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On the $1/n$ Expansion for Some Unitary Invariant Ensembles of Random Matrices

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Dedicated to Joel L. Lebowitz on the occasion of his 70th birthday

Abstract: We present a version of the $1/n$-expansion for random matrix ensembles known as matrix models. The case where the support of the density of states of an ensemble consists of one interval and the case where the density of states is even and its support consists of two symmetric intervals is treated. In these cases we construct the expansion scheme for the Jacobi matrix determining a large class of expectations of symmetric functions of eigenvalues of random matrices, prove the asymptotic character of the scheme and give an explicit form of the first two terms. This allows us, in particular, to clarify certain theoretical physics results on the variance of the normalized traces of the resolvent of random matrices. We also find the asymptotic form of several related objects, such as smoothed squares of certain orthogonal polynomials, the normalized trace and the matrix elements of the resolvent of the Jacobi matrices, etc.

1. Introduction. Problem and Main Results

Random matrix theory is an actively developing field that has a wide variety of applications (see e.g. the review works [20,16,23] and references therein). Among numerous random matrix ensembles studied by the theory and which have important applications the ensembles with the unitary invariant probability distributions (known also as matrix models) play a significant role [15,22]. This is, in particular, because of numerous links of the ensembles with the theory of orthogonal polynomials, potential theory, the theory of integrable systems, and other domains and techniques of analysis and mathematical physics. These ensembles consist of $n \times n$ Hermitian matrices and are defined by the distribution

$$P_n(M)dM = Z_n^{-1} \exp\{-n\mathrm{Tr}V(M)\}dM, \tag{1.1}$$

where Z_n is the normalizing constant, $V : \mathbf{R} \to \mathbf{R}_+$ satisfies the conditions

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(i) for some $\epsilon > 0$ there exists $L_1 > 0$ such that

$$|V(\lambda)| \geq (2+\epsilon)\log|\lambda|, \ |\lambda| \geq L_1, \tag{1.2}$$

(ii) for any $0 < L_2 < \infty$ there exists $\gamma > 0$ such that

$$|V(\lambda_1) - V(\lambda_2)| \leq C|\lambda_1 - \lambda_2|^{\gamma}, \ |\lambda_{1,2}| \leq L_2, \tag{1.3}$$

(iii) there exists $m > 0$ such that

$$\int |V'(\lambda)|e^{-mV(\lambda)}d\lambda < \infty, \tag{1.4}$$

and

$$dM = \prod_{j=1}^{n} dM_{jj} \prod_{j<k} d\Im M_{jk} d\Re M_{jk}. \tag{1.5}$$

The asymptotic regime that we study is intermediate between the global regime (see e.g. [8,12]) and the local regime (see e.g. [24,13]) and the respective results are important in studying the central limit theorem [17], universal conductance fluctuations [6], and the universality of the local eigenvalue statistics at the edges of the support of the Integrated Density of States (IDS) of the ensembles [25].

Let us recall the definition of the IDS. Denote by $\lambda_1^{(n)}, \ldots, \lambda_n^{(n)}$ the eigenvalues of a matrix M of the ensemble and define the eigenvalue counting measure (NCM) of the matrix as

$$N_n(\Delta) = \natural\{\lambda_l^{(n)} \in \Delta\} \cdot n^{-1}, \tag{1.6}$$

where Δ is an interval of the spectral axis. According to [8] the NCM tends weakly in probability as $n \to \infty$ to the nonrandom limiting measure N known as the Integrated Density of States (IDS) of the ensemble. The IDS is normalized to unity and is absolutely continuous if V' satisfies the Lipshitz condition (1.3) (with possibly different constants C and γ) [27]:

$$N(\mathbf{R}) = 1, \quad N(\Delta) = \int_{\Delta} \rho(\lambda)d\lambda. \tag{1.7}$$

The non-negative function ρ in (1.7) is called the Density of States (DOS) of the ensemble. The IDS can be found as the unique solution of a certain variational problem [8,12, 27]. The IDS is one of the main outputs of the study of the global regime.

Let us state now our main conditions.

Condition C1. *The support σ of the IDS of the ensemble consists of either*

(i) *a single interval:*

$$\sigma = [a, b], \quad -\infty < a < b < \infty,$$

or

(ii) *of two symmetric intervals:*

$$\sigma = [-b, -a] \cup [a, b], \quad -\infty < a < b < \infty,$$

and V is even: $V(\lambda) = V(-\lambda)$, $\lambda \in \mathbf{R}$.

Remark 1. It is easy to see that changing the variables according to $M' = M - \dfrac{a+b}{2} I$ in case (i) we can always take the support σ to be symmetric with respect to the origin. Therefore without loss of generality we can assume that in this case

$$\sigma = (-a, a). \tag{1.8}$$

Condition C2. *The DOS $\rho(\lambda)$ is strictly positive in all internal points of σ and behaves asymptotically as* $\text{const}\,|\lambda - c|^{1/2}$, $\lambda \to c$, *in a neighborhood of each edge c of the support. Besides, the function*

$$u(\lambda) = 2 \int \log|\mu - \lambda| d\mu - V(\lambda) \tag{1.9}$$

achieves its maximum if and only if $\lambda \in \sigma$. We will call this behavior generic (see e.g. [19] for results, justifying the term).

Condition C3. *$V(\lambda)$ is real analytic on σ, i.e. there exists an open domain $\mathbf{D} \subset \mathbf{C}$ such that $\sigma \subset \mathbf{D}$ and an analytic in $\mathbf{D}$ function $V(z)$, $z \in \mathbf{D}$ such that*

$$V(\lambda + i0) = V(\lambda), \quad \lambda \in \sigma.$$

Note that we always have the one-interval case (i) if V is convex [8,17], or if it has a unique absolute minimum and sufficiently large amplitude [19], and we always have the two interval case (ii) if V has two equal absolute minima and sufficiently large amplitude [19].

As for the condition C3, it is the case in many of the quantum field theory [15] and of the condensed matter theory [6] applications.

The following statement, known in several contexts, provides a sufficiently explicit form of the IDS in our cases.

Proposition 1. *Let an ensemble of form (1.1)–(1.5) satisfy conditions C1–C3 above. Then its density of states ρ has the form*

$$\rho(\lambda) = \frac{1}{2\pi} \chi_\sigma(\lambda) P(\lambda) X_+(\lambda), \tag{1.10}$$

where $\chi_\sigma(\lambda)$ is the indicator of the support σ of the IDS, $P(\lambda)$ is analytic in $\mathbf{D}$ (including σ) and

$$X_+(\lambda) = \begin{cases} \sqrt{a^2 - \lambda^2}, & |\lambda| \le a \text{ in the case (i)}, \\ \operatorname{sign} \lambda \sqrt{(\lambda^2 - a^2)(b^2 - \lambda^2)}, & a \le |\lambda| \le b \text{ in the case (ii)}. \end{cases} \tag{1.11}$$

Besides, the Stieltjes transform

$$g(z) = \int_\sigma \frac{\rho(\mu) d\mu}{z - \mu}, \quad \Im z \neq 0, \tag{1.12}$$

of the IDS can be represented in the form

$$g(z) = \frac{1}{2}(V'(z) - X(z)P(z)), \quad z \in \mathbf{D}, \tag{1.13}$$

with

$$X(z) = \begin{cases} \sqrt{z^2 - a^2}, & \textit{in the case (i)}, \\ \sqrt{(z^2 - a^2)(z^2 - b^2)}, & \textit{in the case (ii)}, \end{cases} \tag{1.14}$$

and we take the branches of the square roots, which are analytic everywhere except σ *and have the asymptotic* $X(z) = z^p(1 + O(z^{-1}))$, $z \to \infty$ *with* $p = 1, 2$ *for the one interval and for the two interval cases respectively.* $P(z)$ *in (1.13) and in (1.10) can be represented in the form*

$$P(z) = \frac{1}{2\pi i}\int_L Q(z,\zeta)X^{-1}(\zeta)d\zeta, \tag{1.15}$$

where $L \subset \mathbf{D}$ *is a closed contour encircling* σ, *and*

$$Q(z,\zeta) \equiv \frac{V'(z) - V'(\zeta)}{z - \zeta}. \tag{1.16}$$

The proof of the proposition will be given in Sect. 3. Here we remark that in the two-interval case the contour L consists of two connected components encircling each of the intervals of σ.

We need also several facts on ensembles (1.1)–(1.5) (see e.g. [7, 20, 22]).

Denote by $p_n(\lambda_1, \dots, \lambda_n)$ the joint eigenvalue probability density which we assume to be symmetric without loss of generality. It is known that [20]

$$p_n(\lambda_1, \dots \lambda_n) = Z_n^{-1} \prod_{1\le j<k\le n} (\lambda_j - \lambda_k)^2 \exp\left\{ -n\sum_{j=1}^n V(\lambda_j)\right\}, \tag{1.17}$$

where Z_n is the respective normalization factor. Let

$$p_l^{(n)}(\lambda_1, \dots, \lambda_l) = \int p_n(\lambda_1, \dots, \lambda_l, \lambda_{l+1}, \dots \lambda_n)d\lambda_{l+1}\dots d\lambda_n \tag{1.18}$$

be the l^{th} marginal distribution density of (1.17). The link with orthogonal polynomials is provided by the formula [20, 7]

$$p_l^{(n)}(\lambda_1, \dots, \lambda_l) = \frac{(n-l)!}{n!} \det ||k_n(\lambda_j, \lambda_k)||_{j,k=1}^l, \tag{1.19}$$

where

$$k_n(\lambda, \mu) = \sum_{l=1}^n \psi_l^{(n)}(\lambda)\psi_l^{(n)}(\mu) \tag{1.20}$$

is known as the reproducing kernel of the orthonormalized system

$$\psi_l^{(n)}(\lambda) = \exp\{-nV(\lambda)/2\}p_{l-1}^{(n)}(\lambda), \quad l = 1, \dots, \tag{1.21}$$

in which $p_l^{(n)}(\lambda),\ \ l = 1, \dots$ are orthogonal polynomials on $\mathbf{R}$ associated with the weight

$$w_n(\lambda) = e^{-nV(\lambda)}, \tag{1.22}$$

i.e.

$$\int p_l^{(n)}(\lambda) p_m^{(n)}(\lambda) w_n(\lambda) d\lambda = \delta_{l,m}. \tag{1.23}$$

The polynomial $p_l^{(n)}(\lambda)$ has the degree l and the positive coefficient in front of λ^l. The orthonormalized functions $\psi_l^{(n)}(\lambda)$ verify the recurrent relations

$$J_l(n)\psi_{l+1}^{(n)}(\lambda) + q_l(n)\psi_l^{(n)}(\lambda) + J_{l-1}(n)\psi_{l-1}^{(n)}(\lambda) = \lambda\psi_l^{(n)}(\lambda), \quad l = 1, \dots, \tag{1.24}$$

where $J_0(n) = 0$. In other words, we have here a semi-infinite real symmetric Jacobi matrix

$$\begin{aligned} \mathcal{J}(n) &= \{J_{l,m}(n)\}_{l,m=1}^{\infty}, \\ J_{l,m}(n) &= q_l(n)\delta_{l,m} + J_l(n)\delta_{l+1,m} + J_{l-1}(n)\delta_{l-1,m}. \end{aligned} \tag{1.25}$$

Note that if $V(\lambda)$ is even, then $q_l(n) = 0, l = 1, \dots$..

As in statistical mechanics the symmetrized marginal densities (1.18) allow us to compute the expectation with respect to measure (1.1) of random variables of the form

$$\omega_m(\lambda_1, \dots, \lambda_n) = \sum_{ch} \varphi_m(\lambda_{i_1}, \dots, \lambda_{i_m}), \tag{1.26}$$

where $\varphi_m(t_1, \dots, t_m)$ is symmetric in its arguments and $\sum_{\text{ch}}$ denotes the sum over all choices of m λ's from the set $(\lambda_1, \dots, \lambda_n)$. By using (1.19) and noting that the semi-infinite matrix $\{\psi_j^{(n)}(\lambda)\psi_k^{(n)}(\lambda)\}_{j,k=1}^{\infty}$ is the density of the resolution of identity of $\mathcal{J}(n)$, it is easy to show that $\mathbf{E}\{\omega_m\}$ is a linear combination of the matrix elements of $\varphi_m((\mathcal{J}(n))^{\otimes m})$ (see e.g. formula (2.78)). This observation makes $\mathcal{J}(n)$ an important object of the theory. That is why our main result (Theorem 1 below) yields the $1/n$-expansion of entries of $\mathcal{J}(n)$. Besides, we give the $1/n$-expansion of the expectation

$$g_n(z) = \mathbf{E}\{n^{-1}\mathrm{Tr}(z - M)^{-1}\}, \quad \Im z \neq 0,$$

of the normalized trace of the resolvent of the random matrix M and of the variance of the trace. These quantities are of considerable interest by themselves and are also important technical ingredients of the theory.

Theorem 1. *Let the ensemble of the form (1.1)–(1.5) satisfy conditions (1.3), (1.2), and C1–C3 above. Take a sequence of positive integers $N(n)$ and an integer $m > 0$ such that*

$$N(n)n^{-1/(m+1)} \to 0,\ \ N(n)\log^{-2} n \to \infty,\ \ \text{as } n \to \infty. \tag{1.27}$$

Then there exist coefficients $q_{k,n}^{(0)}, \dots, q_{k,n}^{(m)}, J_{k,n}^{(0)}, \dots, J_{k,n}^{(m)}$, $|k-n| \le N(n)$ *and analytic outside of* σ *functions* $g_n^{(0)}(z), \dots, g_n^{(m)}(z)$ *such that for any* $k \in [n - N(n), n + N(n)]$ *we have the following asymptotic formulas:*

$$q_k(n) = \sum_{j=0}^{m} n^{-j} q_{k,n}^{(j)} + n^{-m} \tilde{r}_{k,n}^{(m,q)}, \quad J_k(n) = \sum_{j=0}^{m} n^{-j} J_{k,n}^{(j)} + n^{-m} \tilde{r}_{k,n}^{(m,J)}, \tag{1.28}$$

where

$$|q_{k,n}^{(j)}|,\ |J_{k,n}^{(j)}| \le \text{ const } (|k-n|^j + 1), \quad j = 0, \dots, p, \tag{1.29}$$

$$|\tilde{r}_{k,n}^{(m,J)}|, |\tilde{r}_{k,n}^{(m,q)}| \le \varepsilon_n^{(m)}, \quad \varepsilon_n^{(m)} \to 0, \quad n \to \infty; \tag{1.30}$$

and

$$g_n(z) = \sum_{j=0}^{m} n^{-j} g_n^{(j)}(z) + n^{-m} \tilde{r}_n^{(m,g)}(z), \tag{1.31}$$

where

$$|g_n^{(j)}(z)| \le \text{const}, \tag{1.32}$$

$$\tilde{r}_n^{(m,g)}(z) \to 0, \quad n \to \infty$$

uniformly in any compact set in $\{z : \delta(z) \ge d\}$, *where* $\delta(z) = \text{dist}(z, \sigma)$, $d > 0$ *is an arbitrary fixed number and* const *does not depend on* k, n.

