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Mean-Field Criticality for Percolation on Planar Non-Amenable Graphs

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Abstract: The critical exponents β, γ, δ and Δ are proved to exist and to take their mean-field values for independent percolation on the following classes of infinite, locally finite, connected transitive graphs: (1) Non-amenable planar with one end. (2) Unimodular with infinitely many ends.

1. Introduction

1.1. Results. A great deal of attention has been given recently to the study of statistical mechanics and related systems on various classes of graphs. The reader is invited to consult [Lyo] and [Sch] for introductions to the subject, and for references to the literature. This paper can be seen as a continuation of [Sch], and we refer the reader to that paper for background and motivation. Basic terminology, definitions and notation will be reviewed later in this introduction.

We will consider independent bond percolation on an infinite, locally finite, connected transitive graph $G = (V, E)$. Results similar to the ones presented here hold also for independent site percolation, with similar proofs. The same remark can be made about the extension from transitive to quasi-transitive graphs. Conjecture 1.2 in [Sch], combined with Conjecture 6 in [BS1] (reproduced as Conjectures 1.1 in [Sch]), state that if the graph is non-amenable, critical exponents exist and take their mean-field values. In [Sch], Theorem 1.1, this was proved for various critical exponents in the case in which the graph is unimodular and the edge-isoperimetric constant (Cheeger constant) is a sufficiently large fraction of the degree of the graph (previously, a special case had been handled in [Wu]). Here we prove similar results in two cases.

Theorem 1.1. *For independent bond percolation on the following classes of infinite, locally finite, connected transitive graphs the critical exponents β, γ, δ and Δ exist and take their mean field values:*

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(i) *Graphs which are planar non-amenable and have one end.*
(ii) *Graphs which are unimodular and have infinitely many ends.*

At the end of the next subsection, after introducing the necessary notation, we review the meaning of the exponents addressed in this theorem and recall what their mean-field values are.

It is worth pointing out that while Theorem 1.1 in [Sch] was proved by verifying the triangle condition of [AN] (or, more precisely, the open triangle condition of [BA]), in the present paper we will follow a somewhat different route, based nevertheless also on the work of [AN, BA], and [Ngu]. We do not know whether the triangle condition holds in the cases treated here.

The fact that there is no percolation at the critical point, which is a feature of mean-field criticality, is known to hold for independent percolation on any infinite, locally finite, connected transitive unimodular graph. This was proved in [BLPS1], and a simpler proof was provided in [BLPS2]. Unfortunately, the methods from these papers do not provide information on critical exponents.

Part (i) of Theorem 1.1 is the main contribution in this paper. This is one more instance in which the extra techniques resulting from planarity allow one to prove that results on percolation conjectured to hold with greater generality are true at least in the planar case. In the classical study of percolation (and other statistical mechanics processes) on transitive amenable graphs, and especially on the graphs $\mathbb{Z}^d$, this is a well known fact: planarity allows one to make much faster progress, and much more has been proved in the case of $\mathbb{Z}^2$ than in the more general case of $\mathbb{Z}^d$ (see, e.g., [Gri]). In the context of percolation on transitive non-amenable graphs, a similar pattern has been followed. The paper [Lal1] anticipated for certain transitive non-amenable planar graphs some of the results which would later be proved for more general transitive non-amenable graphs. The study of percolation on transitive non-amenable planar graphs was later greatly developed in the papers [Lal2] and [BS2]. For instance, the fundamental Conjecture 6 in [BS1], which states that for independent bond or site percolation on transitive non-amenable graphs there is always a regime with infinitely many infinite clusters, was proved to hold under the extra assumption of planarity.

In contrast to Theorem 1.1(i), independent percolation on transitive amenable planar graphs with one end is expected to have critical exponents with non-mean-field values. The case in which the graph is $\mathbb{Z}^2$ is extensively discussed in [Gri]. In the case of site percolation on the triangular lattice, various critical exponents have recently been proved to indeed take their conjectured, non-mean-field, values. This is a result of the rapid progress on conformal invariance, in combination with earlier work by H. Kesten relating various critical exponents in the two dimensional case (see [LSW, SW] and references therein).

1.2. Terminology and notation. We will consider independent bond percolation on an infinite, locally finite, connected graph $G = (V, E)$, where V is the set of vertices (sites) and E is the set of edges (bonds). A site $r \in V$ will be singled out and denoted the root of G. The cardinality of a set $S \subset V$ will be denoted by $|S|$. The edge boundary of a set $S \subset V$ is $\partial_{\mathrm{E}} S = \{\{x, y\} \in E : x \in S, y \in S^c\}$ and its inner vertex boundary is $\partial_{\mathrm{in}} S = \{x \in S : \{x, y\} \in \partial_{\mathrm{E}} S \text{ for some } y \in S^c\}$. The **edge-isoperimetric constant** (**Cheeger constant**) of G is defined as

$$i_{\mathrm{E}}(G) = \inf \left\{ \frac{|\partial_{\mathrm{E}} S|}{|S|} : S \subset V, 0 \neq |S| < \infty \right\}.$$

