\bar{t}) + s_-(t)(b^n K_{s_- b^{2n}}^{\alpha_- \mu^{-n}})(t). \tag{2.14}$$

By $\tilde{J}$ we denote the operator multiplication by $z(t)$ in $L^2_{s_+}$ with respect to $\{\tilde{e}^+(n, t)\}$. By Lemma 1.5,

$$s(0)\tilde{e}^+(0, 0)K^{\alpha-\mu}_{s_-b^{-2}}(0) = b'(0).$$

Thus (see (2.12)), $\tilde{e}^+(0, 0) \neq e^+(0, 0)$. Due to the uniqueness part of Theorem 0.5, $\tilde{J} \neq J$. The "only if" part is proved.

Now, let (2.11) hold, and let J be a Jacobi matrix of Szegö class and $\mathcal{F}^{\pm}$ be its representations in $L^2_{s_{\pm}}$. By Lemma 1.7,

$$K^{\alpha_{\pm}}_{s_{\pm}}(0) \leq e^{\pm}(0, 0) = \frac{b'(0)}{s(0)}\frac{1}{(be^{\mp})(-1, 0)} \leq \frac{b'(0)}{s(0)}\frac{1}{K^{\alpha_{\mp}\mu}_{s_{\mp}b^{-2}}(0)}.$$

Then (2.11) implies that, in fact, $e^{\pm}(0, 0) = K^{\alpha_{\pm}}_{s_{\pm}}(0)$ and $(be^{\mp})(-1, 0) = K^{\alpha_{\mp}\mu}_{s_{\mp}b^{-2}}(0)$, thus, due to a conclusion of Lemma 1.7,

$$e^{\pm}(0, t) = K^{\alpha_{\pm}}_{s_{\pm}}(t), \quad e^{\mp}(-1, t) = b^{-1}(t)K^{\alpha_{\mp}\mu}_{s_{\mp}b^{-2}}(t).$$

Recall that these functions determine the functions $r_{\pm}(z)$ and the coefficient p_0 (see (2.3)), and they, in their turn, determine J. The theorem is proved. □

Corollary 2.1. *Let J be a Jacobi matrix of Szegö class with a homogeneous spectrum E. Let $\rho(x)$ be the density of its spectral measure and $S(t)$ be its scattering matrix-function. If*

$$\int_E \rho^{-1}(x)\,dx < \infty, \tag{2.15}$$

then there is no other Jacobi matrix of Szegö class with the same scattering matrix-function $S(t)$.

Proof. By virtue of (1.29), (2.15) is equivalent to

$$\int_{\mathbb{E}} \tilde{\Phi}(t)\tilde{\Phi}^*(t)\,dm < \infty,$$

that is $e^{\pm}(0, t)$ and $e^{\pm}(-1, t)$ belong to $L^2_{dm|\mathbb{E}}$. Then word by word repetition of arguments in the proof of Lemma 1.6 gives us

$$(I + \mathcal{H}_{s_{\pm}})e^{\pm}(0, t)e^{\pm}(0, 0) = k^{\alpha_{\pm}}(t),$$
$$(I + \mathcal{H}_{s_{\pm}b^{-2}})(be^{\pm})(-1, t)(be^{\pm})(-1, 0) = k^{\alpha_{\pm}\mu}(t).$$

Thus, $e^{\pm}(0, 0) = K^{\alpha_{\pm}}_{s_{\pm}}(0)$ and $(be^{\pm})(-1, 0) = K^{\alpha_{\pm}\mu}_{s_{\pm}b^{-2}}(0)$. Since, generally,

$$s(0)e^{\pm}(0, 0)(be^{\pm})(-1, 0) = b'(0),$$

(2.11) holds, the corollary is proved. □

To finish this section we give an example of a scattering matrix-function, which does not determine a Jacobi matrix of Szegö class. Moreover, in this example, the associated operators $(I + \mathcal{H}_{s_{\pm}})$ are invertible.

Example. Let $v_\pm \in H^\infty(\Gamma)$, $\|v_\pm\| < 1$, $\overline{v_\pm(t)} = v(t)$, $v_\pm(0) = 0$. Define outer functions $u_\pm$, $u_\pm(0) > 0$, by

$$|u_\pm|^2 + |v_\pm|^2 = 1.$$

Then, we put

$$s_\pm^0 = -\bar{v}_\pm u_\pm / \bar{u}_\pm.$$

At last,

$$S(t) = \begin{bmatrix} s_- & s \\ s & s_+ \end{bmatrix} = \begin{bmatrix} s_-^0 & 0 \\ 0 & s_+^0 \end{bmatrix} + \begin{bmatrix} u_- & 0 \\ 0 & u_+ \end{bmatrix} \mathcal{E} \left\{ I - \begin{bmatrix} v_- & 0 \\ 0 & v_+ \end{bmatrix} \mathcal{E} \right\}^{-1} \begin{bmatrix} u_- & 0 \\ 0 & u_+ \end{bmatrix},$$

where

$$\mathcal{E} = \begin{bmatrix} \frac{1+\Delta}{2} & \frac{1-\Delta}{2} \\ \frac{1-\Delta}{2} & \frac{1+\Delta}{2} \end{bmatrix},$$

and Δ is an inner function from $H^\infty(\Gamma)$, $\overline{\Delta(t)} = \Delta(t)$.

In this case $\mathcal{H}_{s_\pm} = \mathcal{H}_{s_\pm^0}$, since their symbols differ by functions from $H^\infty(\Gamma, \alpha_\pm^{-2})$, and therefore $(I + \mathcal{H}_{s_\pm})$ are invertible. On the other hand, the coefficient s is of the form

$$s = \frac{u_+ u_- (1 - \Delta)/2}{1 - (v_+ + v_-)(1 + \Delta)/2 + v_+ v_- \Delta},$$

and because of the factor $(1 - \Delta)/2$, $1/s$ does not belong to $H^\infty(\Gamma, \alpha_+ \alpha_-)$.

