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A Riemann–Roch Theorem for One-Dimensional Complex Groupoids

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Abstract: We consider a smooth groupoid of the form $\Sigma \rtimes \Gamma$, where Σ is a Riemann surface and Γ a discrete pseudogroup acting on Σ by local conformal diffeomorphisms. After defining a K-cycle on the crossed product $C_0(\Sigma) \rtimes \Gamma$ generalising the classical Dolbeault complex, we compute its Chern character in cyclic cohomology, using the index theorem of Connes and Moscovici. This involves in particular a generalisation of the Euler class constructed from the modular automorphism group of the von Neumann algebra $L^\infty(\Sigma) \rtimes \Gamma$.

1. Introduction

In a series of papers [4,5], Connes and Moscovici proved a general index theorem for transversally (hypo)elliptic operators on foliations. After constructing K-cycles on the algebra crossed product $C_0(M) \rtimes \Gamma$, where Γ is a discrete pseudogroup acting on the manifold M by local diffeomorphisms [4], they developed a theory of characteristic classes for actions of Hopf algebras that generalise the usual Chern–Weil construction to the non-commutative case [5,6]. The Chern character of the concerned K-cycles is then captured in the periodic cyclic cohomology of a particular Hopf algebra encoding the action of the diffeomorphisms on M. The nice thing is that this cyclic cohomology can be completely exhausted as Gelfand–Fuchs cohomology and renders the index computable.

We shall illustrate these methods with a specific example, namely the crossed product of a Riemann surface Σ by a discrete pseudogroup Γ of local conformal mappings. We find that the relevant characteristic classes are the fundamental class $[\Sigma]$ and a cyclic 2-cocycle on $C_c^\infty(\Sigma) \rtimes \Gamma$ generalising the (Poincaré dual of the) usual Euler class. When applied to the K-cycle represented by the Dolbeault operator of $\Sigma \rtimes \Gamma$, this yields a non-commutative version of the Riemann–Roch theorem. Throughout the text we also stress the crucial role played by the modular automorphism group of the von Neumann algebra $L^\infty(\Sigma) \rtimes \Gamma$.

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2. The Dolbeault K-Cycle

Let Σ be a Riemann surface without boundary and Γ a pseudogroup of local conformal mappings of Σ into itself. We want to define a K-cycle on the algebra $C_0(\Sigma) \rtimes \Gamma$ generalising the classical Dolbeault complex. Following [4], the first step consists in lifting the action of Γ to the bundle P over Σ, whose fiber at point x is the set of Kähler metrics corresponding to the complex structure of Σ at x. By the obvious correspondence metric $\leftrightarrow$ volume form, P is the $\mathbb{R}_+^*$-principal bundle of densities on Σ. The pseudogroup Γ acts canonically on P and we consider the crossed product $C_0(P) \rtimes \Gamma$.

Let ν be a smooth volume form on Σ. As in [2], this gives a weight on the von Neumann algebra $L^\infty(\Sigma) \rtimes \Gamma$ together with a representative σ of its modular automorphism group. Moreover σ leaves $C_0(\Sigma) \rtimes \Gamma$ globally invariant and one has

$$C_0(P) \rtimes \Gamma = (C_0(\Sigma) \rtimes \Gamma) \rtimes_\sigma \mathbb{R}, \tag{1}$$

where the space P is identified with $\Sigma \times \mathbb{R}$ thanks to the choice of the global section ν. Therefore one has a Thom-Connes isomorphism [1]

$$K_i(C_0(\Sigma) \rtimes \Gamma) \to K_{i+1}(C_0(P) \rtimes \Gamma), \quad i = 0, 1, \tag{2}$$

and we shall obtain the desired K-homology class on $C_0(P) \rtimes \Gamma$. The reason for working on P rather than Σ is that P carries quasi Γ-invariant metric structures, allowing the construction of K-cycles represented by differential hypoelliptic operators [4].

