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Smoothing Property for Schrödinger Equations with Potential Superquadratic at Infinity

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Dedicated to Jean-Michel Combes on the occasion of his Sixtieth Birthday

Abstract: We prove a smoothing property for one dimensional time dependent Schrödinger equations with potentials which satisfy $V(x) \sim C|x|^k$ at infinity, $k > 2$. As an application, we show that the initial value problem for certain nonlinear Schrödinger equations with such potentials is L^2 well-posed. We also prove a sharp asymptotic estimate of the L^p-norm of the normalized eigenfunctions of $H = -\Delta + V$ for large energy.

1. Introduction

We consider the initial value problem for a Schrödinger equation on the line $\mathbb{R}$:

$$\begin{cases} i\dfrac{\partial u}{\partial t} = (D^2 + V(x))u, & x \in \mathbb{R}^1,\ t \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^1, \end{cases} \tag{1.1}$$

where $D = -i\partial/\partial x$. We assume that $V(x)$ satisfies the following assumption. Let $V^{(j)}(x)$ be the j^{th} derivative of $V(x)$ and $\langle A\rangle = (1+|A|^2)^{\frac{1}{2}}$ for a self-adjoint operator A.

Assumption 1.1. *The potential $V(x)$ is real valued and of C^3-class. There exists a constant $R > 0$ such that the following conditions are satisfied for $|x| \geq R$:*

(1) *$V(x)$ is convex.*
(2) *For $j = 1, 2, 3$, $|V^{(j)}(x)| \leq C_j\langle x\rangle^{-1}|V^{(j-1)}(x)|$ for some constants C_j.*
(3) *For $k > 2$, $D_1\langle x\rangle^k \leq V(x) \leq D_2\langle x\rangle^k$, where $0 < D_1 \leq D_2 < \infty$.*

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We say that V is superquadratic (at infinity) if it satisfies (1), (2) *and* (3).

Under Assumption 1.1 the operator $D^2 + V(x)$ defined on $C_0^\infty(\mathbb{R})$ is essentially self-adjoint in $L^2(\mathbb{R})$ and we denote its closure by H. Thus, H is self-adjoint with the domain $D(H) = \{u \in L^2(\mathbb{R}) : D^2u + Vu \in L^2(\mathbb{R})\}$ and the solution in $L^2(\mathbb{R})$ of (1.1) is given by $u(t,x) = e^{-itH}u_0(x)$ in terms of the exponential function of H. In this paper, we prove a smoothing property for Eq. (1.1). We then apply it to prove that the initial value problem for nonlinear Schrödinger equations with superquadratic potentials is time locally L^2 well-posed, if the nonlinearities are suffciently mild and spatially localized. We define $\theta(k,p)$ as follows, for $2 \le p \le \infty$ and $2 < k < \infty$:

$$\theta(k,p) = \begin{cases} \dfrac{1}{k}\left(\dfrac{1}{2} - \dfrac{1}{p}\right), & \text{if} \quad 2 \le p < 4; \\ \left(\dfrac{1}{4k}\right)_-, & \text{if} \quad p = 4; \\ \dfrac{1}{4} - \dfrac{1}{3}\left(1 - \dfrac{1}{p}\right)\left(1 - \dfrac{1}{k}\right), & \text{if} \quad 4 < p \le \infty, \end{cases}$$

where a_- denotes any number $< a$. We write $B^s(\mathbb{R})$ for the Besov space $B^s_{2,1}(\mathbb{R})$.

Theorem 1.2. *Let V satisfy* Assumption 1.1. *Let $2 \le p \le \infty$ and let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha + \beta \le \theta(k,p)$. Then, there exists a constant $C > 0$ such that*

$$\|g(t)\langle i\partial/\partial t\rangle^{\alpha}\langle H\rangle^{\beta}e^{-itH}u_0(x)\|_{L^p(\mathbb{R}_x, L^2(\mathbb{R}_t))} \le C\|g\|_{B^{\frac{1}{4}+\frac{1}{2k}}(\mathbb{R})}\|u_0\|_{L^2(\mathbb{R}_x)}, \qquad (1.2)$$

for any $g \in B^{\frac{1}{4}+\frac{1}{2k}}(\mathbb{R})$ and $u_0 \in L^2(\mathbb{R})$.

The next theorem shows that the order $\theta(k,p)$ of Theorem 1.2 may be replaced by $\dfrac{1}{2k}$ for all $2 \le p \le \infty$ if the spatial variable x is restricted to a compact interval of $\mathbb{R}$.

Theorem 1.3. *Let V satisfy* Assumption 1.1. *Let $K \subset \mathbb{R}$ be compact and let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha + \beta \le \dfrac{1}{2k}$. Then, there exists a constant $C > 0$ such that*

$$\sup_{x \in K}\|g(t)\langle i\partial/\partial t\rangle^{\alpha}\langle H\rangle^{\beta}e^{-itH}u_0(x)\|_{L^2(\mathbb{R}_t)} \le C\|g\|_{B^{\frac{1}{4}+\frac{1}{2k}}(\mathbb{R}_t)}\|u_0\|_{L^2(\mathbb{R}_x)} \qquad (1.3)$$

for any $g \in B^{\frac{1}{4}+\frac{1}{2k}}(\mathbb{R})$ and $u_0 \in L^2(\mathbb{R})$.

Note that $\langle i\partial/\partial t\rangle^{\alpha}\langle H\rangle^{\beta}e^{-itH} = \langle i\partial/\partial t\rangle^{\alpha+\beta}e^{-itH} = \langle H\rangle^{\alpha+\beta}e^{-itH}$ in (1.2) and (1.3). The following corollary is readily obtained from Theorem 1.2 and Theorem 1.3 with the help of elliptic estimates and interpolation theory.

Corollary 1.4. *Suppose V satisfies* Assumption 1.1. *Let $2 \le p < \infty$ and $K \subset \mathbb{R}$ be a compact interval. Then, there exists a constant $C > 0$ such that the following estimates are satisfied:*

$$\|\langle D_x\rangle^{2\theta(k,p)}e^{-itH}u_0\|_{L^p(\mathbb{R}_x, L^2((-T,T)_t))} + \|\langle x\rangle^{k\theta(k,p)}e^{-itH}u_0\|_{L^p(\mathbb{R}_x, L^2((-T,T)_t))} \le C\|u_0\|_{L^2(\mathbb{R}_x)}. \qquad (1.4)$$

$$\|\langle D_x\rangle^{1/k}e^{-itH}u_0\|_{L^2((-T,T)_t, L^2(K_x))} \le C\|u_0\|_{L^2(\mathbb{R})}. \qquad (1.5)$$

One consequence of (1.5) is that $e^{-itH}u_0(\cdot) \in H^{1/k}_{\mathrm{loc}}(\mathbb{R})$, a.e. t for $u_0 \in L^2(\mathbb{R})$ and the solution $u(t,x)$ of (1.1) is smoother than the initial function u_0 by the order $1/k$ for almost all t. This is a manifestation of the smoothing property of Eq. (1.1). We remark that we may obtain a series of estimates of the form $\|g(t)e^{-itH}u_0(x)\|_{L^p(\mathbb{R}_x, L^q(\mathbb{R}_t))} \le C\|u_0\|_2$ from (1.2) with the help of the Sobolev embedding theorem and elliptic estimates. In this case we always need $q < p$.

Since Kato's remarkable discovery ([K1] and [K2]), the smoothing property of linear and nonlinear dispersive equations has been intensively studied by many authors in conjunction with applications mainly to the convergence problem and to the initial value problem for nonlinear equations. There is a large number of references, e.g. [St, P, Br, GV1, Y1, V, CS, KY, Sj, KPV, BAD, GV2, BT, HK, Su, H]. Most of these papers are concerned with equations with coefficients which are either constant or asymptotically constant at spatial infinity.