The simplest choice of parameters:

$$E = [-2, 2], \quad v_\pm(t) = a_\pm t, \ a_\pm \in (0, 1);$$

$\Delta(t)$ is a Blaschke product, deg $\Delta > 1$, gives us an example where $\tilde{e}^+(-1, t)$, defined by (2.14), does not belong to L^2 (this is direct calculation), at the same time $e^+(-1, t)$, defined by (2.13), belongs to L^2.

3. A Weighted Hilbert Transform

By $\mathfrak{H}$ we denote the transform

$$(\mathfrak{H}g)(z) = \int_E \frac{g(x)}{z - x} dx, \quad z \in \mathbb{C} \setminus E, \tag{3.1}$$

primarily defined on integrable 2D vector-functions.

Lemma 3.1. *Let J be of Szegö class and $\mathcal{F}^\pm$ give its scattering representation in the model spaces $L^2_{s_\pm}$. Then*

$$\begin{bmatrix} \mathcal{F}^- f^- \\ \mathcal{F}^+ f^+ \end{bmatrix}(\zeta) = p_0 \Phi(\zeta) \{ \mathfrak{H}(\rho \hat{f}) \}(z(\zeta)), \tag{3.2}$$

for any finite vector $f = f^- \oplus f^+ \in l^2(\mathbb{Z}) = l^2(\mathbb{Z}_-) \oplus l^2(\mathbb{Z}_+)$.

Proof. Let $\tilde{\mathcal{P}}_n(z)$ denote the nth matrix orthonormal polynomial with respect the spectral measure $d\sigma$. Recall that

$$\tilde{\mathcal{P}}_n(z) = \left[\hat{e}_{-n-1}(z)\ \hat{e}_n(z) \right] = \begin{bmatrix} P_n^-(z) & -p_0 Q_n^+(z) \\ -p_0 Q_n^-(z) & P_n^+(z) \end{bmatrix},$$

and, analogically to the scalar case,

$$\tilde{\mathcal{Q}}_n(z) := \int d\sigma(x) \frac{\tilde{\mathcal{P}}_n(z) - \tilde{\mathcal{P}}_n(x)}{z - x} = \begin{bmatrix} Q_n^-(z) & 0 \\ 0 & Q_n^+(z) \end{bmatrix}. \tag{3.3}$$

Based on (3.3), we have

$$p_0 \Phi(\zeta) \int d\sigma(x) \frac{\tilde{\mathcal{P}}_n(x)}{z(\zeta) - x} = p_0 \Phi(\zeta) \left\{ \int \frac{d\sigma(x)}{z(\zeta) - x} \tilde{\mathcal{P}}_n(z(\zeta)) \right.$$
$$\left. - \int d\sigma(x) \frac{\tilde{\mathcal{P}}_n(z(\zeta)) - \tilde{\mathcal{P}}_n(x)}{z(\zeta) - x} \right\}$$
$$= -p_0 \Phi(\zeta) R(z(\zeta)) \tilde{\mathcal{P}}_n(z(\zeta)) - p_0 \Phi(\zeta) \tilde{\mathcal{Q}}_n(z(\zeta)).$$

Using (2.8) and Definition (2.10), we get

$$p_0 \Phi(\zeta) \int \frac{\rho(x) \left[\hat{e}_{-n-1}(x)\ \hat{e}_n(x) \right]}{z(\zeta) - x} dx = \Psi(\zeta) \tilde{\mathcal{P}}_n(z(\zeta)) - p_0 \Phi(\zeta) \tilde{\mathcal{Q}}_n(z(\zeta))$$
$$= \begin{bmatrix} e^-(n, \zeta) & 0 \\ 0 & e^+(n, \zeta) \end{bmatrix}$$
$$= \begin{bmatrix} (\mathcal{F}^- e_{-n-1})(\zeta) & 0 \\ 0 & (\mathcal{F}^+ e_n)(\zeta) \end{bmatrix}.$$

In fact, this finishes the proof. $\square$

Theorem 3.1. *Let $\rho(x)$ be the spectral density of a Jacobi matrix J of Szegö class and $s_+(t)$ be the reflection coefficient. Then the following statements are equivalent:*

1. *There exist $C < \infty$ such that*

$$\int_E (\mathfrak{H}g)^*(x - i0)\rho^{-1}(x)(\mathfrak{H}g)(x - i0)\, dx$$
$$+ \int_E (\mathfrak{H}g)^*(x + i0)\rho^{-1}(x)(\mathfrak{H}g)(x + i0)\, dx$$
$$\leq C \int_E g^*(x)\rho^{-1}(x)g(x)\, dx. \tag{3.4}$$

2. *s_+ determines J and the operators $(I + \mathcal{H}_{s_\pm})$ are invertible.*

Proof. $1 \Rightarrow 2$. Since (see (1.30))

$$||\mathcal{F}^- f^-||^2 + ||\mathcal{F}^+ f^+||^2 = \int_\mathbb{E} \{\mathfrak{H}(\rho\hat{f})\}^*(z(t))(p_0\Phi)^*(t)(p_0\Phi)(t)\{\mathfrak{H}(\rho\hat{f})\}(z(t))\,dm$$

$$= \int_\mathbb{E} \{\mathfrak{H}(\rho\hat{f})\}^*(z(t))\frac{1}{2\pi}\rho^{-1}(z(t))\{\mathfrak{H}(\rho\hat{f})\}(z(t))\frac{|z'(t)||dt|}{2\pi}$$

$$= \left(\frac{1}{2\pi}\right)^2 \int_E \{\mathfrak{H}(\rho\hat{f})\}^*(x-i0)\rho^{-1}(x)\{\mathfrak{H}(\rho\hat{f})\}(x-i0)\,dx$$

