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Quantum Dynamical Yang–Baxter Equation Over a Nonabelian Base

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Abstract: In this paper we consider dynamical r-matrices over a nonabelian base. There are two main results. First, corresponding to a fat reductive decomposition of a Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, we construct geometrically a non-degenerate triangular dynamical r-matrix using symplectic fibrations. Second, we prove that a triangular dynamical r-matrix $r : \mathfrak{h}^* \longrightarrow \wedge^2 \mathfrak{g}$ naturally corresponds to a Poisson manifold $\mathfrak{h}^* \times G$. A special type of quantization of this Poisson manifold, called compatible star products in this paper, yields a generalized version of the quantum dynamical Yang–Baxter equation (or Gervais–Neveu–Felder equation). As a result, the quantization problem of a general dynamical r-matrix is proposed.

1. Introduction

Recently, there has been growing interest in the so-called quantum dynamical Yang–Baxter equation:

$$R_{12}(\lambda)R_{13}(\lambda + \hbar h^{(2)})R_{23}(\lambda) = R_{23}(\lambda + \hbar h^{(1)})R_{13}(\lambda)R_{12}(\lambda + \hbar h^{(3)}). \quad (1)$$

This equation arises naturally from various contexts in mathematical physics. It first appeared in the work of Gervais–Neveu in their study of quantum Liouville theory [24]. Recently it reappeared in Felder's work on the quantum Knizhnik–Zamolodchikov–Bernard equation [23]. It also has been found to be connected with the quantum Caloger–Moser systems [4]. As the quantum Yang–Baxter equation is connected with quantum groups, the quantum dynamical Yang–Baxter equation is known to be connected with elliptic quantum groups [23], as well as with Hopf algebroids or quantum groupoids [20, 32, 33].

The classical counterpart of the quantum dynamical Yang–Baxter equation was first considered by Felder [23], and then studied by Etingof and Varchenko [19]. This is the

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so-called classical dynamical Yang–Baxter equation, and a solution to such an equation (plus some other reasonable conditions) is called a classical dynamical r-matrix. More precisely, given a Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ (or over $\mathbb{C}$) with an Abelian Lie subalgebra $\mathfrak{h}$, a classical dynamical r-matrix is a smooth (or meromorphic) function $r : \mathfrak{h}^* \longrightarrow \mathfrak{g}\otimes\mathfrak{g}$ satisfying the following conditions:

(i) (zero weight condition) $[h\otimes 1 + 1\otimes h, r(\lambda)] = 0, \quad \forall h \in \mathfrak{h}$;
(ii) (normal condition) $r_{12} + r_{21} = \Omega$, where $\Omega \in (S^2\mathfrak{g})^{\mathfrak{g}}$ is a Casimir element;
(iii) (classical dynamical Yang–Baxter equation[1])

$$\mathrm{Alt}\,(dr) - ([r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]) = 0, \tag{2}$$

where $\mathrm{Alt}\,dr = \sum(h_i^{(1)} \frac{\partial r_{23}}{\partial \lambda^i} - h_i^{(2)} \frac{\partial r_{13}}{\partial \lambda^i} + h_i^{(3)} \frac{\partial r_{12}}{\partial \lambda^i})$.

A fundamental question is whether a classical dynamical r-matrix is always quantizable. There has appeared a lot of work in this direction, for example, see [2,25,18]. In the triangular case (i.e., r is skew-symmetric: $r_{12}(\lambda) + r_{21}(\lambda) = 0$), a general quantization scheme was developed by the author using the Fedosov method, which works for a vast class of dynamical r-matrices, called splittable triangular dynamical r-matrices [34]. Recently, Etingof and Nikshych, using the vertex-IRF transformation method, proved the existence of quantizations for the so-called completely degenerate triangular dynamical r-matrices [21].

Interestingly, although the quantum dynamical Yang–Baxter equation in [23] only makes sense when the base Lie algebra $\mathfrak{h}$ is Abelian, its classical counterpart admits an immediate generalization for any base Lie algebra $\mathfrak{h}$ which is not necessarily Abelian. Indeed, all one needs to do is to change the first condition (i) to:

(i') $r : \mathfrak{h}^* \longrightarrow \mathfrak{g}\otimes\mathfrak{g}$ is H-equivariant, where H acts on $\mathfrak{h}^*$ by coadjoint action and on $\mathfrak{g}\otimes\mathfrak{g}$ by adjoint action.

There exist many examples of such classical dynamical r-matrices. For instance, when $\mathfrak{g}$ is a simple Lie algebra and $\mathfrak{h}$ is a reductive Lie subalgebra containing the Cartan subalgebra, there is a classification due to Etingof–Varchenko [19]. In particular, when $\mathfrak{h} = \mathfrak{g}$, an explicit formula was discovered by Alekseev and Meinrenken in their study of non-commutative Weil algebras [1]. Later, this was generalized by Etingof and Schiffermann [17] to a more general context. Moreover, under some regularity condition, they showed that the moduli space of dynamical r-matrices essentially consists of a single point once the initial value of the dynamical r-matrices is fixed. A natural question arises as to what should be the quantum counterpart of these r-matrices. And more generally, is any classical dynamical r-matrix (with nonabelian base) quantizable?

A basic question is what the quantum dynamical Yang–Baxter equation should look like when $\mathfrak{h}$ is nonabelian. In this paper, as a toy model, we consider the special case of triangular dynamical r-matrices and their quantizations. As in the Abelian case, these r-matrices naturally correspond to some invariant Poisson structures on $\mathfrak{h}^* \times G$. It is standard that quantizations of Poisson structures correspond to star products [8]. The special form of the Poisson bracket relation on $\mathfrak{h}^* \times G$ suggests a specific form that their star products should take. This leads to our definition of compatible star products. The compatibility condition (which, in this case, is just the associativity) naturally leads to a quantum dynamical Yang–Baxter equation: Eq. (33). As we shall see, this equation

[1] Throughout the paper, we follow the sign convention in [4] for the definition of a classical dynamical r-matrix in order to be consistent with the quantum dynamical Yang–Baxter equation (1). This differs in sign from the one used in [19].

indeed resembles the usual quantum dynamical Yang–Baxter equation (unsymmetrized version). The only difference is that the usual pointwise multiplication on $C^{\infty}(\mathfrak{h}^*)$ is replaced by the PBW-star product, which is indeed the deformation quantization of the canonical Lie–Poisson structure on $\mathfrak{h}^*$. Although Eq. (33) is derived by considering triangular dynamical r-matrices, it makes perfect sense for non-triangular ones as well. This naturally leads to our definition of quantization of dynamical r-matrices over an arbitrary base Lie subalgebra which is not necessary Abelian. The problem is that such an equation only makes sense for $R : \mathfrak{h}^* \longrightarrow U\mathfrak{g}\otimes U\mathfrak{g}[[\hbar]]$. In the Abelian case, it appears that one may consider R valued in a deformed universal enveloping algebra $U_{\hbar}\mathfrak{g}$, but in most cases $U_{\hbar}\mathfrak{g}$ is isomorphic to $U\mathfrak{g}[[\hbar]]$ as an algebra. So Eq. (33), in a certain sense, is general enough to include all the interesting cases. However, the physical meaning of this equation remains mysterious.

Another main result of the paper is to give a geometric construction of triangular dynamical r-matrices. More precisely, we give an explicit construction of a triangular dynamical r-matrix from a fat reductive decomposition of a Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ (see Sect. 2 for the definition). This includes those examples of triangular dynamical r-matrices considered in [19]. Our main purpose is to show that triangular dynamical r-matrices (with nonabelian base) do rise naturally from symplectic geometry. This gives us another reason why it is important to consider their quantizations. Discussion of this part occupies Sect. 2. Section 3 is devoted to the discussion of compatible star products, whose associativity leads to a “twisted-cocycle” condition. In Sect. 4, we will derive the quantum dynamical Yang–Baxter equation from this twisted-cocycle condition. The last section contains some concluding remarks and open questions.

Finally, we note that in this paper, by a dynamical r-matrix, we always mean a dynamical r-matrix over a general base Lie subalgebra unless specified. Also Lie algebras are normally assumed to be over $\mathbb{R}$, although most results can be easily modified for complex Lie algebras. For simplicity, in this paper we assume that a dynamical r-matrix is defined on $\mathfrak{h}^*$. In reality, it may only be defined on an open submanifold $U \subseteq \mathfrak{h}^*$.

2. Classical Dynamical *r*-Matrices

In this section, we will give a geometric construction of triangular dynamical r-matrices. As we shall see, these r-matrices do arise naturally from symplectic geometry. We will show some interesting examples, which include triangular dynamical r-matrices for simple Lie algebras constructed by Etingof–Varchenko [19].

Below let us recall the definition of a classical triangular dynamical r-matrix. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$ (or $\mathbb{C}$) and $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra. A classical dynamical r-matrix $r : \mathfrak{h}^* \longrightarrow \mathfrak{g}\otimes\mathfrak{g}$ is said to be *triangular* if it is skew symmetric: $r_{12} + r_{21} = 0$. In other words, a classical triangular dynamical r-matrix is a smooth function (or meromorphic function in the complex case) $r : \mathfrak{h}^* \longrightarrow \wedge^2\mathfrak{g}$ such that

(i) $r : \mathfrak{h}^* \longrightarrow \wedge^2\mathfrak{g}$ is H-equivariant, where H acts on $\mathfrak{h}^*$ by coadjoint action and acts on $\wedge^2\mathfrak{g}$ by adjoint action.

(ii)

$$\sum_i h_i \wedge \frac{\partial r}{\partial \lambda^i} - \frac{1}{2}[r, r] = 0, \tag{3}$$

where the bracket $[\cdot\,,\cdot]$ refers to the Schouten type bracket: $\wedge^k\mathfrak{g}\otimes \wedge^l\,\mathfrak{g} \longrightarrow \wedge^{k+l-1}\mathfrak{g}$ induced from the Lie algebra bracket on $\mathfrak{g}$, $\{h_1, \ldots, h_l\}$ is a basis of $\mathfrak{h}$, and $(\lambda^1, \ldots, \lambda^l)$ its induced coordinate system on $\mathfrak{h}^*$.

The following proposition gives an alternative description of a classical triangular dynamical r-matrix.