G is said to be **amenable** in case $i_E(G) = 0$. The **number of ends** of the graph G is

$$\mathcal{E}(\mathcal{G}) = \sup_{\substack{S \subset V \\ |S| < \infty}} \{\text{number of infinite connected components of } G \backslash S\},$$

where $G \backslash S$ is the graph obtained from the graph G by removing the vertices which belong to S and the edges incident to these vertices. (The definition of ends of a graph is being omitted because it is not needed in this paper. Those familiar with that concept will note that $\mathcal{E}(\mathcal{G})$ coincides with the cardinality of the set of ends of the graph G in case this cardinality is finite and that $\mathcal{E}(\mathcal{G}) = \infty$ when this cardinality is infinite, but $\mathcal{E}(\mathcal{G})$ does not distinguish between different infinite cardinalities. While this is a drawback of $\mathcal{E}(\mathcal{G})$, its definition is simpler than that of the set of ends of a graph, and is sufficient for various purposes including those in this paper.) Informally, a graph is **transitive** (same as vertex-transitive or homogeneous) if all its vertices play exactly the same role. More precisely, this means that for each pair $x, y \in V$ there is an automorphism of the graph which maps x to y. A graph is said to be **quasi-transitive** if there is a finite set of vertices, V_0, with the property that each vertex of the graph can be mapped into one of the vertices of V_0 by an automorphism. Informally, a graph is quasi-transitive if there is a finite number of types of vertices, and vertices of the same type play the same role. The number of ends of an infinite, locally finite, connected transitive graph is 1,2, or ∞; moreover, when the number of ends is 2, the graph is amenable and when the number of ends is ∞ the graph is non-amenable (see Sect. 6 of [Moh]). The **stabilizer**, $S(x)$, of a vertex $x \in V$ is the set of automorphisms of G which fix x. A transitive graph is **unimodular** if for each $x, y \in V$, $|\{\gamma(y) : \gamma \in S(x)\}| = |\{\gamma(x) : \gamma \in S(y)\}|$. A graph is said to be **planar** if it can be embedded in $\mathbb{R}^2$ with vertices being represented by points and edges being represented by lines which connect the corresponding vertices and can only intersect at their end-points.

The probability measure according to which each edge is occupied with probability p and vacant with probability $1 - p$, independently of the others, will be denoted by $\mathbb{P}_p$. The corresponding expectation will be denoted by $\mathbb{E}_p$. Given $A, B \subset V$, we will write $\{A \leftrightarrow B\}$ for the event that there is a path of occupied bonds connecting A to B (if $A = \{x\}$, we write $\{x \leftrightarrow B\}$, rather than $\{\{x\} \leftrightarrow B\}$, and will use similar conventions systematically). Given also $S \subset V$ we will write $\{A \overset{S}{\longleftrightarrow} B\}$ for the event that there is a path of occupied bonds connecting A to B with all the sites which appear in this path belonging to S. We will set $\{A \not\leftrightarrow B\} = \{A \leftrightarrow B\}^c$, $\{A \overset{S}{\not\longleftrightarrow} B\} = \{A \overset{S}{\longleftrightarrow} B\}^c$. For $x \in V$, $\mathcal{C}(x) = \{y \in V : x \leftrightarrow y\}$ will denote the cluster of the site x. The probability of percolation is defined as $\theta(p) = \mathbb{P}_p(|\mathcal{C}(r)| = \infty)$. The susceptibility is defined as $\chi(p) = \mathbb{E}_p(|\mathcal{C}(r)|) = \sum_{x \in V} \mathbb{P}_p(r \leftrightarrow x)$. The threshold for percolation is the critical point $p_c = \inf\{p \in [0, 1] : \theta(p) > 0\}$. From the methods of [AB], we know that for quasi-transitive graphs $p_c = \sup\{p \in [0, 1] : \chi(p) < \infty\}$. The threshold for uniqueness of the infinite cluster is $p_u = \inf\{p \in [0, 1] : \mathbb{P}_p(\text{there is a unique infinite cluster}) = 1\}$.

In order to define the critical exponent δ, we introduce a "ghost field". Each site is painted green, independently of anything else, with probability q. $\mathbb{P}_{p,q}$ will denote the corresponding probability measure in this enlarged probability space, and $\mathbb{E}_{p,q}$ will be the corresponding expectation. The random set of green sites will be denoted by Q. One defines $\theta(p, q) = \mathbb{P}_{p,q}(r \leftrightarrow Q)$, and $\chi(p, q) = \mathbb{E}_{p,q}(|\mathcal{C}(r)|; \mathcal{C}(r) \cap Q = \emptyset) = \sum_{x \in V} \mathbb{P}_{p,q}(r \leftrightarrow x, r \not\leftrightarrow Q)$.

Next we review what is meant by saying that each one of the critical exponents which appears in Theorem 1.1 exists and takes its mean-field value. The labels on the left indicate the way one usually refers to each statement, and provide the corresponding mean-field value of each critical exponent:

[$\gamma = 1$] $C_1(p_c - p)^{-1} \le \chi(p) \le C_2(p_c - p)^{-1}$, for $p < p_c$,
[$\beta = 1$] $C_1(p - p_c)^1 \le \theta(p) \le C_2(p - p_c)^1$, for $p > p_c$,
[$\delta = 2$] $C_1 q^{1/2} \le \theta(p_c, q) \le C_2 q^{1/2}$, for $q > 0$,
[$\Delta = 2$] For $m = 1, 2, \ldots$ $C_1(p_c - p)^{-2} \le \mathbb{E}_p(|\mathcal{C}(r)|^{m+1})/\mathbb{E}_p(|\mathcal{C}(r)|^m) \le C_2(p_c - p)^{-2}$, for $p < p_c$,

where in each case $C_1, C_2 \in (0, \infty)$.

