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A Spin-Statistics Theorem for Quantum Fields on Curved Spacetime Manifolds in a Generally Covariant Framework

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Abstract: A model-independent, locally generally covariant formulation of quantum field theory over four-dimensional, globally hyperbolic spacetimes will be given which generalizes similar, previous approaches. Here, a generally covariant quantum field theory is an assignment of quantum fields to globally hyperbolic spacetimes with spin-structure where each quantum field propagates on the spacetime to which it is assigned. Imposing very natural conditions such as local general covariance, existence of a causal dynamical law, fixed spinor- or tensor type for all quantum fields of the theory, and that the quantum field on Minkowski spacetime satisfies the usual conditions, it will be shown that a spin-statistics theorem holds: If for some of the spacetimes the corresponding quantum field obeys the "wrong" connection between spin and statistics, then all quantum fields of the theory, on each spacetime, are trivial.

1. Introduction

The spin-statistics theorem of quantum field theory in Minkowski spacetime asserts that elementary particles with integer spin must obey Bose-statistics ("spacelike commutativity"), while those of half-integer spin must obey Fermi-statistics ("spacelike anti-commutativity"). Although this behaviour of elementary particles is often taken as an experimental fact of life, it is remarkable that in quantum field theory such a connection between two at first sight apparently unrelated properties of particles can be deduced from a few very basic principles: (1) Relativistic covariance, (2) stability of matter (spectrum condition and existence of a vacuum state), (3) localization properties of charges and (4) locality (spacelike commutativity of observable quantities).

This deeply rooted connection between the covariance properties of elementary particles and the behaviour under exchange of their positions has attracted the attention of numerous researchers in quantum field theory, and has a long history with a fair number of general and rigorous results. Among the first are the investigations by Pauli [38] and by Fierz [20] who proved the spin-statistics theorem for quantum fields of arbitrary spin obeying linear hyperbolic wave-equations in Minkowski-spacetime. The first results on the connection between spin and statistics in quantum field theory in a completely general, model-independent approach (for quantum fields in the Wightman framework) were then obtained by Burgoyne [11] and by Lüders and Zumino [36]. They have subsequently been further extended and refined, particularly to cover the situation of having several fields of different spinor types in a quantum field theory; these theorems are presented in the textbooks by Jost [33], by Streater and Wightman [44], and by Bogoliubov, Logunov, Todorov and Oksak [5], to which we refer the reader for further discussion and references.

The Wightman-framework takes as fundamental objects pointlike quantum fields which may be charge-carrying and need not represent observable quantities. The operator-algebraic approach to quantum field theory [30, 29] uses, instead, observable quantities as the basic objects describing a theory of elementary particles and, at the same time, abandons their pointlike localizability. The charge-carrying objects and the global gauge group are, in this approach, not put in by hand, but can be reconstructed from the observables together with sets of states distinguished by certain localization properties (representing the localization properties of the charges in a quantum field theory). This is a deep result by Doplicher and Roberts [16] arising from the profound analysis of the charge superselection structure by Doplicher, Haag and Roberts (see [15, 16, 29] and references given therein). Spin-statistics theorems have also been derived in the operator-algebraic approach to quantum field theory, beginning with works by Epstein [19] and by Doplicher, Haag and Roberts [15] for the case of strictly localizable charges. Generalizations of spin-statistics theorems to the case of charges that can be localized in spacelike cones have been obtained by Buchholz and Epstein [10].

A new line of development has been introduced by the Tomita–Takesaki modular theory of von Neumann algebras [46] and its connection to Lorentz-transformations which was first established in two articles by Bisognano and Wichmann [4]; see the recent review by Borchers [6] for more information on this nowadays very important area of activity in algebraic quantum field theory. In this context, there are spin-statistics theorems by Guido and Longo [26] and by Kuckert [35] in algebraic quantum field theory which take a certain geometric action of the Tomita–Takesaki modular objects associated with the vacuum state and distinguished algebras of quantum field observables as the starting point.

The results just summarized concern quantum field theory on four-dimensional Minkowski spacetime. The present article focusses on quantum field theory on four-dimensional curved spacetimes, but before turning to that topic, we just mention that spin-statistics connections have also been investigated in other settings. Among those are, in particular, quantum field theories on flat two-dimensional spacetime and chiral conformal quantum field theories on one-dimensional spacetimes (e.g. the circle S^1), see e.g. the articles [40] for the case of two dimensions and [27] for chiral conformal quantum field theory. A spin-statistics connection for so-called "topological geons" has been investigated within a diffeomorphism-covariant approach to quantum gravity [17, 2] which is not directly related to the quantum field theoretical framework. For the sake of completeness we mention that the spin-statistics connection may also be violated e.g. for quantum fields having infinitely many components; at this point we refer to [5] and references cited there.

While the spin-statistics connection is well-explored in quantum field theory on flat spacetime, offering a wealth of results, there is little analogous to be found so far for quantum field theory on curved spacetime manifolds. We recall that in quantum field theory on curved spacetime one considers quantum fields propagating on a curved, classically described spacetime background; the standard references on that subject, from a more mathematical point of view, include [21, 52]. Clearly, the reason for lacking results on the spin-statistics connection in curved spacetime is that the spin-statistics theorem on Minkowski spacetime rests significantly on Poincaré-covariance which possesses no counterpart in generic curved spacetimes. In general, the isometry group of a curved spacetime will even be trivial. Thus it is not at all clear if a spin-statistics theorem can be established on curved spacetime in a model-independent quantum field theoretical framework.

The situation is, of course, better when the spacetimes on which quantum fields propagate possess still large enough isometry groups. Such a setting has been considered recently in [28]. In that article, the charge superselection theory in the operator-algebraic approach to quantum field theory has been generalized from the familiar case of Minkowski spacetime to arbitrary, globally hyperbolic spacetimes. Moreover, if a spacetime admits a spatial rotation-symmetry with isometry group SO(3), and also a certain time-space reflection symmetry, then a spin-statistics theorem has been shown to hold for covariant charges, where the spin is defined via the SU(2)-covering of the spatial rotation group SO(3). A certain geometric action of Tomita–Takesaki modular objects associated with an isometry-invariant state and distinguished algebras of observables has been taken as input. (We refer to [28] for further details and discussion.) Such a spin-statistics theorem applies e.g. for quantum field theories on Schwarzschild–Kruskal black hole spacetimes.

However, when one is confronted with the question if there is a connection between spin and statistics for quantum fields on general spacetime manifolds, one finds scarcely any results. The only results known to us have been obtained in papers by Parker and Wang [37], and by Wald [50], and they apply to the case of quantum fields obeying linear equations of motion. The situation considered in these two papers is, roughly speaking, as follows: A linear quantum field propagates in the background of a (globally hyperbolic) spacetime consisting of three regions: A "past" region and a "future" region, both of which are isomorphic to flat Minkowski spacetime, and an intermediate region lying between the two (i.e. lying to the future of the "past" region, and to the past of the "future" region) which is assumed to be non-flat. (Actually, only particular types of spacetimes of this form are considered in [37] and [50].) Then it is shown in the mentioned articles that a quantum field of integer spin (≤ 2) obeying a linear wave-equation won't satisfy canonical anti-commutation relations in the "future" region if canonical anti-commutation relations were fulfilled in the "past" region. In other words, the "wrong" commutation relations are unstable under the dynamical evolution of the quantum field in the presence of a curved spacetime background. Likewise, a quantum field of half-integer spin ($\leq 3/2$) will no longer satisfy canonical commutation relations in the "future" region if it did so in the "past" region. It should be noted that these results don't make reference to states (e.g., the vacuum state in any of the flat regions), so that it is really the non-trivial spacetime curvature in the intermediate region inducing dynamical instability of the "wrong" connection between spin and statistics at the level of the commutation relations. In that respect, the line of argument in [37] and [50] seems to be restricted to free fields.

Nevertheless, there are some aspects of it which are worth pointing out since they can be generalized to model-independent quantum field theoretical settings. So one notes that the quantum field theories in the flat, "past" and "future" regions are "the same" regarding field content and dynamics; otherwise it would be difficult to formulate that their commutation relations are unstable under the dynamical evolution. There is another aspect in form of the well-posedness of the Cauchy-problem for linear fields in globally hyperbolic spacetime, entailing that field operators located in the "future" are dynamically determined by the field operators located in the "past" region. This property is sometimes referred to as *strong Einstein causality*, or *existence of a causal dynamical law*, and not restricted to free field theories. Thus one may extract from the setting investigated by Parker and Wang, and by Wald, the two following important ingredients for a quantum field theory on curved spacetime: The parts of the theory restricted to isomorphic spacetime regions should themselves be isomorphic (i.e., copies of each other), and there should exist a causal dynamical law. One may then interpret the results of [37] and [50] as saying that, for a certain class of curved spacetimes and for a certain class of quantum field theories, the two said ingredients are incompatible with assuming the "wrong" connection between spin and statistics.

On the basis of the mentioned ingredients, we can now abstract from the setting of [37] and [50]. We shall consider families $\{\mathbf{\Phi_M}\}_{\mathbf{M}\in\mathcal{G}}$ of quantum field theories indexed by the elements of $\mathcal{G}$, the set of all four-dimensional, globally hyperbolic spacetimes with spin-structures $\mathbf{M}$. Each $\mathbf{\Phi_M}$ is a quantum field propagating on the background spacetime $\mathbf{M}$, and it is assumed that for each $\mathbf{M}$, the quantum field $\mathbf{\Phi_M}$ is of a specific spinor- or tensor-type (the same for all $\mathbf{M}$). The picture is that one can, for each spinor- or tensor-type, formulate field equations that depend on the spacetime metrics in a covariant manner. (A very simple example is $(\Box_g + m^2)\mathbf{\Phi_M} = 0$ for a scalar field $\mathbf{\Phi_M}$ on $\mathbf{M} = (M, g)$, where $\Box_g$ is the d'Alembertian associated with the metric g on the spacetime-manifold M.) Then there should be an isomorphism α_Θ between the algebras $\mathcal{F}_{\mathbf{M}_1}(O_1)$ and $\mathcal{F}_{\mathbf{M}_2}(O_2)$ formed by the field operators $\mathbf{\Phi}_{\mathbf{M}_1}(f_1)$ and $\mathbf{\Phi}_{\mathbf{M}_2}(f_2)$ with supp $f_j \subset O_j$ ($j = 1, 2$), respectively, [1] as soon as the subregions $O_j \subset \mathbf{M}_j$ are isomorphic, i.e. whenever there is a local isomorphism (of metrics and spin-structures) $\Theta : \mathbf{M}_1 \supset O_1 \to O_2 \subset \mathbf{M}_2$. Moreover, α_Θ should be a net-isomorphism in the sense that it respects localized inclusions, meaning that

$$\alpha_\Theta(\mathcal{F}_{\mathbf{M}_1}(O)) = \mathcal{F}_{\mathbf{M}_2}(\Theta(O))$$

holds for all $O \subset O_1$. This is the *principle of general covariance*. It is worth noting that our concept of general covariance is a "local" one, in contrast to a similar, but global notion of general covariance for quantum field theories which has been developed by Dimock [13,14]. Apart from that (and apart from the fact that we need the net-isomorphisms at the level of von Neumann algebras, while in existing literature they have been looked at as C^*-algebraic net-isomorphisms), our concept of general covariance is very close to that suggested by Dimock, and also similar to ideas in [3,34,32].

The principle of existence of a causal dynamical law can then be expressed by demanding that, for each $\mathbf{M}$, there holds

$$\mathcal{F}_{\mathbf{M}}(O_1) \subset \mathcal{F}_{\mathbf{M}}(O)$$

whenever the subregion O_1 of $\mathbf{M}$ lies in the domain of dependence of the subregion O of $\mathbf{M}$ (that is, O_1 is causally determined by O, see Sect. 2 for details).

There is another principle that is also most naturally imposed. Minkowski spacetime $\mathbf{M}_0$ is also a member of $\mathcal{G}$, and clearly the quantum field theory $\mathbf{\Phi}_{\mathbf{M}_0}$ should satisfy the

[1] The precise mathematical sense in which the algebras are formed by the field operators will be explained in Sect. 4. The $\mathbf{\Phi_M}$ are viewed as operator-valued distributions and the f_j are test-spinors or test-tensors (smooth sections of compact support in an appropriate spinor-bundle or tensor-bundle).

usual properties assumed for a quantum field theory (e.g., in the Wightman framework), like Poincaré-covariance, spectrum condition, existence of a vacuum state and, in order that a spin-statistics theorem can be expected, the Bose–Fermi alternative.

If these conditions – fixed spinor- or tensor-type, general covariance, existence of a causal dynamical law and the usual properties for the theory $\mathbf{\Phi}_{\mathbf{M}_0}$ on Minkowski spacetime – are satisfied, we call the family $\{\mathbf{\Phi}_{\mathbf{M}}\}_{\mathbf{M}\in\mathcal{G}}$ *a generally covariant quantum field theory* over $\mathcal{G}$. For such generally covariant quantum field theories over $\mathcal{G}$ we shall establish in the present article a spin-statistics theorem. Roughly speaking, the contents of that theorem are as follows (see Thm. 5.1 for the precise statement): If there is some $\mathbf{M} \in \mathcal{G}$ and a pair of causally separated regions O_1 and O_2 in $\mathbf{M}$ so that pairs of field operators of the quantum field $\mathbf{\Phi}_{\mathbf{M}}$ localized in O_1 and O_2, respectively, fulfill the "wrong" connection between spin and statistics (i.e. they anti-commute if $\mathbf{\Phi}_{\mathbf{M}}$ is of integer spin-type (tensorial), or they commute if $\mathbf{\Phi}_{\mathbf{M}}$ is of half-integer spin type (spinorial)), then this entails that all field operators $\mathbf{\Phi}_{\tilde{\mathbf{M}}}$ are mutliples of the unit operator for all $\tilde{\mathbf{M}} \in \mathcal{G}$, thus the theory is trivial.

Our method of proof is to show with the help of a spacetime deformation argument (Lemma 2.1) that under the said assumptions the "wrong" connection between spin and statistics in any of the theories $\mathbf{\Phi}_{\mathbf{M}}$ leads to the "wrong" spin-statistics connection for the theory $\mathbf{\Phi}_{\mathbf{M}_0}$ on Minkowski spacetime; hence the known spin-statistics theorem for quantum field theory on Minkowski spacetime shows that $\mathbf{\Phi}_{\mathbf{M}_0}$ must be trivial. Using the spacetime deformation argument once more, this will then be shown to imply that all theories $\mathbf{\Phi}_{\tilde{\mathbf{M}}}$ are trivial.

The framework we use is in a sense a mixture of the Wightman-type quantum field theoretical setting and of the operator-algebraic approach to quantum field theory. This seems to have some technical advantages. Upon making some changes, one could reformulate the arguments so that they apply either to a purely Wightman-type quantum field theoretical setting, or to a purely operator-algebraic approach; however in the latter case it wouldn't be so clear how to assign to a theory a spinor- or tensor-type on a curved spacetime. This has resulted in the framework we shall be employing here.