More precisely, consider the product $P \times \mathbb{R}$, viewed as a bundle over Σ with 2-dimensional fiber. The action of Γ extends to $P \times \mathbb{R}$ by making $\mathbb{R}$ invariant. Up to another Thom isomorphism, the K-cycle may be defined on $C_0(P \times \mathbb{R}) \rtimes \Gamma = (C_0(P) \rtimes \Gamma) \otimes C_0(\mathbb{R})$. By a choice of horizontal subspaces on the bundle $P \times \mathbb{R}$, one can lift the Dolbeault operator $\overline{\partial}$ of Σ. This yields the horizontal operator $Q_H = \overline{\partial} + \overline{\partial}^*$, where the adjoint $\overline{\partial}^*$ is taken relative to the L^2-norm given by the canonical invariant measure on $P \times \mathbb{R}$ (see [4] for details). Finally, consider the signature operator of the fibers, $Q_V = d_V d_V^* - d_V^* d_V$, where d_V is the vertical differential. Then the sum $Q = Q_H + Q_V$ is a hypoelliptic operator representing our Dolbeault K-cycle.

This construction ensures that the principal symbol of Q is completely canonical, because it is related only to the fibration of $P \times \mathbb{R}$ over Σ, and hence is invariant under Γ. Another choice of horizontal subspaces does not change the leading term of the symbol of Q. This is basically the reason why Q allows one to construct a spectral triple (of even parity) for the algebra $C_c^\infty(P \times \mathbb{R}) \rtimes \Gamma$.

If $\Gamma = \mathrm{Id}$, then $C_0(P \times \mathbb{R}) \rtimes \Gamma = C_0(\Sigma) \otimes C_0(\mathbb{R}^2)$ and the addition of Q_V to Q_H is nothing else but a Thom isomorphism in K-homology

$$K^*(C_0(\Sigma)) \to K^*(C_0(P \times \mathbb{R})) \tag{3}$$

sending the classical Dolbeault elliptic operator $\overline{\partial} + \overline{\partial}^*$ to Q.

Now we want to compute the Chern character of Q in the periodic cyclic cohomology $H^*(C_c^\infty(P \times \mathbb{R}) \rtimes \Gamma)$ using the index theorem of [5]. We need first to construct an *odd* cycle by tensoring the Dolbeault complex with the spectral triple of the real line $(C_c^\infty(\mathbb{R}), L^2(\mathbb{R}), i\frac{\partial}{\partial x})$. In this way we get a differential operator $Q' = Q + i\frac{\partial}{\partial x}$ whose Chern character lives in the cyclic cohomology of $(C_c^\infty(P) \rtimes \Gamma) \otimes C_c^\infty(\mathbb{R}^2)$. By Bott periodicity it is just the cup product

$$\mathrm{ch}_*(Q') = \varphi \# [\mathbb{R}^2] \tag{4}$$

of a cyclic cocycle $\varphi \in HC^*(C_c^\infty(P) \rtimes \Gamma)$ by the fundamental class of $\mathbb{R}^2$. The main theorem of [5] states that φ can be computed from Gelfand–Fuchs cohomology, after transiting through the cyclic cohomology of a particular Hopf algebra. We perform the explicit computation in the remainder of the paper.

3. The Hopf Algebra and Its Cyclic Cohomology

First we reduce to the case of a flat Riemann surface, since for any groupoid $\Sigma \rtimes \Gamma$ one can find a flat surface Σ' and a pseudogroup Γ' acting by conformal transformations on Σ' such that $C_0(\Sigma') \rtimes \Gamma'$ is Morita equivalent to $C_0(\Sigma) \rtimes \Gamma$ (see [5] and Sect. 5 below).

Let then Σ be a flat Riemann surface and $(z, \overline{z})$ a complex coordinate system corresponding to the complex structure of Σ. Let F be the $Gl(1, \mathbb{C})$-principal bundle over Σ of frames corresponding to the conformal structure. F is gifted with the coordinate system $(z, \overline{z}, y, \overline{y})$, $y, \overline{y} \in \mathbb{C}^*$. A point of F is the frame

$$(y\partial_z, \overline{y}\partial_{\overline{z}}) \quad \text{at } (z, \overline{z}). \tag{5}$$

