Commun. Math. Phys. 226, 205 – 220 (2002)

Communications in
Mathematical Physics

On the Existence of the Absolutely Continuous Component for the Measure Associated with Some Orthogonal Systems

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Received: 26 June 2001 / Accepted: 18 October 2001

Abstract: In this article, we consider two orthogonal systems: Sturm–Liouville operators and Krein systems. For Krein systems, we study the behavior of generalized polynomials at the infinity for spectral parameters in the upper half-plain. That makes it possible to establish the presence of absolutely continuous component of the associated measure. For Sturm–Liouville operator on the half-line with bounded potential q, we prove that essential support of absolutely continuous component of the spectral measure is $[m, \infty)$ if $\limsup_{x\to\infty} q(x) = m$ and $q' \in L^2(R^+)$. That holds for all boundary conditions at zero. This result partially solves one open problem stated recently by S. Molchanov, M. Novitskii, and B. Vainberg. We consider also some other classes of potentials.

1. Introduction

The contents of the paper is as follows. In the first section, we prove the asymptotics of generalized polynomials for Krein systems with coefficients of a special kind. We also establish the presence of absolutely continuous component of the associated measure. In the second section, results obtained for Krein system are applied to Sturm–Liouville operators.

In this introductory section, we will remind some results for the Krein systems. The Krein systems are defined by the equations

$$\begin{cases} \dfrac{dP(x,\lambda)}{dx} = i\lambda P(x,\lambda) - \overline{A(x)}P_*(x,\lambda), & P(0,\lambda)=1, \\ \dfrac{dP_*(x,\lambda)}{dx} = \qquad -A(x)P(x,\lambda), & P_*(0,\lambda)=1, \end{cases} \tag{1}$$

where $A(x)$ is locally summable function on R^+.

In his famous work [6], M. G. Krein showed that $P(x,\lambda)$ have many properties of polynomials orthogonal on the unit circle[1]. For example, there exists a non-decreasing function $\sigma(\lambda)$ (spectral measure) defined on the whole line such that mapping $U_P(f) = \int\limits_0^\infty f(x)P(x,\lambda)dx$ is isometry from $L^2(R^+)$ to $L^2(\sigma, R)$. We will call $P(x,\lambda)$ the generalized polynomials. The following Theorem was stated in [6]. It was proved later in [12].

Theorem 1 ([6,12]). *The following statements are equivalent*

(1) *The integral* $\int\limits_{-\infty}^{\infty} \frac{\ln\sigma'(\lambda)}{1+\lambda^2}d\lambda$ *is finite.*

(2) *At least at some* λ, $\Im\lambda > 0$, *the integral*

$$\int\limits_0^\infty |P(x,\lambda)|^2\,dx \tag{2}$$

converges.

(3) *At least at some* λ, $\Im\lambda > 0$, *the function* $P_*(x,\lambda)$ *is bounded.*

(4) *On any compact set in the open upper half-plane, integral* (2) *converges uniformly. That is equivalent to the existence of uniform limit* $\Pi(\lambda) = \lim_{x_n\to\infty} P_*(x_n,\lambda)$ [19].

Consider some measure μ on R. Let I be the finite union of intervals on R. Let us assume that for any measurable set $\Omega \subset I$ with positive Lebesgue measure ($|\Omega| > 0$), we have $\mu(\Omega) > 0$. That already means that μ has nontrivial absolutely continuous component. Though not any measure with nontrivial absolutely continuous component has this property. If this condition holds, we say that the essential support of absolutely continuous component of measure μ is I ($I \subseteq \text{essupp}\{\mu_{ac}\}$).

Thus, if one of the conditions (2)–(4) holds, then the essential support of $\sigma_{ac}(\lambda)$ is R. It is easy to show [6] that $A(x) \in L^2(R)$ or $A(x) \in L^1(R)$ yields (3). In [3], we proved the criterion for (3) to hold in terms of coefficients $A(x)$ from the so-called Stummel class. In the next section, we will study another class of perturbations that does not shrink the essential support for the absolutely continuous component of the measure. The similar problems for Sturm–Liouville operators were studied in numerous publications (see, for example, [7] and the bibliography there). But first, let us outline some relations between Krein systems and some other orthogonal systems. Consider the Dirac system

$$\begin{cases} \phi' = -\lambda\psi - a_1\phi + a_2\psi, \ \phi(0) = 1, \\ \psi' = \ \ \lambda\phi + a_2\phi + a_1\psi, \quad \psi(0) = 0, \end{cases} \tag{3}$$

where $a_1 = 2\Re A(2x)$, $a_2 = 2\Im A(2x)$. It turns out that $e^{-i\lambda x}P(2x,\lambda) = \phi(x,\lambda) + i\psi(x,\lambda)$. That allows us to say [6] that $\rho_{\text{Dir}}(\lambda) = 2\sigma(\lambda)$, where $\rho_{\text{Dir}}(\lambda)-$ spectral measure of Dirac systems (3).

In case $a_2 = 0$ and a_1- absolutely continuous, we have

$$\begin{aligned} &\psi'' - q\psi + \lambda^2\psi = 0, \ \psi(0) = 0, \qquad \psi'(0) = \lambda, \\ &\phi'' - q_1\phi + \lambda^2\phi = 0, \ \phi(0) = 1, \ \phi'(0) + a_1(0)\phi(0) = 0, \end{aligned} \tag{4}$$

[1] The spectral theory of Krein systems was developed further in [1,3,11–14].

where $q = a_1^2 + a_1',\ q_1 = a_1^2 - a_1'$.