2. Sufficient Conditions for Mean-Field Criticality

From the arguments in [AN] (modified in the fashion of Sect. 3.1 of [BA]) and [Ngu], we have:

Lemma 2.1.A. *Suppose that $G = (V, E)$ is an infinite, locally finite, connected transitive unimodular graph such that $p_c < 1$. Suppose also that there are $\epsilon, c > 0$ and sites $x_1, x_2 \in V$ such that for every $p \in (p_c - \epsilon, p_c)$,*

$$\sum_{z_1, z_2 \in V} \mathbb{P}_p(x_1 \leftrightarrow z_1, x_2 \leftrightarrow z_2, x_1 \not\leftrightarrow x_2) \ge c(\chi(p))^2.$$

Then $\gamma = 1$ and $\Delta = 2$.

From the arguments in [BA] and [New] we have:

Lemma 2.1.B. *Suppose that $G = (V, E)$ is an infinite, locally finite, connected transitive unimodular graph such that $p_c < 1$. Suppose also that there are $\epsilon, c > 0$ and sites $x_1, x_2, x_3 \in V$ such that for every $p \in (p_c - \epsilon, p_c)$ and $q \in (0, \epsilon)$,*

$$\sum_{z \in V} \mathbb{P}_{p,q}(x_1 \leftrightarrow z, x_1 \not\leftrightarrow Q, x_2 \leftrightarrow Q, x_3 \leftrightarrow Q, x_2 \not\leftrightarrow x_3) \ge c\chi(p, q)(\theta(p, q))^2.$$

Then $\delta = 2$ and $\beta = 1$.

The role of unimodularity in the derivation of the two lemmas above is explained in Section 3.2 of [Sch].

In the remainder of this section, we will reduce the lemmas above to further sufficient conditions for statements of mean-field criticality. The reader can either study these lemmas in the order in which they will be presented, or alternatively, study first the lemmas labeled with “A”, which refer to the exponents γ and Δ, and later study the lemmas labeled with “B”, which refer to the exponents δ and β, and which have more involved proofs.

Lemma 2.2.A. *Suppose that $G = (V, E)$ is an infinite, locally finite, connected transitive unimodular graph. Suppose also that there are $\epsilon, c > 0$, disjoint sets of sites $V_1, V_2 \subset V$ and sites $x_1 \in V_1$, $x_2 \in V_2$ such that for every $p \in (p_c - \epsilon, p_c)$,*

$$\mathbb{P}_p(V_1 \leftrightarrow V_2) \le 1 - c,$$

$$\sum_{z \in V_i} \mathbb{P}_p(x_i \overset{V_i}{\longleftrightarrow} z) \ge c\chi(p) \qquad (i = 1, 2).$$

Then $\gamma = 1$ and $\Delta = 2$.

Proof. For $i = 1, 2$, the events $\{x_i \overset{V_i}{\longleftrightarrow} z\}$ depend only on the state of occupancy of the edges which have both endpoints in V_i, while the event $\{V_1 \leftrightarrow V_2\}$ depends only on the state of occupancy of the other edges. Therefore, by independence,

$$\begin{aligned}
\sum_{z_1,z_2\in V} &\mathbb{P}_p(x_1 \leftrightarrow z_1, x_2 \leftrightarrow z_2, x_1 \not\leftrightarrow x_2) \\
&\geq \sum_{z_1,z_2\in V} \mathbb{P}_p(x_1 \overset{V_1}{\longleftrightarrow} z_1, x_2 \overset{V_2}{\longleftrightarrow} z_2, V_1 \not\leftrightarrow V_2) \\
&= \sum_{z_1,z_2\in V} \mathbb{P}_p(x_1 \overset{V_1}{\longleftrightarrow} z_1)\mathbb{P}_p(x_2 \overset{V_2}{\longleftrightarrow} z_2)\mathbb{P}_p(V_1 \not\leftrightarrow V_2) \\
&= \left(\sum_{z_1\in V_1} \mathbb{P}_p(x_1 \overset{V_1}{\longleftrightarrow} z_1)\right)\left(\sum_{z_2\in V_2} \mathbb{P}_p(x_2 \overset{V_2}{\longleftrightarrow} z_2)\right)\mathbb{P}_p(V_1 \not\leftrightarrow V_2) \\
&\geq c^3(\chi(p))^2.
\end{aligned}$$

And the claim follows from Lemma 2.1.A. (The hypothesis in that lemma that $p_c < 1$ must hold, since otherwise $\mathbb{P}_p(V_1 \leftrightarrow V_2) \to 1$, as $p \nearrow p_c$.) □

Lemma 2.2.B. *Suppose that $G = (V, E)$ is an infinite, locally finite, connected transitive unimodular graph. Suppose also that there are $\epsilon, c > 0$, disjoint sets of sites $V_1, V_2, V_3 \subset V$ and sites $x_1 \in V_1, x_2 \in V_2, x_3 \in V_3$ such that for every $p \in (p_c - \epsilon, p_c)$ and $q \in (0, \epsilon)$,*

$$\mathbb{P}_{p,q}(V_i \leftrightarrow V_j) \leq 1 - c \qquad (i \neq j),$$

$$\sum_{z\in V_1} \mathbb{P}_{p,q}(x_1 \overset{V_1}{\longleftrightarrow} z, x_1 \not\leftrightarrow Q) \geq c\chi(p, q),$$

$$\mathbb{P}_{p,q}(x_i \overset{V_i}{\longleftrightarrow} Q) \geq c\theta(p, q) \qquad (i = 2, 3).$$

Then $\delta = 2$ and $\beta = 1$.