In particular:

$$g_n^{(0)}(z) = g(z), \quad g_n^{(1)}(z) = 0; \tag{1.33}$$

in the case (i)

$$q_{k,n}^{(0)} = 0, \quad J_{k,n}^{(0)} = \frac{1}{2}a, \quad q_{k,n}^{(1)} = 0, \quad J_{k,n}^{(1)} = \frac{(k-n)}{a}\left(\frac{1}{P(a)} + \frac{1}{P(-a)}\right), \tag{1.34}$$

i.e. the zero order coefficients for $k = n(1 + o(1))$ *are independent of* k;

in the case (ii)

$$q_{k,n}^{(j)} = 0, \quad j = 0, 1, \dots,$$

and

$$J_{k,n}^{(0)} = \frac{1}{2}(b + (-1)^k a), \quad \textit{or} \quad J_{k,n}^{(0)} = \frac{1}{2}(b - (-1)^k a), \tag{1.35}$$

$$J_{k,n}^{(1)} = \frac{(k-n)}{a^2 - b^2}\left(-\frac{1}{P(b)} \pm \frac{(-1)^k}{P(a)}\right), \tag{1.36}$$

where the sign corresponds to the chosen sign in (1.35)

Besides, for $m \le 1$ *formulas (1.28)–(1.35) are valid for any* $k : |k - n| \le n^{2/3}$ *with*

$$\begin{aligned} |\tilde{r}_{k,n}^{(0,J)}| &\le \text{ const } \frac{|k-n|+1}{n}, \quad |\tilde{r}_{k,n}^{(0,q)}| \le \text{ const } \frac{|k-n|^2+1}{n^2}, \\ |\tilde{r}_{k,n}^{(1,q)}|, |\tilde{r}_{k,n}^{(1,J)}| &\le \text{ const } \frac{|k-n|^2+1}{n}. \end{aligned} \tag{1.37}$$

Remark 2. According to (1.35) the 2-periodic function $J_{k,n}^{(0)}$ is determined by our method up to the shift by 1. By using recent results of [13] on the form of the leading term of the asymptotics of the orthogonal polynomials (1.23) it can be shown that

$$J_{k,n}^{(0)} = \frac{1}{2}(b - (-1)^k a), \quad k = n(1 + o(1)). \tag{1.38}$$

Moreover, all subsequent coefficients $J_{k,n}^{(j)}$ and $g_n^{(j)}$, $j = 1, \ldots, N(n)$ of the asymptotic expansions (1.28) and (1.31) are uniquely determined by the choice (1.35) and by the recurrence procedure described in the proof of the theorem.

Remark 3. The zero order coefficients $q_{k,n}^{(0)}$ and $J_{k,n}^{(0)}$ were found in [2,17]. The first order coefficients $J_{k,n}^{(1)}$ (1.34) of the one-interval case were found in [14] in a somewhat different context.

Theorem 1 allows us, in particular, to find the $1/n$-expansion of the covariance

$$\begin{aligned}\mathcal{D}_n(z_1, z_2) \equiv \mathbf{E}\left\{\frac{1}{n}\mathrm{Tr}(z_1 - M)^{-1}\frac{1}{n}\mathrm{Tr}(z_2 - M)^{-1}\right\}\\ - \mathbf{E}\left\{\frac{1}{n}\mathrm{Tr}(z_1 - M)^{-1}\right\}\mathbf{E}\left\{\frac{1}{n}\mathrm{Tr}(z_2 - M)^{-1}\right\},\end{aligned} \tag{1.39}$$

which is important in a number of questions of the random matrix theory and of its applications.

Here and below the symbol $\mathbf{E}\{\ldots\}$ denotes the expectation with respect to measure (1.1)–(1.5).

In the paper [24] it is proven that for any V satisfying (1.2)–(1.3) we have the bound

$$|\mathcal{D}_n(z_1, z_2)| \leq \frac{\mathrm{const}}{n^2(\Im z_1)^2(\Im z_2)^2}.$$

Hence, the $1/n$-expansion of $\mathcal{D}_n(z_1, z_2)$ has the form

$$\mathcal{D}_n(z_1, z_2) = \sum_{j=2}^{\infty} d_n^{(k)}(z_1, z_2)n^{-j} + o(n^{-p}), \quad n \to \infty \tag{1.40}$$

in which the leading term is of the order n^{-2}.

Theorem 1 implies

Corollary 1. *Under the conditions of Theorem 1 we have:*
in the case (i) the n*-independent*

$$d^{(2)}(z_1, z_2) = -\frac{1}{2(z_1 - z_2)^2}\left(1 + \frac{a^2 - z_1 z_2}{X(z_1)X(z_2)}\right), \tag{1.41}$$

where $X(z)$ *is defined in the first line of (1.14);*
in the case (ii) the 2-periodic in n

$$d_n^{(2)}(z_1, z_2) = -\frac{1}{2(z_1 - z_2)^2}\left(1 + \frac{(a^2 - z_1 z_2)(b^2 - z_1 z_2)}{X(z_1)X(z_2)}\right) - \frac{(-1)^n ab}{2X(z_1)X(z_2)}, \tag{1.42}$$

where $X(z)$ *is defined in the second line of (1.14).*

Remark 4. The covariance $\mathcal{D}_n(z_1, z_2)$ of the traces of the resolvent is of considerable interest in the random matrix theory since the beginning of the 90s, when its study was motivated by matrix models of quantum field theory [1,3–5,9,10] and later by solid state theory (see review [6] and references therein). Initially only the one interval case was studied but later the many interval case was also analyzed. In particular, in [3,1] a version of the large-n expansion procedure was proposed. In the case (ii) of the two-interval symmetric potential the procedure leads to an expression for the leading term amplitude $d_n^{(2)}(z_1, z_2)$ that does not depend on n and contains elliptic integrals, while our expression (1.42) is 2-periodic in n and contains only elementary functions. By using recent results of paper [13] on the asymptotic form of the leading term of orthogonal polynomials (1.22)–(1.23) and our formula (2.78) below for the covariance $\mathcal{D}_n(z_1, z_2)$, it can be shown that in the general case of a two-interval non-symmetric potential the leading term amplitude $d_n^{(2)}(z_1, z_2)$ is quasi-periodic in n and contains Jacobi elliptic functions that disappear when one passes to a two-interval symmetric potential. Moreover, by using the same results, it can be shown that in the case of a potential leading to a p-interval support of the density of states the amplitude $d_n^{(2)}(z_1, z_2)$ is a quasi-periodic function. Its frequency module contains generically $p-1$ incommensurable frequencies (but can reduce to a p-periodic function in some special cases [11]), and its form includes the Riemann θ-function of $p-1$ variables. The frequencies are determined by the density of states, and the θ-function are determined by the endpoints of the support of the density of states of the ensemble.

Remark 5. Formulas (1.41) and (1.42) for the leading terms amplitude $d^{(2)}(z_1, z_2)$ of the covariance $\mathcal{D}_n(z_1, z_2)$ depend on the ensemble only via the number of intervals of the IDS support and via the endpoints of the support. This is why this property of the covariance is often referred to as the long-range universality [10] in contradistinction with the short range (or microscopic) universality that manifests itself in $1/n$ - neighborhoods of the interior points of σ and is valid independently of the number of connected components of σ (see e.g. papers [13,24]). Thus under conditions of these papers all the unitary invariant ensembles belong to the same short range universality class. On the other hand, since according to (1.41) and (1.42) the leading terms of the covariance $\mathcal{D}_n(z_1, z_2)$ are different in the one and in the two-interval cases, the long range universality classes depend on the number of intervals of the IDS support and on its endpoints.

Corollary 2. *Under the conditions of Theorem 1 we have the following expressions for the weak limits of squares of the orthonormalized functions $\psi_k^{(n)}(\lambda)$ with $|k-n| \le N(n)$:*

$$w - \lim_{n\to\infty} \left(\psi_k^{(n)}(\lambda)\right)^2 = \frac{\chi_\sigma(\lambda)}{\pi X_+(\lambda)} \begin{cases} 1, & \textit{in case } \text{(i)}, \\ \lambda, & \textit{in case } \text{(ii)}, \end{cases} \tag{1.43}$$

where $X_+(\lambda)$ is defined in (1.11).

The proofs of these assertions will be given in the next section.

2. Proofs of Main Results

Proof of Theorem 1. We introduce an eigenvalue distribution which is more general than (1.17), making different the number of the variable and the large parameter in front of

V in the exponent of the r.h.s of (1.17):

$$p_{k,n}(\lambda_1, \dots \lambda_k) = Z_{k,n}^{-1} \prod_{1 \le j < m \le k} (\lambda_j - \lambda_m)^2 \exp\left\{ -n \sum_{j=1}^{k} V(\lambda_j) \right\}, \tag{2.1}$$

where $Z_{k,n}$ is the normalizing factor. For $k = n$ this probability distribution density coincides with (1.17). Let

$$\tilde{\rho}_{k,n}(\lambda_1) = \int d\lambda_2 \dots d\lambda_k \, p_{k,n}(\lambda_1, \dots \lambda_k), \tag{2.2}$$

$$\tilde{\rho}_{k,n}(\lambda_1, \lambda_2) = \int d\lambda_3 \dots d\lambda_k \, p_{k,n}(\lambda_1, \dots \lambda_k) \tag{2.3}$$

be the first and the second marginal densities of (2.1). By standard arguments [20, 7] we have

$$\begin{aligned} \tilde{\rho}_{k,n}(\lambda) &= \tilde{K}_{k,n}(\lambda, \lambda), \\ \tilde{\rho}_{k,n}(\lambda, \mu) &= \frac{k}{k-1}[\tilde{K}_{k,n}(\lambda, \lambda)\tilde{K}_{k,n}(\mu, \mu) - \tilde{K}_{k,n}^2(\lambda, \mu)], \end{aligned} \tag{2.4}$$

where

$$\tilde{K}_{k,n}(\lambda, \mu) = k^{-1} \sum_{l=1}^{k} \psi_l^{(n)}(\lambda)\psi_l^{(n)}(\mu), \tag{2.5}$$

and $\psi_l^{(n)}(\lambda)$ is defined by (1.21). We will use the notations

$$\begin{aligned} K_{k,n}(\lambda, \mu) &\equiv n^{-1} \sum_{l=1}^{k} \psi_l^{(n)}(\lambda)\psi_l^{(n)}(\mu) = \frac{k}{n}\tilde{K}_{k,n}(\lambda, \mu), \\ \rho_{k,n}(\lambda) &\equiv K_{k,n}(\lambda, \lambda) = \frac{k}{n}\tilde{\rho}_{k,n}(\lambda). \end{aligned} \tag{2.6}$$

Consider now the quantity $\mathbf{E}_k\left\{\frac{V'(\lambda_1)}{z - \lambda_1}\right\}$ for $z, \Im z \neq 0$, where $\mathbf{E}_k\{\dots\}$ denotes the expectation with respect to the probability distribution (2.1). It is well defined in view of condition (1.4) above. It is easy to find that

$$\mathbf{E}_k\left\{\frac{V'(\lambda_1)}{z - \lambda_1}\right\} = \int \frac{V'(\lambda)\tilde{\rho}_{k,n}(\lambda)}{z - \lambda} d\lambda. \tag{2.7}$$

On the other hand, integrating by parts the r.h.s. in (2.7) and using (2.3), we obtain that

$$\mathbf{E}_k\left\{\frac{V'(\lambda_1)}{z - \lambda_1}\right\} = \frac{1}{n}\int \frac{\tilde{\rho}_{k,n}(\lambda)}{(z - \lambda)^2} d\lambda + 2\frac{k-1}{n}\int \frac{\tilde{\rho}_{k,n}(\lambda, \mu)}{(z - \lambda)(\lambda - \mu)} d\lambda d\mu.$$

Combining these two expressions, we come to the identity

$$\int \frac{V'(\lambda)\tilde{\rho}_{k,n}(\lambda)}{z - \lambda} d\lambda = \frac{1}{n}\int \frac{\tilde{\rho}_{k,n}(\lambda)}{(z - \lambda)^2} d\lambda + 2\frac{k-1}{n}\int \frac{\tilde{\rho}_{k,n}(\lambda, \mu)}{(z - \lambda)(\lambda - \mu)} d\lambda d\mu, \tag{2.8}$$

The symmetry property $\tilde\rho_{k,n}(\lambda,\mu)=\tilde\rho_{k,n}(\mu,\lambda)$ of (2.3) implies

$$\int \frac{\tilde\rho_{k,n}(\lambda,\mu)}{(z-\lambda)(\lambda-\mu)}d\lambda d\mu = -\int \frac{\tilde\rho_{k,n}(\lambda,\mu)}{(z-\mu)(\lambda-\mu)}d\lambda d\mu.$$

This allows us to rewrite (2.8) in the form

$$\int \frac{V'(\lambda)\tilde\rho_{k,n}(\lambda)}{z-\lambda}d\lambda = \frac{1}{n}\int \frac{\tilde\rho_{k,n}(\lambda)}{(z-\lambda)^2}d\lambda + \frac{k-1}{n}\int \frac{\tilde\rho_{k,n}(\lambda,\mu)}{(z-\lambda)(z-\mu)}d\lambda d\mu. \tag{2.9}$$

Now, by using (2.4)–(2.6), we can rewrite (2.9) as

$$\begin{aligned}\int \frac{V'(\lambda)\rho_{k,n}(\lambda)}{z-\lambda}d\lambda = n^{-1}\int \frac{\rho_{k,n}(\lambda)}{(z-\lambda)^2}d\lambda \\ +\int \frac{\rho_{k,n}(\lambda)\rho_{k,n}(\mu)-(K_{k,n}(\lambda,\mu))^2}{(z-\lambda)(z-\mu)}d\lambda d\mu.\end{aligned} \tag{2.10}$$

This relation is a version of the well known loop equation of the matrix models of the Quantum Field Theory [15].

