Commun. Math. Phys. 227, 421 – 460 (2002)

Communications in
Mathematical Physics

Hyperbolic Low-Dimensional Invariant Tori and Summations of Divergent Series

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Received: 9 July 2001 / Accepted: 26 October 2001

Abstract: We consider a class of a priori stable quasi-integrable analytic Hamiltonian systems and study the regularity of low-dimensional hyperbolic invariant tori as functions of the perturbation parameter. We show that, under natural nonresonance conditions, such tori exist and can be identified through the maxima or minima of a suitable potential. They are analytic inside a disc centered at the origin and deprived of a region around the positive or negative real axis with a quadratic cusp at the origin. The invariant tori admit an asymptotic series at the origin with Taylor coefficients that grow at most as a power of a factorial and a remainder that to any order N is bounded by the $(N+1)$-st power of the argument times a power of $N!$. We show the existence of a summation criterion of the (generically divergent) series, in powers of the perturbation size, that represent the parametric equations of the tori by following the renormalization group methods for the resummations of perturbative series in quantum field theory.

1. Introduction

1.1. The model. Consider the Hamiltonian

$$\mathcal{H} = \boldsymbol{\omega}\cdot\mathbf{A} + \frac{1}{2}\mathbf{A}\cdot\mathbf{A} + \frac{1}{2}\mathbf{B}\cdot\mathbf{B} + \varepsilon f(\boldsymbol{\alpha},\boldsymbol{\beta}), \tag{1.1}$$

where $(\boldsymbol{\alpha},\mathbf{A}) \in \mathbb{T}^r \times \mathbb{R}^r$ and $(\boldsymbol{\beta},\mathbf{B}) \in \mathbb{T}^s \times \mathbb{R}^s$ are conjugated variables, $\cdot$ denotes the inner product both in $\mathbb{R}^r$ and in $\mathbb{R}^s$, and $\boldsymbol{\omega}$ is a vector in $\mathbb{R}^r$ satisfying the Diophantine condition

$$|\boldsymbol{\omega}\cdot\boldsymbol{\nu}| > C_0|\boldsymbol{\nu}|^{-\tau} \qquad \forall\boldsymbol{\nu}\in\mathbb{Z}^r\setminus\{\mathbf{0}\}, \tag{1.2}$$

with $C_0 > 0$ and $\tau \ge r-1$; we shall define by $D_\tau(C_0)$ the set of rotation vectors in $\mathbb{R}^r$ satisfying (1.2). We also write

$$f(\boldsymbol{\alpha},\boldsymbol{\beta}) = \sum_{\boldsymbol{\nu}\in\mathbb{Z}^r} e^{i\boldsymbol{\nu}\cdot\boldsymbol{\alpha}} f_{\boldsymbol{\nu}}(\boldsymbol{\beta}), \tag{1.3}$$

and set $d = r + s$. We shall suppose that f is analytic in a strip of width $\kappa > 0$ around the real axis of the variables $\boldsymbol{\alpha}, \boldsymbol{\beta}$, so that there exists a constant F such that $|f_{\boldsymbol{\nu}}(\boldsymbol{\beta})| \le F\, e^{-\kappa|\boldsymbol{\nu}|}$ for all $\boldsymbol{\nu} \in \mathbb{Z}^r$ and all $\boldsymbol{\beta} \in \mathbb{T}^s$.

1.2. Low-dimensional tori. The equations of motion for the system (1.1) are

$$\begin{cases} \dot{\boldsymbol{\alpha}} = \boldsymbol{\omega} + \mathbf{A}, \\ \dot{\boldsymbol{\beta}} = \mathbf{B}, \\ \dot{\mathbf{A}} = -\varepsilon \partial_{\boldsymbol{\alpha}} f(\boldsymbol{\alpha}, \boldsymbol{\beta}), \\ \dot{\mathbf{B}} = -\varepsilon \partial_{\boldsymbol{\beta}} f(\boldsymbol{\alpha}, \boldsymbol{\beta}). \end{cases} \tag{1.4}$$

For $\varepsilon = 0$ the system of Eqs. (1.4), with initial data $(\boldsymbol{\alpha}_0, \boldsymbol{\beta}_0, \mathbf{0}, \mathbf{0})$, admits the solution

$$\begin{cases} \boldsymbol{\alpha}(t) = \boldsymbol{\alpha}_0 + \boldsymbol{\omega} t, \\ \boldsymbol{\beta}(t) = \boldsymbol{\beta}_0, \\ \mathbf{A}(t) = \mathbf{0}, \\ \mathbf{B}(t) = \mathbf{0}, \end{cases} \tag{1.5}$$

which corresponds to a r-dimensional torus (KAM torus): the first r angles rotate with angular velocity $\omega_1, \dots, \omega_r$, while the remaining s remain fixed to their initial values.

Note that (1.4) can be written as

$$\begin{cases} \ddot{\boldsymbol{\alpha}} = -\varepsilon \partial_{\boldsymbol{\alpha}} f(\boldsymbol{\alpha}, \boldsymbol{\beta}), \\ \ddot{\boldsymbol{\beta}} = -\varepsilon \partial_{\boldsymbol{\beta}} f(\boldsymbol{\alpha}, \boldsymbol{\beta}), \end{cases} \tag{1.6}$$

so that we obtain closed equations for the angle variables: once a solution has been found for them, it can be used to find the action components by a simple integration.

We look for solutions of (1.6), for $\varepsilon \neq 0$, conjugated to (1.5), i.e. we look for solutions of the form

$$\begin{cases} \boldsymbol{\alpha}(t) = \boldsymbol{\psi} + \mathbf{a}(\boldsymbol{\psi}, \boldsymbol{\beta}_0; \varepsilon), \\ \boldsymbol{\beta}(t) = \boldsymbol{\beta}_0 + \mathbf{b}(\boldsymbol{\psi}, \boldsymbol{\beta}_0; \varepsilon), \end{cases} \tag{1.7}$$

for some functions $\mathbf{a}$ and $\mathbf{b}$, real analytic and 2π-periodic in $\boldsymbol{\psi} \in \mathbb{T}^r$, such that the motion in the variable $\boldsymbol{\psi}$ is $\dot{\boldsymbol{\psi}} = \boldsymbol{\omega}$.

We shall prove the following result.

Theorem 1.1. *Consider the equations of motion* (1.6) *for* $\boldsymbol{\omega} \in D_\tau(C_0)$, *and suppose* $\boldsymbol{\beta}_0$ *to be such that*

$$\begin{aligned} &\partial_{\boldsymbol{\beta}} f_{\mathbf{0}}(\boldsymbol{\beta}_0) = \mathbf{0}, \\ &\partial^2_{\boldsymbol{\beta}} f_{\mathbf{0}}(\boldsymbol{\beta}_0) \text{ is negative definite.} \end{aligned} \tag{1.8}$$

There exist a constant $\varepsilon_0 > 0$ *and, for all* $\varepsilon \in (0, \varepsilon_0)$, *two functions* $\mathbf{a}(\boldsymbol{\psi}, \boldsymbol{\beta}_0; \varepsilon)$ *and* $\mathbf{b}(\boldsymbol{\psi}, \boldsymbol{\beta}_0; \varepsilon)$, *real analytic and* 2π*-periodic in* $\boldsymbol{\psi} \in \mathbb{T}^r$, *such that* (1.7) *is a solution of* (1.6).

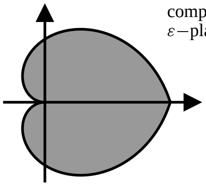

Fig. 1. Analyticity domain D_0 for the hyperbolic invariant torus. The cusp at the origin is a second order cusp. The figure corresponds to the case in (1.8) of the Theorem 1.1

Remarks. (1) As it is well known, and as it will appear from the proof, the solutions whose existence is stated by the theorem cannot be expected to be analytic in ε at $\varepsilon = 0$. Furthermore, if the second condition in (1.8) is replaced with

$$\partial^2_{\boldsymbol{\beta}} f_{\mathbf{0}}(\boldsymbol{\beta}_0) \textit{ is positive definite,} \tag{1.9}$$

then the same conclusions hold for $\varepsilon \in (-\varepsilon_0, 0)$.

(2) The proof will yield more detailed information on the regularity of the considered tori, as we shall point out. In particular the analyticity domain is much larger, see the heart-like domain D_0 in Fig. 1 below (and the discussion in the forthcoming Sect. 5.3). In fact we think that our technique can lead to prove existence of many elliptic invariant tori, i.e. for a large set of negative ε's, and to understand some of their analyticity properties; see Sect. 6 for further remarks and results.

1.3. Contents of the paper. The paper is organized as follows. In Sect. 2 we introduce the main graph techniques which will be used, and we prove through them the formal solubility of the equations of motions (known from [JLZ]): this is enough if one wants to prove existence and analyticity of periodic solutions, (see Remark 2.3 below), but it requires new arguments to obtain existence of quasi-periodic solutions. Such arguments are developed in the following sections: in Sect. 3 we introduce the concept of self-energy graph, which will play a crucial rôle, and we describe the basic cancellation mechanisms which will be used in Sect. 4 to perform a suitable resummation of the series. In Sect. 5 we shall use such results in order to prove the convergence of the resummed series and its analyticity properties. Finally in Sect. 6 we make some conclusive remarks, and briefly discuss possible generalizations and extensions of the results. The main technical aspects of the proof will be relegated to the Appendices.

1.4. Comparison with other papers. The problem considered here is *a priori stable* in the sense of [CG]: the low-dimensional invariant tori are degenerate in absence of perturbations. Hamiltonians of the form (1.1) were explicitly studied in [T], in a more general formulation (see Sect. 6 below), and in [JLZ], in a more particular case.

The problem usually considered in the literature essentially corresponds to a Hamiltonian of the form

$$\frac{1}{2}\mathbf{A}^2 + \boldsymbol{\omega}\cdot\mathbf{A} + \frac{1}{2}\mathbf{B}^2 - \frac{1}{2}\Lambda\boldsymbol{\beta}^2 + f(\mathbf{A},\mathbf{B},\boldsymbol{\alpha},\boldsymbol{\beta}), \quad f_{\mathbf{0}}(\mathbf{A},\mathbf{B},\boldsymbol{\beta}) = O(\mathbf{A}^3+\mathbf{B}^3+\boldsymbol{\beta}^3), \tag{1.10}$$

where Λ is a, *a priori fixed*, nondegenerate matrix (so that before the perturbation is switched on the invariant torus at $\mathbf{A} = \mathbf{0}$ has a priori a well defined stability property, i.e. its elliptic or hyperbolic stability is already well defined); this case is called *a priori unstable* in [CG]. The system (1.10) has been widely studied in the hyperbolic case $\Lambda > 0$, in the elliptic case $\Lambda < 0$, and in the mixed case. The general hyperbolic problem has been studied in [Mo]; in [Gr] the stable and unstable manifolds of the tori are also determined. The elliptic and mixed cases have been considered in great detail in several papers starting with [Me]; the reader will find, besides original results, a complete description of the subsequent results and the relevant references in the recent paper [R], with some very recent further results, on a subject that remains under intense study, in [XY,BKS,Y].

Our case is of the form of (1.10) with Λ replaced by $\varepsilon\Lambda$; in our case the perturbation is small because it is proportional to ε, while in (1.10) one makes also use of the possibility of taking $\mathbf{A}$, $\mathbf{B}$, $\boldsymbol{\beta}$ small to obtain a small perturbation. By classical perturbation analysis our case can be reduced to the theory of (1.10).

We consider the novelty of this paper to be the technical analysis of the analyticity, in ε, of the resonant (i.e. of dimension lower than maximal) hyperbolic invariant tori of (1.1) in a region as large as Figure 1 based on the Lindstedt series method; the same analyticity domain can be obtained by a careful analysis and a nontrivial extension of the methods of [Mo].

This is partially done in [T], where C^∞ dependence on $\sqrt{\varepsilon}$ was proved at $\varepsilon = 0$. And it is done in a more special case in the paper [JLZ], where a scenario very similar to the one provided by our conjecture (see below) emerges.

Closer to our approach is the analysis in [CF]: however the model studied there differs from ours (see (2.24) of [CF]), and existence of hyperbolic low-dimensional tori can be obtained for it without the need of performing the resummations which are on the contrary essential in our case. The technique of [CF] can be extended to cover also our case (which coincides with Eq. (2.22) of [CF]), but it would still make reference to the coordinate changes which are characteristic of the methods of [Mo] (called “classical transformation theory” in [CF]).

In fact one is also interested in asking whether the analyticity region in ε can be extended further to reach *some* points on the negative real axis and whether the analytic continuation to $\varepsilon < 0$ of the parametric equations of the hyperbolic tori can be interpreted as the parametric equations of elliptic tori. We do not address the latter question: the analysis performed in the present paper at first suggests to us that by the same methods it should be possible to prove the following.

Conjecture 1.1. Consider the equations of motion (1.6) for $\boldsymbol{\omega} \in D_\tau(C_0)$, and suppose $\boldsymbol{\beta}_0$ to be such that

$$\begin{aligned} &\partial_{\boldsymbol{\beta}} f_{\mathbf{0}}(\boldsymbol{\beta}_0) = \mathbf{0}, \\ &\partial^2_{\boldsymbol{\beta}} f_{\mathbf{0}}(\boldsymbol{\beta}_0) \text{ is negative definite.} \end{aligned} \tag{1.11}$$

Is it possible that there exist constants $\varepsilon_0 > 0, \xi > 0$ and a subset $I_{\varepsilon_0} \subset [-\varepsilon_0, 0]$ with length $\geq (1 - \varepsilon_0^\xi)\,\varepsilon_0$ such that the functions of Theorem 1.1 above are analytically continuable outside the domain D_0 along vertical lines which start at points interior to D_0 and end on I_{ε_0}, where their boundary value is real and gives the parametric equations of an invariant torus for all $\varepsilon \in I_{\varepsilon_0}$ on which the motion according to (1.6) is $\dot{\boldsymbol{\psi}} = \boldsymbol{\omega}$?

The extended domain shape, near the origin, suggested in the above conjecture is illustrated in the following Fig. 1′.

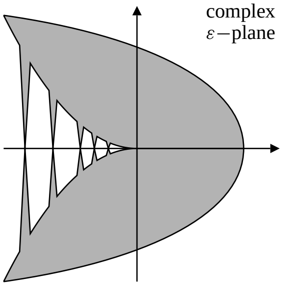

Fig. 1′. The domain D_0 of Fig. 1 can be further extended? The conjecture above asks whether the extended analyticity domain could possibly be represented (close to the origin) as here: the domain reaches the real axis at cusp points which are in I_{ε_0} and correspond, in the complex ε-plane, to the elliptic tori which are the analytic continuations of the hyperbolic tori. The analytic continuation would be continuous through the real axis at the points of I_{ε_0}. The cusps would be at least quadratic

2. Formal Solubility of the Equations of Motion

2.1. Formal expansion and recursive equations. We look for a formal expansion

$$\begin{aligned}\mathbf{a}(\boldsymbol{\psi};\varepsilon) &= \sum_{k=1}^{\infty}\varepsilon^k\mathbf{a}^{(k)}(\boldsymbol{\psi}) = \sum_{\boldsymbol{\nu}\in\mathbb{Z}^r} e^{i\boldsymbol{\nu}\cdot\boldsymbol{\psi}}\mathbf{a}_{\boldsymbol{\nu}}(\varepsilon) = \sum_{k=1}^{\infty}\varepsilon^k\sum_{\boldsymbol{\nu}\in\mathbb{Z}^r} e^{i\boldsymbol{\nu}\cdot\boldsymbol{\psi}}\,\mathbf{a}_{\boldsymbol{\nu}}^{(k)},\\ \mathbf{b}(\boldsymbol{\psi};\varepsilon) &= \sum_{k=1}^{\infty}\varepsilon^k\mathbf{b}^{(k)}(\boldsymbol{\psi}) = \sum_{\boldsymbol{\nu}\in\mathbb{Z}^r} e^{i\boldsymbol{\nu}\cdot\boldsymbol{\psi}}\mathbf{b}_{\boldsymbol{\nu}}(\varepsilon) = \sum_{k=1}^{\infty}\varepsilon^k\sum_{\boldsymbol{\nu}\in\mathbb{Z}^r} e^{i\boldsymbol{\nu}\cdot\boldsymbol{\psi}}\,\mathbf{b}_{\boldsymbol{\nu}}^{(k)},\end{aligned} \tag{2.1}$$

where we have not explicitly written the dependence on $\boldsymbol{\beta}_0$.

Then to order k the equations of motion (1.6) become

$$\begin{aligned}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2\,\mathbf{a}_{\boldsymbol{\nu}}^{(k)} &= [\partial_{\boldsymbol{\alpha}} f]_{\boldsymbol{\nu}}^{(k-1)},\\ (\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2\,\mathbf{b}_{\boldsymbol{\nu}}^{(k)} &= \left[\partial_{\boldsymbol{\beta}} f\right]_{\boldsymbol{\nu}}^{(k-1)},\end{aligned} \tag{2.2}$$

where, given any function F admitting a formal expansion

$$F(\boldsymbol{\psi};\varepsilon) = \sum_{k=1}^{\infty}\varepsilon^k\sum_{\boldsymbol{\nu}\in\mathbb{Z}^r} e^{i\boldsymbol{\nu}\cdot\boldsymbol{\psi}}F_{\boldsymbol{\nu}}^{(k)}, \tag{2.3}$$

we denote by $[F]_{\boldsymbol{\nu}}^{(k)}$ the coefficient with Taylor label k and Fourier label $\boldsymbol{\nu}$.

We can write

$$[\partial_{\boldsymbol{\alpha}} f]_{\boldsymbol{\nu}}^{(k-1)} = \sum_{p\ge 0}\sum_{q\ge 0}\frac{1}{p!}\frac{1}{q!}\sum\nolimits^{*}(i\boldsymbol{\nu}_0)^{p+1}\,\partial_{\boldsymbol{\beta}}^{q} f_{\boldsymbol{\nu}_0}(\boldsymbol{\beta}_0)\Big(\prod_{j=1}^{p}\mathbf{a}_{\boldsymbol{\nu}_j}^{(k_j)}\Big)\Big(\prod_{j=p+1}^{p+q}\mathbf{b}_{\boldsymbol{\nu}_j}^{(k_j)}\Big), \tag{2.4}$$

where $0 < k_j < k$ for all $j = 1, \dots, p+q$ and the $*$ denotes that the sum has to be performed with the constraints

$$1 + \sum_{j=1}^{p+q} k_j = k, \qquad \boldsymbol{\nu}_0 + \sum_{j=1}^{p+q} \boldsymbol{\nu}_j = \boldsymbol{\nu}, \tag{2.5}$$

and, analogously,

$$\big[\partial_{\boldsymbol{\beta}} f\big]_{\boldsymbol{\nu}}^{(k-1)} = \sum_{p\ge0}\sum_{q\ge0}\frac{1}{p!}\frac{1}{q!}\sum\nolimits^{*} (i\boldsymbol{\nu}_0)^p\,\partial_{\boldsymbol{\beta}}^{q+1} f_{\boldsymbol{\nu}_0}(\boldsymbol{\beta}_0)\Big(\prod_{j=1}^{p}\mathbf{a}_{\boldsymbol{\nu}_j}^{(k_j)}\Big)\Big(\prod_{j=p+1}^{p+q}\mathbf{b}_{\boldsymbol{\nu}_j}^{(k_j)}\Big), \tag{2.6}$$

with the same meaning of the symbols.