For Schrödinger equations, the smoothing property has been extended to the case when potentials increase at most quadratically at infinity ([K3,Y2]) viz. $|D^\beta V(x)| \le C_\beta$ for $|\beta| \ge 2$, and the following estimates:

$$\|e^{-itH}u_0\|_{L^\theta((-T,T)_t, L^p(\mathbb{R}^n_x))} \le C\|u_0\|_{L^2(\mathbb{R}^n_x)}, \tag{1.6}$$

$$\|\Phi(x)(1-\Delta)^{\alpha/2}e^{-itH}u(x)\|_{L^\theta((-T,T)_t, L^p(\mathbb{R}^n_x))} \le C\|u\|_{L^2(\mathbb{R}^n_x)} \tag{1.7}$$

have long been known ([Y2]) (see also [Y3] for Schrödinger equations with magnetic potentials which increase at most linearly at infinity). Here $T > 0$ is any finite number, $\Phi \in C_0^\infty(\mathbb{R}^n)$ and $p, \theta \ge 2$ and $\alpha \ge 0$ are such that $0 \le \dfrac{2}{\theta} = 2\alpha + n\left(\dfrac{1}{2} - \dfrac{1}{p}\right) < 1$ and $p < \infty$. Estimates of the type (1.6) are called the L^p smoothing property as they imply that $u(t,\cdot)$ is smoother than $u_0 \in L^2(\mathbb{R}^n)$ for almost all t in the sense $u(t,\cdot)$ belongs to $L^p(\mathbb{R}_x)$ for $p > 2$. Estimates of the type (1.7) are called the differentiability improving property by obvious reason. Note that (1.7) with $p = \theta = 2$ and $\alpha = 1/2$ is equivalent to (1.5) with $k = 2$.

When potentials are superquadratic at infinity, however, no estimates of this kind can be found in the literature to the best of the authors' knowledge. This situation may be related to the fact that the smoothness and boundedness properties of the distribution kernel $E(t,x,y)$ of e^{-itH}, the fundamental solution or FDS for short, has a sharp transition when the growth rate at infinity of the potential passes that of $C|x|^2$ ([Y4]): $E(t,x,y)$ is smooth and spatially bounded for all $t \ne 0$ if $V(x) = o(|x|^2)$. If $V(x) = O(\langle x\rangle^2)$ these results hold for small $|t| > 0$. However, if $V(x) \ge C\langle x\rangle^{2+\varepsilon}$, $\varepsilon > 0$, $E(t,x,y)$ is nowhere C^1 and can be unbounded at spatial infinity ([MY]). Recall that (1.6) is a consequence of the bound $|E(t,x,y)| \le C|t|^{-n/2}$ for small $|t|$, and (1.7) of the fact that $\displaystyle\int_{-T}^{T} \Phi^2(x(t))dt$ is a peudo-differential operator of order -1, where $x(t) = e^{itH}xe^{-itH}$ is the Heisenberg position operator. These two properties hold for potentials with $|V(x)| \le C|x|^2$ but not for superquadratic potentials. One of our motivations to this work was to examine whether or not this transition is inherited by the smoothing property of Eq. (1.1).

Recall that $E(t,x,y)$ under Assumption 1.1 satisfies, for arbitrary $\rho \in C_0^\infty(\mathbb{R}^3)$,

$$|\widehat{\rho E}(\tau,\xi,\eta)| \le C(|\tau| + |\xi|^2 + |\eta|^2)^{-1/k}, \tag{1.8}$$

where $\hat{}$ stands for the Fourier transform ([Y4], Remark 1.2). We should remember here a celebrated theorem of Zygmund ([Z], see also [B]) that $\|e^{-itH}u_0\|_{L^4(\mathbb{T}\times\mathbb{T})} \le$

$C\|u_0\|_{L^2(\mathbb{T})}$ for $H = D^2$ on the torus $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. Notice that the FDS for this H, $E_0(t,x,y) = \sum_{n=-\infty}^{\infty} e^{-in^2t+in(x-y)}$ is nowhere locally integrable with respect to (t,x,y) and functions which satisfy (1.8) are smoother than E_0. This indicates, therefore, that Schrödinger equations with superquadratic potentials satisfy a certain smoothing property. Our result shows that this is indeed the case and, moreover, such a transition as in the smoothness of $E(t,x,y)$ *does not* appear in the smoothing property. Note that estimates (1.2) and (1.3) differ from (1.6) or (1.7) by the change of the order of integrations by x and t, in particular. Nonetheless, we continue to refer to such estimates as (1.2) and (1.3) as the smoothing property.

We mention that the estimate of the form (1.2) appears already in [K1] in a slightly disguised form: For $M \in L^{n+\varepsilon}(\mathbb{R}^n) \cap L^{n-\varepsilon}(\mathbb{R}^n)$, $\varepsilon > 0$,

$$\|Me^{it\Delta}u_0\|_{L^2(\mathbb{R}^{n+1}_{t,x})} \le C(\|M\|_{L^{n-\varepsilon}(\mathbb{R}^n)} + \|M\|_{L^{n+\varepsilon}(\mathbb{R}^n)})\|u_0\|_{L^2(\mathbb{R}^n_x)},$$

(see also [KY] where the right side is replaced by $C\|M\|_{L^n(\mathbb{R}^n)}\|u_0\|_{L^2(\mathbb{R}^n_x)}$) and that [KPV] elaborated and applied it to nonlinear Schrödinger equations. We also remark that there is a micro-local version of (1.7) and the following is known: When H is Schrödinger operators on certain Riemannian minifolds, (1.7) holds with $\alpha = 1/2$ for $u_0 \in L^2$ supported by U if all bicharacteristics starting from U are non-trapping for all $t < 0$ ([CKS]) and it *does not* hold if they are trapping ([D1, D2]).

It is well-known that the operator H is bounded from below and its spectrum consists of simple eigenvalues $\lambda_1 < \lambda_2 < \cdots \to \infty$. We denote the corresponding normalized eigenfunctions by $\psi_1, \psi_2 \ldots$. The proof of Theorem 1.2 and Theorem 1.3 heavily depends upon the following theorem on the asymptotic behavior as $\lambda_n \to \infty$ of L^p norm of ψ_n which we think is of interest in its own right. For the quantities A and B, we write $A \sim B$ if there exist two positive constants c_1 and c_2 such that $c_1 A \le B \le c_2 A$.

Theorem 1.5. *Let* Assumption 1.1 *be satisfied. Let $\psi(x,E)$ be the normalized eigenfunction of $H = -\Delta + V(x)$ with the eigenvalue E. Then:*

(1) *For $1 \le p \le \infty$, we have*

$$\|\psi(x,E)\|_{L^p} \sim \begin{cases} C_p E^{-\theta(k,p)}, & \text{if} \quad p \neq 4; \\ C E^{-\frac{1}{4k}}(\log E)^{\frac{1}{4}}, & \text{if} \quad p = 4, \end{cases} \tag{1.9}$$

for large E, where C_p can be taken independent of p, $p \notin (4-\varepsilon, 4+\varepsilon)$, $\varepsilon > 0$.

(2) *For compact interval $K \subset \mathbb{R}$, $\sup_{x\in K} |\psi(x,E)| \sim E^{-\frac{1}{2k}}$ for large E.*

Remark 1.1. If we set $u_0(x) = \psi_n(x)$ in (1.2), we have

$$\|g(t)\langle i\partial/\partial t\rangle^\alpha \langle H\rangle^\beta e^{-itH}u_0(x)\|_{L^p(\mathbb{R}_x, L^2(\mathbb{R}_t))} = \|g\|_{L^2}\langle\lambda_n\rangle^{\theta(k,p)}\|\psi_n\|_{L^p(\mathbb{R})}.$$

Hence, Theorem 1.5 (1) implies that the condition $\alpha + \beta \le \theta(k,p)$ in (1.2) cannot be relaxed. Likewise Theorem 1.5 (2) implies that the exponent $1/2k$ of Theorem 1.3 is sharp. The exponents $\theta(k,p)$ and $1/2k$ are decreasing functions of k and this matches the fact that the FDS is more singular for larger k ([Y4]). With respect to p on the other hand, $\theta(k,p)$ is increasing for $2 \le p < 4$ and decreasing for $4 < p$. The proof of Theorem 1.5 will show that the p dependence of $\theta(k,p)$ is related to the behavior of $\psi_n(x)$ near the turning point S.