The action of a discrete pseudogroup Γ of conformal transformations on Σ can be lifted to an action on F by pushforward on frames. More precisely, a holomorphic transformation $\psi \in \Gamma$ acts on the coordinates by

$$z \to \psi(z), \qquad \mathrm{Dom}\psi \subset F, \tag{6}$$
$$y \to \psi'(z)y, \quad \psi'(z) = \partial_z \psi(z). \tag{7}$$

Let $C_c^\infty(F)$ be the algebra of smooth complex-valued functions with compact support on F, and consider the crossed product $\mathcal{A} = C_c^\infty(F) \rtimes \Gamma$. $\mathcal{A}$ is the associative algebra linearly generated by elements of the form fU_ψ^* with $\psi \in \Gamma$, $f \in C_c^\infty(F)$, $\mathrm{supp} f \subset \mathrm{Dom}\psi$. We adopt the notation $U_\psi \equiv U_{\psi^{-1}}^*$ for the inverse of U_ψ^*. The multiplication rule

$$f_1 U_{\psi_1}^* f_2 U_{\psi_2}^* = f_1\,(f_2 \circ \psi_1) U_{\psi_2\psi_1}^* \tag{8}$$

makes good sense thanks to the condition $\mathrm{supp} f_i \subset \mathrm{Dom}\psi_i$. We introduce now the differential operators

$$X = y\partial_z, \quad Y = y\partial_y, \quad \overline{X} = \overline{y}\partial_{\overline{z}}, \quad \overline{Y} = \overline{y}\partial_{\overline{y}}, \tag{9}$$

forming a basis of the set of smooth vector fields viewed as a module over $C^\infty(F)$. These operators act on $\mathcal{A}$ in a natural way:

$$X.(fU_\psi^*) = (X.f)U_\psi^*, \quad Y.(fU_\psi^*) = (Y.f)U_\psi^* \tag{10}$$

and similarly for $\overline{X}, \overline{Y}$. We remark that the system $(z, \overline{z})$ determines a smooth volume form $\frac{dz\wedge d\overline{z}}{2i}$ on Σ. This in turn gives a representative σ of the modular automorphism group of $L^\infty(\Sigma) \rtimes \Gamma$, whose action on $C_c^\infty(\Sigma) \rtimes \Gamma$reads (cf. [3] chap. III)

$$\sigma_t(fU_\psi^*) = |\psi'|^{2it} fU_\psi^*, \quad t \in \mathbb{R}. \tag{11}$$

We let D be the derivation corresponding to the infinitesimal action of σ:

$$D = -i\frac{d}{dt}\sigma_t|_{t=0} \quad D(fU_\psi^*) = \ln|\psi'|^2 fU_\psi^*. \tag{12}$$

The operators $\delta_n, \overline{\delta}_n, n \geq 1$ are defined recursively

$$\delta_n = \underbrace{[X, \ldots [X}_{n}, D] \ldots] \quad \overline{\delta}_n = \underbrace{[\overline{X}, \ldots [\overline{X}}_{n}, D] \ldots]. \tag{13}$$

Their action on $\mathcal{A}$ are explicitely given by

$$\delta_n(fU_\psi^*) = y^n \partial_z^n (\ln \psi') fU_\psi^*, \quad \overline{\delta}_n(fU_\psi^*) = y^n \partial_z^n (\ln \overline{\psi'}) fU_\psi^*. \tag{14}$$

Thus $\delta_n, \overline{\delta}_n$ represent in some sense the Taylor expansion of D. All these operators fulfill the commutation relations

$$\begin{aligned} [Y, X] &= X, & [Y, \delta_n] &= n\delta_n, \\ [X, \delta_n] &= \delta_{n+1}, & [\delta_n, \delta_m] &= 0, \end{aligned} \tag{15}$$

and similarly for the conjugates $\overline{X}, \overline{Y}, \overline{\delta}_n$. Thus $\{X, Y, \delta_n, \overline{X}, \overline{Y}, \overline{\delta}_n\}_{n\geq 1}$ form a basis of a (complex) Lie algebra. Let $\mathcal{H}$ be its enveloping algebra. The remarkable fact is that $\mathcal{H}$ is a Hopf algebra. First, the coproduct $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ is determined by the action of $\mathcal{H}$ on $\mathcal{A}$:

$$\Delta h(a_1 \otimes a_2) = h(a_1 a_2) \quad \forall\, h \in \mathcal{H}, a_i \in \mathcal{A}. \tag{16}$$