$$+ \left(\frac{1}{2\pi}\right)^2 \int_E \{\mathfrak{H}(\rho\hat{f})\}^*(x+i0)\rho^{-1}(x)\{\mathfrak{H}(\rho\hat{f})\}(x+i0)\,dx$$

$$\leq \frac{C}{(2\pi)^2} \int_E \hat{f}^*(x)\rho(x)\hat{f}(x)\,dx = \frac{C}{(2\pi)^2}||f||^2, \qquad (3.5)$$

we get $\mathcal{F}^\pm f^\pm \in A_1^2(\Gamma, \alpha_\pm)$. Thus, $\mathcal{F}^\pm\{l^2(Z_\pm)\} = H_{s_\pm}^2(\alpha_\pm)$. By Lemma 1.7 and Theorem 2.1 we come to the conclusion that s_+ determines J. Further, by (3.5)

$$||\mathcal{F}^- f^-||^2 + ||\mathcal{F}^+ f^+||^2 \leq \frac{C}{(2\pi)^2}\{||f^-||^2 + ||f^+||^2\}$$

$$= \frac{C}{(2\pi)^2}\{||\mathcal{F}^- f^-||_{s_-}^2 + ||\mathcal{F}^+ f^+||_{s_+}^2\}.$$

Using again $\mathcal{F}^\pm f^\pm \in A_1^2(\Gamma, \alpha_\pm)$, we can represent the last norms in the form

$$||\mathcal{F}^- f^-||^2 + ||\mathcal{F}^+ f^+||^2$$

$$\leq \frac{C}{(2\pi)^2}\{\langle(I + \mathcal{H}_{s_-})\mathcal{F}^- f^-, \mathcal{F}^- f^-\rangle + \langle(I + \mathcal{H}_{s_+})\mathcal{F}^+ f^+, \mathcal{F}^+ f^+\rangle\}. \quad (3.6)$$

This proves the second statement in 2.

$2 \Rightarrow 1$. Recall that $H_{s_\pm}^2(\alpha_\pm) = \mathrm{clos}_{L_{s_\pm}^2} A_1^2(\Gamma, \alpha_\pm)$, but in the case under consideration, the norm in $H_{s_\pm}^2(\alpha_\pm)$ is equivalent to the norm in $A_1^2(\Gamma, \alpha_\pm)$, i.e.:

$$h \in H_{s_\pm}^2(\alpha_\pm) \Rightarrow h \in A_1^2(\Gamma, \alpha_\pm).$$

Further, since s_+ determines J, by Lemma 1.7, we have $\mathcal{F}^\pm\{l^2(Z_\pm)\} = H_{s_\pm}^2(\alpha_\pm)$. So, starting with (3.6) we obtain (3.4). $\quad\square$

4. Matrix A_2 on Homogeneous Sets

In this section our goal is to show that one can substitute (3.4) by the A_2 condition. We do this in a bit more general setting than we need.

Let E be a homogeneous set. Throughout this section P_+ denotes the orthoprojector from the vector–valued $L^2(\mathbb{C}^n)$ onto $H^2(\mathbb{C}^n)$ in the upper halfplane. We are interested in the boundedness of the weighted transform

$$W^{1/2}P_+W^{-1/2} : \chi_E L^2(\mathbb{C}^n) \to \chi_E L^2(\mathbb{C}^n), \qquad (4.1)$$

where W is a weight on E and χ_E is the characteristic function of the set E.

Here is an analog of the matrix A_2 condition

$$\sup_{x \in E, 0 < \delta < 1} ||\langle W \rangle_{I_{(x,\delta)}}^{1/2} \langle W^{-1} \rangle_{I_{(x,\delta)}}^{1/2}|| < \infty, \tag{4.2}$$

where $I_{(x,\delta)} := (x - \delta, x + \delta)$ and

$$\langle W \rangle_{I_{(x,\delta)}} := \frac{1}{|I_{(x,\delta)}|} \int_{I_{(x,\delta)} \cap E} W(t) \, dt.$$

This supremum will be called $Q_{2,E}(W)$.

Theorem 4.1. *The operator (4.1) is bounded if and only if $Q_{2,E}(W) < \infty$.*

Proof of necessity. With an arbitrary $z_0 \in \mathbb{C}_+$ we associate a subspace $K_{b_{z_0}} = H^2(\mathbb{C}^n) \ominus b_{z_0} H^2(\mathbb{C}^n)$ of the Hardy space, $b_{z_0}(z) = \frac{z - z_0}{z - \overline{z_0}}$. It is well known, that

$$P_{K_{b_{z_0}}} = P_+ - b_{z_0} P_+ \overline{b_{z_0}}$$

and

$$\langle P_{K_{b_{z_0}}} f, g \rangle_{L^2(\mathbb{C}^n)} = \langle (P_+ f)(z_0), (P_+ g)(z_0) \rangle_{\mathbb{C}^n}.$$

Because of the first of these relations we have

$$|\langle W^{1/2} P_{K_{b_{z_0}}} W^{-1/2} f, g \rangle| \leq 2Q ||\chi_E f|| \, ||\chi_E g||.$$

Now, using the second one we get

$$|\langle (P_+ W^{-1/2} f)(z_0), (P_+ W^{1/2} g)(z_0) \rangle| \leq 2Q ||\chi_E f|| \, ||\chi_E g||. \tag{4.3}$$