Proof. Set

$$A_1^z = \{x_1 \overset{V_1}{\to}\longleftrightarrow z\}, \qquad \tilde{A}_1 = \{x_1 \not\leftrightarrow Q\},$$

$$\tilde{A}_1^z = \{x_1 \overset{V_1}{\to}\longleftrightarrow z, x_1 \not\leftrightarrow Q\},$$

$$A_2 = \{x_2 \overset{V_2}{\to}\longleftrightarrow Q\}, \qquad A_3 = \{x_3 \overset{V_3}{\to}\longleftrightarrow Q\},$$

$$B = \{V_1 \not\leftrightarrow V_2, V_2 \not\leftrightarrow V_3, V_3 \not\leftrightarrow V_1\}.$$

For each set $E' \subset E$, we will denote by $\mathcal{F}_{E'}$ the σ-field generated by the state of occupancy of the edges in E'. Let $E_1 = \{\{u, v\} \in E : u, v \in V_1\}$, and let $(E_1^k)_{k\geq 1}$ be an increasing sequence of subsets of E_1 which converges to this set (i.e., $\cup_k E_1^k = E_1$).

For any k, and any configuration $\omega_1 \in \{0, 1\}^{E_1^k}$, the set of configurations in $\{0, 1\}^{E\setminus E_1^k} \times \{0, 1\}^V$ which in combination with ω_1 produce a configuration in $\tilde{A}_1$ is a decreasing set. Similarly for B. Therefore, by the Harris' inequality,

$$\mathbb{P}_{p,q}(\tilde{A}_1 B|\mathcal{F}_{E_1^k}) \geq \mathbb{P}_{p,q}(\tilde{A}_1|\mathcal{F}_{E_1^k})\mathbb{P}_{p,q}(B|\mathcal{F}_{E_1^k}) = \mathbb{P}_{p,q}(\tilde{A}_1|\mathcal{F}_{E_1^k})\mathbb{P}_{p,q}(B),$$

where in the last step we used the fact that B depends only on the state of occupancy of the edges which have at least one endpoint in $(V_1)^c$ and therefore is independent of $\mathcal{F}_{E_1^k}$. Letting $k \to \infty$, and using (5.9) on p. 264 of [Dur], yields

$$\mathbb{P}_{p,q}(\tilde{A}_1 B | \mathcal{F}_{E_1}) \geq \mathbb{P}_{p,q}(\tilde{A}_1 | \mathcal{F}_{E_1}) \mathbb{P}_{p,q}(B).$$

Integration over $A_1^z \in \mathcal{F}_{E_1}$, yields now

$$\mathbb{P}_{p,q}(\tilde{A}_1^z B) = \mathbb{P}_{p,q}(\tilde{A}_1 A_1^z B) \geq \mathbb{P}_{p,q}(\tilde{A}_1 A_1^z) \mathbb{P}_{p,q}(B) = \mathbb{P}_{p,q}(\tilde{A}_1^z) \mathbb{P}_{p,q}(B).$$

Therefore,

$$\begin{aligned}
&\sum_{z \in V} \mathbb{P}_{p,q}(x_1 \leftrightarrow z, x_1 \not\leftrightarrow Q, x_2 \leftrightarrow Q, x_3 \leftrightarrow Q, x_2 \not\leftrightarrow x_3) \geq \sum_{z \in V_1} \mathbb{P}_{p,q}(\tilde{A}_1^z A_2 A_3 B) \\
&\quad = \sum_{z \in V_1} \mathbb{P}_{p,q}(\tilde{A}_1^z B) \mathbb{P}_{p,q}(A_2) \mathbb{P}_{p,q}(A_3) \geq \sum_{z \in V_1} \mathbb{P}_{p,q}(\tilde{A}_1^z) \mathbb{P}_{p,q}(A_2) \mathbb{P}_{p,q}(A_3) \mathbb{P}_{p,q}(B) \\
&\quad \geq c^6 \chi(p,q) (\theta(p,q))^2.
\end{aligned}$$

In the second step above we used the fact that $\tilde{A}_1^z B$ depends only on the state of occupancy of the edges which have at least one endpoint in $(V_2 \cup V_3)^c$ and on the state (green or not) of the vertices in $(V_2 \cup V_3)^c$, while, for $i = 2, 3$, A_i depends only on the state of occupancy of the edges which have both endpoints in V_i and on the state (green or not) of the vertices in V_i. In the last step above we used Harris' inequality to obtain $\mathbb{P}_{p,q}(B) \geq c^3$. The claim follows now from Lemma 2.1.B. (The hypothesis in that lemma that $p_c < 1$ must hold, since otherwise $\mathbb{P}_{p,q}(V_i \leftrightarrow V_j) \to 1$, as $p \nearrow p_c$.) $\square$

Lemma 2.3.A. *Suppose that $G = (V, E)$ is an infinite, locally finite, connected transitive unimodular graph. Suppose also that there are disjoint sets of sites $V_1, V_2 \subset V$ and sites $x_1 \in V_1$, $x_2 \in V_2$ such that*

$$\mathbb{P}_{p_c}(V_1 \leftrightarrow V_2) < 1,$$

$$\sum_{v \in \partial_{in} V_i} \mathbb{P}_{p_c}(x_i \leftrightarrow v) < 1 \qquad (i = 1, 2).$$

Then $\gamma = 1$ and $\Delta = 2$.