We will use also

Proposition 2. *Consider any unitary invariant ensemble of the form (1.1)–(1.5) and assume that $V(\lambda)$ possess two bounded derivatives in some neighborhood of the support σ of the density of states ρ and that $\rho(\lambda)$ satisfies Condition C2. Denote by σ_ε the ε-neighborhood of σ for some $\varepsilon>0$. Then there exist n-independent quantities C, $C_0,\varepsilon_0>0$ such that for any positive n-independent $\varepsilon<\varepsilon_0$ there exists $\varepsilon_1>0$ such that for any integer k satisfying inequality $\frac{|k-n|}{n}\le\varepsilon_1$ we have the bounds*

$$\int_{\mathbf{R}\setminus\sigma_\varepsilon}\rho_{k,n}(\lambda)d\lambda\le e^{-nC\varepsilon},\quad \int_{\mathbf{R}\setminus\sigma_\varepsilon}(\psi_k^{(n)}(\lambda))^2d\lambda\le e^{-nC\varepsilon}. \tag{2.11}$$

Remark 6. The proof of Proposition 2, given in the next section, does not use the fact that ensemble (1.1)–(1.5) consists of Hermitian matrices. Therefore Proposition 2 is valid also for real symmetric and quaternion real matrices, i.e. for orthogonal and symplectic ensembles, satisfying (1.2), (1.3), and Condition C2.

Let us fix now a sufficiently small ε such that $\sigma_\varepsilon\subset\mathbf{D}$ and all the zeros of the function $P(z)$ are outside of σ_ε. Then (2.11) allows us to replace the integrals over the whole line by the integrals over σ_ε in (2.10). Therefore, denoting

$$\begin{aligned}g_{k,n}(z)\equiv\int_{\sigma_\varepsilon}\frac{\rho_{k,n}(\lambda)d\lambda}{z-\lambda},\quad R_{j,m}(z)\equiv\int_{\sigma_\varepsilon}\frac{\psi_j^{(n)}(\lambda)\psi_m^{(n)}(\lambda)d\lambda}{z-\lambda},\\ R'_{j,m}(z)\equiv-\int_{\sigma_\varepsilon}\frac{\psi_j^{(n)}(\lambda)\psi_m^{(n)}(\lambda)d\lambda}{(z-\lambda)^2},\quad \tilde V(z,\zeta)\equiv\frac{V'(\zeta)}{z-\zeta},\end{aligned} \tag{2.12}$$

we get from (2.10):

$$\begin{aligned}(g_{k,n}(z))^2-\int_{\sigma_\varepsilon}\tilde V(z,\lambda)\rho_{k,n}(\lambda)d\lambda \\ -\frac{1}{n^2}\sum_{m=1}^{k}R'_{m,m}(z)-\frac{1}{n^2}\sum_{m,j=1}^{k}R^2_{m,j}(z)=e_n(z),\end{aligned} \tag{2.13}$$

where $e_n(z)$ is the remainder function which appears because of our replacement of the integrals over the whole line by the integrals over σ_ε. Note that since the l.h.s. of (2.13) is an analytic function in $\mathbf{C} \setminus \sigma_\varepsilon$, $e_n(z)$ is also analytic in $\mathbf{C} \setminus \sigma_\varepsilon$, and admits the bound:

$$|e_n(z)| \leq \frac{C_0}{|\delta_\varepsilon(z)|^l}, \tag{2.14}$$

where

$$\delta_\varepsilon(z) \equiv \text{dist}\{z, \sigma_\varepsilon\} \tag{2.15}$$

and $l = 2$. Besides, it follows from (2.11) that

$$e_n(z) \leq \frac{C_1 e^{-nC_2}}{|\Im z|^2 |\delta_\varepsilon(z)|^{l'}} \tag{2.16}$$

with $l' = 0$.

We will denote below by $\{e_n(z)\}_{n=1}^\infty$ sequences of functions (may be different in different formulas) which are analytic everywhere in $\mathbf{C} \setminus \sigma_\varepsilon$ and satisfy the estimates (2.14) and (2.16) with some nonnegative l, l' and some positive n-independent C's.

According to our conditions $\tilde{V}(z, \zeta)$ in (2.12) is analytic with respect to ζ inside $\mathbf{D}$, except for the point $\zeta = z$. Hence, we can write that

$$\begin{aligned}\int_{\sigma_\varepsilon} \tilde{V}(z, \lambda)\rho_{k,n}(\lambda)d\lambda &= \frac{1}{2\pi i}\int_{\sigma_\varepsilon} d\lambda \int_L d\zeta \tilde{V}(z, \zeta)\frac{\rho_{k,n}(\lambda)}{\zeta - \lambda} \\ &= \frac{1}{2\pi i}\int_L d\zeta \tilde{V}(z, \zeta) g_{k,n}(\zeta),\end{aligned} \tag{2.17}$$

where $L \subset \mathbf{D}$ is an arbitrary closed contour which contains σ_ε and does not contain z. This allows us to rewrite (2.13) as

$$\begin{aligned}(g_{k,n}(z))^2 - \frac{1}{2\pi i}\int_L \tilde{V}(z, \zeta) g_{k,n}(\zeta) d\zeta - \frac{1}{n^2}\sum_{m=1}^{k} R'_{m,m}(z) \\ - \frac{1}{n^2}\sum_{m,j=1}^{k} R_{m,j}^2(z) = e_n(z).\end{aligned} \tag{2.18}$$

Now, subtracting from (2.18) the relation obtained from (2.18) by the replacement $k \to (k-1)$, we obtain:

$$\begin{aligned}2R_{k,k}(z) g_{k-1,n}(z) - \frac{1}{2\pi i}\int_L \tilde{V}(z, \zeta) R_{k,k}(\zeta) d\zeta \\ - \frac{1}{n} R'_{k,k}(z) - \frac{2}{n}\sum_{j=1}^{k-1} R_{k,j}^2(z) = e_n(z).\end{aligned} \tag{2.19}$$

Relations (2.18) and (2.19) are our main technical tools in constructing the $1/n$ expansion given in the theorem. We will consider (2.18) and (2.19) as a system of equations with respect to the functions $g_{k,n}(z)$ and $R_{j,m}(z)$ and solve them by iterations in $1/n$.

We will need two more facts on ensembles (1.1)–(1.5).

(a) The function $g_{k,n}(z)$ from (2.12) and $g(z)$ from (1.12) are related as

$$|g_{k,n}(z) - g(z)| \le \text{const}\frac{\log^{1/2} n}{\sqrt{n}\delta_\varepsilon^2(z)} + \frac{|k-n|}{n\delta_\varepsilon(z)}. \tag{2.20}$$

This relation follows from (2.12), (2.6), (2.4), and from the bound valid for any function $\phi(\mu)$, which grows not faster than $e^{bV(\mu)}$, $b > 0$ as $|\mu| \to \infty$,

$$\left|\int \phi(\mu)\rho_n(\mu)d\mu - \int \phi(\mu)\rho(\mu)d\mu\right| \le \text{const}||\phi'||_2^{1/2}||\phi||_2^{1/2}n^{-1/2}\log^{1/2} n, \quad (2.21)$$

where the symbol $||\dots||_2$ denotes the L_2-norm on a compact set of $\mathbf{R}$ containing σ_ε (the bound was proved in [8], Lemma 4, see also [24]).

(b)

$$g^2(z) - V'(z)g(z) + Q(z) = 0, \quad z \in \mathbf{D},\ \Im z \neq 0, \tag{2.22}$$

$$Q(z) = \frac{1}{2\pi i}\int_L Q(z,\zeta)g(\zeta)d\zeta = \int_\sigma \frac{V'(z) - V'(\lambda)}{z-\lambda}\rho(\lambda)d\lambda, \tag{2.23}$$

and $Q(z,\zeta)$ is defined by (1.16). The relations follow from (2.20), and identity (2.10) for $n = k$. Indeed, in view of (2.4) the r.h.s. of (2.10) is

$$g_n^2 + \mathbf{E}\left\{\left[n^{-1}\sum_{l=1}^n (z-\lambda_l)^{-1} - \mathbf{E}\left\{n^{-1}\sum_{l=1}^n (z-\lambda_l)^{-1}\right\}\right]^2\right\}.$$

The second term here is the variance of $n^{-1}\operatorname{Tr}(z-M)^{-1}$, and according to [24], Lemma 3, the variance is of the order $O(n^{-2})$. This and (2.20) imply (2.22).

It follows from the above that the zero order approximation for $g_{k,n}(z)$ coincides with $g(z)$.

To find the zero order approximations for $R_{k,k}(z)$ for $|k-n| \le N(n)$, where $N(n)$ is defined in (1.27), let us note that (2.12) leads to the bounds

$$|R'_{k,k}(z)|,\ |\sum_{j=1}^{k-1} R_{k,j}^2(z)| \le \frac{\text{const}}{\delta_\varepsilon^2(z)}.$$

The first bound follows from the definition of $R_{k,i}(z)$ in (2.12). To prove the second bound we view $R_{k,i}(z)$ of (2.12) as the generalized Fourier coefficients of the function $\chi_\varepsilon(\lambda)\psi_k^{(n)}(\lambda)(z-\lambda)^{-1}$ with respect to the orthonormal system $\{\psi_l^{(n)}(\lambda)\}_{l=1}^\infty$. Then the Bessel inequality gives us the second bound.

These bounds imply that the last two terms in the l.h.s. of (2.19) have the order n^{-1}. Hence, the zero order equations for $R_{kk}(z)$ have the form

$$2g(z)R_{k,k}(z) = \frac{1}{2\pi i}\int_L d\zeta\,\tilde{V}(z,\zeta)R_{k,k}(\zeta) - r_{k,n}^{(0,R)}(z) + e_n(z), \tag{2.24}$$

where the remainder

$$r_{k,n}^{(0,R)}(z) \equiv -\frac{1}{n}R'_{k,k}(z) - \frac{2}{n}\sum_{j=1}^{k-1} R_{k,j}^2(z) + 2R_{k,k}(z)(g_{k-1,n}(z) - g(z)) \to 0, \quad n \to \infty, \tag{2.25}$$

is analytic in $\mathbf{C} \setminus \sigma_\varepsilon$ and tends to zero uniformly on any compact set for which $\mathrm{dist}\,(z, \sigma_\varepsilon) \geq \mathrm{d} > 0$. Besides, since by definition (1.21)

$$\int (\psi_k^{(n)})^2(\lambda)d\lambda = 1,$$

we have from (2.11), that

$$R_{k,k}(z) = \frac{1}{z}(1 + O(\frac{1}{z})) + e_n(z), \quad z \to \infty. \tag{2.26}$$

Equation (2.24) was already considered in [2]. However we will use here a bit different way to analyze the equation, which is based on the following lemma:

Lemma 1. *Consider the equation*

$$2g(z)R(z) - \frac{1}{2\pi i}\int_L d\zeta \tilde{V}(z,\zeta)R(\zeta) = 0, \quad z \in \mathbf{D} \setminus \sigma_\varepsilon, \tag{2.27}$$

where $V(\tilde{z}, \zeta)$ is defined in (2.12), and a closed contour $L \in \mathbf{D}$ contains σ_e and does not contain the point z. Set for $z \notin \sigma$,

$$\Psi(z) = \begin{cases} X^{-1}(z), & \textit{in the case } \text{(i)}, \\ zX^{-1}(z), & \textit{in the case } \text{(ii)}, \end{cases} \tag{2.28}$$

where $X(z)$ is defined by (1.14). Then the following statements are valid under the conditions of Theorem 1:

1. *In the case* (i) *Eq. (2.27) has the unique solution $R(z) = \Psi(z)$ in the class of functions analytic in $\mathbf{C} \setminus \sigma_\varepsilon$ and behaving as*

$$R(z) = z^{-1}(1 + o(1)), \quad z \to \infty. \tag{2.29}$$

In the case (ii) *Eq. (2.27) has the unique solution $R(z) = \Psi(z)$ in the class (2.29), under the additional symmetry condition $R(-z) = -R(z)$.*

2. *In both cases Eq. (2.27) has no solutions in the class of functions $R(z)$ analytic in $\mathbf{C} \setminus \sigma_\varepsilon$ and satisfying the condition*

$$\lim_{|z|\to\infty} |z^2 R(z)| \leq \text{ const } < \infty. \tag{2.30}$$

3. *For any analytic in* $\mathbf{C}\setminus\sigma_\varepsilon$ *function* $F(z)$, *satisfying condition (2.30) and even in the case* (ii), *the inhomogeneous equation*

$$2g(z)R(z) = \frac{1}{2\pi i}\int_L d\zeta\,\tilde V(z,\zeta)R(\zeta) - F(z) \tag{2.31}$$

has the unique solution of the form

$$R(z) = \frac{1}{2\pi i X(z)}\int_L d\zeta\,\frac{F(\zeta)}{P(\zeta)(z-\zeta)}, \tag{2.32}$$

in the class of functions analytic in $\mathbf{C}\setminus\sigma_\varepsilon$, *satisfying condition (2.30) and odd in the case* (ii).
Here $P(z)$ *is defined by (1.15) and a closed contour* L *should be taken sufficiently close to* σ, *to have* z *and all zeros of* $P(z)$ *outside of* L. *In particular, in the case* (ii) *the contour consists of two components, encircling each interval of the support.*

The proof of the lemma will be given in the next section.