We should like to point out that the assumptions imposed on a generally covariant quantum field theory $\{\mathbf{\Phi}_{\mathbf{M}}\}_{\mathbf{M}\in\mathcal{G}}$ over $\mathcal{G}$ are quite general. They are fulfilled for free field theories on curved spacetimes in representations induced by Hadamard states as we will indicate by sketching some examples in Sect. 6. Our current understanding is, however, that these assumptions aren't restricted to the case of free field theories but apply in fact to a larger class of quantum field theories. At any rate, they reflect a few very natural and general principles.

Our work is organized as follows. In Sect. 2 we summarize a few properties of globally hyperbolic spacetimes. Lemma 2.1 will be of importance later for proving the spin-statistics theorem; it states that one can deform a globally hyperbolic spacetime into another globally hyperbolic spacetime which is partially flat, and partially isomorphic to the original spacetime. Section 3 contains the technical definition of local isomorphisms between spacetimes with spin structures. In Sect. 4 we give the full definition of a generally covariant quantum field theory over $\mathcal{G}$. The main result on the connection between spin and statistics for such generally covariant quantum field theories over $\mathcal{G}$ is presented in Sect. 5. In Sect. 6 we sketch the construction of three theories that provide examples for generally covariant quantum field theories over $\mathcal{G}$: The free scalar Klein–Gordon field, the Proca field and the Majorana-Dirac field in representations induced by quasifree Hadamard states.

There are three appendices. Appendix A contains the proof of Lemma 2.1, and in Appendix B we summarize the standard assumptions for a quantum field theory on Minkowski spacetime and quote the corresponding spin-statistics theorem from the literature. In Appendix C we briefly indicate (generalizing similar ideas in [14]) that generally covariant quantum field theories over $\mathcal{G}$ may be viewed as covariant functors from the category $\mathcal{G}$ of globally hyperbolic spacetimes with a spin-structure to the category $\mathcal{N}$ of nets of von Neumann algebras over manifolds, both categories being equipped with suitable local isomorphisms as morphisms.
(See also the "Note added in proof" at the end of the article.)

2. Globally Hyperbolic Spacetimes

We begin the technical discussion by collecting some basics on globally hyperbolic spacetimes. This section will be brief, and serves mainly for fixing our notation. The reader is referred to the monographs [31,51] for further explanations and proofs.

A spacetime is a pair (M, g) where M is a four-dimensional smooth manifold (connected, Hausdorff, paracompact, without boundary) and g is a Lorentzian metric with signature $(+, -, -, -)$ on M. It will be assumed that (M, g) is orientable and time-orientable, meaning that there exists a smooth timelike vectorfield v on M. (Then $g(v, v) > 0$ everywhere on M, so v is nowhere vanishing). A continuous, piecewise smooth causal curve $\mathbb{R} \supset (a, b) \ni t \mapsto \gamma(t)$ is future-directed (past-directed) if $g(\dot{\gamma}, v) > 0$ ($g(\dot{\gamma}, v) < 0$), where $\dot{\gamma} = \frac{d}{dt}\gamma$ is the tangent vector. Henceforth, it will be assumed that an orientation and a time-orientation have been chosen. Then one defines the following regions of causal dependence for any given set $O \subset M$:

(i) $J^{\pm}(O)$ is the set of all points lying on future(+)/past(–)-directed causal curves emanating from O,
(ii) $J(O) = J^{+}(O) \cup J^{-}(O)$,
(iii) $D^{\pm}(O)$ is the set of all points p in $J^{\pm}(O)$ such that each past(+)/future(–)-directed causal curve starting at p passes through O unless it has a past/future endpoint,
(iv) $D(O) = D^{+}(O) \cup D^{-}(O)$,
(v) $O^{\perp} = M \backslash \overline{J(O)}$ is the *causal complement* of O.

The set $D(O)$ is called the *domain of dependence* of O. If $O_1 \subset \text{int}\, D(O)$, then we say that O_1 is *causally determined* by O, and denote this by $O_1 \lhd O$.

A time-orientable spacetime (M, g) is called *globally hyperbolic* if M possesses a smooth hypersurface which is intersected exactly once by each inextendible causal curve. Such a hypersurface is called a *Cauchy-surface*. It is known that globally hyperbolic spacetimes possess C^{∞}-foliations into Cauchy-surfaces, in other words, for each globally hyperbolic spacetime (M, g) there exists a smooth 3-dimensional manifold Σ_0 together with a diffeomorphism $F : \mathbb{R} \times \Sigma_0 \to M$ such that for all $t \in \mathbb{R}$, $F(\{t\} \times \Sigma)$ is a Cauchy-surface in (M, g) and such that, for each $x \in \Sigma_0$, $\mathbb{R} \ni t \mapsto F(t, x)$ is an endpointles timelike curve. While this may at first sight appear to be quite restrictive, it is known that the set of globally hyperbolic spacetimes is quite large and contains many spacetimes of physical interest. Moreover it should be noted that global hyperbolicity isn't connected to the existence of spacetime symmetries.

When N is an open, connected subset of M, then $(N, g \restriction N)$ is again an oriented and time-oriented spacetime. We call it a *globally hyperbolic sub-spacetime* of (M, g) if the following conditions are satisfied (cf. [31]Sect. 6.6): (1) the strong causality assumption holds on $(N, g \restriction N)$, (2) for any two points $p, q \in N$, the set $J^{+}(p) \cap J^{-}(q)$, if non-empty, is compact and contained in N. This entails that $(N, g \upharpoonright N)$ is a globally hyperbolic spacetime in its own right, but also when seen as embedded into (M, g). We give two types of examples for subsets N of M so that $(N, g \upharpoonright N)$ is a globally hyperbolic sub-spacetime: First, if $p, q \in M$ with $p \in \mathrm{int}\, J^+(q)$, then the "double cone" $N = \mathrm{int}(J^-(p) \cap J^+(q))$ gives rise to a globally hyperbolic sub-spacetime. And secondly, suppose that C_1, C_2, C_3 are three Cauchy-surfaces in (M, g) with $C_2 \subset \mathrm{int}\, J^+(C_1)$ and $C_3 \subset \mathrm{int}\, J^+(C_2)$, and let $\mathcal{G}$ be a connected open subset of C_2. Then the "truncated diamond" $\mathcal{N} = \mathrm{int}(D(\mathcal{G}) \cap J^+(C_1) \cap J^-(C_3))$ yields, equipped with the appropriate restriction of g, again a globally hyperbolic sub-spacetime of (M, g).

For the purposes of the present paper, a particularly important property of globally hyperbolic spacetimes is the following: A globally hyperbolic spacetime (M, g) can be "deformed" into another globally hyperbolic spacetime $(\widetilde{M}, \widetilde{g})$ in such a way that certain regions of (M, g) remain unchanged in $(\widetilde{M}, \widetilde{g})$, while other regions in $(\widetilde{M}, \widetilde{g})$ are isomorphic to parts of flat Minkowski spacetime. This will be made more precise in the subsequent statement, whose proof, given in Appendix A, is an extension of methods used in [22].

Lemma 2.1. *Let (M, g) be a globally hyperbolic spacetime and let $p_1, p_2 \in M$ be a pair of causally separated points (i.e. $p_1 \in \{p_2\}^\perp$). Then there is a globally hyperbolic spacetime $(\widetilde{M}, \widetilde{g})$, together with a collection of subsets $U_j, \widetilde{U}_j, \widehat{U}_j$ ($j = 1, 2$) and $G, \widehat{G}$, with the following properties:*

(a) *There are Cauchy-surfaces Σ in (M, g), and $\widetilde{\Sigma}$ in $(\widetilde{M}, \widetilde{g})$, so that with $N_+ = \mathrm{int}\, J^+(\Sigma) \subset M$ and $\widetilde{N}_+ = \mathrm{int}\, J^+(\widetilde{\Sigma})$, $(N_+, g \upharpoonright N_+)$ is isomorphic to $(\widetilde{N}_+, \widetilde{g} \upharpoonright \widetilde{N}_+)$.*

(b) *$p_1, p_2 \in N_+$. The isomorphic images of p_1 and p_2 in $\widetilde{N}_+$ will be denoted by $\widetilde{p}_1$ and $\widetilde{p}_2$.*

(c) *$\widehat{G} \subset \widetilde{N}_- = \mathrm{int}\, J^-(\widetilde{\Sigma})$ is simply connected, and $(\widehat{G}, \widetilde{g} \upharpoonright \widehat{G})$ is a globally hyperbolic sub-spacetime of $(\widetilde{M}, \widetilde{g})$ isomorphic to a globally hyperbolic sub-spacetime $(G_0, \eta \upharpoonright G_0)$ of flat Minkowski-spacetime $(M_0, \eta) \sim (\mathbb{R}^4, \mathrm{diag}(+, -, -, -))$.*

(d) *$G \subset \widetilde{N}_+$ is simply connected and $(G, \widetilde{g} \upharpoonright G)$ is a globally hyperbolic sub-spacetime of $(\widetilde{M}, \widetilde{g})$ containing $\widetilde{p}_1$ and $\widetilde{p}_2$.*

(e) *The sets $U_j, \widetilde{U}_j, \widehat{U}_j$ are, when equipped with the appropriate restrictions of $\widetilde{g}$ as a metric, globally hyperbolic, relatively compact sub-manifolds of $(\widetilde{M}, \widetilde{g})$ which are, respectively, causally separated for different indices, and $\widetilde{p}_j \in U_j \subset G$, $\widetilde{U}_j, \widehat{U}_j \subset \widehat{G}$ ($j = 1, 2$).*

(f) *$\widetilde{U}_j$ is causally determined by U_j, and U_j is causally determined by $\widehat{U}_j$ ($j = 1, 2$).*

Figure 2.1 may help to illustrate the relations between the sets involved in Lemma 2.1.

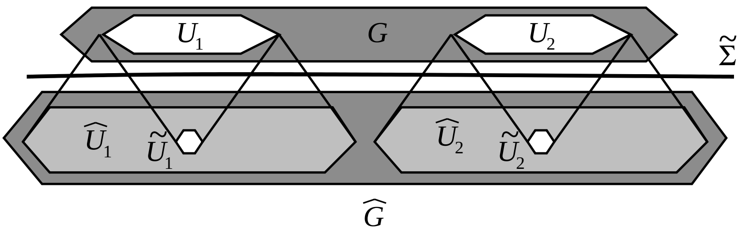

Fig. 2.1. Sketch of the causal relations of the sets $U_j, \widehat{U}_j, \widetilde{U}_j, G, \widehat{G}$

3. Spacetimes with Spin-Structures

Let (M, g) be a globally hyperbolic spacetime where an orientation and a time-orientation have been chosen. Then let $F(M, g)$ be the bundle of oriented and time-oriented (and future-directed) g-orthonormal frames on M. That is, an element $e = (e_0, \dots, e_3)$ in $F(M, g)$ is a collection of four vectors in T_pM, $p \in M$, with $g(e_a, e_b) = \eta_{ab}$, where $(\eta_{ab}) = \mathrm{diag}(+, -, -, -)$ is the Minkowski metric, e_0 is a future-directed timelike vector, and the frame $(e_0, \dots, e_3)$ is oriented according to the chosen orientation on M. The bundle projection $\pi_F : F(M, g) \to M$ assigns to e the base point p to which the vectors $e_0, \dots, e_3$ are affixed. The proper orthochronous Lorentz group $\mathcal{L}_+^\uparrow$ operates smoothly on the right on $F(M, g)$ by $(R_\Lambda e)_a = e_b \Lambda^b{}_a$ and thus $F(M, g)$ is a principal fibre bundle with fibre group $\mathcal{L}_+^\uparrow$ over M. A *spin structure* for (M, g) is a pair $(S(M, g), \psi)$, where $S(M, g)$ is an $\mathrm{SL}(2, \mathbb{C})$-principal fibre bundle over M and $\psi : S(M, g) \to F(M, g)$ is a base-point preserving bundle homomorphism (that is, $\pi_F \circ \psi = \pi_S$ where π_S is the base projection of $S(M, g)$) with the property

$$\psi \circ R_{\mathbf{s}} = R_{\Lambda(\mathbf{s})} \circ \psi .$$

Here, $R_{\mathbf{s}}$ denotes the right action of $\mathbf{s} \in \mathrm{SL}(2, \mathbb{C})$ on $S(M, g)$, and $\mathrm{SL}(2, \mathbb{C}) \ni \mathbf{s} \mapsto \Lambda(\mathbf{s}) \in \mathcal{L}_+^\uparrow$ is the covering projection; recall that $\mathrm{SL}(2, \mathbb{C})$ is the universal covering group of $\mathcal{L}_+^\uparrow$.

Two spin-structures $(S^{(1)}(M, g), \psi^{(1)})$ and $(S^{(2)}(M, g), \psi^{(2)})$ are called (globally) *equivalent* if there is a base-point preserving bundle-isomorphism $\Theta : S^{(1)}(M, g) \to S^{(2)}(M, g)$ so that $\Theta \circ \psi^{(2)} = \psi^{(1)}$. It is known that each 4-dimensional globally hyperbolic spacetime admits spin-structures and that all such spin-structures are equivalent if the spacetime manifold is simply connected (cf. [25]).

From now on, we will abbreviate by $\mathbf{M} = ((M, g), S(M, g), \psi)$ an oriented and time-oriented globally hyperbolic spacetime endowed with a spin-structure, and we shall also use the notation $\mathbf{M}_j = ((M_j, g_j), S_j(M_j, g_j), \psi_j)$ if we have labels j distiguishing several such objects. We denote by $\mathcal{G}$ the set of all 4-dimensional, oriented and time-oriented globally hyperbolic spacetimes with a spin-structure. One may view $\mathcal{G}$ as a category; of interest are then "local morphisms" between its objects, or more properly, morphisms between sub-objects. We will introduce the "local morphisms" as follows. For more details, see Appendix C.

Definition 3.1. *Let* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *be in* $\mathcal{G}$. *Then we say that* $\boldsymbol{\Theta} = (\Theta, \vartheta)$ *is a local isomorphism between* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *if:*

(a) *There are simply connected, oriented and time-oriented globally hyperbolic subspacetimes* $(N_j, g_j \restriction N_j)$ *of* (M_j, g_j) $(j = 1, 2)$ *so that* $\vartheta : (N_1, g_1 \restriction N_1) \to (N_2, g_2 \restriction N_2)$ *is an orientation and time-orientation preserving isomorphism. Then* N_1 *will be called the initial localization of* $\boldsymbol{\Theta}$, *denoted by* $\ell_{\mathrm{ini}}(\boldsymbol{\Theta})$, *and* N_2 *will be called the final localization of* $\boldsymbol{\Theta}$, *denoted by* $\ell_{\mathrm{fin}}(\boldsymbol{\Theta})$.

(b) *When denoting by* $S_j(N_j, g_j)$ *the restriction of* $S_j(M_j, g_j)$ *in its base set (that is,* $S_j(N_j, g_j) = \pi_{S_j}^{-1}(N_j)$*), then*

$$\Theta : S_1(N_1, g_1) \to S_2(N_2, g_2)$$

is a principal fibre bundle isomorphism (so it intertwines the corresponding right actions of the fibre groups) with the following properties:

(i) $\vartheta \circ \pi_{S_1} = \pi_{S_2} \circ \Theta$ *on* $S_1(N_1, g_1)$,
(ii) $\vartheta_F \circ \psi_1 = \psi_2 \circ \Theta$ *on* $S_1(N_1, g_1)$.
Here, $\vartheta_F : F(N_1, g_1) \to F(N_2, g_2)$ *is induced by the tangent map corresponding to* $\vartheta : N_1 \to N_2$.

