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Infinite Random Matrices and Ergodic Measures

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Abstract: We introduce and study a 2-parameter family of unitarily invariant probability measures on the space of infinite Hermitian matrices. We show that the decomposition of a measure from this family on ergodic components is described by a determinantal point process on the real line. The correlation kernel for this process is explicitly computed.

At certain values of parameters the kernel turns into the well-known sine kernel which describes the local correlation in Circular and Gaussian Unitary Ensembles. Thus, the random point configuration of the sine process is interpreted as the random set of "eigenvalues" of infinite Hermitian matrices distributed according to the corresponding measure.

Contents

Introduction

We first introduce some basic notions, and then describe the main results of the paper.

Random point configurations and correlation functions. Let $\mathfrak{X}$ be a locally compact space. A *locally finite point configuration* in $\mathfrak{X}$ is a finite or countably infinite collection of points in $\mathfrak{X}$, also called *particles*, such that any compact set contains finitely many particles. The ordering of the particles is unessential. For the sake of brevity, we will omit the adjective "locally finite". A *point process* on $\mathfrak{X}$ is a probability measure on the space $\operatorname{Conf}(\mathfrak{X})$ of point configurations. Given a point process, we can speak about the *random* point configuration. The n^{th} *correlation measure* of a point process ($n = 1, 2, \dots$) is a symmetric measure ρ_n on $\mathfrak{X}^n$, which is determined by the relation

$$\langle \rho_n, F \rangle = \mathbb{E}\left(\sum F(x_1, \dots, x_n) \right), \tag{0.1}$$

where F is a compactly supported test function on $\mathfrak{X}^n$, $\mathbb{E}$ is the symbol of expectation, and the summation is taken over all ordered n-tuples of particles chosen from the random point configuration. The n^{th} *correlation function* is the density of ρ_n with respect to the n^{th} power of a certain reference measure on $\mathfrak{X}$. Usually, the reference measure is the Lebesgue measure. The first correlation function is also called the *density function*. See [Len], [DVJ, Ch. 5][1], [So].

The Dyson circular unitary ensemble. Let $\mathbb{T} \subset \mathbb{C}$ be the unit circle and $\mathbb{T}^N/S(N)$ be the set of orbits of the symmetric group $S(N)$ of degree N acting on the torus $\mathbb{T}^N$, where $N = 1, 2, \dots$ Consider the following probability measure on $\mathbb{T}^N/S(N)$:

$$\text{const} \cdot \prod_{1 \le j < k \le N} |u_j - u_k|^2 \prod_{j=1}^{N} d\varphi_j, \qquad u_j = e^{2\pi i \varphi_j} \in \mathbb{T}, \quad \varphi_j \in [-\tfrac{1}{2}, \tfrac{1}{2}], \tag{0.2}$$

where const is the normalizing factor, $i = \sqrt{-1}$. This measure defines a point process on $\mathfrak{X} = \mathbb{T}$ living on the N-point configurations, which is called the N^{th} *Dyson circular unitary ensemble* or simply the Dyson ensemble for short. Note that the Dyson ensemble is invariant under rotations of $\mathbb{T}$.

Let $U(N)$ be the group of $N \times N$ unitary matrices. Consider the natural projection $U(N) \to \mathbb{T}^N/S(N)$ assigning to a matrix $U \in U(N)$ the collection of its eigenvalues. Note that the fibers of this projection are exactly the conjugacy classes of the group $U(N)$. The measure (0.2) coincides with the pushforward of the normalized Haar measure on $U(N)$ under this projection. In other terms, (0.2) is the *radial part* of the Haar measure. It follows that the Dyson ensemble is formed by spectra of random unitary matrices $U \in U(N)$ distributed according to the Haar measure. See [Dys, Me].

[1] In the book [DVJ] the correlation measures are called the "factorial moment measures".

The sine process. This is a translationally invariant point process on $\mathfrak{X} = \mathbb{R}$. Its correlation functions (with respect to the Lebesgue measure on $\mathbb{R}$) are given by

$$\rho_n(y_1, \dots, y_n) = \det \left[\frac{\sin(\pi(y_j - y_k))}{\pi(y_j - y_k)} \right]_{j,k=1}^n, \qquad n = 1, 2, \dots, \qquad y_1, \dots, y_n \in \mathbb{R}. \tag{0.3}$$

The function $\frac{\sin(\pi(y-y'))}{\pi(y-y')}$ on $\mathbb{R} \times \mathbb{R}$ is called the *sine kernel.*

The correlation functions of the sine process can be obtained from the correlation functions of the N^{th} Dyson ensemble by the following scaling limit as $N \to \infty$. Fix an arbitrary point $u_0 \in \mathbb{T}$ and rescale the angular coordinate φ about the point u_0 by writing $u = u_0 e^{2\pi i y/N}$. Then, for any fixed n, the n^{th} correlation function of the N^{th} Dyson ensemble, expressed in terms of the y-variables, converges, as $N \to \infty$, to the function (0.3). See [Dys, Me].

A substitute of the Haar measure. A natural question is whether the sine process can be interpreted as a radial part of an infinite–dimensional analog of the Haar measure. In this paper we suggest such an interpretation.

It is convenient to pass from unitary matrices to Hermitian matrices. Let $H(N)$ be the linear space of $N \times N$ complex Hermitian matrices. Consider the Cayley transform

$$H(N) \ni X \mapsto U = \frac{i - X}{i + X} \in U(N), \qquad N = 1, 2, \dots. \tag{0.4}$$

The map (0.4) is one-to-one, and the complement of its image in $U(N)$ is a negligible set. Thus, we can transfer the normalized Haar measure from $U(N)$ to $H(N)$. The result has the following form:

$$\text{const} \cdot \det(1 + X^2)^{-N} \times (\text{the Lebesgue measure}). \tag{0.5}$$

Let H be the space of all infinite Hermitian matrices $X = [X_{jk}]_{j,k=1}^{\infty}$. A remarkable fact is that the measures (0.5) with different values of N are consistent with natural projections $H(N) \to H(N-1)$ and, therefore, determine a probability measure m on H. We view m as a substitute of the Haar measure on $U(N)$ for $N = \infty$.

Ergodic measures. Assume that we have a group acting on a Borel space. An invariant probability Borel measure is called *ergodic* if any invariant mod 0 set has measure 0 or 1. Ergodic measures coincide with extreme points of the convex set of all invariant probability measures, see [Ph]. For continuous actions of compact groups ergodic measures are exactly orbital measures, i.e., invariant probability measures supported by orbits. According to the general philosophy of the ergodic theory, the concept of ergodic measure is a right generalization of that of orbital measure.

We are interested in a special situation when the space is H and the group is an infinite–dimensional version $U(\infty)$ of the groups $U(N)$. By definition, $U(\infty)$ is the union of the groups $U(N)$. Its elements are infinite unitary matrices $[U_{jk}]_{j,k=1}^{\infty}$ with finitely many entries U_{jk} not equal to δ_{jk}. The group $U(\infty)$ acts on the space H by conjugations.

Consider the space Ω whose elements ω are given by 2 infinite sequences

$$\alpha_1^+ \ge \alpha_2^+ \ge \cdots \ge 0, \quad \alpha_1^- \ge \alpha_2^- \ge \cdots \ge 0, \quad \text{where} \quad \sum_{j=1}^{\infty}(\alpha_j^+)^2 + \sum_{j=1}^{\infty}(\alpha_j^-)^2 < \infty, \tag{0.6}$$

and 2 extra real parameters γ_1, γ_2, where $\gamma_2 \ge 0$.

It is known that the ergodic measures on H can be parametrized by the points $\omega \in \Omega$. We consider Ω as a substitute of the space $\mathbb{T}^N/S(N)$ for $N = \infty$.

Let us explain the asymptotic meaning of the parameters $\alpha_j^{\pm}, \gamma_1, \gamma_2$. According to a general result, each ergodic measure M on H can be approximated by a sequence $\{M^{(N)} \mid N = 1, 2, \dots\}$, where $M^{(N)}$ is an orbital measure on $H(N)$ with respect to the action of $U(N)$ by conjugations. Any such measure $M^{(N)}$ is specified by a collection $\lambda^{(N)}$ of eigenvalues. Then the parameters of ω describe the asymptotic behavior of $\lambda^{(N)}$ as $N \to \infty$:

$$\lambda^{(N)} = (\lambda_1^{(N)} \ge \cdots \ge \lambda_N^{(N)}) \sim (N\alpha_1^+, N\alpha_2^+, \dots, -N\alpha_2^-, -N\alpha_1^-),$$
$$\frac{\lambda_1^{(N)} + \cdots + \lambda_N^{(N)}}{N} \to \gamma_1,$$
$$\frac{(\lambda_1^{(N)})^2 + \cdots + (\lambda_N^{(N)})^2}{N^2} \to \gamma_2 + (\alpha_1^+)^2 + (\alpha_2^+)^2 + \cdots + (\alpha_1^-)^2 + (\alpha_2^-)^2 + \dots. \tag{0.7}$$

For more details, see [Pi2,OV], and references therein.

From spectral measures to point processes. It can be proved that any $U(\infty)$-invariant probability measure on H can be decomposed on ergodic components. I.e., it can be written as a continual convex combination of ergodic measures. This decomposition is unique, we call it the *spectral decomposition.* It is determined by a probability measure on Ω, which we call the *spectral measure* of the initial invariant measure.

We map the space Ω to the space $\mathrm{Conf}(\mathbb{R}^*)$ of point configurations on the punctured real line $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ as follows:

$$\Omega \ni \omega = (\{\alpha_j^+\}, \{\alpha_j^-\}, \gamma_1, \gamma_2) \mapsto C = (-\alpha_1^-, -\alpha_2^-, \dots, \alpha_2^+, \alpha_1^+) \in \mathrm{Conf}(\mathbb{R}^*), \tag{0.8}$$

where we omit possible zeros among the numbers $\alpha_j^{\pm}$. The map (0.8) transforms any spectral measure (which is a probability measure on Ω) to a point process on $\mathbb{R}^*$. This makes it possible to describe spectral measures in terms of the correlation functions. However, the map (0.8) ignores the parameters γ_1, γ_2.

Note that each configuration $C \in \mathrm{Conf}(\mathbb{R}^*)$ of the form (0.8) is contained in a sufficiently large interval $|x| \le$ const. It follows that C^{-1} (the image of C under the inversion map $x \mapsto 1/x$) is a well-defined configuration on the whole line $\mathbb{R}$.

An interpretation of the sine process. Applying the procedure described above to the measure m on H we prove the following result.

Theorem I. *Let P be the spectral measure of the $U(\infty)$-invariant measure m and let $\mathcal{P}$ be the corresponding point process on $\mathbb{R}^*$. Then the point process on $\mathbb{R}$ obtained from $\mathcal{P}$ under the transform $x \mapsto y = -\frac{1}{\pi x}$ coincides with the sine process.*

A simple explanation of this result follows from the comparison of two approximation procedures: that for the correlation functions of the sine process and that for the ergodic measures. Indeed, the eigenvalues in (0.7) grow linearly in N, so that we rescale them according to the rule $\lambda = Nx$. Under the Cayley transform $u = \frac{i-\lambda}{i+\lambda}$ the scaling takes the form

$$u = \frac{i - Nx}{i + Nx} = -1 + \frac{2i}{Nx} + O\left(\frac{1}{N^2}\right) = (-1)e^{2\pi i y/N} + O\left(\frac{1}{N^2}\right), \qquad y = -\frac{1}{\pi x}, \tag{0.9}$$

which means that the variable y is consistent with the scaling of the Dyson ensemble near the point $u_0 = -1$.

Thus, the statement of Theorem I is not surprising. However, the justification of the formal limit transition made on the level of correlation functions requires certain efforts.

Note also that dividing the eigenvalues $\lambda \in \mathbb{R}$ by N corresponds in terms of $u = \frac{i-\lambda}{i+\lambda}$ to the fractional–linear transformation of $\mathbb{T}$ of the form

$$u \mapsto \frac{(N+1)u + (N-1)}{(N-1)u + (N+1)}\,. \tag{0.10}$$

This transformation has two fixed points, $+1$ and -1. Near the point -1 it looks like the expansion by the factor of N while near the point $+1$ it looks like the contraction by the factor of N. Using (0.10) as a scaling transformation one can define a scaling limit for the correlation functions of the Dyson ensembles staying on the circle $\mathbb{T}$.

Theorem I is complemented by

Theorem II. *The spectral measure P of the measure m is concentrated on the subset* $\{\omega \in \Omega \mid \gamma_2 = 0\}$.

Thus, the parameter γ_2 (which is ignored by the map (0.8)) is actually irrelevant for the measure m. In a certain sense, this means that the measure m does not involve Gaussian components (see Sect. 4 about the connection of the parameter γ_2 with Gaussian measures).

A generalization: The main result. Let $s \in \mathbb{C}$, $\Re s > -\frac{1}{2}$, be a parameter. Consider the following probability measure on $\mathbb{T}^N/S(N)$:

$$\begin{gathered} \text{const} \cdot \prod_{1 \le j < k \le N} |u_j - u_k|^2 \prod_{j=1}^{N} (1+u_j)^{\bar{s}} (1+\bar{u}_j)^s d\varphi_j, \\ u_j = e^{2\pi i \varphi_j} \in \mathbb{T}, \quad \varphi_j \in [-\tfrac{1}{2}, \tfrac{1}{2}]. \end{gathered} \tag{0.11}$$

When $s = 0$, we get (0.2). Thus, this is a deformation of the measure (0.2) depending on two real parameters, $\Re s$ and $\Im s$. The measure (0.11) is the radial part of the probability measure on $U(N)$ of the form

$$\text{const} \cdot \det((1+U)^{\bar{s}}) \det((1+U^{-1})^s) \times (\text{the Haar measure on } U(N)). \tag{0.12}$$

Transferring the measure (0.12) from the group $U(N)$ to the space $H(N)$ by means of the Cayley transform (0.4) we get the following measure on $H(N)$, which is a deformation of the measure (0.5):

$$\text{const} \cdot \det((1+iX)^{-s-N}) \det((1-iX)^{-\bar{s}-N}) \times (\text{the Lebesgue measure on } H(N)). \tag{0.13}$$

When s is real, the expression (0.13) takes a simpler form:

$$\begin{gathered}\text{const}\cdot\det((1+X^2)^{-s-N})\times(\text{the Lebesgue measure on } H(N)),\\ s\in\mathbb{R},\quad s>-\tfrac{1}{2}\,.\end{gathered}\tag{0.14}$$

Again, it turns out that the measures (0.13) are consistent with the projections $H(N)\to H(N-1)$, and they determine a $U(\infty)$-invariant probability measure on the space H. We denote it by $m^{(s)}$. Note that $m^{(0)}=m$.