Let us substitute

$$f = W^{-1/2} \frac{\xi}{x - \overline{z_0}}, \quad g = W^{1/2} \frac{\eta}{x - \overline{z_0}}, \quad \xi, \eta \in \mathbb{C}^n$$

in (4.3). This gives us

$$|\langle \langle W^{-1} \rangle_{z_0} \xi, \langle W \rangle_{z_0} \eta \rangle| \leq 2Q ||\langle W^{-1} \rangle_{z_0}^{1/2} \xi|| \, ||\langle W \rangle_{z_0}^{1/2} \eta||,$$

where $\langle W \rangle_{z_0}$ denotes an average with the Poisson kernel,

$$\langle W \rangle_{z_0} := \frac{1}{\pi} \int W \frac{\operatorname{Im} z_0}{|x - z_0|^2} \, dx.$$

Thus we proved an inequality with the Poisson averages

$$\langle W \rangle_{z_0} \leq 2Q \langle W^{-1} \rangle_{z_0}^{-1}. \tag{4.4}$$

At last let us note that

$$\frac{\operatorname{Im} z_0}{|x - z_0|^2} \geq \frac{c}{|I|} \chi_I, \quad I = I_{(\operatorname{Re} z_0, \operatorname{Im} z_0)},$$

with an absolute and positive constant c. Therefore (4.4) implies (4.2).

Lemma 4.1. *If I is a centered at the E interval and z_0 is the center of the square built on I, then*

$$W \in A_2(E) \Rightarrow \quad \langle W \rangle_{z_0} \leq C(E, Q_{2,E}(W)) \langle W \rangle_I. \tag{4.5}$$

Proof. First we note that for $\lambda = 2/\eta$,

$$|\lambda I \cap E| \geq \eta |\lambda I| \geq 2|I| \geq 2|I \cap E|,$$

and therefore $|(\lambda I \setminus I) \cap E| \geq |I \cap E|$. Let us show that

$$W(\lambda I) \geq \left(1 + \frac{\eta^2}{\lambda^2 Q^2}\right) W(I) \quad \text{for } W \in A_2(E). \tag{4.6}$$

Integrating the inequality

$$\begin{bmatrix} W^{-1} & 1 \\ 1 & W \end{bmatrix} \geq 0$$

over $(\lambda I \setminus I) \cap E$ we get

$$\begin{bmatrix} W^{-1}(\lambda I \setminus I) & |(\lambda I \setminus I) \cap E| \\ |(\lambda I \setminus I) \cap E| & W(\lambda I \setminus I) \end{bmatrix} \geq 0.$$

Therefore

$$\begin{bmatrix} W^{-1}(\lambda I) & |(\lambda I \setminus I) \cap E| \\ |(\lambda I \setminus I) \cap E| & W(\lambda I) - W(I) \end{bmatrix} \geq 0,$$

or

$$W(\lambda I) - W(I) \geq |(\lambda I \setminus I) \cap E|^2 \{W^{-1}(\lambda I)\}^{-1} \geq |I \cap E|^2 \{W^{-1}(\lambda I)\}^{-1}.$$

Using (4.2) we obtain

$$W(\lambda I) - W(I) \geq \frac{|I \cap E|^2}{Q^2 |\lambda I|^2} W(\lambda I) \geq \frac{\eta^2}{Q^2 \lambda^2} W(I).$$

To prove (4.5), using

$$\frac{\operatorname{Im} z_0}{|x - z_0|^2} \leq \frac{c}{|I|} \sum \frac{1}{\lambda^{2k}} \chi_{\lambda^k I}, \quad \lambda^k I = I_{(\operatorname{Re} z_0, \lambda^k \operatorname{Im} z_0)},$$

we write the following chain of inequalities:

$$\begin{aligned}
\langle W \rangle_{z_0} &\leq \frac{c}{|I|} \sum \frac{1}{\lambda^{2k}} W(\lambda^k I) \\
&\leq \frac{c}{|I|} \sum \frac{Q^2 |\lambda^k I|^2}{\lambda^{2k}} \{W^{-1}(\lambda^k I)\}^{-1} \\
&\leq \frac{cQ^2}{|I|} \sum \left(1 + \frac{\eta^2}{\lambda^2 Q^2}\right)^{-k} |I|^2 \{W^{-1}(I)\}^{-1} \\
&\leq \frac{cQ^2}{\eta^2} \sum \left(1 + \frac{\eta^2}{\lambda^2 Q^2}\right)^{-k} \langle W \rangle_I.
\end{aligned}$$

Proof of sufficiency. We want to prove that (4.2) suffices for $W^{1/2}P_+W^{-1/2}$ in (4.1) to be bounded. Fix $f, g \in \chi_E L^2(\mathbb{C}^n)$. We need to show

$$\int_{\mathbb{C}_+} \left| \langle (P_+W^{-1/2}f)'(z), (P_+W^{1/2}g)'(z)\rangle_{\mathbb{C}^n} \right| \operatorname{Im} z \, dA(z) \leq C\|f\|\,\|g\|.$$

In other words, introducing a Stolz cone Γ_t and

$$S(t) = \int_{\Gamma_t} \left| \langle (P_+W^{-1/2}f)'(z), (P_+W^{1/2}g)'(z)\rangle_{\mathbb{C}^n} \right| dA(z)$$

one needs to prove that

$$\int S(t) \, dt \leq C\|f\|\,\|g\|. \tag{4.7}$$

We follow closely the lines of the proof in [13]. Let us consider a nonnegative function $h(t)$ and

$$S_{h(t)}(t) = \int_{\Gamma_{t,h(t)}} \left| \langle (P_+W^{-1/2}f)'(z), (P_+W^{1/2}g)'(z)\rangle_{\mathbb{C}^n} \right| dA(z),$$

where

$$\Gamma_{t,h(t)} = \Gamma_t \cap \{z : \operatorname{Im} z \leq h(t)\}.$$

Let us note that

$$\int S(t) \, dt \leq c \int S_{h(t)} \, dt \tag{4.8}$$

if the function $h(t)$ has the following property:

$$\forall I \subset \mathbb{R} \quad |\{t \in I : h(t) \geq |I|\}| \geq a|I|. \tag{4.9}$$

Let us choose h to be maximal such that

$$S_{h(t)}(t) \leq B(M\|f\|^{p_*})^{1/p_*}(t)(M\|g\|^{p_*})^{1/p_*}(t), \tag{4.10}$$

where $B, p_* \in (1, 2)$ will be chosen a bit later and M denotes the maximal function

$$(Mf)(x) = \sup_{\delta > 0} \frac{1}{|I(x, \delta)|} \int_{I(x,\delta)} |f(t)| \, dt.$$

If this h satisfies (4.9), then (4.8) and (4.10) imply what we need.