Therefore, the spectral measure $\rho_d(\lambda)$ of Sturm–Liouville operator $l(u) = -u'' + qu$ with Dirichlet boundary condition $u(0) = 0$ is related to $\sigma(\lambda)$ by

$$\rho_d(\lambda) = 4 \int_0^{\sqrt{\lambda}} \xi^2 d\sigma(\xi),\ \lambda > 0. \tag{5}$$

2. Krein Systems with Coefficients of a Special Kind

In this section, we will prove the following theorem.

Theorem 2. *If the bounded coefficient $A(x)$ of the Krein system is real valued,* $0 < l_1 = \liminf_{x\to\infty} A(x) \le \limsup_{x\to\infty} A(x) = l_2$*, and* $A' \in L^2(R^+)$*, then* $(-\infty, -2l_2] \cup [2l_2, \infty) \subset \text{essupp}\{\sigma_{ac}\}$.

But first we will prove one auxiliary lemma. Consider the Krein system

$$\begin{cases} P' = i\lambda P - AP_*, & P(0) = 1, \\ P_*' = \quad -AP, & P_*(0) = 1, \end{cases} \tag{6}$$

where real valued A is bounded, $0 < l_1 = \liminf_{x\to\infty} A \le \limsup_{x\to\infty} A = l_2$, and $A' \in L^2(R^+)$.

Matrix $\aleph = \begin{bmatrix} i\lambda & -A \\ -A & 0 \end{bmatrix}$ has eigenvalues

$$\mu_1 = \frac{i\lambda - \sqrt{4A^2 - \lambda^2}}{2}, \quad \mu_2 = \frac{i\lambda + \sqrt{4A^2 - \lambda^2}}{2}.$$

Consider $\lambda = \tau + ik$, $\tau > 2l_2$ is fixed, $k > 0$. Assume for simplicity A is such that $l_1 < 2A < \tau/2 + l_2$ for all $x \in R^+$. We will explain later why this assumption can always be made without loss of generality. Symbol C is reserved for positive constants whose value might change from one formula to another. It is easy to verify that $\Re\mu_1 > 0$ for all $x > 0$, $k > 0$.

The following Lemma is true.

Lemma 1. *The asymptotics holds at infinity,*

$$P_*(x, \tau + ik) = \exp\left[\int_0^x \mu_1(s,k)ds\right] O(x,k), \tag{7}$$

where $|O(x,k)| < \exp(C/k)$ *for all* $x > 0, k > 0$.

Proof. Many estimates in this proof are very crude, but they will be good enough for our purposes. Let J be a 2×2 matrix that satisfies equation $J' = \aleph J$. We will find J in the form $J = LQ$, where $L = \begin{bmatrix} -\mu_1/A & -\mu_2/A \\ 1 & 1 \end{bmatrix}$ consists of eigenvectors of $\aleph$. We have the following equation for Q: $Q' = L^{-1}\aleph LQ - L^{-1}L'Q$. Multiplying matrixes, we have

$$Q' = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix} Q + VQ, \tag{8}$$

where

$$V=\begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}=\frac{1}{\sqrt{4A^2-\lambda^2}}\begin{bmatrix} -\frac{A'}{A}\mu_1+\mu_1' & -\frac{A'}{A}\mu_2+\mu_2' \\ \frac{A'}{A}\mu_1-\mu_1' & \frac{A'}{A}\mu_2-\mu_2' \end{bmatrix}. \tag{9}$$

Let us notice that the following inequality:

$$\int_0^\infty \|V\|^2 dx < C \tag{10}$$

holds uniformly in k. Introduce the matrix

$$Q_\circ=\begin{bmatrix} \exp\Big[\int_0^x(\mu_1(s,k)+v_{11}(s,k))ds\Big] & 0 \\ 0 & \exp\Big[\int_0^x(\mu_2(s,k)+v_{22}(s,k))ds\Big] \end{bmatrix}. \tag{11}$$

If $Q=Q_\circ S$, then for $S=\begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}$, $S(0)=I$, we have

$$S'=\begin{bmatrix} 0 & v_{12}\exp\Big[\int_0^x \nu(s,k)ds\Big] \\ v_{21}\exp\Big[-\int_0^x \nu(s,k)ds\Big] & 0 \end{bmatrix} S, \tag{12}$$

where $\nu=\sqrt{4A^2-\lambda^2}+\dfrac{i\lambda A'}{A\sqrt{4A^2-\lambda^2}}$.

For s_{11} and s_{21}, we have

$$\begin{cases} s_{11}'= v_{12}\exp\Big[\int_0^x \nu(s,k)ds\Big]s_{21}, & s_{11}(0)=1, \\ s_{21}'=v_{21}\exp\Big[-\int_0^x \nu(s,k)ds\Big]s_{11}, & s_{21}(0)=0. \end{cases} \tag{13}$$

Consider the system of the corresponding integral equations

$$\begin{cases} s_{11}(x)=1+\int_0^x v_{12}(\tau)\exp\Big[\int_0^\tau \nu(s,k)ds\Big]s_{21}(\tau)d\tau, \\ s_{21}(x)= \int_0^x v_{21}(t)\exp\Big[-\int_0^t \nu(s,k)ds\Big]s_{11}(t)dt. \end{cases} \tag{14}$$

Substituting the second formula into the first one and changing the order of integration, we have the following integral equation for s_{11}:

$$s_{11}(x)=1+\int_0^x s_{11}(t)v_{21}(t)\int_t^x v_{12}(s)\exp\Big[\int_t^s \nu(\xi)d\xi\Big]dsdt.$$

This equation yields the integral inequality

$$|s_{11}| \le 1 + \int_0^x |s_{11}(t)||v_{21}(t)| \int_t^\infty |v_{12}(s)|| \exp\Big[\int_t^s \nu(\xi)d\xi\Big]|dsdt. \tag{15}$$

Notice that $\left|\int_t^s \frac{i\lambda A'}{A\sqrt{4A^2-\lambda^2}}d\xi\right| < C$ uniformly in k, t, s.