To our knowledge, the finite–dimensional measures (0.14) were first studied by Hua. He calculated the normalizing constant factor in (0.14) using a recurrence relation in N, and his argument proves the consistency property (although he did not state it explicitly), see [Hua, Theorem 2.1.5]. Much later Pickrell [Pi1] considered analogs of the measures (0.12) and (0.13) (with real s), which live on complex Grassmannians and on the spaces of all complex matrices, respectively. He proved the consistency property and considered the analogs of the measures $m^{(s)}$ on the space of all complex matrices of infinite order. His paper also contains a few other important ideas and results. Apparently, Pickrell was unaware of Hua's work. Note also Shimomura's paper [Shim], where an analog of the measure $m^{(0)}$ for the infinite-dimensional orthogonal group was constructed (more general measures depending on a parameter are not discussed in [Shim]). The possibility of introducing a complex parameter (in the case of Hermitian matrices) was discovered by Neretin [Ner2]. He also examined further generalizations of the measures $m^{(s)}$.

We propose to call the measures $m^{(s)}$ the *Hua–Pickrell measures.*

Theorem III. *The Hua–Pickrell measures $m^{(s)}$ on H are pairwise disjoint. I.e., for any two different values s', s'' of the parameter there exist two disjoint Borel subsets in H supporting $m^{(s')}$ and $m^{(s'')}$, respectively.*

The next claim is the main result of the paper.

Theorem IV. *Let $P^{(s)}$ be the spectral measure of a Hua–Pickrell measure $m^{(s)}$. The corresponding point process $\mathcal{P}^{(s)}$ on $\mathbb{R}^*$ can be described in terms of its correlation functions. They have the determinantal form*

$$\rho_n^{(s)}(x_1,\dots,x_n)=\det[K^{(s)}(x_j,x_k)]_{j,k=1}^n\,,\tag{0.15}$$

where $K^{(s)}(x,x')$ is a certain kernel on $\mathbb{R}^*\times\mathbb{R}^*$ which can be expressed through the confluent hypergeometric function or, for real values of s, through the Bessel function.

We give explicit expressions for the kernel in Theorem 2.1 below. As in Theorem I, one can use the transformation $C\mapsto C^{-1}$ to pass from $\mathbb{R}^*$ to $\mathbb{R}$.

Pseudo-Jacobi polynomials. The proof of Theorem IV, similarly to that of Theorem I, consists of three steps: the calculation of the correlation functions for the finite–dimensional measures (0.13), the scaling limit transition as $N\to\infty$, and a justification. However, the first step is more involved comparing to the Dyson ensemble. We show that the correlation functions are expressed through the Christoffel–Darboux kernel for the so-called pseudo-Jacobi polynomials. This family of orthogonal polynomials, which is not widely known, has interesting features. It is defined by a weight function on $\mathbb{R}$ with only finitely many moments, so that the system of orthogonal polynomials is finite.

Organization of the paper. In Sect. 1 we introduce the pseudo-Jacobi ensemble and obtain its correlation functions. In Sect. 2 we compute the scaling limit of these correlation functions as the number of particles goes to infinity. The limit correlation functions are given by a determinantal formula and we write down the correlation kernel explicitly. In Sect. 3 we define the Hua–Pickrell measures $m^{(s)}$ and show that they are pairwise disjoint. Section 4 provides a brief summary of known results about the ergodic $U(\infty)$-invariant probability measures on H. In Sect. 5 we show that the spectral measure for any $U(\infty)$-invariant probability measure M on H can be approximated by finite-dimensional projections of M. Section 6 contains the proof of our main result (Theorem IV above). In Sect. 7 we prove that the sine process has no Gaussian component (Theorem II above). Section 8 contains remarks concerning the connections of our work with other subjects as well as several open problems. Section 9 is an appendix where we prove the existence and uniqueness of the decomposition of $U(\infty)$-invariant probability measures on H on ergodic measures.

1. The Pseudo-Jacobi Ensemble

In this section we define the pseudo-Jacobi ensemble and compute its correlation functions.

Consider the radial part of the Haar measure on $U(N)$ which determines the Dyson ensemble, see (0.2). Under the inverse Cayley transform $\mathbb{T} \to \mathbb{R}$ which takes $u \in \mathbb{T}$ to $x = i\,\frac{1-u}{1+u} \in \mathbb{R}$, the measure (0.2) turns into the following measure on $\mathbb{R}^N/S(N) = \mathrm{Conf}_N(\mathbb{R})$, the set of N-point configurations on $\mathbb{R}$:

$$\text{const} \prod_{1\le j<k\le N} (x_j - x_k)^2 \cdot \prod_{j=1}^{N} (1+x_j^2)^{-N} dx_j\,. \tag{1.1}$$

More generally, let s be a complex parameter. We introduce the following deformation of the measure (1.1) depending on s:

$$\begin{aligned} &\text{const} \prod_{1\le j<k\le N} (x_j - x_k)^2 \cdot \prod_{j=1}^{N} (1+ix_j)^{-s-N}(1-ix_j)^{-\bar{s}-N} dx_j \\ &= \text{const} \prod_{1\le j<k\le N} (x_j - x_k)^2 \cdot \prod_{j=1}^{N} (1+x_j^2)^{-\Re s-N} e^{2\Im s\,\mathrm{Arg}(1+ix_j)} dx_j\,. \end{aligned} \tag{1.2}$$

Here we assume that the function $\mathrm{Arg}(\dots)$ takes values in $(-\pi, \pi)$ (actually, $\mathrm{Arg}(1 + ix_j) \in (-\frac{\pi}{2}, \frac{\pi}{2})$).

Proposition 1.1. *The measure* (1.2) *is finite provided that* $\Re s > -\frac{1}{2}$.

Proof. This follows from the estimate

$$(1+x^2)^{-\Re s-N} e^{2\Im s\,\mathrm{Arg}(1+ix)} \asymp |x|^{-2\Re s-2N}, \qquad x \in \mathbb{R}, \quad |x| \gg 0, \tag{1.3}$$

and the fact that the expansion of $\prod_{1\le j<k\le N}(x_j - x_k)^2$ involves only monomials of degree less then or equal to $2N-2$ in each variable. □

Henceforth we assume the condition $\Re s > -\frac{1}{2}$ to be satisfied, and we choose the normalizing constant in (1.2) in such a way that (1.2) defines a probability measure. About the case $\Re s \le -\frac{1}{2}$ see Sect. 8 below.

Note that (1.2) corresponds, via the Cayley transform, to the measure (0.11).

For real values of the parameter s the expression (1.2) takes a simpler form

$$\text{const} \prod_{1\le j<k\le N} (x_j - x_k)^2 \cdot \prod_{j=1}^{N} (1 + x_j^2)^{-s-N}, \qquad s \in \mathbb{R}.$$

Our aim is to compute the correlation functions of the measure (1.2). We remark that (1.2) is an orthogonal polynomial ensemble (see [Me, NW]) corresponding to the weight function

$$\phi(x) = (1+ix)^{-s-N}(1-ix)^{-\bar{s}-N} = (1+x^2)^{-\Re s-N} e^{2\Im s \,\mathrm{Arg}(1+ix)}, \qquad x \in \mathbb{R}. \quad (1.4)$$

We call it the N^{th} *pseudo-Jacobi ensemble.* The reason why we use this term is explained below. For real s, this ensemble was also considered in [WF] where it was called the *unitary Cauchy ensemble.* The reason is that for real s, the weight function (1.4) is proportional to the density of the classical Cauchy distribution. For generalities about orthogonal polynomial ensembles, see, e.g., [Me, NW].

Let $\boldsymbol{p}_0 \equiv 1, \boldsymbol{p}_1, \boldsymbol{p}_2, \dots$ denote the monic orthogonal polynomials on $\mathbb{R}$ associated with the weight function (1.4). Since for any s, $\phi(x)$ has only finitely many moments, this system of orthogonal polynomials is finite. Specifically, it follows from (1.3) that the polynomial $\boldsymbol{p}_m(x)$ exists if $m < \Re s + N - \frac{1}{2}$.

According to a well-known general principle (see, e.g., [Me]), the correlation functions in question are given by determinantal formulas involving the Christoffel–Darboux kernel

$$\sum_{m=0}^{N-1} \frac{\boldsymbol{p}_m(x')\boldsymbol{p}_m(x'')}{\|\boldsymbol{p}_m\|^2}. \quad (1.5)$$

By the assumption $\Re s > -\frac{1}{2}$, the polynomials up to the order $m = N - 1$ exist, so that this kernel is well-defined.

The orthogonal polynomials $\boldsymbol{p}_m$ are known. They were introduced by V. Romanovski in 1929, see [Ro], and studied by R. Askey [A] and P. A. Lesky [Les1, §5], [Les2, §1.4]. Following P. A. Lesky we call them *pseudo-Jacobi polynomials,* which explains our choice of the name for the ensemble (1.2).

Let

$${}_2F_1\left[\begin{matrix} a,\ b \\ c \end{matrix}\,\middle|\, z\right] = \sum_{n=0}^{\infty} \frac{a(a+a)\dots(a+n-1)\cdot b(b+1)\dots(b+n-1)}{c(c+1)\dots(c+n-1)\cdot n!}\, z^n$$

denote the Gauss hypergeometric function.

Proposition 1.2. *Let $m < \Re s + n - \frac{1}{2}$, so that the m^{th} monic orthogonal polynomial $\boldsymbol{p}_m$ with the weight function* (1.4) *exists. Then it is given by the explicit formula*

$$\boldsymbol{p}_m(x) = (x-i)^m\, {}_2F_1\left[\begin{matrix} -m,\ s+N-m \\ 2\Re s + 2N - 2m \end{matrix}\,\middle|\, \frac{2}{1+ix}\right] \quad (1.6)$$

and its norm is given by

$$\|\boldsymbol{p}_m(x)\|^2 = \int_{-\infty}^{\infty} \boldsymbol{p}_m^2(x)\phi(x)dx$$
$$= \frac{\pi\, 2^{-2\Re s}}{2^{2(N-m-1)}}\, \Gamma\begin{bmatrix} 2\Re s + 2(N-m)-1,\ 2\Re s + 2(N-m),\ m+1 \\ s + N - m,\ \bar{s} + N - m,\ 2\Re s + 2N - m \end{bmatrix}, \quad (1.7)$$

where we use the notation

$$\Gamma\begin{bmatrix} a,\ b,\ \dots \\ c,\ d,\ \dots \end{bmatrix} = \frac{\Gamma(a)\Gamma(b)\dots}{\Gamma(c)\Gamma(d)\dots}.$$

Proof. These formulas can be extracted from [A], [Les1, §5], [Les2, §1.4]. Another way to get them is to use a general method described in [NU]. This method works for any orthogonal polynomials of hypergeometric type and allows to compute all the data starting from the differential equation. In our case the differential equation has the form

$$-(1+x^2)\boldsymbol{p}_m'' + 2(-\Im s + (\Re s + N - 1)x)\boldsymbol{p}_m' + m(m+1-2\Re s - 2N)\boldsymbol{p}_m = 0. \quad (1.8)$$

□

Note the symmetry property

$$\boldsymbol{p}_m(-x) = (-1)^m \boldsymbol{p}_m(x)\ |_{s\leftrightarrow\bar{s}}\ . \quad (1.9)$$

It follows from the symmetry of the weight function

$$\phi(-x) = \phi(x)\ |_{s\leftrightarrow\bar{s}}$$

and can be verified directly from the expression (1.6).

To compute the Christoffel–Darboux kernel we will use the classical formula

$$\sum_{m=0}^{N-1} \frac{\boldsymbol{p}_m(x')\boldsymbol{p}_m(x'')}{\|\boldsymbol{p}_m\|^2} = \frac{1}{\|\boldsymbol{p}_{N-1}\|^2}\, \frac{\boldsymbol{p}_N(x')\boldsymbol{p}_{N-1}(x'') - \boldsymbol{p}_{N-1}(x')\boldsymbol{p}_N(x'')}{x' - x''}. \quad (1.10)$$

If the parameter s satisfies the stronger condition $\Re s > \frac{1}{2}$ then the polynomial $\boldsymbol{p}_N(x)$ exists and the formula holds. Since all the terms in the left-hand side depend analytically on s and $\bar{s}$, we can use the formula for s with $\frac{1}{2} \geq \Re s > -\frac{1}{2}$ as well with the understanding that the kernel is obtained by analytic continuation in s and $\bar{s}$ viewed as independent variables (or, equivalently, by analytic continuation in the variables s and $s + \bar{s}$).

Note that the trick with analytic continuation is actually needed only for the values of s on the vertical line $\Re s = 0$, because a singularity in the expression (1.6) for $m = N$ arises for $\Re s = 0$ only.

The next lemma makes it possible to get an alternative expression for the Christoffel–Darboux kernel. The advantage of this new formula is that all its terms have no singularities in the whole region $\Re s > -\frac{1}{2}$.

Lemma 1.3. *Set*

$$\widetilde{\boldsymbol{p}}_N(x) = \boldsymbol{p}_N(x) - \frac{2iNs}{2\Re s(2\Re s+1)}\boldsymbol{p}_{N-1}(x). \tag{1.11}$$

This polynomial, initially defined for $\Re s > \frac{1}{2}$, *makes sense for* $\Re s > -\frac{1}{2}$, *as follows from the explicit formula:*

$$\widetilde{\boldsymbol{p}}_N(x) = (x-i)^N {}_2F_1\left[\begin{matrix}-N,\ s\\ 2\Re s+1\end{matrix}\middle| \frac{2}{1+ix}\right]. \tag{1.12}$$

Proof. Indeed, using the power series expansion of the hypergeometric function it is readily verified that the following general relation holds:

$${}_2F_1\left[\begin{matrix}a,\ b\\ c\end{matrix}\middle| z\right] = {}_2F_1\left[\begin{matrix}a,\ b\\ c+1\end{matrix}\middle| z\right] + \frac{abz}{c(c+1)}\,{}_2F_1\left[\begin{matrix}a+1,\ b+1\\ c+2\end{matrix}\middle| z\right]. \tag{1.13}$$

From (1.13) and (1.6) we easily get (1.12). □

We summarize the above results in the following

Theorem 1.4. *The correlation functions of the* N^{th} *pseudo-Jacobi ensemble* (1.2) *have the form*

$$\boldsymbol{\rho}_n^{(s,N)}(x_1,\dots,x_n) = \det[\boldsymbol{K}^{(s,N)}(x_i,x_j)]_{i,j=1}^n \tag{1.14}$$

with a kernel $\boldsymbol{K}^{(s,N)}(x',x'')$ *defined on* $\mathbb{R}\times\mathbb{R}$.