2.2. Tree formalism. By iterating (2.2), (2.4) and (2.6), one finds that one can represent graphically $\mathbf{a}_{\boldsymbol{\nu}}^{(k)}$ and $\mathbf{b}_{\boldsymbol{\nu}}^{(k)}$ in terms of *trees*. The definition and usage of graphical tools based on tree graphs in the context of KAM theory has been advocated recently in the literature as an interpretation of the work [E]; see for instance [G1,GG,BGGM] and [BaG].

A tree θ (see Fig. 2 below) is defined as a partially ordered set of points, connected by *lines*. The lines are oriented toward the *root*, which is the leftmost point; the line entering the root is called the *root line*. If a line ℓ connects two points v_1 and v_2 and is oriented from v_2 to v_1 we say that $v_2 \prec v_1$ and we shall write $v_1' = v_2$ and $\ell = \ell_{v_2}$; we shall say also that ℓ exits from v_2 and enters v_1.

There will be two kinds of points: the *nodes* and the *leaves*. The leaves can only be endpoints, i.e. they have no lines entering them, but an endpoint can be either a node or a leaf. The lines exiting from the leaves play a very different rôle with respect to the lines exiting from the nodes, as we shall see below. We shall denote by v_0 the last (i.e. leftmost) node of the tree, and by ℓ_0 the root line; for future convenience we shall write $v_0' = r$ but r will not be considered a node.

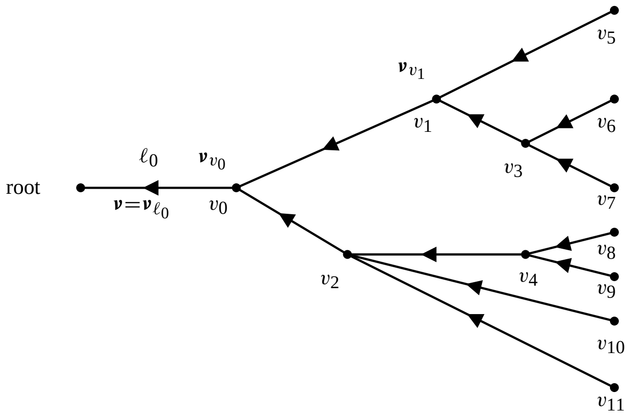

Fig. 2. A tree θ with 12 nodes; one has $p_{v_0}=2, p_{v_1}=2, p_{v_2}=3, p_{v_3}=2, p_{v_4}=2$. The length of the lines should be the same but it is drawn of arbitrary size

We shall denote by $V(\theta)$ the set of nodes, by $L(\theta)$ the set of leaves and by $\Lambda(\theta)$ the set of lines.

For any $\ell_v \in \theta$ fixed, we shall say that the subset of θ containing ℓ_v as well as all nodes $w \preceq v$ and all lines connecting them is a *subtree* of θ with root v': of course a subtree is a tree.

Given a tree, with each node v we associate a *mode label* $\boldsymbol{\nu}_v \in \mathbb{Z}^r$, and to each leaf v a *leaf label* $\kappa_v \in \mathbb{N}$. The quantity

$$k = |V(\theta)| + \sum_{v \in L(\theta)} \kappa_v \tag{2.7}$$

is called the *order* of the tree θ.

With any line ℓ exiting from a node v we associate two labels γ_ℓ, γ'_ℓ assuming the symbolic values α, β and a *momentum label* $\boldsymbol{\nu}_\ell \in \mathbb{Z}^r$, which is defined as

$$\boldsymbol{\nu}_\ell \equiv \boldsymbol{\nu}_{\ell_v} = \sum_{\substack{w \in V(\theta) \\ w \preceq v}} \boldsymbol{\nu}_w, \tag{2.8}$$

while with any line ℓ exiting from a leaf v we associate only the labels $\gamma_\ell = \gamma'_\ell = \beta$.

We can associate with each node also some labels depending on the entering lines and on the exiting one: the *branching labels* p_v and q_v, denoting how many lines ℓ having the label $\gamma_\ell = \alpha$ and, respectively, $\gamma_\ell = \beta$ enter v, and the label δ_v, defined as

$$\delta_v = \begin{cases} 1, & \text{if } \gamma_{\ell_v} = \beta, \\ 0, & \text{if } \gamma_{\ell_v} = \alpha. \end{cases} \tag{2.9}$$

Then with each node v we associate a *node factor*

$$F_v = \frac{1}{p_v!} \frac{1}{q_v!} \Big(i \boldsymbol{\nu}_v \Big)^{p_v + (1-\delta_v)} \partial_{\boldsymbol{\beta}}^{q_v + \delta_v} f_{\boldsymbol{\nu}_v}(\boldsymbol{\beta}_0), \tag{2.10}$$

which is a tensor of rank $p_v + q_v + 1$, while with each leaf v we associate a *leaf factor* (to be defined recursively, see below)

$$L_v = \mathbf{b}_{\mathbf{0}}^{(\kappa_v)}, \tag{2.11}$$

which is a tensor of rank 1 (i.e. a vector); to each line ℓ exiting from a node v we associate a *propagator*

$$G_\ell \equiv \delta_{\gamma_\ell, \gamma_{\ell'}} \frac{\mathbb{1}}{(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell)^2}, \tag{2.12}$$

which is a (diagonal) $d \times d$ matrix, while no small divisor is associated with the lines exiting from the leaves. For consistency we can define

$$G_\ell \equiv \delta_{\gamma_\ell, \gamma_{\ell'}} \, \delta_{\gamma_{\ell'}, \beta} \;\; \mathbb{1}, \tag{2.13}$$

for lines exiting from leaves, so that a propagator G_ℓ is in fact associated with each line.

Remark 2.1. Note that we can write (2.12) in the form

$$G_\ell = \begin{pmatrix} G_{\ell,\alpha\alpha} & G_{\ell,\alpha\beta} \\ G_{\ell,\beta\alpha} & G_{\ell,\beta\beta} \end{pmatrix}, \tag{2.14}$$

where $G_{\ell,\alpha\alpha}$, $G_{\ell,\alpha\beta}$, $G_{\ell,\beta\alpha}$ and $G_{\ell,\beta\beta}$ are $r \times r$, $r \times s$, $s \times r$ and $s \times s$ matrices. By construction one has $G_{\ell,\alpha\beta} = G^T_{\ell,\beta\alpha} = 0$, and

$$G_\ell = G(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell), \qquad G^T(-x) = G^\dagger(x) = G(x); \tag{2.15}$$

here and henceforth T and $\dagger$ denote, respectively, the transposed and the adjoint of a matrix.

2.3. Tree values and reduced tree values. Call $\Theta_{k,\boldsymbol{\nu},\gamma}$ the set of all trees of order k with $\boldsymbol{\nu}_{\ell_0} = \boldsymbol{\nu}$ and $\gamma_{\ell_0} = \gamma$, if ℓ_0 is the root line. Set

$$d_\gamma = \begin{cases} r, & \text{for } \gamma = \alpha, \\ s, & \text{for } \gamma = \beta; \end{cases} \tag{2.16}$$

we can define an application $\mathbf{Val} \colon \Theta_{k,\boldsymbol{\nu},\gamma} \to \mathbb{R}^{d_\gamma}$, defined as

$$\mathbf{Val}(\theta) = \Big(\prod_{v \in V(\theta)} F_v \Big) \Big(\prod_{v \in L(\theta)} L_v \Big) \Big(\prod_{\ell \in \Lambda(\theta)} G_\ell \Big), \tag{2.17}$$

which is called the *value* of the tree θ.

We can define also

$$\mathbf{Val}'(\theta) = \Big(\prod_{v \in V(\theta)} F_v \Big) \Big(\prod_{v \in L(\theta)} L_v \Big) \Big(\prod_{\ell \in \Lambda(\theta) \setminus \ell_0} G_\ell \Big), \tag{2.18}$$

where, as usual, ℓ_0 denotes the root line; $\mathbf{Val}'(\theta)$ is called the *reduced value* of the tree θ.

The following cancellation is proved in Appendix A1.

Lemma 2.1. *Suppose that for all trees $\theta \in \Theta_{k,\boldsymbol{\nu},\gamma}$ the set $\Lambda(\theta) \setminus \ell_0$ does not contain any lines ℓ with momentum $\boldsymbol{\nu}_\ell = \mathbf{0}$. Then $\mathbf{Val}'(\theta)$ is well defined and*

$$\sum_{\theta \in \Theta_{k,\mathbf{0},\alpha}} \mathbf{Val}'(\theta) = \mathbf{0}. \tag{2.19}$$

2.4. Existence of formal solutions. The following result states the existence of formal solutions to (1.6) which are conjugated to the unperturbed motion (1.5), provided the value $\boldsymbol{\beta}_0$ is suitably fixed.

Lemma 2.2. *One can write, formally, for all $\boldsymbol{\nu} \in \mathbb{Z}^r \setminus \{\mathbf{0}\}$,*

$$\begin{aligned} \mathbf{a}^{(k)}_{\boldsymbol{\nu}} &= \sum_{\theta \in \Theta_{k,\boldsymbol{\nu},\alpha}} \mathbf{Val}(\theta), \\ \mathbf{b}^{(k)}_{\boldsymbol{\nu}} &= \sum_{\theta \in \Theta_{k,\boldsymbol{\nu},\beta}} \mathbf{Val}(\theta), \end{aligned} \tag{2.20}$$

while $\mathbf{a}_{\mathbf{0}}^{(k)} \equiv \mathbf{0}$ *and*

$$\mathbf{b}_{\mathbf{0}}^{(k)} = -\left[\partial_{\boldsymbol{\beta}}^2 f_{\mathbf{0}}(\boldsymbol{\beta}_0)\right]^{-1} \sum_{\theta \in \Theta^*_{k+1,\mathbf{0},\beta}} \mathbf{Val}'(\theta), \tag{2.21}$$

where the quantities $\mathbf{Val}(\theta)$ *and* $\mathbf{Val}'(\theta)$ *are defined by* (2.17) *and* (2.18), *respectively, and* $*$ *imposes the constraint that the tree whose reduced value is given by* $\partial_{\boldsymbol{\beta}}^2 f_{\mathbf{0}}(\boldsymbol{\beta}_0)\, \mathbf{b}_{\mathbf{0}}^{(k)}$ *has to be discarded from the set* $\Theta_{k+1,\mathbf{0},\beta}$. *If one has*

$$\begin{aligned} &\partial_{\boldsymbol{\beta}} f_{\mathbf{0}}(\boldsymbol{\beta}_0) = \mathbf{0}, \\ &\det \partial_{\boldsymbol{\beta}}^2 f_{\mathbf{0}}(\boldsymbol{\beta}_0) \neq 0, \end{aligned} \tag{2.22}$$

then there exists a unique way to fix $\mathbf{b}_{\mathbf{0}}^{(k)}$ *for all* $k \in \mathbb{N}$ *such that* $\mathbf{a}_{\boldsymbol{\nu}}^{(k)}$ *and* $\mathbf{b}_{\boldsymbol{\nu}}^{(k)}$ *are finite for all* $\boldsymbol{\nu} \in \mathbb{Z}^p \setminus \{\mathbf{0}\}$ *to all perturbative orders* k.

About the proof. The proof of (2.20) is by induction. In order to show that it is possible to fix uniquely $\mathbf{b}_{\mathbf{0}}^{(k)}$ so that the existence of a formal solution follows, the key is to realize that no division by zero occurs in the recursive solution of (2.2): the coefficients $\mathbf{b}_{\mathbf{0}}^{(k)}$ are determined precisely by imposing the validity of this property for the lines ℓ with $\gamma_\ell = \gamma'_\ell = \beta$. In fact the condition to avoid dividing by zero takes, to all orders k in ε, the form $\partial_{\boldsymbol{\beta}}^2 f_{\mathbf{0}}(\boldsymbol{\beta}_0)\, \mathbf{b}_{\mathbf{0}}^{(k)}$ = some vector determined recursively, so that $\mathbf{b}_{\mathbf{0}}^{(k)}$ is defined by exploiting the assumption (2.22). A further key point is to realize that the lines ℓ with $\gamma_\ell = \gamma'_\ell = \alpha$ and carrying $\boldsymbol{\nu}_\ell = \mathbf{0}$ never appear, and the previous lemma is enough to imply this. Details of the proof are given in Appendix A2. □

Remark 2.2. By (2.2) and by Lemma 2.2 one has

$$\begin{aligned} \left[\partial_{\boldsymbol{\alpha}} f\right]_{\boldsymbol{\nu}}^{(k)} &= \sum_{\theta \in \Theta_{k,\boldsymbol{\nu},\alpha}} \mathbf{Val}'(\theta), \\ \left[\partial_{\boldsymbol{\beta}} f\right]_{\boldsymbol{\nu}}^{(k)} &= \sum_{\theta \in \Theta_{k,\boldsymbol{\nu},\beta}} \mathbf{Val}'(\theta), \end{aligned} \tag{2.23}$$

as one realizes by comparing (2.17) with (2.18).

Remark 2.3. As it will follow from the analysis performed in the next sections, the tools described above are sufficient to prove the convergence (hence the analyticity) of the perturbative expansions (2.1), for ε small enough, in the case of periodic solutions (i.e. $r = 1$): in fact we shall see that the main technical difficulties shall arise from the problem of bounding the propagators, while, in the case of periodic solutions, we can simply bound G_ℓ by the inverse of the rotation vector ω (which is a number in such a case).

3. Self-Energy Graphs

3.1. Trimmed trees. With respect to the papers [GG, BGGM] and [BaG], the trees here carry also "leaves": each leaf can be decomposed in terms of trees, because $\mathbf{b}_{\mathbf{0}}^{(k)}$ is given by

$$\mathbf{b}_{\mathbf{0}}^{(k)} = -\left[\partial_{\boldsymbol{\beta}}^2 f_{\mathbf{0}}(\boldsymbol{\beta}_0)\right]^{-1} \mathbf{G}_{\mathbf{0}}^{(k+1)}, \tag{3.1}$$

with $\mathbf{G}_{\mathbf{0}}^{(k+1)}$ expressed as the sum of reduced values of trees of order $k+1$ (see Appendix A2); more precisely

$$\mathbf{G}_{\mathbf{0}}^{(k+1)} = \sum_{\theta\in\Theta_{k+1,\mathbf{0},\beta}} \mathbf{Val}'(\theta) - \partial_{\boldsymbol{\beta}}^2 f_{\mathbf{0}}(\boldsymbol{\beta}_0)\mathbf{b}_{\mathbf{0}}^{(k)} \equiv \sum_{\theta\in\Theta^*_{k+1,\mathbf{0},\beta}} \mathbf{Val}'(\theta), \tag{3.2}$$

where $*$ has been defined after (2.21): it recalls that the tree whose reduced value is given by $\partial_{\boldsymbol{\beta}}^2 f_{\mathbf{0}}(\boldsymbol{\beta}_0)\,\mathbf{b}_{\mathbf{0}}^{(k)}$ has to be discarded from the set $\Theta_{k+1,\mathbf{0},\beta}$.

Of course each leaf can contain other leaves and so on. If each time a leaf is encountered, it can be decomposed into trees, at the end we have that the value of a tree θ can be expressed as product of factors which are values of trees without leaves, that we can call, as in [Ge], *trimmed trees*. The sum of the orders of all the so obtained trimmed trees is equal to k, if the tree θ belonged to $\Theta_{k,\boldsymbol{\nu},\gamma}$; moreover for all trimmed trees the order equals exactly the number of nodes, as it follows from (2.7) by using that a trimmed tree has no leaves.

3.2. Multi-scale decomposition and clusters. Given a vector $\boldsymbol{\omega}\in D_\tau(C_0)$, define $\boldsymbol{\omega}_0 = 2^\tau C_0^{-1}\boldsymbol{\omega}$. Then there exists a sequence $\{\gamma_n\}_{n\in\mathbb{Z}_+}$, with $\gamma_n\in[2^{n-1},2^n]$, such that

$$\Big||\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}| - \gamma_p\Big| \ge 2^{n+1} \qquad \text{if } 0<|\boldsymbol{\nu}|\le 2^{-(n+3)/\tau}, \tag{3.3}$$

for all $n\le 0$ and for all $p\ge n$, and $|\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}|\ne\gamma_n$ for all $\boldsymbol{\nu}\in\mathbb{Z}$ and for all $n\le 0$; the existence of such a sequence (depending on $\boldsymbol{\omega}$) is proved by proposition in Sect. 3 of [GG].

Given a line ℓ with momentum $\boldsymbol{\nu}_\ell$ we say that ℓ has scale label $n_\ell = 1$ if

$$|\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}_\ell| \ge \gamma_0, \tag{3.4}$$

and scale label $n_\ell = n\in\mathbb{Z}\setminus\mathbb{Z}_+$ if

$$\gamma_{n-1} \le |\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}_\ell| < \gamma_n. \tag{3.5}$$

Once the scale labels have been assigned to the lines one has a natural decomposition of the tree into clusters. A *cluster* T on scale n is a maximal set of nodes and lines connecting them such that all the lines have scales $n'\ge n$ and there is at least one line with scale n; if a cluster T' is contained inside a cluster T we shall say that T' is a subcluster of T. The $m_T\ge 0$ lines entering the cluster T and the possible exiting line (unique if existing at all) are called the *external lines* of the cluster T; given a cluster T on scale n, we shall denote by $n_T = n$ the scale of the cluster. We call $\mathcal{T}(\theta)$ the set of all clusters in a tree θ.

Given a cluster $T \in \mathcal{T}(\theta)$, call $V(T)$, $L(T)$ and $\Lambda(T)$ the set of nodes, the set of leaves and the set of lines of T, respectively. Let us define also

$$\boldsymbol{\nu}_T = \sum_{v \in V(T)} \boldsymbol{\nu}_v, \tag{3.6}$$

and denote by $\mathcal{T}_{\mathbf{0}}(\theta)$ the set of all clusters T with $\boldsymbol{\nu}_T = \mathbf{0}$. Given a cluster T call T_0 the subset of T obtained from T by eliminating all the nodes and lines of the subclusters $T' \subset T$ such that $\boldsymbol{\nu}_{T'} = \mathbf{0}$, and denote by $V(T_0)$ and $\Lambda(T_0)$ the set of nodes and lines, respectively, in T_0.