As an application of Theorem 1.2 and Theorem 1.3, we show that the initial value problem for nonlinear Schrödinger equations with superquadratic potentials and with spatially localized mild nonlinearities

$$\begin{cases} i\dfrac{\partial u}{\partial t} = -\triangle u + V(x)u + f(x,u), & x \in \mathbb{R}, \quad t \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R} \end{cases} \tag{1.10}$$

is L^2 well-posed. Define, for $r \geq 1$ and $\delta > 0$,

$$\begin{aligned} X &= L^4(\mathbb{R}_x; L^{2r}_{loc}(\mathbb{R}_t)) \cap C(\mathbb{R}_t, L^2(\mathbb{R}_x)), \\ X_\delta &= L^4(\mathbb{R}_x; L^{2r}((-\delta,\delta)_t)) \cap C((-\delta,\delta)_t, L^2(\mathbb{R}_x)); \\ Y &= L^{2r}_{loc}(\mathbb{R}_t \times \mathbb{R}_x) \cap C(\mathbb{R}_t, L^2(\mathbb{R}_x)), \\ Y_\delta &= L^{2r}((-\delta,\delta)_t, L^{2r}_{loc}(\mathbb{R}_x)) \cap C((-\delta,\delta)_t, L^2(\mathbb{R}_x)). \end{aligned}$$

Theorem 1.6. *Let* V *satisfy* Assumption 1.1. *Let* $1 \leq r < \dfrac{2k}{2k-1}$ *and let* $\phi(x) \in L^{\frac{4}{2-r}}(\mathbb{R})$. *Suppose that* $f(x,u)$ *satisfies*

$$|f(x,u)| \leq C|\phi(x)||u|^r, \qquad x \in \mathbb{R}, \quad u \in \mathbb{C}, \tag{1.11}$$

$$|f(x,u) - f(x,v)| \leq C|\phi(x)||u-v|(|u|^{r-1} + |v|^{r-1}), \quad x \in \mathbb{R}, \quad u, v \in \mathbb{C}. \tag{1.12}$$

Then, the problem (1.10) *is locally well-posed in* X *for any* $u_0 \in L^2(\mathbb{R})$, *viz. there exists* $\delta > 0$ *such that* (1.10) *admits a unique solution* $u(t,x)$ *in* X_δ *and* $L^2(\mathbb{R}) \ni u_0 \mapsto u \in X_\delta$ *is continuous. If* f *further satisfies*

$$f(x,u)\overline{u} \text{ is real for } x \in \mathbb{R}, \quad u \in \mathbb{C}, \tag{1.13}$$

then (1.10) *is globally well-posed in* X, *viz. the solution* $u(t,x)$ *uniquely extends to the whole real line* $\mathbb{R}$ *and* $L^2(\mathbb{R}) \ni u_0 \mapsto u \in X_T$ *is continuous for all* $T > 0$.

Theorem 1.7. *Let* V *satisfy* Assumption 1.1. *Let* $1 \leq r \leq \dfrac{k}{k-1}$ *and let* $K \subset \mathbb{R}$ *be a compact interval. Suppose* f *satisfies* $f(x,u) = 0$ *for* $x \notin K$ *and*

$$|f(x,u)| \leq C|u|^r, \qquad x \in K, \quad u \in \mathbb{C}, \tag{1.14}$$

$$|f(x,u) - f(x,v)| \leq C|u-v|(|u|^{r-1} + |v|^{r-1}), \quad x \in K, \quad u, v \in \mathbb{C}. \tag{1.15}$$

Then, (1.10) *is locally well-posed in* Y *for any* $u_0 \in L^2(\mathbb{R})$. *If* f *further satisfies* (1.13), *then* (1.10) *is globally well-posed in* Y.

We outline here the plan of the paper, briefly explaining how Theorem 1.2 may be derived from Theorem 1.5. In Sect. 2, we prove Theorem 1.5 by applying Langer's turning point theory as presented in Titchmarsh's monograph [T1]. Theorem 1.2 will be proved in Sect. 3. We expand $u(t,x) = e^{-itH}u_0$ in terms of the eigenfunctions $\psi_1, \psi_2, \ldots$ of H in the form $u(t,x) = \sum_{n=1}^{\infty} \hat{u}_0(n) e^{-it\lambda_n} \psi_n(x)$, where $\hat{u}_0(n) = (u_0, \psi_n)$ is the n^{th} generalized Fourier coefficient. Then, the Plancherel formula implies

$$\int_{\mathbb{R}} |g(t)u(t,x)|^2 dt = \int_{\mathbb{R}} |\sum_{n=1}^{\infty} \hat{u}_0(n)\psi_n(x)\hat{g}(\lambda - \lambda_n)|^2 d\lambda.$$

If $\hat{g}$ is supported by a sufficiently small interval, then, for any fixed $\lambda \in \mathbb{R}$, there is only one eigenvalue such that $\hat{g}(\lambda - \lambda_n) \neq 0$ because $\lambda_{n+1} - \lambda_n \to \infty$ as $n \to \infty$. Hence the right-hand side becomes $\sum_{n=1}^{\infty} |\hat{u}_0(n)|^2 |\psi_n(x)|^2 \|g\|_{L^2}^2$ and Minkowski's inequality implies

$$\|g(t)e^{-itH}u_0(x)\|_{L^p(\mathbb{R}_x, L^2(\mathbb{R}_t))} \le \|g\|_{L^2} \left(\sum_{n=1}^{\infty} |\hat{u}_0(n)|^2 \|\psi_n(x)\|_{L^p}^2 \right)^{1/2}.$$

The right hand side is bounded by $C\left(\sum_{n=1}^{\infty} |\hat{u}_0(n)|^2 \lambda_n^{-2\theta(k,p)}\right)^{1/2} = \|H^{-\theta(k,p)}u_0\|$ by virtue of Theorem 1.5, then, (1.2) follows for such g. For general g we use the standard "cutting and pasting" by the dyadic decomposition of the unity. Theorem 1.3 is also proved in Sect. 3 using a similar idea. We prove Theorem 1.6 and Theorem 1.7 in Sect. 4 via the standard contraction mapping theorem by applying Theorem 1.2 and Theorem 1.3, respectively.

2. L^p Estimate of Eigenfunctions

In this section we prove Theorem 1.5. We denote by $\psi(x, E)$ the eigenfunction of

$$-\psi''(x) + V(x)\psi(x) = E\psi(x) \tag{2.1}$$

such that $\|\psi(\cdot, E)\|_{L^2(\mathbb{R})} = 1$. We use the following estimates (2.3) and (2.4) due to Titchmarsh ([T1,T2]). For large $E > 0$, we write X for the positive root X of $V(X) = E$. We have $V(x) > E$ for $x > X$ and $V(x) < E$ for $0 \le x < X$. We set

$$\zeta(x) = \int_X^x \sqrt{E - V(t)}dt, \tag{2.2}$$

where the branch of the square root is chosen in such a way that arg $\zeta(x) = \pi/2$ for $x > X$, and arg $\zeta(x) = -\pi$ for $x < X$.

Lemma 2.1. *Let the notation be as above. Then, there exists a constant C_{E+} such that*

$$\psi(x, E) = C_{E+}^{-1}[E - V(x)]^{-\frac{1}{4}} \left\{ (\pi\zeta/2)^{\frac{1}{2}} H_{1/3}^{(1)}(\zeta) + O\left(\frac{E^{-\frac{1}{2}}X^{-1}e^{-\mathrm{Im}\zeta}|\zeta|^{1/6}}{1 + |\zeta|^{1/6}} \right) \right\} \tag{2.3}$$

as $E \to \infty$ uniformly with respect to $x > 0$. We have the estimate

$$C_{E+} \sim (XE^{-\frac{1}{2}})^{\frac{1}{2}}. \tag{2.4}$$

Similar statement holds for $x < 0$.