Because $\Re\mu_1 > 0$, we have the estimate

$$\Re\sqrt{4A^2-\lambda^2} < -k. \tag{16}$$

Consequently, the Gronwall Lemma, being applied to (15), yields

$$|s_{11}| \le \exp\left\{C\int_0^x |v_{21}(t)| \int_t^\infty |v_{12}(s)| \exp\Big(-k[s-t]\Big)dsdt\right\} \le \exp(C/k). \tag{17}$$

At the final step, we used (10) and the Young inequality for convolutions.

Therefore, for s_{21}, we have the estimate

$$|s_{21}| \le \exp(C/k) \int_0^x |v_{21}(s)|| \exp\Big[-\int_0^s \nu(\xi,k)d\xi\Big]|ds$$

which follows from the second equation of system (13) and the estimate on s_{11}. In the same way, the estimates for s_{12}, s_{22} can be obtained. They are as follows:

$$|s_{12}| \le \frac{C}{\sqrt{k}}\exp\left(\frac{C}{k}\right), \tag{18}$$

$$|s_{22}| \le \frac{C}{\sqrt{k}}\exp\left(\frac{C}{k}\right)\int_0^x |v_{21}(s)|| \exp\Big[-\int_0^s \nu(\xi,k)d\xi\Big]|ds. \tag{19}$$

If J is such that $J(0) = I$, then $J = LQ_\circ SL^{-1}(0)$. Therefore, for P_*, we will have

$$P_* = \exp\Big[\int_0^x (\mu_1(t) + v_{11}(t))dt\Big]\Big\{\alpha s_{11} + \beta s_{12} \tag{20}$$

$$+ \exp\left(\int_0^x (\mu_2(s) - \mu_1(s) + v_{22}(s) - v_{11}(s))ds\right)(s_{21}\alpha + s_{22}\beta)\Big\}. \tag{21}$$

Constants α and β are chosen in such a way that the initial condition $P_*(0,k) = 1$ is satisfied. We have

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \frac{1}{\sqrt{4A^2(0)-\lambda^2}}\begin{pmatrix}A(0) & \mu_2(0)\\ -A(0) & -\mu_1(0)\end{pmatrix} \times \begin{bmatrix}1\\ 1\end{bmatrix}, \tag{22}$$

therefore, α and β are bounded uniformly in k. Denote by $O(x,k)$,

$$O = \exp\Big[\int_0^x v_{11}(t)dt\Big]\Big\{\alpha s_{11} + \beta s_{12} \tag{23}$$

$$+ \exp\left(\int_0^x (\mu_2(s) - \mu_1(s) + v_{22}(s) - v_{11}(s))ds\right)(s_{21}\alpha + s_{22}\beta)\Big\}. \tag{24}$$

Notice that $\int_0^x v_{11}(t)dt$ and $\int_0^x v_{22}(t)dt$ are bounded uniformly in x and k. Due to the estimates on s_{21} and s_{22}, we have

$$\left|\exp\left(\int_0^x (\mu_2(s) - \mu_1(s) + v_{22}(s) - v_{11}(s))ds\right)(s_{21}\alpha + s_{22}\beta)\right| \tag{25}$$

$$\le \exp(C/k)\left|\exp\Big[\int_0^x \nu(\eta,k)d\eta\Big]\right| \tag{26}$$

$$\times \int_0^x |v_{21}(s)|\left|\exp\Big[-\int_0^s \nu(\xi,k)d\xi\Big]\right| ds \tag{27}$$

$$\le \exp(C/k)\int_0^x |v_{21}(s)|\left|\exp\Big[\int_s^x \nu(\xi,k)d\xi\Big]\right| ds \le \exp(C/k)/\sqrt{k}. \tag{28}$$

To get the last estimate, we used the Cauchy inequality. Finally, bounds on s_{11}, s_{12} lead to $|O(x,k)| < \exp(C/k)$. That finishes the proof of the lemma. □

Remark. More accurate estimates on α, β, s_{11} allow us to write inequality

$$|O(x,z) - 1| < 1/2, \tag{29}$$

where $x \in R^+$, $\tau \le \Re z \le \tau + 1$, $\Im z > k_0$, k_0– some positive constant.

Indeed, it is easy to verify that $\alpha \to 1$ and $|\beta| < \dfrac{C}{\Im z}$ if $\Im z \to +\infty$, $\tau \le \Re z \le \tau + 1$. From (13), by the Cauchy inequality we have

$$|s_{11} - 1| \le \exp(C/k)/k \tag{30}$$

uniformly in $x \in R^+$, $\tau \le \Re z \le \tau + 1$, where $k = \Im z > k_0$.

One can verify that $\int_0^x v_{11}(t)dt \to 0$ uniformly in x if $\Im z \to \infty$.

Therefore, from (23) and (26)–(28), we infer (29).

Let us prove Theorem 2 now.

Proof of Theorem 2. Fix any $\tau > 2l_2$. We will show that $[\tau, \infty) \subset \text{essupp}\{\sigma_{ac}\}$. Because τ is chosen arbitrarily larger than $2l_2$ and σ is odd (since A is real), this inclusion is sufficient for Theorem 2 to be true. Once τ is fixed, we can assume that $l_1 < 2A < \tau/2 + l_2$ for all $x \in R^+$. Due to the standard trace-class perturbation argument applied to the Dirac system (3) [9], we can always make this assumption. Indeed, since $l_1 = \liminf_{x\to\infty} A \le \limsup_{x\to\infty} A = l_2$ as $x \to \infty$, it suffices to multiply A by some smooth function which is equal to 0 on $[0, M_A]$ and 1 on $[M_A + 1, \infty)$. The absolutely continuous part of ρ_{Dir} will not change because of the trace-class argument. On the other hand, if M_A is sufficiently large, we will satisfy the imposed condition.