3.3. Self-energy graphs. We call *self-energy graphs* of a tree θ the clusters $T \in \mathcal{T}(\theta)$ such that

(1) T has only one entering line ℓ_T^2 and one exiting line ℓ_T^1,
(2) $T \in \mathcal{T}_{\mathbf{0}}(\theta)$, i.e.

$$\boldsymbol{\nu}_T \equiv \sum_{v \in V(T)} \boldsymbol{\nu}_v = \mathbf{0}, \tag{3.7}$$

(3) one has

$$\sum_{v \in V(T_0)} |\boldsymbol{\nu}_v| \le 2^{-(n+3)/\tau}, \tag{3.8}$$

where $n_{\ell_T^2} = n$ is the scale of the line ℓ_T^2.

We say that the line ℓ_T^1 exiting a self-energy graph T is a *self-energy line*; we call a *normal line* any line of the tree which is not a self-energy line.

Given a self-energy graph $T \in \mathcal{T}(\theta)$ we say that a self-energy graph $T' \in \mathcal{T}(\theta)$ contained in T is *maximal* if there are no other self-energy graphs internal to T and containing T'. We say that a self-energy graph T has *height* $D_T = 0$ if it does not contain any other self-energy graphs, and that it has height $D_T = D \in \mathbb{Z}_+$, recursively, if it contains maximal self-energy graphs with height $D - 1$.

Given a line $\ell \in \Lambda(T_0)$ with momentum $\boldsymbol{\nu}_\ell$, its *reduced momentum* $\boldsymbol{\nu}_\ell^0$ is defined as

$$\boldsymbol{\nu}_\ell^0 = \sum_{\substack{w \in V(T_0) \\ w \preceq v}} \boldsymbol{\nu}_w, \qquad \ell \equiv \ell_v, \tag{3.9}$$

and it can be given a scale n_ℓ^0 such that

$$\gamma_{n_\ell^0 - 1} \le \left| \boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_\ell^0 \right| < \gamma_{n_\ell^0}; \tag{3.10}$$

we call n_ℓ^0 the *reduced scale* of the line ℓ.

Remark 3.1. (1) Given a self-energy graph T, for all lines $\ell \in \Lambda(T)$, one can write, by setting $\ell = \ell_v$,

$$\boldsymbol{\nu}_\ell = \boldsymbol{\nu}_\ell^0 + \sigma_\ell \boldsymbol{\nu}, \tag{3.11}$$

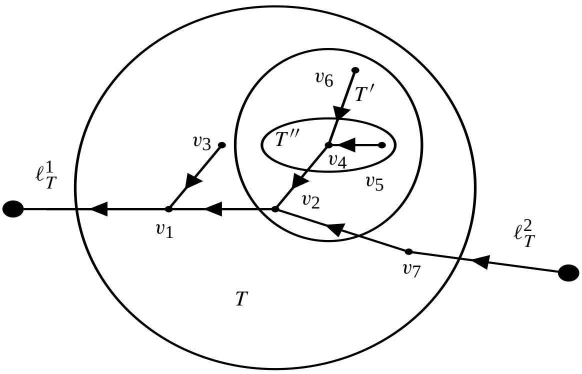

Fig. 3. An example of three clusters symbolically delimited by circles, as visual aids, inside a tree (whose remaining lines and clusters are not drawn and are indicated by the bullets); not all labels are explicitly shown. The scales (not marked) of the lines increase as one crosses inward the circles boundaries: recall, however, that the scale labels are ≤ 0. If the mode labels of (v_4, v_5) add up to $\mathbf{0}$ the cluster T'' is a self-energy graph. If the mode labels of (v_4, v_5, v_2, v_6) add up to $\mathbf{0}$ the cluster T' is a self-energy graph and such is T if the mode labels of $(v_1, v_2, v_7, v_4, v_5, v_2, v_6)$ add up to $\mathbf{0}$. The graph T' is maximal in T. If the three clusters T, T', T'' are self-energy graphs then their heights are respectively 2, 1, 0

where $\boldsymbol{\nu} \equiv \boldsymbol{\nu}_{\ell_T^2}$ is the momentum flowing through the line ℓ_T^2 entering T, while σ_ℓ is defined as follows: writing $\ell \equiv \ell_v$ then $\sigma_\ell = 1$ if ℓ_T^2 enters a node $w \preceq v$ and $\sigma_\ell = 0$ otherwise.
(2) Note that the entering line ℓ_T^2 must have, by the condition (3.7), the same momentum as the exiting line ℓ_T^1, hence, by construction, the same scale $n_{\ell_T^2} = n_{\ell_T^1}$.
(3) The notion of self-energy graphs has been introduced by Eliasson who named them "resonances", [E]. We change the name here not only to avoid confusion with the notion of mechanical resonance (which is related to a rational relation between frequencies of a quasi-periodic motion) but also because the "tree expansions" that we use here (also basically due to Eliasson) can be interpreted, [GGM], as Feynman graphs of a suitable field theory. As such they correspond to classes of self-energy graphs: we use here the correspondence to perform resummation operations typical of renormalization theory.

3.4. Value of a self-energy graph. Given a self-energy graph T and denoting by $|V(T)|$ the number of nodes in T, define the *self-energy value* as

$$\mathcal{V}_T(\boldsymbol{\omega}\cdot\boldsymbol{\nu}) = \varepsilon^{|V(T)|}\Big(\prod_{v\in V(T)} F_v\Big)\Big(\prod_{v\in L(T)} L_v\Big)\Big(\prod_{\ell\in\Lambda(T)} G_\ell\Big), \qquad |V(T)|\ge 1, \quad (3.12)$$

seen as a function of $\boldsymbol{\omega}\cdot\boldsymbol{\nu}$, if $\boldsymbol{\nu} \equiv \boldsymbol{\nu}_{\ell_T^2} = \boldsymbol{\nu}_{\ell_T^1}$ is the momentum flowing through the external lines of the self-energy graph T. Recall that we are considering trimmed trees, so that no leaves can appear; see Sect. 3.1.

We can have four types of self-energy graphs depending on the types α or β of the labels $\gamma'_{\ell_T^1}$ and $\gamma_{\ell_T^2}$:

$$\begin{array}{ccc} & \gamma'_{\ell_T^1} & \gamma_{\ell_T^2} \\ 1. & \alpha & \alpha \\ 2. & \alpha & \beta \\ 3. & \beta & \alpha \\ 4. & \beta & \beta \end{array} \tag{3.13}$$

Given a tree θ, define

$$N_n(\theta) = \{\ell \in \Lambda(\theta) \ : \ n_\ell = n\}\,, \tag{3.14}$$

and

$$M(\theta) = \sum_{v \in V(\theta)} |\boldsymbol{\nu}_v|\,. \tag{3.15}$$

Call $N_n^*(\theta)$ the number of normal lines on scale n and call $R_n(\theta)$ the number of self-energy lines on scale n. Of course

$$N_n(\theta) = N_n^*(\theta) + R_n(\theta). \tag{3.16}$$

Then the following result holds; it is a version of the key estimate of Siegel's theory in the interpretation of Bryuno [B] and Pöschel [P]. This is proved as in [G1] or [BaG], for instance; however, for completeness, a proof is also given in Appendix A3.

Lemma 3.1. *For any tree $\theta \in \Theta_{\kappa,\boldsymbol{\nu},\gamma}$ one has*

$$N_n^*(\theta) \le c\, M(\theta)\, 2^{n/\tau}, \tag{3.17}$$

for some constant c.

3.5. Localization operators. For any self-energy graph T we define

$$\mathcal{L}\mathcal{V}_T(\boldsymbol{\omega}\cdot\boldsymbol{\nu}) \equiv \mathcal{V}_T(0) + (\boldsymbol{\omega}\cdot\boldsymbol{\nu})\,\partial\mathcal{V}_T(0), \tag{3.18}$$

where $\partial\mathcal{V}_T$ denotes the first derivative of $\mathcal{V}_T$ with respect to its argument; the quantity $\mathcal{V}_T(0)$ is obtained from $\mathcal{V}_T(\boldsymbol{\omega}\cdot\boldsymbol{\nu})$ by replacing $\boldsymbol{\nu}_\ell$ with $\boldsymbol{\nu}_\ell^0$ in the argument of each propagator G_ℓ, while $\partial\mathcal{V}_T(0)$ is obtained from $\mathcal{V}_T(\boldsymbol{\omega}\cdot\boldsymbol{\nu})$ by differentiating it with respect to $x = \boldsymbol{\omega}\cdot\boldsymbol{\nu}$, and thence replacing $\boldsymbol{\nu}_\ell$ with $\boldsymbol{\nu}_\ell^0$ in the argument of each propagator G_ℓ.

We shall call $\mathcal{L}$ the *localization operator* and $\mathcal{L}\mathcal{V}_T(\boldsymbol{\omega}\cdot\boldsymbol{\nu})$ the *localized part* of the self-energy value.

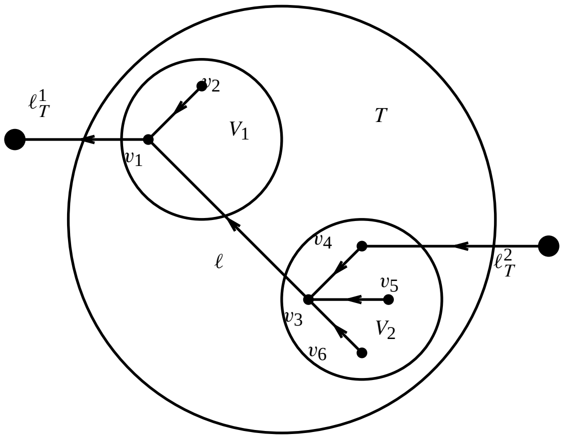

Fig. 4. The sets V_1 and V_2 in a self-energy graph T; note that, even if they are drawn like circles, the sets V_1 and V_2 are not clusters. One has $\boldsymbol{\nu}_\ell^0=\boldsymbol{\nu}_{v_3}+\boldsymbol{\nu}_{v_4}+\boldsymbol{\nu}_{v_5}+\boldsymbol{\nu}_{v_6}$ and $\boldsymbol{\nu}=\boldsymbol{\nu}_{\ell_T^2}$; of course $\boldsymbol{\nu}_{\ell_T^1}=\boldsymbol{\nu}_{\ell_T^2}$ and $\boldsymbol{\nu}_\ell^0=-(\boldsymbol{\nu}_{v_1}+\boldsymbol{\nu}_{v_2})$ by definition of self-energy graph. The black balls represent the remaining parts of the trees. The labels are not explicitly shown.

3.6. Families of self-energy graphs. Given a tree θ containing a self-energy graph T, we can consider all trees obtained by changing the location of the nodes in T_0 (note that T_0 is defined after (3.6)) which the external lines of T are attached to: we denote by $\mathcal{F}_{T_0}(\theta)$ the set of trees so obtained, and call it *the self-energy family* associated with the self-energy graph T. And we shall refer to the operation of detaching and reattaching the external lines, by saying that we are *shifting* such lines.

Of course shifting the external lines of a self-energy graph produces a change of the propagators of the trees. In particular since all arrows have to point toward the root, some lines can revert their arrows.

Moreover the momentum can change, as a reversal of the arrow implies a change of the partial ordering of the nodes inside the self-energy graph and a shifting of the entering line can add or subtract the contribution of the momentum flowing through it. More precisely, if the external lines of a self-energy graph T are detached then reattached to some other nodes in $V(T)$, the momentum flowing through the line $\ell \in \Lambda(T)$ can be changed into $\pm\boldsymbol{\nu}_\ell^0 + \sigma\boldsymbol{\nu}$, with $\sigma \in \{0, 1\}$: if we call V_1 and V_2 the two disjoint sets into which ℓ divides T (see Fig. 2), such that the arrow superposed on ℓ is directed from V_2 to V_1 (before detaching the external lines), then the sign is $+$ if the exiting line is reattached to a node inside V_1 and it is $-$ otherwise, while $\sigma = 1$ if the entering line is reattached to a node inside V_2 when the sign is $+$ and to a node inside V_1 when the sign is $-$, and $\sigma = 0$ otherwise.

Referring to (3.13) for the notion of type of self-energy graph one shows the existence of the following cancellations (the proof is in Appendix A4).

Lemma 3.2. *Given a tree θ, for any self-energy graph $T \in \mathcal{T}(\theta)$ one has*

$$\sum_{\theta' \in \mathcal{F}_{T_0}(\theta)} \mathcal{L}\mathcal{V}_T(\boldsymbol{\omega}\cdot\boldsymbol{\nu}) = \begin{cases} 0, & \text{if } T \text{ is of type 1,} \\ (\boldsymbol{\omega}\cdot\boldsymbol{\nu}) B'_{\mathcal{F}_{T_0}(\theta)}, & \text{if } T \text{ is of type 2,} \\ (\boldsymbol{\omega}\cdot\boldsymbol{\nu}) B''_{\mathcal{F}_{T_0}(\theta)}, & \text{if } T \text{ is of type 3,} \\ A_{\mathcal{F}_{T_0}(\theta)}, & \text{if } T \text{ is of type 4,} \end{cases} \tag{3.19}$$

where $\boldsymbol{\nu} = \boldsymbol{\nu}_{\ell_T^2}$, the sum is over the self-energy family associated with T, and $A_{\mathcal{F}_{T_0}(\theta)}$, $B'_{\mathcal{F}_{T_0}(\theta)}$ and $B''_{\mathcal{F}_{T_0}(\theta)}$ are matrices $s \times s$, $r \times s$ and $s \times r$, respectively, depending only on the self-energy graph T; in particular they are independent of the quantity $\boldsymbol{\omega}\cdot\boldsymbol{\nu}$.

3.7. First step toward the resummation of self-energy graphs. Let θ be a tree $\theta \in \Theta_{\boldsymbol{\nu},k,\gamma}$ with a self-energy graph T. Define $\theta_0 = \theta \setminus T$ as the set of nodes and lines of θ outside T (of course θ_0 is not a tree). Consider simultaneously all trees such that the structure θ_0 outside of the self-energy graph is the same, while the self-energy graph itself can be arbitrary, i.e. T can be replaced by any other self-energy graph T' with $k_{T'} \geq 1$. This allows us to define as a formal power series the matrix

$$M(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon) = \sum_{\theta=\theta_0 \cup T'} \mathcal{V}_{T'}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}), \tag{3.20}$$

where the sum is over all trees θ such that $\theta \setminus T$ is fixed to be θ_0 and the mode labels of the nodes $v \in V(T)$ have to satisfy the conditions (1)–(3) in Sect. 3.3 defining the self-energy graphs.

The following property holds (the proof is in Appendix A5) as an algebraic identity between formal power series.

Lemma 3.3. *The following two properties hold: (1) $(M(x;\varepsilon))^T = M(-x;\varepsilon)$, and (2) $(M(x;\varepsilon))^\dagger = M(x;\varepsilon)$; the latter means that the matrix $M(x;\varepsilon)$ is self-adjoint.*

Remark 3.2. (1) The function $M(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)$ depends on ε but, by construction, it is independent of θ_0: hence we can rewrite (3.20) as

$$M(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon) = \sum_{T'} \mathcal{V}_{T'}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}), \tag{3.21}$$

where the sum is over all self-energy graphs of order $k \geq 1$ with external lines with momentum $\boldsymbol{\nu}$.
(2) In (3.20) or (3.21), if $\gamma_{n-1} \leq |\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}| < \gamma_n$, the sum has to be restricted to the self-energy graphs T' on scale $n_{T'} \geq n+3$. Writing, for any line $\ell \in T'_0$, $\boldsymbol{\nu}_\ell$ as in (3.11) one has

$$\left|\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}_\ell^0\right| > 2^\tau C_0^{-1} C_0 \left|\boldsymbol{\nu}_\ell^0\right|^{-\tau} \geq 2^\tau \Big(\sum_{v\in V(T_0)} |\boldsymbol{\nu}_v| \Big)^{-\tau} \geq 2^\tau 2^{n+3}, \tag{3.22}$$

while $|\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}| < 2^n$, so that, by using again (3.11), one obtains

$$|\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_\ell| > 2^\tau 2^{n+3} - 2^n > 2^{n+2}, \tag{3.23}$$

which implies $n_\ell \geq n+3$.
(3) The matrix $M(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)$ can be written as

$$M(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon) = \begin{pmatrix} M_{\alpha\alpha}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon) & M_{\alpha\beta}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon) \\ M_{\beta\alpha}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon) & M_{\beta\beta}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon) \end{pmatrix}, \tag{3.24}$$

where $M_{\alpha\alpha}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)$, $M_{\alpha\beta}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)$, $M_{\beta\alpha}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)$ and $M_{\beta\beta}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)$ are $r\times r$, $r\times s$, $s\times r$ and $s\times s$ matrices. It is easy to realize that (up to convergence problems to be discussed in Sect. 5)

$$\begin{aligned} M_{\alpha\alpha}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon) &= O(\varepsilon^2(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2)), \\ M_{\alpha\beta}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon) &= O(\varepsilon^2(\boldsymbol{\omega}\cdot\boldsymbol{\nu})), \\ M_{\beta\beta}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon) &= O(\varepsilon) + O(\varepsilon^2(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2). \end{aligned} \tag{3.25}$$

The proportionality of $M_{\alpha\alpha}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)$ to $(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2$ and of $M_{\alpha\beta}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)$ to $\boldsymbol{\omega}\cdot\boldsymbol{\nu}$ is a consequence of Lemma 3.2. First order computations already give, for instance,

$$\begin{aligned} M_{\beta\beta}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon) &= \varepsilon\partial^2_{\boldsymbol{\beta}} f_{\mathbf{0}}(\boldsymbol{\beta}_0) + O(\varepsilon^2) + O(\varepsilon^2(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2), \\ M_{\alpha\beta}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon) &= -2\varepsilon^2 i\,(\boldsymbol{\omega}\cdot\boldsymbol{\nu}) \sum_{\substack{\boldsymbol{\nu}_1+\boldsymbol{\nu}_2=\mathbf{0} \\ |\boldsymbol{\nu}_1|+|\boldsymbol{\nu}_2|<2^{-(n+3)/\tau}}} \frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_2)^3} , \\ &\quad \Big[\boldsymbol{\nu}_1\partial_{\boldsymbol{\beta}} f_{\boldsymbol{\nu}_1}(\boldsymbol{\beta}_0)\partial^2_{\boldsymbol{\beta}} f_{\boldsymbol{\nu}_2}(\boldsymbol{\beta}_0) - \boldsymbol{\nu}_1^2 f_{\boldsymbol{\nu}_1}(\boldsymbol{\beta}_0)\boldsymbol{\nu}_2\partial_{\boldsymbol{\beta}} f_{\boldsymbol{\nu}_2}(\boldsymbol{\beta}_0)\Big] + O(\varepsilon^3(\boldsymbol{\omega}\cdot\boldsymbol{\nu})) \end{aligned} \tag{3.26}$$

where $\gamma_{n-1} \leq |\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}| < \gamma_n$. Therefore $M_{\beta\beta}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon) \neq 0$ by hypothesis (see (1.8)), and $M_{\alpha\beta}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)$ is generically nonvanishing.
(4) Lemma 3.3 implies that, by defining the matrices $B'_{\mathcal{F}_{T_0}(\theta)}$ and $B''_{\mathcal{F}_{T_0}(\theta)}$ as in Lemma 3.2, one has

$$B'_{\mathcal{F}_{T_0}(\theta)} = -\left(B''_{\mathcal{F}_{T_0}(\theta)}\right)^T. \tag{3.27}$$

3.8. Changing scales. When shifting the lines external to the self-energy graphs, the momenta of the internal lines can change. As a consequence in principle also the scale labels could change; however this does not happen, as the following result shows; for the proof see Appendix A6.