Proof. We will verify the hypothesis of Lemma 2.2.A with $\epsilon = p_c$ and $c = \min\{1 - \mathbb{P}_{p_c}(V_1 \leftrightarrow V_2), 1 - \sum_{v \in \partial_{\text{in}} V_1} \mathbb{P}_{p_c}(x_1 \leftrightarrow v), 1 - \sum_{v \in \partial_{\text{in}} V_2} \mathbb{P}_{p_c}(x_2 \leftrightarrow v)\}$. By monotonicity in p, only the second display in the hypothesis of Lemma 2.2.A requires any nontrivial argumentation. To verify it, we note that if $\{x_i \leftrightarrow z\}$ occurs, then either $\{x_i \stackrel{V_i}{\longleftrightarrow} z\}$ occurs, or else there is some vertex $v \in \partial_{\text{in}} V_i$ for which the event $\{x_i \leftrightarrow v\} \Box \{v \leftrightarrow z\}$ occurs. From the van den Berg–Kesten–Fiebig–Reimer inequality, we obtain then, for $p < p_c$,

$$\begin{aligned}
\mathbb{P}_p(x_i \leftrightarrow z) &\leq \mathbb{P}_p(x_i \stackrel{V_i}{\longleftrightarrow} z) + \sum_{v \in \partial_{\text{in}} V_i} \mathbb{P}_p(x_i \leftrightarrow v) \mathbb{P}_p(v \leftrightarrow z) \\
&\leq \mathbb{P}_p(x_i \stackrel{V_i}{\longleftrightarrow} z) + \sum_{v \in \partial_{\text{in}} V_i} \mathbb{P}_{p_c}(x_i \leftrightarrow v) \mathbb{P}_p(v \leftrightarrow z).
\end{aligned}$$

Summing over $z \in V$,

$$\chi(p) \leq \sum_{z \in V_i} \mathbb{P}_p(x_i \overset{V_i}{\longleftrightarrow} z) + \sum_{v \in \partial_{\text{in}} V_i} \mathbb{P}_{p_c}(x_i \leftrightarrow v)\chi(p).$$

Therefore,

$$\sum_{z \in V_i} \mathbb{P}_p(x_i \overset{V_i}{\longleftrightarrow} z) \geq c\chi(p). \qquad \square$$

Lemma 2.3.B. *Suppose that $G = (V, E)$ is an infinite, locally finite, connected transitive unimodular graph. Suppose also that there are disjoint sets of sites $V_1, V_2, V_3 \subset V$ and sites $x_1 \in V_1$, $x_2 \in V_2$, $x_3 \in V_3$ such that*

$$\mathbb{P}_{p_c}(V_i \leftrightarrow V_j) < 1 \qquad (i \neq j),$$

$$\sum_{v \in \partial_{in} V_i} \mathbb{P}_{p_c}(x_i \leftrightarrow v) < 1 \qquad (i = 1, 2, 3).$$

Then $\delta = 2$ and $\beta = 1$.

Proof. We will verify the hypothesis of Lemma 2.2.B with $\epsilon = p_c$ and $c = \min\{1 - \mathbb{P}_{p_c}(V_1 \leftrightarrow V_2), 1 - \mathbb{P}_{p_c}(V_2 \leftrightarrow V_3), 1 - \mathbb{P}_{p_c}(V_3 \leftrightarrow V_1), 1 - \sum_{v \in \partial_{\text{in}} V_1} \mathbb{P}_{p_c}(x_1 \leftrightarrow v), 1 - \sum_{v \in \partial_{\text{in}} V_2} \mathbb{P}_{p_c}(x_2 \leftrightarrow v), 1 - \sum_{v \in \partial_{\text{in}} V_3} \mathbb{P}_{p_c}(x_3 \leftrightarrow v)\}$. By monotonicity in p, only the second and third displays in the hypothesis of Lemma 2.2.B require any non-trivial argumentation.

To verify the third display, we note that if $\{x_i \leftrightarrow Q\}$ occurs, then either $\{x_i \overset{V_i}{\longleftrightarrow} Q\}$ occurs, or else there is some vertex $v \in \partial_{\text{in}} V_i$ for which the event $\{x_i \leftrightarrow v\} \square \{v \leftrightarrow Q\}$ occurs. From the van den Berg–Kesten–Fiebig–Reimer inequality, we obtain then, for $p < p_c$ and $q \in (0, 1]$,

$$\begin{aligned}\theta(p, q) = \mathbb{P}_{p,q}(x_i \leftrightarrow Q) &\leq \mathbb{P}_{p,q}(x_i \overset{V_i}{\longleftrightarrow} Q) + \sum_{v \in \partial_{\text{in}} V_i} \mathbb{P}_{p,q}(x_i \leftrightarrow v)\mathbb{P}_{p,q}(v \leftrightarrow Q) \\ &\leq \mathbb{P}_{p,q}(x_i \overset{V_i}{\longleftrightarrow} Q) + \sum_{v \in \partial_{\text{in}} V_i} \mathbb{P}_{p_c}(x_i \leftrightarrow v)\theta(p, q).\end{aligned}$$