Omitting in (2.24) the error terms, we deduce from the obtained homogeneous equation and from (2.26) on the basis of Assertion 1 of Lemma 1 that the zero order approximation $R^{(0)}_{k,k}(z)$ of $R_{k,k}(z)$ is $\Psi(z)$ from (2.28). Moreover, the difference $R_{k,k}(z)-\Psi(z)$ decays at infinity as z^{-2} at least, and the error terms in the r.h.s. of (2.24) decays also as z^{-2}, as $z\to\infty$. Thus on the basis of Assertion 3 of the lemma we can write that

$$R_{k,k}(z) = \Psi(z) + \tilde r^{(0,R)}_{k,n}(z) + e_n(z). \tag{2.33}$$

Here $\tilde r^{(0,R)}_{k,n}(z)$ is obtained from formula (2.32) with $F(z) = r^{(0,R)}_{k,n}(z)$ given by (2.25)). Using the fact that $|r^{(0,R)}_{k,n}(z)|\to 0$ as $|z|\to\infty$ and that $P(z)$ has no zeros on L we obtain the bound

$$\begin{aligned} |\tilde r^{(0,R)}_{k,n}(z)| &\le \left|\frac{1}{2\pi i P(z)X(z)}\int_L d\zeta\,\frac{r^{(0,R)}_{k,n}(\zeta)}{(z-\zeta)}\right| \\ &\quad + \left|\frac{1}{2\pi i X(z)}\int_L d\zeta\, r^{(0,R)}_{k,n}(\zeta)\frac{P^{-1}(\zeta)-P^{-1}(z)}{(z-\zeta)}\right| \\ &\le \frac{\text{const}}{|X(z)|}\left(\left|r^{(0,R)}_{k,n}(z)\right| + \max_{\zeta\in L}\left|r^{(0,R)}_{k,n}(\zeta)\right|\right)\to 0,\quad n\to\infty. \end{aligned} \tag{2.34}$$

Thus, for all k such that $|k-n|\le N(n)$, where $N(n)$ is given in (1.27) for $m=0$, we have

$$R^{(0)}_{k,k} \equiv \lim_{n\to\infty} R_{k,k}(z) = \Psi(z). \tag{2.35}$$

We have also the relations following from (1.21), (1.24), (2.11) and (2.12):

$$\begin{aligned} q_k &= \int \lambda\psi_k^2(\lambda)d\lambda = \frac{1}{2\pi i}\int_L \zeta R_{k,k}(\zeta)d\zeta + O(e^{-nC\varepsilon}), \\ q_k^2 + J_k^2 + J_{k-1}^2 &= \int \lambda^2\psi_k^2(\lambda)d\lambda = \frac{1}{2\pi i}\int_L \zeta^2 R_{k,k}(\zeta)d\zeta + O(e^{-nC\varepsilon}), \\ (q_k^2 + J_k^2 + J_{k-1}^2)^2 &+ (q_k+q_{k+1})^2J_k^2 + (q_k+q_{k-1})^2J_{k-1}^2 + J_k^2J_{k+1}^2 \\ +J_{k-1}^2J_{k-2}^2 &= \int \lambda^4\psi_k^2(\lambda)d\lambda = \frac{1}{2\pi i}\int_L \zeta^4 R_{k,k}(\zeta)d\zeta + O(e^{-nC\varepsilon}). \end{aligned} \tag{2.36}$$

In what follows we omit the subindex n in the coefficients $q_{k,n}^{(j)}$ and $J_{k,n}^{(j)}$, introduced in (1.28).

By using (2.35), and (2.28) for the case (i), we find from the first of the above relations that the zero order term $q_k^{(0)}$ is zero. Then, combining the second relation of (2.36) for $k, k-1$, and $k+1$ and the third relation of (2.36), we find that $J_k^{(0)} = a/2$. In the case (ii) the same scheme carried out for even and odd k leads to the coefficients $J_k^{(0)}$ of (1.35). In other words we have proved that in the zero order in $1/n$ the coefficients of the Jacobi matrix $J(n)$ defined in (1.25) do not depend on k, $|k-n| \leq N(n)$ in the case (i) of a one interval support of the density of states and are 2-periodic functions of k in the case (ii) of a two interval symmetric support.

To find the first order terms for these coefficients, we will study the first order versions of Eqs. (2.18). Note first that we have the bound

$$\frac{1}{n}\left|\sum_{j=1}^{k}\left[-R'(z)_{j,j} - \sum_{j,m=1}^{k} R_{j,m}^2(z)\right]\right| \leq \frac{\text{const}}{n\delta_\varepsilon^4(z)} + |e_n(z)|, \tag{2.37}$$

where const does not depend on n, z. Indeed, by using the orthonormality of system (1.21) we can write the l.h.s. as

$$\frac{n}{2}\int_{\sigma_\varepsilon} d\lambda \int_{\sigma_\varepsilon} d\mu(\phi(\lambda)-\phi(\mu))^2 K_{k,n}^2(\lambda,\mu) + n\int_{\sigma_\varepsilon} d\lambda \int_{\mathbf{R}\setminus\sigma_\varepsilon} d\mu\phi^2(\lambda)K_{k,n}^2(\lambda,\mu),$$

where $\phi(\lambda) = (z-\lambda)^{-1}$ and $K_{k,n}(\lambda,\mu)$ is defined in (2.6). According to Lemma 3 of [24] the first term here is bounded by const $\cdot \sup|\phi'(\lambda)|^2/n \leq \text{const}/n\delta_\varepsilon^4(z)$, and according to Proposition 2, the second term is $e_n(z)$.

We conclude that the first order equation for the function

$$g_{k,n}^{(1)}(z) \equiv n(g_{k,n}(z) - g(z)) \tag{2.38}$$

has the form

$$2g(z)g_{k,n}^{(1)}(z) = \frac{1}{2\pi i}\int V(z,\zeta)g_{n,k}^{(1)}(\zeta)d\zeta - r_{k,n}^{(1,g)}(z) + e_n(z), \tag{2.39}$$

with

$$\begin{aligned} r_{k,n}^{(1,g)}(z) &\equiv \frac{1}{n}(g_{k,n}^{(1)}(z))^2 + \frac{1}{n}\sum_{j=1}^{k}\left[-R'(z)_{j,j} - \sum_{m=1}^{k} R_{j,m}^2(z)\right] \\ &\equiv \frac{1}{n}(g_{k,n}^{(1)}(z))^2 + \overline{r}_{k,n}^{(1,g)}(z), \quad \left|\overline{r}_{k,n}^{(1,g)}(z)\right| \leq \frac{\text{const}}{n\delta_\varepsilon^4(z)}. \end{aligned} \tag{2.40}$$

Besides, we have the normalization condition

$$g_{k,n}^{(1)}(z) = (k-n)z^{-1}\left(1 + O\left(\frac{1}{z}\right)\right) + e_n(z), \; z\to\infty, \; |k-n| \leq N(n), \tag{2.41}$$

which follows from Definition (2.12) of the function $g_{k,n}(z)$. Then, according to Lemma 1, we get

$$g_{k,n}^{(1)}(z) = (k-n)\Psi(z) + \tilde{r}_{k,n}^{(1,g)}(z) + e_n(z), \tag{2.42}$$

where the remainder $\tilde{r}^{(1,g)}_{k,n}(z)$ has the form

$$\tilde{r}^{(1,g)}_{k,n}(z) = \frac{1}{2\pi i X(z)} \int_L \frac{n^{-1}(g^{(1)}_{k,n}(\zeta))^2 + \overline{r}^{(1,g)}_{k,n}(\zeta)}{P(\zeta)(z-\zeta)} d\zeta. \tag{2.43}$$

Thus, denoting

$$m^{(1)}_{k,n}(d) \equiv \max_{\{z:\delta_\varepsilon(z)\geq d\}} |g^{(1)}_{k,n}(z)|,$$

where d is a positive constant, we obtain from relations (2.42) and (2.43) the inequality

$$m^{(1)}_{k,n}(d) \leq \frac{|k-n|}{d^{1/2}} + C\left(\frac{(m^{(1)}_{k,n}(d))^2}{nd^{3/2}} + \frac{1}{nd^{9/2}}\right),$$

where C is independent of n, k, and d. This inequality implies that either

$$m^{(1)}_{k,n}(d) \leq \frac{2|k-n|}{d^{1/2}}, \quad \text{or} \quad m^{(1)}_{k,n}(d) \geq nd^{3/2}C^{-1} + O(1).$$

But the second inequality here cannot be true, because it was proved above that

$$n^{-1}m^{(1)}_{k,n}(d) = \max_{\{z:\delta_\varepsilon(z)\geq d\}} |g_{k,n}(z) - g(z)| \to 0$$

for any k such that $|k-n| = N(n)$, where $N(n)$ is given in (1.27) for $m = 0$. Hence in view of (2.43) we get that for $\{z : \delta_\varepsilon(z) \geq d\}$,

$$|\tilde{r}^{(1,g)}_{k,n}(z)| \leq \text{const}\left(\frac{|k-n|^2}{nd} + \frac{1}{nd^{9/4}}\right). \tag{2.44}$$

Substituting now representation (2.42) in the r.h.s. of (2.43), and using bound (2.44), we get finally

$$\tilde{r}^{(1,g)}_{k,n}(z) = \frac{(k-n)^2}{n} Y(z) + O(|k-n|^3 n^{-2} d^{-5/2}) + O((nd^5)^{-1}), \tag{2.45}$$

where

$$\begin{aligned} Y(z) &\equiv \frac{1}{2\pi i X(z)} \int_L d\zeta \frac{\Psi^2(\zeta)}{P(\zeta)(z-\zeta)} \\ &= \frac{1}{X(z)} \begin{cases} \dfrac{1}{2a}\left(\dfrac{1}{P(a)(z-a)} - \dfrac{1}{P(-a)(z+a)}\right), & \text{(i)}, \\ \dfrac{1}{(a^2-b^2)}\left(\dfrac{az}{P(a)(z^2-a^2)} - \dfrac{bz}{P(b)(z^2-b^2)}\right), & \text{(ii)}. \end{cases} \end{aligned} \tag{2.46}$$

We have obtained the first order term in the $1/n$-expansion for $g_{n,k}(z)$.

Now we need a lemma that will allow us to replace $R_{k,j}(z)$ in (2.18), (2.19) by a certain simpler expression constructed from the coefficients $q^{(j)}_{k,n}$, $J^{(j)}_{k,n}$, $j = 0, \ldots, p$ found during the previous p steps of our expansion process and to estimate the error of this replacement.

Lemma 2. *Take* $\tilde{N}(n) = [\log^2 n]$ *and let* $N_1(n)$ *be such that*

$$N_1(n)n^{-1/(p+1)} \to 0, \ (N_1(n))^{-1}\tilde{N}(n) \to 0, \quad \text{as } n \to \infty. \tag{2.47}$$

Assume that for any $k : |k-n| \le N_1(n)$ *we have found the coefficients* $q_k^{(0)}, \dots, q_k^{(p)}$, $J_k^{(0)}, \dots, J_k^{(p)}$, *satisfying bound (1.29), and such that (1.28) is fulfilled for* $m = p$. *Here and below we omit the subindex* n *in the coefficients* $q_{k,n}^{(j)}$, $J_{k,n}^{(j)}$ *of the asymptotic formula (1.28) of Theorem 1.*

For any s *such that* $|s| \le 2/n$ *consider the* $(2N_1+1)$*-periodic symmetric Jacobi matrix* $\tilde{J}^{(p)}(s)$ *defined by the entries*

$$\tilde{J}_{k,k}^{(p)} \equiv \tilde{q}_k^{(p)} = \sum_{j=0}^{p} s^j q_k^{(j)}, \quad \tilde{J}_{k,k+1}^{(p)} \equiv \tilde{J}_k^{(p)} = \sum_{j=0}^{p} s^j J_k^{(j)}, \quad |k-n| \le N_1(n). \tag{2.48}$$

Denote by $\tilde{R}^{(p)}(z,s)$ *the resolvent of* $\tilde{J}^{(p)}(s)$, *and set*

$$R^{(j)}(z) \equiv \frac{1}{j!}\frac{\partial^j}{\partial s^j}\tilde{R}^{(p)}(z,s)|_{s=0}, \quad S^{(p)}(z) \equiv \sum_{j=0}^{p} n^{-j} R^{(j)}. \tag{2.49}$$

Then for any $L > 0$ *there exist positive* n*-independent quantities* C_1 *and* C_2 *such that for any* k *satisfying the inequality:*

$$|k-n| \le N_1 - 2\tilde{N} \equiv N_2(n), \tag{2.50}$$

and for any $z \notin \sigma_\varepsilon$, $|z| < L$,

$$\begin{aligned}\big|R_{k,k}(z) - S_{k,k}^{(p)}(z)\big|, \big| - R'_{k,k}(z) - (S^{(p)} \cdot S^{(p)})_{k,k}(z)\big| \\ \le \frac{2\varepsilon_n^{(p)}}{\delta_\varepsilon^2(z)n^p} + \frac{C_1 N_1^{p+1}}{\delta_\varepsilon^{p+1}(z)n^{p+1}} + \frac{e^{-C_2\delta_\varepsilon(z)\tilde{N}}}{\delta_\varepsilon(z)|\Im z|^2},\end{aligned} \tag{2.51}$$

$$\Big|\sum_{m=1}^{k} R_{k,m}^2(z) - \sum_{m=1}^{k}(S_{k,m}^{(p)}(z))^2\Big| \le \frac{2\varepsilon_n^{(p)}}{\delta_\varepsilon^2(z)n^p} + \frac{C_1 N_1^{p+1}}{\delta_\varepsilon^{p+1}(z)n^{p+1}} + \frac{e^{-C_2\delta_\varepsilon(z)\tilde{N}}}{\delta_\varepsilon(z)|\Im z|^2}, \tag{2.52}$$

$$\begin{aligned}\Big|\frac{1}{n}\sum_{j=1}^{k}\Big[- R'(z)_{j,j} - \sum_{m=1}^{k} R_{j,m}^2(z)\Big] - \frac{1}{n}\sum_{j=1}^{k}\Big[(S^{(p)} \cdot S^{(p)})_{j,j}(z) - \sum_{m=1}^{k}(S_{j,m}^{(p)}(z))^2\Big]\Big| \\ \le \frac{2\varepsilon_n^{(p)} N_1}{\delta_\varepsilon^2(z)n^{p+1}} + \frac{C_1 N_1^{p+2}}{\delta_\varepsilon^{p+1}(z)n^{p+2}} + \frac{e^{-C_2\delta_\varepsilon(z)\tilde{N}/2}}{|\Im z|^3},\end{aligned} \tag{2.53}$$

where $\delta_\varepsilon(z) \equiv \operatorname{dist}\{z, \sigma_\varepsilon\}$ *and* $\varepsilon_n^{(p)} = o(1)$, $n \to \infty$ *(see (1.30)).*

The proof of the lemma will be given in the next section.