This kernel is given by the formulas

$$\boldsymbol{K}^{(s,N)}(x',x'') = \frac{2^{2\Re s}}{\pi}\,\Gamma\left[\begin{matrix}2\Re s+N+1,\ s+1,\ \bar s+1\\ N,\ 2\Re s+1,\ 2\Re s+2\end{matrix}\right] \times \frac{\boldsymbol{p}_N(x')\boldsymbol{p}_{N-1}(x'')-\boldsymbol{p}_{N-1}(x')\boldsymbol{p}_N(x'')}{x'-x''}\sqrt{\phi(x')\phi(x'')} \tag{1.15}$$

or, equivalently,

$$\boldsymbol{K}^{(s,N)}(x',x'') = \frac{2^{2\Re s}}{\pi}\,\Gamma\left[\begin{matrix}2\Re s+N+1,\ s+1,\ \bar s+1\\ N,\ 2\Re s+1,\ 2\Re s+2\end{matrix}\right] \times \frac{\widetilde{\boldsymbol{p}}_N(x')\boldsymbol{p}_{N-1}(x'')-\boldsymbol{p}_{N-1}(x')\widetilde{\boldsymbol{p}}_N(x'')}{x'-x''}\sqrt{\phi(x')\phi(x'')}, \tag{1.16}$$

where

$$\phi(x) = (1+ix)^{-s-N}(1-ix)^{-\bar s-N} = (1+x^2)^{-\Re s-N}e^{2\Im s\,\mathrm{Arg}(1+ix)}, \quad x\in\mathbb{R}, \tag{1.17}$$

and

$$\boldsymbol{p}_N(x) = (x-i)^N\,{}_2F_1\left[\begin{matrix}-N,\ s\\ 2\Re s\end{matrix}\middle| \frac{2}{1+ix}\right], \tag{1.18}$$

$$\boldsymbol{p}_{N-1}(x) = (x-i)^{N-1}\,{}_2F_1\left[\begin{matrix}-N+1,\ s+1\\ 2\Re s+2\end{matrix}\middle| \frac{2}{1+ix}\right], \tag{1.19}$$

$$\widetilde{\boldsymbol{p}}_N(x) = (x-i)^N {}_2F_1\left[\begin{matrix}-N,\ s\\ 2\Re s+1\end{matrix}\middle| \frac{2}{1+ix}\right]. \tag{1.20}$$

Note that the expression (1.15) is directly applicable when the parameter s does not lie on the line $\Re s = 0$ while the expression (1.16) makes sense for any s with $\Re s > -\frac{1}{2}$.

Proof. A standard argument from the Random Matrix Theory, see, e.g., [Me] shows that the correlation functions are given by the determinantal formula (1.14), where the kernel is equal to the Christoffel–Darboux kernel (1.5) multiplied by the factor $\sqrt{\phi(x')\phi(x'')}$. Together with (1.6), (1.7), (1.10) this leads to the expression (1.15) for the kernel. The alternative formula (1.16) then follows from Lemma 1.3. $\square$

Remark 1.5. For $s = 0$ the polynomial $\boldsymbol{p}_N$ can be defined by taking the limit as $s \to 0$ *along the real line.* Taking the limit in the hypergeometric series it is easy to get the following expression:

$$\boldsymbol{p}_N(x)\ |_{s=0} = \frac{(x+i)^N + (x-i)^N}{2}\,.$$

Likewise, we get

$$\boldsymbol{p}_{N-1}(x)\ |_{s=0} = \frac{(x+i)^N - (x-i)^N}{2iN}\,.$$

It follows that the Christoffel–Darboux kernel (1.10) is an elementary expression. This agrees with the fact that for $s = 0$ our ensemble is related (via the Cayley transform) to the Dyson ensemble.

2. The Scaling Limit of the Correlation Functions

In this section we compute the scaling limit of the correlation functions of the pseudo-Jacobi ensemble as the number of particles goes to infinity. The limit correlation functions have a determinantal form, and we express the correlation kernel through the confluent hypergeometric function.

Recall the definition of the confluent hypergeometric function:

$${}_1F_1\left[\begin{matrix} a \\ c \end{matrix}\middle| z\right] = \sum_{n=0}^{\infty} \frac{a(a+1)\dots(a+n-1)}{c(c+1)\dots(c+n-1)\cdot n!}\, z^n\,,$$

see, e.g., [Er, 6.1].

Let us rescale the correlation functions $\boldsymbol{\rho}_n^{(s,N)}$ of the pseudo-Jacobi ensemble (see (1.14)) by setting

$$\rho_n^{(s,N)}(x_1,\dots,x_n) = N^n \cdot \boldsymbol{\rho}_n^{(s,N)}(Nx_1,\dots,Nx_n).$$

Note that the factor N^n comes from the transformation of the reference (Lebesgue) measure $dx_1 \dots dx_n$. We will assume that the variables range over the punctured real line $\mathbb{R}^*$, not the whole line $\mathbb{R}$, as before.

Theorem 2.1. *Let $\Re s > -\frac{1}{2}$, as before. For any $n = 1, 2, \dots$ and $x_1, \dots, x_n \in \mathbb{R}^*$ there exists a limit of the scaled n^{th} correlation functions $\rho_n^{(s,N)}$ as $N \to \infty$:*

$$\lim_{N\to\infty} \rho_n^{(s,N)}(x_1,\dots,x_n) = \det\left[K^{(s,\infty)}(x_i,x_j)\right]_{1\le i,j\le n}.$$

Here the kernel $K^{(s,\infty)}(x',x'')$ on $\mathbb{R}^*\times\mathbb{R}^*$ is as follows:

$$\begin{aligned} K^{(s,\infty)}(x',x'') &= \frac{1}{2\pi}\,\Gamma\begin{bmatrix} s+1,\ \bar{s}+1 \\ 2\Re s+1,\ 2\Re s+2 \end{bmatrix} \frac{P(x')Q(x'')-Q(x')P(x'')}{x'-x''}, \\ P(x) &= \left|\frac{2}{x}\right|^{\Re s} e^{-i/x+\pi\Im s\cdot \operatorname{sgn}(x)/2}\,{}_1F_1\left[\begin{matrix} s \\ 2\Re s \end{matrix}\,\middle|\,\frac{2i}{x}\right], \\ Q(x) &= \frac{2}{x}\left|\frac{2}{x}\right|^{\Re s} e^{-i/x+\pi\Im s\cdot \operatorname{sgn}(x)/2}\,{}_1F_1\left[\begin{matrix} s+1 \\ 2\Re s+2 \end{matrix}\,\middle|\,\frac{2i}{x}\right]. \end{aligned} \tag{2.1}$$

Or, equivalently,

$$\begin{aligned} K^{(s,\infty)}(x',x'') &= \frac{1}{2\pi}\,\Gamma\begin{bmatrix} s+1,\ \bar{s}+1 \\ 2\Re s+1,\ 2\Re s+2 \end{bmatrix} \frac{\widetilde{P}(x')Q(x'')-Q(x')\widetilde{P}(x'')}{x'-x''}, \\ \widetilde{P}(x) &= \left|\frac{2}{x}\right|^{\Re s} e^{-i/x+\pi\Im s\cdot \operatorname{sgn}(x)/2}\,{}_1F_1\left[\begin{matrix} s \\ 2\Re s+1 \end{matrix}\,\middle|\,\frac{2i}{x}\right]. \end{aligned} \tag{2.2}$$

The limit is uniform provided that the variables $x_1,\dots,x_n$ range over any compact subset of $\mathbb{R}^*$.

Comments. 1. As in Theorem 1.4, the formula (2.1) is directly applicable provided that s does not lie on the line $\Re s=0$, while the formula (2.2) holds for any s with $\Re s>-\frac{1}{2}$.

2. The kernel $K^{(s,\infty)}(x',x'')$ can be expressed through the M-Whittaker functions, see [Er, 6.9] for the definition. Namely,

$$P(x)=e^{-\frac{i\pi\bar{s}\,\operatorname{sgn}(x)}{2}}M_{-i\Im s,\Re s-\frac{1}{2}}\left(\frac{2i}{x}\right),\qquad Q(x)=e^{-\frac{i\pi(\bar{s}+1)\operatorname{sgn}(x)}{2}}M_{-i\Im s,\Re s+\frac{1}{2}}\left(\frac{2i}{x}\right). \tag{2.3}$$

3. The symmetry property (1.9) of the pseudo-Jacobi polynomials implies that

$$P(-x)=P(x)\,|_{s\leftrightarrow\bar{s}}\,,\qquad Q(-x)=-Q(x)\,|_{s\leftrightarrow\bar{s}}\,, \tag{2.4}$$

which can also be verified directly from (2.3) using the formula [Er, 6.9(7)]:

$$M_{\kappa,\mu}(t)=e^{i\epsilon\pi\left(\mu+\frac{1}{2}\right)}M_{-\kappa,\mu}(-t),\qquad \epsilon=\begin{cases}1, & \Im t>0,\\ -1, & \Im t<0.\end{cases}$$

It follows that the correlation kernel $K^{(s,\infty)}(x',x'')$ remains invariant when x',x'',s are replaced by $-x',-x'',\bar{s}$ (there is one more change of sign in the denominator $(x'-x'')$).

4. Formula (2.4) implies that the functions $P(x)$ and $Q(x)$ are real–valued, which agrees with the fact that the pseudo-Jacobi polynomials have real coefficients. Hence, the kernel $K^{(s,\infty)}(x',x'')$ is real symmetric.

5. When s is real, the confluent hypergeometric function ${}_1F_1$ turns into the Bessel function, and the expressions for P and Q can be written as follows:

$$\begin{aligned} P(x) &= 2^{2s-1/2}\Gamma(s+1/2)|x|^{-1/2}J_{s-1/2}\left(\frac{1}{|x|}\right), \\ Q(x) &= \operatorname{sgn}(x)2^{2s+1/2}\Gamma(s+3/2)|x|^{-1/2}J_{s+1/2}\left(\frac{1}{|x|}\right). \end{aligned}$$

6. For $s = 0$ the Bessel functions with indices $\pm\frac{1}{2}$ degenerate to trigonometric functions, and we get

$$P(x)\mid_{s=0} = \cos(\tfrac{1}{x}), \qquad Q(x)\mid_{s=0} = 2\sin(\tfrac{1}{x}),$$
$$K^{(0,\infty)}(x', x'') = \frac{1}{\pi}\,\frac{\sin(\frac{1}{x''} - \frac{1}{x'})}{x' - x''}\,.$$

Changing the variable, $y = \frac{1}{\pi x}$, and taking into account the corresponding transformation of the differential dx we get the sine kernel, in accordance with (0.9).

Proof of Theorem 2.1. We will show that

$$\lim_{N\to\infty} (\operatorname{sgn}(x')\operatorname{sgn}(x''))^N N \cdot K^{(s,N)}(Nx', Nx'') = K^{(s,\infty)}(x', x''), \quad x', x'' \in \mathbb{R}^*,$$

uniformly on compact sets in $\mathbb{R}^*$. Note that the factor $(\operatorname{sgn}(x')\operatorname{sgn}(x''))^N$ does not affect the determinantal formula.

We start with the formula (1.15). First of all, we remark that

$$\frac{\Gamma(2\Re s + N + 1)}{\Gamma(N)} \sim N^{2\Re s + 1},$$

which easily follows from the Stirling formula.

Next, we will examine the asymptotics of

$$\boldsymbol{p}_N(Nx)\sqrt{\phi(Nx)}, \quad \boldsymbol{p}_{N-1}(Nx)\sqrt{\phi(Nx)}, \qquad N \to \infty.$$

Here we will assume that x is not a real but a complex variable ranging in a neighborhood of a point $x_0 \in \mathbb{R}^*$. This will allow us to overcome the difficulty related to the singularity $x' - x'' = 0$ in the denominator of (1.15) by making use of the Cauchy integral formula.

The asymptotics of the hypergeometric functions entering the formulas (1.18) and (1.19) are as follows:

$$\lim_{N\to\infty} {}_2F_1\left[\begin{matrix}-N,\ s\\ 2\Re s\end{matrix}\,\middle|\,\frac{2}{1+iNx}\right] = {}_1F_1\left[\begin{matrix}s\\ 2\Re s\end{matrix}\,\middle|\,\frac{2i}{x}\right],$$
$$\lim_{N\to\infty} {}_2F_1\left[\begin{matrix}-N+1,\ s+1\\ 2\Re s + 2\end{matrix}\,\middle|\,\frac{2}{1+iNx}\right] = {}_1F_1\left[\begin{matrix}s+1\\ 2\Re s + 2\end{matrix}\,\middle|\,\frac{2i}{x}\right].$$

Indeed, this is a special case of the well-known limit relation

$$\lim_{|a|\to\infty} {}_2F_1\left[\begin{matrix}a,\ b\\ c\end{matrix}\,\middle|\,\frac{z}{a}\right] = {}_1F_1\left[\begin{matrix}b\\ c\end{matrix}\,\middle|\,z\right], \qquad z \in \mathbb{C}.$$

This can be readily verified using the integral representation of the hypergeometric function written in the form

$${}_2F_1\left[\begin{matrix}a,\ b\\ c\end{matrix}\,\middle|\,\frac{z}{a}\right] = \Gamma(c)\left\langle \frac{t_+^{b-1}}{\Gamma(b)}\,\frac{(1-t)_+^{c-b-1}}{\Gamma(c-b)}\,,\ \frac{1}{(1-tz/a)^a}\right\rangle,$$

where the brackets denote the pairing between a generalized function (which in the present case is supported by $[0, 1]$) and a test function, and t is the argument of both functions. Note that the limit is uniform provided that z ranges over a bounded subset of $\mathbb{C}$.

The asymptotics of the remaining terms look as follows:

$$\lim_{N\to\infty}(\pm 1)^N(Nx-i)^N\sqrt{\phi(Nx)}\sim N^{-\Re s}(\pm x)^{-\Re s}e^{-i/x}e^{\pm\pi\Im s},$$

where $\pm$ is the sign of $\Re x$ and the limit is uniform on compact subsets in the open right or left half-plane. Indeed, assume $\Re x>0$. In the transformations below any expression of the form z^c with $c\in\mathbb{C}$ is understood as a holomorphic function in the domain $\mathbb{C}\setminus(-\infty,0]$. We have

$$\begin{aligned}(Nx-i)^N\sqrt{\phi(Nx)}&=(Nx-i)^N(1+iNx)^{-(s+N)/2}(1-iNx)^{-(\bar s+N)/2}\\&=(Nx)^N(iNx)^{-(s+N)/2}(-iNx)^{-(\bar s+N)/2}\\&\quad\times\left(1-\frac{i}{Nx}\right)^N\left(1+\frac{1}{iNx}\right)^{-(s+N)/2}\left(1-\frac{1}{iNx}\right)^{-(\bar s+N)/2}\\&=N^{-\Re s}x^{-\Re s}i^{-(s+N)/2}(-i)^{-(\bar s+N)/2}\\&\quad\times\left(1-\frac{i}{Nx}\right)^N\left(1+\frac{1}{iNx}\right)^{-(s+N)/2}\left(1-\frac{1}{iNx}\right)^{-(\bar s+N)/2}\\&\sim N^{-\Re s}x^{-\Re s}e^{\pi\Im s}e^{-i/x}.\end{aligned}$$

For $\Re x<0$ the argument is similar.

Combining all these asymptotics we get the desired result. □

3. The Hua–Pickrell Measures

In this section we define the Hua–Pickrell measures. They form a 2-parameter family of $U(\infty)$-invariant probability measures on the space of infinite Hermitian matrices.

Let $H(N)$ denote the real vector space formed by complex Hermitian $N\times N$ matrices, $N=1,2,\dots$ Let H stand for the space of all infinite Hermitian matrices $X=[X_{i,j}]_{i,j=1}^{\infty}$. For $X\in H$ and $N=1,2,\dots$, we denote by $\theta_N(X)\in H(N)$ the upper left $N\times N$ corner of X. Using the projections $\theta_N\ H\to H(N)$, $N=1,2,\dots$, we may identify H with the projective limit space $\varprojlim H(N)$. We equip H with the corresponding projective limit topology. We will also use the Borel structure on H generated by this topology.