One has

$$\begin{aligned} &\Delta X = 1 \otimes X + X \otimes 1 + \delta_1 \otimes Y, \\ &\Delta Y = 1 \otimes Y + Y \otimes 1, \quad \Delta\delta_1 = 1 \otimes \delta_1 + \delta_1 \otimes 1. \end{aligned} \tag{17}$$

$\Delta\delta_n$ for $n > 1$ is obtained recursively from (13) using the fact that Δ is an algebra homomorphism, $\Delta(h_1 h_2) = \Delta h_1 \Delta h_2$. Similarly for the conjugate elements.
The counit $\varepsilon : \mathcal{H} \to \mathbb{C}$ satisfies simply $\varepsilon(1) = 1$, $\varepsilon(h) = 0\ \forall\, h \neq 1$. Finally, $\mathcal{H}$ has an antipode $S : \mathcal{H} \to \mathcal{H}$, determined uniquely by the condition $m \circ S \otimes \mathrm{Id} \circ \Delta = m \circ \mathrm{Id} \otimes S \circ \Delta = \eta\varepsilon$, where $m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ is the multiplication and $\eta : \mathbb{C} \to \mathcal{H}$ the unit of $\mathcal{H}$. One finds

$$S(X) = -X + \delta_1 Y, \quad S(Y) = -Y, \quad S(\delta_1) = -\delta_1. \tag{18}$$

Since S is an antiautomorphism: $S(h_1 h_2) = S(h_2)S(h_1)$, the values of $S(\delta_n)$, $n > 1$ follow.

We are interested now in the cyclic cohomology of $\mathcal{H}$ [5,6]. As a space, the cochain complex $C^*(\mathcal{H})$ is the tensor algebra over $\mathcal{H}$:

$$C^*(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n}. \tag{19}$$

The crucial step is the construction of a characteristic map

$$\gamma : \mathcal{H}^{\otimes n} \to C^n(\mathcal{A}, \mathcal{A}^*) \tag{20}$$

from the cochain complex of $\mathcal{H}$ to the Hochschild complex of $\mathcal{A}$ with coefficients in $\mathcal{A}^*$ [3]. First F has a canonical Γ-invariant measure $dv = dz d\overline{z} \frac{dy d\overline{y}}{(y\overline{y})^2}$. This yields a trace τ on $\mathcal{A}$:

$$\begin{aligned} \tau(f) &= \int_F f\, dv, \quad f \in C_c^\infty(F), \\ \tau(fU_\psi^*) &= 0 \quad \text{if } \psi \neq 1. \end{aligned} \tag{21}$$

Then the characteristic map sends the n-cochain $h_1 \otimes \cdots \otimes h_n \in \mathcal{H}^{\otimes n}$ to the Hochschild cochain $\gamma(h_1 \otimes \cdots \otimes h_n) \in C^n(\mathcal{A}, \mathcal{A}^*)$ given by

$$\gamma(h_1 \otimes \cdots \otimes h_n)(a_0, \ldots, a_n) = \tau(a_0 h_1(a_1) \ldots h_n(a_n)) , \quad a_i \in \mathcal{A}. \tag{22}$$

The cyclic cohomology of $\mathcal{H}$ is defined such that γ is a morphism of cyclic complexes. One introduces the face operators $\delta^i : \mathcal{H}^{\otimes(n-1)} \to \mathcal{H}^{\otimes n}$ for $0 \le i \le n$:

$$\begin{aligned}
&\delta^0(h_1 \otimes \cdots \otimes h_{n-1}) = 1 \otimes h_1 \otimes \cdots \otimes h_{n-1},\\
&\delta^i(h_1 \otimes \cdots \otimes h_{n-1}) = h_1 \otimes \cdots \otimes \Delta h_i \otimes \cdots \otimes h_{n-1}, \quad 1 \le i \le n-1,\\
&\delta^n(h_1 \otimes \cdots \otimes h_{n-1}) = h_1 \otimes \cdots \otimes h_{n-1} \otimes 1,
\end{aligned} \tag{23}$$

as well as the degeneracy operators $\sigma_i : \mathcal{H}^{\otimes(n+1)} \to \mathcal{H}^{\otimes n}$,

$$\sigma_i(h_1 \otimes \cdots \otimes h_{n+1}) = h_1 \otimes \ldots \varepsilon(h_{i+1}) \cdots \otimes h_{n+1}, \quad 0 \le i \le n. \tag{24}$$