Outline of the proof. For the readers' convenience, we outline the proof here. It is based upon Langer's turning point theory as presented in Chapter 22.27 of [T2]. We make a change of independent variable $x \to \zeta(x)$ and dependent variable $\psi \to G$ in (2.1), where

$$\psi(x) = [E - V(x)]^{-\frac{1}{4}} G(\zeta). \tag{2.5}$$

We sometimes write $G(x)$ for $G(\zeta(x))$. Then $G(\zeta)$ satisfies

$$\frac{d^2G}{d\zeta^2} + \left(1 + \frac{5}{36\zeta^2}\right) G = f(x)G, \tag{2.6}$$

where $f(x)$ is defined by

$$f(x) = \frac{5}{36\zeta^2} - \frac{V''(x)}{4(E - V(x))^2} - \frac{5V'(x)^2}{16(E - V(x))^3}.$$

We then transform Eq. (2.6) into the integral equation of the form

$$\begin{aligned} G(x) = (\frac{\pi\zeta}{2})^{\frac{1}{2}} H^{(1)}_{1/3}(\zeta) + \frac{\pi i}{2} \int_x^\infty & \left\{ H^{(1)}_{1/3}(\zeta) J_{1/3}(\theta) - J_{1/3}(\zeta) H^{(1)}_{1/3}(\theta) \right\} \times \\ & \times \zeta^{1/2}\theta^{1/2} f(t)(E - V(t))^{1/2} G(t) dt. \end{aligned} \tag{2.7}$$

Here $J_\nu(\zeta)$ and $H^{(j)}_\nu(\zeta)$ are the Bessel and Hankel functions, respectively, and we wrote $\zeta = \zeta(x)$ and $\theta = \zeta(t)$. $(\frac{\pi\zeta}{2})^{\frac{1}{2}} J_{1/3}(\zeta)$ and $(\frac{\pi\zeta}{2})^{\frac{1}{2}} H^{(1)}_{1/3}(\zeta)$ are linearly independent solutions of the associate homogeneous equation

$$\frac{d^2G}{d\zeta^2} + \left(1 + \frac{5}{36\zeta^2}\right) G = 0,$$

and the inhomonegenous term is chosen in such a way that the solution of (2.7) decays as $x \to \infty$. The functions $\zeta^{\frac{1}{2}} H^{(1)}_{1/3}(\zeta) e^{\mathrm{Im}\ \zeta}$ and $\zeta^{\frac{1}{2}} J_{1/3}(\zeta) e^{-\mathrm{Im}\ \zeta}$ are bounded for $x \in (0, \infty)$, and Im $(\zeta - \theta) > 0$ in the integrand of (2.7). It can be proven ([T2, Lemma 22.27]) that

$$\begin{aligned} \int_0^\infty |f(x)||E - V(x)|^{1/2} dx &= O\left(\frac{1}{XE^{1/2}}\right), \qquad E \to \infty, \\ \int_x^\infty |f(x)||E - V(x)|^{1/2} dx &= O\left(\frac{1}{xV(x)^{1/2}}\right), \qquad x \to \infty. \end{aligned}$$

It follows that (2.7) can be uniquely solved by iteration in the function space

$$\mathcal{G} = \{G : e^{\ \zeta(\mathrm{x})} G(x) \text{ is bounded and continuous}\}$$

and the solution $G(x, E)$ satisfies, as $E \to \infty$,

$$G(x, E) = (\pi\zeta/2)^{\frac{1}{2}} H^{(1)}_{1/3}(\zeta) + O(E^{-\frac{1}{2}} X^{-1} e^{-\mathrm{Im}\zeta} |\zeta|^{1/6}/(1 + |\zeta|^{1/6})) \tag{2.8}$$

uniformly with respect to $x \in (0, \infty)$ and that, for fixed E, as $x \to \infty$,

$$G(x, E) = (\pi\zeta/2)^{\frac{1}{2}} H^{(1)}_{1/3}(\zeta)(1 + O(x^{-1} V(x)^{-1/2})).$$

Since the linear space of solutions of (2.1) which decay as $x \to \infty$ is one dimensional, we have $\psi(x, E) = C^{-1}_{E+}[E - V(x)]^{-\frac{1}{4}} G(x, E)$ for a constant C_{E+}. Titchmarsh ([T1, pp. 170–171]) shows $C_{E+} \sim (XE^{-\frac{1}{2}})^{\frac{1}{2}}$. □

We write the right side of (2.3) in the form $C_{E+}^{-1}\psi^+(x,E)$ and we let $C_{E-}^{-1}\psi^-(x,E)$ be the corresponding expression for $x \in (-\infty, 0)$. It follows from Lemma 2.1 that

$$\begin{aligned}\|\psi(x,E)\|_{L^p(\mathbb{R})} &\sim \|\psi(x,E)\|_{L^p(\mathbb{R}^+)} + \|\psi(x,E)\|_{L^p(\mathbb{R}^-)}\\ &\sim X^{-\frac{1}{2}}E^{\frac{1}{4}}(\|\psi^+(x,E)\|_{L^p(\mathbb{R}^+)} + \|\psi^-(x,E)\|_{L^p(\mathbb{R}^-)}). \qquad (2.9)\end{aligned}$$

We estimate the L^p-norm of $\psi^+(x,E)$. The estimate for $\psi^-(x,E)$ is similar. We define $q(y)$ and $Q(y)$ by

$$q(y) = \frac{V(yX)}{V(X)}, \quad Q(y) = \begin{cases} -\int_y^1 \sqrt{1-q(s)}ds, & \text{if } \ y<1;\\ i\int_1^y \sqrt{q(s)-1}ds, & \text{if } \ y>1. \end{cases} \qquad (2.10)$$

We have

$$\zeta(x) = E^{\frac{1}{2}} X Q(x/X).$$

Under the assumptions, we have $V(x) \sim xV'(x) \sim |x|^k$ for $|x| \geq R$.

Lemma 2.2. *Let V satisfy* Assumption 1.1 *and $K > 1$. Then there exists a constant L such that the following estimates are satisfied uniformly with respect to $|X| \geq L$:*

$$\begin{aligned} 1-q(y) &\sim 1-y, && \text{for } \ 0 \leq y \leq 1,\\ q(y)-1 &\sim y-1, && \text{for } \ 1 \leq y \leq K,\\ q(y)-1 &\sim y^k, && \text{for } \ y \geq K, \end{aligned} \qquad (2.11)$$

and

$$\begin{aligned} Q(y) &\sim -(1-y)^{3/2}, && \text{for } \ 0 \leq y \leq 1,\\ -iQ(y) &\sim (y-1)^{3/2}, && \text{for } \ 1 \leq y \leq K,\\ -iQ(y) &\sim y^{1+k/2}, && \text{for } \ y \geq K. \end{aligned} \qquad (2.12)$$

Proof. Take sufficiently large $L > 2R$, R being the constant of Assumption 1.1. Then, we have for $1/2 \leq y \leq 1$, uniformly with respect to $|X| \geq L$,

$$1-q(y) = \frac{V(X)-V(yX)}{V(X)} = (1-y)\frac{XV'(\theta X)}{V(X)} \sim 1-y, \quad y \leq \theta \leq 1.$$

Let $0 \leq y \leq 1/2$ and $R \leq yX$. We have $0 < V(yX) \leq V(R) + y(V(X)-V(R)) \leq yV(X)$ since $V(x)$ is convex for $|x| \geq R$, and $1-q(y) \geq 1-y$. If $yX \leq R$, $|V(yX)| \leq \sup_{|x|\leq R}|V(x)| \leq 10^{-1}V(X)$ and $1-q(y) \sim 1-y$ is obvious for $|X| \geq L$ and large L. This proves the first estimate. Estimates for $q(y)-1$, $y > 1$, may be obtained similarly. Estimates (2.12) for $Q(y)$ may be obtained by integrating (2.11). □

Hereafter we let E large enough such that the corresponding X satisfies the condition $|X| \geq L$ of Lemma 2.2. Writing $\psi^+(x, E)$ in the form

$$\psi^+(x, E) = E^{-\frac{1}{4}}[1 - q(x/X)]^{-\frac{1}{4}} G(E^{\frac{1}{2}} X Q(x/X), E)$$

and changing variable, we have

$$\int_0^\infty |\psi^+(x, E)|^p dx = X E^{-\frac{p}{4}} \int_0^\infty |1 - q(y)|^{-\frac{p}{4}} |G(E^{\frac{1}{2}} X Q(y), E)|^p dy.$$

We insert (2.8) for $G(x, E)$. This produces two integrals, the one with $(\pi\zeta/2)^{\frac{1}{2}} H^{(1)}_{1/3}(\zeta)$ and the other with the remainder term $O(\dots)$ in place of $G(\zeta, E)$. We estimate the latter first as it is simpler. We define

$$\delta(p) = \begin{cases} (4-p)^{-1}, & \text{if } p < 4; \\ \log(E^{\frac{1}{2}} X), & \text{if } p = 4; \\ (p-4)^{-1} (E^{\frac{1}{2}} X)^{\frac{p-4}{6}}, & \text{if } p > 4. \end{cases}$$