Lemma 3.4. *For all lines $\ell \in \Lambda(\theta)$ one has $n_\ell = n^0_\ell$; in particular this implies that, when shifting the lines external to the self-energy graphs of a tree θ, the scale labels n_ℓ of all line $\ell \in \Lambda(\theta)$ do not change.*

4. Resummations of Self-Energy Graphs: Renormalized Propagators

4.1. Renormalized trees. So far we considered formal power expansions in ε. By introducing the function $\mathbf{h} = (\mathbf{h}_\alpha, \mathbf{h}_\beta) = (\mathbf{a}, \mathbf{b})$, we can write the function $\mathbf{h}(\boldsymbol{\psi}, \boldsymbol{\beta}_0; \varepsilon) \equiv \mathbf{h}(\boldsymbol{\psi}; \varepsilon)$ as

$$\mathbf{h}(\boldsymbol{\psi}; \varepsilon) = \sum_{\boldsymbol{\nu} \in \mathbb{Z}^r} e^{i\boldsymbol{\nu}\cdot\boldsymbol{\psi}} \mathbf{h}_{\boldsymbol{\nu}}(\varepsilon), \tag{4.1}$$

because we are looking for a solution periodic in $\boldsymbol{\psi} \in \mathbb{T}^r$.

In terms of the formal power expansion envisaged in Sect. 3, we can define a solution "approximated to order k" as

$$\mathbf{h}_{\boldsymbol{\nu}}^{(\leq k)}(\varepsilon) = \sum_{k'=1}^{k} \varepsilon^{k'} \mathbf{h}_{\boldsymbol{\nu}}^{(k')}, \qquad \mathbf{h}_{\gamma\boldsymbol{\nu}}^{(k')} = \sum_{\theta \in \Theta_{k',\boldsymbol{\nu},\gamma}} \mathbf{Val}(\theta), \tag{4.2}$$

where $\Theta_{k',\boldsymbol{\nu},\gamma}$ is defined in 2.3, and $\mathbf{Val}(\theta)$ is given by (2.17).

However we can define a different sequence of *approximating functions* $\overline{\mathbf{h}}^{[k]}(\boldsymbol{\psi}; \varepsilon)$, formally converging to the formal solution (as we shall see in Proposition 5.4 below), by defining it iteratively as follows.

Denote by $\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},\gamma}$ the set of all trees of order k without self-energy graphs and with labels $\boldsymbol{\nu}_{\ell_0} = \boldsymbol{\nu}$ and $\gamma_{\ell_0} = \gamma$ associated with the root line; we shall call $\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},\gamma}$ the set of *renormalized trees* of order k (and with labels $\boldsymbol{\nu}$ and γ associated with the root line). Given a tree $\theta \in \Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},\gamma}$ and a cluster $T \in \mathcal{T}(\theta)$, by extension we shall say that T is a *renormalized cluster*.

We can also consider a self-energy graph which does not contain any other self-energy graph: we shall say that such a self-energy graph is a *renormalized self-energy graph*; of course no one of such clusters can appear in any tree in $\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},\gamma}$.

For a renormalized tree θ of arbitrary order k', define

$$\overline{\mathbf{Val}}^{[k]}(\theta) = \Big(\prod_{v \in V(\theta)} F_v \Big) \Big(\prod_{v \in L(\theta)} L_v \Big) \Big(\prod_{\ell \in \Lambda(\theta)} \overline{G}_\ell^{[k-1]} \Big), \tag{4.3}$$

with the *dressed propagators* given by

$$\begin{cases} \overline{G}_\ell^{[0]} & = (\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell)^{-2} \ \mathbb{1} \delta_{\gamma_\ell, \gamma_{\ell'}}, \\ \overline{G}_\ell^{[k]} & = \left[(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell)^2 \ \mathbb{1} - M^{[k]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell; \varepsilon) \right]^{-1}, \qquad \text{for } k \geq 1, \end{cases} \tag{4.4}$$

where the sequence $\{M^{[k]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}; \varepsilon)\}_{k \in \mathbb{N}}$ is iteratively defined as the sum of the values of all renormalized self-energy graphs which can be obtained by using the propagators $\overline{G}_\ell^{[k-1]}$, i.e. as

$$\begin{aligned} M^{[k]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}; \varepsilon) &= \sum_{\text{renormalized } T} \mathcal{V}_T^{[k]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}), \\ \mathcal{V}_T^{[k]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}) &= \varepsilon^{|V(T)|} \Big(\prod_{v \in V(T)} F_v \Big) \Big(\prod_{v \in L(T)} L_v \Big) \Big(\prod_{\ell \in \Lambda(T)} \overline{G}_\ell^{[k-1]} \Big), \end{aligned} \tag{4.5}$$

where $|V(T)|$ is the number of nodes in T; we can also define $M^{[0]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)\equiv 0$. The leaf factors L_v in (4.3) are recursively defined as

$$L_v = \mathbf{b}_0^{[k,\kappa_v]} = -\left[\partial^2_{\boldsymbol{\beta}} f_{\mathbf{0}}(\boldsymbol{\beta}_0)\right]^{-1} \sum_{\theta\in\Theta^{\mathcal{R}*}_{\kappa_v,\mathbf{0},\beta}} \overline{\mathbf{Val}}^{[k]\prime}(\theta), \tag{4.6}$$

where

$$\overline{\mathbf{Val}}^{[k]\prime}(\theta) = \Big(\prod_{v\in V(\theta)} F_v\Big)\Big(\prod_{v\in L(\theta)} L_v\Big)\Big(\prod_{\ell\in\Lambda(\theta)\setminus\ell_0} \overline{G}_\ell^{[k-1]}\Big), \tag{4.7}$$

and $*$ has the same meaning as after (3.2).

To avoid confusing the value of a renormalized tree with the tree value introduced in (2.13), we shall call (4.3) the *renormalized value* of the (renormalized) tree.

Then we shall write

$$\begin{aligned}
&\overline{\mathbf{h}}^{[k]}(\boldsymbol{\psi};\varepsilon) = \sum_{\boldsymbol{\nu}\in\mathbb{Z}^r} e^{i\boldsymbol{\nu}\cdot\boldsymbol{\psi}}\,\overline{\mathbf{h}}^{[k]}_{\boldsymbol{\nu}}(\varepsilon),\\
&\overline{\mathbf{h}}^{[k]}_{\boldsymbol{\nu}}(\varepsilon) = \sum_{k'=1}^{\infty} \varepsilon^{k'}\,\overline{\mathbf{h}}^{[k,k']}_{\boldsymbol{\nu}}(\varepsilon), \qquad \overline{\mathbf{h}}^{[k,k']}_{\gamma\boldsymbol{\nu}}(\varepsilon) = \sum_{\theta\in\Theta^{\mathcal{R}}_{k',\boldsymbol{\nu},\gamma}} \overline{\mathbf{Val}}^{[k]}(\theta),
\end{aligned} \tag{4.8}$$

where the last formula holds for $\boldsymbol{\nu}\neq\mathbf{0}$, because for $\boldsymbol{\nu}=\mathbf{0}$, one has (4.6) for $\gamma_\ell=\beta$, while $\overline{\mathbf{h}}^{[k,k']}_{\gamma\mathbf{0}}\equiv 0$.

Remark 4.1. Note that if we expand the quantity $M^{[k]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)$ in powers of ε, by expanding the propagators $\overline{G}_\ell^{[k-1]}$, we reconstruct the sum of the values of all self-energy graphs containing only self-energy graphs with height $D\le k$. Therefore if we expand $M^{[k+1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)$ in powers of ε we obtain the same terms as if expanding $M^{[k]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)$, plus the sum of the values of all the self-energy graphs containing also self-energies graphs with height $k+1$, which are absent in the self-energy graphs contributing to $M^{[k]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)$. Such a result will be used in Appendix A7 in order to prove the following result.

Lemma 4.1. *The power series defining the functions* $\overline{\mathbf{h}}^{[k]}_{\boldsymbol{\nu}}(\varepsilon)$*, truncated at order* $k'\le k$*, coincide with the functions* $\mathbf{h}^{(\le k')}_{\boldsymbol{\nu}}(\varepsilon)$ *given by* (4.2).

5. Convergence of the Renormalized Perturbative Expansion

5.1. Domains of analyticity and norms. Consider in the complex ε-plane the domain $D_{\varepsilon_0}(\varphi)$ in Fig. 5 below: if φ denotes the half-opening of the sector $D_{\varepsilon_0}(\varphi)$, then the radius of the circle delimiting $D_{\varepsilon_0}(\varphi)$ will be of the form $\varepsilon(\varphi)=(\pi-\varphi)\varepsilon_0$ (see below).

We shall define $\|\cdot\|$ an algebraic matrix norm (i.e. a norm which verifies $\|AB\|\le\|A\|\,\|B\|$ for all matrices A and B); for instance $\|\cdot\|$ can be the uniform norm.

The propagators $\overline{G}_\ell^{[k]}$ in (4.4) satisfy interesting k-independent bounds described and proved below.

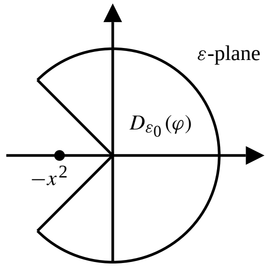

Fig. 5. The domain $D_{\varepsilon_0}(\varphi)$ in the complex ε-plane: the half-opening angle of the sector is $\varphi<\pi$ but, otherwise, arbitrary, and the radius of the circle delimiting $D_{\varepsilon_0}(\varphi)$ is given by $\varepsilon(\varphi) = (\pi - \varphi)\varepsilon_0$

Proposition 5.1. *Let $D_{\varepsilon_0}(\varphi)$ be obtained from the disk of diameter $\varepsilon_0 > 0$ in the complex ε-plane by taking out a sector of half-opening $\pi - \varphi$ around the negative real axis. Assume that the propagators $\overline{G}_\ell^{[k]} \equiv \overline{G}^{[k]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell;\varepsilon)$ satisfy*

$$\left(\overline{G}^{[k]}(x;\varepsilon)\right)^T = \overline{G}^{[k]}(-x;\varepsilon), \qquad \left\|\overline{G}^{[k]}(x;\varepsilon)\right\| < \frac{2}{\pi-\varphi}\frac{1}{x^2} \tag{5.1}$$

for all $|\varepsilon| < (\pi-\varphi)\varepsilon_0$, if ε_0 is small enough. Then there is a constant B_f such that, summing over all renormalized trees θ with $|V(\theta)| = V$ nodes the values $|\overline{\mathbf{Val}}^{[k]}(\theta)|$, one has

$$\begin{aligned}
\overline{\mathbf{h}}_{\gamma\boldsymbol{\nu}}^{[k,V]} \le \sum_{\theta\in\Theta^{\mathcal{R}}_{V,\boldsymbol{\nu},\gamma}} \left|\overline{\mathbf{Val}}^{[k]}(\theta)\right| &\le \left(\frac{|\varepsilon| B_f}{\pi-\varphi}\right)^V, \\
\sum_{\substack{\theta\in\Theta^{\mathcal{R}}_{V,\boldsymbol{\nu},\gamma} \\ M(\theta)=s}} \left|\overline{\mathbf{Val}}^{[k]}(\theta)\right| &\le \left(\frac{|\varepsilon| B_f}{\pi-\varphi}\right)^V e^{-\kappa s/8}, \\
\left|\mathbf{b}_{\mathbf{0}}^{[k,V]}\right| &\le \left\|\left(\partial_{\boldsymbol{\beta}}^2 f_{\mathbf{0}}(\boldsymbol{\beta}_0)^{-1}\right)\right\| \left(\frac{B_f}{\pi-\varphi}\right)^{V+1} |\varepsilon|^V,
\end{aligned} \tag{5.2}$$

for all $s > 0$.

Remark 5.1. Note that, although the propagators are no longer diagonal, they still satisfy the same property as (2.15), which is the crucial one which is used in order to prove both the Lemmata 2.1 and 2.2 about the formal solubility of the equations of motion and the Lemmata 3.2 and 3.3 about the formal cancellations between tree values.

Proof of Proposition 5.1. We can consider first trees without leaves, so that the tree values are given by (4.3) with $L(\theta) = \emptyset$.

The hypothesis (5.1) implies that for all propagators $\overline{G}_\ell^{[k]}$ one has

$$\left\|\overline{G}_\ell^{[k]}\right\| \le C_1 2^{-2n_\ell}, \qquad C_1 = \frac{2}{\pi-\varphi}\left(\frac{2^{\tau+2}}{C_0}\right)^2. \tag{5.3}$$

Therefore the contribution from a single tree (see Sect. 2.3) is bounded for all $n_0 \le 0$ by

$$|\varepsilon|^V \left(C_1\, 2^{-2(n_0-1)}\right)^V \prod_{v\in V(\theta)} \left[\frac{1}{p_v! q_v!} |\boldsymbol{\nu}_v|^{p_v+1} \left|\partial_{\boldsymbol{\beta}}^{q_v+1} f_{\boldsymbol{\nu}_v}(\boldsymbol{\beta}_0)\right| \left(\prod_{n=-\infty}^{n_0-1} C_1 2^{-2c|\boldsymbol{\nu}_v| n 2^{n/\tau}}\right)\right], \tag{5.4}$$

where $V = |V(\theta)|$, having used that, for all trees $\theta \in \Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},\gamma}$, the number $N_n(\theta)$ of lines with scale n in θ satisfy the bound

$$N_n(\theta) \le c\, M(\theta)\, 2^{n/\tau} = c\, 2^{n/\tau} \sum_{v\in V(\theta)} |\boldsymbol{\nu}_v|, \tag{5.5}$$

for some constant c: an estimate which follows from the proof of Lemma 3.1 (see Sect. A3.3). The bound (5.5) is used in deriving (5.3) for all lines $\ell \in \Lambda(\theta)$ with scale $n_\ell < n_0$, while for the lines ℓ with scale $n_\ell \ge n_0$ we have used simply that the propagators are bounded by $C_1 2^{-2(n_0-1)}$.

If we use (recall that we are supposing that there are no leaves)

$$\begin{aligned} \frac{1}{p!} |\boldsymbol{\nu}|^{p+1} &\le (p+1) \left(\frac{8}{\kappa}\right)^{p+1} e^{\kappa|\boldsymbol{\nu}|/8}, \\ \frac{1}{q!} \left|\partial_{\boldsymbol{\beta}}^{q+1} f_{\boldsymbol{\nu}}(\boldsymbol{\beta}_0)\right| &\le C_2^q F e^{-\kappa|\boldsymbol{\nu}|}, \\ \sum_{v\in V(\theta)} (p_v + q_v) &= k-1, \end{aligned} \tag{5.6}$$

for some constant C_2, and if we choose n_0 so that

$$\frac{\kappa}{8} + 2c \sum_{n=|n_0|+1}^{\infty} n 2^{-n/\tau} \le \frac{\kappa}{4}, \tag{5.7}$$

(e.g. we can choose $n_0 = \min\{0, -2\tau \log 2 \log((1-2^{-1/\tau})\kappa/(16c\log 2))$, then we obtain the first bound in (5.2), where we can take $B_f = D_f$, with

$$D_f = D_0 C_0^2 2^{-(n_0-1)} F \sum_{\boldsymbol{\nu}\in\mathbb{Z}^r} e^{-\kappa|\boldsymbol{\nu}|/4}, \tag{5.8}$$

for some positive constant D_0. This follows after summing over all renormalized trees with V nodes and without leaves: this can be easily done. The sum over the mode labels can be performed by using the decay factors $e^{-\kappa|\boldsymbol{\nu}_v|/8}$, while the sum over all the possible tree shapes gives a constant to the power k.

Furthermore the value $\kappa/8$ is so small that with our choices of the constants an extra factor $\exp[-\kappa M(\theta)/8]$ has been bounded by 1 so that if, instead, the value of $M(\theta)$ is fixed we obtain the second bound of (5.2).

So far we considered only trees without leaves. If we want to consider also tree with leaves, we can proceed in the following way.

Given a tree θ (with leaves) of order k', we can write its renormalized value $\overline{\mathbf{Val}}^{[k]}(\theta)$ as the product of the value of a trimmed tree $\overline{\theta}$ times the factors of its leaves: simply look at (4.3)), and interpret the renormalized value of the trimmed tree $\overline{\theta}$ as

$$\overline{\mathbf{Val}}^{[k]}(\overline{\theta}) = \Big(\prod_{v\in V(\theta)} F_v\Big)\Big(\prod_{\ell\in\Lambda(\theta)} \overline{G}_\ell^{[k-1]}\Big), \tag{5.9}$$

while the product

$$\Big(\prod_{v\in L(\theta)} L_v\Big) \tag{5.10}$$

represents the product of the leaf factors (4.6) associated to the $|L(\theta)|$ leaves of θ; note that in (5.9) we can completely neglect the propagators associated to the lines exiting from the leaves, as (2.13) trivially implies.

The only effect of the leaves on $\overline{\mathbf{Val}}^{[k]}(\overline{\theta})$ is through the presence of some extra derivatives $\partial_{\boldsymbol{\beta}}$ acting on the node factors corresponding to some nodes $v \in V(\overline{\theta})$; in particular the momenta of the lines $\ell \in \Lambda(\overline{\theta})$ are completely independent of the leaves (which contribute $\mathbf{0}$ to such momenta).

Each leaf whose factor contributes to (5.10) can be written as a sum of values of renormalized trees $\theta_1, \dots, \theta_{|L(\theta)|}$, according to (4.6); for each such tree, say θ_j, we can write $\overline{\mathbf{Val}}^{[k]}(\theta_j)$ as a product of the renormalized value of the trimmed tree $\overline{\theta}_j$ times the product of the factors of its $|L(\theta_j)|$ leaves. And so on: we iterate until only trimmed trees are left. The sum of the orders of all trimmed trees equals the order k' of the tree θ.