Consider the Krein system with coefficients $A_{(n)}(x) = \begin{cases} A(x), & x < n, \\ 0, & x \ge n. \end{cases}$ Denote the corresponding measure by σ_n. We have the formula ((3.14) from [12]):

$$\sqrt{2\pi}\, P_*(\infty, z) = \sqrt{2\pi}\, P_*(n, z) = e^{i\alpha_n} \exp\left[\int_{-\infty}^{\infty} \frac{1}{2\pi i} \frac{(1 + tz)\ln \sigma_n'(t)}{(z - t)(1 + t^2)} dt\right], \Im z > 0, \tag{31}$$

where α_n are some real constants. Because $A(x)$ is real, functions σ_n are odd. Therefore, if we take $z = i$, the left-hand side of (31), together with exponent from the right-hand side, are real valued. Thus, α_n can be chosen equal to zero.

The asymptotics of $P_*(n, z)$ as $|z| \to \infty$, $\Im z > 0$ is $P_*(n, \infty) = 1$ [11]. Therefore, we can rewrite this formula as follows:

$$-2\pi i \ln P_*(n, z) = \int_{-\infty}^{\infty} \frac{(1 + tz)\ln(2\pi\sigma_n'(t))}{(t - z)(1 + t^2)} dt \tag{32}$$

if $\Im z > 0$. Here we used the identity

$$\sqrt{2\pi} = \exp\left[\int_{-\infty}^{\infty} -\frac{1}{2\pi i} \frac{(1 + tz)\ln(2\pi)}{(z - t)(1 + t^2)} dt\right], \ \Im z > 0.$$

Also we used the fact that $2\pi\sigma_n'(t) \to 1$ if $|t| \to \infty$ [11].

The right-hand side $RHS(z)$ of (32) satisfies the condition $\overline{RHS(z)} = RHS(\bar{z})$. Let us define the left-hand side for $\Im z < 0$ according to this rule. So we have some analytic function $LHS(z)$ defined in the region: $\Im z \neq 0$. Recall the asymptotic formula for $P_*(x, z)$, ($\Im z > 0, \Re z = \tau > 2l_2$):

$$P_*(x, z) = \exp\left[\int_0^x \mu_1(s, z) ds\right] O(x, z). \tag{33}$$

Then,

$$LHS(z) = -2\pi i \int\limits_0^n \mu_1(s,z)ds - 2\pi i \ln O(n,z)$$

for $\Im z > 0$.

It is easy to see that the continuation according to the chosen rule for the function $-2\pi i \int\limits_0^n \mu_1(s,z)ds$ is exactly the Schwarz analytic continuation. It follows from the fact that the value of this function on the half-line $\Im z = 0, \Re z \geq \tau$ is real.

Consider $z = \tau + ik$. Let $\ln O(n,z) = r_1(n,k) + ir_2(n,k)$. From the definition of $LHS(z)$ in $\Im z < 0$, we have $\overline{i(r_1(n,k) + ir_2(n,k))} = -ir_1(n,k) - r_2(n,k) = ir_1(n,-k) - r_2(n,-k)$. Thus, $r_1(n,k)$ is odd and $r_2(n,k)$ is even. For fixed n, they are well defined functions for all $k \neq 0$ with finite right and left limits at $k = 0$. Notice also that for $k > 0$, $r_1(n,k) = \ln|O(n,\tau+ik)|$ and $r_2(n,k) = Arg O(n,\tau+ik)$. For r_1, r_2, we have $r_1(n,\infty) = 0$, $r_2(n,\infty) = 0$ because $P_*(n,\infty) = 1$ and $\mu_1(s,\infty) = 0$.

Then, integrate both sides of (32), together with some auxiliary function $\frac{(z-\tau)^3}{((z-\tau)^2+m^2)^4}$, along the contour Con. We choose Con as the contour that consists of the complex numbers $\tau + ik$ ($|k| \leq m-1$, $|k| \geq m+1$) and two right semicircles with radii 1 centered at $\tau \mp im$. The direction of integration is upward.

For the right-hand side, we have

$$\begin{aligned}
&\int\limits_{\text{Con}} \frac{(z-\tau)^3}{((z-\tau)^2+m^2)^4} \int\limits_{-\infty}^{\infty} \frac{(1+tz)\ln(2\pi\sigma_n'(t))}{(1+t^2)(t-z)} dt dz \\
&= \int\limits_{-\infty}^{\infty} \frac{\ln(2\pi\sigma_n'(t))}{1+t^2} \int\limits_{\text{Con}} \frac{(1+tz)(z-\tau)^3}{((z-\tau)^2+m^2)^4(t-z)} dz dt \\
&= 2\pi i \int\limits_{\tau}^{\infty} \frac{(t-\tau)^3}{((t-\tau)^2+m^2)^4} \ln(2\pi\sigma_n'(t)) dt.
\end{aligned}$$

Here we changed the order of integration by the Fubini Theorem and then used the Cauchy formula.