Proposition 2.1. *A smooth function $r : \mathfrak{h}^* \longrightarrow \wedge^2\mathfrak{g}$ is a triangular dynamical r-matrix iff*

$$\pi = \pi_{\mathfrak{h}^*} + \sum_i \frac{\partial}{\partial\lambda_i} \wedge \overrightarrow{h_i} + \overrightarrow{r(\lambda)}$$

is a Poisson tensor on $M = \mathfrak{h}^* \times G$, where $\pi_{\mathfrak{h}^*}$ denotes the standard Lie (also known as Kirillov–Kostant) Poisson tensor on the Lie algebra dual $\mathfrak{h}^*$, $\overrightarrow{h_i} \in \mathfrak{X}(M)$ is the left invariant vector field on M generated by $h_i \in \mathfrak{h}$, and similarly $\overrightarrow{r(\lambda)} \in \Gamma(\wedge^2 TM)$ is the left invariant bivector field on M corresponding to $r(\lambda)$.

Proof. Set

$$\pi_1 = \pi_{\mathfrak{h}^*} + \sum_i \frac{\partial}{\partial\lambda_i} \wedge \overrightarrow{h_i}\,.$$

Then $\pi = \pi_1 + \overrightarrow{r(\lambda)}$. Note that, for any (λ, x), $\pi_1|_{(\lambda,x)}$ is tangent to $\mathfrak{h}^* \times xH$, on which it is isomorphic to the standard Poisson (symplectic) structure on the cotangent bundle T^*H (see, e.g., [27]). Here T^*H is identified with $\mathfrak{h}^* \times H$ (hence with $\mathfrak{h}^* \times xH$) via left translations. It thus follows that $[\pi_1, \pi_1] = 0$. Therefore

$$[\pi, \pi] = 2[\pi_1, \overrightarrow{r(\lambda)}] + [\overrightarrow{r(\lambda)}, \overrightarrow{r(\lambda)}].$$

Now

$$\begin{aligned}[\pi_1,\ \overrightarrow{r(\lambda)}] &= [\pi_{\mathfrak{h}^*},\ \overrightarrow{r(\lambda)}] + \sum_i [\frac{\partial}{\partial\lambda_i} \wedge \overrightarrow{h_i}\,,\ \overrightarrow{r(\lambda)}] \\ &= [\pi_{\mathfrak{h}^*},\ \overrightarrow{r(\lambda)}] + \sum_i [\overrightarrow{r(\lambda)},\ \frac{\partial}{\partial\lambda_i}] \wedge \overrightarrow{h_i} - \sum_i \frac{\partial}{\partial\lambda_i} \wedge [\overrightarrow{r(\lambda)},\ \overrightarrow{h_i}\,].\end{aligned}$$

Hence $[\pi, \pi] = I_1 + I_2$, where

$$I_1 = 2\sum_i [\overrightarrow{r(\lambda)},\ \frac{\partial}{\partial\lambda_i}] \wedge \overrightarrow{h_i} + [\overrightarrow{r(\lambda)}, \overrightarrow{r(\lambda)}], \quad \text{and}$$

$$I_2 = 2[\pi_{\mathfrak{h}^*}, \overrightarrow{r(\lambda)}] - 2\sum_i \frac{\partial}{\partial\lambda_i} \wedge [\overrightarrow{r(\lambda)},\ \overrightarrow{h_i}\,].$$

With respect to the natural bigrading on $\wedge^3 T(\mathfrak{h}^* \times G)$, I_1 and I_2 correspond to the (0, 3) and (1, 2)-terms of $[\pi, \pi]$, respectively. It thus follows that $[\pi, \pi] = 0$ iff $I_1 = 0$ and $I_2 = 0$.

It is simple to see that

$$I_1 = -2\sum_i \overrightarrow{h_i} \wedge \frac{\partial \overrightarrow{r}}{\partial\lambda^i} + \overrightarrow{[r(\lambda), r(\lambda)]}.$$

Hence $I_1 = 0$ is equivalent to Eq. (3).

To find out the meaning of $I_2 = 0$, let us write $\pi_{\mathfrak{h}^*} = \frac{1}{2}\sum_{ij} f_{ij}(\lambda)\frac{\partial}{\partial\lambda^i} \wedge \frac{\partial}{\partial\lambda^j}$ $(f_{ij} = -f_{ji})$. A simple computation yields that

$$I_2 = 2\sum_i \frac{\partial}{\partial\lambda_i} \wedge \sum_j f_{ij}(\lambda)\frac{\partial \overrightarrow{r}}{\partial\lambda^j} + 2\sum_i \frac{\partial}{\partial\lambda_i} \wedge \overrightarrow{[h_i, r(\lambda)]}.$$

Thus $I_2 = 0$ is equivalent to

$$[h_i, r(\lambda)] = -\sum_j f_{ij}(\lambda)\frac{\partial r(\lambda)}{\partial\lambda^j} = \frac{d}{dt}\bigg|_{t=0} r(\mathrm{Ad}^*_{\exp^{-1}\mathrm{th_i}}\lambda),\ \forall \mathrm{i},$$

which means exactly that r is H-equivariant. This concludes the proof. □

Remark. Note that $M(= \mathfrak{h}^* \times G)$ admits a left G-action and a right H-action defined as follows: $\forall(\lambda, x) \in \mathfrak{h}^* \times G$,

$$y \cdot (\lambda, x) = (\lambda, yx), \quad \forall y \in G;$$
$$(\lambda, x) \cdot y = (\mathrm{Ad}^*_{\mathrm{y}}\lambda, \mathrm{xy}), \quad \forall \mathrm{y} \in \mathrm{H}.$$

It is clear that the Poisson structure π is invariant under both actions. And, in short, we will say that π is $G \times H$-invariant.

Definition 2.2. *A classical triangular dynamical r-matrix* $r : \mathfrak{h}^* \longrightarrow \wedge^2\mathfrak{g}$ *is said to be* **non-degenerate** *if the corresponding Poisson structure* π *on* M *is non-degenerate, i.e., symplectic.*

In what follows, we will give a geometric construction of non-degenerate dynamical r-matrices. To this end, let us first recall a useful construction of a symplectic manifold from a fat principal bundle [26,31]. A principal bundle $P(M, H)$ with a connection is called *fat* on an open submanifold $U \subseteq \mathfrak{h}^*$ if the scalar-valued form $< \lambda, \Omega >$ is non-degenerate on each horizontal space in TP for $\lambda \in U$. Here Ω is the curvature form, which is a tensorial form of type $\mathrm{Ad_H}$ on P (i.e., it is horizontal, $\mathfrak{h}$-valued, and $\mathrm{Ad_H}$-equivariant).

Given a fat bundle $P(M, H)$ with a connection, one has a decomposition of the tangent bundle $TP = \mathrm{Vert(P)} \oplus \mathrm{Hor(P)}$. We may identify Vert(P) with a trivial bundle with fiber $\mathfrak{h}$. Thus

$$\mathrm{Vert^*P} \cong \mathfrak{h}^* \times \mathrm{P}.$$

On the other hand, $\mathrm{Vert^*P} \cong \mathrm{Hor}^{\perp}(\mathrm{P}) \subset \mathrm{T^*P}$. Thus, by pulling back the canonical symplectic structure on T^*P, one can equip Vert*P, hence $\mathfrak{h}^* \times P$, an H-invariant presymplectic structure, where H acts on $\mathfrak{h}^* \times P$ by $(\lambda, x) \cdot h = (\mathrm{Ad}^*_{\mathrm{h}}\lambda, \mathrm{x} \cdot \mathrm{h}), \forall h \in H$ and $(\lambda, x) \in \mathfrak{h}^* \times P$. If $U \subseteq \mathfrak{h}^*$ is an open submanifold on which $P(M, H)$ is fat, then we obtain an H-invariant symplectic manifold $U \times P$. In fact, the presymplectic form ω can be described explicitly. Note that Vert*P admits a natural fibration with T^*H being the fibers, and the connection on P induces a connection on this fiber bundle. In other words, Vert*P is a symplectic fibration in the sense of Guillemin–Lerman–Sternberg [26]. At any point $(\lambda, x) \in \mathfrak{h}^* \times P \cong \mathrm{Vert^*P}$, the presymplectic form ω can be described as follows: it restricts to the canonical two-form on the fiber; the vertical subspace is ω-orthogonal to the horizontal subspace; and the horizontal subspace is isomorphic to the horizontal subspace of T_xP and the restriction of ω to this subspace is the two form

$- < \lambda, \Omega(x) >$ obtained by pairing the curvature form with λ (see Examples 2.2–2.3 in [26]).

Now assume that

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \tag{4}$$

is a reductive decomposition of a Lie algebra $\mathfrak{g}$, i.e., $\mathfrak{h}$ is a Lie subalgebra and $\mathfrak{m}$ is stable under the adjoint action of $\mathfrak{h}$: $[\mathfrak{h}, \ \mathfrak{m}] \subset \mathfrak{m}$. By G, we denote a Lie group with Lie algebra $\mathfrak{g}$, and H the Lie subgroup corresponding to $\mathfrak{h}$. It is standard [28] that the decomposition (4) induces a left G-invariant connection on the principal bundle $G(G/H, H)$, where the curvature is given by

$$\Omega(X, Y) = -[X, \ Y]_{\mathfrak{h}}, \quad \mathfrak{h} - \text{component of } [X, \ Y] \in \mathfrak{g}. \tag{5}$$

Here X and Y are arbitrary left invariant vector fields on G belonging to $\mathfrak{m}$.

A reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is said to be *fat* if the corresponding principal bundle $G(G/H, H)$ is fat on an open submanifold $U \subseteq \mathfrak{h}^*$. As a consequence, a fat decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ gives rise to a $G \times H$-invariant symplectic structure on $M = U \times G$, where the symplectic structure is the restriction of the canonical symplectic form on T^*G. In other words, M is a symplectic submanifold of T^*G. Here the embedding $U \times G \subseteq \mathfrak{h}^* \times G \longrightarrow \mathfrak{g}^* \times G \ (\cong T^*G)$ is given by the natural inclusion $(\lambda, x) \longrightarrow (pr^*\lambda, x)$, where $pr : \mathfrak{g} \longrightarrow \mathfrak{h}$ is the projection along the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Since the symplectic structure ω on $U \times G$ is left invariant, in order to describe ω explicitly, it suffices to specify it at a point $(\lambda, 1)$. Now $T_{(\lambda,1)}(U \times G) \cong \mathfrak{h}^* \oplus \mathfrak{g} = \mathfrak{h}^* \oplus \mathfrak{h} \oplus \mathfrak{m}$. Under this identification, we have $\omega = \omega_1 \oplus \omega_2$, where $\omega_1 \in \Omega^2(\mathfrak{h}^* \oplus \mathfrak{h})$ is the canonical symplectic two-form on T^*H at the point $(\lambda, 1) \in \mathfrak{h}^* \times H \ (\cong T^*H)$, and $\omega_2 \in \Omega^2(\mathfrak{m})$ is given by

$$\omega_2(X, Y) = \langle \lambda, [X, \ Y]_{\mathfrak{h}} \rangle, \quad \forall X, Y \in \mathfrak{m}.$$

Let $r(\lambda) \in \wedge^2\mathfrak{m}$ be the inverse of ω_2, which always exists for $\lambda \in U$ since ω_2 is assumed to be non-degenerate on U. It thus follows that the Poisson structure on $U \times G$ is

$$\pi = \pi_{\mathfrak{h}^*} + \sum_i \frac{\partial}{\partial \lambda_i} \wedge \overrightarrow{h_i} + \overrightarrow{r(\lambda)}.$$

According to Proposition 2.1, $r : U \longrightarrow \wedge^2\mathfrak{m} \subset \wedge^2\mathfrak{g}$ is a non-degenerate triangular dynamical r-matrix. Thus we have proved

Theorem 2.3. *Assume that* $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ *is a reductive decomposition which is fat on an open submanifold* $U \subseteq \mathfrak{h}^*$*. Then the dual of the linear map* $\varphi : \wedge^2\mathfrak{m} \longrightarrow \mathfrak{h} : (X, Y) \longrightarrow [X, \ Y]_{\mathfrak{h}}, \ \forall X, Y \in \mathfrak{m}$ *defines a non-degenerate triangular dynamical r-matrix* $r : U(\subseteq \mathfrak{h}^*) \longrightarrow \wedge^2\mathfrak{m} \subset \wedge^2\mathfrak{g}, \forall\lambda \in U$*. Here* $\mathfrak{m}^*$ *is identified with* $\mathfrak{m}$ *using the non-degenerate bilinear form* $\varphi^*(\lambda) \in \wedge^2\mathfrak{m}^*$*.*

It is often more useful to express $r(\lambda)$ explicitly in terms of a basis. To this end, let us choose a basis $\{e_1, \dots, e_m\}$ of $\mathfrak{m}$. Let $a_{ij}(\lambda) = \langle \lambda, [e_i, e_j]_{\mathfrak{h}} \rangle, \ \ i, j = 1, \dots, m$. By $(c_{ij}(\lambda))$ we denote the inverse of the matrix $(a_{ij}(\lambda)), \ \ \forall\lambda \in U$. Then one has

$$r(\lambda) = \frac{1}{2} \sum_{ij} c_{ij}(\lambda) e_i \wedge e_j. \tag{6}$$

Remark. (i) After the completion of the first draft, we learned that a similar formula is also obtained independently by Etingof [15]. Note that this dynamical r-matrix r is always singular at 0. To remove this singularity, one needs to make a shift of the dynamical parameter $\lambda \to \lambda - \mu$.
(ii) It would be interesting to compare our formula with Theorem 3 in [17].

We end this section with some examples.

Example 2.1. Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ and $\mathfrak{h}$ a Cartan subalgebra. Let

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$$

be the root space decomposition, where Δ_+ is the set of positive roots with respect to $\mathfrak{h}$. Take $\mathfrak{m} = \oplus_{\alpha\in\Delta_+}(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$. Then $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is clearly a reductive decomposition. Let $e_\alpha \in \mathfrak{g}_\alpha$ and $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$ be dual vectors with respect to the Killing form: $(e_\alpha, e_{-\alpha}) = 1$. For any $\lambda \in \mathfrak{h}^*$, set $a_{\alpha\beta}(\lambda) = \langle \lambda, [e_\alpha, e_\beta]_{\mathfrak{h}} \rangle, \ \ \forall \alpha, \beta \in \Delta_+ \cup (-\Delta_+)$. It is then clear that $a_{\alpha\beta}(\lambda) = 0$, whenever $\alpha + \beta \neq 0$; and

$$\begin{aligned} a_{\alpha,-\alpha}(\lambda) &= \langle \lambda, [e_\alpha, e_{-\alpha}]_{\mathfrak{h}} \rangle \\ &= (\lambda, \alpha)(e_\alpha, e_{-\alpha}) \\ &= (\lambda, \alpha). \end{aligned}$$

Therefore, from Theorem 2.3 and Eq. (6), it follows that

$$r(\lambda) = -\sum_{\alpha \in \Delta_+} \frac{1}{(\lambda, \alpha)} e_\alpha \wedge e_{-\alpha}$$

is a non-degenerate triangular dynamical r-matrix, so we have recovered this standard example in [19].

Example 2.2. As in the above example, let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ with a fixed Cartan subalgebra $\mathfrak{h}$, and $\mathfrak{l}$ a reductive Lie subalgebra containing $\mathfrak{h}$. There is a subset $\Delta(\mathfrak{l})_+$ of Δ_+ such that

$$\mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{l})_+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}).$$

Let $\overline{\Delta}_+ = \Delta_+ - \Delta(\mathfrak{l})_+, \ \ \Delta(\mathfrak{l}) = \Delta(\mathfrak{l})_+ \cup (-\Delta(\mathfrak{l})_+)$, and $\overline{\Delta} = \overline{\Delta}_+ \cup (-\overline{\Delta}_+)$, and denote by $\mathfrak{m}$ the subspace of $\mathfrak{g}$:

$$\mathfrak{m} = \sum_{\alpha \in \overline{\Delta}_+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}).$$

It is simple to see that $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{m}$ is indeed a fat reductive decomposition, and therefore induces a non-degenerate triangular dynamical r-matrix $r : \mathfrak{l}^* \longrightarrow \wedge^2 \mathfrak{g}$. To describe r explicitly, we note that the dual space $\mathfrak{l}^*$ admits a natural decomposition

$$\mathfrak{l}^* = \mathfrak{h}^* \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{l})_+} (\mathfrak{g}^*_\alpha \oplus \mathfrak{g}^*_{-\alpha}).$$

Hence any element $\mu \in \mathfrak{l}^*$ can be written as $\mu = \lambda \oplus \oplus_{\alpha \in \Delta(\mathfrak{l})} \xi_\alpha$, where $\lambda \in \mathfrak{h}^*$ and $\xi_\alpha \in \mathfrak{g}_\alpha^*$. Let $a_{\alpha\beta}(\mu) =< \mu, [e_\alpha, e_\beta]_\mathfrak{l} >, \ \ \forall \alpha, \beta \in \overline{\Delta}$. It is easy to see that

$$a_{\alpha\beta}(\mu) = \begin{cases} (\lambda, \alpha), & \text{if } \alpha + \beta = 0; \\ < \xi_\gamma, [e_\alpha, e_\beta] >, & \text{if } \alpha + \beta = \gamma \in \Delta(\mathfrak{l}); \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

By $(c_{\alpha\beta}(\mu))$, we denote the inverse matrix of $(a_{\alpha\beta}(\mu))$. According to Eq. (6),

$$r(\mu) = \frac{1}{2} \sum_{\alpha, \beta \in \overline{\Delta}} c_{\alpha\beta}(\mu) e_\alpha \wedge e_\beta$$

is a non-degenerate triangular dynamical r-matrix over $\mathfrak{l}^*$. In particular, if $\mu = \lambda \in \mathfrak{h}^*$, it follows immediately that

$$r(\lambda) = -\sum_{\alpha \in \overline{\Delta}_+} \frac{1}{(\lambda, \alpha)} e_\alpha \wedge e_{-\alpha}. \tag{8}$$

Equation (8) was first obtained by Etingof–Varchenko in [19].

The following example was pointed out to us by D. Vogan.

Example 2.4. Let $\mathfrak{g} = \mathbb{R}^{m+n} \oplus \mathbb{R}^{m+n} \oplus \mathbb{R}$ be a $2(m+n)+1$ dimensional Heisenberg Lie algebra and $\mathfrak{h} = \mathbb{R}^n \oplus \mathbb{R}^n \oplus \mathbb{R}$ its standard Heisenberg Lie subalgebra. By $\{p_i, q_i, c\}, \ i = 1, \ldots, n+m$, we denote the standard generators of $\mathfrak{g}$ and $\{p_{m+i}, q_{m+i}, c\}, \ i = 1, \ldots, n$, the generators of $\mathfrak{h}$. Let $\mathfrak{m}$ be the subspace of $\mathfrak{g}$ generated by $\{p_i, q_i\}, \ i = 1, \ldots, m$. It is then clear that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is a reductive decomposition. Let $\{p_i^*, q_i^*, c^*\}, \ i = 1, \ldots, n+m$, be the dual basis corresponding to the standard generators of $\mathfrak{g}$. For any $\lambda \in \mathfrak{h}^*$, write $\lambda = \sum_{i=1}^n (a_i p_{m+i}^* + b_i q_{m+i}^*) + x c^*$. This induces a coordinate system on $\mathfrak{h}^*$, and therefore a function on $\mathfrak{h}^*$ can be identified with a function with variables (a_i, b_i, x). It is clear that

$$\omega(p_i, q_j)(\lambda) = \langle \lambda, [p_i, q_j]_\mathfrak{h} \rangle = x\delta_{ij};$$
$$\omega(p_i, p_j) = \omega(q_i, q_j) = 0, \quad \forall i, \ j = 1, \ldots, m.$$

It thus follows that

$$r(a_i, b_i, x) = -\frac{1}{x} \sum_{i=1}^m p_i \wedge q_i : \ \ \mathfrak{h}^* \longrightarrow \wedge^2 \mathfrak{g}$$

is a non-degenerate triangular dynamical r-matrix.

3. Compatible Star Products

From Proposition 2.1, we know that a triangular dynamical r-matrix $r : \mathfrak{h}^* \longrightarrow \wedge^2 \mathfrak{g}$ is equivalent to a special type of Poisson structure on $\mathfrak{h}^* \times G$. It is thus very natural to expect that quantization of r can be derived from a certain special type of star-product on $\mathfrak{h}^* \times G$. It is simple to see that the Poisson brackets on $C^\infty(\mathfrak{h}^* \times G)$ can be described as follows:

(i) for any $f, g \in C^\infty(\mathfrak{h}^*)$, $\{f, g\} = \{f, g\}_{\pi_{\mathfrak{h}^*}}$;

(ii) for any $f \in C^\infty(\mathfrak{h}^*)$ and $g \in C^\infty(G)$, $\{f, g\} = \sum_i (\frac{\partial f}{\partial \lambda^i})(\overrightarrow{h_i}\, g)$;

(iii) for any $f, g \in C^\infty(G)$, $\{f, g\} = \overrightarrow{r(\lambda)}(f, g)$.