To choose B, p_* and to prove that h satisfies (4.9) we follow the algorithm below. Let I_0 be an arbitrary interval on the real axis. We will consider two cases: $2I_0 \cap E \neq \emptyset$ and $2I_0 \cap E = \emptyset$.

In the first case we fix an interval I centered at E such that $I_0 \subset I$ and $|I| \leq 3|I_0|$. Let $f_1 = f \cdot \chi_{2I}, g_1 = g \cdot \chi_{2I}$ and $f_2 = f - f_1, g_2 = g - g_1$. Denote $A_I = \langle W \rangle_I^{1/2}$. Consider

$$t \in I, \quad S^{A_I}(f_i)(t) = \left(\int_{\Gamma_{t,|I|}} \|(P_+A_IW^{-1/2}f_i)'\|^2 \, dA(z) \right)^{1/2}, \quad i = 1, 2;$$

$$t \in I, \quad S^{A_I^{-1}}(g_i)(t) = \left(\int_{\Gamma_{t,|I|}} \|(P_+A_I^{-1}W^{1/2}g_i)'\|^2 \, dA(z) \right)^{1/2}, \quad i = 1, 2.$$

We will fix later $\alpha = \alpha(Q_{2,E}(W), n) > 1$. Now,

$$\frac{1}{|I|} \int_I \left(S^{A_I^{-1}}(g_1)(t) \right)^\alpha dt \leq \frac{C(\alpha)}{|I|} \int_{2I} ||A_I^{-1} W^{1/2} g(t)||^\alpha dt$$

$$\leq \frac{C_1(\alpha, n)}{|I|} \int_{2I} \left(||g(t)|| \sum_{i=1}^n ||W^{1/2}(t) A_I^{-1} e_i|| \right)^\alpha dt$$

$$\leq \frac{C_2(\alpha, n)}{|I|} \left(\int_{2I} ||g(t)||^{(2-\tilde{\epsilon})\alpha} dt \right)^{\frac{1}{2-\tilde{\epsilon}}} \sum_{i=1}^n \left(\int_{2I} ||W^{1/2}(t) A_I^{-1} e_i||^{(2+\epsilon)\alpha} dt \right)^{\frac{1}{2+\epsilon}}.$$
$$(4.11)$$

Here $(2+\epsilon)^{-1} + (2-\tilde{\epsilon})^{-1} = 1$. Notice that for every vector $x \in \mathbb{C}^n$ the scalar function $t \to ||W(t)^{1/2}x||$ is uniformly in the scalar $A_2(E)$. In particular, there exists such an $\epsilon_0 > 0$ that we have the inverse Hölder inequality for all such functions uniformly:

$$\forall I \text{ centered at } x \in E \quad \left(\frac{1}{|I|} \int_I ||W(t)^{1/2}x||^{2+\epsilon_0} dt \right)^{\frac{1}{2+\epsilon_0}}$$

$$\leq C \left(\frac{1}{|I|} \int_I ||W(t)^{1/2}x||^2 dt \right)^{\frac{1}{2}}. \quad (4.12)$$

Let us choose $\epsilon = \frac{\epsilon_0}{2}$ ($\tilde{\epsilon} = \frac{\epsilon_0}{2+\epsilon_0}$), $\alpha = 1 + \frac{\epsilon_0}{2(2+\epsilon_0)}$, then we have

$$(2+\epsilon)\alpha < 2 + \epsilon_0, \quad (4.13)$$

$$(2-\tilde{\epsilon})\alpha < 2. \quad (4.14)$$

We use (4.13) and the inverse Hölder inequality (4.12) in (4.11) to rewrite it as

$$\left(\frac{1}{|I|} \int_I \left(S^{A_I^{-1}}(g_1)(t) \right)^\alpha dt \right)^{1/\alpha}$$

$$\leq C(\alpha, n) \left(\frac{1}{|I|} \int_{2I} ||g(t)||^{(2-\tilde{\epsilon})\alpha} dt \right)^{\frac{1}{(2-\tilde{\epsilon})\alpha}} \sum_{i=1}^n \left(\frac{1}{|I|} \int_{2I} ||W^{1/2}(t) A_I^{-1} e_i||^2 dt \right)^{\frac{1}{2}}$$

$$\leq C_1(\alpha, n) \left(\frac{1}{|I|} \int_{2I} ||g(t)||^{(2-\tilde{\epsilon})\alpha} dt \right)^{\frac{1}{(2-\tilde{\epsilon})\alpha}} \sum_{i=1}^n \left\langle \langle W \rangle_I^{-1/2} \langle W \rangle_{2I} \langle W \rangle_I^{-1/2} e_i, e_i \right\rangle^{\frac{1}{2}}$$

$$\leq C_3(\alpha, n, Q_{2,E}(W)) \inf_{x \in I} \left(M||g||^{p_*} \right)^{\frac{1}{p_*}} (x),$$

where $p_* = (2-\tilde{\epsilon})\alpha < 2$. We used the doubling property of W: $\langle W \rangle_I^{-1/2} \langle W \rangle_{2I}$ $\langle W \rangle_I^{-1/2} \leq 2 \frac{Q^2}{\eta^2}$, the inequality which can be proved in the same way as (4.6).