Lemma 2.3. *There exists a constant $C > 0$ such that for large $E \geq E_0$,*

$$\int_0^\infty |1 - q(y)|^{-\frac{p}{4}} \left(E^{-\frac{1}{2}} X^{-1} e^{-E^{1/2} X \operatorname{Im} Q(y)} \frac{|E^{1/2} X Q(y)|^{\frac{1}{6}}}{(1 + |E^{1/2} X Q(y)|)^{\frac{1}{6}}} \right)^p dy$$
$$\leq C^p (E^{\frac{1}{2}} X)^{-p} \delta(p). \quad (2.13)$$

Proof. We split the integral into three parts by using the constant K of Lemma 2.2,

$$\int_0^1 + \int_1^K + \int_K^\infty \dots dy \equiv I_1 + I_2 + I_3.$$

By virtue of (2.11) and (2.12), we have

$$I_1 \leq C^p \int_0^1 (1-y)^{-\frac{p}{4}} (E^{\frac{1}{2}} X)^{-p} \left[\frac{E^{1/2} X (1-y)^{3/2}}{1 + E^{1/2} X (1-y)^{3/2}} \right]^{p/6} dy$$

$$= C^p (E^{\frac{1}{2}} X)^{-p} (E^{\frac{1}{2}} X)^{\frac{p-4}{6}} \int_0^{(E^{\frac{1}{2}} X)^{\frac{2}{3}}} \frac{1}{(1 + y^{3/2})^{p/6}} dy \leq C^p (E^{\frac{1}{2}} X)^{-p} \delta(p).$$

Since $|e^{-X E^{1/2} \operatorname{Im} Q(y)}| \leq 1$ for $1 \leq y \leq K$, we likewise have

$$I_2 \leq C^p \int_1^K |y - 1|^{-\frac{p}{4}} \left(E^{-\frac{1}{2}} X^{-1} \frac{|E^{1/2} X Q(y)|^{\frac{1}{6}}}{(1 + |E^{1/2} X Q(y)|)^{\frac{1}{6}}} \right)^p dy \quad (2.14)$$
$$\leq C^p (E^{\frac{1}{2}} X)^{-p} \delta(p).$$

For $K \le y < \infty$, we have $|1-q(y)|^{-\frac{p}{4}} \sim y^{-\frac{kp}{4}} \le C^p$, $-iQ(y) \sim y^{1+\frac{k}{2}} \ge cy$ and

$$I_3 \le C^p \int_K^\infty e^{-cpXE^{1/2}y}dy \le C^p e^{-cpE^{1/2}X} \le C^p \delta(p). \tag{2.15}$$

Combining estimates (2.14) and (2.15), we obtain (2.13). □

Recall that $H^{(1)}_{\frac{1}{3}}(\zeta)$ satisfies the following (cf. [T1, (7.1.8), (7.8.5) and (7.8.7))]:

(1) When $\zeta = -z < 0$, $H^{(1)}_{\frac{1}{3}}(\zeta) = \dfrac{2}{\sqrt{3}} e^{-\frac{1}{6}\pi i}\{J_{\frac{1}{3}}(z) + J_{-\frac{1}{3}}(z)\}$ and

$$\zeta^{\frac{1}{2}} H^{(1)}_{\frac{1}{3}}(\zeta) = \begin{cases} 2^{\frac{3}{2}}\pi^{-\frac{1}{2}} e^{\frac{1}{3}\pi i}\{\cos(z-(\pi/4)) + O(z^{-1})\} & (z \to \infty), \\ \dfrac{2^{\frac{2}{3}}}{\sqrt{3}} \dfrac{e^{\frac{1}{3}\pi i}}{\Gamma(2/3)} z^{\frac{1}{6}}(1+O(z)) & (z \to 0). \end{cases} \tag{2.16}$$

(2) When $\zeta = iw$ and $w \ge 0$, $H^{(1)}_{\frac{1}{3}}(\zeta) = \dfrac{2}{\pi} e^{-\frac{2}{3}\pi i} K_{\frac{1}{3}}(w)$ and

$$\zeta^{\frac{1}{2}} H^{(1)}_{\frac{1}{3}}(\zeta) = \begin{cases} O(e^{-w}) & (w \to \infty), \\ 2^{\frac{1}{3}} e^{-\frac{1}{6}\pi}\pi^{-1}\Gamma(1/3) w^{\frac{1}{6}} + O(w^{\frac{3}{2}}) & (w \to 0). \end{cases} \tag{2.17}$$

Lemma 2.4. *There exists a constant $C > 0$ such that for large $E \ge E_0$,*

$$\int_0^\infty |(1-q(y))|^{-\frac{p}{4}} |\zeta^{\frac{1}{2}} H^{(1)}_{\frac{1}{3}}(\zeta)|^p dy \le C^p \delta(p), \quad \zeta = E^{1/2} X Q(y). \tag{2.18}$$

Proof. We split the integral into four parts

$$\int_0^1 + \int_1^K + \int_K^\infty \cdots dy = \mathrm{II}_1 + \mathrm{II}_2 + \mathrm{II}_3$$

and estimate them separately. When $0 \le y \le 1$, $\zeta = E^{1/2}XQ(y) \sim -E^{1/2}X(1-y)^{3/2} < 0$. We take large $N > 0$ and split the integral II_1 into two parts $\mathrm{II}_1 = \mathrm{II}_{11} + \mathrm{II}_{12}$. II_{11} is the integral over the part of the interval $(0, 1)$ where $N < E^{1/2}X(1-y)^{3/2}$ and II_{12} over the complement. Applying the first relation of (2.16) to II_{11} and the second to II_{12}, we obtain

$$\mathrm{II}_{11} \le C^p \int_0^{1-N^{\frac{2}{3}}(E^{\frac{1}{2}}X)^{-\frac{2}{3}}} (1-y)^{-\frac{p}{4}} dy \le C^p \delta(p), \tag{2.19}$$

$$\begin{aligned} \mathrm{II}_{12} &\le C^p (E^{1/2}X)^{\frac{p}{6}} \int_0^{N^{\frac{2}{3}}(E^{1/2}X)^{-\frac{2}{3}}} y^{-\frac{p}{4}} y^{\frac{p}{4}} dy \\ &= C^p N^{\frac{2}{3}} (E^{1/2}X)^{\frac{p-4}{6}} \le C^p \delta(p). \end{aligned} \tag{2.20}$$

When $1 \le y \le K$, we have $q(y)-1 \sim y-1$ and $w = -i\zeta \sim E^{1/2}XQ(y)(y-1)^{3/2} > 0$. We split the integral

$$\mathrm{II}_2 = \left(\frac{2}{\pi}\right)^p \int_1^K |1-q(y)|^{-\frac{p}{4}}|w^{\frac{1}{2}}K_{\frac{1}{3}}(w)|^p dy = \mathrm{II}_{21} + \mathrm{II}_{22}$$

into the part II_{21} over $w \ge 1$ and II_{22} over $0 \le w \le 1$. We apply the first of (2.17) to II_{21} and the second to II_{22} and obtain

$$\mathrm{II}_{21} \le C^p \int_{1+C(E^{1/2}X)^{-2/3}}^{K} (y-1)^{-\frac{p}{4}} dy \le C^p\, \delta(p). \tag{2.21}$$

$$\mathrm{II}_{22} \le C^p \int_0^{C(E^{1/2}X)^{-2/3}} y^{-\frac{p}{4}}(E^{1/2}Xy^{3/2})^{\frac{p}{6}} dy \le C^p (E^{1/2}X)^{\frac{p-4}{6}} \le C^p\delta(p). \tag{2.22}$$

For $K \le y < \infty$, $q(y)-1 \sim y^k$, $w \sim E^{\frac{1}{2}}Xy^{1+\frac{k}{2}}$ and (2.17) yields

$$\mathrm{II}_3 \le C^p \int_K^\infty y^{-\frac{kp}{4}} e^{-cpE^{1/2}Xy^{1+\frac{k}{2}}} dy \le C^p e^{-cpE^{1/2}X} \le C^p\delta(p). \tag{2.23}$$

Combining estimates (2.19), (2.20), (2.21), (2.22) and (2.23), we obtain (2.18). □

Lemma 2.5. *There exists a constant $C > 0$ such that we have following lower bound*

$$\int_0^1 |(1-q(y))^{-\frac{p}{4}}|\zeta^{\frac{1}{2}}H^{(1)}_{\frac{1}{3}}(\zeta)|^p dy \ge C^p\delta(p), \quad \zeta = E^{1/2}XQ(y)$$

for sufficiently large $E \ge E_0$.