These Poisson bracket relations naturally motivate the following:

Definition 3.1. *A star product $*_\hbar$ on $M = \mathfrak{h}^* \times G$ is called a compatible star product if*

(i) *for any $f, g \in C^\infty(\mathfrak{h}^*)$,*

$$f(\lambda) *_\hbar g(\lambda) = f(\lambda) * g(\lambda); \tag{9}$$

(ii) *for any $f(x) \in C^\infty(G)$ and $g(\lambda) \in C^\infty(\mathfrak{h}^*)$,*

$$f(x) *_\hbar g(\lambda) = f(x) g(\lambda); \tag{10}$$

(iii) *for any $f(\lambda) \in C^\infty(\mathfrak{h}^*)$ and $g(x) \in C^\infty(G)$,*

$$f(\lambda) *_\hbar g(x) = \sum_{k=0}^{\infty} \frac{\hbar^k}{k!} \frac{\partial^k f}{\partial \lambda^{i_1} \cdots \partial \lambda^{i_k}} \overrightarrow{h_{i_1}} \cdots \overrightarrow{h_{i_k}} g; \tag{11}$$

(iv) *for any $f(x),\ g(x) \in C^\infty(G)$,*

$$f(x) *_\hbar g(x) = \overrightarrow{F(\lambda)}(f, g), \tag{12}$$

where $F(\lambda)$ is a smooth function $F : \mathfrak{h}^* \longrightarrow U\mathfrak{g} \otimes U\mathfrak{g}[[\hbar]]$ such that $F = 1 + \hbar F_1 + O(\hbar^2)$.

Here $*$ denotes the standard PBW-star product on $\mathfrak{h}^*$ quantizing the canonical Lie–Poisson structure (see [12]), whose definition is recalled below. Let $\mathfrak{h}_\hbar = \mathfrak{h}[[\hbar]]$ be a Lie algebra with the Lie bracket $[X, Y]_\hbar = \hbar[X, Y]$, $\forall X, Y \in \mathfrak{h}[[\hbar]]$, and

$$\sigma : S(\mathfrak{h})[[\hbar]] \cong U\mathfrak{h}_\hbar$$

be the Poincaré–Birkhoff–Witt map, which is a vector space isomorphism. Thus the multiplication on $U\mathfrak{h}_\hbar$ induces a multiplication on $S(\mathfrak{h})[[\hbar]]$ ($\cong \mathrm{Pol}(\mathfrak{h}^*)[[\hbar]]$), hence on $C^\infty(\mathfrak{h}^*)[[\hbar]]$, which is denoted by $*$. It is easy to check that $*$ satisfies

$$f * g = fg + \frac{1}{2}\hbar\{f, g\}_{\pi^*_{\mathfrak{h}}} + \sum_{k \geq 0} \hbar^k B_k(f, g) + \cdots, \quad \forall f, g \in C^\infty(\mathfrak{h}^*),$$

where B_k's are bidifferential operators. In other words, $*$ is indeed a star product on $\mathfrak{h}^*$, which is called the PBW-star product.

The following proposition is quite obvious.

Proposition 3.2. *The classical limit of a compatible star product is the Poisson structure $\pi = \pi_{\mathfrak{h}^*} + \sum_i \frac{\partial}{\partial \lambda_i} \wedge \overrightarrow{h_i} + \overrightarrow{r(\lambda)}$, where $r(\lambda) = F_{12}(\lambda) - F_{21}(\lambda)$.*

Below we will study some important properties of compatible star products.

Proposition 3.3. *A compatible star product is always invariant under the left G-action. It is right H-invariant iff $F : \mathfrak{h}^* \longrightarrow U\mathfrak{g} \otimes U\mathfrak{g}[[\hbar]]$ is H-equivariant, where H acts on $\mathfrak{h}^*$ by the coadjoint action and on $U\mathfrak{g}\otimes U\mathfrak{g}$ by the adjoint action.*

Proof. First of all, note that Eqs. (9–12) completely determine a star product. It is clear from these equations that $*_\hbar$ is left G-invariant.

As for the right H-action, it is obvious from Eq. (10) that $*_\hbar$ is invariant for $f(x) *_\hbar g(\lambda)$. It is standard that $*$ is invariant under the coadjoint action, so it follows from Eq. (9) that $f(\lambda) *_\hbar g(\lambda)$ is also H-invariant.

For any $h \in \mathfrak{h}$, $g(x) \in C^\infty(G)$ and any fixed $y \in H$,

$$\begin{aligned} \overrightarrow{h}(R_y^* g)(x) &= (L_x h)(R_y^* g) \\ &= (R_y L_x h)(g) \\ &= (L_{xy} \mathrm{Ad}_{y^{-1}} h)(g) \\ &= (\overrightarrow{\mathrm{Ad}_{y^{-1}} h} g)(xy) \\ &= [R_y^*(\overrightarrow{\mathrm{Ad}_{y^{-1}} h} g)](x). \end{aligned}$$

Thus it follows that

$$\overrightarrow{h_{i_1}} \cdots \overrightarrow{h_{i_k}} (R_y^* g) = R_y^* (\overrightarrow{h'_{i_1}} \cdots \overrightarrow{h'_{i_k}} g), \tag{13}$$

where $h'_i = \mathrm{Ad}_{y^{-1}} h_i$, $i = 1, \ldots, n$. Let $\xi'_i = \mathrm{Ad}_y^* \xi_i$, $i = 1, \ldots, n$. Then $\{\xi'_1, \ldots, \xi'_l\}$ is a dual basis for $\{h'_1, \ldots, h'_l\}$. Let $(\lambda'^1, \ldots, \lambda'^l)$ be its corresponding induced coordinates on $\mathfrak{h}^*$. Then

$$\begin{aligned} \frac{\partial}{\partial \lambda^i}((\mathrm{Ad}_y^*)^* f) &= \left.\frac{d}{dt}\right|_{t=0} ((\mathrm{Ad}_y^*)^* f)(\lambda + t\xi_i) \\ &= \left.\frac{d}{dt}\right|_{t=0} f(\mathrm{Ad}_y^* \lambda + t \mathrm{Ad}_y^* \xi_i) \\ &= \left.\frac{d}{dt}\right|_{t=0} f(\mathrm{Ad}_y^* \lambda + t\xi'_i) \\ &= \frac{\partial f}{\partial \lambda'^i}(\mathrm{Ad}_y^* \lambda) \\ &= (\mathrm{Ad}_y^*)^* \frac{\partial f}{\partial \lambda'^i}. \end{aligned}$$

Hence

$$\frac{\partial^k [(\mathrm{Ad}_y^*)^* f]}{\partial \lambda^{i_1} \cdots \partial \lambda^{i_k}} = (\mathrm{Ad}_y^*)^* [\frac{\partial^k f}{\partial \lambda'^{i_1} \cdots \partial \lambda'^{i_k}}]. \tag{14}$$

From Eq. (11), it follows that for any $f(\lambda) \in C^\infty(\mathfrak{h}^*)$ and $g(x) \in C^\infty(G)$,

$$\begin{aligned} (R_y^* f)(\lambda) *_\hbar (R_y^* g)(x) &= \sum_{k=0}^\infty \frac{\hbar^k}{k!} \frac{\partial^k [(\mathrm{Ad}_y^*)^* f]}{\partial \lambda^{i_1} \cdots \partial \lambda^{i_k}} \overrightarrow{h_{i_1}} \cdots \overrightarrow{h_{i_k}} (R_y^* g) \quad \text{(by Eqs. (13–14))} \\ &= \sum_{k=0}^\infty \frac{\hbar^k}{k!} (\mathrm{Ad}_y^*)^* [\frac{\partial^k f}{\partial \lambda'^{i_1} \cdots \partial \lambda'^{i_k}}] R_y^* [\overrightarrow{h'_{i_1}} \cdots \overrightarrow{h'_{i_k}} g] \\ &= R_y^*(f(\lambda) *_\hbar g(x)). \end{aligned}$$

I.e., $f(\lambda) *_\hbar g(x)$ is also right H-invariant.

Finally, $\forall f(x),\ g(x) \in C^\infty(G)$,

$$\begin{aligned}(R_y^*(f *_\hbar g))(\lambda, x) &= (f *_\hbar g)(\mathrm{Ad}_{\mathrm{y}}^*\lambda, \mathrm{xy}) \\ &= \overrightarrow{F(\mathrm{Ad}_{\mathrm{y}}^*\lambda)}(f, g)(xy) \\ &= [L_{xy}(F(\mathrm{Ad}_{\mathrm{y}}^*\lambda))](\mathrm{f}, \mathrm{g}).\end{aligned}$$

On the other hand,

$$\begin{aligned}(R_y^* f *_\hbar R_y^* g)(\lambda, x) &= \overrightarrow{F(\lambda)}(R_y^* f, R_y^* g)(x) \\ &= (L_x F(\lambda))(R_y^* f, R_y^* g) \\ &= (R_y L_x F(\lambda))(f, g).\end{aligned}$$

Therefore $R_y^*(f *_\hbar g) = R_y^* f *_\hbar R_y^* g$ iff $L_{xy}(F(\mathrm{Ad}_{\mathrm{y}}^*\lambda)) = \mathrm{R_y L_x F}(\lambda)$. The latter is equivalent to that $F(\mathrm{Ad}_{\mathrm{y}}^*\lambda) = \mathrm{Ad}_{\mathrm{y}^{-1}}\mathrm{F}(\lambda)$, or F is H-equivariant. This concludes the proof. □

In order to give an explicit formula for $*_\hbar$, let us write

$$F(\lambda) = \sum a_{\alpha\beta}(\lambda) U_\alpha \otimes U_\beta, \tag{15}$$

where $a_{\alpha\beta}(\lambda) \in C^\infty(\mathfrak{h}^*)[[\hbar]]$ and $U_\alpha \otimes U_\beta \in U\mathfrak{g} \otimes U\mathfrak{g}$. Using this expression, indeed one can describe $*_\hbar$ explicitly.