Then we can see how the analysis performed above in the case of trees without leaves can be modified when trees with leaves are also taken into account.

First of all note that if, when considering the trees whose renormalized values contribute to the leaf factor (4.6), we retain only the trees without leaves, we can repeat the analysis leading to (5.8), with the only difference that (as it can be read from (4.6)) one has a matrix $\left[\partial^2_{\boldsymbol{\beta}} f_{\mathbf{0}}(\boldsymbol{\beta}_0)\right]^{-1}$ acting on the reduced value $\overline{\mathbf{Val}}^{[k]\prime}(\theta)$ and the tree θ has order $\kappa_v + 1$ (hence $V+1$, if V is the number of nodes of θ, as we are supposing that θ has no leaves), so that the first bound in (5.2) has to be replaced with

$$\begin{aligned} \left|\mathbf{b}_{\mathbf{0}}^{[k,V]}\right| &\le \left| -\left[\partial^2_{\boldsymbol{\beta}} f_{\mathbf{0}}(\boldsymbol{\beta}_0)\right]^{-1} \sum_{\theta\in\Theta^{\mathcal{R}*}_{V,\boldsymbol{\nu},\gamma}} \overline{\mathbf{Val}}^{[k]\prime}(\theta)\right| \\ &\le \left\|\left(\partial^2_{\boldsymbol{\beta}} f_{\mathbf{0}}(\boldsymbol{\beta}_0)^{-1}\right)\right\| \left(\frac{D_f}{\pi-\varphi}\right)^{V+1} |\varepsilon|^V, \end{aligned} \tag{5.11}$$

so that we have an extra factor

$$C_3 = \left\|\left(\partial^2_{\boldsymbol{\beta}} f_{\mathbf{0}}(\boldsymbol{\beta}_0)^{-1}\right)\right\| \left(\frac{D_f}{\pi-\varphi}\right) \tag{5.12}$$

with respect to the bound one obtains for (5.2): this yields the third bound in (5.2) for leaves arising from trees which do not contain other leaves.

Now we consider any tree of order k', and we decompose it in a collection of trimmed trees (as described above) $\overline{\theta}_0, \overline{\theta}_1, \overline{\theta}_2, \dots$, such that the root of $\overline{\theta}_0$ is the root r of θ, while the root r_i of each other trimmed tree $\overline{\theta}_i$, $i \ge 1$, coincides with a node of some other trimmed tree. Moreover the propagators of the root lines of the trimmed trees $\overline{\theta}_j$, $j \geq 0$, can be neglected by the definition (2.13). Then the value of the tree θ becomes the product of (factorising) values of trimmed trees.

Then we can define the clusters as done in Sect. 3, with the further constraint that all lines internal to a cluster have to belong to the same trimmed tree. Then for each trimmed tree the cancellation mechanisms described in the previous sections apply, and for each of them the same bound as before is obtained.

Therefore for $\overline{\theta}_0$ we can repeat the same analysis as for trees without leaves with the only difference that the third of (5.6) does not hold anymore, and it has to be replaced with

$$\sum_{v \in V(\theta)} (p_v + q_v) = k - 1 + |L(\theta)|; \tag{5.13}$$

as noted before the presence of the leaves implies that, for each of them, there is a derivative $\partial_{\boldsymbol{\beta}}$ acting on the node factor of some node $v \in V(\overline{\theta})$, so that, with respect to the bound (5.2), we obtain an extra factor $C_2^{|L(\theta)|}$ (one for leaf).

Now we can consider the trimmed trees $\overline{\theta}_1, \ldots, \overline{\theta}_{|L(\theta)|}$, and proceed in the same way. With respect to the previous case, for each trimmed tree $\overline{\theta}_j$ we obtain an extra factor C_2 for each leaf attached to some node of $\overline{\theta}_j$. Furthermore, as all the trimmed trees except $\overline{\theta}_0$ contribute to leaves, there is also an extra factor C_3 for each of them.

At the end, instead of the first bound in (5.2) with D_f given by (5.8), we obtain

$$\left(\frac{|\varepsilon| D_f}{\pi - \varphi} \right)^{k'} (C_2 C_3)^L \, ; \tag{5.14}$$

as the total number of leaves is less than the total number of lines with vanishing momentum (hence less than k'), we obtain the first bound in (5.2), provided that one replaces the previous value (5.8) for B_f with

$$B_f = D_f C_2 C_3. \tag{5.15}$$

The sum over the trees can be performed exactly as in the previous case.

In the same way one discusses the second and the third bound in (5.2), which follow with the constant B_f given by (5.15)). This completes the proof. □

Proposition 5.2. *Let $D_{\varepsilon_0}(\varphi)$ be as in Proposition 5.1; then the matrices $M^{[k]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}; \varepsilon)$ satisfy for $\varepsilon \in D_{\varepsilon_0}(\varphi)$ the relation*

$$\left(M^{[k]}(x; \varepsilon) \right)^T = M^{[k]}(-x; \varepsilon). \tag{5.16}$$

Let also

$$M^{[k]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}; \varepsilon) = \begin{pmatrix} M^{[k]}_{\alpha\alpha}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}; \varepsilon) & M^{[k]}_{\alpha\beta}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}; \varepsilon) \\ M^{[k]}_{\beta\alpha}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}; \varepsilon) & M^{[k]}_{\beta\beta}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}; \varepsilon) \end{pmatrix}; \tag{5.17}$$

then, if $\gamma_{q-1} \le |\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}| < \gamma_q$ *and* ε_0 *is small enough, the submatrices* $M^{[k]}_{\gamma\gamma'}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)$ *can be analytically continued in the full disk* $|\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}| \le \gamma_q$ *and satisfy the bounds*

$$\begin{aligned} \left\| M^{[k]}_{\alpha\alpha}(x;\varepsilon) \right\| &\le (|\varepsilon|/(\pi-\varphi))^2 \; C\,x^2, \\ \left\| M^{(k)}_{\alpha\beta}(x;\varepsilon) \right\| &\le (|\varepsilon|/(\pi-\varphi))^2 \; C\;|x|\,, \\ \left\| M^{[k]}_{\beta\beta}(x;\varepsilon) - \varepsilon\,\partial^2_{\boldsymbol{\beta}} f_{\mathbf{0}}(\boldsymbol{\beta}_0) \right\| &\le (|\varepsilon|/(\pi-\varphi))^2 \; C\,x^2, \end{aligned} \tag{5.18}$$

for all $k \in \mathbb{N}$ *and for a suitable constant* C*. As a consequence* $\overline{G}^{[k]}_\ell$ *verify* (5.1) *for all* $k \ge 1$*, and therefore* (5.2) *holds for all* $k \ge 1$.

Proof of Proposition 5.2. We consider the matrices $M^{[k]}$ defined in (4.5) and suppose *inductively* that $M^{[k]}$ verifies (5.18) and the analyticity property preceding it for $0 \le k \le p-1$; note that the assumption holds trivially for $k=0$. Note also that (5.18) imply that the propagators $\overline{G}^{[k]}_\ell$ verify (5.1) for ε_0 small enough and $\varepsilon \in D_{\varepsilon_0}(\varphi)$.

To define $M^{[k]}$ we must consider the renormalized self-energy graphs T and evaluate their values by using the propagators $\overline{G}^{[p-1]}_\ell$, according to (4.5).

Given $x = \boldsymbol{\omega}\cdot\boldsymbol{\nu}$ such that $\gamma_{q-1} \le |x| < \gamma_q$ for some $q \le 0$, the propagators $\overline{G}^{[p-1]}_\ell$ have an analytic extension to the disk $|x| < \gamma_{q+2}$ and, under the hypotheses (5.16) and (5.18), verify the symmetry property and the bound in (5.1), as is shown in Appendix A8.

We have (see (4.5))

$$M^{[p]}(x;\varepsilon) = \sum_{h=q+3}^{0} \sum_{\substack{\text{renormalized } T \\ n_T = h}} \mathcal{V}^{[p]}_{T,h}(x), \tag{5.19}$$

where by appending the label h to $\mathcal{V}^{[p]}_T(x)$ we distinguish the contributions to $M^{[p]}(x;\varepsilon)$ coming from self-energy graphs T on scale h (which is constrained to be $\ge q+3$; see Remark 3.2, (2)).

The value $\mathcal{V}^{[p]}_T(x)$ is analytic in x for $|x| \le \gamma_{h+2}$ and the sum over all T's with V nodes is bounded by

$$\sum_{\substack{T \\ |V(T)|=V}} \left| \mathcal{V}^{[p]}_{T,h}(x) \right| \le \frac{(|\varepsilon| B_f)^V}{1-e^{-\kappa/8}} e^{-\kappa 2^{-h/\tau}/8}, \tag{5.20}$$

because the mode labels $\boldsymbol{\nu}_v$ of the nodes $v \in V(T)$ must satisfy $\sum_{v\in V(T)} |\boldsymbol{\nu}_v| > 2^{-h/\tau}$ (recall that we are dealing with renormalized trees, so that for all clusters $T \in \mathcal{T}(\theta)$ one has $T_0 = T$, and use (3.8) and use Remark 3.2, (2)).

Since the symmetry property expressed by (5.1) for $k=p$ is implied by (5.16) and this is the only property of the propagators that one needs in order to check the algebraic Lemmata 3.2 and 3.3 (see Remark 5.1), we can conclude that the same cancellation mechanisms extend to the renormalized self-energy values $\mathcal{V}^{[p]}_T(\boldsymbol{\omega}\cdot\boldsymbol{\nu})$ (see Remarks A4.6 and A5.3). Therefore we see that $\mathcal{V}^{[p]}_{T,h,\gamma\gamma'}(x)$ will vanish at $x=0$ to order $\sigma_{\gamma\gamma'}$, if we set

$$\sigma_{\gamma\gamma'} = \begin{cases} 2, & \text{for } \gamma = \alpha \text{ and } \gamma' = \alpha, \\ 1, & \text{for } \gamma = \alpha \text{ and } \gamma' = \beta, \\ 1, & \text{for } \gamma = \beta \text{ and } \gamma' = \alpha, \\ 0, & \text{for } \gamma = \beta \text{ and } \gamma' = \beta; \end{cases} \tag{5.21}$$

moreover $\mathcal{V}^{[p]}_{T,h,\beta\beta}(x) - \mathcal{V}^{[p]}_{T,h,\beta\beta}(0)$ vanishes to order 2 at $x = 0$.

By the analyticity in x for $|x| \le \gamma_{h+2}$ and by the maximum principle (Schwarz's lemma) we deduce from (5.10) that one has

$$\begin{aligned} \sum_{\substack{T \\ |V(T)|=V}} \left| \mathcal{V}^{[p]}_{T,h,\gamma\gamma'}(x) \right| &\le \frac{(|\varepsilon| B_f)^V}{1 - e^{-\kappa/8}} e^{-\kappa 2^{-h/\tau}/8} \left(\frac{x}{\gamma_{h+2}} \right)^{\sigma_{\gamma\gamma'}}, \\ \sum_{\substack{T \\ |V(T)|=V}} \left| \mathcal{V}^{[p]}_{T,h,\beta\beta}(x) - \mathcal{V}^{[p]}_{T,h,\beta\beta}(0) \right| &\le \frac{(|\varepsilon| B_f)^V}{1 - e^{-\kappa/8}} e^{-\kappa 2^{-h/\tau}/8} \left(\frac{x}{\gamma_{h+2}} \right)^2. \end{aligned} \tag{5.22}$$

Therefore we can use that $\sum_{h=q+3}^{0} e^{-\kappa 2^{-h/\tau}} 2^{-2h} < B_1 < \infty$ and that $V \ge 2$ for $(\gamma, \gamma') \in \{(\alpha, \alpha), (\alpha, \beta), (\beta, \alpha)\}$, while $V \ge 1$ for $(\gamma, \gamma') = (\beta, \beta')$, and the proof is complete. $\square$

5.2. Convergence of the sequence $\{M^{[k]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}; \varepsilon)\}_{k\in\mathbb{N}}$. It also follows that there exists the limit

$$\lim_{k\to\infty} M^{[k]}(x; \varepsilon) = M^{[\infty]}(x; \varepsilon), \tag{5.23}$$

with $M^{[\infty]}(x; \varepsilon)$ analytic in ε in $D_{\varepsilon_0}(\varphi)$: in fact the following result holds (the proof is in Appendix 6.3).

Lemma 5.1. *For all* $k \ge 1$ *one has*

$$\left\| M^{[k+1]}(x; \varepsilon) - M^{[k]}(x; \varepsilon) \right\| \le \hat{B}_1 \hat{B}_2^k \varepsilon_0^{2k}, \tag{5.24}$$

for some constants $\hat{B}_1$ *and* $\hat{B}_2$.

5.3. Fully renormalized expansion. We can now define the "fully renormalized" expansion of the parametric equations of the invariant torus as the sum of the values of the renormalized trees evaluated according to (4.3) with $\overline{G}^{[k-1]}_\ell$ replaced by

$$\overline{G}^{[\infty]}(x; \varepsilon) = \left(x^2 \, \mathbb{1} - M^{[\infty]}(x; \varepsilon) \right)^{-1}, \qquad x = \boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell. \tag{5.25}$$

The above discussion shows that the series converges for all $\varepsilon \in D_{\varepsilon_0}$ and that it coincides with the limit for $k \to \infty$ of $\overline{\mathbf{h}}^{[k]}(\boldsymbol{\psi}; \varepsilon)$, which therefore exists.

The radius of the domain $D_{\varepsilon_0}(\varphi)$ is $(\pi - \varphi)\,\varepsilon_0$, if φ is the half-opening of the sector $D_{\varepsilon_0}(\varphi)$, because the norms of the propagators $\overline{G}^{[\infty]}(x;\varepsilon)$ are bounded by $2/(x^2(\pi-\varphi))$ (see A8.3).

Therefore the functions $\overline{\mathbf{h}}^{[k]}(\boldsymbol{\psi};\varepsilon)$ converge in a heart-like domain

$$\bigcup_{-\pi\le\varphi<\pi} D_{\varepsilon_0}(\varphi) = D_0, \tag{5.26}$$

whose boundary, for negative ε close to 0, is such that Im (ε) is proportional to (Re $(\varepsilon))^2$.

Proposition 5.3. *There exist positive constants ε_0, B, $\tilde{B}_1$ and $\tilde{B}_2$, such that if*

$$\begin{aligned}
\mathbf{h}^{(k)}_{\mathcal{R}\gamma\boldsymbol{\nu}}(\varepsilon) &= \sum_{\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},\gamma}} \overline{\mathbf{Val}}^{[\infty]}(\theta),\\
\overline{\mathbf{Val}}^{[\infty]}(\theta) &= \Big(\prod_{v\in V(\theta)} F_v\Big)\Big(\prod_{v\in\Lambda(\theta)} F_v\Big)\Big(\prod_{\ell\in L(\theta)} \overline{G}^{[\infty]}_{\ell}\Big),
\end{aligned}\tag{5.27}$$

the renormalized series

$$\overline{\mathbf{h}}^{[\infty]}(\boldsymbol{\psi};\varepsilon) = \sum_{k=1}^{\infty}\varepsilon^k \sum_{\boldsymbol{\nu}\in\mathbb{Z}^r} e^{i\boldsymbol{\nu}\cdot\boldsymbol{\psi}}\mathbf{h}^{(k)}_{\mathcal{R}\boldsymbol{\nu}}(\varepsilon) \tag{5.28}$$

converges in the heart-shaped domain (5.26) *and its coefficients are bounded by*

$$\left|\mathbf{h}^{(k)}_{\mathcal{R}\boldsymbol{\nu}}(\varepsilon)\right| \le \tilde{B}_1\tilde{B}_2^k, \qquad N!^{-1}\left|\partial_\varepsilon^N \mathbf{h}^{(k)}_{\mathcal{R}\boldsymbol{\nu}}(\varepsilon)\right| < N!^{2\tau+1}B^N\,\tilde{B}_1\tilde{B}_2^k, \qquad \text{for } N\ge 0, \tag{5.29}$$

uniformly in $\varepsilon\in D_0$.

5.4. Comments about (5.29). We leave out, for simplicity, the proof that the $N!^{2\tau+1}$ is the appropriate power of $N!$ that follows from our analysis. Although it is quite clear that one has obtained a remainder bound proportional to a power of $N!$, derived already in [JLZ], we evaluated it explicitly in the hope that the power series expansion of $\overline{\mathbf{h}}^{[\infty]}(\boldsymbol{\psi};\varepsilon)$ (hence of $\mathbf{h}(\boldsymbol{\psi};\varepsilon)$, see Proposition 5.4 below) at $\varepsilon = 0$ could be shown to be summable in the sense of the Borel transforms or of its extensions. Since we have analyticity of $\mathbf{h}$ in the domain D_0 of Fig. 1 in Sect. 1 we would need that the remainder in (5.29) behaves at most as $N!^2$, see [CGM]. Since $\tau \ge r-1$ and $r\ge 2$ (in order to have quasi-periodic solutions) we see that (5.29) is not compatible with the general theory. Therefore one needs more information than just (5.29) in order to be able to reconstruct from the power series at the origin the full equation of the invariant torus.

5.5. Conclusions. In Appendix A10 we show that the function $\overline{\mathbf{h}}^{[\infty]}(\boldsymbol{\psi};\varepsilon)$, i.e. the limit for $k\to\infty$ of the approximated functions $\overline{\mathbf{h}}^{[k]}(\boldsymbol{\psi};\varepsilon)$, solves the equations of motion (1.6), so proving the following proposition: this concludes the proof of Theorem 1.1.

Proposition 5.4. *One has, formally (i.e. order by order in the expansion in ε around $\varepsilon = 0$)*

$$\overline{\mathbf{h}}^{[\infty]}(\boldsymbol{\psi};\varepsilon) \equiv \lim_{k\to\infty} \overline{\mathbf{h}}^{[k]}(\boldsymbol{\psi};\varepsilon) = \mathbf{h}(\boldsymbol{\psi};\varepsilon), \tag{5.30}$$

where $\mathbf{h}(\boldsymbol{\psi};\varepsilon)$ is the formal power series which solves Eq. (1.6).

6. Concluding Remarks

6.1. Some extensions. The case of more general Hamitonians of the form

$$\mathcal{H} = h_0(\mathbf{A}) + \varepsilon f(\boldsymbol{\alpha},\mathbf{A}), \tag{6.1}$$

with $(\boldsymbol{\alpha},\mathbf{A}) \in \mathbb{T}^d \times \mathcal{A}$, where $\mathcal{A}$ is an open domain in $\mathbb{R}^d$, should be easily studied as the case treated here to show existence and regularity of invariant tori associated with rotation vectors $\boldsymbol{\omega} \in \mathbb{R}^d$ among whose components there are s rational relations, while the independent ones verify a Diophantine condition.