Next, the cyclic structure is provided by the antipode S and the multiplication of $\mathcal{H}$. Consider the twisted antipode $\tilde{S} = (\delta \otimes S) \circ \Delta$, where $\delta : \mathcal{H} \to \mathbb{C}$ is a character such that

$$\tau(h(a)b) = \tau(a\tilde{S}(h)(b)) \quad \forall\, a, b \in \mathcal{A}. \tag{25}$$

This last formula plays the role of ordinary integration by parts. One finds:

$$\begin{aligned}
&\delta(1) = 1, \quad \delta(Y) = \delta(\overline{Y}) = 1,\\
&\delta(X) = \delta(\overline{X}) = \delta(\delta_n) = \delta(\bar{\delta}_n) = 0 \quad \forall\, n \ge 1.
\end{aligned} \tag{26}$$

The definition implies $\tilde{S}^2 = 1$. Connes and Moscovici proved in [6] that the latter identity is sufficient to ensure the existence of a cyclicity operator $\tau_n : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n}$,

$$\tau_n(h_1 \otimes \cdots \otimes h_n) = (\Delta^{n-1}\tilde{S}(h_1)) \cdot h_2 \otimes \cdots \otimes h_n \otimes 1, \tag{27}$$

with $(\tau_n)^{n+1} = 1$. Now $C^*(\mathcal{H})$ endowed with $\delta^i, \sigma_i, \tau_n$ defines a cyclic complex. The Hochschild coboundary operator $b : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes(n+1)}$ is

$$b = \sum_{i=0}^{n+1} (-)^i \delta^i \tag{28}$$

and Connes' operator $B : \mathcal{H}^{\otimes(n+1)} \to \mathcal{H}^{\otimes n}$ is

$$B = \sum_{i=0}^{n} (-)^{ni} (\tau_n)^i B_0 \quad B_0 = \sigma_n \tau_{n+1} + (-)^n \sigma_n. \tag{29}$$

They fulfill the usual relations $B^2 = b^2 = bB + Bb = 0$, so that $C^*(\mathcal{H}, b, B)$ is a bicomplex. We define the cyclic cohomology $HC^*(\mathcal{H})$ as the b-cohomology of the subcomplex of cyclic cochains. The corresponding *periodic* cyclic cohomology $H^*(\mathcal{H})$ is isomorphic to the cohomology of the bicomplex $C^*(\mathcal{H}, b, B)$ [3]. Furthermore, the definitions of $\delta^i, \sigma_i, \tau_n$ imply that γ is a morphism of cyclic complexes. Consequently, γ passes to cyclic cohomology

$$\gamma : HC^*(\mathcal{H}) \to HC^*(\mathcal{A}), \tag{30}$$

as well as to periodic cyclic cohomology

$$\gamma : H^*(\mathcal{H}) \to H^*(\mathcal{A}). \tag{31}$$

In fact we are not interested in the frame bundle F but rather in the bundle of metrics $P = F/SO(2)$, where $SO(2) \subset Gl(1, \mathbb{C})$ is the group of rotations of frames. P is gifted with the coordinate chart $(z, \bar{z}, r)$, where the radial coordinate r is obtained from the decomposition

$$y = e^{-r+i\theta}, \quad r \in \mathbb{R}, \theta \in [0, 2\pi[. \tag{32}$$

The pseudogroup Γ still acts on P by

$$\begin{aligned} z \to \psi(z), \quad \bar{z} \to \overline{\psi(z)}, \\ r \to r - \frac{1}{2} \ln |\psi'(z)|^2. \end{aligned} \tag{33}$$