Integrating the $LHS(z)$ with the same function, we have

$$\begin{aligned}
&-2\pi i \int\limits_{\text{Con}} \int\limits_0^n \mu_1(s,z)ds \frac{(z-\tau)^3}{((z-\tau)^2+m^2)^4} dz - 2\pi i \int\limits_{\text{Con}} \frac{(z-\tau)^3}{((z-\tau)^2+m^2)^4} \ln O(n,z) dz \\
&= 0 - 2\pi i \int\limits_{\text{Con}} \frac{(z-\tau)^3}{((z-\tau)^2+m^2)^4} \ln O(n,z) dz
\end{aligned}$$

because $\int\limits_0^n \mu_1(s,z)ds$ is analytic and bounded in $\Re z \geq \tau$. Thus, we have the equality

$$-\int\limits_{\text{Con}} \frac{(z-\tau)^3}{((z-\tau)^2+m^2)^4} \ln O(n,z) dz = \int\limits_{\tau}^{\infty} \frac{(t-\tau)^3}{((t-\tau)^2+m^2)^4} \ln(2\pi\sigma_n'(t)) dt. \quad (34)$$

We have the following inequality [12]

$$\int_{-\infty}^{\infty} \frac{d\sigma_n(t)}{1+t^2} < C \tag{35}$$

uniformly in n. Define $\ln^+ h = \ln h$ for $h > 1$ and zero otherwise, $\ln^- h = -\ln h$ if $0 < h < 1$ and zero otherwise. The trivial inequality $\ln^+ h < h$, together with (35), guarantee that the right-hand side of (34) is bounded above uniformly in n. Our goal is to show that it is bounded from below as well.

Thus, we want to show that

$$\int_{\text{Con}} \frac{(z-\tau)^3}{((z-\tau)^2+m^2)^4} \ln O(n,z)dz < C$$

uniformly in n. The left-hand side is equal to $I_1 + I_2$, where

$$I_1 = \int_{|k|<m-1,|k|>m+1} i\frac{(ik)^3}{(m^2-k^2)^4}(r_1(n,k)+ir_2(n,k))dk$$
$$= 2\int_{0<k<m-1,k>m+1} \frac{k^3}{(m^2-k^2)^4} r_1(n,k)dk \le C$$

uniformly in n due to the estimates on $\ln|O(n,k)|$. We also used the fact that $r_2(n,k)$ is even. It is crucial since we do not have control over the argument of $O(n,k)$. We only know the upper bound on $\ln|O(n,k)|$. The term I_2 corresponds to integration along the semicircles. Choose m sufficiently large. Then, for I_2, we have the estimate

$$|I_2| \le C$$

uniformly in n because of the estimate (29).

To finish the proof, we will use one argument from [2]. Consider any compact set $Co \in (\tau,\infty)$ of positive Lebesgue measure. Then, as it follows from (35) and boundedness of right-hand side in (34), $\int_{Co} \ln^- \sigma_n'(t)dt$ is bounded uniformly in n. Jensen's inequality yields

$$\ln^- \left\{ \frac{1}{|Co|} \int_{Co} \sigma_n'(t)dt \right\} \le \frac{1}{|Co|} \int_{Co} \ln^- \sigma_n'(t)dt.$$

Therefore, $\sigma_n(Co)$ is greater than some positive constant $d(Co)$ for all n. The Weyl–Titchmarsh function of the system with coefficients $A_{(n)}$ converges to the Weyl–Titchmarsh function of a system with coefficient A [12]. This convergence is uniform on any compact set of upper half-plane. By the Stone–Weierstrass Theorem, we have weak convergence of σ_n to σ. Consequently, $\sigma(Co) \ge \limsup_{n\to\infty} \sigma_n(Co) > 0$ for each compact Co with $|Co| > 0$. □

In the next Theorem, we consider a different class of coefficients.

Theorem 3. *If $A(x) = -l - v(x)$, where $l \in R$, $v(x)$ is real valued, and $v(x) \in L^2(R^+)$, then for the corresponding Krein system* $\operatorname{essupp}\{\sigma_{ac}\} = (-\infty, -2|l|] \cup [2|l|, \infty)$.

For $l = 0$, the result follows from [6]. Therefore, we will consider the case $l \neq 0$. To make calculations more simple, let $l = -1/2$. The idea of the proof is the same as that of Theorem 2. We need the asymptotics on $P_*(x, \lambda)$. Consider the Krein system written in the following way:

$$\begin{cases} P' = i\lambda P + P_*/2 + vP_*, & P(0) = 1, \\ P_*' = \qquad P/2 + vP, & P_*(0) = 1. \end{cases} \tag{36}$$

The matrix $\begin{bmatrix} i\lambda & 1/2 \\ 1/2 & 0 \end{bmatrix}$ has eigenvalues $\mu_\pm = (i\lambda \pm \sqrt{1-\lambda^2})/2$. Consider $\lambda = \tau + ik,\ \tau > 1$ is fixed, $k > 0$. One can verify that $\Re\mu_- > 0$ for all $k > 0$.

Lemma 2. *The following asymptotics holds at the infinity*

$$P_*(x, \tau + ik) = \exp\left[\mu_-(\lambda)x - \frac{1}{\sqrt{1-\lambda^2}}\int_0^x v(s)ds\right] O(x, k), \tag{37}$$

where $|O(x, k)| < \exp(C/k)$ for all $x > 0, k > 0$.

Proof. We will give only the sketch of the proof, because it repeats essentially the proof of Lemma 1. Consider the following matrix differential system:

$$X' = \begin{bmatrix} i\lambda & 1/2 \\ 1/2 & 0 \end{bmatrix} X + v(x)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} X, \quad X(0) = E, \tag{38}$$

where E is 2×2 identity matrix.