Theorem 3.4. *Given a compatible star product $*_\hbar$ as in Definition 3.1, for any $f(\lambda, x)$, $g(\lambda, x) \in C^\infty(\mathfrak{h}^* \times G)[[\hbar]]$,*

$$f(\lambda, x) *_\hbar g(\lambda, x) = \sum_{\alpha\beta} \sum_{k=0}^{\infty} \frac{\hbar^k}{k!} a_{\alpha\beta}(\lambda) * \overrightarrow{U_\alpha} \frac{\partial^k f}{\partial \lambda^{i_1} \cdots \partial \lambda^{i_k}} * \overrightarrow{U_\beta} \overrightarrow{h_{i_1}} \cdots \overrightarrow{h_{i_k}} g. \tag{16}$$

We need a couple of lemmas first.

Lemma 3.5. *Under the same hypothesis as in Theorem 3.4,*

(i) *for any $f(\lambda, x) \in C^\infty(\mathfrak{h}^* \times G)$ and $g(\lambda) \in C^\infty(\mathfrak{h}^*)$,*

$$f(\lambda, x) *_\hbar g(\lambda) = f(\lambda, x) * g(\lambda); \tag{17}$$

(ii) *for any $f(x) \in C^\infty(G)$ and $g(\lambda, x) \in C^\infty(\mathfrak{h}^* \times G)$,*

$$f(x) *_\hbar g(\lambda, x) = \sum_{\alpha\beta} a_{\alpha\beta}(\lambda) * \overrightarrow{U_\alpha} f(x) \overrightarrow{U_\beta} g(\lambda, x); \tag{18}$$

(iii) *for any $f(\lambda, x) \in C^\infty(\mathfrak{h}^* \times G)$ and $g(x) \in C^\infty(G)$,*

$$f(\lambda, x) *_\hbar g(x) = \sum_{\alpha\beta} \sum_{k=0}^{\infty} \frac{\hbar^k}{k!} a_{\alpha\beta}(\lambda) * \overrightarrow{U_\alpha} \frac{\partial^k f(\lambda, x)}{\partial \lambda^{i_1} \cdots \partial \lambda^{i_k}} \overrightarrow{U_\beta} \overrightarrow{h_{i_1}} \cdots \overrightarrow{h_{i_k}} g(x). \tag{19}$$

Proof. (i) It suffices to show this identity for $f(\lambda, x) = f_1(x) f_2(\lambda), \forall f_1(x) \in C^\infty(G)$ and $f_2(\lambda) \in C^\infty(\mathfrak{h}^*)$. Now

$$\begin{aligned} f(\lambda, x) *_\hbar g(\lambda) &= (f_1(x) f_2(\lambda)) *_\hbar g(\lambda) \quad \text{(by Eq. (10))} \\ &= (f_1(x) *_\hbar f_2(\lambda)) *_\hbar g(\lambda) \\ &= f_1(x) *_\hbar (f_2(\lambda) *_\hbar g(\lambda)) \quad \text{(by Eqs. (9–10))} \\ &= f_1(x)(f_2(\lambda) * g(\lambda)) \\ &= (f_1(x) f_2(\lambda)) * g(\lambda) \\ &= f(\lambda, x) * g(\lambda). \end{aligned}$$

(ii) Similarly, we may assume that $g(\lambda, x) = g_1(x) g_2(\lambda)$, $\forall g_1(x) \in C^\infty(G)$ and $g_2(\lambda) \in C^\infty(\mathfrak{h}^*)$. Then,

$$\begin{aligned} f(x) *_\hbar g(\lambda, x) &= f(x) *_\hbar (g_1(x) g_2(\lambda)) \\ &= f(x) *_\hbar (g_1(x) *_\hbar g_2(\lambda)) \\ &= (f(x) *_\hbar g_1(x)) *_\hbar g_2(\lambda) \quad \text{(by Eq. (12))} \\ &= \sum_{\alpha\beta} [a_{\alpha\beta}(\lambda)(\overrightarrow{U_\alpha} f(x))(\overrightarrow{U_\beta} g_1(x))] * g_2(\lambda) \\ &= \sum_{\alpha\beta} a_{\alpha\beta}(\lambda) * \overrightarrow{U_\alpha} f(x) \overrightarrow{U_\beta} g(\lambda, x). \end{aligned}$$

(iii) Assume that $f(\lambda, x) = f_1(x) f_2(\lambda)$, $\forall f_1(x) \in C^\infty(G)$ and $f_2(\lambda) \in C^\infty(\mathfrak{h}^*)$. Then

$$\begin{aligned} f(\lambda, x) *_\hbar g(x) &= (f_1(x) f_2(\lambda)) *_\hbar g(x) \\ &= (f_1(x) *_\hbar f_2(\lambda)) *_\hbar g(x) \\ &= f_1(x) *_\hbar (f_2(\lambda) *_\hbar g(x)) \quad \text{(using Eq. (18))} \\ &= \sum_{\alpha\beta} a_{\alpha\beta}(\lambda) * \overrightarrow{U_\alpha} f_1(x) \overrightarrow{U_\beta} (f_2(\lambda) *_\hbar g(x)) \\ &= \sum_{\alpha\beta} \sum_{k=0}^\infty \frac{\hbar^k}{k!} a_{\alpha\beta}(\lambda) * [\overrightarrow{U_\alpha} f_1(x) \overrightarrow{U_\beta} (\frac{\partial^k f_2(\lambda)}{\partial\lambda^{i_1} \cdots \partial\lambda^{i_k}} \overrightarrow{h_{i_1}} \cdots \overrightarrow{h_{i_k}} g(x))] \\ &= \sum_{\alpha\beta} \sum_{k=0}^\infty \frac{\hbar^k}{k!} a_{\alpha\beta}(\lambda) * [\overrightarrow{U_\alpha} f_1(x) \frac{\partial^k f_2(\lambda)}{\partial\lambda^{i_1} \cdots \partial\lambda^{i_k}} \overrightarrow{U_\beta} \overrightarrow{h_{i_1}} \cdots \overrightarrow{h_{i_k}} g(x)] \\ &= \sum_{\alpha\beta} \sum_{k=0}^\infty \frac{\hbar^k}{k!} a_{\alpha\beta}(\lambda) * [\overrightarrow{U_\alpha} \frac{\partial^k (f_1(x) f_2(\lambda))}{\partial\lambda^{i_1} \cdots \partial\lambda^{i_k}} \overrightarrow{U_\beta} \overrightarrow{h_{i_1}} \cdots \overrightarrow{h_{i_k}} g(x)] \\ &= \sum_{\alpha\beta} \sum_{k=0}^\infty \frac{\hbar^k}{k!} a_{\alpha\beta}(\lambda) * \overrightarrow{U_\alpha} \frac{\partial^k f(\lambda, x)}{\partial\lambda^{i_1} \cdots \partial\lambda^{i_k}} \overrightarrow{U_\beta} \overrightarrow{h_{i_1}} \cdots \overrightarrow{h_{i_k}} g(x). \end{aligned}$$

This concludes the proof of the lemma. □

Now we are ready to prove the main result of this section.

Proof of Theorem 3.4. Again, we may assume that $g(\lambda, x) = g_1(x)g_2(\lambda)$, $\forall g_1(x) \in C^\infty(G)$ and $g_2(\lambda) \in C^\infty(\mathfrak{h}^*)$. Then

$$\begin{aligned}
f(\lambda, x) *_\hbar g(\lambda, x) &= f(\lambda, x) *_\hbar (g_1(x)g_2(\lambda)) \\
&= f(\lambda, x) *_\hbar (g_1(x) *_\hbar g_2(\lambda)) \\
&= (f(\lambda, x) *_\hbar g_1(x)) *_\hbar g_2(\lambda) \quad \text{(by Eq. (17))} \\
&= (f(\lambda, x) *_\hbar g_1(x)) * g_2(\lambda) \quad \text{(by Eq. (19))} \\
&= \sum_{\alpha\beta} \sum_{k=0}^{\infty} \frac{\hbar^k}{k!} [a_{\alpha\beta}(\lambda) * \overrightarrow{U_\alpha} \frac{\partial^k f(\lambda, x)}{\partial \lambda^{i_1} \cdots \partial \lambda^{i_k}} \overrightarrow{U_\beta} \overrightarrow{h_{i_1}} \cdots \overrightarrow{h_{i_k}} g_1(x)] * g_2(\lambda) \\
&= \sum_{\alpha\beta} \sum_{k=0}^{\infty} \frac{\hbar^k}{k!} a_{\alpha\beta}(\lambda) * \overrightarrow{U_\alpha} \frac{\partial^k f(\lambda, x)}{\partial \lambda^{i_1} \cdots \partial \lambda^{i_k}} * \overrightarrow{U_\beta} \overrightarrow{h_{i_1}} \cdots \overrightarrow{h_{i_k}} (g_1(x)g_2(\lambda)) \\
&= \sum_{\alpha\beta} \sum_{k=0}^{\infty} \frac{\hbar^k}{k!} a_{\alpha\beta}(\lambda) * \overrightarrow{U_\alpha} \frac{\partial^k f(\lambda, x)}{\partial \lambda^{i_1} \cdots \partial \lambda^{i_k}} * \overrightarrow{U_\beta} \overrightarrow{h_{i_1}} \cdots \overrightarrow{h_{i_k}} g(\lambda, x). \quad \square
\end{aligned}$$

As a consequence of Theorem 3.4, we will see that if a function $F(\lambda) : \mathfrak{h}^* \longrightarrow U\mathfrak{g} \otimes U\mathfrak{g}[[\hbar]]$ defines a compatible star product, it must satisfy a "twisted-cocycle" type condition. To describe this condition explicitly, we need to introduce some notations.