Remark. In [14], Dimock has introduced the category $\mathcal{G}$, and global isomorphisms between pairs of objects in $\mathcal{G}$ as morphisms. Since each globally hyperbolic sub-spacetime of a globally hyperbolic spacetime with spin-structure is itself a member of $\mathcal{G}$, the definition of local isomorphisms can be regarded as introducing morphisms between sub-objects of objects in $\mathcal{G}$. It should be noted that the class of local isomorphisms between elements of $\mathcal{G}$ is clearly larger than the class of global isomorphisms as considered in [14], and therefore covariance properties imposed on quantum systems with respect to the class of local isomorphisms are more restrictive than those using only global isomorphisms. Further below we will see the implications of that.

Let ρ be a linear representation of $\mathrm{SL}(2, \mathbb{C})$ on some finite-dimensional vector-space V_ρ (which may be real or complex). Then, given a spacetime-manifold with spin-structure $\mathbf{M} = ((M, g), S(M, g), \psi) \in \mathcal{G}$, one can form the vector bundle

$$\mathcal{V}_\rho = S(M, g) \ltimes_\rho V_\rho$$

associated with the principal fibre bundle $S(M, g)$ and the representation ρ. $\mathcal{V}_\rho$ is a vector bundle over the base-manifold $\mathcal{M}$, and we recall that the elements of $(\mathcal{V}_\rho)_p$, the fibre of $\mathcal{V}_\rho$ at a base point $p \in M$, are the orbits $\{(R_{\mathbf{s}^{-1}} s_p, \rho(\mathbf{s})v) : \mathbf{s} \in \mathrm{SL}(2, \mathbb{C})\}$ of pairs $(s_p, v) \in S(M, g)_p \times V_\rho$ under the action

$$\mathbf{s} \mapsto (R_{\mathbf{s}^{-1}} s_p, \rho(\mathbf{s})v) \tag{3.1}$$

of the structure group $\mathrm{SL}(2, \mathbb{C})$ of $S(M, g)$. This action induces a linear representation $\check{\rho}$ of $\mathrm{SL}(2, \mathbb{C})$ on each $(\mathcal{V}_\rho)_p$. We say that $\mathcal{V}_\rho$ is the vector bundle of (spin-) representation type ρ.

Now let $\mathbf{M}_1$ and $\mathbf{M}_2$ be in $\mathcal{G}$ and let $\mathcal{V}_1$ and $\mathcal{V}_2$ be associated vector bundles of representation type ρ_1 and ρ_2, respectively. Suppose that ρ_1 and ρ_2 are equivalent, i.e. there is some bijective linear map $T : V_1 \to V_2$ so that

$$T\rho_1(\,.\,)T^{-1} = \rho_2(\,.\,). \tag{3.2}$$

One finds from these assumptions that any local isomorphism $\mathbf{\Theta} = (\Theta, \vartheta)$ between $\mathbf{M}_1$ and $\mathbf{M}_2$ lifts to a local isomorphism $\check{\Theta}$ between $\mathcal{V}_1$ and $\mathcal{V}_2$ in a way we shall now indicate. Let $\check{\pi}_j$ denote the base projections of $\mathcal{V}_j$ $(j = 1, 2)$ and, with $N_1 = \ell_{\mathrm{ini}}(\mathbf{\Theta})$, $N_2 = \ell_{\mathrm{fin}}(\Theta)$, let $\mathcal{V}_j(N_j) = \check{\pi}_j^{-1}(N_j)$ denote the restrictions of the vector bundles in the base sets. Then define $\check{\Theta} : \mathcal{V}_1(N_1) \to \mathcal{V}_2(N_2)$ by assigning to any element (s_p, v) in $S(M_1, g_1)_p \times V_1$, with $p \in N_1$, the element $((\Theta s)_{\vartheta(p)}, Tv)$ in $S(M_2, g_2)_{\vartheta(p)} \times V_2$, and form the orbits under the corresponding structure group actions (3.1). It is not difficult to check that this assignment indeed induces a well-defined map between $\mathcal{V}_1(N_1)$ and $\mathcal{V}_2(N_2)$ which is linear in the fibres and fulfills

$$\vartheta \circ \check{\pi}_1 = \check{\pi}_2 \circ \check{\Theta}$$

on $\mathcal{V}_1(N_1)$. Moreover, $\check{\Theta}$ intertwines the representations $\check{\rho}_j$ in the sense that

$$\check{\Theta} \circ \check{\rho}_1(\mathbf{s}) = \check{\rho}_2(\mathbf{s}) \circ \check{\Theta}$$

for all $\mathbf{s} \in \mathrm{SL}(2, \mathbb{C})$.

4. Generally Covariant Quantum Fields

In the present section we introduce a concept of generally convariant quantum field theories on curved spacetimes with spin-structures. Moreover, we will make the assumption that these quantum field theories fulfill the condition of strong Einstein causality, or synonymously, that there exists a causal dynamical law. The combination of these two assumptions – general covariance and existence of a causal dynamical law – will lead to the connection between spin and statistics shown in the subsequent section.

It should be remarked that there are several possible formulations of these two assumptions at the technical level. Here, we have chosen to use a framework which is in a sense a mixture of the Wightman-approach to "pointlike" quantum fields (operator-valued distributions) and the Haag-Kastler approach which emphasizes local algebras of bounded operators. Therefore, some technical assumptions have to be made in order to match these two approaches; yet we feel that the resulting framework is more general and more flexible than e.g. a framework using only Wightman fields, since then we would have to make even more stringent technical assumptions, for instance fairly detailed assumptions on the domains of field operators, or we would have to impose a very restrictive form of general covariance and strong Einstein causality. Since we don't wish to impose conditions of such kind, we regard the approach to be presented in this section as reasonable and fairly general.

The relevant assumptions will be listed next.

(a) Quantum fields of a spin representation type and their (local) von Neumann algebras. Let $\mathbf{M} = ((M, g), S(M, g), \psi) \in \mathcal{G}$ be a globally hyperbolic spacetime with spin-structure. Moreover, let ρ be a representation of $\mathrm{SL}(2, \mathbb{C})$ on the finite-dimensional vector-space V_ρ. We will say that a triple of objects $(\Phi, \mathcal{D}, \mathcal{H})$ is a *quantum field of spin representation type* ρ on $\mathbf{M}$ if: $\mathcal{H}$ is a Hilbert-space, $\mathcal{D}$ is a dense linear subspace of $\mathcal{H}$, and Φ is a linear map taking elements $f \in \Gamma_0(\mathcal{V}_\rho)$, the space of C^∞-sections in $\mathcal{V}_\rho$ with compact support, to closable operators $\Phi(f)$ in $\mathcal{H}$ having domain $\mathcal{D}$. In addition, it will be assumed that $\mathcal{D}$ is invariant under application of the operators $\Phi(f)$, and that $\mathcal{D}$ is also an invariant domain for the adjoint field operators $\Phi(f)^*$. It will also be assumed that there are cyclic vectors in $\mathcal{D}$, where $\chi \in \mathcal{D}$ is called cyclic if the space generated by χ and all $F_1 \cdots F_n \chi$, $n \in \mathbb{N}$, where $F_j \in \{\Phi(f_j), \Phi(f_j)^*\}$[2] with $f_j \in \Gamma_0(\mathcal{V}_\rho)$, is dense in $\mathcal{H}$.

We write $\mathrm{orc}(M)$ to denote the set of open, relatively compact subsets of M. Let $O \in \mathrm{orc}(M)$, then denote by $\mathcal{F}(O)$ the von Neumann algebra which is generated by all $e^{i\lambda|\Phi(f)|}$, $\lambda \in \mathbb{R}$, and J_f, with $\mathrm{supp}\, f \subset O$, where

$$\overline{\Phi(f)} = J_f |\Phi(f)|$$

denotes the polar decomposition of a field operator's closure. Thus the quantum field $(\Phi, \mathcal{D}, \mathcal{H})$ induces a net of von Neumann algebras $\{\mathcal{F}(O)\}_{O \in \mathrm{orc}(M)}$ fulfilling the isotony condition

$$O_1 \subset O_2 \Rightarrow \mathcal{F}(O_1) \subset \mathcal{F}(O_2).$$

In the following, we shall abbreviate a quantum field $(\Phi, \mathcal{D}, \mathcal{H})$ by the symbol $\mathbf{\Phi}$.

(b) Existence of a causal dynamical law. Let $\mathbf{\Phi}$ be a quantum field of some spin-representation type ρ on $\mathbf{M}$. We say that there exists a *causal dynamical law* for the

[2] $\{\Phi(f_j), \Phi(f_j)^*\}$ denotes the set containing the operators in the curly brackets, and not their anti-commutator. In this work, we will never use curly brackets to denote anti-commutators.

quantum field (or that the quantum field fulfills *strong Einstein causality*) if for the net $\{\mathcal{F}(O)\}_{O\in\mathrm{orc}(M)}$ of local von Neumann algebras it holds that

$$O_1 \lhd O_2 \Rightarrow \mathcal{F}(O_1) \subset \mathcal{F}(O_2).$$

(c) Local morphisms. Assume that we have two representations ρ_1 and ρ_2 on finite-dimensional vector spaces V_1 and V_2, respectively, and suppose that these representations are isomorphic, i.e. (3.2) holds with some bijective linear map $T : V_1 \to V_2$. Let $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_2$ be quantum fields of spin-representation type ρ_1 and ρ_2 on $\mathbf{M}_1$ and $\mathbf{M}_2$, respectively, where $\mathbf{M}_j \in \mathcal{G}$ $(j = 1, 2)$. Moreover, suppose that there is a local isomorphism $\mathbf{\Theta} = (\Theta, \vartheta)$ between $\mathbf{M}_1$ and $\mathbf{M}_2$.

Then we say that the local morphism $\mathbf{\Theta}$ between $\mathbf{M}_1$ and $\mathbf{M}_2$ is *covered by local isomorphisms* between the quantum field theories $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_2$ if the following holds: Given any relatively compact subset $N_i \subset \ell_{\mathrm{ini}}(\mathbf{\Theta})$ and writing $N_f = \vartheta(N_i)$, and denoting by $\{\mathcal{F}_1(O_i)\}_{O_i\in\mathrm{orc}(N_i)}$ and $\{\mathcal{F}_2(O_f)\}_{O_f\in\mathrm{orc}(N_f)}$ the von Neumann algebraic nets induced by the quantum fields $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_2$ restricted to N_i and N_f, respectively, there is a von Neumann algebraic isomorphism $\alpha_{\mathbf{\Theta},N_i} : \mathcal{F}_1(N_i) \to \mathcal{F}_2(N_f)$ fulfilling the covariance property

$$\alpha_{\mathbf{\Theta},N_i}(\mathcal{F}_1(O_i)) = \mathcal{F}_2(\vartheta(O_i)), \quad O_i \in \mathrm{orc}(N_i). \tag{4.1}$$

Comments and Remarks. (i) In (a), the property of a quantum field to be a spinor field of a certain type is just specified by requiring that it acts linearly on the test-spinors of the corresponding type. This is a quite common approach to defining spinor fields on curved spacetime. An algebraic transformation property, e.g. that a (local) spinor-transformation $\rho(\mathbf{s})$ on $\mathcal{V}_\rho$ induces an endomorphism on the $*$-algebra of quantum field operators, holds in general only when the underlying spacetime has a flat metric. One may regard the properties of Def. 4.1 below as a weak replacement of such an algebraic transformation property.

(ii) Existence of a causal dynamical law is a typical feature of quantum fields obeying linear hyperbolic equations of motion, but is expected to hold also for interacting quantum field theories as long as the mass spectrum behaves moderately. For free field theories, the existence of a causal dynamical law is commonly fulfilled in the following stricter form (see [13] for the case of the scalar field, but the argument generalizes to more general types of fields, cf. e.g. [42]): Given $O_1 \lhd O_2$, then for each $f_1 \in \Gamma_0(\mathcal{V}_\rho)$ with supp $f_1 \subset O_1$ there is $f_2 \in \Gamma_0(\mathcal{V}_\rho)$ with supp $f_2 \subset O_2$ such that $\Phi(f_2) = \Phi(f_1)$. Our formulation given in (b) is more general.

(iii) It is of some importance in (c) that N_i and N_f are assumed to be relatively compact subsets of $\ell_{\mathrm{ini}}(\mathbf{\Theta})$ and $\ell_{\mathrm{fin}}(\mathbf{\Theta})$, respectively, as otherwise it is known from free field examples that a von Neumann algebraic isomorphism $\alpha_{\mathbf{\Theta},N_i} : \mathcal{F}_1(N_i) \to \mathcal{F}_2(N_f)$ with the covariance property (4.1) cannot be expected to exist. In typical cases, the von Neumann algebras $\mathcal{F}_j(O)$ are of properly infinite type, and then $\alpha_{\mathbf{\Theta},N_i}$ is implemented by a unitary operator $U_{\mathbf{\Theta},N_i} : \mathcal{H}_1 \to \mathcal{H}_2$.

The subsequent definition will fix the notion of general covariance for quantum fields on curved spacetimes.

Definition 4.1. *Let ρ be a linear representation of $SL(2,\mathbb{C})$ on a finite dimensional vector space V. By $\mathcal{G}$ we denote, as before, the set of all oriented and time-oriented, 4-dimensional, globally hyperbolic spacetimes equipped with a spin-structure. A family*

$\{\mathbf{\Phi_M}\}_{\mathbf{M}\in\mathcal{G}}$ *will be called a* **generally covariant quantum field theory over** $\mathcal{G}$ *of spin representation type* ρ *if the following properties are fulfilled:*

(A) *For each* $\mathbf{M} \in \mathcal{G}$, $\mathbf{\Phi_M} = (\Phi_\mathbf{M}, \mathcal{D}_\mathbf{M}, \mathcal{H}_\mathbf{M})$ *is a quantum field theory on* $\mathbf{M}$ *of spin representation type* ρ *(the same for all* $\mathbf{M}$*) such that the properties (a) and (b) stated above are satisfied.*

(B) *For the case that* $\mathbf{M} = \mathbf{M}_0$ *is Minkowski spacetime with its usual spin-structure, we demand that the corresponding quantum field theory* $\mathbf{\Phi}_{\mathbf{M}_0}$ *fulfills the Wightman axioms, including the Bose–Fermi alternative (or normal commutation relations); see Appendix B for details.*

(C) *If for a pair* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *in* $\mathcal{G}$ *there is a local isomorphism* $\mathbf{\Theta}$ *between* M_1 *and* M_2*, then it is covered by local isomorphisms between the corresponding quantum field theories* $\mathbf{\Phi}_{\mathbf{M}_1}$ *and* $\mathbf{\Phi}_{\mathbf{M}_2}$.