Let $U(N)$ be the group of unitary $N\times N$ matrices, $N=1,2,\dots$ For any N, we embed $U(N)$ into $U(N+1)$ using the mapping $u\mapsto\begin{bmatrix}u&0\\0&1\end{bmatrix}$. Let $U(\infty)=\varinjlim U(N)$ denote the corresponding inductive limit group. We regard $U(\infty)$ as the group of infinite unitary matrices $U=[U_{ij}]_{i,j=1}^{\infty}$ with finitely many entries $U_{ij}\neq\delta_{ij}$. The group $U(\infty)$ acts on the space H by conjugations.

Proposition 3.1. *For any $s\in\mathbb{C}$, $\Re s>-\frac{1}{2}$, there exists a probability Borel measure $m^{(s)}$ on H, characterized by the following property: for any $N=1,2,\dots$, the image of*

$m^{(s)}$ under the projection θ_N is the probability measure $m^{(s,N)}$ on $H(N)$ defined by

$$m^{(s,N)}(dX) = (\text{const}_N)^{-1} \det((1+iX)^{-s-N}) \det((1-iX)^{-\bar{s}-N}) \times \prod_{j=1}^{N} dX_{jj} \prod_{1\le j<k\le N} d(\Re X_{jk}) d(\Im X_{jk}), \tag{3.1}$$

$$\textit{where} \quad \text{const}_N = \prod_{j=1}^{N} \frac{\pi^j \Gamma(s+\bar{s}+j)}{2^{s+\bar{s}+2j-2}\Gamma(s+j)\Gamma(\bar{s}+j)}.$$

The measure $m^{(s)}$ is invariant under the action of $U(\infty)$.

Comments. 1. For $X \in H(N)$ and $z \in \mathbb{C}$ we define the matrix $(1 \pm iX)^z$ by means of the functional calculus. This makes the expression

$$f_N(X) = \det((1+iX)^{-s-N}) \det((1-iX)^{-\bar{s}-N}), \qquad X \in H(N)$$

meaningful. Equivalently, denoting by $x_1, \dots, x_N$ the eigenvalues of X,

$$f_N(X) = \prod_{j=1}^{N} (1+ix_j)^{-s-N} (1-ix_j)^{-\bar{s}-N}, \tag{3.2}$$

where we use the analytic continuation of the function $(\dots)^z$ from the positive axis to the region $\mathbb{C} \setminus (-\infty, 0]$.

2. When s is real, the expression (3.2) takes a simpler form

$$f_N(X) = (\det(1+X^2))^{-s-N}, \qquad X \in H(N), \quad s \in \mathbb{R}.$$

Proof. Step 1. First of all, note that $f_N(X) \ge 0$. Therefore, if f_N is integrable then it defines a finite measure on $H(N)$.

Fix $N \ge 2$ and write an arbitrary matrix $X \in H(N)$ in the block form

$$X = \begin{bmatrix} Y & \xi \\ \xi^* & t \end{bmatrix}, \qquad Y \in H(N-1), \quad \xi \in \mathbb{C}^{N-1}, \quad t \in \mathbb{R}.$$

We will prove that for any $Y \in H(N-1)$ the integral of f_N over ξ, t is finite and it is equal to

$$\int_{(\xi,t)\in\mathbb{C}^{N-1}\times\mathbb{R}} f_N\left(\begin{bmatrix} Y & \xi \\ \xi^* & t \end{bmatrix}\right) \cdot \prod_{j=1}^{N} d(\Re \xi_j) d(\Im \xi_j) \cdot dt = f_{N-1}(Y) \cdot \frac{\pi^N \Gamma(s+\bar{s}+N)}{2^{s+\bar{s}+2N-2}\Gamma(s+N)\Gamma(\bar{s}+N)}. \tag{3.3}$$

For $N = 1$, Y and t disappear, and the claim is that the integral of f_1 over $\mathbb{R}$ is finite and it is given by

$$\int_{t\in\mathbb{R}} f_1(t)dt = \int_{-\infty}^{\infty} (1+it)^{-s-1}(1-it)^{-\bar{s}-1} dt = \frac{\pi \Gamma(s+\bar{s}+1)}{2^{s+\bar{s}}\Gamma(s+1)\Gamma(\bar{s}+1)}. \tag{3.4}$$

Let us show (3.3) and (3.4) imply the proposition. Indeed, using induction on N we see that the integral of f_N over $H(N)$ is finite and equals const_N. Thus, the measure $m^{(s,N)}$ is correctly defined for any N.

Next, (3.3) implies that the measures $m^{(s,N)}$ and $m^{(s,N-1)}$ are consistent with the projection $X \mapsto Y$ from $H(N)$ to $H(N-1)$. [2] Since H coincides with the projective limit of the spaces $H(N)$ as $N \to \infty$, we conclude that the measure $m^{(s)}$ exists and is unique.

Finally, $m^{(s)}$ is invariant under the action of $U(\infty)$, because each $m^{(s,N)}$ is invariant under the action of $U(N)$ for all $N = 1, 2, \dots$

Step 2. We proceed to the proof of (3.3) and (3.4). The latter formula follows from formula (3.9) in Lemma 3.3 below. The former formula is proved in [Hua, Theorem 2.1.5] for real s, and we employ his argument with slight modifications. Applying Lemma 3.2 (see below) we get

$$f_N(X) = \det((1+iY)^{-s-N})(1+it+\xi^*(1+iY)^{-1}\xi)^{-s-N} \times \det((1-iY)^{-\bar{s}-N})(1-it+\xi^*(1-iY)^{-1}\xi)^{-\bar{s}-N}. \quad (3.5)$$

Next, note that the integral (3.3) is invariant under the conjugation of Y by a matrix $V \in U(N-1)$. Indeed to see this, we use the invariance of the function f_N and make a change of a variable, $V\xi \mapsto \xi$. Therefore, without loss of generality we may assume that Y is a diagonal matrix. Denoting its diagonal entries (which are real numbers) as $y_1, \dots, y_{N-1}$ and using (3.5) we reduce the integral (3.3) to

$$\prod_{j=1}^{N-1}(1+iy_j)^{-s-N}(1-iy_j)^{-\bar{s}-N} \times \int_{(\xi,t)\in\mathbb{C}^{N-1}\times\mathbb{R}} \left(1+\sum_{j=1}^{N-1}\frac{|\xi_j|^2}{1+y_j^2}+i\left(t-\sum_{j=1}^{N-1}\frac{|\xi_j|^2 y_j}{1+y_j^2}\right)\right)^{-s-N}$$
$$\times \left(1+\sum_{j=1}^{N-1}\frac{|\xi_j|^2}{1+y_j^2}-i\left(t-\sum_{j=1}^{N-1}\frac{|\xi_j|^2 y_j}{1+y_j^2}\right)\right)^{-\bar{s}-N} \prod_{j=1}^{N} d(\Re\xi_j)d(\Im\xi_j)\cdot dt. \quad (3.6)$$

This integral is easily simplified. First, assuming that the variables $\xi_1, \dots, \xi_{N-1}$ are fixed, we make a change of variable

$$t - \sum_{j=1}^{N-1}\frac{|\xi_j|^2 y_j}{1+y_j^2} \mapsto t.$$

Next, we change the variables ξ_j,

$$\frac{\xi_j}{\sqrt{1+y_j^2}} \mapsto \xi_j, \qquad j = 1, \dots, N-1,$$

[2] For real s, this fact was discovered by Hua Loo–Keng [Hua]. As we learnt from Peter Forrester, it was also discussed in the physics literature, see [Br].

which gives rise to the factor $\prod(1+y_j)^2$. Then (3.6) is reduced to

$$\prod_{j=1}^{N-1}(1+iy_j)^{-s-N+1}(1-iy_j)^{-\bar{s}-N+1}\cdot\int_{(\xi,t)\in\mathbb{C}^{N-1}\times\mathbb{R}}\left(1+\sum_{j=1}^{N-1}|\xi_j|^2+it\right)^{-s-N} \times\left(1+\sum_{j=1}^{N-1}|\xi_j|^2-it\right)^{-\bar{s}-N}\prod_{j=1}^{N}d(\Re\xi_j)d(\Im\xi_j)\cdot dt. \quad (3.7)$$

Setting $r=\sum|\xi_j|^2$ we readily reduce (3.7) to

$$\prod_{j=1}^{N-1}(1+iy_j)^{-s-N+1}(1-iy_j)^{-\bar{s}-N+1} \cdot\frac{\pi^{N-1}}{\Gamma(N-1)}\int_{r\geq 0}\int_{t\in\mathbb{R}}(1+r+it)^{-s-N}(1+r-it)^{-\bar{s}-N}r^{N-2}drdt.$$

By Lemma 3.3, the double integral is finite and its value is given by (3.9), where we substitute $a=s+N$, $b=\bar{s}+N$ (the assumption of Lemma 3.3 is satisfied because $\Re s>-\frac{1}{2}$). This implies (3.3). □

We proceed to the proof of two lemmas which were used in Proposition 3.1.

Lemma 3.2. *Consider the $N\times N$ matrix analog of the right halfplane in $\mathbb{C}$:*

$$\mathrm{Mat}(N,\mathbb{C})_+=\{A\in\mathrm{Mat}(N,\mathbb{C})\mid A+A^*>0\}.$$

Write $N\times N$ matrices in the block form according to a partition $N=N_1+N_2$,

$$A=\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}.$$

Then for $z\in\mathbb{C}$ and $A\in\mathrm{Mat}(N,\mathbb{C})_+$ the following relation holds

$$\det(A^z)=\det(A_{11}^z)\det((A_{22}-A_{21}A_{11}^{-1}A_{12})^z). \quad (3.8)$$

Proof. First of all, we show that both sides in (3.8) make sense. Note that if $A\in\mathrm{Mat}(N,\mathbb{C})_+$ then any eigenvalue λ of A lies in the open right halfplane (indeed, if $\xi\in\mathbb{C}^N$ is an eigenvector with the eigenvalue λ then $0<((A+A^*)\xi,\xi)=2\Re\lambda(\xi,\xi)$, which implies $\Re\lambda>0$). Therefore, we can define the matrix A^z by means of the functional calculus. Next, note that the matrices A_{11} and $A_{22}-A_{21}A_{11}^{-1}A_{12}$ also belong to the matrix right halfplanes. Indeed, for the former matrix this is evident, and for the latter matrix this follows from the fact that $A^{-1}\in\mathrm{Mat}(N,\mathbb{C})_+$ and

$$A_{22}-A_{21}A_{11}^{-1}A_{12}=((A^{-1})_{22})^{-1}.$$

Thus, the expressions $(\dots)^z$ in the right-hand side of (3.8) are well-defined.

Since both sides of (3.8) are holomorphic functions in A in the connected region $\mathrm{Mat}(N,\mathbb{C})_+$, we may assume, without loss of generality, that A lies in a small neighborhood of the matrix 1. Then we may interchange the symbol of determinant and exponentiation. This reduces (3.8) to the classical formula for the determinant of a block matrix,

$$\det A=\det A_{11}\cdot\det(A_{22}-A_{21}A_{11}^{-1}A_{12}),$$

see, e.g. [Ga, Ch. II, §5.3]. □

Lemma 3.3. *We have*

$$\frac{\pi^{N-1}}{\Gamma(N-1)}\int_{r\ge 0}\int_{t\in\mathbb{R}}(1+r+it)^{-a}(1+r-it)^{-b}r^{N-2}drdt$$
$$=\frac{\pi^N\Gamma(a+b-N)}{2^{a+b-2}\Gamma(a)\Gamma(b)},\qquad a,b\in\mathbb{C},\quad \Re(a+b)>N,\quad N>1,\qquad (3.9)$$

and

$$\int_{t\in\mathbb{R}}(1+it)^{-a}(1-it)^{-b}dt=\frac{\pi\Gamma(a+b-1)}{2^{a+b-2}\Gamma(a)\Gamma(b)},\qquad a,b\in\mathbb{C},\quad \Re(a+b)>1. \qquad (3.10)$$

Proof. The integral (3.10) is readily reduced to a known integral, see [Er, 1.5 (30)].

To evaluate the integral (3.9), we make a change of variable, $t\mapsto(1+r)t$. The integral splits into a product of two integrals, one of which is (3.10) and the other one is the integral

$$\int_{r\ge 0}(1+r)^{-a-b+1}\frac{r^{N-2}}{\Gamma(N-1)}dr=\frac{\Gamma(a+b-N)}{\Gamma(a+b-1)}.$$

This proves (3.9).

Note also that (3.10) is a degeneration of (3.9), because $r_+^{N-2}/\Gamma(N-1)$ degenerates to the delta function $\delta(r)$ at $N=1$. □

Let $\mathbb{C}_+$ denote the right halfplane. Following Neretin [Ner2] we define a map

$$H\ni X=[X_{jk}]_{j,k=1}^{\infty}\mapsto(\zeta_1,\zeta_2,\dots)\in\mathbb{R}\times\mathbb{C}_+^{\infty} \qquad (3.11)$$

as follows. For any $N=2,3,\dots$, write the matrix $\theta_N(X)=[X_{jk}]_{j,k=1}^N$ in the block form

$$\theta_N(X)=\begin{bmatrix}\theta_{N-1}(X) & \xi\\ \xi^* & t\end{bmatrix}$$

and then set

$$\zeta_N=it+\xi^*(1+i\theta_{N-1})^{-1}\xi\in\mathbb{C}_+.$$

Finally, set $\zeta_1=X_{11}\in\mathbb{R}$.

Proposition 3.4. *The pushforward of the measure $m^{(s)}$ under the map* (3.11) *is a product measure $\mu_1\times\mu_2\times\dots$ on the space $\mathbb{R}\times\mathbb{C}_+^{\infty}$. Here $\mu_1,\mu_2,\dots$ are the following probability measures:*

$$\mu_1(dt)=\frac{2^{s+\bar s}\Gamma(s+1)\Gamma(\bar s+1)}{\pi\Gamma(s+\bar s+1)}(1+it)^{-s-1}(1-it)^{-\bar s-1}dt$$

and, for $N\ge 2$, $\zeta=r+it\in\mathbb{C}_+$,

$$\mu_N(d\zeta)=\frac{2^{s+\bar s+2N-2}\Gamma(s+N)\Gamma(\bar s+N)}{\pi\Gamma(s+\bar s+N)}(1+\zeta)^{-s-N}(1+\bar\zeta)^{-\bar s-N}\frac{r^{N-2}}{\Gamma(N-1)}drdt. \qquad (3.12)$$

Proof. This follows from the proof of Proposition 3.1. □

Theorem 3.5. *The Hua–Pickrell measures $m^{(s)}$ are pairwise disjoint. I.e., if s', s'' are two distinct values of the parameter s then there exist two disjoint Borel sets in H supporting the measures $m^{(s')}$ and $m^{(s'')}$, respectively.*

Proof. We will apply Kakutani's theorem [Ka]. Assume first we are given two probability measures, μ' and μ'', defined on the same Borel space. Take any measure ν such that both μ' and μ'' are absolutely continuous with respect to ν. For instance, $\nu = \mu' + \mu''$. Denote by μ'/ν and μ''/ν the respective Radon-Nikodym derivatives. The measure $\sqrt{\frac{\mu'}{\nu}\frac{\mu''}{\nu}} \cdot \nu$ does not depend on the choice of ν. Denote it by $\sqrt{\mu'\mu''}$ and set

$$\langle \mu', \mu'' \rangle = \int \sqrt{\mu'\mu''}.$$

We have $0 \leq \langle \mu', \mu'' \rangle \leq 1$. Moreover, $\langle \mu', \mu'' \rangle = 1$ is equivalent to $\mu' = \mu''$ while $\langle \mu', \mu'' \rangle = 0$ exactly means that μ' and μ'' are disjoint.