Consider the function

$$R^{(1,n)}_{k,k}(z) \equiv n\left(R_{k,k}(z) - R^{(0)}_{k,k}(z)\right), \tag{2.54}$$

with $R^{(0)}_{k,k}(z)$ defined in (2.35). From (2.19) and (2.42) we get the first order equation for R_{kk}:

$$2g(z)R^{(1,n)}_{k,k}(z) = \frac{1}{2\pi i}\int_L d\zeta\, \tilde{V}(z,\zeta)R^{(1,n)}_{kk}(\zeta) - F^{(1,R)}_k(z) - r^{(1,R)}_{k,n}(z) + e_n(z). \tag{2.55}$$

Here

$$F^{(1,R)}_k(z) \equiv 2R^{(0)}_{k,k}(z)g^{(1)}_{k-1}(z) + (R^{(0)}\cdot R^{(0)})_{k,k}(z) - 2\sum_{j=1}^{k-1}(R^{(0)}_{k,j}(z))^2,$$

$R^{(0)}$ denotes the resolvent of the double infinite Jacobi matrix $J^{(0)}$ of the zero order coefficients $\{J^{(0)}_k\}_{k\in\mathbf{Z}}$, and

$$\begin{aligned} r^{(1,R)}_{k,n}(z) \equiv{}& 2R^{(0)}_{k,k}(z)\tilde{r}^{(1,g)}_{k,n}(z) + \frac{2}{n}R^{(1,n)}_{k,k}(z)g^{(1)}_{k,n}(z)\\ &+\left[-R'_{k,k}(z) - (R^{(0)}\cdot R^{(0)})_{k,k}(z)\right]\\ &-2\left[\sum_{j=1}^{k-1}(R_{k,j}(z))^2 - \sum_{j=1}^{k-1}(R^{(0)}_{k,j}(z))^2\right]. \end{aligned} \tag{2.56}$$

By using the translational symmetry of the resolvent $R^{(0)}$ and the exponential decay of its matrix elements $R^{(0)}_{jm}$ in $|j-m|$, as $|j-m|\to\infty$, it is easy to show that

$$(R^{(0)}\cdot R^{(0)})_{k,k}(z) - 2\sum_{j=1}^{k-1}(R^{(0)}_{k,j}(z))^2 = \begin{cases} (R^{(0)}_{k,k}(z))^2 + e_n(z), & \text{(i)},\\ (R^{(0)}_{k,k}(z))^2 + \dfrac{(J^{(0)}_k)^2 - (J^{(0)}_{k-1})^2}{X^2(z)} + e_n(z), & \text{(ii)}, \end{cases}$$

This relation, and formulas (2.42), and (2.54) imply that

$$F^{(1,R)}_k = \begin{cases} [2(k-n)-1]\,\Psi^2(z), & \text{(i)},\\ [2(k-n)-1]\,\Psi^2(z) \pm \dfrac{(-1)^k ab}{X^2(z)}, & \text{(ii)}, \end{cases}$$

where the sign in the case (ii) corresponds to that in (1.35).

In addition, bound (2.45), and the fact that $n^{-1}R^{(1,n)}_{k,k}(z) \to 0$, as $n\to\infty$ (see formulas (2.54) and (2.33)–(2.35)) imply that the first two terms in the r.h.s. of (2.56) tend to zero as $n\to\infty$. And on the basis of Lemma 2, one can conclude that the last two terms there also vanish as $n\to\infty$. Therefore $r^{(1,R)}_{k,n}(z)\to 0$ as $n\to\infty$. Then on the basis of Lemma 1, and similarly to (2.38)–(2.46) we get for the first order term $R^{(1)}_{k,k}(z)$ all k such that $|k-n| \le N_1(n)$, where $N_1(n)$ is given in (2.47):

$$R^{(1)}_{k,k}(z) = \begin{cases} [2(k-n)-1]Y(z), & \text{(i)}, \\ [2(k-n)-1]\,Y(z) \pm (-1)^k Y^{(\pm)}(z), & \text{(ii)}, \end{cases} \tag{2.57}$$

where $Y(z)$ is defined in (2.46),

$$\begin{aligned} Y^{(\pm)}(z) &\equiv \frac{ab}{2\pi i X(z)} \int_L \frac{d\zeta}{P(\zeta)X^2(\zeta)(z-\zeta)} \\ &= \frac{z}{X(z)(a^2-b^2)}\left(\frac{b}{P(a)(z^2-a^2)} - \frac{a}{P(b)(z^2-b^2)}\right), \end{aligned}$$

and the remainder function $\tilde{r}^{(1,R)}_{k,n}(z)$ is

$$\tilde{r}^{(1,R)}_{k,n}(z) = \begin{cases} \dfrac{2(k-n)[3(k-n)-1]}{n}\tilde{Y}(z) + O\left(\dfrac{|k-n|^4}{n^3}\right) + O\left(\dfrac{1}{n}\right), & \text{(i)}, \\ \dfrac{2(k-n)[3(k-n)-1]}{n}\tilde{Y}(z) \pm 2(-1)^k(k-n)\tilde{Y}^{\pm}(z) & \\ +O\left(\dfrac{|k-n|^4}{n^3}\right) + O\left(\dfrac{1}{n}\right), & \text{(ii)} \end{cases} \tag{2.58}$$

where

$$\begin{aligned} \tilde{Y}(z) &\equiv \frac{1}{2\pi i X(z)} \int_L d\zeta\, \frac{Y(\zeta)\Psi(\zeta)}{P(\zeta)(z-\zeta)}, \\ \tilde{Y}^{\pm}(z) &\equiv \frac{1}{2\pi i X(z)} \int_L d\zeta\, \frac{Y^{\pm}(\zeta)\Psi(\zeta)}{P(\zeta)(z-\zeta)}. \end{aligned} \tag{2.59}$$

Now in the case (ii) we take the first order terms with respect to n^{-1} in Eqs. (2.36) (recall that the diagonal coefficients $q^{(0)}_k$ are zero for all k). We obtain the relations

$$2(J^{(0)}_{2q}J^{(1)}_{2q} + J^{(0)}_{2q-1}J^{(1)}_{2q-1}) = \frac{1}{2\pi i}\int_L \zeta^2 R^{(1)}_{2q,2q}(\zeta)d\zeta + r^{(1,J,2)}_{2q},$$

$$\begin{aligned} &4(J^{(0)}_{2q}J^{(1)}_{2q} + J^{(0)}_{2q-1}J^{(1)}_{2q-1})((J^{(0)}_{2q})^2 + (J^{(0)}_{2q-1})^2) \\ &\quad + 2J^{(0)}_{2q}J^{(0)}_{2q-1}(J^{(0)}_{2q-1}J^{(1)}_{2q} + J^{(0)}_{2q}J^{(1)}_{2q+1} + J^{(0)}_{2q}J^{(1)}_{2q-1} + J^{(0)}_{2q-1}J^{(1)}_{2q-2}) \\ &= \frac{1}{2\pi i}\int_L \zeta^4 R^{(1)}_{2q,2q}(\zeta)d\zeta + r^{(1,J,4)}_{2q}, \end{aligned} \tag{2.60}$$

where $k = 2q,\ |k-n| \le N_1(n),\ N_1(n)$ is defined in (2.47) for $p = 0$, and:

$$\begin{aligned} r^{(1,J,2)}_{k,n} &\equiv \int_L \zeta^2 \tilde{r}^{(1,R)}_{k,n}(\zeta)d\zeta \to 0, \quad n \to \infty, \\ r^{(1,J,4)}_{k,n} &\equiv \int_L \zeta^4 \tilde{r}^{(1,R)}_{k,n}(\zeta)d\zeta \to 0, \quad n \to \infty. \end{aligned} \tag{2.61}$$

Consider also the two analogs of the first equation in (2.60) with $2q$ replaced by $2q-1$ and by $2q+1$. These relations and (2.60) comprise a linear system with the unknowns $J^{(1)}_{2q-2}$, $J^{(1)}_{2q-1}$, $J^{(1)}_{2q}$ and $J^{(1)}_{2q+1}$. The system is uniquely soluble for $J^{(0)}_{2q} \neq J^{(0)}_{2q-1}$, and its solution is specified by (1.36), and its remainder terms satisfy the bounds (1.37).

However, for $J^{(0)}_{2q} = J^{(0)}_{2q-1}$ this system is degenerated. Thus, in the case (i) we cannot use the system to find coefficients $J^{(1)}_{k,n}$. In this case we use first identity (2.36) that yields the following relation in the first order:

$$q^{(1)}_k = r^{(1,q,1)}_{k,n} \equiv \int_L \zeta \tilde{r}^{(1,R)}_{k,n}(\zeta) d\zeta.$$

This and (2.57) yield that $q^{(1)}_k = 0$. Furthermore, the first equation in (2.60) for $J^{(0)}_{2q} = J^{(0)}_{2q-1} = a/2$, in view of (2.57) and (2.58), has the form

$$a(J^{(1)}_k + J^{(1)}_{k-1}) = [2(k-n)-1]I^{(i)} + r^{(1,J,2)}_{k,n}, \\ I^{(i)} \equiv \frac{1}{2}\left(\frac{1}{P(a)} + \frac{1}{P(-a)}\right). \tag{2.62}$$

Iterating this relation starting from $k = n$ it is easy to obtain the one-parameter family of solutions

$$aJ^{(1)}_k = (k-n)I^{(i)} - c(-1)^{k-n} + \tilde{r}^{(1,J)}_{k,n}, \tag{2.63}$$

where

$$\tilde{r}^{(1,J)}_{k,n} = \sum_{j=0}^{k-n} (-1)^{k-n-j} r^{(1,J,2)}_{n+j,n}.$$

Substituting expression (2.58) for $\tilde{r}^{(1,R)}_{k,n}(z)$ in (2.61) and using the resulting $r^{(1,J,2)}_{k,n}(z)$ in the last relations, we obtain the bound

$$|\tilde{r}^{(1,J)}_{k,n}| \leq \text{const}\left(\frac{|k-n|^2+1}{n} + \frac{|k-n|^5}{n^3}\right). \tag{2.64}$$

This leads to (1.37) for the case (i), if $|k-n| \leq n^{2/3}$.

To fix the parameter c in (2.63) we use the relation known in random matrix theory as the string equation (see e.g. [15]):

$$J_k \int V'(\lambda)\psi^{(n)}_k(\lambda)\psi^{(n)}_{k+1}(\lambda)d\lambda = \frac{k}{n}.$$

The relation can be easily obtained from the identity

$$\int \left(e^{-nV(\lambda)} p^{(n)}_{k-1}(\lambda) p^{(n)}_k(\lambda)\right)' d\lambda = 0.$$

We use this relation in the form

$$\frac{J_n}{2\pi i}\int_L V'(\zeta)R_{n,n+1}(\zeta)d\zeta = 1 + O(e^{-nC}), \tag{2.65}$$

following from Proposition 2. The first order equation which follows from (2.65) has the form

$$\frac{J_n^{(0)}}{2\pi i}\int_L V'(\zeta)R_{n,n+1}^{(1)}(\zeta)d\zeta + \frac{J_n^{(1)}}{2\pi i}\int_L V'(\zeta)R_{n,n+1}^{(0)}(\zeta)d\zeta = 0.$$

By using (1.34), (2.33), (2.57), and (2.63), we get a linear equation with respect to c:

$$D^{(i)}c - A^{(i)} = 0, \tag{2.66}$$

with

$$\begin{aligned} D^{(i)} &\equiv J_n^{\pm}\int_L V'(\zeta)R_{n,n+1}^{(0)}(\zeta)d\zeta + \frac{a}{2}\int_L V'(\zeta)(R^{(0)}J^{\pm}R^{(0)})_{n,n+1}(\zeta)d\zeta,\\ A^{(i)} &\equiv J_n^{(1*)}\int_L V'(\zeta)R_{n,n+1}^{(0)}(\zeta)d\zeta + \frac{a}{2}\int_L V'(\zeta)(R^{(0)}\cdot J^{(1*)}\cdot R^{(0)})_{n,n+1}(\zeta)d\zeta, \end{aligned} \tag{2.67}$$

where $J^{\pm}$ is the symmetric Jacobi matrix with coefficient $J_k^{\pm} = (-1)^{n-k}$ and $J^{(1*)}$ is the symmetric Jacobi matrix with coefficients defined by (1.34).

Lemma 3. *Under conditions of the theorem $A^{(i)} = 0$, $D^{(i)} \neq 0$ and Eq. (2.66) has the unique solution $c = 0$.*

The proof of this lemma is given in the next section.

By using the lemma we find the first order terms of our expansion in the case (i) given in (1.34).

Now we will prove (1.31) and (1.28) by induction. The scheme of the induction procedure will be as follows. Assume that we have found coefficients $q_k^{(0)}, \dots, q_k^{(p)}$ and $J_k^{(0)}, \dots, J_k^{(p)}$. Then we can find the $p+1$ correction $g_k^{(p+1)}(z)$ and estimate the respective remainder $\widetilde{r}_{k,n}^{(p+1,g)}$ from the $(p+1)$ form of Eq. (2.18) (see Eq. (2.70) below), in which we use the functions $g_k^{(0)}(z), \dots, g_k^{(p)}(z)$ and $R_{kk}^{(0)}(z), \dots, R_{kk}^{(p)}(z)$ found previously. Then, by using the $(p+1)$ form of Eq. (2.19) (see Eq. (2.73) below), we determine $R_{kk}^{(p)}(z)$ and estimate the respective remainder $\widetilde{r}_{k,n}^{(p+1,R)}$. Finally, we find the coefficients $q_k^{(p+1)}$, and $J_k^{(p+1)}$ and estimate the respective remainder by using the $(p+1)$ form of relations (2.36) and (2.65).

To realize this scheme we first write the asymptotic relation:

$$g_{k,n}(z) = \sum_{j=0}^{p} n^{-j}g_k^{(j)}(z) + n^{-p}\tilde{r}_{k,n}^{(p,g)}(z), \quad \tilde{r}_{k,n}^{(p,g)}(z) \to 0, \quad \text{as } n \to \infty, \tag{2.68}$$

valid for all k such that $|k-n| \le N_1(n)$. Let matrices $R^{(j)}(z)$, $j = 0, \dots, p$ be defined as in Lemma 2 (see formula (2.48), (2.49)). Then, denoting

$$g_{k,n}^{(p+1)}(z) \equiv n^{p+1}\left(g_{k,n}(z) - \sum_{j=0}^{p} n^{-j}g_k^{(j)}(z)\right), \tag{2.69}$$

we obtain from (2.18) the equation of the $(p+1)^{\text{th}}$ order for $g_{k,n}^{(p+1)}(z)$:

$$2g(z)g_{k,n}^{(p+1)}(z) = \frac{1}{2\pi i}\int \tilde{V}(z,\zeta)g_{k,n}^{(p+1)}(\zeta)d\zeta \\ - F_k^{(p+1,g)}(z) - r_{k,n}^{(p+1,g)}(z) + e_n(z), \tag{2.70}$$

where

$$F_k^{(p+1,g)}(z) = \sum_{l=1}^{p} g_k^{(p+1-l)}(z)g_k^{(l)}(z) + \sum_{m=1}^{k}\sum_{j=k+1}^{\infty}\sum_{l=0}^{p-1} R_{m,j}^{(p-l-1)}(z)R_{m,j}^{(l)}(z),$$

$$r_{k,n}^{(p+1,g)}(z) = n^{-p-1}(g_{k,n}^{(p+1)}(z))^2 + 2g_{k,n}^{(p+1)}(z)\sum_{l=1}^{p} n^{-l}g_k^{(l)}(z) \\ + \sum_{l,l'=1,l+l'>p+1}^{p} n^{p+1-l-l'}g_k^{(l)}(z)g_k^{(l')}(z) \\ \cdot n^p\left[\frac{1}{n}\sum_{j=1}^{k}\left(-R'(z)_{j,j} - \sum_{m=1}^{k}R_{j,m}^2(z)\right)\right. \\ \left. - \frac{1}{n}\sum_{j=1}^{k}\left((S^{(p)}\cdot S^{(p)})_{j,j}(z) - \sum_{m=1}^{k}(S_{j,m}^{(p)}(z))^2\right)\right], \tag{2.71}$$

with $S_{j,m}^{(p)}(z)$ defined by (2.49). On the basis of (2.68), (1.28), and Lemma 2 we conclude that the relations

$$\left|F_k^{(p+1,g)}(z)\right| \le \text{ const } (|k-n|^{p+1}+1),$$

and

$$r_{k,n}^{(p+1,g)}(z) \to 0, \quad as \quad n\to\infty,$$

are valid uniformly in $\{z : \delta_\varepsilon(z) \ge d\}$, for any fixed $d > 0$, because by the induction assumption (2.68) we have that $n^{-1}g_{k,n}^{(p+1)}(z) \equiv \tilde{g}_{k,n}^{(p)}(z) \to 0$ as $n\to\infty$. Then Lemma 1 leads to the relations

$$g_{k,n}^{(p+1)}(z) = g_k^{(p+1)}(z) + \tilde{r}_{k,n}^{(p+1,g)}(z), \tag{2.72}$$

where for $\delta_\varepsilon(z) \ge d > 0$,

$$g_k^{(p+1)}(z) = \frac{1}{2\pi i}\int_L \frac{F_k^{(p+1,g)}(\zeta)}{P(\zeta)(\zeta-z)}d\zeta, \quad |g_k^{(p+1)}(z)| \le \text{ const } (|k-n|^{p+1}+1)$$

and

$$|\tilde{r}_{k,n}^{(p+1,g)}(z)| \le \frac{\text{const}}{|X(z)|}((|r_{k,n}^{(p+1,g)}(z)| + \max_{\{\zeta\in L\}}|r_{k,n}^{(p+1,g)}(\zeta)|).$$