Let us discuss some features of that definition in a further set of

Comments and Remarks. (iv) Readers familiar with the articles of Dimock [13,14] will notice that our definition is very much inspired by the concept of general covariance introduced in those works for quantum field theories on curved spacetimes. The main difference, as we have mentioned already in the Remark below Def. 3.1, is that the isomorphisms between the spacetimes with spin-structures, and accordingly between the corresponding quantum field theories, are here assumed to be local, whereas in [13,14] they are assumed to be global. To allow local isomorphisms in the condition of general covariance (C) leads, in combination with the conditions (A) and (B), to restrictions which apparently are not present when using only global isomorphisms.

The significance of that point has, in a somewhat different context, been noted by Kay [34]. Our definition of a generally covariant quantum field theory resembles an approach taken by Kay in his investigation of “F-locality” in [34]. The main difference (apart from differences of technical detail) is that Kay considers a much larger class $\widehat{\mathcal{G}}$ of spacetimes which need not be globally hyperbolic, and he essentially investigates the question of what the largest class $\widehat{\mathcal{G}}$ of spacetimes might be so that a quantum field theory over $\widehat{\mathcal{G}}$ is compatible with the covariance property (C) once certain properties are assumed for the quantum fields on the individual spacetimes in $\widehat{\mathcal{G}}$. For the case of the scalar Klein–Gordon field, he finds that restrictions on the class of spacetimes $\widehat{\mathcal{G}}$ arise in order to obtain compatibility, see [34] for further discussion.

(v) Given a local isomorphism $\mathbf{\Theta}$ between $\mathbf{M}_1$ and $\mathbf{M}_2$ in $\mathcal{G}$, then it is known for free fields that typically the identification

$$\Phi_{\mathbf{M}_2} \circ \check{\Theta}^\star(f) = \Phi_{\mathbf{M}_1}(f), \quad \operatorname{supp} f \in \ell_{\mathrm{ini}}(\mathbf{\Theta}), \quad \text{with } \check{\Theta}^\star f = \check{\Theta} \circ f \circ \vartheta^{-1},$$

preserves CAR or CCR and thereby gives rise to a (C^*-algebraic) local isomorphism $\alpha_{\mathbf{\Theta}}$ covering $\mathbf{\Theta}$ between the quantum field theories. In [52] (pp. 89–91 of that reference), such a covariance property has been proposed as a condition on the (renormalized) stress-energy tensor of a quantum field on curved spacetimes, and more recently, Hollands and Wald have defined the notion of a local, covariant quantum field by means of such a covariance behaviour of the quantum field and have shown that one may construct, essentially uniquely, Wick-polynomials of the free scalar field in such a way that they become local, covariant quantum fields [32]. Our conditions on a local isomorphism between quantum field theories are much less detailed; indeed, the slightly complicated definition of a local isomorphism between quantum field theories serves the purpose of keeping this notion as general as possible and yet to transfer enough algebraic information for making it a useful (i.e. sufficiently restrictive) concept in combination with existence of a causal dynamical law formulated in (b).

(vi) We have required that the spin-representation ρ be the same for all members $\boldsymbol{\Phi}_{\mathbf{M}}$ of a generally covariant quantum field theory over $\mathcal{G}$, expressing that all these quantum field theories on the various spacetimes have the same field content. (Of course, it would be sufficient just to require that the various $\rho_{\mathbf{M}}$ be isomorphic; to demand equality is just a simplification of notation.) We think that this is necessary in order that (C) can be fulfilled, but a proof of that remains to be given.

(vii) It should be noted that each element $\mathbf{M} \in \mathcal{G}$ comes equipped with an orientation and a time-orientation. The local isomorphisms have been assumed to preserve orientation and time-orientation, so the condition of general covariance imposes no restrictions on quantum field theories $\boldsymbol{\Phi}_{\mathbf{M}_1}$ and $\boldsymbol{\Phi}_{\mathbf{M}_2}$ when $\mathbf{M}_1$ and $\mathbf{M}_2$ are connected by a local isomorphism that reverses orientation and time-orientation. In fact, if $\boldsymbol{\Theta}$ is an (appropriately defined) local isomorphism between $\mathbf{M}_1$ and $\mathbf{M}_2$ reversing both time-orientation and orientation, one would expect that for any relatively compact $N_i \subset \ell_{\text{ini}}(\boldsymbol{\Theta})$, writing $N_f = \vartheta(N_i)$, there is an *anti-linear* von Neumann algebraic isomorphism $\alpha_{\boldsymbol{\Theta},N_i}$ having the covariance property (4.1). It would be quite interesting to see if one could deduce the existence of such anti-linear local von Neumann algebraic isomorphisms at least for a distinguished class of time-orientation and orientation reversing local isomorphisms $\boldsymbol{\Theta}$ from the assumptions on $\{\boldsymbol{\Phi}_{\mathbf{M}}\}_{\mathbf{M}\in\mathcal{G}}$ of Def. 4.1. That would correspond to a PCT-theorem in the present general setting.

(viii) The assignment of quantum field theories $\boldsymbol{\Phi}_{\mathbf{M}}$ to each $\mathbf{M} \in \mathcal{G}$ fulfilling the condition of general covariance allows a functorial description which will be indicated in Appendix C.

5. Spin and Statistics

In the present section we state and prove a spin-statistics theorem for generally covariant quantum field theories over $\mathcal{G}$. Before we can start to formulate the result, it is in order to briefly recapitulate the terminology referring to "integer" and "half-integer" spin.

Let $\circledS^k\mathbb{C}^2$ denote the k-fold symmetrized tensor product of $\mathbb{C}^2$. Then an irreducible complex linear representation $D^{(k,l)}$ of $\mathrm{SL}(2,\mathbb{C})$ for $k, l \in \mathbb{N}_0$ is given on the vectorspace $V_{k,l} = (\circledS^k\mathbb{C}^2) \otimes (\circledS^l\mathbb{C}^2)$ by

$$D^{(k,l)}(\mathbf{s}) = (\circledS^k\mathbf{s}) \otimes (\circledS^l\bar{\mathbf{s}}),$$

where $\mathbf{s} \in \mathrm{SL}(2,\mathbb{C})$ acts like a matrix on column vectors in $\mathbb{C}^2$, and $\bar{\mathbf{s}}$ is the matrix with complex conjugate entries.[3] All finite-dimensional complex linear irreducible representations of $\mathrm{SL}(2,\mathbb{C})$ arise in this way. Such an irreducible representation is said to be of *integer type* (or simply *integer*) if $k + l$ is even and of *half-integer type* (or simply *half-integer*) if $k+l$ is odd. There also the (finite dimensional) real linear irreducible representations $D^{(k,l)} \oplus D^{(l,k)}$ for $k \neq l$, and $D^{(l,l)}$. They are called real-linear irreducible because it is possible to select real-linear subspaces in $V_{k,l} \oplus V_{l,k}$ and in $V_{l,l}$, respectively, on which these representations act irreducibly as real-linear representations. As complex linear representations they are, however, reducible except for the case $D^{(l,l)}$. The classification of these representations as being of "integer" or "half-integer" type is analogous to that of complex linear irreducible representations.

[3] By convention, the case $k = 0$ and $l = 0$ corresponds to a scalar field, with the trivial one-dimensional representation of $\mathrm{SL}(2,\mathbb{C})$.

Theorem 5.1. *Let $\{\mathbf{\Phi_M}\}_{\mathbf{M}\in\mathcal{G}}$ be a generally covariant quantum field theory over $\mathcal{G}$ of spin representation type ρ, where ρ is assumed to be a complex linear irreducible, or real linear irreducible, finite dimensional representation of* $\mathrm{SL}(2,\mathbb{C})$.

(I) *If ρ is of half-integer type, and if there exist an $\mathbf{M} \in \mathcal{G}$ and a pair of non-empty $O_1, O_2 \in \mathrm{orc}(M)$ with $O_1 \subset O_2^{\perp}$ so that $\mathcal{F}_{\mathbf{M}}(O_1) \subset \mathcal{F}_{\mathbf{M}}(O_2)'$ (where by $\mathcal{F}_{\mathbf{M}}(O)$ we denote the local von Neumann algebras generated by $\mathbf{\Phi_M}$ and by $\mathcal{F}_{\mathbf{M}}(O)'$ the commutant algebras[4]) then it follows for all $\hat{\mathbf{M}} \in \mathcal{G}$ that $\Phi_{\hat{\mathbf{M}}}(f) = c_f \cdot 1$ for some $c_f \in \mathbb{C}$, i.e. the quantum field operators of all quantum fields of the generally covariant theory are multiples of the unit operator.*

(II) *If ρ is of integer type, and if there exist an $\mathbf{M} \in \mathcal{G}$, a pair of causally separated points p_1 and p_2 in M and for each pair of open neighbourhoods O_j of p_j with $O_1 \subset O_2^{\perp}$ a pair $f_j \in \Gamma_0(\mathcal{V}_\rho)$ with* supp $f_j \subset O_j$ *and $\Phi_{\mathbf{M}}(f_j) \neq 0$ $(j = 1, 2)$ so that*

$$\begin{gathered}\Phi_{\mathbf{M}}(f_1)\Phi_{\mathbf{M}}(f_2) + \Phi_{\mathbf{M}}(f_2)\Phi_{\mathbf{M}}(f_1) = 0 \ \text{ or} \\ \Phi_{\mathbf{M}}(f_1)\Phi_{\mathbf{M}}(f_2)^* + \Phi_{\mathbf{M}}(f_2)^*\Phi_{\mathbf{M}}(f_1) = 0,\end{gathered} \tag{5.1}$$

then it follows again for all $\hat{\mathbf{M}} \in \mathcal{G}$ that all field operators $\Phi_{\hat{\mathbf{M}}}(f)$ are multiples of the unit operator.

We note that $\mathcal{F}_{\mathbf{M}}(O_1) \subset \mathcal{F}_{\mathbf{M}}(O_2)'$ means that the field operators $\Phi_{\mathbf{M}}(f_1)$ and $\Phi_{\mathbf{M}}(f_2)$ for supp $f_j \subset O_j$ commute strongly in the sense that the operators appearing in their polar decompositions commute strongly. This stronger form of commutativity at causal separation is expected to hold in physically relevant theories. In Appendix B we give a few more comments on this point. If the stronger forms of general covariance at the level of invidual field operators as indicated in Remarks (ii) and (iv) of Sect. 4 were assumed, the statement for the half-integer case could be strengthened to resemble the integer case more closely; namely, then one would conclude for the half-integer case that the relations $\Phi_{\mathbf{M}}(f_1)\Phi_{\mathbf{M}}(f_2) - \Phi_{\mathbf{M}}(f_2)\Phi_{\mathbf{M}}(f_1) = 0$ or $\Phi_{\mathbf{M}}(f_1)\Phi_{\mathbf{M}}(f_2)^* - \Phi_{\mathbf{M}}(f_2)^*\Phi_{\mathbf{M}}(f_1) = 0$ for some $\mathbf{M}$ and a pair of test-spinors f_1 and f_2 with causally separated supports so that $\Phi_{\mathbf{M}}(f_j) \neq 0$ already imply that the field operators $\Phi_{\hat{\mathbf{M}}}(f)$ are multiples of unity for all $\hat{\mathbf{M}} \in \mathcal{G}$.

Proof of Theorem 4.1. We begin with part (I) of the statement involving a theory of half-integer type, and we suppose that $\mathcal{F}(O_1) \subset \mathcal{F}(O_2)'$ for a pair of causally separated $O_1, O_2 \in \mathrm{orc}(M)$, where we use the notation $\mathcal{F}(O) = \mathcal{F}_{\mathbf{M}}(O)$. Then let $p_j \in O_j$, and choose for this pair of causally separated points in M a globally hyperbolic spacetime $(\widetilde{M}, \widetilde{g})$ with neighbourhoods U_j, $\widehat{U}_j$, $\widetilde{U}_j$, G, $\widehat{G}$, as in Lemma 2.1, which can be done in such a way that $\vartheta^{-1}(U_j) \subset O_j$, where ϑ is the isomorphism $M \supset N \to \widetilde{N} \subset \widetilde{M}$. Now we equip $(\widetilde{M}, \widetilde{g})$ with any spin-structure and denote the resulting spacetime with spin-structure by $\widetilde{\mathbf{M}}$. The neighbourhoods G and $\widehat{G}$ are simply connected. Thus, since all spin-structures over simply connected globally hyperbolic spacetimes are equivalent, there is a local isomorphism $\mathbf{\Theta}$ between $\mathbf{M}$ and $\widetilde{\mathbf{M}}$ with $\ell_{\mathrm{fin}}(\mathbf{\Theta}) = G$, and also a local isomorphism $\mathbf{\Theta}_0$ between $\widetilde{\mathbf{M}}$ and $\widetilde{\mathbf{M}}_0$, where $\widetilde{\mathbf{M}}_0$ is Minkowski spacetime with its standard spin-structure. This is due to the fact that G is isomorphic to a subset $\vartheta^{-1}(G)$ in M and $\widehat{G}$ is isomorphic to a subset in Minkowski-spacetime M_0, cf. Lemma 2.1.

Let us now introduce the notation $\widetilde{\mathcal{F}}(U) = \mathcal{F}_{\widetilde{\mathbf{M}}}(U)$ and $\mathcal{F}_0(U) = \mathcal{F}_{\mathbf{M}_0}(U)$ for the local von Neumann algebras corresponding to the theories $\mathbf{\Phi}_{\widetilde{\mathbf{M}}}$ and $\mathbf{\Phi}_{\mathbf{M}_0}$, respectively.

[4] I.e. $\mathcal{F}_{\mathbf{M}}(O)' = \{A' \in B(\mathcal{H}_{\mathbf{M}}) : A'A = AA',\ \forall\, A \in \mathcal{F}_{\mathbf{M}}(O)\}$.

Then choose two globally hyperbolic, relatively compact submanifolds N_f and $\widehat{N}_i$ of G and $\widehat{G}$, respectively, with the additional property that $U_j \subset N_f$ and $\widetilde{U}_j, \widehat{U}_j \subset \widehat{N}_i$ ($j = 1, 2$). Denote $N_i = \vartheta^{-1}(N_f)$. According to the general covariance assumption (C) there are local isomorphisms $\alpha_{\mathbf{\Theta}, N_i}$ between $\mathbf{\Phi_M}$ and $\mathbf{\Phi}_{\widetilde{\mathbf{M}}}$ and $\alpha_{\mathbf{\Theta}_0, \widehat{N}_i}$ between $\mathbf{\Phi}_{\widetilde{\mathbf{M}}}$ and $\mathbf{\Phi}_{\mathbf{M}_0}$ so that

$$\alpha_{\mathbf{\Theta}, N_i}(\mathcal{F}(\vartheta^{-1}(U))) = \widetilde{\mathcal{F}}(U), \quad U \in \text{orc}(N_f), \tag{5.2}$$

$$\alpha_{\mathbf{\Theta}_0, \widehat{N}_i}(\widetilde{\mathcal{F}}(\widehat{U})) = \mathcal{F}_0(\vartheta_0(\widehat{U})), \quad \widehat{U} \in \text{orc}(\widehat{N}_i), \tag{5.3}$$

where ϑ_0 is the isomorphism embedding $\widehat{G}$ into M_0. Since we have supposed initially that $\mathcal{F}(O_1) \subset \mathcal{F}(O_2)'$, and since $\vartheta^{-1}(U_j) \subset O_j$, relation (5.2) implies that $\widetilde{\mathcal{F}}(U_1) \subset \widetilde{\mathcal{F}}(U_2)'$. Moreover, $U_j \rhd \widetilde{U}_j$ and hence, by the existence of a causal dynamical law, it follows that

$$\widetilde{\mathcal{F}}(\widetilde{U}_1) \subset \widetilde{\mathcal{F}}(\widetilde{U}_2)'.$$

Exploiting also (5.3), one obtains

$$\mathcal{F}_0(\vartheta_0(\widetilde{U}_1)) \subset \mathcal{F}_0(\vartheta_0(\widetilde{U}_2))', \tag{5.4}$$

where $\vartheta_0(\widetilde{U}_1)$ and $\vartheta_0(\widetilde{U}_2)$ are a pair of open, causally separated subsets of Minkowski spacetime. Since the quantum field theory $\mathbf{\Phi}_{\mathbf{M}_0}$ on Minkowski spacetime has been assumed to fulfill the usual assumptions, and is, by assumption, of half-integer spin-type, the last relation (5.4) implies by the known spin-statistics theorem for quantum field theories on Minkowski spacetime that $\mathcal{F}_0(U_0) = \mathbb{C} \cdot 1$ holds for all $U_0 \in \text{orc}(M_0)$. (See Appendix B for details.)