Next, assume $\mu' = \mu'_1 \times \mu'_2 \times \dots$ and $\mu'' = \mu''_1 \times \mu''_2 \times \dots$ are two product probability measures defined on the same countably infinite product space. Kakutani's theorem [Ka] says that μ' and μ'' are disjoint if the infinite product $\prod_{N=1}^{\infty} \langle \mu'_N, \mu''_N \rangle$ is divergent, i.e., the partial products tend to 0.

Finally, consider the product space $\mathbb{R} \times \mathbb{C}_+^{\infty}$ and take as μ' and μ'' the pushforwards of measures $m^{(s')}$ and $m^{(s'')}$, respectively, as explained in Proposition 3.4. We prove that μ' and μ'' are disjoint. Then this immediately implies that the initial measures $m^{(s')}$ and $m^{(s'')}$ are disjoint.

We omit the value $N = 1$ which plays a special role and calculate the integral defining $\langle \mu'_N, \mu''_N \rangle$ for $N \geq 2$. By (3.12) and (3.9) we get

$$\langle \mu'_N, \mu''_N \rangle = \sqrt{\frac{\Gamma(s'+N)\Gamma(\overline{s'}+N)\Gamma(s''+N)\Gamma(\overline{s''}+N)}{\Gamma(s'+\overline{s'}+N)\Gamma(s''+\overline{s''}+N)}} \, \frac{\Gamma(s+\bar{s}+N)}{\Gamma(s+N)\Gamma(\bar{s}+N)},$$
$$s = \frac{s'+s''}{2}.$$

The classical asymptotic formula for the ratio of two Γ-functions, see [Er, 1.18(4)], implies that

$$\frac{\Gamma(z+N)\Gamma(\bar{z}+N)}{\Gamma(z+\bar{z}+N)\Gamma(N)} \sim 1 - \frac{z\bar{z}}{N} + O\left(\frac{1}{N^2}\right).$$

It follows that

$$\langle \mu'_N, \mu''_N \rangle \sim 1 - \frac{|s'-s''|^2}{4N} + O\left(\frac{1}{N^2}\right).$$

Thus, the product of $\langle \mu'_N, \mu''_N \rangle$'s is divergent. □

4. Ergodic Measures

In this section we recall the classification theorem and some other known results on $U(\infty)$-invariant ergodic probability measures on the space of infinite Hermitian matrices.

Consider the natural embeddings

$$H(N) \to H(N+1), \qquad A \mapsto \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix},$$

and denote by $H(\infty)$ the corresponding inductive limit space $\varinjlim H(N)$. Then $H(\infty)$ is identified with the space of infinite Hermitian matrices with finitely many nonzero entries. We equip $H(\infty)$ with the inductive limit topology. In particular, a function $f\colon H(\infty) \to \mathbb{C}$ is continuous if its restriction to $H(N)$ is continuous for any N.

There is a natural pairing

$$H(\infty) \times H \to \mathbb{R}, \qquad (A, X) \mapsto \operatorname{tr}(AX).$$

H is the algebraic dual space of $H(\infty)$ with respect to this pairing.

Using the map

$$H \ni X \mapsto \{X_{ii}\}_{i=1}^{\infty} \sqcup \{\Re X_{ij}, \Im X_{ij}\}_{i<j}$$

we can identify H, as a topological vector space, with the infinite product space $\mathbb{R}^\infty = \mathbb{R} \times \mathbb{R} \times \cdots$. Under this identification, $H(\infty) \subset H$ turns into $\mathbb{R}_0^\infty := \bigcup_{n\ge 1} \mathbb{R}^n$, and the pairing defined above turns into the standard pairing between $\mathbb{R}_0^\infty$ and $\mathbb{R}^\infty$.

Given a Borel probability measure M on H, we define its Fourier transform, or characteristic function, as the following function on $H(\infty)$:

$$A \mapsto \int_H e^{i\,\operatorname{tr}(AX)} M(dX). \tag{4.1}$$

The group $U(\infty)$ acts by conjugations both on $H(\infty)$ and H, and the pairing between these two spaces is, clearly, $U(\infty)$-invariant. Each matrix from $H(\infty)$ can be brought by a conjugation to a diagonal matrix $\operatorname{diag}(r_1, r_2, \dots)$ with finitely many nonzero entries. It follows that the Fourier transform of a $U(\infty)$-invariant measure on H is uniquely determined by its values on the diagonal matrices from $H(\infty)$.

Set

$$\Omega = \{\omega = (\alpha^+, \alpha^-, \gamma_1, \delta) \in \mathbb{R}^{2\infty+2} = \mathbb{R}^\infty \times \mathbb{R}^\infty \times \mathbb{R} \times \mathbb{R} \mid$$

$$\alpha^+ = (\alpha_1^+ \ge \alpha_2^+ \ge \cdots \ge 0), \qquad \alpha^- = (\alpha_1^- \ge \alpha_2^- \ge \cdots \ge 0),$$
$$\gamma_1 \in \mathbb{R}, \qquad \delta \ge 0, \qquad \sum (\alpha_i^+)^2 + \sum (\alpha_i^-)^2 \le \delta\}.$$

This is a closed subset of $\mathbb{R}^{2\infty+2}$.

Denote

$$\gamma_2 = \delta - \sum (\alpha_i^+)^2 - \sum (\alpha_i^-)^2 \ge 0.$$

In this notation we have

Proposition 4.1. *There exists a parametrization of ergodic $U(\infty)$-invariant probability measures on the space H by the points of the space Ω. Given ω, the Fourier transform* (4.1) *of the corresponding ergodic measure M^ω is given by*

$$\int_{X\in H} e^{i\,\operatorname{tr}(\operatorname{diag}(r_1,\dots,r_n,0,0,\dots)\,X)} M^\omega(dX)$$
$$= \prod_{j=1}^{n} \left\{ e^{i\gamma_1 r_j - \gamma_2 r_j^2} \prod_{k=1}^{\infty} \frac{e^{-i\alpha_k^+ r_j}}{1 - i\alpha_k^+ r_j} \prod_{k=1}^{\infty} \frac{e^{i\alpha_k^- r_j}}{1 + i\alpha_k^- r_j} \right\}.$$

Proof. See [Pi2, Proposition 5.9] and [OV, Theorem 2.9]. □

Remark 4.2. If only one of the parameters $\alpha_i^{\pm}$, γ_1, γ_2 is distinct from 0 then the corresponding ergodic measure is called *elementary*. A description of the elementary measures can be found in [OV, Corollaries 2.5–2.7]. Note, in particular, that the elementary measures corresponding to the parameter γ_2 are standard Gaussian measures on H, see [OV, Corollary 2.6]. Since the expression of Proposition 4.1 is multiplicative with respect to the coordinates of ω, any ergodic measure is a convolution product of elementary ergodic measures.

For $N = 1, 2, \dots$, let $\mathbb{S}_N \subset \mathbb{R}^N$ denote the set of N-tuples of weakly decreasing real numbers

$$\lambda = (\lambda_1 \geq \cdots \geq \lambda_N).$$

Given $\lambda \in \mathbb{S}_N$, let $\mathrm{Orb}(\lambda)$ denote the set of matrices $X \in H(N)$ with eigenvalues $\lambda_1, \dots, \lambda_N$. The sets of the form $\mathrm{Orb}(\lambda)$ are exactly the $U(N)$-orbits in $H(N)$.

Given $\lambda \in \mathbb{S}_N$, we set

$$a_i^+(\lambda) = \begin{cases} \dfrac{\max(\lambda_i, 0)}{N}, & i = 1, \dots, N, \\ 0, & i = N+1, N+2, \dots, \end{cases}$$

$$a_i^-(\lambda) = \begin{cases} \dfrac{\max(-\lambda_{N+1-i}, 0)}{N}, & i = 1, \dots, N, \\ 0, & i = N+1, N+2, \dots. \end{cases}$$

Equivalently, if k and l denote the numbers of strictly positive terms in $\{a_i^+\}$ and $\{a_i^-\}$, respectively then

$$\lambda = (a_1^+(\lambda), \dots, a_k^+(\lambda), 0, \dots, 0, -a_l^-(\lambda), \dots, -a_1^-(\lambda)).$$

Further, we set

$$c(\lambda) = \sum_{i=1}^{\infty} a_i^+(\lambda) - \sum_{i=1}^{\infty} a_i^-(\lambda) = \frac{\lambda_1 + \cdots + \lambda_N}{N},$$

$$d(\lambda) = \sum_{i=1}^{\infty} (a_i^+(\lambda))^2 + \sum_{i=1}^{\infty} (a_i^-(\lambda))^2 = \frac{\lambda_1^2 + \cdots + \lambda_N^2}{N^2}.$$

By virtue of [OV, Theorem 3.3], any ergodic measure can be approximated by orbital measures on the spaces $H(N)$ as $N \to \infty$. The next result provides an explicit description of the approximating orbital measures. It also clarifies the meaning of the parameters in Proposition 4.1.

Proposition 4.3. *Let* $\{\mathrm{Orb}(\lambda^{(N)}) \mid \lambda^{(N)} \in \mathbb{S}_N\}$ *be a sequence of orbits and let* $\{M^{(N)}\}$ *be the sequence of the corresponding orbital measures on the spaces* $H(N)$, $N = 1, 2, \dots$. *We view each* $M^{(N)}$ *as a measure on* H.

The measures $M^{(N)}$ *weakly converge to a measure* M *on* H, *i.e.,* $\langle f, M^{(N)} \rangle \to \langle f, M \rangle$ *for any bounded continuous function* f *on* H, *if and only if there exist limits*

$$\alpha_i^{\pm} = \lim_{N\to\infty} a_i^{\pm}(\lambda^{(N)}), \qquad i = 1, 2, \dots,$$

$$\gamma_1 = \lim_{N\to\infty} c(\lambda^{(N)}),$$

$$\delta = \lim_{N\to\infty} d(\lambda^{(N)}).$$

If this condition holds then the collection $\omega = (\{\alpha_i^+\}, \{\alpha_i^-\}, \gamma_1, \delta)$ *is a point of* Ω *and the limit measure* M *coincides with the ergodic measure* M^ω.

Proof. See [OV, Theorem 4.1]. □

Proposition 4.4. *For any* $U(\infty)$*-invariant probability measure* M *on* H *there exists a probability measure* P *on* Ω *such that*

$$M = \int_\Omega M^\omega P(d\omega),$$

which means that for any bounded Borel function f *on* H,

$$\langle f, M\rangle = \int_\Omega \langle f, M^\omega\rangle P(d\omega). \tag{4.2}$$

Such measure P *is unique. Conversely, any probability measure* P *on* Ω *arises in this way from a certain measure* M.

Proof. This follows from Theorem 9.1 and Proposition 9.4. □

We will call P the *spectral measure* for M.

5. Approximation of Spectral Measures

In this section we show that the spectral measure for a $U(\infty)$-invariant probability measure M on H can be obtained as a certain limit of finite–dimensional projections of M.

For $X \in H$, let $\lambda^{(N)}(X) \in \mathbb{S}_N$ denote the spectrum of the finite matrix $\theta_N(X) \in H(N)$. Let us say that $X \in H$ is *regular* if there exist limits

$$\begin{aligned} \alpha_i^\pm(X) &= \lim_{N\to\infty} a_i^\pm(\lambda^{(N)}(X)), \qquad i = 1, 2, \ldots, \\ \gamma_1(X) &= \lim_{N\to\infty} c(\lambda^{(N)}(X)), \\ \delta(X) &= \lim_{N\to\infty} d(\lambda^{(N)}(X)). \end{aligned} \tag{5.1}$$

Let $H_{\mathrm{reg}} \subset H$ denote the subset of regular matrices in H. Since $\lambda^{(N)}(X)$ is a continuous function in X for any N, the functions $a_i^\pm(\lambda^{(N)}(X))$, $c(\lambda^{(N)}(X))$, and $d(\lambda^{(N)}(X))$ are also continuous. It follows that H_{reg} is a Borel subset of H (more precisely, a subset of type $F_{\sigma\delta}$).

Theorem 5.1. *Any* $U(\infty)$*-invariant probability measure on* H *is supported by* H_{reg}.

Proof. First, let M be an ergodic $U(\infty)$-invariant probability measure on H. By Vershik's ergodic theorem (see [OV, Theorem 3.2]), M is concentrated on the set of those $X \in H$ for which the orbital measures $\mathrm{Orb}(\lambda^{(N)}(X))$ weakly converge to M. By Proposition 4.3, this set consists of exactly those X for which the limits (5.1) exist and coincide with the parameters of M given in Proposition 4.1. All such matrices X belong to H_{reg}, so that M is supported by H_{reg}. Thus, the claim of the theorem holds for ergodic measures.

Now let M be an arbitrary $U(\infty)$-invariant probability measure on H and P be its spectral measure. Apply (4.2) by taking as f the characteristic function of the set $H_{\mathrm{reg}} \subset H$. We have $\langle f, M^\omega\rangle = 1$ for any $\omega \in \Omega$. Since P is a probability measure, we get from (4.2) that $\langle f, M\rangle = 1$. Therefore, H_{reg} is of full measure with respect to M. □

Let $\pi : H_{\text{reg}} \to \Omega$ denote the map sending $X \in H_{\text{reg}}$ to the point ω with the coordinates defined by (5.1). This is a Borel map, because it is a pointwise limit of a sequence of continuous maps.

Theorem 5.2. *Let M be a $U(\infty)$-invariant probability measure on H and let $M \mid_{H_{\text{reg}}}$ be the restriction of M to H_{reg}, which is correctly defined by Theorem 5.1.*

The pushforward of the measure $M \mid_{H_{\text{reg}}}$ under the Borel map π introduced above coincides with the spectral measure P.

Proof. Let F be an arbitrary bounded Borel function on Ω and f be its pullback on H_{reg}. We must prove that $\langle f, M\rangle = \langle F, P\rangle$.

By definition of P, we have

$$\langle f, M\rangle = \int_{\Omega} \langle f, M^{\omega}\rangle P(d\omega).$$

On the other hand, we know that for any $\omega \in \Omega$, the measure M^{ω} is supported by $\pi^{-1}(\omega) \subset H_{\text{reg}}$ (see the beginning of the proof of Theorem 5.1). Finally, by the definition of f, we have $f \mid_{\pi^{-1}(\omega)} \equiv F(\omega)$, so that $\langle f, M^{\omega}\rangle = F(\omega)$.