Proof. Denote the integral on the left by II_{11} as in the proof of the previous lemma. We take N large enough so that $|O(1/z)| \le 1/10$ in the first of (2.16) for $z \ge N$. Take a large $C > 0$ such that $z = -\zeta \sim E^{\frac{1}{2}}X(1-y)^{\frac{3}{2}} \ge N$ when $CN^{2/3}(E^{1/2}X)^{-2/3} < 1-y < 1$. Then, by virtue of (2.16), we have, for $E \ge E_0$,

$$\begin{aligned}\mathrm{II}_{11} &\ge C^p \int_{N^{2/3}(E^{1/2}X)^{-2/3}<1-y<1} (1-y)^{-p/4} \left|\cos\left(\zeta - \frac{\pi}{4}\right) + O\left(\frac{1}{\zeta}\right)\right|^p dy \\ &\ge (C/2)^p \int_{N^{2/3}(E^{1/2}X)^{-2/3}<1-y<1, |\cos(\zeta-\pi/4)|>\sqrt{2}/2} (1-y)^{-p/4} dy \ge \varepsilon_N C^p \delta(p)\end{aligned}$$

with some $\varepsilon_N > 0$. □

Proof of Theorem 1.5. We first prove (1). We have

$$\|\psi(x,E)\|_{L^p(0,\infty)} \sim X^{\frac{1}{p}-\frac{1}{2}} \left(\int_0^\infty |1-q(y)|^{-\frac{p}{4}}|G(E^{\frac{1}{2}}XQ(y),E)|^p dy\right)^{\frac{1}{p}}.$$

It follows from (2.13) and (2.18) that

$$\|\psi(x,E)\|_{L^p(0,\infty)} \sim X^{\frac{1}{p}-\frac{1}{2}}\delta(p)^{\frac{1}{p}} \sim \begin{cases} C_p E^{-\frac{1}{k}(\frac{1}{2}-\frac{1}{p})}, & \text{if } \ p<4; \\ C E^{-\frac{1}{4k}}(\log E)^{1/4}, & \text{if } \ p=4; \\ C_p E^{\frac{k-4}{12k}-\frac{k-1}{3pk}}, & \text{if } \ p>4, \end{cases} \tag{2.24}$$

where C_p is taken independent of p for $p \notin (4-\varepsilon, 4+\varepsilon)$, $\varepsilon > 0$. An entirely similar argument produces the corresponding estimate for $\|\psi(x,E)\|_{L^p(\mathbb{R}^-)}$ and we obtain the upper bound of (1.9). The lower bound readily follows from Lemma 2.5.

For proving the second statement, we remark that the estimate (2.3) remains to hold for $x \in K$ uniformly. It is obvious from (2.4) that

$$\left| C_{E+}^{-1}(E-V(x))^{-\frac{1}{4}} O\left(\frac{E^{-\frac{1}{2}}X^{-1}e^{-\mathrm{Im}\zeta}|\zeta|^{1/6}}{1+|\zeta|^{1/6}}\right)\right| \le CX^{-\frac{1}{2}}(E^{-\frac{1}{2}}X^{-1}). \tag{2.25}$$

Since $\zeta = -z \sim -E^{\frac{1}{2}}X$ for large E uniformly for $x \in K$, we have from the first relation of (2.16) that

$$C_{E+}^{-1}[E-V(x)]^{-\frac{1}{4}}(\pi\zeta/2)^{\frac{1}{2}}H_{1/3}^{(1)}(\zeta) \sim X^{-\frac{1}{2}}\left\{\cos\left(z-\frac{\pi}{4}\right)+O(E^{-\frac{1}{2}}X^{-1})\right\}. \tag{2.26}$$

The second statement follows by combining (2.25) and (2.26) because $X \sim E^{\frac{1}{k}}$. □

3. Smoothing Properties

In this section we prove Theorem 1.2 and Theorem 1.3 by using estimates obtained in Sect. 2. We write $\hat{g}$ for the Fourier transform of g. In terms of the eigenvalues $\lambda_1 < \lambda_2 < \dots$ of H and the corresponding normalized eigenfunctions $\psi_1(x), \psi_2(x), \dots$, we may write

$$e^{-itH}u_0(x) = \sum_{n=1}^{\infty} e^{-it\lambda_n}\hat{u}_0(n)\psi_n(x), \tag{3.1}$$

where $\hat{u}_0(n) = \int_{\mathbb{R}} u_0(x)\psi_n(x)dx$, $n = 1, 2, \dots$ are the generalized Fourier coefficients. Under Assumption 1.1 we know that there exists a constant $C > 0$ such that

$$\Delta\lambda_n \equiv \lambda_{n+1} - \lambda_n \ge C\lambda_n^{\frac{k-2}{2k}}, \tag{3.2}$$

hence $\lambda_n \ge Cn^{\frac{2k}{k+2}}$ for $n = 1, 2, \dots$ (cf. e.g. [Y4]).

Lemma 3.1. *Suppose $u_0 \in D(H^\ell)$ for sufficiently large ℓ, then*

$$\|g(t)e^{-itH}u_0(x)\|^2_{L^2(\mathbb{R}_t)} \le C\|g\|^2_{B^{\frac{1}{4}+\frac{1}{2k}}(\mathbb{R})}\sum_{n=1}^{\infty}|\hat{u}_0(n)\psi_n(x)|^2, \quad \forall x \in \mathbb{R}. \tag{3.3}$$

Proof. By virtue of Theorem 1.5, (3.1) converges uniformly with respect to (t, x). If the support of $\hat{g}$ has a diameter $< 2^j$, then, by virtue of (3.2), there exist at most $C2^{j\left(\frac{1}{2}+\frac{1}{k}\right)}$ number of λ_n such that $\hat{g}(\lambda+\lambda_n) \neq 0$ for every fixed λ. It follows by Plancherel theorem that for such g,

$$\int_{-\infty}^{\infty} |g(t)e^{-itH}u_0(x)|^2 dt = \int_{-\infty}^{\infty} |\sum_{n=1}^{\infty} \hat{g}(\lambda+\lambda_n)\hat{u}_0(n)\psi_n(x)|^2 d\lambda$$

$$\leq C2^{j\left(\frac{1}{2}+\frac{1}{k}\right)} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} |\hat{g}(\lambda+\lambda_n)\hat{u}_0(n)\psi_n(x)|^2 d\lambda \tag{3.4}$$

$$\leq C2^{j\left(\frac{1}{2}+\frac{1}{k}\right)} \|\hat{g}\|_{L^2}^2 \sum_{n=1}^{\infty} |\hat{u}_0(n)\psi_n(x)|^2,$$

where in the second step we used Schwarz' inequality. If g is not compactly supported, we decompose it by using a dyadic decomposition of the unity $\sum_{j=-\infty}^{\infty} \hat{h}_j(\lambda) = 1$ such that

$$\text{supp } \hat{h}_0 \subset \{\lambda : |\lambda| < 1\}, \quad \text{supp } \hat{h}_{\pm j} \subset \{\lambda : \pm 2^{|j|-2} < \lambda < \pm 2^{|j|}\}, \ j = 1, 2, \dots .$$

in the form $g = \sum_{j=-\infty}^{\infty} g_j$ so that $\hat{g}_j = \hat{g}\hat{h}_j$ has a support whose diameter is less than $2^{|j|}$. Then, (3.4) implies

$$\|g(t)e^{-itH}u_0(x)\|_{L^2(\mathbb{R}_t)}^2 \leq C \left(\sum_{j=0}^{\infty} \|\hat{g}_j\|_{L^2(\mathbb{R})} 2^{\frac{j}{2}\left(\frac{1}{2}+\frac{1}{k}\right)} \right)^2 \sum_{n=1}^{\infty} |\hat{u}_0(n)\psi_n(x)|^2$$

$$\leq C\|g\|_{B^{\frac{1}{4}+\frac{1}{2k}}(\mathbb{R})}^2 \sum_{n=1}^{\infty} |\hat{u}_0(n)\psi_n(x)|^2. \qquad \square$$