In a next step we will show how that conclusion implies that all other quantum field theories $\mathbf{\Phi}_{\hat{\mathbf{M}}}$ are likewise trivial. Let $\hat{\mathbf{M}} = ((\hat{M}, \hat{g}), S(\hat{M}, \hat{g}), \hat{\psi}) \in \mathcal{G}$ and choose any point $p_1 \in \hat{M}$ (and any other causally separated point $p_2 \in \hat{M}$, which actually plays no role). Then choose a spacetime $(\widetilde{M}, \widetilde{g})$ with subsets $U_j, \widetilde{U}_j, \widehat{U}_j, G, \widehat{G}$ as in Lemma 2.1 for these data, $(\hat{M}, \hat{g})$ now playing the role of (M, g). Identifying $\mathcal{F}(O) = \mathcal{F}_{\hat{\mathbf{M}}}(O)$ and making similar adaptations, Eqs. (5.2) and (5.3) hold accordingly. Then $\mathcal{F}_0(\vartheta_0(\widehat{U}_1)) = \mathbb{C} \cdot 1$ implies, by (5.3), $\widetilde{\mathcal{F}}(\widehat{U}_1) = \mathbb{C} \cdot 1$, and since $\widehat{U}_1 \rhd U_1$ it follows that $\widetilde{\mathcal{F}}(U_1) = \mathbb{C} \cdot 1$. Hence (5.2) leads to $\mathcal{F}(\vartheta^{-1}(U_1)) = \mathbb{C} \cdot 1$, implying that $\Phi_{\hat{M}}(f)$ is a multiple of the unit operator for all f with supp $f \subset \vartheta^{-1}(U_1)$. As $\vartheta^{-1}(U_1)$ is an open neighbourhood of an arbitrary point $p_1 \in \hat{M}$, and since the quantum field $f \mapsto \Phi_{\hat{\mathbf{M}}}(f)$ is linear, a partition of unity argument shows that therefore one must have $\Phi_{\hat{\mathbf{M}}}(f) = c_f \cdot 1$ with suitable $c_f \in \mathbb{C}$ for all test-spinors f on $\hat{M}$.

Now we turn to the proof of statement (II) of the theorem. According to the assumptions, there are two points p_1 and p_2 in M which are causally separated, and moreover, when choosing a deformation $(\widetilde{M}, \widetilde{g})$ of (M, g) with neighbourhoods $U_j, \widetilde{U}_j, \widehat{U}_j, G, \widehat{G}$ as in Lemma 2.1, there are a pair of testing spinors f_j supported in $\vartheta^{-1}(U_j)$ so that $\Phi_{\mathbf{M}}(f_j) \neq 0$ and such that one of the relations (5.1) holds. We shall, for the sake of simplicity of notation, assume that

$$\Phi_{\mathbf{M}}(f_1)\Phi_{\mathbf{M}}(f_2) + \Phi_{\mathbf{M}}(f_2)\Phi_{\mathbf{M}}(f_1) = 0 \tag{5.5}$$

holds, and we will show that these properties are in conflict with Bosonic commutation relations for the theory $\mathbf{\Phi}_{\mathbf{M}_0}$ on Minkowski spacetime. The other case of (5.1) can be treated by similar arguments. The proof proceeds indirectly, so we suppose that $\mathbf{\Phi}_{\mathbf{M}_0}$

possesses Bosonic commutation relations. As before in the proof of (I) above, we can find local isomorphisms $\alpha_{\mathbf{\Theta},N_i}$ and $\alpha_{\mathbf{\Theta}_0,\widehat{N}_i}$ fulfilling the relations (5.2) and (5.3) for the von Neumann algebraic nets corresponding to the quantum field theories on $\mathbf{M}$, $\widetilde{\mathbf{M}}$ and $\mathbf{M}_0$. Having supposed Bosonic commutation relations for the quantum field theory on Minkowski spacetime, it follows by (5.3) that $\widetilde{\mathcal{F}}(\widehat{U}_1) \subset \widetilde{\mathcal{F}}(\widehat{U}_2)'$. Now $U_j \lhd \widehat{U}_j$ and thus, by the existence of a causal dynamical law, it holds that $\widetilde{\mathcal{F}}(U_1) \subset \widetilde{\mathcal{F}}(U_2)'$. By (5.2) we obtain $\mathcal{F}(\vartheta^{-1}(U_1)) \subset \mathcal{F}(\vartheta^{-1}(U_2))'$. Since the operators $\Phi_{\mathbf{M}}(f_j)$ are affiliated to the von Neumann algebras $\mathcal{F}(\vartheta^{-1}(U_j))$, one concludes that

$$\Phi_{\mathbf{M}}(f_1)\Phi_{\mathbf{M}}(f_2) - \Phi_{\mathbf{M}}(f_2)\Phi_{\mathbf{M}}(f_1) = 0. \tag{5.6}$$

Comparing (5.5) and (5.6) yields

$$\Phi_{\mathbf{M}}(f_1)\Phi_{\mathbf{M}}(f_2) = 0.$$

It is clear that this relation entails $\Phi_{\mathbf{M}}(f_1)^*\Phi_{\mathbf{M}}(f_1)\Phi_{\mathbf{M}}(f_2)\Phi_{\mathbf{M}}(f_2)^* = 0$. The operators $A_1 = \Phi_{\mathbf{M}}(f_1)^*\Phi_{\mathbf{M}}(f_1)$ and $A_2 = \Phi_{\mathbf{M}}(f_2)\Phi_{\mathbf{M}}(f_2)^*$ are positive and possess selfadjoint extensions affiliated to $\mathcal{F}(\vartheta^{-1}(U_1))$ and $\mathcal{F}(\vartheta^{-1}(U_2))$, respectively. Denoting by $E_j(a)$ their spectral projections corresponding to the spectral interval $(-a, a)$, the operators $A_j(a) = E_j(a)A_j$ are contained in $\mathcal{F}(\vartheta^{-1}(\widehat{U}_1))$ and it holds that $A_1(a)A_2(a) = 0$ for all $a > 0$. Repeating the arguments that led to Eq. (5.6), one can see that the $A_j(a)$ possess isomorphic images $\widehat{A}_j(a)$ in $\mathcal{F}_0(\vartheta_0(\widehat{U}_j))$ so that $\widehat{A}_1(a)\widehat{A}_2(a) = 0$ for all $a > 0$. But since the net $\{\mathcal{F}_0(U)\}_{U\in \mathrm{orc}(M_0)}$ was assumed to fulfill Bosonic commutations relations, and since it fulfills the usual assumptions for a quantum field theory on Minkowski-spacetime, including the spectrum condition and the existence of a vacuum state, it follows that the Schlieder property [43] holds for this net. This property states that the relations $\widehat{A}_j(a) \in \mathcal{F}_0(\vartheta_0(\widehat{U}_j))$, $\mathrm{cl}\,\vartheta_0(\widehat{U}_1) \subset \vartheta_0(\widehat{U}_2)^\perp$ and $\widehat{A}_1(a)\widehat{A}_2(a) = 0$ imply $\widehat{A}_1(a) = 0$ or $\widehat{A}_2(a) = 0$. Hence one obtains that, for all $a > 0$, $A_1(a) = 0$ or $A_2(a) = 0$, and this entails $A_1 = 0$ or $A_2 = 0$, which in turn enforces $\Phi_{\mathbf{M}}(f_1) = 0$ or $\Phi_{\mathbf{M}}(f_2) = 0$. Thus one arrives at a contradiction since both operators $\Phi_{\mathbf{M}}(f_1)$ and $\Phi_{\mathbf{M}}(f_2)$ are by assumption different from 0. One concludes that Bosonic commutation relations are an impossible option for the theory $\mathbf{\Phi}_{\mathbf{M}_0}$ on Minkowski spacetime and thus, due to the Bose–Fermi-alternative, that theory must fulfill Fermionic commutation relations. Since the theory is of integer spin-type, this implies that the von Neumann algebras $\mathcal{F}_0(U_0)$ of the theory on Minkowski spacetime consist only of multiples of the unit operator because of the spin-statistics theorem on flat spacetime (cf. Appendix B). Repeating the argument given for part (I) above, it follows that for each $\hat{\mathbf{M}} \in \mathcal{G}$ the quantum field operators $\Phi_{\hat{\mathbf{M}}}(f)$ are multiples of the unit operator for all test-tensors f. $\square$

6. Examples

In this section we briefly indicate examples of linear quantum field theories which fulfill the properties required for a generally covariant quantum field theory over $\mathcal{G}$ in Sect. 4.

1. The free scalar field. The simplest example is the free scalar field, although its significance for a spin-statistics theorem is, naturally, quite limited.

For each globally hyperbolic spacetime $\mathbf{M} = (M, g) \in \mathcal{G}$ (endowed with a spin-structure whose explicit appearance is now suppressed since it is irrelevant for the scalar field) we consider the scalar Klein–Gordon equation

$$(\Box_g + m^2)\varphi = 0$$

for real-valued functions φ on M, where $m \geq 0$ is a constant independent of $\mathbf{M}$ and $\Box_g$ is the scalar d'Alembertian for (M, g). Following Dimock [13], one can construct a C^*-algebraic quantization of this field as follows. There are uniquely determined, continuous linear maps $E^{\pm}_{\mathbf{M}} : C_0^{\infty}(M, \mathbb{R}) \to C^{\infty}(M, \mathbb{R})$ with the properties

$$(\Box_g + m^2)E^{\pm}_{\mathbf{M}} = f = E^{\pm}_{\mathbf{M}}(\Box_g + m^2)f \quad \text{and}$$
$$\operatorname{supp} E^{\pm}_{\mathbf{M}} f \subset J^{\pm}(\operatorname{supp} f),$$
$$f \in C_0^{\infty}(M, \mathbb{R}).$$

Their difference $E^{\pm}_{\mathbf{M}} = E^{+}_{\mathbf{M}} - E^{-}_{\mathbf{M}}$, called the (causal) propagator, induces a symplectic form

$$\kappa_{\mathbf{M}}([f], [h]) = \int_M d\eta\, f \cdot E_{\mathbf{M}} h, \quad [f], [h] \in K_{\mathbf{M}},$$

on $K_{\mathbf{M}} = C_0^{\infty}(M, \mathbb{R})/\ker E_{\mathbf{M}}$, where $f \mapsto [f] = [f]_{\mathbf{M}}$ denotes the quotient map and $d\eta$ is the metric-induced volume-form on (M, g). To the resulting symplectic space $(K_{\mathbf{M}}, \kappa_{\mathbf{M}})$ there corresponds the CCR-Weyl algebra $\mathfrak{A}[K_{\mathbf{M}}, \kappa_{\mathbf{M}}]$, defined as the (up to C^*-isomorphisms unique) C^*-algebra generated by unitary elements $W_{\mathbf{M}}(x)$, $x \in K_{\mathbf{M}}$, fullfilling the Weyl-relations, or "exponentiated" canonical commutation relations (see [9])

$$W_{\mathbf{M}}(x)W_{\mathbf{M}}(y) = \exp(-i\kappa_{\mathbf{M}}(x, y)/2)W_{\mathbf{M}}(x + y),$$
$$W_{\mathbf{M}}(x)^* = W_{\mathbf{M}}(-x), \quad x, y \in K_{\mathbf{M}}.$$

Dimock has shown that any isometry $\theta : \mathbf{M}_1 \to \mathbf{M}_2$ induces a C^*-algebraic isomorphism $\alpha_\theta : \mathfrak{A}[K_{\mathbf{M}_1}, \kappa_{\mathbf{M}_1}] \to \mathfrak{A}[K_{\mathbf{M}_2}, \kappa_{\mathbf{M}_2}]$ with the property that

$$\alpha_\theta(W_{\mathbf{M}_1}([f]_{\mathbf{M}_1})) = W_{\mathbf{M}_2}([\theta^* f]_{\mathbf{M}_2}), \quad f \in C_0^{\infty}(M_1, \mathbb{R}), \tag{6.1}$$

where $\theta^* f = f \circ \theta^{-1}$. If $\mathbf{M}_1 \subset \mathbf{M}'_1$ and $\mathbf{M}_2 \subset \mathbf{M}'_2$ are globally hyperbolic sub-spacetimes of a pair of globally hyperbolic spacetimes $\mathbf{M}'_1$ and $\mathbf{M}'_2$, then $W_{\mathbf{M}_j} = W_{\mathbf{M}'_j} \restriction K_{\mathbf{M}_j}$ $(j = 1, 2)$ holds up to C^*-isomorphisms as a consequence of the uniqueness of the causal propagators, thus there is always a C^*-algebraic Weyl-algebra isomorphism covering a local isomorphism between members of $\mathcal{G}$. Furthermore, Dimock has also shown in [13] that, upon denoting by $\mathfrak{A}_{\mathbf{M}}(O)$ the C^*-subalgebra of $\mathfrak{A}[K_{\mathbf{M}}, \kappa_{\mathbf{M}}]$ generated by all $W_{\mathbf{M}}([f]_{\mathbf{M}})$, $\operatorname{supp} f \subset O$, there holds

$$O_1 \lhd O_2 \Rightarrow \mathfrak{A}_{\mathbf{M}}(O_1) \subset \mathfrak{A}_{\mathbf{M}}(O_2) \tag{6.2}$$

for all $O_1, O_2 \subset M$.