Therefore, the integral in the right-hand side is equal to $\langle F, P\rangle$. $\square$

For $N = 1, 2, \dots$, let $\pi_N : H \to \Omega \subset \mathbb{R}^{2\infty+2}$ denote the composition of the maps $H \ni X \mapsto \lambda^{(N)}(X) \in \mathbb{S}_N$ and $\mathbb{S}_N \ni \lambda \mapsto (\{a_i^+(\lambda)\}, \{a_i^-(\lambda)\}, c(\lambda), d(\lambda)) \in \Omega$.

Theorem 5.3. *Let M be a $U(\infty)$-invariant probability measure on H, P be its spectral measure, and P_N be the pushforward of M under the map $\pi_N : H \to \Omega$ defined above.*

Then P_N weakly converge to P as $N \to \infty$. That is, for any continuous bounded function F on Ω,

$$\lim_{N\to\infty} \langle F, P_N\rangle = \langle F, P\rangle.$$

Proof. By Theorem 5.1, $H_{\text{reg}} \subset H$ is of full measure with respect to M, so that we may view (H_{reg}, M) as a probability space.

We have

$$\pi_N(M) = P_N, \qquad \pi(M) = P.$$

Indeed, the first equality follows from the definition of P_N and the fact that H_{reg} is of full measure, and the second equality is given by Theorem 5.2.

Next, by the very definition of H_{reg}, we have $\pi_N(t) \to \pi(t)$ for any $t \in H_{\text{reg}}$ as $N \to \infty$, where the limit is taken with respect to the coordinatewise convergence on the space $\mathbb{R}^{2\infty+2}$. Since F is continuous, we get $F(\pi_N(t)) \to F(\pi(t))$. That is, $F \circ \pi_N$ converges to $F \circ \pi$ at any point $t \in H_{\text{reg}}$. Since these functions are uniformly bounded, it follows that

$$\int_{H_{\text{reg}}} (F \circ \pi_N)(t) M(dt) \to \int_{H_{\text{reg}}} (F \circ \pi)(t) M(dt).$$

Since $\pi_N(M) = P_N$ and $\pi(M) = P$,

$$\int_{H_{\text{reg}}} (F \circ \pi_N)(t) M(dt) = \langle F, P_N\rangle, \qquad \int_{H_{\text{reg}}} (F \circ \pi)(t) M(dt) = \langle F, P\rangle.$$

Consequently, $\langle F, P_N\rangle \to \langle F, P\rangle$. $\square$

6. The Main Result

Let $s \in \mathbb{C}$, $\Re s > -\frac{1}{2}$. Consider the Hua–Pickrell measure $m^{(s)}$. Let $P^{(s)}$ be its spectral measure and $\mathcal{P}^{(s)}$ be the corresponding point process on $\mathbb{R}^*$, see (0.8).

In this section we prove the following theorem which is our main result.

Theorem 6.1. *The correlation functions of the process $\mathcal{P}^{(s)}$ exist and coincide with the limit correlation functions from Theorem 2.1.*

Let X range over H_{reg}. Recall that in Sect. 5 we attached to X two monotone sequences $\{\alpha_i^+(X)\}$, $\{\alpha_i^-(X)\}$ and also, for any $N = 1, 2, \dots$, two monotone sequences

$$\{a_{i,N}^+(X) = a_i^+(\lambda^{(N)}(X))\}, \qquad \{a_{i,N}^-(X) = a_i^-(\lambda^{(N)}(X))\}.$$

From these data we form point configurations

$$\mathcal{C}(X) = \{\alpha_i^+(X)\} \sqcup \{-\alpha_i^-(X)\}, \qquad \mathcal{C}_N(X) = \{a_{i,N}^+(X)\} \sqcup \{-a_{i,N}^-(X)\},$$

where we omit the zero coordinates.

Let M be a $U(\infty)$-invariant probability measure on H. We restrict M to H_{reg}, which is a subset of full measure, and view (H_{reg}, M) as a probability space. Then any quantity depending on X becomes a random variable.

Let P be the spectral measure of M and let P_N be the finite–dimensional measures defined in Theorem 5.3. Recall that P_N's approximate P as $N \to \infty$.

Let $\mathcal{P}_N$ and $\mathcal{P}$ be the point processes on $\mathbb{R}^*$ corresponding to P_N and P, respectively. We may view $\mathcal{P}_N$ and $\mathcal{P}$ as the random point configurations $\mathcal{C}_N(X)$ and $\mathcal{C}(X)$, where X is viewed as the point of the probability space (H_{reg}, M).

By $\rho_k^{(N)}$ and ρ_k we denote the k^{th} correlation measures of the processes $\mathcal{P}_N$ and $\mathcal{P}$, respectively. Note that the very existence of the measures ρ_k is not evident.

For a compact set $A \subset \mathbb{R}^*$ we set

$$\mathcal{N}_{A,N}(X) = \mathrm{Card}(\mathcal{C}_N(X) \cap A), \qquad \mathcal{N}_A(X) = \mathrm{Card}(\mathcal{C}(X) \cap A).$$

These are random variables.

We know that for any fixed X and for any index $i = 1, 2, \dots, a_{i,N}^\pm(X)$ tends to $\alpha_i^\pm(X)$ as $N \to \infty$. We would like to conclude from this that $\rho_k^{(N)}$ converges to ρ_k as $N \to \infty$. The next lemma says that, under a reasonable technical assumption, this is indeed true.

Lemma 6.2. *Assume that for any compact set $A \subset \mathbb{R}^*$ there exist uniform in N estimates*

$$\mathbb{E}[\mathcal{N}_{A,N}^l] \le C_l\,, \qquad l = 1, 2, \dots,$$

where the symbol $\mathbb{E}$ stands for the expectation.

Then for any $k = 1, 2, \dots$, the correlation measure ρ_k exists and coincides with the weak limit of the measures $\rho_k^{(N)}$ as $N \to \infty$. The limit is understood in the following sense: for any continuous compactly supported function F on $(\mathbb{R}^*)^k$,

$$\lim_{N\to\infty} \langle F, \rho_k^{(N)} \rangle = \langle F, \rho_k \rangle.$$

Proof. Fix a continuous compactly supported function F on $(\mathbb{R}^*)^k$. It will be convenient to assume that F is nonnegative (this does not mean any loss of generality). Introduce random variables f and f_N as follows:

$$f(X) = \sum_{x_1,\dots,x_k \in \mathcal{C}(X)} F(x_1, \dots, x_k), \quad f_N(X) = \sum_{x_1,\dots,x_k \in \mathcal{C}_N(X)} F(x_1, \dots, x_k), \quad (6.1)$$

where the sums are taken over ordered k-tuples of points with pairwise distinct labels. Any such sum is actually finite because F is compactly supported and the point configurations are locally finite.

By the definition of the correlation measures,

$$\langle F, \rho_k \rangle = \mathbb{E}[f], \qquad \langle F, \rho_k^{(N)} \rangle = \mathbb{E}[f_N].$$

The correlation measure ρ_k exists if $\mathbb{E}[f]$ is finite for any f as above, see, e.g., [Len].

Thus, we have to prove that $\mathbb{E}[f_N] \to \mathbb{E}[f] < \infty$ as $N \to \infty$. By a general theorem (see [Shir, Ch. II, §6, Theorem 4]), it suffices to check the following two conditions:

Condition 1. $f_N(X) \to f(X)$ for any $X \in H_{\text{reg}}$.

Condition 2. The random variables f_N are uniformly integrable, that is,

$$\sup_N \int_{\{X \mid f_N(X) \ge c\}} f_N(X) M(dX) \to 0, \quad \text{as } c \to +\infty.$$

Let us check Condition 1. This condition does not depend on M, it is a simple consequence of the regularity property. Indeed, let us fix $X \in H_{\text{reg}}$. For any $\varepsilon > 0$ set $\mathbb{R}^\varepsilon = \mathbb{R} \setminus (-\varepsilon, \varepsilon)$. Choose ε so small that the function F is supported by $(\mathbb{R}^\varepsilon)^k$. Fix j so large that $\alpha_j^\pm(X) < \varepsilon$. Since $a_{j,N}^\pm(X) \to \alpha_j^\pm(X)$, we have $a_{N,j}^\pm < \varepsilon$ for all N large enough. By monotonicity, the same inequality holds for the indices $j+1, j+2, \dots$ as well.

Recall that each point $x \in \mathcal{C}_N(X)$ has the form $x = a_{i,N}^+(X)$ or $x = -a_{i,N}^-(X)$ for a certain index i. It follows that in the sums (6.1), only the points with indices $i = 1, \dots, j-1$ may really contribute. Then, using the continuity of F we conclude that $f_N(X) \to f(X)$.

Let us check Condition 2. Choose a compact set A such that F is supported by A^k. The supremum of F (let us denote it by $\sup F$) is finite. We have

$$f_N(X) \le \sup F \cdot \mathcal{N}_{A,N}(X)(\mathcal{N}_{A,N}(X)-1)\dots(\mathcal{N}_{A,N}(X)-k+1) \le \sup F \cdot (\mathcal{N}_{A,N}(X))^k.$$

Therefore, the random variables f_N are uniformly integrable provided that this is true for the random variables $(\mathcal{N}_{A,N})^k$ for any fixed k. But the latter fact follows from the assumption of the theorem and Chebyshev's inequality. □

Assume that $\mathcal{P}_N$ is a determinantal process given by a symmetric nonnegative integral operator K_N on $\mathbb{R}^*$. That is, the correlation functions have determinantal form with the kernel K_N. For a compact set $A \subset \mathbb{R}^*$ we denote by $K_{A,N}$ the restriction of the kernel K_N to A.

Lemma 6.3. *Assume that for any compact set $A \subset \mathbb{R}^*$ we have an estimate* $\operatorname{tr} K_{A,N} \le \text{const}$, *where the constant does not depend on N. Then the assumption of Lemma 6.2 is satisfied.*

Proof. Instead of ordinary moments we can deal with factorial moments. Given $l = 1, 2, \dots$, the l^{th} factorial moment of $\mathcal{N}_{A,N}$ is equal to

$$\rho_l^{(N)}(A^l) = \int_{A^l} \det[K_{A,N}(x_i, x_j)]_{1 \le i,j \le l}\, dx_1 \dots dx_l = l!\, \mathrm{tr}(\wedge^l K_{A,N}).$$

Since $K_{A,N}$ is nonnegative, we have

$$\mathrm{tr}(\wedge^l K_{A,N}) \le \mathrm{tr}(\otimes^l K_{A,N}) = (\mathrm{tr}(K_{A,N}))^l.$$

This concludes the proof, because we have a uniform bound for the traces by the assumption. □

Proof of Theorem 6.1. Take $M = m^{(s)}$ and denote the correlation measure $\rho_k^{(N)}$ by $\rho_k^{(s,N)}$. The latter measure is calculated in Sect. 1: it coincides with a scaling of the kth correlation function $\boldsymbol{\rho}_k^{(s,N)}(x_1, \dots, x_k)$ for the N^{th} pseudo-Jacobi ensemble. In terms of the corresponding correlation functions,

$$\rho_k^{(s,N)}(x_1, \dots, x_k) = N^k \boldsymbol{\rho}_k^{(s,N)}(Nx_1, \dots, Nx_k), \qquad x_1, \dots, x_k \in \mathbb{R}^*.$$

By Theorem 2.1, for each $k = 1, 2, \dots$, there exists a limit

$$\lim_{N \to \infty} \rho_k^{(s,N)}(x_1, \dots, x_k) = \rho_k^{(s,\infty)}(x_1, \dots, x_k), \tag{6.2}$$

uniformly on compact subsets in $(\mathbb{R}^*)^k$. Moreover, the correlation functions have determinantal form. It follows that the assumptions of Lemma 6.3 are satisfied (indeed, $\mathrm{tr}\, K_{A,N}$ is simply the integral of the first correlation function $\rho_1^{(s,N)}(x)$ over A). Consequently, we may apply Lemma 6.2. By this lemma, the correlation measures of the process $\mathcal{P}^{(s)}$ exist and coincide with limits of the measures $\rho_k^{(s,N)}$ as $N \to \infty$. Therefore, they are nothing else than the measures $\rho_k^{(s,\infty)}$ defined by the limit correlation functions (6.2). □

7. Vanishing of the Parameter γ_2

In this section we show that the parameter γ_2 which is responsible for the presence of the Gaussian component vanishes for the measure $m^{(0)}$.

We start with a general result concerning an abstract $U(\infty)$-invariant probability measure M. As in Sect. 6, let $\mathcal{P}_N$ and $\mathcal{P}$ denote the corresponding point processes on $\mathbb{R}^*$, and let $\rho_1^{(N)}$ and ρ_1 be their first correlation measures. We assume that $\rho_1^{(N)}$ approach ρ_1, as $N \to \infty$, in the sense of Lemma 6.2:

$$\langle G, \rho_1^{(N)} \rangle \to \langle G, \rho_1 \rangle \qquad \text{for any } G \in C_0(\mathbb{R}^*), \tag{7.1}$$

where $C_0(\mathbb{R}^*)$ denotes the space of continuous functions with compact support on $\mathbb{R}^*$. In Sect. 6 we verified that the condition (7.1) holds when M is a Hua–Pickrell measure.

Proposition 7.1. *Let M satisfy the condition* (7.1). *Further, assume that*

$$\lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon} x^2\rho_1^{(N)}(dx)=0 \qquad \textit{uniformly in } N. \tag{7.2}$$

Then the spectral measure P of the measure M is concentrated on the subset $\gamma_2=0$ of Ω.

Comment. The density of the measure ρ_1 may have a singularity at 0. For instance, when $M=m^{(0)}$, the density function is proportional to $1/x^2$. The condition (7.2) means that the densities of the measures $\rho_1^{(N)}$, multiplied by x^2, are uniformly integrable about $x=0$.

We need a simple lemma.

Lemma 7.2. *Assume we are given sequences*

$$a_{1,N}^{+}\ge a_{2,N}^{+}\ge\cdots\ge 0,\qquad a_{1,N}^{-}\ge a_{2,N}^{-}\ge\cdots\ge 0,\qquad N=1,2,\dots,$$

such that

$$\lim_{N\to\infty} a_{i,N}^{\pm}=\alpha_i^{\pm},\qquad i=1,2,\dots$$

and

$$\lim_{N\to\infty}\sum_{i=1}^{\infty}((a_{i,N}^{+})^2+(a_{i,N}^{-})^2)=\delta<+\infty,\qquad N=1,2,\dots .$$

Further, let $F(x)$ be an arbitrary continuous function on $\mathbb{R}_+$ such that

$$F(x)=x^2\qquad \textit{for } |x|<\varepsilon$$

with a certain $\varepsilon>0$. Set $\gamma_2=\delta-\sum_{i=1}^{\infty}((\alpha_i^{+})^2+(\alpha_i^{-})^2)$ and note that $\gamma_2\ge 0$.