For any $f(\lambda) \in C^\infty(\mathfrak{h}^*)$, define $f(\lambda + \hbar h) \in C^\infty(\mathfrak{h}^*) \otimes U\mathfrak{h}[[\hbar]]$ by

$$\begin{aligned}
f(\lambda + \hbar h) = f(\lambda) \otimes 1 + \hbar \sum_i \frac{\partial f}{\partial \lambda_i} \otimes h_i + \frac{1}{2!} \hbar^2 \sum_{i_1 i_2} \frac{\partial^2 f}{\partial \lambda_{i_1} \partial \lambda_{i_2}} \otimes h_{i_1} h_{i_2} \\
+ \cdots + \frac{\hbar^k}{k!} \sum \frac{\partial^k f}{\partial \lambda_{i_1} \cdots \partial \lambda_{i_k}} \otimes h_{i_1} \cdots h_{i_k} + \cdots . \qquad (20)
\end{aligned}$$

The correspondence $C^\infty(\mathfrak{h}^*) \longrightarrow C^\infty(\mathfrak{h}^*) \otimes U\mathfrak{h}[[\hbar]] : f(\lambda) \longrightarrow f(\lambda + \hbar h)$ extends naturally to a linear map from $C^\infty(\mathfrak{h}^*) \otimes U\mathfrak{g} \otimes U\mathfrak{g}[[\hbar]]$ to $C^\infty(\mathfrak{h}^*) \otimes U\mathfrak{h} \otimes U\mathfrak{g} \otimes U\mathfrak{g}[[\hbar]] \subseteq C^\infty(\mathfrak{h}^*) \otimes U\mathfrak{g} \otimes U\mathfrak{g} \otimes U\mathfrak{g}[[\hbar]]$, which is denoted by $F(\lambda) \longrightarrow F_{23}(\lambda + \hbar h^{(1)})$. More explicitly, assume that $F(\lambda) = \sum_{\alpha\beta} f_{\alpha\beta}(\lambda) U_\alpha \otimes U_\beta$, where $f_{\alpha\beta}(\lambda) \in C^\infty(\mathfrak{h}^*)[[\hbar]]$ and $U_\alpha \otimes U_\beta \in U\mathfrak{g} \otimes U\mathfrak{g}$. Then

$$F_{23}(\lambda + \hbar h^{(1)}) = \sum_{\alpha\beta} f_{\alpha\beta}(\lambda + \hbar h) \otimes U_\alpha \otimes U_\beta. \qquad (21)$$

By a suitable permutation, one may define $F_{12}(\lambda + \hbar h^{(3)})$ and $F_{13}(\lambda + \hbar h^{(2)})$ similarly. Note that $U\mathfrak{g}$ is a Hopf algebra. By $\Delta : U\mathfrak{g} \longrightarrow U\mathfrak{g} \otimes U\mathfrak{g}$ and $\epsilon : U\mathfrak{g} \longrightarrow \mathbb{R}$, we denote its co-multiplication and co-unit, respectively. Then Δ naturally extends to a map $C^\infty(\mathfrak{h}^*) \otimes U\mathfrak{g}[[\hbar]] \longrightarrow C^\infty(\mathfrak{h}^*) \otimes U\mathfrak{g} \otimes U\mathfrak{g}[[\hbar]]$, which will be denoted by the same symbol.

Corollary 3.6. *Assume that $F : \mathfrak{h}^* \longrightarrow U\mathfrak{g} \otimes U\mathfrak{g}[[\hbar]]$ defines a compatible star product $*_\hbar$ as in Definition 3.1. Then*

$$(\Delta \otimes id) F(\lambda) * F_{12}(\lambda + \hbar h^{(3)}) = (id \otimes \Delta) F(\lambda) * F_{23}(\lambda); \qquad (22)$$

$$(\epsilon \otimes id) F(\lambda) = 1; \quad (id \otimes \epsilon) F(\lambda) = 1. \qquad (23)$$

Proof. Equation (23) follows from the fact that $1 *_\hbar f(x) = f(x) *_\hbar 1 = f(x)$, $\forall f(x) \in C^\infty(G)$.

As for Eq. (22), note that for any $f_1(x)$, $f_2(x)$ and $f_3(x) \in C^\infty(G)$, according to Eq. (19), we have

$$\begin{aligned}&(f_1(x) *_\hbar f_2(x)) *_\hbar f_3(x)\\ &\quad = \sum_{\alpha\beta}\sum_{k=0}^{\infty}\frac{\hbar^k}{k!}a_{\alpha\beta}(\lambda) * \overrightarrow{U_\alpha}\frac{\partial^k(f_1(x) *_\hbar f_2(x))}{\partial\lambda^{i_1}\cdots\partial\lambda^{i_k}}\overrightarrow{U_\beta}\overrightarrow{h_{i_1}}\cdots\overrightarrow{h_{i_k}}f_3(x).\end{aligned}$$

Now

$$\begin{aligned}&(\Delta\otimes id)F(\lambda) * F_{12}(\lambda + \hbar h^{(3)})\\ &\quad = \sum_{k=0}^{\infty}\frac{\hbar^k}{k!}(\Delta\otimes id)F(\lambda) * (\frac{\partial^k F}{\partial\lambda^{i_1}\cdots\partial\lambda^{i_k}}\otimes h_{i_1}\cdots h_{i_k})\\ &\quad = \sum_{\alpha\beta}\sum_{k=0}^{\infty}\frac{\hbar^k}{k!}a_{\alpha\beta}(\lambda) * \Delta U_\alpha\frac{\partial^k F}{\partial\lambda^{i_1}\cdots\partial\lambda^{i_k}}\otimes U_\beta h_{i_1}\cdots h_{i_k}.\end{aligned}$$

It thus follows that

$$\begin{aligned}&\overrightarrow{(\Delta\otimes id)F(\lambda) * F_{12}(\lambda + \hbar h^{(3)})}(f_1(x),\ f_2(x),\ f_3(x))\\ &\quad = \sum_{\alpha\beta}\sum_{k=0}^{\infty}\frac{\hbar^k}{k!}a_{\alpha\beta}(\lambda) * \overrightarrow{U_\alpha}(\frac{\partial^k\overrightarrow{F(\lambda)}(f_1(x), f_2(x))}{\partial\lambda^{i_1}\cdots\partial\lambda^{i_k}})\overrightarrow{U_\beta}\overrightarrow{h_{i_1}}\cdots\overrightarrow{h_{i_k}}f_3(x)\\ &\quad = \sum_{\alpha\beta}\sum_{k=0}^{\infty}\frac{\hbar^k}{k!}a_{\alpha\beta}(\lambda) * \overrightarrow{U_\alpha}\frac{\partial^k(f_1(x) *_\hbar f_2(x))}{\partial\lambda^{i_1}\cdots\partial\lambda^{i_k}}\overrightarrow{U_\beta}\overrightarrow{h_{i_1}}\cdots\overrightarrow{h_{i_k}}f_3(x)\\ &\quad = (f_1(x) *_\hbar f_2(x)) *_\hbar f_3(x).\end{aligned}$$

On the other hand,

$$\begin{aligned}f_1(x) *_\hbar (f_2(x) *_\hbar f_3(x)) &= f_1(x) *_\hbar \overrightarrow{F(\lambda)}(f_2(x), f_3(x))\ \ \text{(by Eq. (18))}\\ &= \sum_{\alpha\beta}a_{\alpha\beta}(\lambda) * \overrightarrow{U_\alpha}f_1(x)\overrightarrow{U_\beta}(\overrightarrow{F(\lambda)}(f_2(x), f_3(x)))\\ &= \overrightarrow{(id\otimes\Delta)F(\lambda) * F_{23}(\lambda)}(f_1(x),\ f_2(x),\ f_3(x)).\end{aligned}$$

Now Eq. (22) follows from the associativity of $*_\hbar$. □

To end this section, as a special case, let us consider $M = \mathfrak{h}^* \times H \cong T^*H$, which is equipped with the canonical cotangent symplectic structure. The following proposition describes an explicit formula for a compatible star-product on it.

Proposition 3.7. *For any $f(\lambda, x),\ g(\lambda, x) \in C^\infty(\mathfrak{h}^* \times H)[[\hbar]]$, the following equation*

$$f(\lambda, x) *_\hbar g(\lambda, x) = \sum_{k=0}^{\infty}\frac{\hbar^k}{k!}\frac{\partial^k f}{\partial\lambda^{i_1}\cdots\partial\lambda^{i_k}} * \overrightarrow{h_{i_1}}\cdots\overrightarrow{h_{i_k}}g \tag{24}$$

defines a compatible star product on $M = \mathfrak{h}^* \times H \cong T^*H$, which is in fact a deformation quantization of its canonical cotangent symplectic structure.

Proof. As earlier in this section, let $\mathfrak{h}_\hbar = \mathfrak{h}[[\hbar]]$ be equipped with the Lie bracket $[X, Y]_\hbar = \hbar[X, Y], \forall X, Y \in \mathfrak{h}_\hbar$, and $\sigma : S(\mathfrak{h})[[\hbar]] \longrightarrow U\mathfrak{h}_\hbar$ the PBW-map. Note that $\mathfrak{h}_\hbar$ is isomorphic to $\mathfrak{h}$ as a Lie algebra. Hence $U\mathfrak{h}_\hbar$ is canonically isomorphic to $U\mathfrak{h}[[\hbar]]$, whose elements can be considered as left invariant (formal) differential operators on H. To each polynomial function on $T^*H \cong \mathfrak{h}^* \times H$, we assign a (formal) differential operator on H according to the following rule. For $f \in C^\infty(H)$, we assign the operator multiplying by f; for $f \in \mathrm{Pol}(\mathfrak{h}^*) \cong S(\mathfrak{h})$, we assign the left invariant differential operator $\overrightarrow{\sigma(f)}$; in general, for $f(x)g(\lambda)$ with $f(x) \in C^\infty(H)$ and $g(\lambda) \in \mathrm{Pol}(\mathfrak{h}^*)$, we assign the differential operator $f(x)\overrightarrow{\sigma(g)}$. Then the multiplication on the algebra of differential operators induces an associative product $*_\hbar$ on $\mathrm{Pol}(T^*H)[[\hbar]]$, hence a star product on T^*H. It is simple to see from the above construction that

(i) for any $f(\lambda), g(\lambda) \in C^\infty(\mathfrak{h}^*)$,

$$f(\lambda) *_\hbar g(\lambda) = f(\lambda) * g(\lambda); \tag{25}$$

(ii) for any $f(x) \in C^\infty(H)$ and $g(\lambda) \in C^\infty(\mathfrak{h}^*)$,

$$f(x) *_\hbar g(\lambda) = f(x)g(\lambda); \tag{26}$$

(iii) for any $f(\lambda) \in C^\infty(\mathfrak{h}^*)$ and $g(x) \in C^\infty(H)$,

$$f(\lambda) *_\hbar g(x) = \sum_{k=0}^{\infty} \frac{\hbar^k}{k!} \frac{\partial^k f(\lambda)}{\partial\lambda^{i_1} \cdots \partial\lambda^{i_k}} \overrightarrow{h_{i_1}} \cdots \overrightarrow{h_{i_k}} g(x); \tag{27}$$

(iv) for any $f(x),\ g(x) \in C^\infty(H)$,

$$f(x) *_\hbar g(x) = f(x)g(x). \tag{28}$$

In other words, this is indeed a compatible star product with $F \equiv 1$. Equation (24) thus follows immediately from Theorem 3.4. $\square$

Remark. It would be interesting to compare Eq. (24) with the general construction of star products on cotangent symplectic manifolds in [10, 11].