The last inequality ensures that for any τ, $\tau \in (0, 1)$, using Kolmogorov-type inequalities we can find a subset $E(\tau, I_0) \subset I_0$, $|E(\tau, I_0)| \geq |I_0| - \tau^\alpha |I| \geq (1 - 3\tau^\alpha)|I_0|$ such that

$$t \in E(\tau, I_0) \Rightarrow \quad S^{A_I^{-1}}(g_1)(t) \leq \frac{C_3(\alpha, n, Q_{2,E}(W))}{\tau} \inf_{x \in I} \left(M||g||^{p_*} \right)^{\frac{1}{p_*}} (x). \quad (*)$$

Similarly, for every τ there exists a set $E(\tau, I_0)$, $|E(\tau, I_0)| \geq (1 - 3\tau^\alpha)|I_0|$ such that

$$t \in E(\tau, I_0) \implies \quad S^{A_I}(f_1)(t) \leq \frac{C(\alpha, n, Q_{2,E}(W))}{\tau} \inf_{x \in I} \left(M\|f\|^{p_*} \right)^{\frac{1}{p_*}}(x). \quad (*)$$

Here we use the same calculations and the fact that for any I centered at E,

$$\langle W \rangle_I^{1/2} \langle W^{-1} \rangle_{2I} \langle W \rangle_I^{1/2} \leq 2Q^2.$$

Now let us work with f_2, g_2. Let c_I be the center of the square built on $2I$. Using the representation

$$\left(P_+ A_I W^{-1/2} f_2 \right)'(z) = \frac{1}{2\pi i} \int \frac{(A_I W^{-1/2} f_2)(x)}{(x - z)^2} dx, \quad \operatorname{Im} z > 0, \quad (4.15)$$

clearly, we obtain for every $t \in I$,

$$\left(\int_{\Gamma_{t,|I|}} \| \left(P_+ A_I W^{-1/2} f_2 \right)'(z)\|^2 \, dA(z) \right)^{1/2} \leq C \int \frac{\operatorname{Im} c_I}{|x - c_I|^2} \|(A_I W^{-1/2} f_2)(x)\| \, dx. \tag{4.16}$$

Therefore, using the inverse Hölder inequality (4.12), we have again

$$\left(\int_{\Gamma_{t,|I|}} \| \left(P_+ A_I W^{-1/2} f_2 \right)'(z)\|^2 \, dA(z) \right)^{1/2}$$

$$\leq C \sum_{i=1}^n \int \frac{\operatorname{Im} c_I}{|x - c_I|^2} \|W^{-1/2} A_I e_i\| \, \|f_2\| \, dx$$

$$\leq C_1 \sum_{i=1}^n \left(\int \frac{\operatorname{Im} c_I}{|x - c_I|^2} \|W^{-1/2} A_I e_i\|^{2+\epsilon} \, dx \right)^{\frac{1}{2+\epsilon}} \left(\int \frac{\operatorname{Im} c_I}{|x - c_I|^2} \|f_2\|^{2-\tilde{\epsilon}} \, dx \right)^{\frac{1}{2-\tilde{\epsilon}}}$$

$$\leq C_2 \sum_{i=1}^n \left\langle \langle W \rangle_I^{1/2} \langle W^{-1} \rangle_{c_I} \langle W \rangle_I^{1/2} e_i, e_i \right\rangle^{\frac{1}{2}} \inf_{x \in I} \left(M\|f\|^{2-\tilde{\epsilon}}(x) \right)^{\frac{1}{2-\tilde{\epsilon}}}. \tag{4.17}$$

Here $2 + \epsilon$ is close to 2 ($\epsilon \leq \epsilon_0$). Finally, using Lemma 4.1 we estimate the last sum by a constant:

$$\left(\int_{\Gamma_{t,|I|}} \| \left(P_+ A_I W^{-1/2} f_2 \right)'(z)\|^2 \, dA(z) \right)^{1/2} \leq C(n, E, Q) \inf_{x \in I} \left(M\|f\|^{2-\tilde{\epsilon}}(x) \right)^{\frac{1}{2-\tilde{\epsilon}}}.$$

That is

$$S^{A_I}(f_2)(t) \leq C(n, E, Q) \inf_{x \in I} \left(M\|f\|^{2-\tilde{\epsilon}}(x) \right)^{\frac{1}{2-\tilde{\epsilon}}}, \quad \forall t \in I, \quad (*)$$

the same for $S^{A_I^{-1}}(g_2)(t)$.

Combining all (*) inequalities we obtain that with a suitable $C = C(n, E, W)$,

$$S_I(t) := \int_{\Gamma_{t,|I|}} \left| \langle (P_+ W^{-1/2} f)'(z), (P_+ W^{1/2} g)'(z) \rangle_{\mathbb{C}^n} \right| dA(z)$$

$$\leq S^{A_I}(f)(t) S^{A_I^{-1}}(g)(t) \leq C^2 \left(M \|f\|^{p_*}(t) \right)^{\frac{1}{p_*}} \left(M \|g\|^{p_*}(t) \right)^{\frac{1}{p_*}} \qquad (4.18)$$

at least on a quarter of I_0. Of course, $S_{I_0}(t) \leq S_I(t)$.