Therefore,

$$\mathbb{P}_{p,q}(x_i \overset{V_i}{\longleftrightarrow} Q) \geq c\theta(p, q) \qquad (i = 2, 3).$$

To verify the second display in the hypothesis of Lemma 2.2.B note that if $\{x_1 \leftrightarrow z, x_1 \not\leftrightarrow Q\}$ occurs, then either $\{x_1 \overset{V_1}{\longleftrightarrow} z, x_1 \not\leftrightarrow Q\}$ occurs, or else there is some vertex $v \in \partial_{\text{in}} V_i$ for which the event $\{x_1 \leftrightarrow v\} \square \{v \leftrightarrow z, v \not\leftrightarrow Q\}$ occurs (this is slightly subtle; recall that our sample space is $\{0, 1\}^E \times \{0, 1\}^V$ and, using the notation in [Gri], p. 38, take for the set K in the definition of $\square$ a set of edges which produce a path from x_1 to v – do not include any vertex in K). From the van den Berg–Kesten–Fiebig–Reimer inequality, we obtain then, for $p < p_c$ and $q \in (0, 1]$,

$$\begin{aligned}&\mathbb{P}_{p,q}(x_1 \leftrightarrow z, x_1 \not\leftrightarrow Q) \\ &\quad \leq \mathbb{P}_{p,q}(x_1 \overset{V_1}{\longleftrightarrow} z, x_1 \not\leftrightarrow Q) + \sum_{v \in \partial_{\text{in}} V_1} \mathbb{P}_{p,q}(x_1 \leftrightarrow v)\mathbb{P}_{p,q}(v \leftrightarrow z, v \not\leftrightarrow Q) \\ &\quad \leq \mathbb{P}_{p,q}(x_1 \overset{V_1}{\longleftrightarrow} z, x_1 \not\leftrightarrow Q) + \sum_{v \in \partial_{\text{in}} V_1} \mathbb{P}_{p_c}(x_1 \leftrightarrow v)\mathbb{P}_{p,q}(v \leftrightarrow z, v \not\leftrightarrow Q).\end{aligned}$$

Summing over $z \in V$,

$$\chi(p,q) \le \sum_{z \in V_1} \mathbb{P}_{p,q}(x_1 \overset{V_1}{\longleftrightarrow} z, x_1 \not\leftrightarrow Q) + \sum_{v \in \partial_{\text{in}} V_1} \mathbb{P}_{p_c}(x_1 \leftrightarrow v)\chi(p,q).$$

Therefore,

$$\sum_{z \in V_1} \mathbb{P}_{p,q}(x_1 \overset{V_1}{\longleftrightarrow} z, x_1 \not\leftrightarrow Q) \ge c\chi(p,q). \qquad \square$$

3. The Case of Planar Graphs

In this section we suppose that $G = (V, E)$ is an infinite, locally finite, connected transitive non-amenable planar single-ended graph. Proposition 2.1 of [BS2] states that G is unimodular and that it can be embedded in the hyperbolic plane $\mathbb{H}^2$ in the following way. Each vertex of G is mapped into a point of $\mathbb{H}^2$ and each edge of G is mapped into a geodesic line segment with endpoints at the points of $\mathbb{H}^2$ which are images of its endpoints; moreover the group of automorphisms of G is mapped in this way into a group of isometries of $\mathbb{H}^2$. It is clear that, by adjusting the length scale, such an embedding can be chosen so that each face in the embedding has diameter less than 1. In particular any point of $\mathbb{H}^2$ is then within distance 1 of a point which represents a vertex of G, and all the geodesic line segments which represent edges of G have length at most 1. We will refer to such an embedding as a "nice embedding".

One convenient way to describe the dual $G^\dagger = (V^\dagger, E^\dagger)$ of G is to represent each element of $V^\dagger$ by a face (tile) in the embedding of G, described above, and represent elements of $E^\dagger$ by pairs of faces whose topological boundaries intersect on a non-degenerate geodesic line segment (which represents an edge of G). This establishes a one-to-one correspondence between E and $E^\dagger$, and the image of $e \in E$ under this correspondence will be denoted by $e^\dagger$. Since G is transitive, $G^\dagger$ is quasi-transitive.

Any bond percolation process on G is coupled to a bond percolation process on $G^\dagger$, by declaring each edge $e^\dagger$ vacant (resp. occupied) if e is occupied (resp. vacant). Independent percolation at density p on G is coupled in this fashion to independent percolation at density $1-p$ on $G^\dagger$.

The following lemma is a basic building block in our argumentation in this section. In the statement of this lemma, we identify a path in the dual graph with the union of the tiles that correspond to the endpoints of the dual edges in this path in the embedding.