Equation (27) implies that the element $f(\lambda + \hbar h) \in C^\infty(\mathfrak{h}^*) \otimes U\mathfrak{h}[[\hbar]]$, being considered as a left invariant differential operator on H, admits the following expression:

$$\overrightarrow{f(\lambda + \hbar h)} = f(\lambda) *_\hbar .$$

Thus we have:

Corollary 3.8. *For any* $f, g \in C^\infty(\mathfrak{h}^*)$,

$$(f * g)(\lambda + \hbar h) = f(\lambda + \hbar h) * g(\lambda + \hbar h), \tag{29}$$

where the $*$ on the left hand side stands for the PBW-star product on $\mathfrak{h}^*$, while on the right hand side it refers to the multiplication on the algebra tensor product of $(C^\infty(\mathfrak{h}^*)[[\hbar]], *)$ with $U\mathfrak{h}[[\hbar]]$.

Proof. Let $*_\hbar$ denote the star product on T^*H as in Proposition 3.7. For any $\varphi(x) \in C^\infty(H)$,

$$(f(\lambda) *_\hbar g(\lambda)) *_\hbar \varphi(x) \text{ (by Eqs. (25, 27))} = \sum_{k=0}^{\infty} \frac{\hbar^k}{k!} \frac{\partial^k (f(\lambda) * g(\lambda))}{\partial\lambda^{i_1}\cdots\partial\lambda^{i_k}} \overrightarrow{h_{i_1}} \cdots \overrightarrow{h_{i_k}} \varphi(x)$$
$$= \overrightarrow{(f * g)(\lambda + \hbar h)} \varphi(x).$$

On the other hand,

$$\begin{aligned} &f(\lambda) *_\hbar (g(\lambda) *_\hbar \varphi(x)) \text{ (by Eq. (24))} \\ &= \sum_{k=0}^{\infty} \frac{\hbar^k}{k!} \frac{\partial^k f(\lambda)}{\partial\lambda^{i_1}\cdots\partial\lambda^{i_k}} * \overrightarrow{h_{i_1}} \cdots \overrightarrow{h_{i_k}} (g(\lambda) *_\hbar \varphi(x)) \\ &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{\hbar^k}{k!} \frac{\partial^k f(\lambda)}{\partial\lambda^{i_1}\cdots\partial\lambda^{i_k}} * \overrightarrow{h_{i_1}} \cdots \overrightarrow{h_{i_k}} \Big(\frac{\hbar^l}{l!} \frac{\partial^l g(\lambda)}{\partial\lambda^{j_1}\cdots\partial\lambda^{j_l}} \overrightarrow{h_{j_1}} \cdots \overrightarrow{h_{j_l}} \varphi(x)\Big) \\ &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{\hbar^{k+l}}{k!l!} \frac{\partial^k f(\lambda)}{\partial\lambda^{i_1}\cdots\partial\lambda^{i_k}} * \frac{\partial^l g(\lambda)}{\partial\lambda^{j_1}\cdots\partial\lambda^{j_l}} \overrightarrow{h_{i_1}} \cdots \overrightarrow{h_{i_k}} \overrightarrow{h_{j_1}} \cdots \overrightarrow{h_{j_l}} \varphi(x) \\ &= \overrightarrow{f(\lambda + \hbar h) * g(\lambda + \hbar h)} \varphi(x). \end{aligned}$$

The conclusion thus follows from the associativity of $*_\hbar$. □

Corollary 3.9. *For any* $F, G \in C^\infty(\mathfrak{h}^*) \otimes U\mathfrak{g} \otimes U\mathfrak{g}[[\hbar]]$,

$$(F * G)_{23}(\lambda + \hbar h^{(1)}) = F_{23}(\lambda + \hbar h^{(1)}) * G_{23}(\lambda + \hbar h^{(1)}). \tag{30}$$

In particular, if $F(\lambda) \in C^\infty(\mathfrak{h}^*) \otimes U\mathfrak{g} \otimes U\mathfrak{g}[[\hbar]]$ *is invertible, we have*

$$F_{23}^{-1}(\lambda + \hbar h^{(1)}) = F_{23}(\lambda + \hbar h^{(1)})^{-1}. \tag{31}$$

4. Quantum Dynamical Yang–Baxter Equation

The main purpose of this section is to derive the quantum dynamical Yang–Baxter equation over a nonabelian base $\mathfrak{h}$ from the "twisted-cocycle" condition (22). This was standard when $\mathfrak{h}$ is Abelian (e.g., see [6]). The proof was based on the Drinfel'd theory of quasi-Hopf algebras [13]. In our situation, however, the quasi-Hopf algebra approach does not work any more. Nevertheless, one can carry out a proof in a way completely parallel to the ordinary case.

The main result of this section is the following:

Theorem 4.1. *Assume that* $F : \mathfrak{h}^* \longrightarrow U\mathfrak{g} \otimes U\mathfrak{g}[[\hbar]]$ *satisfies the "twisted-cocycle" condition (22). Then*

$$R(\lambda) = F_{21}(\lambda)^{-1} * F_{12}(\lambda) \tag{32}$$

satisfies the following generalized quantum dynamical Yang–Baxter equation (or Gervais–Neveu–Felder equation):

$$R_{12}(\lambda) * R_{13}(\lambda + \hbar h^{(2)}) * R_{23}(\lambda) = R_{23}(\lambda + \hbar h^{(1)}) * R_{13}(\lambda) * R_{12}(\lambda + \hbar h^{(3)}). \tag{33}$$

Here $*$ *denotes the natural multiplication on* $C^\infty(\mathfrak{h}^*) \otimes (U\mathfrak{g})^n[[\hbar]]$, $\forall n$, *with* $C^\infty(\mathfrak{h}^*)$ *being equipped with the PBW-star product.*

It is simple to see that the usual relation

$$\Delta(a * b) = \Delta a * \Delta b \tag{34}$$

still holds for any $a,\ b \in C^\infty(\mathfrak{h}^*)\otimes U\mathfrak{g}[[\hbar]]$. Define $\widetilde{\Delta} : C^\infty(\mathfrak{h}^*)\otimes U\mathfrak{g}[[\hbar]] \longrightarrow C^\infty(\mathfrak{h}^*)\otimes U\mathfrak{g}\otimes U\mathfrak{g}[[\hbar]]$ by

$$\widetilde{\Delta}a = F(\lambda)^{-1} * \Delta a * F(\lambda), \quad \forall a \in C^\infty(\mathfrak{h}^*)\otimes U\mathfrak{g}[[\hbar]]. \tag{35}$$

It is simple to see, using the associativity of $*$, that

$$\widetilde{\Delta}^{\mathrm{op}}a = R(\lambda) * \widetilde{\Delta}a * R(\lambda)^{-1}. \tag{36}$$

The following is immediate from Corollary 3.9.

Corollary 4.2.

$$R_{23}(\lambda + \hbar h^{(1)}) = F_{32}(\lambda + \hbar h^{(1)})^{-1} * F_{23}(\lambda + \hbar h^{(1)}). \tag{37}$$

Remark. Equation (37) is trivial when $\mathfrak{h}$ is Abelian. It, however, does not seem obvious in general. We can see from the proof of Corollary 3.9 that this equation essentially follows from the associativity of the star product given by Eq. (24).

For any given $F(\lambda) \in C^\infty(\mathfrak{h}^*)\otimes U\mathfrak{g}\otimes U\mathfrak{g}[[\hbar]]$, introduce $\Phi_{123}(\lambda) \in C^\infty(\mathfrak{h}^*)\otimes U\mathfrak{g}\otimes U\mathfrak{g}\otimes U\mathfrak{g}[[\hbar]]$ by

$$\Phi_{123}(\lambda) = F_{23}(\lambda)^{-1} * [(id\otimes\Delta)F(\lambda)^{-1}] * [(\Delta\otimes id)F(\lambda)] * F_{12}(\lambda). \tag{38}$$

Lemma 4.3.

$$(\widetilde{\Delta}\otimes id)R = \Phi_{231} * R_{13} * \Phi_{132}^{-1} * R_{23} * \Phi_{123}; \tag{39}$$

$$(id\otimes\widetilde{\Delta})R = \Phi_{312}^{-1} * R_{13} * \Phi_{213} * R_{12} * \Phi_{123}^{-1}. \tag{40}$$

Proof. By applying the permutation $a_1\otimes a_2\otimes a_3 \longrightarrow a_1\otimes a_3\otimes a_2$ on Eq. (38), one obtains that

$$\begin{aligned}\Phi_{132}(\lambda) &= F_{32}(\lambda)^{-1} * \sigma_{23}[(id\otimes\Delta)F(\lambda)^{-1}] * \sigma_{23}[(\Delta\otimes id)F(\lambda)] * F_{13}(\lambda)\\ &= F_{32}(\lambda)^{-1} * [(id\otimes\Delta)F(\lambda)^{-1}] * \sigma_{23}[(\Delta\otimes id)F(\lambda)] * F_{13}(\lambda),\end{aligned}$$

since Δ is cocommutative. Similarly, applying the permutation $a_1\otimes a_2\otimes a_3 \longrightarrow a_2\otimes a_3\otimes a_1$ on Eq. (38), one obtains that

$$\Phi_{231}(\lambda) = F_{12}(\lambda)^{-1} * [(\Delta\otimes id)F_{21}(\lambda)^{-1}] * \sigma_{23}[(\Delta\otimes id)F(\lambda)] * F_{31}(\lambda). \tag{41}$$