In the case $2I_0 \cap E = \emptyset$ we fix an interval I centered at E such that $I_0 \subset I$ and $\text{dist}(I_0, E) \geq |I|/6$. Let c_I be the center of the square built on I. We can use again a representation of the form (4.15):

$$\left(P_+ A_I W^{-1/2} f \right)'(z) = \frac{1}{2\pi i} \int \frac{(A_I W^{-1/2} f)(x)}{(x - z)^2} dx, \quad \text{Im } z > 0,$$

to obtain an analog of (4.16),

$$\left(\int_{\Gamma_{t,|I|}} \left\| \left(P_+ A_I W^{-1/2} f \right)'(z) \right\|^2 dA(z) \right)^{1/2} \leq C \int \frac{\text{Im } c_I}{|x - c_I|^2} \|(A_I W^{-1/2} f)(x)\| dx$$

for all $t \in I_0$. Continuing in this way we get

$$S^{A_I}(f)(t) \leq C(n, E, Q) \inf_{x \in I} \left(M \|f\|^{p_*}(x) \right)^{\frac{1}{p_*}}, \quad \forall t \in I_0.$$

The same for $S^{A_I^{-1}}(g)(t)$. Thus

$$S_I(t) \leq C(n, E, Q)^2 \left(M \|f\|^{p_*}(t) \right)^{\frac{1}{p_*}} \left(M \|g\|^{p_*}(t) \right)^{\frac{1}{p_*}} \qquad (4.19)$$

everywhere on I_0.

Let B be the largest constant in (4.18), (4.19). We have already chosen $p_* < 2$. Now we introduce the following function $h(t)$:

$$h(t) = \sup\{h : S_h(t) \leq B \left(M \|f\|^{p_*}(t) \right)^{\frac{1}{p_*}} \left(M \|g\|^{p_*}(t) \right)^{\frac{1}{p_*}} \}.$$

What we proved can be summarized in:

$$\begin{aligned} &\text{if } I_0 : 2I_0 \cap E \neq \emptyset \quad \text{then} \quad h(t) \geq |I_0| \quad \text{on a quarter of measure of } I_0, \\ &\text{if } I_0 : 2I_0 \cap E = \emptyset \quad \text{then} \quad h(t) \geq |I_0| \quad \forall t \in I_0. \end{aligned}$$

In any case,

$$\frac{1}{4} \int_{\mathbb{C}_+} \left| \langle (P_+ W^{-1/2} f)'(z), (P_+ W^{1/2} g)'(z) \rangle_{\mathbb{C}^n} \right| \text{Im } z \, dA(z)$$

$$\leq \int_{\mathbb{R}} \int_{\Gamma_{t,h(t)}} \left| \langle (P_+ W^{-1/2} f)'(z), (P_+ W^{1/2} g)'(z) \rangle_{\mathbb{C}^n} \right| dA(z) \, dt$$

$$\leq B \int_{\mathbb{R}} \left(M \|f\|^{p_*}(t) \right)^{\frac{1}{p_*}} \left(M \|g\|^{p_*}(t) \right)^{\frac{1}{p_*}} dt$$

$$\leq B \left(\int_{\mathbb{R}} \left(M \|f\|^{p_*}(t) \right)^{\frac{2}{p_*}} dt \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \left(M \|g\|^{p_*}(t) \right)^{\frac{2}{p_*}} dt \right)^{\frac{1}{2}}$$

$$\leq BC(p_*) \left(\int_{\mathbb{R}} \|f\|^2(t) \, dt \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \|g\|^2(t) \, dt \right)^{\frac{1}{2}}$$

because $\frac{2}{p_*} > 1$, and we can use the Hardy–Littlewood maximal theorem. The theorem is proved. $\square$

5. The Inverse Scattering Problem and a Riemann–Hilbert Problem

Reduction of an inverse scattering problem to Riemann–Hilbert Problem is, maybe, the most popular approach (see, e.g. [14]). In this section, we show what kind of a Riemann–Hilbert problem is associated with the problem under consideration.

Let us define $\sqrt{-b^2 z'}(\zeta)$ as the square root of an outer function such that $\sqrt{-b^2 z'}(0) > 0$. Put

$$\sqrt{-z'}(\zeta) = \frac{\sqrt{-b^2 z'}(\zeta)}{b(\zeta)}. \tag{5.1}$$

In this case, $\overline{\sqrt{-z'}(\bar{\zeta})} = \sqrt{-z'}(\zeta)$. Let $\mathbb{E}_- = \{t \in \mathbb{E} : \mathrm{Im}\, t < 0\}$. Then $-t z'(t) = i |z'(t)|$, $t \in \mathbb{E}_-$. Thus,

$$t\{\sqrt{-z'}(t)\}^2 = i\sqrt{-z'}(t)\overline{\sqrt{-z'}(t)} = i\sqrt{-z'}(t)\sqrt{-z'}(\bar{t}), \quad t \in \mathbb{E}_-,$$

or

$$\bar{t}\sqrt{-z'}(\bar{t}) = -i\sqrt{-z'}(t), \quad t \in \mathbb{E}_-. \tag{5.2}$$

Besides,

$$\sqrt{-z'}|[\gamma] = \epsilon(\gamma)\sqrt{-z'}(\zeta),$$

where $\epsilon \in \Gamma^*$, $\epsilon^2 = 1_{\Gamma^*}$. But, in fact, the group Γ is defined up to a choice of a half–period $\tilde{\epsilon} \in \Gamma^*$. So, we may assume that

$$\sqrt{-z'}|[\gamma] = \sqrt{-z'}(\zeta). \tag{5.3}$$

Proposition 5.1. *Let $E = [b_0, a_0] \setminus \cup_{j \geq 1}(a_j, b_j)$ be a homogeneous set. Then*

$$\Pi(z(\zeta)) := \frac{\Phi(\zeta)}{\sqrt{-z'}(\zeta)}$$

is a holomorphic matrix function in $\bar{\mathbb{C}} \setminus [b_0, a_0]$ satisfying the following RHP:

$$\Pi(x - i0) = \begin{bmatrix} \alpha_{-,j} & 0 \\ 0 & \alpha_{+,j} \end{bmatrix} \Pi(x + i0), \quad x \in (a_j, b_j), \tag{5.4}$$

$$\Pi(x - i0) = -i\Sigma(x)\Pi(x + i0), \quad x \in E, \tag{5.5}$$

where $\Sigma(z(t)) := S(t)$, $t \in \mathbb{E}_-$, with the normalization at infinity:

$$\Pi(z) = \begin{bmatrix} 1 + \cdots & -\frac{a}{z} + \cdots \\ -\frac{b}{z} + \cdots & 1 + \cdots \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{bs(0)}} & 0 \\ 0 & \frac{1}{\sqrt{as(0)}} \end{bmatrix}. \tag{5.6}$$