Then we have

$$\lim_{N\to\infty}\sum_{i=1}^{\infty}(F(a_{i,N}^{+})+F(-a_{i,N}^{-}))=\sum_{i=1}^{\infty}(F(\alpha_i^{+})+F(-\alpha_i^{-}))+\gamma_2.$$

Proof. Fix k so large that $\alpha_{k+1}^{+}<\varepsilon$, $\alpha_{k+1}^{-}<\varepsilon$. Then $a_{k+1,N}^{+}<\varepsilon$, $a_{k+1,N}^{-}<\varepsilon$ for sufficiently large N and, moreover, $a_{i,N}^{+}<\varepsilon$, $a_{i,N}^{-}<\varepsilon$ for all $i\ge k+1$ by monotonicity. Likewise, $\alpha_i^{+}<\varepsilon$, $\alpha_i^{-}<\varepsilon$ for $i\ge k+1$. Therefore,

$$F(\pm a_{i,N}^{\pm})=(a_{i,N}^{\pm})^2\quad\text{(for large } N\text{)},\qquad F(\pm\alpha_i^{\pm})=(\alpha_i^{\pm})^2,\qquad i\ge k+1.$$

It follows that

$$\sum_{i=1}^{\infty}(F(a_{i,N}^{+})+F(-a_{i,N}^{-}))=\sum_{i=1}^{k}(F(a_{i,N}^{+})+F(-a_{i,N}^{-}))+\sum_{i=k+1}^{\infty}((a_{i,N}^{+})^2+(a_{i,N}^{-})^2)$$

and similarly

$$\sum_{i=1}^{\infty}(F(\alpha_i^{+})+F(-\alpha_i^{-}))=\sum_{i=1}^{k}(F(\alpha_i^{+})+F(-\alpha_i^{-}))+\sum_{i=k+1}^{\infty}((\alpha_i^{+})^2+(\alpha_i^{-})^2).$$

As $N \to \infty$, we have

$$\sum_{i=1}^{k}(F(a_{i,N}^{+}) + F(-a_{i,N}^{-})) \to \sum_{i=1}^{k}(F(\alpha_i^{+}) + F(-\alpha_i^{-})),$$

by continuity of F, and

$$\sum_{i=k+1}^{\infty}((a_{i,N}^{+})^2 + (a_{i,N}^{-})^2) \to \sum_{i=k+1}^{\infty}((\alpha_i^{+})^2 + (\alpha_i^{-})^2) + \gamma_2,$$

by the assumption of the lemma. This conludes the proof. □

Proof of Proposition 7.1. Let X range over H_{reg}. Recall the notation $a_{i,N}^{\pm}(X)$ and $\alpha_i^{\pm}(X)$ introduced in Sect. 5 and in the beginning of Sect. 6. Let $\gamma_2(X)$ denote the value of the parameter γ_2 at the point $\pi(X) \in \Omega$, where $\pi : H_{\text{reg}} \to \Omega$ is the projection defined in Sect. 5. Our aim is to prove that $\gamma_2(X) = 0$ almost everywhere with respect to the measure M. This implies the claim of the proposition.

Fix a continuous function $F(x) \geq 0$, with compact support on $\mathbb{R}$ and such that $F(x) = x^2$ near 0. For any $X \in H_{\text{reg}}$ set

$$\varphi_N(X) = \sum_{i=1}^{\infty}(F(a_{i,N}^{+}(X)) + F(-a_{i,N}^{-}(X))),$$
$$\varphi_\infty(X) = \sum_{i=1}^{\infty}(F(\alpha_i^{+}(X)) + F(-\alpha_i^{-}(X))).$$

Applying Lemma 7.2 to the sequences $a_{i,N}^{\pm} = a_{i,N}^{\pm}(X)$ and $\alpha_i^{\pm} = \alpha_i^{\pm}(X)$, we get

$$\varphi_N(X) \to \varphi_\infty(X) + \gamma_2(X), \qquad X \in H_{\text{reg}}.$$

The functions $\varphi_N(X)$, $\varphi_\infty(X)$, $\gamma_2(X)$ are all nonnegative Borel functions. By Fatou's lemma (see, e.g., [Shir, Ch. II, §6, Theorem 2]),

$$\liminf_{N\to\infty} \int_{t\in\mathbb{T}_{\text{reg}}} \varphi_N(X)M(dX) \geq \int_{X\in H_{\text{reg}}} \varphi_\infty(X)M(dX) + \int_{X\in H_{\text{reg}}} \gamma_2(X)M(dX).$$

Recall that in the beginning of Sect. 6 we introduced the point configurations $\mathcal{C}_N(X)$ associated with an arbitrary $X \in H_{\text{reg}}$. We have

$$\varphi_N(X) = \sum_{i=1}^{\infty}(F(a_{i,N}^{+}(X)) + F(-a_{i,N}^{-}(X)) = \sum_{x\in\mathcal{C}_N(X)} F(x),$$

so that

$$\int_{X\in H_{\text{reg}}} \varphi_N(X)M(dX) = \langle F, \rho_1^{(N)} \rangle.$$

Likewise,

$$\int_{X\in H_{\text{reg}}} \varphi_\infty(X)M(dX) = \langle F, \rho_1 \rangle.$$

Therefore,

$$\liminf_{N\to\infty}\langle F,\rho_1^{(N)}\rangle \geq \langle F,\rho_1\rangle + \int_{X\in H_{\rm reg}} \gamma_2(X)M(dX). \tag{7.3}$$

On the other hand, we will prove that

$$\limsup_{N\to\infty}\langle F,\rho_1^{(N)}\rangle \leq \langle F,\rho_1\rangle. \tag{7.4}$$

It will follow from (7.3) and (7.4) that $\gamma_2(X)=0$ for M-almost all X, because $\gamma_2(X)\geq 0$.

To prove (7.4) we represent $F(x)$, for an arbitrary $\varepsilon>0$, in the form

$$F(x)=F_\varepsilon(x)+G_\varepsilon(x),$$

where $0\leq F_\varepsilon(x)\leq x^2$, $\operatorname{supp} F_\varepsilon\subset[-\varepsilon,\varepsilon]$, $F_\varepsilon(x)=x^2$ near 0, $G_\varepsilon\in C_0(\mathbb{R}^*)$. Choosing ε small enough, we can make $\langle F_\varepsilon,\rho_1^{(N)}\rangle$ arbitrarily small, uniformly in N, by virtue of the assumption (7.2). As for $\langle G_\varepsilon,\rho_1^{(N)}\rangle$, it tends to $\langle G_\varepsilon,\rho_1\rangle$, by (7.1). This concludes the proof of Proposition 7.1. □

Theorem 7.3. *The spectral measure of the measure $m^{(0)}$ is concentrated on the subset $\gamma_2=0$ of Ω.*

Proof. By virtue of Proposition 7.1, it suffices to verify the condition (7.2). To do this, we use the fact that in our case the first correlation function $\rho_1^{(N)}(x)=\rho_1^{(0,N)}(x)$ has a very simple expression:

$$\rho_1^{(0,N)}(x)=\frac{1}{\pi}\,\frac{N^2}{1+N^2x^2}\,. \tag{7.5}$$

The simplest way to check (7.5) is to use the relationship to the $N^{\rm th}$ Dyson ensemble, where the first correlation function is identically equal to N.

From (7.5) and the trivial estimate $\frac{N^2x^2}{1+N^2x^2}\leq 1$ we readily conclude that the condition (7.2) is indeed satisfied. □

We expect that Theorem 7.3 holds for any Hua–Pickrell measure.

8. Remarks and Problems

Orthogonal polynomials on the circle. In this paper we deal with the pseudo-Jacobi ensemble (1.1) defined by the weight function (1.4) on the real line. Instead of this, one could work with the orthogonal polynomial ensemble (0.11). Then we need orthogonal polynomials on the unit circle $\mathbb{T}$ with the weight function

$$(1+u)^{\bar s}(1+\bar u)^s=2^a\,(1+\cos\varphi)^a\,e^{b\varphi},$$

where $u=e^{i\varphi}\in\mathbb{T}$, $-\pi<\varphi<\pi$, $s=a+ib$.

For real s, the weight function depends only on $\Re u=\cos\varphi\in[-1,1]$. Then one can use a general trick described in [Sz, §11.5]. It allows one to express the polynomials on $\mathbb{T}$ in terms of two families of orthogonal polynomials on the interval $[-1,1]$, which, in our case, turn out to be certain Jacobi polynomials. This makes it possible to evaluate the Christoffel–Darboux kernel and then pass to a limit as $N\to\infty$, which leads to another derivation of Theorem 2.1 (for real s). Perhaps, such an approach can be used for nonreal values of s as well.

Painlevé V. Consider a kernel of the form

$$K(x', x'') = \frac{P(x')Q(x'') - Q(x')P(x'')}{x' - x''},$$

where the functions P and Q satisfy a differential equation of the form

$$\frac{d}{dx}\begin{bmatrix} P(x) \\ Q(x) \end{bmatrix} = A(x)\begin{bmatrix} P(x) \\ Q(x) \end{bmatrix}$$

with a traceless rational 2×2 matrix $A(x)$. Let J be a union of intervals inside the real line. Then the Fredholm determinant $\det(1 + K|_J)$ satisfies a certain system of partial differential equations with the endpoints of J regarded as variables, see [TW]. In particular, when only one endpoint is moving the corresponding ordinary differential equation often happens to be one of the Painlevé equations.

The kernel $K^{(s,\infty)}$ introduced in Theorem 2.1 is not an exception. In particular, the function

$$\sigma(t) = t\,\frac{d \ln\det\left(1 - K^{(s,\infty)}|_{(t^{-1},+\infty)}\right)}{dt}, \quad t > 0,$$

satisfies a σ-version of the Painlevé V equation:

$$-(t\sigma'')^2 = (2(t\sigma' - \sigma) + (\sigma')^2 + i(\bar{s} - s)\sigma')^2 - (\sigma')^2(\sigma' - 2is)(\sigma' + 2i\bar{s}),$$

see [BD] for details. Note that the approach of [BD] is very different from the machinery developed in [TW].

Infinite measures. The construction of the Hua–Pickrell measures $m^{(s)}$, $\Re s > -\frac{1}{2}$, given in Sect. 3 can be extended to arbitrary complex values of s. However, when $\Re s \le -\frac{1}{2}$, $m^{(s)}$ ceases to be a probability measure and becomes an infinite measure. Its pushforward $m^{(s,N)}$ under the projection $\theta_N : H \to H(N)$ makes sense only for sufficiently large values of N. Specifically, N must be strictly greater than $-2\Re s$. Then the measure $m^{(s,N)}$ is defined, within a constant factor not depending on N, by formula (3.1), where the factor const_N is subject to the recurrence relation

$$\mathrm{const}_N = \mathrm{const}_{N-1}\,\frac{\pi^N \Gamma(s + \bar{s} + N)}{2^{s+\bar{s}+2N-2}\,\Gamma(s + N)\Gamma(\bar{s} + N)}.$$

In other words, even if the measures $m^{(s,N)}$ are infinite, their projective limit $m^{(s)} = \varprojlim m^{(s,N)}$ still exists. The reason is that the fibers of the projection $H(N) \to H(N-1)$ have finite mass with respect to the conditional measures provided that N is large enough.

Problem. *Define and study the spectral decomposition of the infinite measures $m^{(s)}$, $\Re s \le -\frac{1}{2}$.*

Representation-theoretic meaning of $U(\infty)$-invariant measures on H. Let $G(N) = U(N) \ltimes H(N)$ be the semidirect product of the group $U(N)$ acting on the additive group $H(N)$ by conjugations. Similarly, set

$$G = U(\infty) \ltimes H(\infty) = \varinjlim G(N).$$

The groups $G(N)$ are examples of the so-called Cartan motion groups, and the group G is an infinite–dimensional version of the groups $G(N)$.

A unitary representation T of the group G is called *spherical* if it possesses a cyclic unit vector ξ which is invariant with respect to the subgroup $U(\infty) \subset G$. There is a one-to-one correspondence between the classes of equivalence of the pairs (T, ξ) and the $U(\infty)$-invariant probability Borel measures M on H. Given M, the representation T can be realized in the Hilbert space $L^2(H, M)$. Elements $U \in U(\infty)$ and $A \in H(\infty)$ act on functions $f \in L^2(H, M)$ as follows:

$$(T(U)f)(X) = f(U^{-1}XU), \quad (T(A)f)(X) = e^{i\,\mathrm{tr}(AX)} f(X), \qquad X \in H.$$

In this realization, ξ is the constant function 1.

Consider the matrix coefficient $\varphi(g) = (T(g)\xi, \xi)$, called the *spherical function.* Since φ is $U(\infty)$-biinvariant, the function $\varphi\mid_{H(\infty)}$, the restriction of φ to the subgroup $H(\infty) \subset G$, is a $U(\infty)$-invariant positive definite normalized function on $H(\infty)$. It follows that $\varphi\mid_{H(\infty)}$ coincides with the Fourier transform (4.1) of the $U(\infty)$-invariant probability Borel measure M.

Under the correspondence $(T, \xi) \leftrightarrow M$, ergodicity of M is equivalent to irreducibility of T. Note also that for an irreducible spherical representation T, the vector ξ is unique (within a scalar multiple), so that the function φ is an invariant of T.

Thus, irreducible spherical representations of the group $G = U(\infty) \ltimes H(\infty)$ are parametrized by ergodic measures on H. For more details about representations of the group G, see [Ol2, Pi2].

The graph of spectra. Recall that by $\mathbb{S}_N$ we denoted the subset of $\mathbb{R}^N$ formed by vectors λ with weakly decreasing coordinates. For $\mu \in \mathbb{S}_{N-1}$ and $\lambda \in \mathbb{S}_N$ we write $\mu \prec \lambda$ if the coordinates of λ and μ interlace:

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \cdots \geq \lambda_{N-1} \geq \mu_{N-1} \geq \lambda_N.$$

We set

$$q_{N-1,N}(\mu, \lambda) = \begin{cases} \prod_{1\le i<j\le N-1} (\mu_i - \mu_j) \Big/ \prod_{1\le k<l\le N} (\lambda_k - \lambda_l), & \text{if } \mu \prec \lambda, \\ 0, & \text{otherwise.} \end{cases}$$

Note that for any $\lambda \in \mathbb{S}_N$,

$$\int_{\mathbb{S}_{N-1}} q_{N-1,N}(\mu, \lambda) d\mu = 1, \qquad d\mu = d\mu_1 \ldots d\mu_{N-1}.$$

Let M be an arbitrary $U(\infty)$-invariant probability Borel measure and P_N be the radial part of the measure $\theta_N(M)$ (this is a probability measure on $\mathbb{S}_N$). Then the measures $P_1, P_2, \ldots$ satisfy the following consistency relation:

$$\int_{\mathbb{S}_N} q_{N-1,N}(\mu, \lambda) P_N(d\lambda) = \text{the density of } P_{N-1} \text{ at } \mu.$$

Conversely, if a sequence $\{P_N\}$ of probability measures satisfies the above consistency relation for each pair of adjacent indices then this sequence comes from a certain measure M.