6.2. Periodic orbits. The fully resonant case $r = 1$ corresponds to periodic orbits is of course a special case of our theory, but it is well known. Note that in such a case the series expansion envisaged in Sect. 2 is sufficient to prove existence (and analyticity) of the periodic solutions, and no resummation is needed; see Remark 2.3.

6.3. (Lack of) Borel summability. As pointed out in the concluding sentence of Sect. 5 the results that we have are not sufficient to imply (extended) Borel summability of the formal power series at the origin of the parametric equations of the torus, i.e. of $\mathbf{h}(\boldsymbol{\psi};\varepsilon)$. The resummations that lead to the construction of $\mathbf{h}(\boldsymbol{\psi};\varepsilon)$ are therefore of a different type from the well known ones associated with the Borel transforms.

Appendix A1. Proof of Lemma 2.1

A1.1. Proof of the lemma. Part I: Neglecting the factorials. Consider all contributions arising from the trees $\theta \in \Theta_{\mathbf{0},k,\alpha}$: we group together all trees obtained from each other by shifting the root line, i.e. by changing the node which the root line exits and orienting the arrows in such a way that they still point toward the root. We call $\mathcal{F}(\theta)$ such a class of trees (here θ is any element inside the class).

The reduced values $\mathbf{Val}'(\theta')$ of such trees $\theta' \in \mathcal{F}(\theta)$ differ because
(1) there is a factor $i\boldsymbol{\nu}_v$ depending on the node v to which the root line is attached (see the definition (2.9) of F_v), and
(2) some arrows change their directions; more precisely, when the root line is detached from the node v_0 and reattached to the node v, if $\mathcal{P}(v_0,v) = \{w \in V(\theta) \;:\; v_0 \succeq w \succeq v\}$ denotes the path joining the node v_0 to the node v, all the momenta flowing through the lines ℓ along the path $\mathcal{P}(v_0,v)$ change their signs, the factorials of the node factors corresponding to the nodes joined by them can change, and the propagators G_ℓ are replaced transposed.

The change of the signs of the momenta simply follows from the fact that

$$\sum_{v \in V(\theta)} \boldsymbol{\nu}_v = \mathbf{0}, \tag{A1.1}$$

as $\theta \in \Theta_{k,\mathbf{0},\alpha}$: by the property (2.15), the propagator does not change.

The change of the factorials contributing to the node factors is due to the fact that for the nodes along the path $\mathcal{P}(v_0, v)$, an entering line can become an exiting line and *vice versa*, so that the labels p_v and q_v can be transformed into $p_v \pm 1$ and $q_v \pm 1$, respectively: this does not modify the factor $(i\boldsymbol{\nu}_v)^{p_v+(1-\delta_v)} \partial_{\boldsymbol{\beta}}^{q_v+\delta_v} f_{\boldsymbol{\nu}_v}(\boldsymbol{\beta}_0)$ in (2.9) – up to the factor $i\boldsymbol{\nu}_v$ (if the root line is attached to v), which has been already taken into account –, as one immediately checks, but it can produce a change of the factorials.

If we neglect the change of the factorials, i.e. if we assume that all combinatorial factors are the same, by summing the reduced values of all possible trees inside the class $\mathcal{F}(\theta)$ we obtain a common value times i times (A1.1), and the sum gives zero.

A1.2. Proof of the lemma. Part II: Taking into account the factorials. One can easily show that a correct counting of the trees implies that all factorials are in fact equal: to do this it is convenient to use *topological* trees instead of the usual *semitopological* used so far (we follow the discussion in [BeG]).

We briefly outline the differences between the two kinds of trees, deferring to [G2] and [GM] for a more detailed discussion of the differences between what finally amounts to a different way to count trees. Define a group of transformations acting on trees generated by the following operations: fix any node $v \in V(\theta)$ and permute the subtrees entering such a node. We shall call semitopological trees the trees which are superposable up to a continuous deformation of the lines, and topological trees the trees for which the same happens modulo the action of the just defined group of transformations. We define *equivalent* two trees which are equal as topological trees.

Then we can still write (2.15) restricting the sum over the set of all nonequivalent topological trees of order k with labels $\boldsymbol{\nu}_{\ell_0} = \boldsymbol{\nu}$ and $\gamma_{\ell_0} = \gamma$ (we can denote it by $\Theta^{\text{top}}_{k,\boldsymbol{\nu},\gamma}$), provided that to each node $v \in V(\theta)$ we associate a combinatorial factor which is not the $(p_v!q_v!)^{-1}$ appearing in (2.9).

In fact for topological trees the combinatorial factor associated to each node is different, because we have to look now to how the subtrees emerging from each node differ. For semitopological trees we have a factor $(p_v!q_v!)^{-1}$ for each node v, where p_v and q_v are the numbers of lines ℓ with $\gamma_\ell = a$ and $\gamma_\ell = \beta$, respectively, entering v: except for the labels γ'_ℓ, we are disregarding the kinds of the subtrees entering v, so that in this way we are counting as different many trees otherwise identical. On the contrary, in the case of topological trees, we consider one and the same tree those trees that are different as semitopological trees, but have the same value because they just differ in the order in which *identical* subtrees enter each node v: therefore, if $s_{v,1}, \ldots, s_{v,j_v}$ are the number of entering lines to which are attached subtrees of a given shape and with the same labels (so that $s_{v,1} + \cdots + s_{v,j_v} = p_v + q_v$, $1 \le j_v \le p_v + q_v$), the combinatorial factor, for each node, becomes

$$\frac{1}{p_v!q_v!} \cdot \frac{p_v!q_v!}{s_{v,1}! \ldots s_{v,j_v}!} = \frac{1}{s_{v,1}! \ldots s_{v,j_v}!}; \tag{A1.2}$$

note in the second factor in the above formula the multinomial coefficient corresponding to the number of different semitopological trees corresponding to the same topological tree, for each node.

So in terms of topological trees $\mathbf{a}_{\boldsymbol{\nu}}^{(k)}$ and $\mathbf{b}_{\boldsymbol{\nu}}^{(k)}$ can be expressed as a sum of tree values $\mathbf{Val}^{\mathrm{top}}(\theta)$, where

$$\mathbf{Val}^{\mathrm{top}}(\theta) = \Big(\prod_{v \in V(\theta)} F_v^{\mathrm{top}} \Big) \Big(\prod_{v \in L(\theta)} L_v \Big) \Big(\prod_{\ell \in \Lambda(\theta)} G_\ell \Big), \tag{A1.3}$$

where

$$F_v^{\mathrm{top}} = \frac{1}{s_{v,1}! \dots s_{v,j_v}!} \Big(i \boldsymbol{\nu}_v \Big)^{p_v + (1-\delta_v)} \partial_{\boldsymbol{\beta}}^{q_v + \delta_v} f_{\boldsymbol{\nu}_v}(\boldsymbol{\beta}_0). \tag{A1.4}$$

Still, when computing the combinatorial factors inside each family $\mathcal{F}(\theta)$, they do differ. But this is actually an apparent, not a real discrepancy. In fact, due to symmetries in the tree (that is, to the fact that the subtrees emerging from some node are sometimes equal, i.e. that some $s_{v,i}$ are greater than 1), the actual number of topological trees in a given family $\mathcal{F}(\theta)$ is less than the total number of trees obtained by the action of the group of transformations: in other words some trees obtained by the action of the group are equivalent as topological trees. When moving the root line from a node v_0 to another node v_1, so transforming a tree θ into a tree $\theta_1 \in \mathcal{F}(\theta)$, for some nodes w along the path $P(v_0, v_1)$ the factor $1/s_{w,i}!$ can turn into $1/(s_{w,i} - 1)!$, but then this means that the same topological tree could be formed by the action of $s_{w,i}$ different transformations of the group: each of the $s_{w,i}$ equivalent subtrees entering w contains a node such that, by attaching to it the root line, the same topological tree is obtained. Therefore, by counting all trees obtained by the action of the group, the corresponding topological tree value is in fact counted $s_{w,i}$ times, so to avoid overcounting one needs a factor $1/s_{w,i}$: this gives back the same combinatorial factor $1/s_{w,i}!$. Analogously one discusses the case of a factor $1/s_{w,i}!$ turning into $1/(s_{w,i} + 1)!$, simply by noting that the same argument as above can be followed also in this case by changing the rôles of the two nodes v_0 and v_1.

A1.3. Remark. The proof of the lemma relies only on the property (2.15) of the propagators, so that also the function $\overline{\mathbf{h}}^{[k]}(\boldsymbol{\psi}; \varepsilon)$ is well defined for all $k \in \mathbb{N}$.

Appendix A2. Proof of Lemma 2.2

A2.1. Proof. In order to prove the lemma we shall show by induction that $\mathbf{a}_{\mathbf{0}}^{(k)}$ can be arbitrarily fixed and $\mathbf{b}_{\mathbf{0}}^{(k)}$ can be uniquely fixed in order to make formally solvable Eqs. (2.2).

For $k = 1$ it is straightforward to realize that $\mathbf{a}_{\boldsymbol{\nu}}^{(1)}$ and $\mathbf{b}_{\boldsymbol{\nu}}^{(1)}$ are well defined for all $\boldsymbol{\nu} \in \mathbb{Z}^r \setminus \{\mathbf{0}\}$, by using the first condition in (1.8).

Then, for $k > 1$, assume that all $\mathbf{a}_{\mathbf{0}}^{(k')}$ and $\mathbf{b}_{\mathbf{0}}^{(k')}$, with $k' < k - 1$, have been fixed, and that, as a consequence, all $\mathbf{a}_{\boldsymbol{\nu}}^{(k')}$ and $\mathbf{b}_{\boldsymbol{\nu}}^{(k')}$ are well defined for $k' < k$ and for all $\boldsymbol{\nu} \in \mathbb{Z}^r \setminus \{\mathbf{0}\}$.

By (2.16) and by Lemma 2.1, in (2.2) one has $[\partial_{\boldsymbol{\alpha}} f]_{\boldsymbol{\nu}}^{(k-1)} = 0$, so that the equation for $\mathbf{a}^{(k)}$ is formally soluble, and $\mathbf{a}_{\mathbf{0}}^{(k)}$ can be arbitrarily fixed, for instance equal to $\mathbf{0}$.

In the second equation in (2.2) one can write

$$\left[\partial_{\boldsymbol{\beta}} f\right]_{\boldsymbol{\nu}}^{(k)} = \partial_{\boldsymbol{\beta}}^2 f_{\mathbf{0}}(\boldsymbol{\beta}_0)\mathbf{b}_{\boldsymbol{\nu}}^{(k-1)} + \mathbf{G}_{\boldsymbol{\nu}}^{(k)}, \tag{A2.1}$$

where the function $\mathbf{G}_{\boldsymbol{\nu}}^{(k)}$ takes into account all contributions except the one explicitly written, and, by construction, all terms appearing in $\mathbf{G}_{\boldsymbol{\nu}}^{(k)}$ can depend only on factors $\mathbf{b}_{\mathbf{0}}^{(k')}$ of orders $k' \le k-2$. We can choose

$$\mathbf{b}_{\mathbf{0}}^{(k-1)} = -\left[\partial_{\boldsymbol{\beta}}^2 f_{\mathbf{0}}(\boldsymbol{\beta}_0)\right]^{-1} \mathbf{G}_{\mathbf{0}}^{(k)}, \tag{A2.2}$$

where the second condition in (1.8) has been used, so that also the equation for $\mathbf{b}^{(k)}$ becomes formally soluble. Of course also $\mathbf{b}_{\mathbf{0}}^{(k)}$ is left undetermined: it will have to be fixed in the next iterative step.

To complete the proof of the lemma one has still to show that the sums over the Fourier labels can be performed, but this is a trivial fact for $\boldsymbol{\omega} \in D_\tau(C_0)$. $\square$

A2.2. Remark. The same proof applies to the renormalized trees introduced in Sect. 4.1.

Appendix A3. Proof of Lemma 3.1

A3.1. Inductive bounds. We prove inductively on the number of nodes of the trees the bounds

$$N_n^*(\theta) \le \max\{0, 2\, M(\theta)\, 2^{(n+3)/\tau} - 1\}, \tag{A3.1}$$

where $M(\theta)$ is defined in (3.15).

First of all note that if $M(\theta) < 2^{-(n+3)/\tau}$ then $N_n(\theta) = 0$ as in such a case for any line $\ell \in \Lambda(\theta)$ one has

$$|\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_\ell| > 2^\tau |\boldsymbol{\nu}_\ell|^{-\tau} > 2^\tau M(\theta)^{-\tau} > 2^\tau 2^{n+3}, \tag{A3.2}$$

by the Diophantine hypothesis (1.2) and by the definition of $\boldsymbol{\omega}_0$ given in Sect. 3.2.

A3.2. Bound on $N_n^*(\theta)$. If θ has only one node the bound is trivially satisfied because, if v is the only node in $V(\theta)$, one must have $M(\theta) = |\boldsymbol{\nu}_v| \ge 2^{-n/\tau}$ in order that the line exiting from v is on scale $\le n$: then $2\, M(\theta)\, 2^{(n+3)/\tau} \ge 4$.

If θ is a tree with $V > 1$ nodes, we assume that the bound holds for all trees having $V' < V$ nodes. Define $E_n = (2\, 2^{(n+3)/\tau})^{-1}$: so we have to prove that $N_n^*(\theta) \le \max\{0, M(\theta)\, E_n^{-1} - 1\}$.

If the root line ℓ of θ is either on scale $\neq n$ or a self-energy line with scale n, call $\theta_1, \dots, \theta_m$ the $m \ge 1$ subtrees entering the last node v_0 of θ. Then

$$N_n^*(\theta) = \sum_{i=1}^{m} N_n^*(\theta_i), \tag{A3.3}$$

hence the bound follows by the inductive hypothesis.

If the root line ℓ is normal (i.e. it is not a self-energy line) and it has scale n, call $\ell_1, \dots, \ell_m$ the $m \geq 0$ lines on scale $\leq n$ which are the nearest to ℓ (this means that no other line along the paths connecting the lines $\ell_1, \dots, \ell_m$ to the root line is on scale $\leq n$). Note that in such a case $\ell_1, \dots, \ell_m$ are the entering line of a cluster T on scale $> n$.

If θ_i is the subtree with ℓ_i as root line, one has

$$N_n^*(\theta) = 1 + \sum_{i=1}^{m} N_n^*(\theta_i), \tag{A3.4}$$

so that the bound becomes trivial if either $m = 0$ or $m \geq 2$.

If $m = 1$ then one has $T = \theta \setminus \theta_1$, and the lines ℓ and ℓ_1 are both with scales $\leq n$; as ℓ_1 is not entering a self-energy graph, then

$$|\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_\ell| \leq 2^n, \qquad |\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_{\ell_1}| \leq 2^n, \tag{A3.5}$$

and either $\boldsymbol{\nu}_\ell = \boldsymbol{\nu}_{\ell_1}$ and one must have (recall that T_0 is defined after (3.6))

$$\sum_{v \in V(T)} |\boldsymbol{\nu}_v| \geq \sum_{v \in V(T_0)} |\boldsymbol{\nu}_v| > \ 2^{-(n+3)/\tau} = 2E_n > E_n, \tag{A3.6}$$

or $\boldsymbol{\nu}_\ell \neq \boldsymbol{\nu}_{\ell_1}$, otherwise T would be a self-energy graph (see (3.7) and (3.8)). If $\boldsymbol{\nu}_\ell \neq \boldsymbol{\nu}_{\ell_1}$, then, by (A3.5) one has $|\boldsymbol{\omega}_0 \cdot (\boldsymbol{\nu}_\ell - \boldsymbol{\nu}_{\ell_1})| \leq 2^{n+1}$, which, by the Diophantine condition (1.2), implies $|\boldsymbol{\nu}_\ell - \boldsymbol{\nu}_{\ell_1}| > 2\,2^{-(n+1)/\tau}$, so that again

$$\sum_{v \in V(T)} |\boldsymbol{\nu}_v| \geq \left|\boldsymbol{\nu}_\ell - \boldsymbol{\nu}_{\ell_1}\right| > 2\,2^{-(n+2)/\tau} > 2^{1/\tau+2} E_n > E_n, \tag{A3.7}$$

as in (A3.6). Therefore in both cases we get

$$M(\theta) - M(\theta_1) = \sum_{v \in T} |\boldsymbol{\nu}_v| > E_n, \tag{A3.8}$$

which, inserted into (A3.4) with $m = 1$, gives, by using the inductive hypothesis,

$$\begin{aligned} N_n^*(\theta) &= 1 + N_n^*(\theta_1) \leq 1 + M(\theta_1)\, E_n^{-1} - 1 \\ &\leq 1 + \Big(M(\theta) - E_n\Big) E_n^{-1} - 1 \leq M(\theta)\, E_n^{-1} - 1, \end{aligned} \tag{A3.9}$$

hence the bound is proved also if the root line is normal and on scale n.

A3.3 Remark. The same argument proves the bound (5.4) for renormalized trees, by using the observation that there are no self-energy lines in the renormalized trees.

Appendix A4. Proof of Lemma 3.2

A4.1. Factorials. As for the proof of the Lemma 2.2 we ignore the factorials: to take them into account one can reason as said in Appendix A1.

A4.2. Self-energy graphs of type 1. First we prove that $\sum_T \mathcal{V}_T(0) = 0$. Given a tree θ consider all trees which can be obtained by shifting the entering line ℓ_T^2. Note that the trees so obtained are contained in the self-energy graph family $\mathcal{F}_{T_0}(\theta)$.

Corresponding to such an operation $\mathcal{V}_T(0)$ changes by a factor $i\boldsymbol{\nu}_v$ if v is the node which the entering line is attached to, as all node factors and propagators do not change. By (3.6) the sum of all such values is zero.