On the other hand, by definition,

$$R_{13}(\lambda) = F_{31}(\lambda)^{-1} * F_{13}(\lambda), \tag{42}$$

$$R_{23}(\lambda) = F_{32}(\lambda)^{-1} * F_{23}(\lambda). \tag{43}$$

It thus follows that

$$\begin{aligned}&\Phi_{231} * R_{13} * \Phi_{132}^{-1} * R_{23} * \Phi_{123}\\ &\quad = F_{12}(\lambda)^{-1} * (\Delta\otimes id)F_{21}(\lambda)^{-1} * (\Delta\otimes id)F(\lambda) * F_{12}(\lambda) \quad \text{(by Eq. (34))}\\ &\quad = F_{12}(\lambda)^{-1} * (\Delta\otimes id)R(\lambda) * F_{12}(\lambda) \quad \text{(by Eq. (35))}\\ &\quad = (\widetilde{\Delta}\otimes id)R.\end{aligned}$$

Equation (39) can be proved similarly. □

Proof of Theorem 4.1. From Eq. (36), it follows that

$$R_{12} * (\widetilde{\Delta} \otimes id) R = (\widetilde{\Delta}^{op} \otimes id) R * R_{12}.$$

According to Eq. (39), this is equivalent to

$$R_{12} * \Phi_{231} * R_{13} * \Phi_{132}^{-1} * R_{23} * \Phi_{123} = \Phi_{321} * R_{23} * \Phi_{312}^{-1} * R_{13} * \Phi_{213} * R_{12}.$$

Thus,

$$R_{12} * (\Phi_{231} * R_{13} * \Phi_{132}^{-1}) * R_{23} = (\Phi_{321} * R_{23} * \Phi_{312}^{-1}) * R_{13} * (\Phi_{213} * R_{12} * \Phi_{123}^{-1}). \tag{44}$$

Now the twisted-cocycle condition (22) implies that

$$\Phi_{123}(\lambda) = F_{12}(\lambda + \hbar h^{(3)})^{-1} * F_{12}(\lambda). \tag{45}$$

It thus follows that

$$\begin{aligned} &\Phi_{213} * R_{12} * \Phi_{123}^{-1} \\ &\quad = F_{21}(\lambda + \hbar h^{(3)})^{-1} * F_{21}(\lambda) * F_{21}(\lambda)^{-1} * F_{12}(\lambda) * F_{12}(\lambda)^{-1} * F_{12}(\lambda + \hbar h^{(3)}) \\ &\quad = F_{21}(\lambda + \hbar h^{(3)})^{-1} * F_{12}(\lambda + \hbar h^{(3)}) \quad \text{(by Corollary 4.2)} \\ &\quad = R_{12}(\lambda + \hbar h^{(3)}). \end{aligned}$$

Applying the permutations: $a_1 \otimes a_2 \otimes a_3 \longrightarrow a_3 \otimes a_1 \otimes a_2$, and $a_1 \otimes a_2 \otimes a_3 \longrightarrow a_1 \otimes a_3 \otimes a_2$ respectively to the equation above, one obtains

$$\begin{aligned} \Phi_{321} * R_{23} * \Phi_{312}^{-1} &= R_{23}(\lambda + \hbar h^{(1)}) \text{ and} \\ \Phi_{231} * R_{13} * \Phi_{132}^{-1} &= R_{13}(\lambda + \hbar h^{(2)}). \end{aligned}$$

Equation (33) thus follows immediately. □

5. Concluding Remarks

Even though our discussion so far has been mainly confined to triangular dynamical r-matrices, we should point out that there do exist many interesting examples of non-triangular ones. For instance, when the Lie algebra $\mathfrak{g}$ admits an ad-invariant bilinear form and the base Lie algebra $\mathfrak{h}$ equals $\mathfrak{g}$, Alekseev and Meinrenken found an explicit construction of an interesting non-triangular dynamical r-matrix [1] in connection with their study of the non-commutative Weil algebra. In fact, for simple Lie algebras, the existence of AM-dynamical r-matrices was already proved by Etingof and Varchenko [19]. The construction of Alekseev and Meinrenken was later generalized by Etingof and Schiffmann to a more general context [17]. So there is no doubt that there are abundant non-trivial examples of dynamical r-matrices with a nonabelian base. It is therefore desirable to know how they can be quantized. Inspired by the above discussion in the triangular case, we are ready to propose the following quantization problem along the line of Drinfel'd's naive[2] quantization [14].

[2] Drinfeld's original naive quantization was proposed for a classical r-matrix in $A \otimes A$ for an associative algebra A. Here one can consider A as the universal enveloping algebra $U\mathfrak{g}$, and $r \in \mathfrak{g} \otimes \mathfrak{g} \subset U\mathfrak{g} \otimes U\mathfrak{g}$.

Definition 5.1. *Given a classical dynamical r-matrix* $r : \mathfrak{h}^* \longrightarrow \mathfrak{g}\otimes\mathfrak{g}$, *a quantization of* r *is* $R(\lambda) = 1 + \hbar r(\lambda) + O(\hbar^2) \in U(\mathfrak{g})\otimes U(\mathfrak{g})[[\hbar]]$ *which is* H*-equivariant and satisfies the generalized quantum dynamical Yang–Baxter equation (or Gervais–Neveu–Felder equation):*

$$R_{12}(\lambda) * R_{13}(\lambda + \hbar h^{(2)}) * R_{23}(\lambda) = R_{23}(\lambda + \hbar h^{(1)}) * R_{13}(\lambda) * R_{12}(\lambda + \hbar h^{(3)}). \tag{46}$$

Combining Proposition 2.1, Proposition 3.2, Corollary 3.6 and Theorem 4.1, we may summarize the main result of this paper in the following:

Theorem 5.2. *A triangular dynamical* r*-matrix* $r : \mathfrak{h}^* \longrightarrow \wedge^2\mathfrak{g}$ *is quantizable if there exists a compatible star product on the corresponding Poisson manifold* $\mathfrak{h}^* \times G$.

We conclude this paper with a list of questions together with some thoughts.

Question 1. Is every classical triangular dynamical r-matrix quantizable?

According to Theorem 5.2, this question is equivalent to asking whether a compatible star product always exists for the corresponding Poisson manifold $\mathfrak{h}^* \times G$. When the base Lie algebra is Abelian, a quantization procedure was found for splittable classical triangular dynamical r-matrices using Fedosov's method [34]. Recently Etingof and Nikshych [21], using the vertex-IRF transformation method, showed the existence of quantization for the so-called completely degenerate triangular dynamical r-matrices, which leads to the hope that the existence of quantization could be possibly settled by combing both methods in [34] and [21]. However, when the base Lie algebra $\mathfrak{h}$ is nonabelian, the method in [34] does not admit a straightforward generalization. One of the main difficulties is that the Fedosov method uses Weyl quantization, while our quantization here is in normal ordering. Nevertheless, for the dynamical r-matrices constructed in Theorem 2.3, under some mild assumptions a quantization seems feasible by using the generalized Karabegov method [3,5]. This problem will be discussed in a separate publication.

Question 2. What is the symmetrized version of the quantum dynamical Yang–Baxter equation (46)?

We derived Eq. (46) from a compatible star product, which is a normal ordering star product. The reason for us to choose the normal ordering here is that one can obtain a very explicit formula for the star product: Eq. (16). A Weyl ordering compatible star product may exist, but it may be more difficult to work with. For the canonical cotangent symplectic structure T^*H, a Weyl ordering star product was found by Gutt [27], but it is rather difficult to write down an explicit formula [9]. As we can see from the previous discussion, how a quantum dynamical Yang–Baxter equation looks is closely related to the choice of a star product on T^*H. When H is Abelian, there is a very simple operator establishing an isomorphism between these two quantizations, which is indeed the transformation needed to transform a unsymmetrized QDYBE into a symmetrized one. Such an operator also exists for a general cotangent bundle T^*Q [10], but it is much more complicated. Nevertheless, this viewpoint may still provide a useful method to obtain the symmetrized version of a QDYBE.

Question 3. Is every classical dynamical r-matrix quantizable?

This question may be a bit too general. As a first step, it should already be quite interesting to find a quantum analogue of Alekseev–Meinrenken dynamical r-matrices.

Question 4. What is the deformation theory controlling the quantization problem as proposed in Definition 5.1?

If $R = 1 + \hbar r + \cdots + \hbar^i r_i + \cdots$, where $r_i \in C^\infty(\mathfrak{h}^*)\otimes U\mathfrak{g}\otimes U\mathfrak{g},\ i \geq 2$, is a solution to the QDYBE, the $\hbar$-term r must be a solution of the classical dynamical Yang–Baxter equation. Indeed the quantum dynamical Yang–Baxter equation implies a sequence of equations of r_i in terms of lower order terms. One should expect some cohomology theory here just as for any deformation theory [8]. However, in our case, the equation seems very complicated. On the other hand, it is quite surprising that such a theory does not seem to exist in the literature even in the case of quantization of a usual r-matrix.

Finally, we would like to point out that perhaps a more useful way of thinking of quantization of a dynamical r-matrix is to consider the quantum groupoids as defined in [33]. This is in some sense an analogue of the "sophisticated" quantization in terms of Drinfel'd [14]. A classical dynamical r-matrix gives rise to a Lie bialgebroid $(T\mathfrak{h}^* \times \mathfrak{g},\ T^*\mathfrak{h}^* \times \mathfrak{g}^*)$ [7,29]. Its induced Poisson structure on the base space $\mathfrak{h}^*$ is the Lie-Poisson structure $\pi_{\mathfrak{h}^*}$, which admits the PBW-star product as a standard deformation quantization. This leads to the following

Question 5. Does the Lie bialgebroid $(T\mathfrak{h}^* \times \mathfrak{g},\ T^*\mathfrak{h}^* \times \mathfrak{g}^*)$ corresponding to a classical dynamical r-matrix always admit a quantization in the sense of [33], with the base algebra being the PBW-star algebra $C^\infty(\mathfrak{h}^*)[[\hbar]]$?

To connect the quantization problem in Definition 5.1 with that of Lie bialgebroids, it is clear that one needs to consider preferred quantization of Lie bialgebroids: namely, a quantization where the total algebra is undeformed and remains to be $\mathcal{D}(\mathfrak{h}^*)\otimes U\mathfrak{g}[[\hbar]]$.

Question 6. Does the Lie bialgebroid $(T\mathfrak{h}^* \times \mathfrak{g},\ T^*\mathfrak{h}^* \times \mathfrak{g}^*)$ admit a preferred quantization? How is such a preferred quantization related to the quantization of r as proposed in Definition 5.1?

When $\mathfrak{h} = 0$, namely for usual r-matrices, the answer to Question 6 is positive, due to a remarkable theorem of Etingof–Kazhdan [16].

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