By virtue of Minkowski inequality we have

$$\left\| \left(\sum_{n=1}^{\infty} |\hat{u}_0(n)\psi_n(x)|^2 \right)^{1/2} \right\|_{L^p} = \left\| \sum_{n=1}^{\infty} |\hat{u}_0(n)\psi_n(x)|^2 \right\|_{L^{p/2}}^{1/2}$$

$$\leq \left(\sum_{n=1}^{\infty} |\hat{u}_0(n)|^2 \|\psi_n(x)\|_{L^p}^2 \right)^{1/2}.$$

The right-hand side may be estimated by using Theorem 1.5 by

$$C_p \left(\sum_{n=1}^{\infty} |\hat{u}_0(n)|^2 \lambda_n^{-2\theta(k,p)} \right)^{1/2} = C_p \|H^{-\theta(k,p)}u_0\|_{L^2}. \tag{3.5}$$

Combination of (3.3) and (3.5) yields

$$\|g(t)e^{-itH}u_0(x)\|_{L^p(\mathbb{R}_x,L^2(\mathbb{R}_t))} \le C_p\|g\|_{B^{\frac{1}{4}+\frac{1}{2k}}(\mathbb{R})}\|H^{-\theta(k,p)}u_0\|_{L^2(\mathbb{R})}, \tag{3.6}$$

where the constant C_p is taken uniformly with respect to p outside $(4-\varepsilon, 4+\varepsilon)$. Since $D(H^\ell)$ is dense in $L^2(\mathbb{R})$, (3.6) holds for all $u \in L^2(\mathbb{R})$. Theorem 1.2 follows from (3.6). □

Proof of Theorem 1.3. Theorem 1.5 (2) implies that

$$\sup_{x\in K}\sum_{n=1}^{\infty}|\hat{u}_0(n)|^2|\psi_n(x)|^2 \le C\sum_{n=1}^{\infty}|\lambda_n^{-\frac{1}{2k}}\hat{u}_0(n)|^2 = C\|H^{-\frac{1}{2k}}u_0\|^2_{L^2(\mathbb{R})}. \tag{3.7}$$

Thus, Theorem 1.3 follows by combining (3.3) with (3.7). □

4. Applications to Nonlinear Equations

In this section we prove Theorem 1.6 and Theorem 1.7. Since the proofs are quite similar, we prove Theorem 1.7, and only indicate the modifications necessary for the proof of Theorem 1.6. Hereafter, we often omit some of the variables of function $u(t,x)$ and write $u(t)$ or simply u for $u(t,x)$, if no confusions are feared. By taking g such that $g(t) = 1$ for $|t| \le \delta$ in Theorem 1.2 and Theorem 1.3, we have

$$\|\langle i\partial/\partial t\rangle^{\alpha}\langle H\rangle^{\beta}e^{-itH}u_0\|_{L^p(\mathbb{R}_x,L^2([-\delta,\delta]_t))} \le C_\delta\|u_0\|_{L^2}, \quad \alpha+\beta=\theta(k,p), \quad p\ge 2; \tag{4.1}$$

$$\sup_{x\in K}\|\langle i\partial/\partial t\rangle^{1/2k}e^{-itH}u_0\|_{L^2([-\delta,\delta]_t)} \le C_\delta\|u_0\|_{L^2}. \tag{4.2}$$

Proof of Theorem 1.7. We prove Theorem 1.7 for $t \ge 0$ only. The argument for $t \le 0$ is similar. We consider the equivalent integral equation

$$u(t) = e^{-itH}u_0 - i\int_0^t e^{-i(t-s)H}f(x,u(s))ds. \tag{4.3}$$

For $\delta > 0$, we write $K_\delta = [0,\delta]\times K$ and define the Banach space $Y_\delta(K)$ by

$$Y_\delta(K) = C([0,\delta], L^2(\mathbb{R}))\cap L^{2r}(K_\delta), \quad \|u\|_{Y_\delta(K)} \equiv \|u\|_{L^\infty([0,\delta],L^2(\mathbb{R}))} + \|u\|_{L^{2r}(K_\delta)}.$$

We define a nonlinear map $\Psi : Y_\delta(K) \to Y_\delta(K)$ by

$$\Psi(u) = e^{-itH}u_0 - i\Phi(u), \quad \Phi(u) = \int_0^t e^{-i(t-s)H}f(x,u(s))ds. \tag{4.4}$$

Write $B_M = \{u \in Y_\delta(K) : \|u\|_{Y_\delta(K)} \le M\}$.

Lemma 4.1. *The map Ψ is well defined on $Y_\delta(K)$. There exist $M > 0$ and $\delta > 0$ depending only on $\|u_0\|_{L^2(\mathbb{R})}$ such that Ψ maps B_M into itself and*

$$\|\Psi(u)-\Psi(v)\|_{Y_\delta(K)} \le \frac{1}{2}\|u-v\|_{Y_\delta(K)}, \quad u,v\in B_M. \tag{4.5}$$

Proof. For $u_0 \in L^2(\mathbb{R})$, we have $e^{-itH}u_0 \in C(\mathbb{R}, L^2(\mathbb{R}))$. By virtue of (4.2) and the Sobolev embedding theorem, $e^{-itH}u_0 \in L^\infty(K_x, L^{2r}([0,\delta]_t))$. Hence, $e^{-itH}u_0 \in Y_\delta(K)$ and

$$\|e^{-itH}u_0\|_{Y_\delta(K)} \leq c_1\|u_0\|_{L^2}. \tag{4.6}$$

Let $\chi(s<t)$ be such that $\chi(s<t)=1$ if $0<s<t$, and $\chi(s<t)=0$ otherwise. If $u \in Y_\delta(K)$, then, the assumptions that $f(x,u)=0$ for $x \notin K$ and (1.14) imply $f(x,u(t,x)) \in L^2([0,\delta]_t \times \mathbb{R}_x)$ and

$$\|f(x,u(t,x))\|_{L^2(K_\delta)} \leq C\|u\|^r_{L^{2r}(K_\delta)}. \tag{4.7}$$

It then easily follows that $\Phi(u) \in C([0,\delta], L^2(\mathbb{R}))$ and by Schwarz' inequality and

$$\|\Phi(u)\|_{L^\infty([0,\delta];L^2(\mathbb{R}))} \leq C\delta^{\frac{1}{2}}\|u\|^r_{L^{2r}(K_\delta)}. \tag{4.8}$$

By Minkowski's inequality, (4.2) and (4.7), we have

$$\begin{aligned} \|\Phi(u)\|_{L^{2r}(K_\delta)} &\leq \int_0^\delta \|\|\chi(s<t)e^{-itH}\{e^{isH}f(x,u(s,x))\}\|_{L^{2r}(K_\delta)}ds \\ &\leq C\int_0^\delta \|f(x,u(s,x))\|_{L^2(K)}ds \leq C\delta^{\frac{1}{2}}\|f(x,u)\|_{L^2(K_\delta)} \\ &\leq C\delta^{\frac{1}{2}}\|u\|^r_{L^{2r}(K_\delta)}, \end{aligned} \tag{4.9}$$

which with (4.6) and (4.8) implies that Ψ is well-defined on $Y_\delta(K)$.

It follows also from (4.6), (4.8) and (4.9) that, with constants c_1 and c_2 which can be taken independent of small δ,

$$\|\Psi u\|_{Y_\delta(K)} \leq \|e^{-itH}u_0\|_{Y_\delta(K)} + \|f(u)\|_{Y_\delta(K)} \leq c_1\|u_0\|_{L^2} + c_2\delta^{\frac{1}{2}}\|u\|^r_{Y_\delta(K)}. \tag{4.10}$$

Thus, if we take M such that $M > 2c_1\|u_0\|_{L^2}$, $\delta < (2c_2M^{r-1})^{-2}$, then $\|\Psi u\|_{Y_\delta(K)} \leq 2c_1\|u_0\|_{L^2} < M$ whenever $\|u\|_{Y_\delta(K)} \leq M$ and Ψ maps B_M into itself. To show that Ψ satisfies (4.5), we estimate

$$\Psi(u_1) - \Psi(u_2) = -i\int_0^t e^{-i(t-s)H}[f(x,u_1(s)) - f(x,u_2(s))]ds.$$