Then consider $\partial\mathcal{V}_T(0)$. By construction

$$\partial\mathcal{V}_T(0) = \sum_{\ell\in\Lambda(T)} \Big(\prod_{v\in V(T)} F_v \Big) \Big(\partial G_\ell^{(n_\ell)} \prod_{\ell'\in\Lambda(T)\setminus\ell} G_{\ell'}^{(n_{\ell'})} \Big), \tag{A4.1}$$

where all propagators have to be computed for $\boldsymbol{\omega}\cdot\boldsymbol{\nu} = 0$, and

$$\partial G_\ell^{(n_\ell)} = \frac{\mathrm{d}}{\mathrm{d}x} G_\ell^{(n_\ell)}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell^0 + \sigma_\ell x)\Big|_{x=0}, \qquad x = \boldsymbol{\omega}\cdot\boldsymbol{\nu}. \tag{A4.2}$$

The line ℓ divides $V(T)$ into two disjoint set of nodes V_1 and V_2, such that ℓ_T^1 exits from a node inside V_1 and ℓ_T^2 enters a node inside V_2: if $\ell = \ell_v$ one has $V_2 = \{w \in (T) : w \preceq v\}$ and $V_1 = V(T) \setminus V_2$. By (3.4), if

$$\boldsymbol{\nu}_1 = \sum_{v\in V_1} \boldsymbol{\nu}_v, \qquad \boldsymbol{\nu}_2 = \sum_{v\in V_2} \boldsymbol{\nu}_v, \tag{A4.3}$$

one has $\boldsymbol{\nu}_1 + \boldsymbol{\nu}_2 = \mathbf{0}$. Then consider the families $\mathcal{F}_1(\theta)$ and $\mathcal{F}_2(\theta)$ of trees obtained as follows: $\mathcal{F}_1(\theta)$ is obtained from θ by detaching ℓ_T^1 then reattaching to all the nodes $w \in V_1$ and by detaching ℓ_T^2 then reattaching to all the nodes $w \in V_2$, while $\mathcal{F}_2(\theta)$ is obtained from θ by reattaching the line ℓ_T^1 to all the nodes $w \in V_2$ and by reattaching the line ℓ_T^2 to all the nodes $w \in V_2$; note that $\mathcal{F}_1(\theta) \cup \mathcal{F}_2(\theta) \subset \mathcal{F}_{T_0}(\theta)$.

As a consequence of such an operation the arrows of some lines $\ell \in \Lambda(T)$ change their directions: this means that for some line ℓ the momentum $\boldsymbol{\nu}_\ell$ is replaced with $-\boldsymbol{\nu}_\ell$ and the propagators G_ℓ are replaced with their transposed G_ℓ^T. As the propagators satisfy (2.15) no overall change is produced by such factors, except for the differentiated propagator which can change sign: one has a different sign for the trees in $\mathcal{F}_1(\theta)$ with respect to the trees in $\mathcal{F}_2(\theta)$. Then by summing over all the possible trees in $\mathcal{F}_1(\theta)$ we obtain a value $i^2\boldsymbol{\nu}_1\boldsymbol{\nu}_2$ times a common factor, while by summing over all the possible trees in $\mathcal{F}_2(\theta)$ we obtain $-i^2\boldsymbol{\nu}_1\boldsymbol{\nu}_2$ times the same common factor, so that the sum of two sums gives zero.

A4.3. Self-energy graphs of type 2. Given a tree θ with a self-energy graph T consider all trees obtained by detaching the exiting line, then reattaching to all the nodes $v \in V(T)$; note again that the trees so obtained are contained in the self-energy graph family $\mathcal{F}_{T_0}(\theta)$. In such a case again some momenta can change sign, but the corresponding propagator does not change (reason as above for self-energy graphs of type 1). So at the end we obtain a common factor times $i\boldsymbol{\nu}_v$, where v is the node which the exiting line is attached to. By (3.6) again we obtain $\sum_T \mathcal{V}_T(0) = 0$.

A4.4. Self-energy graphs of type 3. To prove that $\sum_T \mathcal{V}_T(0) = 0$ simply reason as for $\sum_T \mathcal{V}_T(0)$ in the case (1), by using that the entering line ℓ_T^2 has $\gamma_{\ell_T^2} = \alpha$.

A4.5. Self-energy graphs of type 4. Given a tree θ with a self-energy graph T consider the contribution to $\partial V_T(0)$ in which a line ℓ is differentiated (see (A4.1)). The line ℓ divides $V(T)$ into two disjoint set of nodes V_1 and V_2, such that ℓ_T^1 exits from a node v_1 inside V_1 and ℓ_T^2 enters a node v_2 inside V_2: if $\ell = \ell_v$ one has $V_2 = \{w \in V(T) : w \preceq v\}$ and $V_1 = V(T) \setminus V_2$. By (3.6), with the notations (A4.3), one has $\boldsymbol{\nu}_1 + \boldsymbol{\nu}_2 = \mathbf{0}$. Then consider the tree obtained by detaching ℓ_T^1 from v_1, then reattaching to the node v_2 and, simultaneously, by detaching ℓ_T^2 from v_2, then reattaching to the node v_1; note that the tree so obtained is inside the class $\mathcal{F}_{T_0}(\theta)$.

As a consequence of such an operation the arrows along the path $\mathcal{P}$ connecting v_1 to v_2 change their directions: this means that for such lines ℓ the momentum $\boldsymbol{\nu}_\ell$ is replaced with $-\boldsymbol{\nu}_\ell$, but the propagators are even in the momentum, so that no overall change is produced by such factors, if not because of the differentiated propagator (which is along the path by construction) which changes sign. For all the other lines (i.e. the lines not belonging to $\mathcal{P}$) the propagator is left unchanged.

Since a derivative with respect to $\boldsymbol{\beta}$ acts on both the nodes v_1 and v_2, the shift of the external lines does not produce any change on the node factors (except for the factorials, that we are not explicitly considering, as said at the beginning of this subsection). Then by summing over the two considered trees we obtain zero because of the change of sign of the differentiated propagator.

A4.6. Remark. To prove the Lemma 3.2 we only use that the propagators satisfy (2.15), so that the same proof applies also to the renormalized self-energy graphs (see the Proposition 5.2), where there are no self-energy lines and the propagators are given by (4.4).

Appendix A5. Proof of Lemma 3.3

A5.1. Proof of the property (1). Given a self-energy graph T with momentum $\boldsymbol{\nu}$ flowing through the entering line ℓ_T^2, call $\mathcal{P}$ the path connecting the exiting line ℓ_T^1 to the entering line ℓ_T^2. Then consider also the self-energy graph T' obtained by taking ℓ_T^1 as the entering line and ℓ_T^2 as the exiting line and by taking $-\boldsymbol{\nu}$ as the momentum flowing through the (new) entering line ℓ_T^1: in this way the arrows of all the lines along the path $\mathcal{P}$ are reverted, while all the subtrees (internal to T) having the root in $\mathcal{P}$ are left unchanged. This implies that the momenta of the lines belonging to $\mathcal{P}$ change signs, while all the other momenta do not change. Since all propagators G_ℓ are transformed into G_ℓ^T the property (2.15) implies that the entry ij of the matrix $M(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)$ corresponding to the self-energy graph T is equal to the entry ji of the matrix $M(-\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)$; then the assertion follows.

A5.2. Proof of the property (2). Given a self-energy graph T, consider also the self-energy graph T' obtained by reverting the sign of the mode labels of the nodes $v \in V(T)$, and by swapping the entering line with the exiting one. In this way the arrows of all the lines along the path $\mathcal{P}$ joining the two external lines are reverted, while all the subtrees (internal to T) having the root in $\mathcal{P}$ are left unchanged. It is then easy to realize that the complex conjugate of $\mathcal{V}_{T'}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})$ equals $\mathcal{V}_T(\boldsymbol{\omega}\cdot\boldsymbol{\nu})$, by using the form of the node factors (2.10), and the fact that one has $f_{\boldsymbol{\nu}}^*(\boldsymbol{\beta}) = f_{-\boldsymbol{\nu}}(\boldsymbol{\beta})$ and $G^\dagger(\boldsymbol{\omega}\cdot\boldsymbol{\nu}) = G(\boldsymbol{\omega}\cdot\boldsymbol{\nu})$ (see (2.15)).

A5.3 Remark. The lemma has been proved without making use of the exact form of the propagator, but only exploiting the fact that it satisfies the property (2.15): therefore, once more, the proof applies also to the renormalized trees as a consequence of the first relation in (5.1), and it gives (5.6).

Appendix A6. Proof of Lemma 3.4

A6.1. Set-up. Given a tree $\theta \in \Theta_{k,\boldsymbol{\nu},\gamma}$, consider a self-energy graph T with height D. Call $T^{(2)} \subset T^{(3)} \subset \ldots \subset T^{(D)}$ the resonances containing T, and set $T = T^{(1)}$; denote by $n = n_1 > n_2 > n_3 > \ldots > n_D$ the scales of the lines entering such resonances.

For any $\ell \in \Lambda(T_0)$, one can write

$$\boldsymbol{\nu}_\ell = \boldsymbol{\nu}_\ell^0 + \sigma_\ell \boldsymbol{\nu}, \tag{eA6.1}$$

where $\boldsymbol{\nu}$ is the momentum of the line ℓ_T^2 (with scale n) entering T (see (3.9)).

A6.2. Proof. By shifting the lines entering all the resonances containing ℓ, the momentum $\boldsymbol{\nu}_\ell$ can change into a new value $\tilde{\boldsymbol{\nu}}_\ell$ (as it can be seen by applying iteratively (3.10)) in such a way that $|\boldsymbol{\omega}_0 \cdot \tilde{\boldsymbol{\nu}}_\ell|$ differs from $\left|\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_\ell^0\right|$ by a quantity bounded by $\gamma_n + \gamma_{n_2} + \ldots + \gamma_{n_D} \le 2^n + 2^{n_2} + \ldots + 2^{n_D} < 2^{n+1}$.

As $\left|\boldsymbol{\nu}_\ell^0\right| < 2^{-(n+3)/\tau}$, by definition of the self-energy graph, we can apply (3.2) and conclude that $|\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_\ell^0|$ has to be contained inside an interval $[\gamma_{p-1}, \gamma_p]$, with $p = n_\ell^0 \ge n+3$ (see Remark 3.2, (2)), at a distance at least 2^{n+1} from the extremes: therefore the quantity $|\boldsymbol{\omega}_0 \cdot \tilde{\boldsymbol{\nu}}_\ell|$ still falls inside the same interval $[\gamma_{p-1}, \gamma_p]$. In particular this implies the identity

$$n_\ell = \tilde{n}_\ell = n_\ell^0, \tag{A6.2}$$

if $\tilde{n}_\ell$ is defined as the integer such that $\gamma_{\tilde{n}_\ell - 1} \le |\boldsymbol{\omega}_0 \cdot \tilde{\boldsymbol{\nu}}_\ell| < \gamma_{\tilde{n}_\ell}$.

Appendix A7. Proof of Lemma 4.1

A7.1. Set-up. We have to prove that, for all $k' \le k$, one has

$$[\overline{\mathbf{h}}^{[k]}(\boldsymbol{\psi}; \varepsilon)]_{\boldsymbol{\nu}}^{(k')} = \mathbf{h}_{\boldsymbol{\nu}}^{(k')}, \tag{A7.1}$$

which yields that the functions $\mathbf{h}$ and $\overline{\mathbf{h}}^{[k]}$ admit the same power series expansions up to order k.

Of course both $\mathbf{h}_{\boldsymbol{\nu}}^{(k')}$ and $[\overline{\mathbf{h}}^{[k]}(\boldsymbol{\psi}; \varepsilon)]_{\boldsymbol{\nu}}^{(k')}$ are given by sums of several contributions; in the same way $[M^{[k]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}; \varepsilon)]^{(k')}$ can be expressed as the sum of several terms. We shall prove that, given any tree value $\mathbf{Val}(\theta)$ contributing to $\mathbf{h}_{\boldsymbol{\nu}}^{(k')}$ and any self-energy value $\mathcal{V}_T(x)$ corresponding to a self-energy graph T of order k', one can find the same terms contributing, respectively, to $[\overline{\mathbf{h}}^{[k]}(\boldsymbol{\psi}; \varepsilon)]_{\boldsymbol{\nu}}^{(k')}$ and to $[M^{[k]}(x; \varepsilon)]^{(k')}$, and *vice versa*. The proof will be by induction on $k' = 1, \ldots, k$.

A7.2. Remarks. (1) Note that $[\overline{\mathbf{h}}^{[k]}(\boldsymbol{\psi};\varepsilon)]_{\boldsymbol{\nu}}^{(k')}$ is not the same as $\overline{\mathbf{h}}_{\boldsymbol{\nu}}^{[k,k']}(\boldsymbol{\psi};\varepsilon)$: the quantity $[\overline{\mathbf{h}}^{[k]}(\boldsymbol{\psi};\varepsilon)]_{\boldsymbol{\nu}}^{(k')}$ is the coefficient to order k' that one obtains by developing $\overline{\mathbf{h}}^{(k)}(\boldsymbol{\psi};\varepsilon)$ in powers of ε.
(2) The contributions to $[\overline{\mathbf{h}}^{[k]}(\boldsymbol{\psi};\varepsilon)]_{\boldsymbol{\nu}}^{(k')}$ and to $[M^{[k]}(x;\varepsilon)]^{(k')}$ can arise only from trees of order $k''\le k'$. Furthermore $[\overline{\mathbf{h}}^{[k]}(\boldsymbol{\psi};\varepsilon)]_{\boldsymbol{\nu}}^{(k')}=[\overline{\mathbf{h}}^{[k']}(\boldsymbol{\psi};\varepsilon)]_{\boldsymbol{\nu}}^{(k')}$ and $[M^{[k]}(x;\varepsilon)]^{(k')}=[M^{[k']}(x;\varepsilon)]^{(k')}$: this simply follows from Remark 4.1 and the trivial observation that, for $k>k'$, the self-energy graphs [trees] whose values contribute to $M^{[k]}(x;\varepsilon)$ $[\overline{\mathbf{h}}^{[k]}(\boldsymbol{\psi};\varepsilon)]$ but not to $M^{[k']}(x;\varepsilon)$ $[\overline{\mathbf{h}}^{[k']}(\boldsymbol{\psi};\varepsilon)]$ are those containing also self-energy graphs with height $D>k'$: such contributions are of order at least D in ε, so that they can not contribute to $[M^{[k]}(x;\varepsilon)]^{(k')}$ $[[\overline{\mathbf{h}}^{[k]}(\boldsymbol{\psi};\varepsilon)]^{(k')}]$.

A7.3. Starting from **h**. The case $k'=1$ is trivial. Suppose that, given $k'\le k$, the assertion is true for all $k''<k'$: then we show that it is true also for k'.

Consider a tree $\theta\in\Theta_{k,\boldsymbol{\nu},\gamma}$ and let $\mathbf{Val}(\theta)$ be its value. Denote by $T_1,\dots,T_N$ the maximal self-energy graphs in θ, and by $k_1,\dots,k_N$ the number of nodes that they contain, respectively; the number of nodes external to the self-energy graphs will be $k_0=k-k_1-\dots-k_N$. Call θ_0 the tree obtained from θ by replacing each chain of self-energy graphs together with their external lines with a new line carrying the same momentum of the external lines. The tree θ_0 will have k_0+1 lines: by construction each line ℓ_i of θ_0 corresponds to a chain of p_i self-energy graphs, with $p_i\ge 0$, of orders $K_{i1},\dots,K_{ip_i}$, such that

$$\sum_{i=1}^{k_0}\sum_{j=1}^{p_i}K_{ij}=k-k_0. \tag{A7.2}$$

Then consider θ_0 as a tree $\theta^{\mathcal{R}}\in\Theta^{\mathcal{R}}_{k_0,\boldsymbol{\nu},\gamma}$; let ℓ_i be the line in $\theta^{\mathcal{R}}$ which corresponds to the line with the same name in θ_0. For each line $\ell_i\in\Lambda(\theta^{\mathcal{R}})$, by setting $x_i=\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_i}$, the propagator is of the form

$$\overline{G}_{\ell_i}^{[k-1]}=\overline{G}^{[k-1]}(x_i;\varepsilon),\qquad \overline{G}^{[k-1]}(x_i;\varepsilon)=\Big[x_i^2\,\mathbb{1}-M^{[k-1]}(x_i;\varepsilon)\Big]^{-1}, \tag{A7.3}$$

and it can be expanded in powers of $M^{[k-1]}(x_i;\varepsilon)$ as

$$\overline{G}^{[k-1]}(x_i;\varepsilon)=\frac{1}{x_i^2}\Bigg(\mathbb{1}+M^{[k-1]}(x_i;\varepsilon)\frac{1}{x_i^2}+M^{[k-1]}(x_i;\varepsilon)\frac{1}{x_i^2}M^{[k-1]}(x_i;\varepsilon)\frac{1}{x_i^2}+\dots\Bigg). \tag{A7.4}$$

We can consider the contribution

$$\frac{1}{x_i^2}[M^{[k-1]}(x_i;\varepsilon)]^{(K_{i1})}\frac{1}{x_i^2}\cdots\frac{1}{x_i^2}[M^{[k-1]}(x_i;\varepsilon)]^{(K_{ip_i})}\frac{1}{x^2} \tag{A7.5}$$

to (A7.4). Note that $K_{ij}<k-1$ for all i,j, by construction, so that by the Remark A7.2, (2), we can write

$$[M^{[k-1]}(x_i;\varepsilon)]^{(K_{ip_i})}=[M^{[k]}(x_i;\varepsilon)]^{(K_{ip_i})}=[M^{[K_{ip_i}]}(x_i;\varepsilon)]^{(K_{ip_i})}, \tag{A7.6}$$

for all $j = 1, \dots, p_i$ and for all $i = 1, \dots, k_0$. Hence by the inductive hypothesis, we can deduce that, for all i, j, there is a contribution to $[M^{[k-1]}(x_i; \varepsilon)]^{(K_{ij})} = [M^{[k]}(x_i; \varepsilon)]^{(K_{ij})}$ which corresponds to the considered resonance in θ. As a consequence we can also conclude that there is a term contributing to $[\overline{\mathbf{h}}^{[k]}(\boldsymbol{\psi}; \varepsilon)]_{\boldsymbol{\nu}}^{(k')}$ which is the same as the considered tree value $\mathbf{Val}(\theta)$.

Of course if instead of a tree value we had considered a self-energy value, the same argument should have applied, so that the assertion follows.

A7.4. Starting from $\overline{\mathbf{h}}^{[k]}$. The construction described in Sect. A7.3 can be used in the opposite direction, in order to prove that each term of order k' in ε which is obtained by truncating $\overline{\mathbf{h}}^{[k]}$ to order k corresponds to a term contributing to $\mathbf{h}^{(k')}$.