Now, denoting (cf. (2.69))

$$R_{k,k}^{(p+1,n)}(z) \equiv n^{p+1}\Big(R_{k,k}(z) - \sum_{j=0}^{p} n^{-j} R_{k,k}^{(j)}(z)\Big),$$

we get from (2.19) the equation of the form (cf. (2.55))

$$\begin{aligned} 2g(z)R_{k,k}^{(p+1,n)}(z) = \frac{1}{2\pi i}\int \tilde{V}(z,\zeta)R_{k,k}^{(p+1)}(\zeta)d\zeta \\ - F_k^{(p+1,R)}(z) - r_{k,n}^{(p+1,R)}(z) + e_n(z), \end{aligned} \tag{2.73}$$

where

$$F_k^{(p+1,R)}(z) = \sum_{l=0}^{p} g_{k-1}^{(p+1-l)}(z)R_{k,k}^{(l)}(z) + \Big(\sum_{j=1}^{k} - \sum_{j=k+1}^{\infty}\Big)\sum_{l=0}^{p} R_{m,j}^{(p-l)}(z)R_{m,j}^{(l)}(z),$$

$$\begin{aligned} r_{k,n}^{(p+1,R)}(z) = {} & 2R_{k,k}^{(p+1)}(z)\sum_{l=1}^{p} n^{-l} g_{k-1}^{(l)}(z) \\ & + \sum_{l,l'=1,l+l'>p+1}^{p} n^{p+1-l-l'} g_{k-1}^{(l)}(z)R_{k,k}^{(l')}(z) \\ & + n^{p-1}\Big[\Big(- R'(z)_{k,k} - 2\sum_{m=1}^{k}(R_{k,m}(z))^2\Big) \\ & - \Big((S^{(p)}\cdot S^{(p)})_{j,j}(z) - 2\sum_{m=1}^{k}(S_{j,m}^{(p)}(z))^2\Big)\Big]. \end{aligned}$$

By the virtue of (2.68), (1.28) and of Lemma 2, we conclude that the relations

$$\big|F_k^{(p+1,R)}(z)\big| \le \text{ const } (|k-n|^{p+1}+1),$$

and

$$r_{k,n}^{(p+1,R)}(z) \to 0, \quad as \quad n\to\infty,$$

are valid uniformly in $\{z : \delta_\varepsilon(z) \ge d\}$, for any fixed $d > 0$. Using again Lemma 1, we get

$$R_{k,k}^{(p+1,n)}(z) = R_{k,k}^{(p+1)}(z) + \tilde{r}_{k,n}^{(p+1,R)}(z), \tag{2.74}$$

where for $\delta_\varepsilon(z) > d$,

$$R_{k,k}^{(p+1)}(z) = \frac{1}{2\pi i}\int_L \frac{F_k^{(p+1,R)}(\zeta)}{P(\zeta)(\zeta - z)}d\zeta,\ \ |R_{k,k}^{(p+1)}(z)| \le \text{ const } (|k-n|^{p+1}+1) \tag{2.75}$$

and

$$\big|\tilde{r}_{k,n}^{(p+1,R)}(z)\big| \le \frac{\text{const}}{|X(z)|}\Big(\big|r_{k,n}^{(p+1,R)}(z)\big| + \max_{\{\zeta\in L\}}\big|r_{k,n}^{(p+1,R)}(\zeta)\big|\Big).$$

Now, as for the first order approximation case, in the case (ii) we take the $(p+1)$ - order terms (with respect to n^{-1}) of Eqs. (2.36) for $k = 2q$:

$$2(J_{2q}^{(0)} J_{2q}^{(p+1)} + J_{2q-1}^{(0)} J_{2q-1}^{(p+1)}) = \frac{1}{2\pi i} \int_L \zeta^2 R_{2q,2q}^{(p+1)}(\zeta) d\zeta + r_{2q}^{(p+1,J,2)},$$

$$\begin{aligned} 4(J_{2q}^{(0)} J_{2q}^{(p+1)} &+ J_{2q-1}^{(0)} J_{2q-1}^{(p+1)})((J_{2q}^{(0)})^2 + (J_{2q-1}^{(0)})^2) \\ &+ 2J_{2q}^{(0)} J_{2q-1}^{(0)} (J_{2q-1}^{(0)} J_{2q}^{(p+1)} + J_{2q}^{(0)} J_{2q+1}^{(p+1)} + J_{2q}^{(0)} J_{2q-1}^{(p+1)} + J_{2q-1}^{(0)} J_{2q-2}^{(p+1)}) \\ = \frac{1}{2\pi i} \int_L \zeta^4 R_{2q,2q}^{(p+1)}(\zeta) d\zeta + r_{2q}^{(p+1,J,4)}, \end{aligned} \tag{2.76}$$

where $F_k^{(p+1,J,2)}$ and $F_k^{(p+1,J,4)}$ are the coefficients at n^{-p-1} in the r.h.s. of the second and the third equations (2.36) which we get, substituting there $J_k = \sum_{j=0}^{p} n^{-j} J_k^{(j)}$, and

$$\begin{aligned} r_k^{(p+1,J,2)} &\equiv \int_L \zeta^2 \tilde{r}_{k,n}^{(p+1,R)}(\zeta) d\zeta \to 0, \quad n \to \infty, \\ r_k^{(p+1,J,4)} &\equiv \int_L \zeta^4 \tilde{r}_{k,n}^{(p+1,R)}(\zeta) d\zeta \to 0, \quad n \to \infty. \end{aligned}$$

Consider also the two analogs of the first relation of (2.76), in which $2q$ is replaced by $2q-1$ and $2q+1$. These relations together with (2.76) comprise a linear system with respect to the variables $J_{2q-2}^{(p+1)}$, $J_{2q-1}^{(p+1)}$, $J_{2q}^{(p+1)}$ and $J_{2q+1}^{(p+1)}$. For $J_{2q}^{(0)} \neq J_{2q-1}^{(0)}$, i.e. in the case (ii), the system is uniquely soluble and the solution satisfies condition (1.29) in view of (2.75).

However, for $J_{2q}^{(0)} = J_{2q-1}^{(0)}$ this system is degenerated and so in the case (i) we cannot find $J_k^{(p+1)}$ from the system. Therefore similarly to (2.62)–(2.64) for the case (i) we obtain the one-parameter family of solutions

$$J_k^{(p+1)} = b_k^{(p+1)} - c(-1)^{k-n} + \tilde{r}_{k,n}^{(p+1,J)}, \tag{2.77}$$

where

$$b_k^{(p+1)} = \sum_{j=0}^{k-n} (-1)^{k-n-j} a_{n+j}^{(p+1)}, \quad \tilde{r}_{k,n}^{(p+1,J)} = \sum_{j=0}^{k-n} (-1)^{k-n-j} r_{n+j,n}^{(p+1,J,2)},$$

with

$$a_k^{(p+1)} \equiv -F_k^{(p+1,J,2)} + \frac{1}{2\pi i} \int_L \zeta^2 R_{k,k}^{(p+1)}(\zeta) d\zeta,$$

To fix the parameter c we use again identity (2.65) and Lemma 2. Then we get the equation for c of the form

$$D^{(i)} c - A_{p+1}^{(i)} = 0,$$

where, as usually in perturbation theory, the coefficient $D^{(i)}$ is the same in each order of the procedure. Thus, in view of Lemma 3, $D^{(i)}$ is nonzero and the parameter c is uniquely defined by this equation. By the same argument as in the case $p = 1$ it is easy to see that in view of (2.75) $q_k^{(p+1)}$ and $J_k^{(p+1)}$satisfy bounds (1.30). Theorem 1 is proven.

Proof of Corollary 1. By using general formulas (1.18)–(1.25), (2.12) (2.14)–(2.16) and the Christoffel–Darboux identity for orthogonal polynomials it can be shown that the covariance (1.39) can be written as

$$\begin{aligned} D_n(z_1,z_2) &= \frac{1}{2n^2}\int \frac{(\lambda-\mu)^2 k_n^2(\lambda,\mu)d\lambda d\mu}{(z_1-\lambda)(z_1-\mu)(z_2-\lambda)(z_2-\mu)} \\ &= \frac{J_n^2}{n^2}\left(\frac{\delta R_{n+1,n+1}}{\delta z}\frac{\delta R_{n,n}}{\delta z} - \left(\frac{\delta R_{n+1,n}}{\delta z}\right)^2\right) + e_n(z_1) + e_n(z_2), \end{aligned} \tag{2.78}$$

where $k_n(\lambda,\mu)$ is defined in (1.20) and we denote $\delta R_{k,j} \equiv R_{k,j}(z_1) - R_{k,j}(z_2)$ and $\delta z \equiv z_1 - z_2$.

Then, on the basis of Lemma 2, we conclude that the amplitude $d_n^{(2)}(z_1,z_2)$ of the asymptotic formula (1.40) is:

$$d_n^{(2)}(z_1,z_2) = (J_n^{(0)})^2\left(\frac{\delta R^{(0)}_{n+1,n+1}}{\delta z}\frac{\delta R^{(0)}_{n,n}}{\delta z} - \left(\frac{\delta R^{(0)}_{n+1,n}}{\delta z}\right)^2\right).$$

According to Theorem 1 and Remark 2 after the theorem the zero-order coefficients $J_k^{(0)}$ of the Jacobi matrix $J(n)$ do not depend on k ($k = n(1+o(1))$) in the case (i) and are 2-periodic functions of k in the case (ii). Thus, we have only to compute the matrix elements of the resolvent of the constant Jacobi matrix and of the 2-periodic Jacobi matrices whose coefficients are given by (1.34) and (1.35) in the cases (i) and (ii) respectively. The computations are standard and lead to (1.41) and to (1.42). □

Proof of Corollary 2. The weak convergence of $(\psi_k^{(n)}(\lambda))^2$ is equivalent to the convergence of its Stieltjes transform

$$\int \frac{(\psi_k^{(n)}(\lambda))^2 d\lambda}{z-\lambda} \tag{2.79}$$

uniformly in z on any compact set of $\mathbf{C}\setminus\mathbf{R}$. According to (2.12) and Proposition 2 the Stieltjes transform (2.79) is $R_{kk}(z) + e_n(z)$. Now the asymptotic formula (2.33) implies that the Stieltjes transform (2.79) converges to $\Psi(z)$ as $n \to \infty$ and $\mathrm{dist}\{z,\sigma_\varepsilon\} \geq \dot{d} > 0$. This fact and the inversion formula (3.2) yield the result. □

3. Auxiliary Results

Proposition 1. For the proof of weak convergence of measures N_n and (1.10) see [8]. Furthermore, it follows from Eq. (2.22) that in $\mathbf{D}$ $g(z)$ can be written as

$$\frac{V'(z)}{2} - \frac{1}{2}\sqrt{(V'(z))^2 - 4Q(z)}, \tag{3.1}$$

where $Q(z)$ is defined in (2.23). Since

$$\rho(\lambda) = -\frac{1}{\pi}\lim_{\varepsilon\to+0}\Im g(\lambda+i\varepsilon), \tag{3.2}$$

we conclude that $\rho(\lambda)$ satisfies the Holder condition. Thus we find from the real parts of (3.1) that:

$$v.p. \int_\sigma \frac{\rho(\mu)d\mu}{\lambda-\mu} = \frac{V'(\lambda)}{2}, \ \lambda \in \sigma.$$

Regarding this relation as a singular integral equation and using standard facts (see [21]), we obtain (1.10) in which

$$P(\lambda) = \frac{1}{\pi} \int_\sigma Q(\lambda,\mu) X_+^{-1}(\mu) d\mu$$

and Q and $X_+^{-1}(\mu)$ are defined in (1.16) and (1.11). It is clear that $P(\lambda)$ can be analytically continued into $\mathbf{D}$ and can be written in form (1.15). Since $g(z)$ is uniquely determined by its boundary values on σ and its asymptotic behaviour $g(z) = z^{-1}(1+o(1))$, as $z \to \infty$, we obtain the assertions of the lemma. □

Proof of Proposition 2. According to the result of [8], and our condition C2, if we consider the function $u(x)$ of the form (1.9), then $u(x) = C^*$ $(x \in \sigma)$ and $u(x) < C^*$ $(x \in \sigma)$. It is easy to see that at all endpoints $a*$ of σ there exist one-side derivatives $u'_\pm(a*)$ (we take the right derivative for the right endpoints $a*$ and the left derivative for the left endpoints), and these derivatives are nonzero. Set $C_1 = \frac{1}{2} \min |u'_\pm(a*)|$ and consider the function

$$V_1(x) = \begin{cases} 0, & x \in \sigma, \\ C_1\varepsilon, & x \in \mathbf{R} \setminus \sigma_\varepsilon, \\ \pm C_1(x-a*), & \sigma_\varepsilon \setminus \sigma. \end{cases} \tag{3.3}$$

In the last line here we take plus for the right endpoints and minus for the left endpoints of the spectrum. It is easy to see that we can always choose ε_0 so small that for any $\varepsilon \le \varepsilon_0$ the function $u_1(x) \equiv u(x) + V_1(x)$ also takes its maximum value C^* on σ.