$$X_0 = \begin{bmatrix} \exp(\mu_+(\lambda)x) & \exp(\mu_-(\lambda)x) \\ (-i\lambda + \sqrt{1-\lambda^2})\exp(\mu_+(\lambda)x) & (-i\lambda - \sqrt{1-\lambda^2})\exp(\mu_-(\lambda)x) \end{bmatrix}$$

is the solution of the equation $X_0' = \begin{bmatrix} i\lambda & 1/2 \\ 1/2 & 0 \end{bmatrix} X_0$. Introduce the matrix $V = X_0^{-1}(0) = \frac{1}{2\sqrt{1-\lambda^2}}\begin{bmatrix} i\lambda + \sqrt{1-\lambda^2} & 1 \\ -i\lambda + \sqrt{1-\lambda^2} & -1 \end{bmatrix}$. We will find X in the form $X = X_0 Y$. Then we have an equation for Y: $Y' = v(x)X_0^{-1}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} X_0 Y$, and $Y(0) = V$. The multiplication of the matrixes yields

$$Y' = \frac{v(x)}{\sqrt{1-\lambda^2}} \times \begin{pmatrix} 1 & (\lambda^2 - i\lambda\sqrt{1-\lambda^2})\exp(-\sqrt{1-\lambda^2}x) \\ -(\lambda^2 + i\lambda\sqrt{1-\lambda^2})\exp(\sqrt{1-\lambda^2}x) & -1 \end{pmatrix} Y. \tag{39}$$

Let $Y = \begin{bmatrix} \exp\left(\frac{1}{\sqrt{1-\lambda^2}}\int\limits_0^x v(s)ds\right) & 0 \\ 0 & \exp\left(-\frac{1}{\sqrt{1-\lambda^2}}\int\limits_0^x v(s)ds\right) \end{bmatrix} T.$

For T, we have the equation

$$T' = \frac{v(x)}{\sqrt{1-\lambda^2}} \times \begin{pmatrix} 0 & (\lambda^2 - i\lambda\sqrt{1-\lambda^2})\exp(-\phi(x,\lambda)) \\ -(\lambda^2 + i\lambda\sqrt{1-\lambda^2})\exp(\phi(x,\lambda)) & 0 \end{pmatrix} T, \quad (40)$$

where $\phi(x,\lambda) = \sqrt{1-\lambda^2}x + \frac{2}{\sqrt{1-\lambda^2}}\int\limits_0^x v(s)ds$. Systems (40) and (12) have the same structure. Therefore, we can use arguments that were applied to the system (12) from Lemma 1. □

Remark. Let us apply the Taylor formula for the square root to μ_1 from (7). We see that

$$\int\limits_0^x \mu_1(s,k)ds = \frac{(i\lambda - \sqrt{1-\lambda^2})x}{2} - \frac{1}{\sqrt{1-\lambda^2}}\int\limits_0^x v(s)ds + O(1).$$

Taking the exponent, we have exactly the main factor in the asymptotics from Lemma 2.

The proof of Theorem 3 is straightforward now.

Proof of the Theorem 3. It suffices to use Lemma 2 and the arguments from the proof of Theorem 2 to establish the inclusion $(-\infty, -2|l|] \cup [2|l|, \infty) \subseteq \operatorname{essupp}\{\sigma_{ac}\}$. The converse inclusion follows from the fact that the essential spectrum of the Dirac system (3) is $(-\infty, -2|l|] \cup [2|l|, \infty)$ due to Weyl Theorem [8]. □

Remark. It is likely that condition $v \in L^2(R^+)$ can be relaxed as it was done in [3].

3. Sturm–Liouville Operators

The main goal of this section is to prove the following theorem. The idea of the proof will follow [4] more or less. But it will require more technical details.

Theorem 4. *Consider the Sturm–Liouville operator on the half-line given by the differential expression*

$$l(u) = -u'' + qu, \quad (41)$$

where q is bounded, $\limsup_{x\to\infty} q = m$, *$q(x)$ is the absolutely continuous, and $q' \in L^2(R^+)$. Then, for any boundary condition at zero, the essential support of absolutely continuous component of the corresponding spectral measure is* $[m, \infty)$.

Problems of this kind for Sturm–Liouville operators were treated in many publications. We mention some of the results obtained. P. Deift and R. Killip proved

Theorem 5 ([2]). *If in (41), $q(x) \in L^2(R^+)$, then the essential support of the absolutely continuous component of the spectral measure is R^+ for all boundary conditions.*

In the paper of S. Molchanov, M. Novitskii, and B. Vainberg [7], some other results were obtained. For example, it was proved that $q \in L^3(R^+)$ and $q' \in L^2(R^+)$ lead to the same property of the spectral measure. Consider the particular case of Theorem 4, where $q \to 0$ as $x \to \infty$, $q' \in L^2(R^+)$. We see that condition $q \in L^3(R^+)$ used in [7] can be relaxed to $q \to 0$ as $x \to \infty$. In [7], the authors also pose the following open problem: Is it true that for all boundary conditions at zero, $\text{essupp}\{\rho_{ac}\} = [m, \infty)$ provided that q is bounded, $\limsup_{x\to\infty} q = m$, and for some integer p, $q^{(p)} \in L^2(R^+)$? Here $q^{(p)}$ means the derivative of order p.

Theorem 4 solves this problem for $p = 1$. Because any bounded function admits a supremum, we actually characterize the absolutely continuous component of the spectrum for bounded potentials with a square summable first derivative. We think that the method developed here can be used to deal with any p. No doubt, it will require more calculations to establish the asymptotics.

Let us prove the following lemma first.

Lemma 3. *If bounded q is such that $\limsup_{x\to\infty} q = m$ and $q' \in L^2(R^+)$, then for some large $\gamma > 0$, there exists bounded $v(x)$ so that $\limsup_{x\to\infty} v = \sqrt{\gamma^2 + m} - \gamma$, $v' \in L^2(R^+)$, and $q = v^2 + 2\gamma v + v'$.*

Proof. Consider the corresponding integral equation

$$v(x) = e^{-2\gamma x} \int_0^x e^{2\gamma s}(q(s) - v^2(s))ds. \tag{42}$$

If $v(x)$ is a solution of this integral equation, then it satisfies the differential equation as well. Write (42) as follows $v = OP_\gamma v$, where OP_γ is the corresponding formal nonlinear operator. Consider the complete metric space $M = \{f \in C(R^+), \|f\|_\infty \le 1\}$ with $\|.\|_\infty$ metric. By $C(R^+)$ we denote continuous functions on R^+. If γ is large enough, the operator OP_γ acts from M to M. Naturally, the choice of γ depends on $\|q\|_\infty$. For large γ, OP_γ has a contracting property. Indeed,

$$|OP_\gamma g_1 - OP_\gamma g_2| \le e^{-2\gamma x} \int_0^x e^{2\gamma s}|g_1 - g_2||g_1 + g_2|ds \le \frac{1}{2}\|g_1 - g_2\|_\infty,$$

for γ large enough. Therefore, there is the unique fixed point from M. Let us call this function v.