Now let $\omega_{\mathbf{M}}$ be an arbitrary quasifree Hadamard state on $\mathfrak{A}[K_{\mathbf{M}}, \kappa_{\mathbf{M}}]$. Such a state is determined by its two-point correlation function which here is required to be of "Hadamard form". The Hadamard form specifies the singular short-distance behaviour in a particular way, see [21,52] and references cited therein for discussion. Equivalently, the Hadamard form of a two-point function can be characterized by a certain form of its wavefront set (see [39,42] for details). It has been shown in [22] that there exists an abundance of Hadamard states on $\mathfrak{A}[K_{\mathbf{M}}, \kappa_{\mathbf{M}}]$. To such a quasifree Hadamard state $\omega_{\mathbf{M}}$ there corresponds its GNS-Hilbertspace representation $(\pi_{\mathbf{M}}, \mathcal{H}_{\mathbf{M}}, \Omega_{\mathbf{M}})$, cf. e.g. [8]. In that representation, we define the local von Neumann algebras

$$\mathcal{F}_{\mathbf{M}}(O) = \pi_{\mathbf{M}}(\mathfrak{A}_{\mathbf{M}}(O))''$$

for each $O \in \mathrm{orc}(M)$. Then (6.2) clearly implies the existence of a causal dynamical law

$$O_1 \lhd O_2 \Rightarrow \mathcal{F}_{\mathbf{M}}(O_1) \subset \mathcal{F}_{\mathbf{M}}(O_2).$$

A vector $\chi \in \mathcal{H}_{\mathbf{M}}$ is defined to be in $\mathcal{D}_{\mathbf{M}}$ if for each choice of $\boldsymbol{x} = (x_1, \ldots, x_n) \in (K_{\mathbf{M}})^n$ the map

$$\boldsymbol{t} \mapsto \pi_{\mathbf{M}}(W_{\mathbf{M}}(t_1 x_1)) \cdots \pi_{\mathbf{M}}(W_{\mathbf{M}}(t_n x_n))\chi, \quad \boldsymbol{t} = (t_1, \ldots, t_n) \in \mathbb{R}^n,$$

is C^∞. One can show that $\mathcal{D}_{\mathbf{M}}$ is a dense domain in $\mathcal{H}_{\mathbf{M}}$ (cf. [9]). One can define for each $f \in C_0^\infty(M, \mathbb{R})$ the quantum field operator $\Phi_{\mathbf{M}}(f)$ by

$$\Phi_{\mathbf{M}}(f)\chi = -i \frac{d}{dt}\bigg|_{t=0} \pi_{\mathbf{M}}(W_{\mathbf{M}}(t[f]_{\mathbf{M}}))\chi, \quad \chi \in \mathcal{D}_{\mathbf{M}}.$$

One can also show that $\mathcal{D}_{\mathbf{M}}$ is left invariant under the action of $\Phi_{\mathbf{M}}(f)$ and that $\Phi_{\mathbf{M}}(f)$ is essentially self-adjoint [9]. It is also obvious that $\Phi_{\mathbf{M}}(f)$ is affiliated to $\mathcal{F}_{\mathbf{M}}(O)$ as soon as $\mathrm{supp}\, f \subset O$.

Moreover, the results of [48] show that the C^*-algebraic isomorphism α_θ in (6.1) can be extended, in representations induced by quasifree Hadamard states, to von Neumann algebraic isomorphisms in the following way. Suppose that between $\mathbf{M}_1$ and $\mathbf{M}_2$ in $\mathcal{G}$ there is a local isomorphism θ, and let $N_i \subset \ell_{\mathrm{ini}}(\theta)$ be a relatively compact subset. Then, writing $N_f = \theta(N_i)$, the Weyl-algebra isomorphism α_θ in (6.1) extends to an isomorphism $\alpha_{\theta,N_i} : \mathcal{F}_{\mathbf{M}_1}(N_i) \to \mathcal{F}_{\mathbf{M}_2}(N_f)$ between von Neumann algebras. Consequently, there holds the covariance property

$$\alpha_{\theta,N_i}(\mathcal{F}_{\mathbf{M}_1}(O_i)) = \mathcal{F}_{\mathbf{M}_2}(\theta(O_i)), \quad O_i \in \mathrm{orc}(N_i).$$

Finally, if $\mathbf{M}_0$ is Minkowski spacetime, we take $\omega_{\mathbf{M}_0}$ to be the vacuum state which is known to be a quasifree Hadamard state. In conclusion, the just constructed family $\{\boldsymbol{\Phi}_{\mathbf{M}}\}_{\mathbf{M}\in\mathcal{G}}$ of Klein–Gordon quantum fields for each $\mathbf{M} \in \mathcal{G}$ satisfies all the assumptions required for a generally covariant quantum field theory over $\mathcal{G}$.

2. The Proca field. The Proca field is a co-vector field, i.e. of tensorial type, corresponding to the $D^{(1,1)}$ irreducible representation of $\mathrm{SL}(2, \mathbb{C})$. For each globally hyperbolic spacetime $\mathbf{M} = (M, g) \in \mathcal{G}$ (where again we suppress the spin-structure in our notation since it is presently not relevant), we denote by d the exterior derivative of differential forms, by $*$ the Hodge-star operator corresponding to the metric g, and define the co-differential $\delta = *d*$. Then the Proca equation reads, for $\varphi \in \Gamma_0(T^*M)$,

$$(\delta d + m^2)\varphi = 0,$$

where $m > 0$ is a constant independent of $\mathbf{M}$. (Note that δd depends on the metric g.) A C^*-algebraic quantization has recently been given by Furlani [23] (cf. also [45], whose notation we follow here). To this end one constructs advanced and retarded fundamental solutions $F^{\pm}_{\mathbf{M}} : \Gamma_0(T^*M) \to \Gamma(T^*M)$ uniquely determined by

$$F^{\pm}_{\mathbf{M}}(\delta d + m^2) f = f = (\delta d + m^2) F^{\pm}_{\mathbf{M}} f, \ \ \mathrm{supp}\, F^{\pm}_{\mathbf{M}} f \subset J^{\pm}(\mathrm{supp}\, f), \quad f \in \Gamma_0(T^*M).$$

As in the case of the scalar Klein–Gordon field, one defines the (causal) propagator $F_{\mathbf{M}} = F^{+}_{\mathbf{M}} - F^{-}_{\mathbf{M}}$ and a symplectic space $(K_{\mathbf{M}}, \kappa_{\mathbf{M}})$ where

$$\kappa_{\mathbf{M}}([f],[h]) = \int_M f \wedge * F_{\mathbf{M}} h, \quad [f],[h] \in K_{\mathbf{M}},$$

on $K_{\mathbf{M}} = \Gamma_0(T^*M)/\ker F_{\mathbf{M}}$ and $f \mapsto [f] = [f]_{\mathbf{M}}$ is the quotient map.

From here onwards, all the arguments leading to the construction of a generally covariant theory $\{\mathbf{\Phi}_{\mathbf{M}}\}_{\mathbf{M}\in\mathcal{G}}$ can be taken over almost literally, except for obvious modifications, from the previous case of the scalar Klein–Gordon field to the present case of the Proca field. There are some provisions which should nevertheless be recorded: Firstly, the existence of Hadamard states for the Proca field has not been demonstrated. However, as mentioned towards the end of Sect. 5.1 in [42], the existence of Hadamard states could be established by using the existence of a ground state for the Proca field on ultrastatic spacetimes [24] in combination with results in [41] and [22] to prove that there exists a large set of quasifree Hadamard states for the Proca field. Secondly, the arguments given in [48] showing that the C^*-algebraic isomorphism (6.1) can be extended to a von Neumann algebraic isomorphism in the above said way apply to the case of the free scalar Klein–Gordon field. But those arguments can obviously be generalized to apply to a far more general class of free fields, including the Proca field. Thus, one may conclude that also the Proca field gives rise to a generally covariant quantum field theory $\{\mathbf{\Phi}_{\mathbf{M}}\}_{\mathbf{M}\in\mathcal{G}}$.

3. The Dirac field. Our last example is the Dirac field, which is a spinorial field of spin $1/2$. We consider it in a Majorana representation; our presentation follows [14] to large extent, with some alterations specific to Majorana representations, see [42] for details. The Majorana representation corresponds to the real linear irreducible representation $D^{(1,0)} \oplus D^{(0,1)}$ of $\mathrm{SL}(2,\mathbb{C})$. This Majorana-Dirac representation will be denoted by ρ. Its representation space is $V_\rho = \mathbb{C}^4$.

Let $\mathbf{M} = (M, g, S(M,g), \psi) \in \mathcal{G}$ be a globally hyperbolic spacetime with spin-structure. The vector bundle $\mathcal{V} = S(M,g) \ltimes_\rho \mathbb{C}^4$ associated with $S(M,g)$ and the representation ρ will be denoted by $D_\rho M$; its sections are called spinors, or spinor fields. The metric-induced connection ∇ on TM lifts to a connection on the frame bundle $F(M,g)$ which in turn lifts to a connection on $S(M,g)$, and this induces also a connection on $D_\rho M$. The corresponding covariant derivative operator will be denoted by $\mathbf{\nabla}$. One can then introduce the spinor-tensor $\boldsymbol{\gamma} \in \Gamma(T^*M \otimes D_\rho M \otimes D^*_\rho M)$ by requiring that its components $\boldsymbol{\gamma}_a{}^A{}_B$ in (appropriate, dual) local frames are equal to the matrix elements $(\gamma_a)^A{}_B$ of the gamma-matrices in the Majorana-representation. This is a set of four 4×4 matrices $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ obeying the relations

$$\gamma_a\gamma_b + \gamma_b\gamma_a = 2\eta_{ab}, \quad \gamma_0^* = \gamma_0, \quad \gamma_k^* = -\gamma_k \ (k = 1,2,3), \quad \overline{\gamma_a} = \gamma_a.$$

Here, γ_a^* means the Hermitian conjugate of γ_a and $\overline{\gamma_a}$ is the transpose of γ_a^*, and $(\eta_{ab}) = \mathrm{diag}(1,-1,-1,-1)$ is the Minkowskian metric. Then the Dirac-operator $\nabla\!\!\!/$ is defined by setting in frame components, for any local section $f = f^A E_A \in \Gamma_0(D_\rho M)$,

$$(\nabla\!\!\!/ f)^A = \eta^{ab} \boldsymbol{\gamma}_a{}^A{}_B (\mathbf{\nabla}_b f)^B.$$

(At this point, we refer to [14,42] for details.) There is a charge conjugation C which operates by complex conjugation of the frame-components in any frame, i.e. $(Cu)^A = \overline{u^A}$ for the components of $u \in D_\rho M$. There is also the Dirac adjoint $u \mapsto u^+$ mapping $D_\rho M$ anti-linearly and base-point preserving onto its dual bundle $D_\rho^* M$; in dual frame components it is defined as $(u^+)_B = \overline{u^A} \gamma_{0\,AB}$.

The Dirac-equation on $\mathbf{M}$ is the differential equation

$$(\nabla\!\!\!/ + im)\varphi = 0$$

for $\varphi \in \Gamma(D_\rho M)$, where $m \geq 0$ is a constant, independent of $\mathbf{M}$. As in the cases considered before, there are uniquely determined advanced and retarded fundamental solutions $S^\pm_{\mathbf{M}} : \Gamma_0(D_\rho M) \to \Gamma(D_\rho M)$ distinguished by the properties

$$S^\pm_{\mathbf{M}}(\nabla\!\!\!/ + im)f = f = (\nabla\!\!\!/ + im)S^\pm_{\mathbf{M}} f, \;\; \text{supp}\, S^\pm_{\mathbf{M}} f \subset J^\pm(\text{supp}\, f), \quad f \in \Gamma_0(D_\rho M).$$

Hence one obtains a distinguished causal propagator $S_{\mathbf{M}} = S^+_{\mathbf{M}} - S^-_{\mathbf{M}}$. It gives rise to a pre-Hilbertspace $(H_{\mathbf{M}}, s_{\mathbf{M}})$, where $H_{\mathbf{M}} = \Gamma_0(D_\rho M)/\ker S_{\mathbf{M}}$ with scalar product

$$s_{\mathbf{M}}([f],[h]) = \int_M d\eta\, (Sf)^+(h), \quad [f],[h] \in H_{\mathbf{M}},$$

where we have denoted the metric-induced measure on M by $d\eta$ and by $f \mapsto [f] = [f]_{\mathbf{M}}$ the quotient map. The charge conjugation C can be shown to induce a conjugation on $(H_{\mathbf{M}}, s_{\mathbf{M}})$ which will be denoted by the same symbol. We shall also notationally identify $H_{\mathbf{M}}$ with its completion to a Hilbertspace.

To the Hilbertspace $(H_{\mathbf{M}}, s_{\mathbf{M}})$ with complex conjugation C there corresponds (uniquely, up to C^*-algebraic equivalence) the self-dual CAR-algebra $\mathfrak{B}[H_{\mathbf{M}}, s_{\mathbf{M}}, C]$ (cf. [1]) which is a C^*-algebra generated by elements $B_{\mathbf{M}}(v)$ depending linearly on $v \in H_{\mathbf{M}}$ and fulfilling the canonical anti-commutation relations

$$B_{\mathbf{M}}(v)^* B_{\mathbf{M}}(w) + B_{\mathbf{M}}(w) B_{\mathbf{M}}(v)^* = s_{\mathbf{M}}(v,w), \;\; B_{\mathbf{M}}(v)^* = B_{\mathbf{M}}(Cv), \quad v,w \in H_{\mathbf{M}}.$$

In [14], Dimock has proven that each (global) isomorphism $\mathbf{\Theta} = (\Theta, \vartheta)$ between members $\mathbf{M}_1$ and $\mathbf{M}_2$ in $\mathcal{G}$ induces a C^*-algebraic isomorphism $\alpha_{\mathbf{\Theta}} : \mathfrak{B}[H_{\mathbf{M}_1}, s_{\mathbf{M}_1}, C] \to \mathfrak{B}[H_{\mathbf{M}_2}, s_{\mathbf{M}_2}, C]$ satisfying

$$\alpha_{\mathbf{\Theta}}(B_{\mathbf{M}_1}([f]_{\mathbf{M}_1})) = B_{\mathbf{M}_2}([\check{\Theta}^\star f]_{\mathbf{M}_2}), \quad f \in \Gamma_0(D_\rho M_1), \tag{6.3}$$

where $\check{\Theta}^\star f = \check{\Theta} \circ f \circ \vartheta^{-1}$, $\check{\Theta}$ being the map $D_\rho M_1 \to D_\rho M_2$ induced by Θ. As in the cases discussed before, this statement has a local version to the effect that for each local isomorphism between members of $\mathcal{G}$ there is a C^*-algebraic isomorphism between the corresponding CAR-algebras covering it.

Moreover it was shown in [49] that strong Einstein causality,

$$O_1 \lhd O_2 \Rightarrow \mathfrak{B}_{\mathbf{M}}(O_2) \subset \mathfrak{B}_{\mathbf{M}}(O_2), \tag{6.4}$$

holds for the local C^*-subalgebras $\mathfrak{B}_{\mathbf{M}}(O)$ of $\mathfrak{B}[H_{\mathbf{M}}, s_{\mathbf{M}}, C]$ which are generated by all $B_{\mathbf{M}}([f]_{\mathbf{M}})$ with supp $f \subset O$.