Consider now the following functions of $(x_1,\ldots,x_n) \in \mathbf{R}^n$ that we will call Hamiltonians because their role below will be analogous to that of Hamiltonians of classical statistical mechanics (see [8] for this analogy):

$$\begin{aligned} H_n(x_1,\ldots,x_n) &= n\sum_{i=1}^n V(x_i) - 2\sum_{1\le i<j\le n} \ln|x_i-x_j|, \\ H_n^{(1)}(x_1,\ldots,x_n) &= n\tilde V(x_1) + n\sum_{i=2}^n V(x_i) - 2\sum_{1\le i<j\le n} \ln|x_i-x_j|, \\ H_n^{(1a)}(x_1,\ldots,x_n) &= \tilde V(x_1) - (n-1)u_1(x_1) + n\sum_{i=2}^n V(x_i) \\ &\quad - 2\sum_{2\le i<j\le n} \ln|x_i-x_j|, \\ H_n^{(a)}(x_1,\ldots,x_n) &= -nV_1(x_1) - n\sum_{i=1}^n u(x_i) \\ &\quad + n(n-1)\int \ln|x-y|\rho(x)\rho(y)dxdy, \end{aligned} \tag{3.4}$$

where

$$\tilde{V}(x) \equiv V(x) - V_1(x),$$

u is defined in (1.9), and $u_1 = u + V_1$. Denote by

$$p_n^{\natural} = (Z^{\natural})^{-1} \exp\{-H_n^{\natural}\}$$

the probability density defined by one of these functions (cf. (1.17)).

We will use the Bogolyubov inequality, valid for any two Hamiltonians $H_{1,2}$ with correspondent normalization constants (partition functions) $Z_{1,2}$,

$$\langle H_2 - H_1 \rangle_{H_2} \le \log Z_1 - \log Z_2 \le \langle H_2 - H_1 \rangle_{H_1}, \tag{3.5}$$

where the symbol $\langle \dots \rangle_H$ denotes the mathematical expectation with respect to the probability density $p = Z^{-1} \exp\{-H\}$.

Using the r.h.s inequality in (3.5) for $H_1 = H_n^{(1)}$ and $H_2 = H_n^{(1a)}$, we get

$$\begin{aligned}
&\log Z_n^{(1)} - \log Z_n^{(1a)} \\
&\quad \le 2(n-1) \int \log|x_1 - x_2| \Big(\rho_n^{(1,1)}(x_1, x_2) - \rho_n^{(1,1)}(x_1)\rho_n^{(1,2)}(x_2)\Big) dx_1 dx_2 \\
&\qquad + 2(n-1) \int \log|x_1 - x_2| \rho_n^{(1,1)}(x_1) \Big(\rho_n^{(1,2)}(x_2) - \rho(x_2)\Big) dx_1 dx_2,
\end{aligned} \tag{3.6}$$

where $\rho_n^{(1,1)}(x_1)$, and $\rho_n^{(1,2)}(x_2)$ are the first marginal densities corresponding to x_1 and x_2 for the Hamiltonian $H_n^{(1)}$ (note that $\rho_n^{(1,1)}(x_1) \ne \rho_n^{(1,2)}(x_1)$ since $H_n^{(1)}$ is not symmetric in x_1 and x_2), and $\rho_n^{(1,1)}(x_1, x_2)$ is the second marginal density, corresponding to x_1, x_2 (note that $\rho_n^{(1,1)}(x_1, x_2)$ is not symmetric because of the same reason). Lemma 4 of [8] (valid for not necessarily symmetric Hamiltonians) implies that the first term in the r.h.s. of (3.6) is $O(\log n)$. To estimate the second term we first take into account that the integral kernel $\log|x - y|^{-1}$ is positive definite, hence by the corresponding Schwartz inequality

$$\begin{aligned}
&\left| \int \log|x - y| \rho_n^{(1,1)}(x) \Big(\rho_n^{(1,2)}(y) - \rho(y)\Big) dx dy \right| \\
&\qquad \le \left| \int \log|x - y| \rho_n^{(1,1)}(x) \rho_n^{(1,1)}(y) dx dy \right|^{1/2} \\
&\qquad \times \left| \int \log|x - y| \Big(\rho_n^{(1,2)}(x) - \rho(x)\Big)\Big(\rho_n^{(1,2)}(y) - \rho(y)\Big) dx dy \right|^{1/2}.
\end{aligned} \tag{3.7}$$

By using the estimate

$$\begin{aligned}&\left|\rho_n^{(1,1)}\Big(x+\frac{\tilde{x}}{n^{3/\gamma}}\Big)-\rho_n^{(1,1)}(x)\right|\\&=(Z_n^{(1)})^{-1}\int dx_2\dots dx_n\left|\exp\left\{-n\tilde{V}\Big(x+\frac{\tilde{x}}{n^{3/\gamma}}\Big)-n\sum_{i=2}^n V\Big(x_i+\frac{\tilde{x}}{n^{3/\gamma}}\Big)\right\}\right.\\&\quad\left.-\exp\left\{-n\tilde{V}(x)-n\sum_{i=2}^n V(x_i)\right\}\right|\cdot\prod_{i=2}^n|x-x_i|^2\prod_{2\le i<j}^n|x_i-x_j|^2\\&\le\frac{\text{const}}{n}\rho_n^{(1,1)}(x),\end{aligned}\tag{3.8}$$

valid for $|\tilde{x}|<1$ in view of the condition (1.3), and the fact that $\int\rho_n^{(1,1)}(x)dx=1$, we obtain that $\rho_n^{(1,1)}(x)\le\text{const}\,n^{3/\gamma}$. Hence we have the following bound for the first factor in the r.h.s. of (3.7):

$$\left|\int\ln|x-y|\rho_n^{(1,1)}(x)\rho_n^{(1,1)}(y)dxdy\right|\le\text{const}\ \log n.$$

To estimate the second factor in the r.h.s. of (3.7) we use the l.h.s inequality in (3.5) for the Hamiltonians $H_1=H_n^{(a)}$ and $H_2=H_n^{(1)}$, where $H_n^{(a)}$ and $H_n^{(1)}$ are defined in (3.4). We obtain the inequality

$$\begin{aligned}&-\frac{(n-1)(n-2)}{n^2}\int\log|x-y|(\rho_n^{(1,2)}(x,y)-\rho_n^{(1,2)}(x)\rho_n^{(1,2)}(y))dxdy\\&-\frac{(n-1)(n-2)}{n^2}\int\log|x-y|(\rho_n^{(1,2)}(x)-\rho(x))(\rho_n^{(1,2)}(y)-\rho(y))dxdy\\&+\frac{2(n-1)}{n^2}\int\log|x-y|(\rho_n^{(1,2)}(x)-\rho(x))\rho(y)dxdy\\&+\frac{2}{n}\int\log|x-y|\rho_n^{(1,1)}(x)\rho(y)dxdy\\&-\frac{2(n-1)}{n^2}\int\log|x-y|\rho_n^{(1,1)}(x,y)dxdy\\&\le\frac{1}{n^2}\log Z_n^{(a)}-\frac{1}{n^2}\log Z_n^{(1)}=\left(\frac{1}{n^2}\log Z_n^{(a)}-\frac{1}{n^2}\log Z_n\right)\\&\quad+\left(\frac{1}{n^2}\log Z_n-\frac{1}{n^2}\log Z_n^{(1)}\right)\le O\left(\frac{\log n}{n}\right)-\frac{2}{n}\int V_1(x)\rho_n(x)dx.\end{aligned}\tag{3.9}$$

In the r.h.s. here we have used the result of [8] to estimate $1/n^2\log Z_n^{(a)}-1/n^2\log Z_n$ and inequality (3.5) to estimate $1/n^2\log Z_n-1/n^2\log Z_n^{(1)}$. Using Lemma 4 of [8] (more precisely, repeating almost literally the arguments of that lemma in the case of the non symmetric Hamiltonian), we obtain that the first and the last terms in the l.h.s. of (3.9) are of the order $O(\log n/n)$. And the third, the fourth and the fifth terms here are evidently of the order $O(n^{-1})$. Therefore finally we get from (3.9),

$$-\int\log|x-y|(\rho_n^{(1,2)}(x)-\rho(x))(\rho_n^{(1,2)}(y)-\rho(y))dxdy\le\text{const}\,\frac{\log n}{n}.\tag{3.10}$$

Substituting this estimate in (3.6) we obtain

$$\log Z_n^{(1)} - \log Z_n^{(1a)} \leq \text{const} \sqrt{n} \log n. \tag{3.11}$$

Now we use the r.h.s. inequality in (3.5) for $H_1 = H_n^{(1a)}$ and $H_2 = H_n$, where $H_n^{(1a)}$ and H_n are defined in (3.4). We get

$$\begin{aligned} \log Z_n^{(1a)} - \log Z_n \leq n \int V_1(x_1) \rho_n^{(a1)}(x_1) dx_1 \\ + (n-1) \int \rho_n^{(a1)}(x_1)(\rho_n^{(a2)}(y) - \rho(y)) dx_1 dy, \end{aligned} \tag{3.12}$$

where $\rho_n^{(a1)}$ and $\rho_n^{(a2)}$ are the first marginal densities of the Hamiltonian $H_n^{(1a)}$, corresponding to x_1 and x_2. On the other hand it is easy to see that

$$\rho_n^{(a1)}(x) = \frac{\exp\{(n-1)u_1(x) - \tilde{V}(x)\}}{\int \exp\{(n-1)u_1(x) - \tilde{V}(x)\} dx},$$

and due to the choice of the function V_1 the density $\rho_n^{(a1)}(x)$ decays exponentially outside of σ. Thus since $V_1(x) = 0$ for $x \in \sigma$ the first term in the r.h.s. of (3.12) is of the order $O(1)$. The second term can be estimated by the Schwartz inequality similarly to (3.7) and then, using the fact that $\rho_n^{(a2)}(x)$ coincides with the first marginal densities of the Hamiltonian,

$$H_n^{'}(x_2, \ldots, x_n) = n \sum_{i=2}^{n} V(x_i) - 2 \sum_{2 \leq i < j} \ln |x_i - x_j|.$$

Therefore the analog of inequality (3.10) for $\rho_n^{(a2)}(x)$ follows directly from the results of [8]. Thus, from (3.12) we derive

$$\log Z_n^{(1a)} - \log Z_n \leq \text{const} \sqrt{n} \log n. \tag{3.13}$$

Bounds (3.11) and (3.13) lead to the relation

$$\int e^{nV_1(x_1)} \rho_n(x_1) dx_1 = \frac{Z_n^{(1)}}{Z_n} \leq e^{C_2 \sqrt{n} \log n}.$$

Taking $C_0 = 2\frac{C_2}{C_1}$, we obtain from the last relation that for any positive ε satisfying the inequality: $C_0 n^{-1/2} \log n \leq \varepsilon \leq \varepsilon_0$ we have

$$\int_{\mathbf{R} \setminus \sigma_e} \rho_n(x_1) dx_1 \leq \exp\{C_2 \sqrt{n} \log n - C_1 \varepsilon n\} \leq e^{-nC_1 \varepsilon / 2}.$$

To obtain this statement for $\rho_{k,n}$ we have to prove now that for any n-independent ε we can choose ε_1 such that for $|k - n| \leq \varepsilon_1 n$ the spectrum of the ensemble with potential $\tilde{V} \equiv \frac{n}{k} V$ is inside of $\sigma_{\varepsilon/2}$. This fact follows from the main result of [8, 12] and also from [19]. Proposition 2 is proven. □

Proof of Lemma 1. Using Proposition 1 we rewrite Eq. (2.27) in **D**:

$$P(z)X(z)R(z) = \frac{1}{2\pi i}\int_L d\zeta\, Q(z,\zeta)R(\zeta), \tag{3.14}$$

with $Q(z,\zeta)$ defined by (1.16). It follows from formula (1.15) for $P(z)$ that the function $\Psi(z)$ of (2.28) solves Eq. (3.14) in the class (2.29). Let us show that the solution is unique. Denoting by $\tilde{Q}(z)$ the r.h.s. of (3.14), we see that $\tilde{Q}(z)$ is an analytic function in **D**. From Eq. (3.14) we derive that zeros of $P(z)$ in **D** coincide with zeros $\tilde{Q}(z)$ and have the same order. Thus, function $R(z)X(z) = \frac{\tilde{Q}(z)}{P(z)}$ is analytic in **D**. In the rest of **C** it is analytic, because we are looking for a solution analytic outside σ_ε. Thus $R(z)X(z)$ is analytic in the whole **C**. Besides, if $R(z) = \frac{1}{z}(1+o(1))$, as $|z| \to \infty$, then in the case (i) $R(z)X(z)$ is bounded, as $|z| \to \infty$. Therefore by the Liouville theorem, $R(z)X(z)$ is a constant. In the case (ii) we get also from the Liouville theorem, that $R(z)X(z) = az+b$. By the symmetry of the function $R(z)$ we get $R(z) = zX^{-1}(z)$. This proves the first statement of the lemma.

To prove the second statement, we notice that under condition (2.30) in the case (i) we have $R(z)X(z) \to 0$, as $|z| \to \infty$. Thus, according to the above conclusions $R(z) = 0$ for all z. In the case (ii) condition (2.30) implies that $R(z)X(z) = \text{ const }$ and we get $R(z)X(z) = 0$ from the symmetry condition.

To prove that (2.32) is a solution of Eq. (2.31) we note first that for any closed contour L that does not contain the zeros of $P(z)$ we can write the relation

$$R(z)X(z) = \frac{1}{2\pi i}\int_L \frac{R(\zeta)X(\zeta)d\zeta}{(\zeta - z)} - \frac{1}{2\pi i}\int_L \frac{\tilde{Q}(\zeta)d\zeta}{P(\zeta)(\zeta - z)}, \tag{3.15}$$

where $\tilde{Q}(z)$ is defined as in the r.h.s. of (3.14). Indeed, under the condition of the lemma $R(z)X(z) = z^{-1}(1+o(1))$, as $z \to \infty$, i.e. the function is analytic outside of contour L. Then, by the Cauchy theorem, the first term in the r.h.s. is $R(z)X(z)$. The second term is zero, because the integrand is analytic inside the contour L and z is outside of L. By using this relation, we can rewrite formula (2.32) for the solution as

$$\frac{1}{2\pi i}\int_{L_1}\Big((V'(\zeta) - P(\zeta)X(\zeta))R(\zeta) - \frac{1}{2\pi i}\int_L V(\zeta,\zeta_1)R(\zeta_1)d\zeta_1 + F(\zeta)\Big)\frac{d\zeta}{P(\zeta)(\zeta - z)} = 0,$$

where the contour L_1 lies outside of L and is close enough to L. According to the condition of the lemma the expression in the brackets is analytic outside of L_1. Thus by the Cauchy theorem, we have

$$(V'(z) - P(z)X(z))R(z) - \frac{1}{2\pi i}\int_L V(z,\zeta)R(\zeta)d\zeta + F = 0.$$

Since $2g(z) = V' - P(z)X(z)$, the last relation proves that (2.32) is the solution of Eq. (2.31).