Differentiate (42). After integration by parts, we will have

$$v' = e^{-2\gamma x}(q(0) - v^2(0)) + e^{-2\gamma x} \int_0^x e^{2\gamma s} q'(s)ds - 2e^{-2\gamma x} \int_0^x e^{2\gamma s} v(s)v'(s)ds.$$

Consider integral equation

$$b = e^{-2\gamma x}(q(0) - v^2(0)) + e^{-2\gamma x} \int_0^x e^{2\gamma s} q'(s)ds - 2e^{-2\gamma x} \int_0^x e^{2\gamma s} v(s)b(s)ds. \tag{43}$$

It has the unique solution from $C(R^+)$. Therefore, its solution is v'. The uniqueness follows from the convergence of the corresponding iterated series. Write (43) as $b = OP1_\gamma b$,

$$e^{-2\gamma x}\int_0^x e^{2\gamma s}q'(s)ds = \int_0^\infty r_\gamma(x-s)q'(s)ds,$$

where $r_\gamma(s) = e^{-2\gamma s}$ for positive s, and zero otherwise. Because $q' \in L^2(R^+)$, the Young inequality for convolutions yields that $OP1_\gamma$ acts from the ball $\|.\|_2 \le 1$ into itself, provided that γ is large enough. For large γ, it has a contracting property. Therefore, there is the unique fixed point b in ball $\|.\|_2 \le 1$ which is equal to v'. We also used the fact that $\|v\|_\infty \le 1$. It is clear that $v'(x) \to 0$ as $x \to \infty$. Therefore, solving equation $v^2 + 2\gamma v + v' - q = 0$ gives the formula

$$v = -\gamma \pm \sqrt{\gamma^2 - v' + q}. \tag{44}$$

We know that q is bounded and $\|v\|_\infty \le 1$ for large $\gamma > 0$. Consequently, to obtain the asymptotics of v at infinity, one should take sign $+$ in (44). Therefore, we have $\limsup_{x\to\infty} v(x) = -\gamma + \sqrt{\gamma^2 + m}$. □

Now the proof of Theorem 4 is straightforward.

Proof of Theorem 4. Notice that the essential support of the absolutely continuous component does not depend on the boundary condition at zero. It follows, for example, from the subordinacy theory [5,15]. Consider (41) with Dirichlet boundary condition at zero. Add to the potential q some constant γ^2. Denote the corresponding self-adjoint operator by H_γ. Obviously, the spectral measure of H_γ is the shift of the spectral measure of the initial operator H_0. On the other hand, for large γ, we can solve equation $q + \gamma^2 = (v+\gamma)^2 + (v+\gamma)'$. That is due to Lemma 3 of this section. Now we can apply results of the first section for Krein systems with coefficient $A(x) = \gamma/2 + v(x/2)/2$. We can use Theorem 2 because for large γ, $0 < \liminf_{x\to\infty} A(x) \le \limsup_{x\to\infty} A(x) = \frac{\sqrt{\gamma^2+m}}{2}$. Let us use formula (5) from the Introduction. Thus, we see that $[\gamma^2 + m, \infty) \subset \text{essupp}\{\rho_{ac}(H_\gamma)\}$ which is equivalent to $[m, \infty) \subset \text{essupp}\{\rho_{ac}(H_0)\}$.

To prove that it is actually an equality, one should use some result from [17] which goes back to [16]. Before formulating this result, let us introduce some notations used in [17]. Consider V – real valued and locally integrable on $(0, \infty)$. Assume that $-\frac{d^2}{dx^2} + V$ is limit point at $+\infty$ and $\int_0^\varepsilon |V(s)|ds < \infty$. Let $T = -\frac{d^2}{dx^2} + V$ in $L^2(R^+)$ with any fixed boundary condition at zero. Consider also $V_\circ$ – integrable and bounded from below function. Denote by $T_\circ$ operator generated by $-\frac{d^2}{dx^2} + V_\circ$ in $L^2(R)$. Then, the following theorem holds.

Theorem 6 ([17], Theorem 2.1). *Assume that* $-\infty \le \alpha < \beta \le +\infty$ *and* $(\alpha,\beta) \cap Spectrum(T_\circ) = \emptyset$ *and that intervals* $I_n \subset [0,\infty)$ *exist such that*

$$|I_n| \to \infty \textit{ as } n \to \infty \textit{ and } \sup_{x\in\cup_n I_n} |V(x) - V_\circ(x)| \le \delta.$$

Then, $\text{essupp}\{\rho_{ac}(T)\} \cap (\alpha+\delta, \beta-\delta) = \emptyset$.