We have by Minkowski's inequality and Hölder's inequality that

$$\begin{aligned} &\|\Psi(u_1) - \Psi(u_2)\|_{L^\infty([0,\delta]_t;L^2(\mathbb{R}_x))} \leq \int_0^\delta \|f(x,u_1(s)) - f(x,u_2(s))\|_{L^2(K)}ds \\ &\leq C\int_0^\delta \||u_1-u_2|(|u_1|^{r-1} + |u_2|^{r-1})\|_{L^2(K)}ds \\ &\leq C\int_0^\delta \|u_1(s)-u_2(s)\|_{L^{2r}(K)}(\|u_1\|^{r-1}_{L^{2r}(K)} + \|u_2\|^{r-1}_{L^{2r}(K)})ds \\ &\leq C\delta^{\frac{1}{2}}(\|u_1\|^{r-1}_{L^{2r}(K_\delta)} + \|u_2\|^{r-1}_{L^{2r}(K_\delta)})\|u_1-u_2\|_{L^{2r}(K_\delta)}. \end{aligned} \tag{4.11}$$

Likewise, by virtue of (4.6), we have by Minkowski's inequality and Hölder's ineqaulity

$$\begin{aligned}\|\Psi(u_1)-\Psi(u_2)\|_{L^{2r}(K_\delta)} &\le \int_0^\delta \|\chi(s<t)e^{-itH}e^{isH}[f(x,u_1)-f(x,u_2)]\|_{L^{2r}(K_\delta)}ds \\ &\le C\int_0^\delta \|f(x,u_1(s,x))-f(x,u_2(s,x))\|_{L^2(\mathbb{R}_x)}ds \\ &\le C\delta^{\frac{1}{2}}(\|u_1\|^{r-1}_{L^{2r}(K_\delta)}+\|u_2\|^{r-1}_{L^{2r}(K_\delta)})\|u_1-u_2\|_{L^{2r}(K_\delta)}.\end{aligned} \tag{4.12}$$

Combining (4.11) with (4.12), we obtain

$$\|\Psi(u_1)-\Psi(u_2)\|_{Y_\delta(K)} \le c_3\delta^{\frac{1}{2}}(\|u_1\|^{r-1}_{Y_\delta(K)}+\|u_2\|^{r-1}_{Y_\delta(K)})\|u_1-u_2\|_{Y_\delta(K)}, \tag{4.13}$$

and (4.5) follows if we choose δ such that $\delta < \min\{(2c_2M^{r-1})^{-2}, (4c_3M^{r-1})^{-2}\}$. □

Continuation of Proof of Theorem 1.7. By virtue of Lemma 4.1, the contraction mapping theorem implies that Ψ has a unique fixed point $u \in B_M$ and (4.3) has a unique solution u in $Y_\delta(K)$. To prove that the solution depends on the initial data u_0 continuously as described in the theorem, we take $u_0, \tilde{u}_0 \in L^2(\mathbb{R})$ and let u and $\tilde{u}$ be the corresponding solutions. Then, the preceding estimates (4.6) and (4.13) show

$$\|u-\tilde{u}\|_{Y_\delta(K)} \le c_1\|u_0-\tilde{u}_0\|_{L^2}+c_3\delta^{\frac{1}{2}}(\|u\|^{r-1}_{Y_\delta(K)}+\|\tilde{u}\|^{r-1}_{Y_\delta(K)})\|u-\tilde{u}\|_{Y_\delta(K)}$$

and $\|u-\tilde{u}\|_{Y_\delta(K)} \le c\|u_0-\tilde{u}_0\|_{L^2}$ for small $\delta > 0$. This shows the desired continuous dependence.

When f satisfies the additional assumption (1.13), we will show $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$. Once this is shown, the solution $u(t)$ extends uniquely to $[0,\infty)$ since the length δ of the interval on which the solution exists depends only on $\|u_0\|_{L^2(\mathbb{R}_x)}$ as has been shown above. Also the map $L^2(\mathbb{R}) \ni u_0 \mapsto u \in C([0,T], L^2(\mathbb{R})) \cap L^{2r}([0,T]_t \times K)$ is continuous for any $T > 0$ because $u(t,\cdot)$ is $L^2(\mathbb{R}_x)$ valued continuous and we will be done. To show $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$, we compute $\|\cdot\|^2_{L^2(\mathbb{R}_x)}$ of both sides of (4.3). Denoting the inner product and the norm of $L^2(\mathbb{R}_x)$ by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively, and writing $f(t,x) = f(t,u(t,x))$, we have

$$\begin{aligned}\|u(t)\| &= \left\|e^{-itH}u_0 - i\int_0^t e^{-i(t-s)H}f(s,x)ds\right\|^2 \\ &= \|u_0\|^2_{L^2} - 2\mathrm{Re}\left(u_0, i\int_0^t e^{isH}f(s,x)ds\right) \\ &\quad + \int_0^t\int_0^t (e^{isH}f(s,x), e^{irH}f(r,x))dsdr.\end{aligned}$$

The last two terms on the right cancel each other because the last integral is equal to

$$\begin{aligned}&\int_0^t\left(f(s,x), \int_0^s e^{-i(s-r)H}f(r,x)dr\right)ds + \int_0^t\left(\int_0^r e^{-i(r-s)H}f(s,x)ds, f(r,x)\right)dr \\ &= \int_0^t (f(s,x), iu(s)-ie^{-isH}u_0))ds + \int_0^t (iu(r)-ie^{-irH}u_0, f(r,x))dr \\ &= 2\mathrm{Re}\left(u_0, i\int_0^t e^{isH}f(s,x)ds\right),\end{aligned}$$

where we used the fact that u is a solution in the first step and (1.13) in the second. This completes the proof. □

Proof of Theorem 1.6. The proof is very similar to that of Theorem 1.7 and we only indicate the necessary modifications. Instead of $Y_\delta(K)$, we use now the Banach space

$$X_\delta = C([0,\delta]_t; L^2(\mathbb{R}_x)) \cap L^4(\mathbb{R}_x; L^{2r}([0,\delta]_t))$$

with the norm

$$\|u\|_{X_\delta} = \|u\|_{L^\infty([0,\delta]_t;L^2(\mathbb{R}_x))} + \|u\|_{L^4(\mathbb{R}_x;L^{2r}([0,\delta]_t))}.$$

(This notation is slightly different from that in the theorem, but no confusion should occur.) We define the nonlinear operators Φ and Ψ by (4.4) as previously and set $B_M = \{u \in X_\delta : \|u\|_{X_\delta} \le M\}$. We show that, for any $u_0 \in L^2(\mathbb{R})$, Ψ is a contraction map from B_M into B_M if the parameters $\delta > 0$ and M are chosen suitably. To show $e^{-itH}u_0 \in X_\delta$ and $\|e^{-itH}u_0\|_{X_\delta} \le C\|u_0\|_{L^2}$, we use (4.1) instead of (4.2) and Sobolev embedding theorem which implies $e^{-itH}u_0 \in L^4(\mathbb{R}_x; L^{2r}([0,\delta]_t))$. By the assumption on f, we have

$$\begin{aligned}
\|\Phi(u)\|_{L^\infty([0,\delta]_t;L^2(\mathbb{R}_x))} &\le \int_0^\delta \|f(x,u(s))\|_{L^2} ds \le C\int_0^\delta \||\phi(x)||u(s)|^r\|_{L^2} ds \\
&\le C\delta^{\frac12}\left\{\int_{[0,\delta]\times\mathbb{R}} |\phi(x)|^2|u(t,x)|^{2r} dt dx\right\}^{\frac12} \\
&= C\delta^{\frac12}\{\int_{\mathbb{R}} |\phi(x)|^2 \|u(t,x)\|^{2r}_{L^{2r}([0,\delta]_t)} dx\}^{\frac12} \\
&\le C\delta^{\frac12}\|\phi\|_{L^{\frac{4}{2-r}}(\mathbb{R})}\|u\|^r_{L^4(\mathbb{R}_x;L^{2r}([0,\delta]_t))} \le C\delta^{\frac12}\|u\|^r_{X_\delta}.
\end{aligned} \tag{4.14}$$

As in the proof of Theorem 1.7, (4.1) and (4.14) imply

$$\|\Phi(u)\|_{L^4(\mathbb{R}_x;L^{2r}([0,\delta]_t))} \le C\delta^{\frac12}\|u\|^r_{X_\delta}. \tag{4.15}$$

It follows that Ψ maps B_M into B_M for suitable M and δ which depend only on $\|u_0\|_{L^2}$. The rest of the proof may be done by repeating the argument of the proof of Theorem 1.7 by using these estimates. We omit the details. □

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