Now let $\omega_{\mathbf{M}}$ be any quasifree Hadamard state on $\mathfrak{B}[H_{\mathbf{M}}, s_{\mathbf{M}}, C]$, and $(\pi_{\mathbf{M}}, \mathcal{H}_{\mathbf{M}}, \Omega_{\mathbf{M}})$ the corresponding GNS-representation, then the local von Neumann algebras will be defined via

$$\mathcal{F}_{\mathbf{M}}(O) = \pi_{\mathbf{M}}(\mathfrak{B}_{\mathbf{M}}(O))'', \quad O \in \text{orc}(M),$$

whereas the field operators are now given as

$$\Phi_{\mathbf{M}}(f) = \pi_{\mathbf{M}}(B_{\mathbf{M}}([f]_{\mathbf{M}})), \quad f \in \Gamma_0(D_\rho M).$$

Owing to the canonical anti-commutation relations, these field operators are bounded, and one may take their domain $\mathcal{D}_{\mathbf{M}}$ to be equal to $\mathcal{H}_{\mathbf{M}}$. The existence of a causal dynamical law at the level of the local von Neumann algebras is then granted by (6.4).

It is to be expected that the arguments of [48] showing that the C^*-algebraic Weyl-algebra isomorphisms (6.1) (when appropriately localized, see above) extend to von Neumann algebraic isomorphisms for the case of the scalar Klein–Gordon field and have generalizations allowing to conclude that the C^*-algebraic CAR-algebra isomorphisms (6.3) extend, in a similar manner, to von Neumann algebraic isomorphisms, so that general covariance is fulfilled. Another provision is that, as in the case of the Proca field, the existence of quasifree Hadamard states for the Dirac field has as yet not been demonstrated. However, the same comment as given above for the case of the Proca field applies here. Anticipating therefore that these provisions are lifted, the just constructed family $\{\mathbf{\Phi}_{\mathbf{M}}\}_{\mathbf{M}\in\mathcal{G}}$ of Dirac quantum fields for each $\mathbf{M} \in \mathcal{G}$ yields another example of a generally covariant quantum field theory over $\mathcal{G}$ upon choosing $\omega_{\mathbf{M}_0}$ as the vacuum state (being quasifree and Hadamard) on Minkowski spacetime $\mathbf{M}_0$.
(See also the "Note added in proof" at the end of the article.)

Appendix A

Proof of Lemma 2.1. Let two causally separated points p_1 and p_2 be given; hence we may form the manifold $M^\vee = M\backslash(J^+(p_1) \cup J^+(p_2))$. Then $(M^\vee, g \restriction M^\vee)$ is again a globally hyperbolic spacetime. This globally hyperbolic spacetime may be smoothly foliated into Cauchy-surfaces and thus one can move Cauchy-surfaces for $(M^\vee, g \restriction M^\vee)$ arbitrarily close to p_1 and p_2. We will use this property in order to construct a Cauchy-surface Σ in (M, g) having the following properties:

(i) $\Sigma \subset M^\vee$.
(ii) There is an open, simply connected neighbourhood $W \subset \Sigma$ which is contained in a coordinate chart (for Σ), and it holds that $J^-(p_j) \cap \Sigma \subset W$ $(j = 1, 2)$.

To this end, let $F : \mathbb{R} \times \Sigma_0 \to M$ be a C^∞-foliation of (M, g) in Cauchy-surfaces. If C is any Cauchy-surface in (M, g), then there is a diffeomorphism $\Psi_C : \Sigma_0 \to C$ which is defined by assigning to $x \in \Sigma_0$ the point $q_x \in C$ so that $F(t_x, x) = q_x$ for some (uniquely determined) $t_x \in \mathbb{R}$. Now let $(t_j, x_j) \in \mathbb{R} \times \Sigma_0$ be such that $F(t_j, x_j) = p_j$, $j = 1, 2$. Then there is clearly a pair S_1, S_2 of open neighbourhoods of x_1, x_2, respectively, in Σ_0 lying in a simply connected chart domain W_0 (of Σ_0), cf. [12], Prop. 16.26.9. Thus, whenever C is a Cauchy-surface in (M, g), then the sets $\Psi_C(S_1)$ and $\Psi_C(S_2)$ are contained in the simply connected chart domain $\Psi_C(W_0)$ of C. On the other hand, $\Psi_C(S_j)$ is the intersection of C with the "tube" $T_j = \bigcup\{F(t, x) : t \in \mathbb{R},\ x \in S_j\}$. It is now fairly easy to see that, if B_j denotes the unit ball in $T_{p_j}M$ with respect to arbitrarily given coordinates, then the sets $V_j(\tau) = \{\exp_{p_j}(v) : v \in \tau \cdot B_j,\ v \text{ past-directed and causal}\}$ of segments of "causal rays" emanating to the past from p_j will be contained in T_j if $\tau > 0$ is small enough. Choosing such a τ and using that $(M^\vee, g \restriction M^\vee)$ is globally hyperbolic, one can thus find a Cauchy-surface Σ in $(M^\vee, g \restriction M^\vee)$ with $(V_j(\tau)\backslash V_j(\tau/2)) \subset \text{int}\, J^-(\Sigma)$; this implies that the intersection of $J^-(p_j)$ with Σ is contained in $T_j \cap \Sigma = \Psi_\Sigma(S_j)$, and since Σ is also a Cauchy-surface for (M, g), one realizes that it has the desired properties (i) and (ii) upon choosing $W = \Psi_\Sigma(W_0)$.

In a next step we note that, since the sets $J^-(p_j) \cap \Sigma$ are closed and contained in the open set W, also the closures of sufficiently small open neighbourhoods of these sets are contained in W. Thus we can choose two sufficiently small sets $U_j = \text{int}(J^-(p_1^+) \cap J^+(p_j^-))$, where $p_j^\pm \in \text{int}\, J^\pm(p_j)$, i.e. they are "double cones" surrounding the points p_j, with $\overline{J^-(U_j)} \cap \Sigma \subset W$. [Note that in Fig. 2.1 we have represented the sets U_j as truncated double cones since this turned out to be easier graphically.] Obviously one may choose the U_j so that they are contained in $N_+ = \text{int}\, J^+(\Sigma)$. Moreover, $J^-(U_j) \cap \Sigma$ will be contained in an open, simply connected subset W_1 of Σ with $\overline{W_1} \subset W$. Then int $D^+(W_1)$ is a simply connected neighbourhood of $\overline{U_1}$ and $\overline{U_2}$, and is globally hyperbolic when endowed with the metric g. Since $(N_+, g \restriction N_+)$ is a globally hyperbolic spacetime, one can choose a Cauchy-surface Σ_+ in $(N_+, g \restriction N_+)$ "sufficiently close to Σ" so that the set

$$G = \text{int}\, D^+(W_1) \cap \text{int}\, J^+(\Sigma_+) \subset N_+$$

is still an open, simply connected neighbourhood of $\overline{U_1}$ and $\overline{U_2}$ which is globally hyperbolic when supplied with g as metric.

The remaining part of the argument proceeds in a similar way as the proof of Appendix C in [22]. We can cover Σ with a system $\{X_\alpha\}_\alpha$ of coordinate patches, choosing one of them, say X_1, to have the property

$$\overline{W}_1 \subset X_1, \quad \overline{X}_1 \subset W. \tag{A.1}$$

Using Gaussian normal coordinates for Σ, one may introduce coordinate patches $(-\varepsilon_\alpha, \varepsilon_\alpha) \times X_a$ covering a neighbourhood N_0 of Σ, on each of which the metric g assumes the form

$$dt^2 - g_{ij}(t, x)dx^i dx^j,$$

where $t \in (-\varepsilon_\alpha, \varepsilon_\alpha)$ and $x = (x^i)_{i=1}^3$ are coordinates on X_α; $(g_{ij}(t, x))$ are the coordinates of the 3-dim. Riemannian metric induced by the metric g on the slices of constant t. Here, the coordinatization is assumed to be such that (t, x) represents a point in N_+ for $t > 0$ and a point in $N_- = \text{int}\, J^-(\Sigma)$ for $t < 0$. Moreover, N_0 may be chosen so that it is, with $g \restriction N_0$ as metric, a globally hyperbolic sub-spacetime of (M, g), and assuming now that N_0 has been chosen in that way, also $N_0 \cap N_-$ is a globally hyperbolic sub-spacetime with the appropriate restriction of g as metric. After a moment of reflection one can see that this implies the existence of a Cauchy-surface Σ_1 in $N_0 \cap N_-$ so that

$$J^-(\overline{W}_1) \cap J^+(\Sigma_1) \subset (-\varepsilon_1, 0) \times X_1$$

by "moving Σ_1 sufficiently close to Σ". Upon moving Σ_1, if necessary, "still closer" to Σ, it is also possible to ensure that the parts of $J^-(\overline{U}_1)$ and $J^-(\overline{U}_2)$ lying in $J^+(\Sigma_1)$ are causally separated. With Σ_1 chosen in that manner, one can now pick some pair of small neighbourhoods $\widetilde{U}_j$ lying relatively compact in $\text{int}(J^+(\Sigma_1) \cap J^-(U_j))$ $(j = 1, 2)$. We may then also select another Cauchy-surface Σ_2 in $N_0 \cap N_1$, with

$$\text{cl}\, \widetilde{U}_j \subset \text{int}\, J^-(\Sigma_2), \quad \Sigma_2 \subset \text{int}\, J^+(\Sigma_1).$$

In the next step, we endow Σ with a complete Riemannian metric γ, which we prescribe to be a flat Euclidean metric on X_1 (which is possible because of (A.1) in view of the fact that W is a coordinate patch). We shall, furthermore, choose γ so that the flat Lorentzian metric η on $(-\varepsilon_1, 0) \times X_1$ given by

$$\eta = dt^2 - \gamma_{ij} dx^i dx^j$$

has for $(t, x) \in (-\varepsilon_1, 0) \times \mathcal{X}_1$ the property that each causal curve for η is also a causal curve for g, i.e. $J_\eta(q) \subset J_g(q)$ on $(-\varepsilon_1, 0) \times X_1$. This may always be realized by rescaling γ by a constant factor.

Now define $\widetilde{M} = \text{int}\, J^+(\Sigma_1)$. Let $f \in C^\infty(\widetilde{M}, \mathbb{R}_+)$ have the following properties: $0 \leq f \leq 1$, $f \equiv 0$ on $J^+(\Sigma)$, $f \equiv 1$ on $J^-(\Sigma_2)$. Then define a metric $\widetilde{g}$ on $N_0 \cap \widetilde{M}$ by setting its coordinate expression to be equal to

$$b(t,x)dt^2 - \big(f(t,x)\gamma_{ij} + (1 - f(t,x))g_{ij}(t,x)\big)\, dx^i dx^j$$

on each coordinate patch $(-\varepsilon_\alpha, \varepsilon_\alpha) \times X_\alpha$. Here, b is a smooth function on $N_0 \cap \widetilde{M}$ with $0 < b \leq 1$ and sufficiently small so that, with the new metric $\widetilde{g}$, N_0 is globally hyperbolic; from the properties of γ mentioned before it is obvious that one can choose such a b so that $b \equiv 1$ on N_+ and $b \equiv 1$ on the set

$$Y = \text{int}\left(\widetilde{M} \cap J^-(\Sigma_2) \cap (-\varepsilon_1, 0) \times X_1\right).$$

With this choice of b, it is moreover clear that $\widetilde{g}$ coincides on N_+ with the metric g, and so $\widetilde{g}$ may be extended from $N_0 \cap \widetilde{M}$ to all of $\widetilde{M}$ by defining $\widetilde{g}$ as g on N_+. Moreover, $\widetilde{g}$ is a flat Lorentzian metric on Y, and viewing U_j, $j = 1, 2$, canonically as subsets of $\widetilde{M}$, the previous constructions entail that there are two globally hyperbolic sub-spacetimes $\widehat{U}$ (with metric $\widetilde{g}$) which are relatively compact in Y, and have the property that $\widehat{U}_j \rhd U_j$ with respect to the metric $\widetilde{g}$.

Finally, one can make Y slightly smaller in order to obtain a globally hyperbolic sub-spacetime $\widehat{G}$ of $(\widetilde{M}, \widetilde{g})$ which is simply connected and still contains $\widetilde{U}_j$ and $\widehat{U}_j$ (if necessary, by making the $\widehat{U}_j$ slightly smaller as well); and $\widetilde{g}$ is flat on $\widehat{G}$. Therefore we have now constructed the required $(\widetilde{M}, \widetilde{g})$ and the subsets U_j, $\widetilde{U}_j$, $\widehat{U}_j$ $(j = 1, 2)$ and G, $\widehat{G}$ with the properties claimed in Lemma 2.1. □

Appendix B

In this appendix we collect the assumptions about a quantum field theory $\mathbf{\Phi}_{\mathbf{M}_0}$ on Minkowski spacetime equipped with its standard spin structure, and quote the spin-statistics theorem for this setting. The assumptions are those given in the book by Streater and Wightman [44], except that in formulating the Bose–Fermi alternative (normal commutation relations), we will posit that Bosonic commutation relations hold in the strong sense, similarly as in the statement of Thm. 4.1. See below for details.

To begin with, write $(M_0, \eta) = (\mathbb{R}^4, \text{diag}(+, -, -, -))$ for Minkowski spacetime. A Lorentzian coordinate frame $(e_0, \ldots, e_3)$ has been chosen by which M_0 is identified with $\mathbb{R}^4$, and which also serves to fix orientation and time-orientation. The framebundle $F(M_0, \eta)$ is isomorphic to $\mathbb{R}^4 \times \mathcal{L}_+^\uparrow$, and for each $x \in \mathbb{R}^4$, $(x, (e_0, \ldots, e_3))$ represents an element in $F(M_0, \eta)$. Then the spin-bundle $S(M_0, \eta)$ is isomorphic to $\mathbb{R}^4 \times \text{SL}(2, \mathbb{C})$, and one obtains a spin-structure $\psi_0 : S(M_0, \eta) \to F(M_0, \eta)$ by assigning to $(x, \mathbf{s}) \in S(M_0, \eta)$ the element $\psi_0(x, \mathbf{s}) = (x, (e_0(\mathbf{s}), \ldots, e_3(\mathbf{s})))$ in $F(M_0, \eta)$ with

$$e_b(\mathbf{s}) = e_a \Lambda^a{}_b(\mathbf{s}),$$

where $SL(2,\mathbb{C}) \ni \mathbf{s} \mapsto \Lambda(\mathbf{s}) \in \mathcal{L}_+^\uparrow$ is the covering projection. Explicitly, the matrix components of $\Lambda(\mathbf{s})$ are given by

$$\Lambda_{ab}(\mathbf{s}) = \frac{1}{2}\mathrm{Tr}(\mathbf{s}^*\sigma_a \mathbf{s}\sigma_b),$$

where $\sigma_0, \dots, \sigma_3$ are the Pauli-matrices.