Lemma 3.1. *Suppose that G is an infinite, locally finite, connected transitive non-amenable planar single-ended graph, nicely embedded in $\mathbb{H}^2$. If $p < p_u$, then there is $C_0 > 0$ such that the following happens. Let $\mathcal{L}$ be an arbitrary geodesic line in $\mathbb{H}^2$, s' and s'' be two points on $\mathcal{L}$, separated by distance $l > 2$, and $\mathcal{L}'$ and $\mathcal{L}''$ be geodesic lines perpendicular to $\mathcal{L}$ through s' and s'', respectively. Then*

$$\mathbb{P}_p(\textit{there is an occupied dual path separating } \mathcal{L}' \textit{ from } \mathcal{L}'') > C_0.$$

Proof. This was proved in a somewhat more restricted setting and for site percolation in [Lal2], Lemma 2.15. The more general case considered here can be handled in the same way, by using results in [BS2]. First, from Theorem 3.7 of [BS2], we learn that there is percolation in the dual process when $p < p_u$. From the generalization of Corollary 4.4 of [BS2] to quasi-transitive tilings of $\mathbb{H}^2$, we learn then that percolation also occurs in this dual process on hyperbolic half-spaces. This enables us to use the arguments in the proofs of Lemma 2.14 and 2.15 in [Lal2] to conclude the proof. $\square$

Given a nice embedding of G in $\mathbb{H}^2$ and a set $\mathcal{S} \subset \mathbb{H}^2$, we will use the notation $\bar{\mathcal{S}}$ for the set of vertices of G which are endpoints of edges represented in the embedding by geodesic line segments which intersect $\mathcal{S}$.

Lemma 3.2. *Suppose that G is an infinite, locally finite, connected transitive non-amenable planar single-ended graph, nicely embedded in $\mathbb{H}^2$. If $p < p_u$, then there are $C_1, C_2 \in (0, \infty)$, such that the following happens. Let $\mathcal{L}$ be an arbitrary geodesic line in $\mathbb{H}^2$, s' and s'' be two points on $\mathcal{L}$, separated by distance L, and $\mathcal{L}'$ and $\mathcal{L}''$ be geodesic lines perpendicular to $\mathcal{L}$ through s' and s'', respectively. Then*

$$\mathbb{P}_p(\bar{\mathcal{L}}' \leftrightarrow \bar{\mathcal{L}}'') \leq C_1 e^{-C_2 L}.$$

Proof. Take $l > 2$ and consider the set of geodesic lines which separate $\mathcal{L}'$ from $\mathcal{L}''$, are perpendicular to $\mathcal{L}$ and cross it at points which are at distance jl, $j = 1, 2, \ldots, \lfloor L/l \rfloor$ from s'. Since any path from $\bar{\mathcal{L}}'$ to $\bar{\mathcal{L}}''$ has to cross all these lines, the claim follows from Lemma 3.1. □

Lemma 3.3. *Suppose that G is an infinite, locally finite, connected transitive non-amenable planar single-ended graph, nicely embedded in $\mathbb{H}^2$. If $p < p_u$, then there are $C_3, C_4 \in (0, \infty)$, such that the following happens. Let $\mathcal{L}$ be an arbitrary geodesic line in $\mathbb{H}^2$, s and s' be two points on $\mathcal{L}$, separated by distance L, and $\mathcal{L}'$ be the geodesic line perpendicular to $\mathcal{L}$ through s'. Let x be a vertex of G which in the embedding is mapped into a point of $\mathbb{H}^2$ at distance at most 1 from s. Then*

$$\mathbb{P}_p(x \leftrightarrow y) \leq C_3 e^{-C_4 L}.$$

Proof. Let $\mathcal{L}'_+$ and $\mathcal{L}'_-$ be the two half-lines into which s' partitions $\mathcal{L}'$. Take some $l > 2$. Set $s_0 = s'$, and for $k \in \{1, 2, \ldots\}$ let s_k (resp. s_{-k}) be the point on $\mathcal{L}'_+$ (resp. $\mathcal{L}'_-$) at distance kl from s'. For $k \in \{1, 2, \ldots\}$ let $\mathcal{I}_k$ (resp. $\mathcal{I}_{-k}$) be the geodesic segment (contained in $\mathcal{L}'$) with endpoints s_{k-1} and s_k (resp. s_{-k+1} and s_{-k}). For $j \in \mathbb{Z}$, let $\mathcal{L}_j$ be the geodesic line perpendicular to $\mathcal{L}'$ through s_j. Then

$$\sum_{y \in \bar{\mathcal{L}}'} \mathbb{P}_p(x \leftrightarrow y) \leq \sum_{j \in \mathbb{Z} \setminus \{0\}} \sum_{y \in \bar{\mathcal{I}}_j} \mathbb{P}_p(x \leftrightarrow y).$$

Let D be the degree of G. It is easy to see that for some small $\epsilon > 0$ any ball of radius ϵ in $\mathbb{H}^2$ can intersect at most D edges of the embedding of G in $\mathbb{H}^2$. Therefore it is also easy to see that any geodesic line segment of length d can intersect at most dD/ϵ such edges. Therefore, from the previous display we obtain, for arbitrary J,

$$\sum_{y \in \bar{\mathcal{L}}'} \mathbb{P}_p(x \leftrightarrow y) \leq \frac{lD}{\epsilon} \sum_{j : |j| > J} \mathbb{P}_p(x \leftrightarrow \bar{\mathcal{I}}_j) + \frac{2lJD}{\epsilon} \mathbb{P}_p(x \leftrightarrow \bar{\mathcal{L}}).$$