Now, consider $V_\circ = m$, $V = q$. Because $\limsup_{x\to\infty} q = m$ and $q' \in L^2(R^+)$, one can easily show that for any $\delta > 0$, $\operatorname{essupp}\{\rho_{ac}(H_0)\} \cap (-\infty, m-\delta) = \emptyset$. Indeed, it suffices to choose $x_n \to \infty$ such that $|q(x_n) - m| < \delta/2$ and I_n as some neighboring intervals. We specify them as follows. Notice that $q(x) - q(x_n) = \int_{x_n}^{x} q'(s)ds$. Therefore,

$$|q(x) - q(x_n)| \le \left(\int_{x_n}^{\infty} \left[q'(s)\right]^2 ds \right)^{1/2} \cdot \sqrt{x - x_n} = \varepsilon_n \sqrt{x - x_n},$$

where $\varepsilon_n \to 0$ as $n \to \infty$. Choose as I_n intervals $[x_n, x_n + \varepsilon_n^{-1}]$ for n so large that $\varepsilon_n < \frac{\delta^2}{4}$. Evidently, $|I_n| \to \infty$ and for each $x \in I_n$, we have $|q(x_n) - m| < \delta$. Therefore, the Stolz Theorem yields $\operatorname{essupp}\{\rho_{ac}(H_0)\} = [m, \infty)$.

Remark. There are many functions that satisfy conditions of Theorem 4. In particular, these are some slowly oscillating functions. One can think about $\cos(x^\mu)$ for $0 < \mu < 1/2$. In that case $\limsup_{x\to\infty} q(x) = 1$, $q' \in L^2(R^+)$. Consequently, $\operatorname{essupp}\{\rho_{ac}\} = [1, \infty)$ for all boundary conditions. In paper [17], the author used the theory of subordinate solutions to show that for any $0 < \mu < 1$, the spectrum is actually purely absolutely continuous on $[1, \infty)$. The interval $[-1, 1]$ is filled by singular spectrum.

Remark. The condition $q' \in L^2(R^+)$ from Theorem 4 is optimal [7]. That means that the statement can be false if $q' \in L^p(R^+),\ p > 2$. The famous von Neumann-Wigner potential [18, 10] satisfies the conditions of Theorem 4. That means that under the conditions of Theorem 4, the singular component of the spectrum can appear on the interval $[m, \infty)$ which supports the absolutely continuous part.

Remark. It should be noted that the class of Sturm–Liouville operators with decaying potentials match very well the class of Krein systems with coefficients that tend to nonzero constant. For example, Sturm–Liouville operators of this kind admit negative eigenvalues with zero as the only possible point of accumulation. For the Krein systems, these eigenvalues of the discrete spectrum might accumulate only near the edges of the corresponding symmetric interval centered at zero. This relation can be explained as follows. Consider the potential from $L^p(R^+)$, $(1 \le p < \infty)$ space and Dirichlet boundary condition for example. Then, for γ large enough, Eq. (42) can be solved so that $v \in L^\infty(R^+) \cap L^p(R^+)$. Consequently, formula (5) from the Introduction allows us to study the Krein system with coefficient $A = \gamma/2 + v(x/2)/2$ instead of initial Sturm–Liouville operator.

Theorem 3 from the first section lets us prove the theorem of P. Deift and R. Killip for more general class of potentials.

Theorem 7. *Consider Sturm–Liouville operator (41) with potential $q(x)$. If q is uniformly square summable functions from $H^{-1}(R^+)$ space, i.e.*

$$\int_x^{x+1} q^2(s)ds < C$$

uniformly in $x \in R^+$, and

$$e^{-x}\int_0^x e^s q(s)ds \in L^2(R^+), \tag{45}$$

then, for any boundary conditions at zero, the essential support of the absolutely continuous component of the corresponding spectral measure is a positive half-line.

This result solves one open problem stated in [4]. The proof is essentially the same as the proof of Theorem 4.

Lemma 4. *Under the conditions imposed on $q(x)$, we can find the absolutely continuous function $a(x)$ which is square summable on the positive half-line and satisfies the equation $q(x) = a'(x) + a(x)$.*

Proof. Consider the function $a(x) = e^{-x}\int_0^x e^s q(s)ds$. Obviously, $a(x) \in AC(R^+)$ and $q = a' + a$. From (45), we have $a(x) \in L^2(R^+)$. □

Proof of Theorem 7. For given $q(x)$, find the corresponding function $a(x)$ and consider the Krein system (1) with $A(x) = a(x/2)/2+1/4$. The absolutely continuous component of the corresponding measure σ has essential support $(-\infty, -1/2] \cup [1/2, \infty)$. That is due to Lemma 4 and Theorem 3. Consequently, for the Sturm–Liouville operator on the half-line with potential $q^* = a^2 + a + 1/4 + a'$ and Dirichlet boundary condition at zero, the essential support of the absolutely continuous component of the spectral measure is $[1/4, \infty)$. The essential support of the absolutely continuous component does not depend on the boundary condition. Therefore, this property holds for all boundary conditions at zero. Because $a \in L^2(R^+)$ and $q = a' + a$, the standard trace-class perturbation argument yields that for operator with potential $q + 1/4$, the essential support of the absolutely continuous component is again the interval $[1/4, \infty)$. It suffices to subtract $1/4$ from the operator to complete the proof of the theorem. □

Remark. Let $q(x)$ be zero for $x < 0$. Then condition (45) means that $q(x)$ is from $H^{-1}(R)$ class.

Indeed, since $q(x) = 0$ for $x < 0$, we can write

$$e^{-x}\int_0^x e^s q(s)ds = \int_{-\infty}^{+\infty} q(s)r(x-s)ds,$$

where $r(t) = e^{-t}$ for $t > 0$ and $r(t) = 0$ otherwise. Taking the Fourier transform, we have $\widehat{q(\omega)}\widehat{r(\omega)} = \frac{1}{\sqrt{2\pi}(1-i\omega)}\widehat{q(\omega)} \in L^2(R)$. That means $q(x) \in H^{-1}(R)$.

Acknowledgements. The author thanks Professor J. Goodman for useful discussion. Most of this work was done during a stay at the Courant Institute of Mathematical Sciences.

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Communicated by B. Simon