Proof of the bound (5.1) **from** (5.8)

A8.1. Set-up. Consider the matrix

$$A(x;\varepsilon) = \left(G^{[k]}(x;\varepsilon)\right)^{-1} = x^2\,\mathbb{1} - M^{[k]}(x;\varepsilon) = \Lambda + \varepsilon^2 x\Delta_1 + \varepsilon^2 x^2 \Delta_2, \quad \text{(A8.1)}$$

with

$$\begin{aligned} \Lambda &= \begin{pmatrix} \Lambda_{\alpha\alpha} & 0 \\ 0 & \Lambda_{\beta\beta} \end{pmatrix} = \begin{pmatrix} x^2\,\mathbb{1} & 0 \\ 0 & x^2\,\mathbb{1} - \varepsilon\partial_{\boldsymbol{\beta}}^2 f_{\mathbf{0}}(\boldsymbol{\beta}_0) \end{pmatrix}, \\ \Delta_1 &= \begin{pmatrix} 0 & B(\varepsilon) \\ -B(\varepsilon) & 0 \end{pmatrix}, \\ \Delta_2 &= \begin{pmatrix} -M_{\alpha\alpha}^{[k]}(x;\varepsilon) & -M_{\alpha\beta}^{[k]}(x;\varepsilon) - \varepsilon^2 x B(\varepsilon) \\ -M_{\beta\alpha}^{[k]}(x;\varepsilon) - \varepsilon^2 x B(\varepsilon) & -M_{\beta\beta}^{[k]}(x;\varepsilon) + \varepsilon\partial_{\boldsymbol{\beta}}^2 f_{\mathbf{0}}(\boldsymbol{\beta}_0) \end{pmatrix}, \end{aligned} \quad \text{(A8.2)}$$

where

$$B(\varepsilon) = \left(\varepsilon^2 x\right)^{-1} \sum_{T'} \mathcal{L}\mathcal{V}_{T'}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}), \quad \text{(A8.3)}$$

with the sum running over all self-energy graphs of type 2, so that one has

$$\|\Delta_1\| \le C, \qquad \|\Delta_2\| \le C, \quad \text{(A8.4)}$$

for some positive constant C.

The matrix Λ is a block matrix which induces a natural decomposition $\mathbb{R}^d = \mathbb{R}^r \oplus \mathbb{R}^s$; the eigenvalues of the block $\Lambda_{\alpha\alpha} \equiv \Lambda|\mathbb{R}^r$ are all equal to x^2, while the eigenvalues of the block $\Lambda_{\beta\beta} = \Lambda|\mathbb{R}^s$ are of the form $\lambda_j = x^2 + a_j\varepsilon$, with $a_j > 0$.

Set $\Lambda_1 = \Lambda + \eta\Delta_1$, with $\eta = \varepsilon^2 x$.

Define

$$\begin{aligned} B(x;\varepsilon) &= e^{\eta X} A(x;\varepsilon) e^{-\eta X} = e^{\eta X}\Lambda_1 e^{-\eta X} + \varepsilon^2 x^2 e^{\eta X}\Delta_2 e^{-\eta X} \\ &\equiv B_0(x;\varepsilon) + \varepsilon^2 x^2 e^{\eta X}\Delta_2 e^{-\eta X}; \end{aligned} \quad \text{(A8.5)}$$

of course $B(x;\varepsilon)$ has the same eigenvalues as $A(x;\varepsilon)$.

A8.2. Block-diagonalization. Consider $B_0(x;\varepsilon)$: we shall fix the matrix X in such a way that $B_0(x;\varepsilon)$ is block-diagonal up to order η^2.

So we look for X such that

$$\left(\mathbb{1}+\eta X+O(\eta^2)\right)(\Lambda+\eta\Delta_1)\left(\mathbb{1}-\eta X+O(\eta^2)\right)=\Lambda+\eta J_1+O(\eta^2), \quad \text{(A8.6)}$$

with

$$J_1=\begin{pmatrix} J_{1,\alpha\alpha} & J_{1,\alpha\beta} \\ J_{1,\beta\alpha} & J_{1,\beta\beta}\end{pmatrix}=\begin{pmatrix} J_{1,\alpha\alpha} & 0 \\ 0 & J_{1,\beta\beta}\end{pmatrix}. \quad \text{(A8.7)}$$

By expanding to first order (A8.6) we obtain

$$[X,\Lambda]+\Delta_1=J_1, \quad \text{(A8.8)}$$

while imposing (A8.7) gives

$$\begin{aligned} X_{\alpha\beta}&=-B(\varepsilon)\left(\Lambda_{\beta\beta}-x^2\,\mathbb{1}\right)^{-1},\\ X_{\beta\alpha}&=-\left(\Lambda_{\beta\beta}-x^2\,\mathbb{1}\right)^{-1}B(\varepsilon), \end{aligned} \quad \text{(A8.9)}$$

where

$$\left\|\left(\Lambda_{\beta\beta}-x^2\,\mathbb{1}\right)^{-1}\right\|=\frac{1}{\varepsilon}\left\|\left(\partial^2_{\boldsymbol{\beta}}f_{\mathbf{0}}(\boldsymbol{\beta}_0)\right)^{-1}\right\|\le\frac{C}{\varepsilon}, \quad \text{(A8.10)}$$

for some constant C. Furthermore, by choosing $X_{\alpha\alpha}=0$ and $X_{\beta\beta}=0$, one obtains $J_1\equiv 0$.

Then it follows that one has

$$B(x;\varepsilon)=\Lambda+O(\eta^2X^2)+O(\varepsilon^2x^2)=\Lambda+O(\varepsilon^2x^2), \quad \text{(A8.11)}$$

so that

$$B^{-1}(x;\varepsilon)=\Lambda^{-1}+O(\varepsilon^2x^2), \quad \text{(A8.12)}$$

where the eigenvalues of Λ^{-1} are of the form either $1/x^2$ or $1/(x^2+a_j\varepsilon)$.

Therefore one has

$$\left\|G^{[k]}(x;\varepsilon)\right\|\le\left\|e^{\eta X}\right\|\left\|B^{-1}\right\|\left\|e^{-\eta X}\right\|\le\frac{2}{x^2}, \quad \text{(A8.13)}$$

which proves the bound in (5.1) for real ε.

A8.3. Bounds in the complex plane. The above analysis applies also for complex values of ε. Consider the domain D_0 represented in Fig. 5, with half-opening angle $\varphi<\pi$. For $\varepsilon\in D_0$ one has that the norms of $\overline{G}^{[k]}(x;\varepsilon))^{-1}$ are bounded from below by

$$\frac{x^2}{2}(\pi-\varphi), \quad \text{(A8.14)}$$

when φ is close to π.

Appendix A9. Proof of Lemma 5.1

A9.1. Set-up. Both $M^{[k]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)$ and $M^{[k+1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)$ can be expressed by (4.5): the only difference is that one has to use the propagators $\overline{G}_\ell^{[k]}$ for $M^{[k+1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)$.

This means that there is a correspondence 1-to-1 between the graphs contributing to $M^{[k]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)$ and those contributing to $M^{[k+1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)$, so that we can write

$$\begin{aligned} M^{[k]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon) - M^{[k+1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon) &= \sum_{\text{renormalized } T} \mathcal{V}_T^{[k,k+1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}),\\ \mathcal{V}_T^{[k,k+1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}) &= \varepsilon^{|V(T)|}\Big(\prod_{v\in V(T)} F_v\Big)\Big[\Big(\prod_{\ell\in\Lambda(T)}\overline{G}_\ell^{[k-1]}\Big) - \Big(\prod_{\ell\in\Lambda(T)}\overline{G}_\ell^{[k]}\Big)\Big]. \end{aligned} \tag{A9.1}$$

For each renormalized self-energy T we can write $\mathcal{V}_T^{[k,k+1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})$ as sum of $V = |V(T)|$ terms corresponding to trees whose lines have all the propagators of the form either $\overline{G}_\ell^{[k-1]}$ or $\overline{G}_\ell^{[k]}$, up to one which has a new propagator given by the difference $\overline{G}_\ell^{[k-1]} - \overline{G}_\ell^{[k]}$.

A9.2. Remark. Note that the scales of all lines are uniquely fixed by the momenta, so that both propagators $\overline{G}_\ell^{[k-1]}$ and $\overline{G}_\ell^{[k]}$ admit the same bounds (see (5.6)).

A9.3. Bounds. We can order the lines in $\Lambda(T)$ and construct a set of V subsets $\Lambda_1(T),\ldots,\Lambda_V(T)$ of $\Lambda(T)$, with $|\Lambda_j(T)| = j$, in the following way. Set $\Lambda_1(T) = \emptyset$, $\Lambda_2(T) = \ell_1$, if ℓ_1 is the root line of θ and, inductively for $2 \le j \le V-1$, $\Lambda_{j+1}(T) = \Lambda_j(T)\cup\ell_j$, where the line $\ell_j \in \Lambda(T)\setminus\Lambda_j(T)$ is connected to $\Lambda_j(T)$; of course $\Lambda_V(T) = \Lambda(T)$. Then

$$\begin{aligned} \mathcal{V}_T^{[k,k+1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}) &= \varepsilon^{|V(T)|}\Big(\prod_{v\in V(T)} F_v\Big)\\ &\sum_{j=1}^{V}\Big[\Big(\prod_{\ell\in\Lambda_j(T)}\overline{G}_\ell^{[k-1]}\Big)\Big(\overline{G}_{\ell_j}^{[k-1]} - \overline{G}_{\ell_j}^{[k]}\Big)\Big(\prod_{\ell\in\Lambda(T)\setminus\Lambda_j(T)}\overline{G}_\ell^{[k]}\Big)\Big], \end{aligned} \tag{A9.2}$$

where, by construction, the sets $\Lambda_j(\theta)$ are connected (while of course the sets $\Lambda(\theta)\setminus\Lambda_j(\theta)$ in general are not).

We can write

$$\overline{G}_{\ell_j}^{[k-1]} - \overline{G}_{\ell_j}^{[k]} = \overline{G}_{\ell_j}^{[k-1]}\Big[M^{[k]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_j};\varepsilon) - M^{[k-1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_j};\varepsilon)\Big]\overline{G}_{\ell_j}^{[k]}, \tag{A9.3}$$

so that in (A9.2) we can bound

$$\begin{aligned} &\Big(\prod_{\ell\in\Lambda_j(T)}\Big\|\overline{G}_\ell^{[k-1]}\Big\|\Big)\Big\|\overline{G}_{\ell_j}^{[k-1]}\Big\|\,\Big\|\overline{G}_{\ell_j}^{[k]}\Big\|\Big(\prod_{\ell\in\Lambda(T)\setminus\Lambda_j(T)}\Big\|\overline{G}_\ell^{[k]}\Big\|\Big)\\ &\quad\le\Big[\Big(2^{2+2\tau}C_0^{-1}\Big)^{2k}\prod_{n=-\infty}^{1}2^{-2nN_n(\theta)}\Big]^2, \end{aligned} \tag{A9.4}$$

where the power 2 (with respect to (5.9)) is due to the fact that in the product two propagators correspond to the line ℓ_j.

This means that $\mathcal{V}_T^{[k,k+1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})$ admits the same bound as the square of $\mathcal{V}_T^{[k]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})$ times the supremum (over $\boldsymbol{\nu}$) of the norms

$$\left\| M^{[k]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon) - M^{[k-1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon) \right\| . \tag{A9.5}$$

If we perform the sum over all self-energy graphs in (A9.1) and we use that the first non-trivial terms correspond to graphs with $V=2$ nodes, we obtain, for $k\geq 1$,

$$\left\| M^{[k+1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon) - M^{[k]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon) \right\| \leq C\varepsilon^2 \sum_{\boldsymbol{\nu}\in\mathbb{Z}^r} \left\| M^{[k]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon) - M^{[k-1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon) \right\| , \tag{A9.6}$$

for some constant C. Therefore the lemma follows.

Appendix 10. Proof of Proposition 5.4

A10.1. Set-up. Define

$$\overline{\mathbf{Val}}^{[\infty]}(\theta) = \Big(\prod_{v\in V(\theta)} F_v\Big)\Big(\prod_{v\in L(\theta)} L_v\Big)\Big(\prod_{\ell\in\Lambda(\theta)} \overline{G}_\ell^{[\infty]}\Big), \tag{A10.1}$$

where the propagators $\overline{G}_\ell^{[\infty]} = \overline{G}^{[\infty]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell;\varepsilon)$ are defined in (5.19); we shall denote by $\overline{G}^{[\infty]}$ the operator with kernel $\overline{G}^{[\infty]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)$ in Fourier space.

Then one has

$$\overline{\mathbf{h}}_\gamma^{[\infty]}(\boldsymbol{\psi};\varepsilon) = \sum_{k=1}^{\infty}\sum_{\boldsymbol{\nu}\in\mathbb{Z}^r} \varepsilon^k e^{i\boldsymbol{\nu}\cdot\boldsymbol{\psi}} \sum_{\theta\in\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},\gamma}} \overline{\mathbf{Val}}^{[\infty]}(\theta), \tag{A10.2}$$

which we can represent, in a more compact notation, as

$$\overline{\mathbf{h}}^{[\infty]}(\boldsymbol{\psi};\varepsilon) = \sum_{\theta\in\Theta^{\mathcal{R}}} \overline{\mathbf{Val}}^{\mathcal{R}}(\theta;\boldsymbol{\psi};\varepsilon), \tag{A10.3}$$

where $\Theta^{\mathcal{R}}$ is the set of all renormalized trees, and, for $\theta\in\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},\gamma}\subset\Theta^{\mathcal{R}}$, we have defined

$$\overline{\mathbf{Val}}^{\mathcal{R}}(\theta;\boldsymbol{\psi};\varepsilon) = \varepsilon^k e^{i\boldsymbol{\nu}\cdot\boldsymbol{\psi}}\overline{\mathbf{Val}}^{[\infty]}(\theta). \tag{A10.4}$$

The function $\mathbf{h}(\boldsymbol{\psi};\varepsilon)$ solving the equations of motion (1.6) is formally defined as the solution of the functional equation

$$\mathbf{h}(\boldsymbol{\psi};\varepsilon) = G\partial_{\boldsymbol{\psi}} f\left(\boldsymbol{\psi} + \mathbf{h}(\boldsymbol{\psi};\varepsilon)\right), \tag{A10.5}$$

where $G = (i\omega\cdot\partial)^{-2} = \overline{G}^{[0]}$ is the operator with kernel $G(x) = x^2$.

We have the following result.

A10.2. Lemma. One has $G(x)\left(M^{[\infty]}(x;\varepsilon)+(G^{[\infty]}(x;\varepsilon))^{-1}\right)=\mathbb{1}$.

A10.3. Proof. By definition one has $\overline{G}^{[\infty]}(x;\varepsilon)=(G^{-1}(x)-M^{[\infty]}(x;\varepsilon))^{-1}$, so that $G^{-1}(x)=(\overline{G}^{[\infty]}(x;\varepsilon))^{-1}+M^{[\infty]}(x;\varepsilon)$; then the assertion follows. □

A10.4. Conclusions. The following result shows that the function $\overline{\mathbf{h}}^{[\infty]}(\boldsymbol{\psi};\varepsilon)$ formally solves the equation of motions (1.6); as the analysis of the previous sections shows that the function $\overline{\mathbf{h}}^{[\infty]}(\boldsymbol{\psi};\varepsilon)$ is well defined and it is, order by order, equal to the formal solution envisged in Sect. 3, we have proved the proposition.

A10.5. Lemma. The function $\overline{\mathbf{h}}^{[\infty]}(\boldsymbol{\psi};\varepsilon)$ defined by (A10.3) formally solves (A10.4).

A10.6. Proof. We shall show that (A10.3) solves (A10.5). One has

$$\begin{aligned} G\partial_{\boldsymbol{\psi}} f\left(\boldsymbol{\psi}+\overline{\mathbf{h}}^{[\infty]}(\boldsymbol{\psi};\varepsilon)\right) &= G\sum_{p=0}^{\infty}\frac{1}{p!}\partial_{\boldsymbol{\psi}}^{p+1} f(\boldsymbol{\psi})\left(\overline{\mathbf{h}}^{[\infty]}(\boldsymbol{\psi};\varepsilon)\right)^{p} \\ &= G\sum_{p=0}^{\infty}\frac{1}{p!}\partial_{\boldsymbol{\psi}}^{p+1} f(\boldsymbol{\psi})\sum_{\theta_1\in\Theta^{\mathcal{R}}}\overline{\mathbf{Val}}^{\mathcal{R}}(\theta_1;\boldsymbol{\psi};\varepsilon)\ldots\sum_{\theta_p\in\Theta^{\mathcal{R}}}\overline{\mathbf{Val}}^{\mathcal{R}}(\theta_p;\boldsymbol{\psi};\varepsilon) \\ &= G\left(\overline{G}^{[\infty]}\right)^{-1}\sum_{\theta\in\Theta^{*}_{\mathcal{R}}}\overline{\mathbf{Val}}^{\mathcal{R}}(\theta;\boldsymbol{\psi};\varepsilon), \end{aligned} \tag{A10.7}$$

where $\Theta^{*}_{\mathcal{R}}$ differs from $\Theta^{\mathcal{R}}$ as it contains also trees which can have only one self-energy graph with exiting line ℓ_0, if, as usual, ℓ_0 denotes the root line of θ; the operator $G(\overline{G}^{[\infty]})^{-1}$ takes into account the fact that, by construction, to the root line ℓ_0 an operator G is associated, while in $\overline{\mathbf{Val}}^{\mathcal{R}}(\theta;\boldsymbol{\psi};\varepsilon)$, by definition, a propagator $\overline{G}^{[\infty]}$ is associated.

Then we can write (A10.5), by explicitly separating the trees containing such a self-energy graph from the others,

$$\begin{aligned} & G\partial_{\boldsymbol{\psi}} f\left(\boldsymbol{\psi}+\overline{\mathbf{h}}^{[\infty]}(\boldsymbol{\psi};\varepsilon)\right) \\ &\quad = G\left(\overline{G}^{[\infty]}\right)^{-1}\left(\overline{G}^{[\infty]}M^{[\infty]}\sum_{\theta\in\Theta^{\mathcal{R}}}\overline{\mathbf{Val}}^{\mathcal{R}}(\theta;\boldsymbol{\psi};\varepsilon)+\sum_{\theta\in\Theta^{\mathcal{R}}}\overline{\mathbf{Val}}^{\mathcal{R}}(\theta;\boldsymbol{\psi};\varepsilon)\right) \\ &\quad = G\Big(M^{[\infty]}\overline{\mathbf{h}}^{[\infty]}(\boldsymbol{\psi};\varepsilon)+(\overline{G}^{[\infty]})^{-1}\overline{\mathbf{h}}^{[\infty]}(\boldsymbol{\psi};\varepsilon)\Big) \\ &\quad = G\Big(M^{[\infty]}+(\overline{G}^{[\infty]})^{-1}\Big)\overline{\mathbf{h}}^{[\infty]}(\boldsymbol{\psi};\varepsilon)=\overline{\mathbf{h}}^{[\infty]}(\boldsymbol{\psi};\varepsilon), \end{aligned} \tag{A10.8}$$

where Lemma A10.2 has been used in the last line.

Note that at each step only absolutely converging series have been dealt with; then the assertion is proved.

Acknowledgements. We are indebted to V. Mastropietro for his criticism in the early stages of this work. We thank also H. Eliasson for useful discussions.

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Communicated by A. Kupiainen