Introduce the set $\mathcal{T}$ formed by all infinite sequences

$$\tau = (\tau^{(1)} \prec \tau^{(2)} \prec \dots), \qquad \tau^{(N)} \in \mathbb{S}_N.$$

Consider the probability measures $\widetilde{P}$ on $\mathcal{T}$ with the following property: for each $N = 2, 3, \dots,$ the probability that $\tau^{(N-1)}$ lies in an infinitesimal region $d\mu$ about a point $\mu \in \mathbb{S}_{N-1}$ provided that $\tau^{(N)} = \lambda$, is $q_{N-1,N}(\mu, \lambda)d\mu$. Any such measure $\widetilde{P}$ is uniquely determined by a sequence $\{P_N\}$ satisfying the consistency relations. Thus, the measures $\widetilde{P}$ are in one-to-one correspondence with the $U(\infty)$-invariant probability measures M on H.

We call the collection of sets $\{\mathbb{S}_N\}$ together with the functions $q_{N-1,N}(\mu, \lambda)$ the *graph of spectra*. This term was suggested by Sergei Kerov. According to the philosophy of [VK] we call the functions $q_{N-1,N}(\mu, \lambda)$ the *cotransition functions* of the graph of spectra. Here the term “graph” should not be understood literally, it only hints at a similarity with some “branching graphs” like the Young graph [VK] or the Gelfand–Tsetlin graph [BO]. For instance, the set $\mathcal{T}$ is an analogue of the set of paths in a branching graph. It can be shown that the graph of spectra can be obtained from the Gelfand–Tsetlin graph via a scaling limit procedure.

Projective limit of the spaces $U(N)$. There exist projections (not group homomorphisms!) $U(N) \to U(N-1)$ which correspond, via the Cayley transform, to the projections $H(N) \to H(N-1)$. This allows one to form the projective limit space $\mathfrak{U} = \varprojlim U(N)$. The space $\mathfrak{U}$ admits a natural two-sided action of the group $U(\infty)$. The space H is embedded into $\mathfrak{U}$, and the measures $m^{(s)}$ are transferred to $\mathfrak{U}$ via this embedding. The resulting measures on $\mathfrak{U}$ are quasiinvariant with respect to the two-sided action of $U(\infty)$. This makes it possible to construct analogs of the biregular representation for the group $U(\infty)$, see [Ner2, Ol5] for more details.

Analogy with the infinite symmetric group and the Poisson–Dirichlet distributions. The construction of the space $\mathfrak{U}$ mentioned above is parallel to the construction of the space $\varprojlim S(n)$ of virtual permutations, see [KOV]. Here $S(n)$ denotes the symmetric group of degree n. The family of the Hua–Pickrell measures should be viewed as a counterpart of a family $\{\mu_t\}_{t>0}$ of probability measures on the space of virtual permutations, see [KOV]. The Hua–Pickrell measures play the same role in harmonic analysis on the group $U(\infty)$ as the measures μ_t do in harmonic analysis on the infinite symmetric group $S(\infty)$. The decomposition of the measures μ_t on ergodic components is described by the Poisson–Dirichlet distributions. These are remarkable probability measures on an infinite–dimensional simplex (see [Kin]), which were studied by many authors. Thus, the spectral measures $P^{(s)}$ may be viewed as counterparts of the Poisson–Dirichlet distributions.

Other examples of group actions. The action of the group $U(\infty)$ on the space H examined in the present paper is connected with a particular series of flat symmetric spaces

$\{G(N)/U(N) = H(N)\}_{N=1,2,\dots}$ which in turn is related to a series of compact symmetric spaces: the unitary groups $U(N)$ with the action of $U(N) \times U(N)$. There exist 10 infinite series of compact symmetric spaces and related flat spaces. With each such series, one can associate an infinite–dimensional group action on a space of infinite matrices (see, e.g., [Pi2]) and a family of "Hua–Pickrell measures" on that space depending on a real or complex parameter (see [Ner2]). We expect that the results of the present paper can be carried over to this more general context.

9. Appendix: Existence and Uniqueness of Decomposition on Ergodic Components

Let $\mathfrak{M}$ be the set of $U(\infty)$-invariant probability Borel measures on H. We equip $\mathfrak{M}$ with the Borel structure generated by the functions of the form $M \mapsto \langle F, M\rangle$, where M ranges over $\mathfrak{M}$ and F is an arbitrary bounded Borel function on H.

Let the symbol ex$(\dots)$ denote the set of extreme points of a convex set. Recall that elements of $\operatorname{ex}\mathfrak{M}$ are called ergodic measures.

Theorem 9.1. (i) $\operatorname{ex}\mathfrak{M}$ *is a Borel subset in* $\mathfrak{M}$.

(ii) *For any* $M \in \mathfrak{M}$ *there exists a probability Borel measure* P *on* $\operatorname{ex}\mathfrak{M}$ *representing* M, *i.e.,*

$$\langle F, M\rangle = \int_{\mathcal{M}\in \operatorname{ex}\mathfrak{M}} \langle F, \mathcal{M}\rangle P(d\mathcal{M}) \tag{9.1}$$

for any bounded Borel function F *on* H.

(iii) *The measure* P *is unique.*

There exist different ways to prove such results, in particular:

(a) Representation–theoretic techniques.
(b) Dynkin's theorem about boundaries of general Markov processes, see [Dyn] and the references therein.
(c) Choquet's theorem about existence and uniqueness of barycentric decomposition in compact metrizable convex sets which are "Choquet simplices", see [Ph].

In (a) we reduce the problem to that of decomposing a spherical representation of the Cartan motion group G (see Sect. 8 above). Here we have to apply the classical desintegration theory for representations of locally compact groups and C^*-algebras (see [Dix]) to groups which are not locally compact but are inductive limits of locally compact groups (see [Ol1, §3.6]). A crucial fact is that $(G, U(\infty))$ is a Gelfand pair in the sense of [Ol4, §6].

In (b) one should use the graph of spectra (see Sect. 8) to reduce Theorem 9.1 to Dynkin's theorem.

We follow (c) below.

Proposition 9.2 (Choquet's theorems). *Let* $\mathfrak{A}$ *be a convex subset of a locally convex topological vector space* E. *Assume that* $\mathfrak{A}$ *is compact and metrizable.*

(i) $\operatorname{ex}\mathfrak{A}$ *is a Borel subset of* $\mathfrak{A}$ *(more precisely, a* G_δ *subset).*

(ii) *For any* $a \in \mathfrak{A}$ *there exists a probability Borel measure* P *on* $\operatorname{ex}\mathfrak{A}$ *representing* a, *i.e.,*

$$f(a) = \int_{b\in \operatorname{ex}\mathfrak{A}} f(b)P(db) \tag{9.2}$$

for any continuous linear functional f *on* E.

(iii) *The measure P is unique if and only if the cone spanned by $\mathfrak{A}$ is a lattice.*

Proof. Claim (i) is an elementary fact, see [Ph, Prop. 1.3]. Claims (ii) and (iii) are Choquet's theorems, see [Ph, Sects. 3 and 9]. □

We need one more general result.

Proposition 9.3. *For any group action on a Borel space, the cone of finite Borel measures is a lattice.*

Proof. See [Ph, Sect. 10]. □

By Proposition 9.3, the set $\mathfrak{M}$ satisfies the lattice condition from the last part of Proposition 9.2. However, there is no apparent way to make $\mathfrak{M}$ a compact space, which is the major obstacle to applying Choquet's theorems. We bypass it by embedding $\mathfrak{M}$ into a larger convex set to which Choquet's theorems can be applied. Here we use an idea borrowed from the proof of Theorem 22.10 in [Ol3] (see also Sect. 6 in [OkOl]).

Proof of Theorem 9.1. For $N = 1, 2, \dots$ let $\mathfrak{M}_N$ denote the set of $U(N)$-invariant probability Borel measures on $H(N)$, and let $\widetilde{\mathfrak{M}}_N$ be the larger set formed by $U(N)$-invariant finite Borel measures of total mass less or equal to 1.

Further, let $C_0(H(N))$ be the Banach space of continuous functions on $H(N)$ vanishing at infinity, and let E_N denote its dual space equipped with the weak star topology. Using the natural pairing between functions from $C_0(H(N))$ and finite measures, we embed $\widetilde{\mathfrak{M}}_N$ into E_N. Note that $\widetilde{\mathfrak{M}}_N$ is a compact metrizable space with respect to the topology of E_N.

For $N = 2, 3 \dots$, let $\theta_{N-1,N}$ denote the projection $H(N) \to H(N-1)$ which consists in removing the N^{th} row and column from a $N \times N$ matrix. This projection sends $\widetilde{\mathfrak{M}}_N$ to $\widetilde{\mathfrak{M}}_{N-1}$ and also sends $\mathfrak{M}_N$ to $\mathfrak{M}_{N-1}$. Moreover, $\mathfrak{M}$ coincides with the projective limit space $\varprojlim \mathfrak{M}_N$.

Note that the map $\theta_{N-1,N} : \widetilde{\mathfrak{M}}_N \to \widetilde{\mathfrak{M}}_{N-1}$ is not continuous. The reason is that the projection $H(N) \to H(N-1)$ is not a proper map. (To illustrate this phenomenon, consider the projection of the plane $\mathbb{R}^2$ onto its first coordinate axis. Take the Dirac measure at a point on the second coordinate axis and move the point to infinity. Then the measure will weakly converge to the zero measure, while its projection will remain fixed.)

However, the map $\theta_{N-1,N} : \widetilde{\mathfrak{M}}_N \to \widetilde{\mathfrak{M}}_{N-1}$ possesses a weaker property: it is semicontinuous from below. (This property does not rely on the specific character of the projection $H(N) \to H(N-1)$, it holds for any continuous map between locally compact spaces.) This implies that for any $N = 2, 3, \dots$ the set

$$A_{N-1,N} = \{(M_{N-1}, M_N) \in \widetilde{\mathfrak{M}}_{N-1} \times \widetilde{\mathfrak{M}}_N \mid M_{N-1} \geq \theta_{N-1,N}(M_N)\} \tag{9.3}$$

is closed.

It is convenient to allow the index N in (9.3) to take the value $\{1\}$. To this end we define $H(0)$ as a one-point set. Then $\theta_{0,1}$ projects $H(1)$ onto a single point, the vector space E_0 is identified with $\mathbb{R}$, $\widetilde{\mathfrak{M}}_0$ is the interval $[0, 1] \in E_0$, and $\mathfrak{M}_0$ is identified with 1.

Next, we take as $\mathfrak{A}$ the subset of $E_0 \times E_1 \times \dots$ formed by infinite sequences $a = (M_0, M_1, \dots)$ such that $M_0 = 1$, $M_N \in \widetilde{\mathfrak{M}}_N$ for $N = 1, 2, \dots$, and for any $N = 1, 2, \dots$, the pair (M_{N-1}, M_N) belongs to the set $A_{N-1,N}$ defined in (9.3). We remark that $\mathfrak{A}$ is a convex compact metrizable set.

For any $N = 0, 1, 2, \ldots$, we define an embedding $\iota : \mathfrak{M}_N \to \mathfrak{A}$ as follows:

$$\mathfrak{M}_N \ni M \mapsto a = (M_0, M_1, \ldots, M_N, 0, 0, \ldots),$$
$$M_N = M, \qquad M_{i-1} = \theta_{i-1,i}(M_i), \quad i = N, \ldots, 1.$$

We also consider the embedding $\iota : \mathfrak{M} \to \mathfrak{A}$ which comes from the identification of $\mathfrak{M}$ with $\varprojlim \mathfrak{M}_N$.

Now, we make the following crucial observation:

(*) *Any element $a \in \mathfrak{A}$ can be written as a convex combination of certain elements $a_N \in \iota(\mathfrak{M}_N)$ and an element $a_\infty \in \iota(\mathfrak{M})$. Moreover, this representation is unique.*

By Proposition 9.3, for any N, the cone in E_N spanned by $\mathfrak{M}_N$ is a lattice, and the same is also true for $\mathfrak{M}$. Together with (*), this implies that the cone generated by $\mathfrak{A}$ is a lattice. Thus, the set $\mathfrak{A}$ satisfies all the assumptions of Choquet's theorem. Applying this theorem, we get that any point $a \in \mathfrak{A}$ is uniquely represented by a probability measure P on $\operatorname{ex}\mathfrak{A}$.

On the other hand, (*) implies the following fact:

(**) $\operatorname{ex}\mathfrak{A}$ *is the disjoint union of the sets* $\iota(\operatorname{ex}\mathfrak{M}), \iota(\operatorname{ex}\mathfrak{M}_0), \iota(\operatorname{ex}\mathfrak{M}_1), \ldots$.

Since $\operatorname{ex}\mathfrak{A}$ is a Borel set by Choquet's theorem, and since all the sets $\iota(\operatorname{ex}\mathfrak{M}_N)$ are evidently Borel sets, we conclude from (**) that $\iota(\operatorname{ex}\mathfrak{M}) \subset \mathfrak{A}$ is a Borel set.

Next, we note that the Borel structure on $\mathfrak{M}$ coming from its embedding into $\mathfrak{A}$ coincides with its initial Borel structure. Indeed, both structures are defined by functions on $\mathfrak{M}$ of the form $M \mapsto \langle F, M\rangle$, the only difference is in the choice of a class $\{F\}$ of functions on the space H. In the latter case, F may be an arbitrary bounded Borel function, while in the former case F belongs to the smaller class of cylindrical functions of the form $G \circ \theta_N$ with $G \in C_0(H(N))$, $N = 1, 2, \ldots$. However, both classes clearly generate the same Borel structure.

This proves claim (i) of Theorem 9.1.

Further, it follows from (**) and the definition of the set $\mathfrak{A}$ that if $a \in \mathfrak{M}$ then its representing measure P is concentrated on $\iota(\operatorname{ex}\mathfrak{M}) \subset \operatorname{ex}\mathfrak{A}$. Comparing (9.1) and (9.2) we get that (9.1) holds for any cylindrical function of the form $F = G \circ \theta_N$ with $G \in C_0(H(N))$. But then it also holds for any bounded Borel function on H, as required. □

Recall that we have an explicit description of the set $\operatorname{ex}\mathfrak{M}$: it is parametrized by the space Ω (Proposition 4.1). The next claim, together with Theorem 9.1, is used in Proposition 4.4 above:

Proposition 9.4. *The "abstract" Borel structure on* $\operatorname{ex}\mathfrak{M}$*, which comes from the standard Borel structures on* $\mathfrak{M}$*, coincides with the "concrete" Borel structure, which comes from the natural Borel structure on* Ω *via the bijection* $\operatorname{ex}\mathfrak{M} \leftrightarrow \Omega$.

Proof. Let us show that for any bounded Borel function f, the expression $\langle f, M^\omega\rangle$ is a Borel function in $\omega \in \Omega$. Indeed, it suffices to check this claim for functions f of the form $f(X) = e^{i\,\operatorname{tr}(AX)}$, where A is an arbitrary fixed matrix from $H(\infty)$. Further, without loss of generality we may assume that A is a diagonal matrix, and then the claim follows from Proposition 4.1.

Consider the correspondence $\operatorname{ex}\mathfrak{M} \leftrightarrow \Omega$ provided by Proposition 4.1. We have just proved that $\Omega \to \operatorname{ex}\mathfrak{M}$ is a Borel map. Since both Ω and $\operatorname{ex}\mathfrak{M}$ are standard Borel spaces, we may apply a general result (see [Ma, Theorem 3.2]) to conclude that our correspondence is an isomorphism of Borel spaces. □

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