Now let ρ denote any of the complex linear irreducible representations $D^{(k,l)}$, or of the real linear irreducible representations $D^{(k,l)} \oplus D^{(l,k)}$ $(k \neq l)$, where $k, l \in \mathbb{N}_0$. The corresponding representation space will be denoted by V_ρ. Then we require that the quantum field theory $\mathbf{\Phi}_{\mathbf{M}_0} = (\Phi_{\mathbf{M}_0}, \mathcal{D}_{\mathbf{M}_0}, \mathcal{H}_{\mathbf{M}_0})$ has the following properties (where in the following, we abbreviate $(\Phi_{\mathbf{M}_0}, \mathcal{D}_{\mathbf{M}_0}, \mathcal{H}_{\mathbf{M}_0})$ by $(\Phi_0, \mathcal{D}_0, \mathcal{H}_0)$):

(1) $\mathcal{H}_0$ is a Hilbertspace and $\mathcal{D}_0 \subset \mathcal{H}_0$ is a dense linear subspace.
(2) Φ_0 is a linear map taking elements f in $\mathcal{S}(\mathbb{R}^4, V_\rho)$ to closable operators $\Phi_0(f)$ all having the common, dense and invariant domain $\mathcal{D}_0$. Here, $\mathcal{S}(\mathbb{R}^4, V_\rho)$ is the set of Schwartz-functions on $\mathbb{R}^4$ taking values in the finite-dimensional representation space V_ρ.[5]
(3) For each pair of vectors $\chi, \chi' \in \mathcal{D}_0$, the map

$$\mathcal{S}(\mathbb{R}^4, V_\rho) \ni f \mapsto (\chi, \Phi_0(f)\chi')$$

is continuous, hence an element in $\mathcal{S}'(\mathbb{R}^4, V_\rho)$.
(4) There is a strongly continuous representation

$$\widetilde{\mathcal{P}}_+^\uparrow \ni (a, \mathbf{s}) \mapsto U(a, \mathbf{s})$$

of $\widetilde{\mathcal{P}}_+^\uparrow = \mathbb{R}^4 \rtimes SL(2,\mathbb{C})$ (the covering group of the proper orthochronous Poincaré group) by unitary operators on $\mathcal{H}_0$; $\mathcal{D}_0$ is left invariant under the action of the $U(a, \mathbf{s})$.
(5) The spectrum of the translation-subgroup $a \mapsto U(a, 1)$ is contained in the closed forward lightcone $\overline{V}_+$, i.e. the relativistic spectrum condition holds. Moreover, there is an up to a phase unique unit vector $\Omega \in \mathcal{H}_0$, the vacuum vector, fulfilling $U(a, \mathbf{s})\Omega = \Omega$ for all $(a, \mathbf{s}) \in \widetilde{\mathcal{P}}_+^\uparrow$. This vector is assumed to be contained in $\mathcal{D}_0$ and to be cyclic for the algebra generated by the field operators in the sense that $\mathcal{D}_0$ coincides with the vector space spanned by Ω and all vectors of the form $F_1 \cdots F_n\Omega$, $n \in \mathbb{N}$, $F_j \in \{\Phi_0(f_j), \Phi_0(f_j)^*\}$, $f_1, \dots, f_n \in \mathcal{S}(\mathbb{R}^4, V_\rho)$.
(6) The quantum field possesses the covariance property

$$U(a, \mathbf{s})\Phi_0(f)U(a, \mathbf{s})^{-1} = \Phi_0({\rho_a}^\star(\mathbf{s})f),$$

where

$${\rho_a}^\star(\mathbf{s})f(y) = \rho(\mathbf{s})(f(\Lambda(\mathbf{s})^{-1}(y-a)))$$

for all $a \in \mathbb{R}^4$, $\mathbf{s} \in SL(2,\mathbb{C})$, $f \in \mathcal{S}(\mathbb{R}^4, V_\rho)$.
(7) Spacelike clustering holds on the vacuum, i.e. if a is any non-zero spacelike vector, then one has

$$(\Omega, F_1 \cdots F_k U(ta, 1) F_{k+1} \cdots F_n\Omega) \underset{t\to\infty}{\longrightarrow} (\Omega, F_1 \cdots F_k\Omega)(\Omega, F_{k+1} \cdots F_n\Omega)$$

for all $F_j \in \{\Phi_0(f_j), \Phi_0(f_j)^*\}$, with $f_1, \dots, f_n \in \mathcal{S}(\mathbb{R}^4, V_\rho)$, $n \in \mathbb{N}$.

[5] In the case of flat Minkowski-spacetime, $S(M_0, \eta) = \mathbb{R}^4 \times SL(2,\mathbb{C})$ and one can canonically identify $\mathcal{V}_\rho$ with $\mathbb{R}^4 \times V_\rho$ and $\check{\rho}$ with $\mathrm{id} \times \rho$.

(8) Finally, the Bose–Fermi alternative is required to hold in the following form. The quantum field fulfills either
Bosonic commutation relations. Given any pair of causally separated subsets $O_1, O_2 \in \mathrm{orc}(\mathbb{R}^4)$, then it holds that

$$\mathcal{F}_0(O_1) \subset \mathcal{F}_0(O_2)',$$

or
Fermionic commutation relations. Given any pair of $f_1, f_2 \in \mathcal{S}(\mathbb{R}^4, V_\rho)$ with spacelike separated supports, then it holds that

$$\Phi_0(f_1)\Phi_0(f_2) + \Phi_0(f_2)\Phi_0(f_1) = 0 \text{ and } \Phi_0(f_1)\Phi_0(f_2)^* + \Phi_0(f_2)^*\Phi_0(f_1) = 0.$$

In formulating the statement of Bosonic commutation relations (or *locality*, as it is also called), $\mathcal{F}_0(O)$ denotes the von Neumann algebra generated via the polar decomposition of the closed field operators $\overline{\Phi_0(f)}$ with supp $f \subset O$ as described in assumption (a) of Sect. 4. The above statement of Bosonic commutation relations is thus equivalent to saying that the field operators $\Phi_0(f_1)$ and $\Phi_0(f_2)$ commute strongly for spacelike separated supports of f_1 and f_2; here we say that a pair of closable operators X_j ($j = 1, 2$) commutes strongly if J_1 and $e^{is|X_1|}$ commute with J_2 and $e^{it|X_2|}$, $s, t \in \mathbb{R}$, where $\overline{X_j} = J_j|X_j|$ denotes polar decomposition. Clearly, the property of field operators to commute strongly at spacelike separation implies their spacelike commutativity in the ordinary sense,

$$\Phi_0(f_1)\Phi_0(f_2) - \Phi_0(f_2)\Phi_0(f_1) = 0 \text{ and } \Phi_0(f_1)\Phi_0(f_2)^* - \Phi_0(f_2)^*\Phi_0(f_1) = 0$$

whenever the supports of f_1 and f_2 are spacelike separated, but without further information one can in general not conclude that this last relation also implies spacelike commutativity of the field operators in the strong sense as usually the field operators will be unbounded. The question as to when this conclusion may nevertheless be drawn for field operators in quantum field theory is a longstanding one; however, several criteria are known. We refer the reader to [7, 18] for further discussion and references. Suffice it to say here that ordinary spacelike commutativity is expected to imply strong spacelike commutativity of field operators in the case of physically relevant theories.

We also mention that in Def. 4.1 the quantum field $\Phi_0 = \Phi_{\mathbf{M}_0}$ has only been assumed to be an operator-valued distribution defined on test-spinors of compact support, which would correspond to elements in $\mathcal{D}(\mathbb{R}^4, V_\rho)$. Thus, we assume here that Φ_0 can be extended to an operator-valued distribution on $\mathcal{S}(\mathbb{R}^4, V_\rho)$ with the above stated properties.

Now we quote the spin-statistics theorem for a quantum field theory on Minkowski spacetime which is proved in [44] for complex linear irreducible ρ and in [33] for real linear irreducible ρ. (In fact, the results in [44, 33] are slightly more general since Bosonic commutation relations are only required in the ordinary sense there.)

Theorem 2.1. *Suppose that $\boldsymbol{\Phi}_{\mathbf{M}_0}$ is a quantum field theory on Minkowski spacetime fulfilling the above listed Conditions (1)–(8). Then the following two cases imply that $\Phi_0(f) = 0$, $f \in \mathcal{S}(\mathbb{R}^4, V_\rho)$, and hence that $\mathcal{F}_0(O) = \mathbb{C} \cdot 1$ holds for all bounded open regions O in Minkowski spacetime:*

(α) Bosonic commutation relations hold and the field is of half-integer spin type ($k + l$ is odd).
(β) Fermionic commutation relations hold and the field is of integer-spin type ($k + l$ is even).

Appendix C

In this appendix we will explain how a generally covariant quantum field theory over $\mathcal{G}$ may be viewed as a covariant functor between the category $\mathcal{G}$ and a category $\mathcal{N}$ of nets of von Neumann algebras over manifolds (more generally, one could consider $\mathcal{N}$ as the category of isotonous families of Neumann algebras indexed by directed index sets, but we don't need that generality here). A similar functorial description has been given by Dimock [14] for the case that the morphisms of $\mathcal{G}$ are global isomorphisms, and that $\mathcal{N}$ is a category of C^*-algebraic nets. Here, we take the morphisms of $\mathcal{G}$ to be the local isomorphisms, and correspondingly we have to consider local morphisms for $\mathcal{N}$.

We now consider $\mathcal{G}$ as a category whose objects are the four-dimensional, globally hyperbolic spacetimes with a spin-structure. Given $\mathbf{M}_1$ and $\mathbf{M}_2$ in $\mathcal{G}$, we define the set of morphisms $\hom(\mathbf{M}_1, \mathbf{M}_2)$ to consist of the local isomorphisms between $\mathbf{M}_1$ and $\mathbf{M}_2$. We also add to $\hom(\mathbf{M}_1, \mathbf{M}_2)$ a trivial morphism $\mathbf{0}$. (In fact, $\mathbf{0}$ should be indexed by $\mathbf{M}_1$ and $\mathbf{M}_2$, but that is inconvenient and will be skipped as there is no danger of confusion.) The composition of two morphisms $\mathbf{\Theta}_a \in \hom(\mathbf{M}_1, \mathbf{M}_2)$ and $\mathbf{\Theta}_b \in \hom(\mathbf{M}_2, \mathbf{M}_3)$ will be defined according to the following rules: If $\mathbf{\Theta}_a = \mathbf{0}$ or $\mathbf{\Theta}_b = \mathbf{0}$, then $\mathbf{\Theta}_b\mathbf{\Theta}_a = \mathbf{0}$. If both $\mathbf{\Theta}_a$ and $\mathbf{\Theta}_b$ are non-trivial, but $\ell_{\rm ini}(\mathbf{\Theta}_b) \cap \ell_{\rm fin}(\mathbf{\Theta}_a) = \emptyset$, then also $\mathbf{\Theta}_b\mathbf{\Theta}_a = \mathbf{0}$. Otherwise, we declare $\mathbf{\Theta}_b\mathbf{\Theta}_a$ to be the local isomorphism between $\mathbf{M}_1$ and $\mathbf{M}_3$ obtained by composing the bundle maps and isometries on their natural domains, so that $\ell_{\rm ini}(\mathbf{\Theta}_b\mathbf{\Theta}_a) = \vartheta_a^{-1}(\ell_{\rm ini}(\mathbf{\Theta}_b) \cap \ell_{\rm fin}(\mathbf{\Theta}_a))$. This is reasonable because it is not difficult to show that the intersection of two globally hyperbolic submanifolds of a globally hyperbolic spacetime yields again a globally hyperbolic submanifold. The identical bundle map gives the unit element in $\hom(\mathbf{M}, \mathbf{M})$, and one can straightforwardly check that also the associativity of morphisms is fulfilled.

The objects of the category $\mathcal{N}$ are families $\mathcal{F} = \{\mathcal{F}(O)\}_{O \in {\rm orc}(X)}$ of von Neumann algebras which are indexed by the open, relatively compact subsets of a manifold X and which are subject to the condition of isotony (cf. Sect. 4, item (a)). The morphisms in $\hom(\mathcal{F}_1, \mathcal{F}_2)$ are local net-isomorphisms. A local net isomorphism is a pair $(\{\alpha_{N_i}\}, \phi)$ with the following properties: $\phi : X_1 \supset N_1 \to N_2 \subset X_2$ is a diffeomorphism between open subsets of the manifolds X_1 and X_2 which relate to the indexing sets of $\mathcal{F}_1$ and $\mathcal{F}_2$ in the obvious manner. $\{\alpha_{N_i}\}_{N_i \in {\rm orc}(N_1)}$ is a family of von Neumann algebraic isomorphisms $\alpha_{N_i} : \mathcal{F}_1(N_i) \to \mathcal{F}_2(N_f)$ with $N_f = \phi(N_i)$ obeying the covariance property

$$\alpha_{N_i}(\mathcal{F}_1(O)) = \mathcal{F}_2(\phi(O)), \quad O \in {\rm orc}(N_i).$$

As before, we add to the local net-isomorphisms in $\hom(\mathcal{F}_1, \mathcal{F}_2)$ a trivial morphism $\mathbf{0}$ (which may here be concretely thought of as the map which sends each algebra element in the net $\mathcal{F}_1$ to the algebraic zero element in the net $\mathcal{F}_2$). The composition rule for morphisms is then analogous as before, we only have to specify the case of two net-isomorphisms $(\alpha_{N_i}, \phi) \in \hom(\mathcal{F}_1, \mathcal{F}_2)$ and $(\beta_{N_i'}, \phi') \in \hom(\mathcal{F}_2, \mathcal{F}_3)$ when $\ell_{\rm ini}(\phi') \cap \ell_{\rm fin}(\phi) \neq \emptyset$. In this situation, we define the composition of the two morphisms as the element (γ_{N_i}, ψ) in $\hom(\mathcal{F}_1, \mathcal{F}_3)$, where ψ is $\phi' \circ \phi$ restricted to $\phi^{-1}(\ell_{\rm ini}(\phi') \cap \ell_{\rm fin}(\phi))$, and for any open, relatively compact subset N_i in $\ell_{\rm ini}(\psi)$ we define

$$\gamma_{N_i} = \beta_{\phi(N_i)} \circ \alpha_{N_i}.$$

Again, each $\hom(\mathcal{F}, \mathcal{F})$ contains the identical map as an identity, and one may check the associativity of the composition rule.

Then the covariance structure (Condition (C) of Def. 4.1) of a generally covariant quantum field theory is that of a covariant functor $\mathsf{F} : \mathcal{G} \to \mathcal{N}$ which assigns to each object $\mathbf{M} \in \mathcal{G}$ an object $\mathsf{F}(\mathbf{M}) = \{\mathcal{F}(O)\}_{O \in \mathrm{orc}(M)}$ in $\mathcal{N}$, and which assigns to each (nontrivial) morphism $\mathbf{\Theta} = (\Theta, \vartheta)$ of $\mathcal{G}$ a morphism $\mathsf{F}(\mathbf{\Theta}) = (\alpha_{\mathbf{\Theta},N_i}, \vartheta)$ of $\mathcal{N}$. Moreover, F maps trivial morphisms to trivial morphisms. Diagrammatically, one has

$$\begin{array}{ccc} \mathbf{M}_1 & \xrightarrow{\ \mathsf{F}\ } & \{\mathcal{F}_1(O)\}_{O \in \mathrm{orc}(M_1)} \\ {\scriptstyle \mathbf{\Theta}}\big\downarrow & & \big\downarrow {\scriptstyle (\{\alpha_{\mathbf{\Theta},N_i}\}, \vartheta)} \\ \mathbf{M}_2 & \xrightarrow{\ \mathsf{F}\ } & \{\mathcal{F}_2(U)\}_{U \in \mathrm{orc}(M_2)} \end{array}$$

Note added in proof.

- A more general and concise functorial decription of the principle of general covariance will appear in [53].
- The required properties concerning Hadamard states mentioned at the end of Sect. 6 have recently been discussed in a preprint by D'Antoni and Hollands [54].

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