Uniqueness follows from the absence of solutions of the homogeneous equation (2.27) in the class (2.30). This fact was proven above. □

Proof of Lemma 2. Consider the "block" symmetric Jacobi matrix $\dot{J}^{(n,N_1)}$ which can be obtained from J if we set $J_{n-N_1-1} = 0$. Let $\dot{R}^{(n,N_1)}(z)$ be its resolvent. We will use the resolvent identity valid for any two selfadjoint operators $J_{1,2}$ with resolvents $R_{1,2}$ respectively,

$$R_1(z) - R_2(z) = R_1(z)(J_2 - J_1)R_2(z). \tag{3.16}$$

Thus taking as $R_1(z)$ the resolvent $\mathcal{R}(z)$ of $J(n)$, and as $R_2(z)$ the resolvent $\dot{R}^{(n,N_1)}(z)$ of $\dot{J}^{(n,N_1)}$ we obtain

$$\begin{aligned}\mathcal{R}_{k,j}(z) - \dot{R}^{(n,N_1)}_{k,j}(z) = \dot{R}^{(n,N_1)}_{k,n-N_1-1} J_{n-N_1-1}\mathcal{R}_{n-N_1,j}(z)\\ + \dot{R}^{(n,N_1)}_{k,n+N_1+1} J_{n+N_1+1}\mathcal{R}_{n+N_1+2,j}(z).\end{aligned} \tag{3.17}$$

Now we use the general fact of the theory of the Jacobi matrices.

Proposition 3. *Let J be the Jacobi matrix with coefficients $J_{k,k+1} = J_{k+1,k} = a_k \in \mathbf{R}$, $|J_{k,k}| \le \varepsilon$, and $|a_k| \le A$. Then there exist positive constants $C_{1,2}$, such that for any $z \in \mathbf{C} \setminus [-2A-\varepsilon, 2A+\varepsilon]$ the matrix elements of the resolvent $G = (zI - J)^{-1}$ satisfy the inequalities:*

$$|G_{k,k'}(z)| \le \frac{C_1}{\delta_\varepsilon(z)} e^{-C_2\delta_\varepsilon(z)|k-k'|}, \tag{3.18}$$

where $\delta_\varepsilon(z) \equiv dist\{z, [-2A-\varepsilon, 2A+\varepsilon]\}$.

The proof of the proposition is similar to that of the well-known Combes- Thomas estimates for the Schroedinger operator (see e.g. [26]) and we omit the proof.

On the basis of the proposition we obtain the bound

$$|\dot{R}^{(n,N_1)}_{j,k}(z)| \le \frac{1}{\delta_\varepsilon(z)} e^{-C_2\delta_\varepsilon(z)|j-k|}. \tag{3.19}$$

Thus, for $(N_1 - 2\tilde{N}) \le |k-n| \le (N_1 - \tilde{N})$ we have

$$|\dot{R}^{(n,N_1)}_{n-N_1-1,k}(z)|, |\dot{R}^{(n,N_1)}_{n+N_1+1,k}(z)| \le \frac{1}{\delta_\varepsilon(z)} e^{-C_2\delta_\varepsilon(z)\tilde{N}}.$$

So, it follows from (3.17) that

$$|\mathcal{R}_{k,j}(z) - \dot{R}^{(n,N_1)}_{k,j}(z)| \le \frac{\text{const}}{|\Im z|\delta_\varepsilon(z)} e^{-C_1\delta_\varepsilon(z)\tilde{N}}. \tag{3.20}$$

Similarly, if we consider the $(2N_1+1)$-periodic symmetric Jacobi matrix $\tilde{J}$ such that

$$\tilde{J}_{k,k+1} = J_{k,k+1} \quad |k-n| \le N_1, \tag{3.21}$$

and denote by $\tilde{R}$ its resolvent, then

$$|\tilde{R}_{k,k} - \dot{R}^{(n,N_1)}_{k,k}(z)| \le \frac{2}{|\Im z|\delta_\varepsilon(z)} e^{-C_2\delta_\varepsilon(z)\tilde{N}}. \tag{3.22}$$

Therefore,

$$|\mathcal{R}_{k,k}(z) - \tilde{R}_{k,k}(z)| \le \frac{\text{const}}{|\Im z|\delta_\varepsilon(z)} e^{-C_2\delta_\varepsilon(z)\tilde{N}}. \tag{3.23}$$

Applying the resolvent identity (3.16) to the matrices $\tilde{J}^{(p)}$ and $\tilde{J}$ we obtain in view of estimate (1.28):

$$|\tilde{R}_{k,j}(z) - \tilde{R}^{(p)}_{k,j}(z, n^{-1})| \leq \frac{2\varepsilon_n^{(p)}}{n^p|\Im z|^2}, \tag{3.24}$$

where $\tilde{R}^{(p)}_{k,j}(z,s)$ is the resolvent of the Jacobi matrix $\tilde{J}^{(p)}(z,s)$ defined in (2.48) and $\varepsilon_n^{(p)}$ is defined in (1.30). Now expanding $\tilde{R}^{(p)}_{k,k}(z,n^{-1})$ with respect to n^{-1} it is easy to find that

$$|\tilde{R}^{(p)}_{k,j}(z, n^{-1}) - S^{(p)}_{k,j}(z)| \leq \frac{C_1 N_1^{p+1}}{\delta_\varepsilon^{p+1}(z) n^{p+1}}. \tag{3.25}$$

From (3.23)–(3.25) we derive that

$$\mathcal{R}_{k,k}(z) - S^{(p)}_{k,k}(z)| \leq \frac{\varepsilon_n^{(p)}}{\delta_\varepsilon^2(z) n^p} + \frac{C_1 N_1^{p+1}}{\delta_\varepsilon^{p+1}(z) n^{p+1}} + \frac{e^{-C_2\delta_\varepsilon(z)\tilde{N}}}{|\Im z|\delta_\varepsilon(z)}.$$

This inequality and (2.11), lead to the first inequality in (2.51).

To prove the second inequality in (2.51) we use again identity (3.17). Taking the second power of the identity, using the bounds

$$|\dot{R}^{(n,N_1)}_{k,j}(z)|,\ |\mathcal{R}_{k,j}(z)| \leq \frac{1}{|\Im z|},$$

valid for resolvents of arbitrary selfadjoint operators, and bound (3.19), we obtain

$$\begin{aligned}
&\left|\sum_{j=1}^{\infty}(\mathcal{R}_{k,j}(z))^2 - \sum_{j=1}^{\infty}(\dot{R}^{(n,N_1)}_{k,j}(z))^2\right| \\
&\qquad \leq \frac{4}{\delta_\varepsilon(z)} e^{-C_2\delta_\varepsilon(z)\tilde{N}}\left(\sum_{j=1}^{\infty}|\mathcal{R}_{n-N_1,j}(z)|^2 + \sum_{j=1}^{\infty}|\mathcal{R}_{n+N_1,j}(z))^2|\right) \\
&\qquad \leq \frac{8}{|\Im z|^2\delta_\varepsilon(z)} e^{-C_2\delta_\varepsilon(z)\tilde{N}}.
\end{aligned} \tag{3.26}$$

To estimate here the sums of the type $\sum_{j=1}^{\infty}|\mathcal{R}_{n-N_1,j}(z)|^2$ we have used the simple inequalities

$$\sum_{j=1}^{\infty}|\mathcal{R}_{n-N_1,j}(z)|^2 = \sum_{j=1}^{\infty}\mathcal{R}_{n-N_1,j}(z)\mathcal{R}_{j,n-N_1}(\overline{z}) \leq (\mathcal{R}(z)\cdot\mathcal{R}(\overline{z}))_{n-N_1,n-N_1} \leq \frac{1}{|\Im z|^2}.$$

Similarly,

$$\left|\sum_{m=k+1}^{\infty}(\tilde{R}_{k,m}(z))^2 - \sum_{m=k+1}^{\infty}(\dot{R}^{(n,N_1)}_{km}(z))^2\right| \leq 2\frac{1}{|\Im z|^2\delta_\varepsilon(z)} e^{-C_2\delta_\varepsilon(z)\tilde{N}}. \tag{3.27}$$

And then, by the same way as in (3.23)-(3.25) we get the second inequality of (2.51).

The proof of (2.52) is similar.

Note that in fact we have proved (2.51) and (2.52) for $|k-n| \le (N_1 - \tilde{N})$.

To prove (2.53) we need to make one more step. Let us prove that for $|k-n| \le (N_1 - 2\tilde{N})$,

$$\Big| \sum_{j=1}^{n-(N_1-\tilde{N})} \Big[- R'_{j,j}(z) - \sum_{m=1}^{k} R^2_{j,m}(z) \Big] \Big| \le \frac{1}{|\Im z|^2 \delta_\varepsilon(z)} e^{-C_2\delta_\varepsilon(z)\tilde{N}/2}. \qquad (3.28)$$

To this end we consider one more ”block” symmetric Jacobi matrix $\dot{J}^{(n,(N_1-2\tilde{N}))}$ which can be obtained from J if we put $J_{n-(N_1-2\tilde{N})-1,n-(N_1-2\tilde{N})} = 0$. Using identity (3.16) for J and $\dot{J}^{(n,(N_1-2\tilde{N}))}$ and (3.19) for $\dot{J}^{(n,(N_1-2\tilde{N}))}$, we obtain similarly to (3.26),

$$\Big| \sum_{j=1}^{n-(N_1-\tilde{N})} \sum_{m=k+1}^{\infty} (\mathcal{R}_{j,m}(z))^2 - \sum_{j=1}^{n-(N_1-\tilde{N})} \sum_{m=k+1}^{\infty} (\dot{R}^{(n,(N_1-2\tilde{N}))}_{j,m}(z))^2 \Big| \le \frac{2n}{|\Im z|^3} e^{-C_2\delta_\varepsilon(z)\tilde{N}} \le \frac{1}{|\Im z|^2 \delta_\varepsilon(z)} e^{-C_2\delta_\varepsilon(z)\tilde{N}/2}. \qquad (3.29)$$

Then, using the estimate (3.19) for $\dot{R}^{(n,(N_1-2\tilde{N}))}_{j,m}(z)$ with $j \le n-(N_1-\tilde{N})$ and $m \ge k+1 > n-(N_1-2\tilde{N})$ we get

$$\Big| \sum_{j=1}^{n-(N_1-\tilde{N})} \sum_{m=k+1}^{\infty} (\dot{R}^{(n,(N_1-2\tilde{N}))}_{j,m}(z))^2 \Big| \le \frac{2n}{\delta^2_\varepsilon(z)} e^{-C_2\delta_\varepsilon(z)\tilde{N}} \le \frac{1}{\delta^2_\varepsilon(z)} e^{-C_2\delta_\varepsilon(z)\tilde{N}/2}.$$

This inequality combined with (3.29) proves that

$$\Big| \sum_{j=1}^{n-(N_1-\tilde{N})} \Big[(\mathcal{R}\cdot\mathcal{R})_{j,j}(z) - \sum_{m=1}^{k} \mathcal{R}^2_{j,m}(z) \Big] \Big| = \Big| \sum_{j=1}^{n-(N_1-\tilde{N})} \sum_{m=k+1}^{\infty} \mathcal{R}^2_{j,m}(z) \Big| \le \frac{1}{|\Im z|^3} e^{-C_2\delta_\varepsilon(z)\tilde{N}/2}.$$

Now, using (2.11), we can replace $(\mathcal{R} * \mathcal{R})_{j,j}(z)$ by $(-R'_{j,j}(z))$ and $\mathcal{R}_{j,m}(z)$ by $R_{j,m}(z)$ to get (3.28). Applying the first and the second line of (2.51) for $|k-n| \le (N_1 - \tilde{N})$ we get (2.53). □

Proof of Lemma 3. To find $D_k^{(i)}$ we first compute the quantity

$$\begin{aligned}(R^{(0)}(\zeta)J^{(\pm)}R^{(0)}(\zeta))_{n,n+1} &= \sum_{j=-\infty}^{\infty}(R^{(0)}_{n,j}(\zeta)(-1)^{n-j}R^{(0)}_{j+1,n+1} + R^{(0)}_{n,j+1}(\zeta)(-1)^{n-j}R^{(0)}_{j+2,n+1}(\zeta) \\ &= \sum_{j=-\infty}^{\infty}\frac{1}{(2\pi)^2}\int_0^{2\pi} dx dy \frac{e^{i(n-j)(x-y-\pi)}(1+e^{-i(x+y)})}{(\zeta - a\cos x)(\zeta - a\cos x)} \\ &= \frac{1}{2\pi}\int_0^{2\pi} dx \frac{1-\cos 2x}{(\zeta^2 - a^2\cos^2 x)} \\ &= 2\left(1-\frac{\zeta^2}{a^2}\right)\frac{1}{2\pi}\int_0^{2\pi} dx \frac{1}{(\zeta^2 - a^2\cos^2 x)} + \frac{1}{\pi a^2} \\ &= \frac{2}{\zeta}\left(1-\frac{\zeta^2}{a^2}\right)X^{-1}(\zeta).\end{aligned}$$

Then using the simple formula $R^{(0)}_{n,n+1}(\zeta) = a^{-1}(\zeta R^{(0)}_{n,n}(\zeta) - 1) = a^{-1}(\zeta X^{-1}(\zeta) - 1)$ we find from (2.65),

$$\begin{aligned}D^{(i)} &= \frac{1}{2\pi i}\int_L V'(\zeta)\left(\frac{\zeta}{a} + \frac{a}{\zeta}\left(1-\frac{\zeta^2}{a^2}\right)\right)X^{-1}(\zeta)d\zeta \\ &= \frac{a}{2\pi i}\int_L \frac{V'(\zeta)}{X(\zeta)\zeta}d\zeta = aP(0) \neq 0.\end{aligned}$$

Here we have used representation (1.15) and the fact that $\int_L d\zeta (X(\zeta)\zeta)^{-1} = 0$.

Similar calculations show us that $A^{(i)} = 0$, so it follows from Eq. (2.66) that $c = 0$ and we get (1.34). □

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