Proof. Equation (5.4) follows from Equation (5.3). (5.5) follows from (5.2) and (2.6). To prove (5.6), we represent $\Pi(z)$ in the form

$$
\Pi(z) = \begin{bmatrix} 1 + \cdots & -\frac{a}{z} + \cdots \\ -\frac{b}{z} + \cdots & 1 + \cdots \end{bmatrix} \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}.
$$

Then, we note that

$$
c_1 c_2 = \det \Pi(\infty) = \left. \frac{e^-(-1, \zeta)e^+(-1, \zeta) - e^-(0, \zeta)e^+(0, \zeta)}{-z'(\zeta)} \right|_{\zeta=0} = \frac{1}{p_0 s(0)},
\tag{5.7}
$$

and

$$
\frac{ac_2}{c_1} = \frac{bc_1}{c_2} = \left. \frac{z(\zeta)e^{\pm}(0, \zeta)}{e^{\pm}(-1, \zeta)} \right|_{\zeta=0} = p_0.
\tag{5.8}
$$

Solving together (5.7), (5.8), we get (5.6). □

We want to finish this section with the following discussion.

As an initial data for the inverse scattering problem in this paper we used a character automorphic function $s_+ \in L^\infty(\Gamma, \alpha_+^{-2})$ and a character $\alpha_+ \in \Gamma^*$. In fact, this set of data can be defined uniquely by a function $\sigma_+(x)$ on the spectral set E ($\sigma_+(z(t)) = s_+(t)$, $t \in \mathbb{E}_-$) and a system of unimodular multipliers $\{\alpha_{+,j}\}$, each factor $\alpha_{+,j}$ is associated with a spectral gap (a_j, b_j). In terms of $\sigma_+(x)$ and $\{\alpha_{+,j}\}$ one can define a 2×2 matrix function $\Sigma(x)$ over interval $[b_0, a_0]$. Then one has to solve a RHP (5.4), (5.5) with a normalization condition (5.6) at infinity. The spectral density $\rho(x)$ (and therefore J itself) is determined via a solution of the RHP by

$$
\rho^{-1} = 2\pi a b \Pi^* \Pi.
$$

However, when solving the RHP, one carefully has to specify a class of analytic functions to which $\Pi(z)$ belongs. Therefore, in any case, one has to introduce this or that analog of the functional space $A_1^2(\Gamma, \alpha)$.

References

1. Aptekarev, A., Nikishin, E.: The scattering problem for a discrete Sturm–Liouville operator. Russian Mat. Sb. (N.S.) **121(163)**, 327–358 (1983)
2. Arov, D., Dym, H.: On matricial Nehari problems, *J*-inner matrix functions and the Muckenhoupt condition. J. Funct. Anal. **181**, 227–299 (2001)
3. Carleson, L.: *On H^∞ in multiply connected domains*. In: Conference on harmonic analysis in honor of Antoni Zygmund. Vol. **I**, **II**. (Chicago, IL, 1981), 349–372, Wadsworth Math. Ser., Belmont, CA: Wadsworth, 1983
4. Geronimo, J., Case, K.: Scattering theory and polynomials orthogonal on the real line. Trans. Amer. Math. Soc. **258**, 467–494 (1980)
5. Gesztesy, F., Nowell, R., Pötz, W.: One-dimensional scattering for quantum systems with nontrivial spatial asymptotics. Diff. Integral Eqs. **10** 521–546 (1997)
6. Guseinov, G.S.: The determination of an infinite Jacobi matrix from the scattering data. Soviet Math. Dokl. **17**, 596–600, 1976
7. Hasumi, M.: *Hardy Classes on Infinitely Connected Riemann Surfaces*. Lecture Notes in Math. 1027, Berlin–New York: Springer-Verlag, 1983
8. Marchenko, V.: *Sturm–Liouville Operators and Applications*. Basel: Birkhäuser, 1986

9. Marchenko, V.: *Nonlinear equations and operator algebras*. Translated from the Russian by V. I. Rublinet-skiĭ. Mathematics and its Applications (Soviet Series), 17. Dordrecht–Boston, MA: D. Reidel Publishing Co., 1988
10. Peherstorfer, F. and Yuditskii, P.: Asymptotic behavior of polynomials orthonormal on a homogeneous set. J. Anal. Math., to appear
11. Sodin, M. and Yuditskii, P.: Almost periodic Jacobi matrices with homogeneous spectrum, infinite dimensional Jacobi inversion, and Hardy spaces of character-automorphic functions. Journ. of Geom. Analysis **7**, 387–435 (1997)
12. Teschl, G.: *Jacobi operators and completely integrable nonlinear lattices*. Mathematical Surveys and Monographs, **72**. Providence, RI: Am. Math. Soc., 2000
13. Volberg, A.: Matrix A_p weights via S-functions. Journ. Am. Math. Soc. **10**, 445–466 (1997)
14. Beals, R., Deift, P. and Tomei, C.: *Direct and inverse scattering on the line*. Mathematical Surveys and Monographs, **28**. Providence, RI: Am. Math. Soc., 1988
15. Nevanlinna, R.: *Analytic functions*. Berlin: Springer Verlag, 1970

Communicated by B. Simon