When $j > 1$ (resp. $j < 1$) any path from x to $\bar{\mathcal{I}}_j$ has to cross the lines $\mathcal{L}_i$, $i = 1, 2, \ldots, j-1$, (resp. $i = -1, -2, \ldots, -j+1$). Hence, Lemma 3.1 implies

$$\mathbb{P}_p(x \leftrightarrow \bar{\mathcal{I}}_j) \leq C_5 e^{-C_6 j},$$

for some $C_5, C_6 \in (0, \infty)$. Therefore, using Lemma 3.2 and taking $J = \lfloor L \rfloor$, we obtain

$$\sum_{y \in \bar{\mathcal{L}}'} \mathbb{P}_p(x \leftrightarrow y) \leq C_7 e^{-C_6 \lfloor L \rfloor} + C_8 \lfloor L \rfloor e^{-C_2 L} \leq C_3 e^{-C_4 L}. \qquad □$$

Proof of Theorem 1.1(i). We will check that the hypothesis of Lemma 2.3.A and Lemma 2.3.B are satisfied (note that the former are contained in the latter).

Suppose that G is nicely embedded in $\mathbb{H}^2$. Let $\mathcal{L}$ be a geodesic line and $s_1, \dots, s_7$ be distinct points on $\mathcal{L}$, such that for $i = 1, \dots, 6$, the distance between s_i and s_{i+1} has the same common value L. For each i, let $\mathcal{L}_i$ be the geodesic line perpendicular to $\mathcal{L}$ through r_i. The removal of $\mathcal{L}_2 \cup \mathcal{L}_3 \cup \mathcal{L}_5 \cup \mathcal{L}_6$ breaks $\mathbb{H}^2$ into 5 connected components. For $i = 1, 4, 7$, let $\mathcal{V}_i$ be the connected component which contains s_i.

Set $V_1 = \bar{\mathcal{V}}_1$, $V_2 = \bar{\mathcal{V}}_4$, $V_3 = \bar{\mathcal{V}}_7$. Let x_1, x_2 and x_3 be vertices of G which in the embedding are mapped into points of $\mathbb{H}^2$ at distance at most 1 from s_1, s_4 and s_7, respectively. With these choices, the hypothesis of Lemma 2.3.B are satisfied, provided that L is large enough, as can be seen from Lemma 3.2, Lemma 3.3 and Theorem 1.1 of [BS2], which states that $p_c < p_u$. $\square$

4. The Case of Graphs with Infinitely Many Ends

We will need some notation and terminology related to the binary homogeneous tree, $\mathbb{T}_2$, i.e., the tree in which every vertex has degree 3. The set of vertices of this tree will be denoted by $V(\mathbb{T}_2)$. Given $i, j, k \in V(\mathbb{T}_2)$ we will say that k is between i and j if the shortest path from i to j passes through k.

The following proposition will be used in this section; it can be easily proved with the arguments in the proof of Propositions 6.1 in [Moh2]. (Compare with Proposition 2.1 in [Sch].) Below $B(u, n)$ will denote the ball of radius n centered at $u \in V$ in the graph $G = (V, E)$.

Proposition 4.1. *Suppose that $G = (V, E)$ is an infinite, locally finite, connected transitive graph. If G has infinitely many ends, then there is a positive integer n and vertices $u_k \in V$, $k \in V(\mathbb{T}_2)$ such that the balls $B(u_k, n)$, $k \in Z$ ar disjoint and have the following property. For each $i, j \in V(\mathbb{T}_2)$ any path from $B(u_i, n)$ to $B(u_j, n)$ intersects each $B(u_k, n)$ with k between i and j.*

Proof of Theorem 1.1(ii). We will check that the hypothesis of Lemma 2.3.A and Lemma 2.3.B are satisfied (note that the former are contained in the latter).

Let $k_0, k_1, k_2, k_3 \in V(\mathbb{T}_2)$ be such that for $1 \le i < j \le 3$, k_0 is between k_i and k_j, and for $i = 1, 2, 3$, the distance in $\mathbb{T}_2$ between k_i and k_0 has a common value l. Using the notation in Proposition 4.1, set $x_i = u_{k_i}$, $i = 0, 1, 2, 3$. Proposition 4.1 implies that $G \backslash B(x_0, n)$ has at least 3 distinct infinite components, which contain respectively x_1, x_2 and x_3. Call them, respectively, V_1, V_2 and V_3.

Since G has infinitely many ends, it is non-amenable and hence, by Theorem 2 of [BS1] (adapted to bond percolation), it has $p_c < 1$.

To verify the hypothesis of Lemma 2.3.B, let K be the number of edges of G which have at least one endpoint in $B(u_0, n)$, and note that, for $1 \le i < j \le 3$,

$$\mathbb{P}_{p_c}(V_i \leftrightarrow V_j) \le 1 - (1 - p_c)^K < 1,$$

and, for $i = 1, 2, 3$,

$$\sum_{v \in \partial_{\text{in}} V_i} \mathbb{P}_{p_c}(x_i \leftrightarrow v) \le |\partial_{\text{in}} V_i| \mathbb{P}_{p_c}(x_i \leftrightarrow \partial_{\text{in}} V_i) \le K(1 - (1 - p_c)^K)^{l-1}.$$

The last expression can be made arbitrarily small by taking l sufficiently large. $\square$

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