Define $\mathcal{A}_1 = \mathcal{A}^{SO(2)} \subset \mathcal{A}$ the subalgebra of elements of $\mathcal{A}$ invariant under the (right) action of $SO(2)$ on F. $\mathcal{A}_1$ is canonically isomorphic to the crossed product $C_c^\infty(P) \rtimes \Gamma$. P carries a Γ-invariant measure $dv_1 = e^{2r} dz d\bar{z} dr$, so that there is a trace on $\mathcal{A}_1$, namely

$$\begin{aligned} \tau_1(f) = \int_P f\, dv_1, \quad f \in C_c^\infty(P), \\ \tau_1(fU_\psi^*) = 0 \quad \text{if } \psi \neq 1. \end{aligned} \tag{34}$$

Thus passing to $SO(2)$-invariants yields an induced characteristic map from the relative cyclic cohomology of $\mathcal{H}$ [5]

$$\gamma_1 : HC^*(\mathcal{H}, SO(2)) \to HC^*(\mathcal{A}_1) \tag{35}$$

given by $\gamma_1(h_1 \otimes \cdots \otimes h_n)(a_0, \ldots, a_n) = \tau_1(a_0 h_1(a_1) \ldots h_1(a_n))$, $a_i \in \mathcal{A}_1$, where $h_1 \otimes \cdots \otimes h_n$ represents an element of $HC^*(\mathcal{H}, SO(2))$. The map γ_1 generalises the classical Chern-Weil construction of characteristic classes from connections and curvatures. In the crossed product case $\Sigma \rtimes \Gamma$, these classes are captured by the periodic cyclic cohomology of $\mathcal{H}$. Connes and Moscovici computed the latter as Gelfand–Fuchs cohomology. This is the subject of the next section.

4. Gelfand–Fuchs Cohomology

Let G be the group of complex analytic transformations of $\mathbb{C}$. G has a unique decomposition $G = G_1 G_2$, where G_1 is the group of affine transformations

$$x \to ax + b, \quad x \in \mathbb{C}, \ a, b \in \mathbb{C} \tag{36}$$

and G_2 is the group of transformations of the form

$$x \to x + o(x). \tag{37}$$

Any element of G is then the composition $k \circ \psi$ for $k \in G_1$, $\psi \in G_2$. Since G_2 is the left quotient of G by G_1, G_1 acts on G_2 from the right: for $k \in G_1$, $\psi \in G_2$, one has $\psi \triangleleft k \in G_2$. Similarly, G_2 acts on G_1 from the left: $\psi \triangleright k \in G_1$.

We remark that G_1 is the crossed product $\mathbb{C} \rtimes Gl(1,\mathbb{C})$. The space $\mathbb{C} \times Gl(1,\mathbb{C})$ is a prototype for the frame bundle F of a flat Riemann surface. This motivates the notation $a = y$, $b = z$ for the coordinates on G_1. Under this identification, the left action of G_2 on G_1 corresponds to the action of G_2 on F: for a holomorphic transformation $\psi \in G_2$, one has

$$z \to \psi(z), \quad y \to \psi'(z)y, \tag{38}$$

with $\psi(0) = 0$, $\psi'(0) = 1$. Furthermore, the vector fields $X, \overline{X}, Y, \overline{Y}$ form a basis of invariant vector fields for the left action of G_1 on itself, i.e. a basis of the (complexified) Lie algebra of G_1. Its dual basis is given by the left-invariant 1-forms (Maurer–Cartan form)

$$\begin{aligned} \omega_{-1} = y^{-1}dz, &\quad \overline{\omega}_{-1} = \overline{y}^{-1}d\overline{z}, \\ \omega_0 = y^{-1}dy, &\quad \overline{\omega}_0 = \overline{y}^{-1}d\overline{y}. \end{aligned} \tag{39}$$

The left action $G_2 \triangleright G_1$ implies a right action of G_2 on forms by pullback. One has in particular, for $\psi \in G_2$,

$$\omega_{-1} \circ \psi = \omega_{-1}, \quad \omega_0 \circ \psi = \omega_0 + y\partial_z \ln \psi' \omega_{-1} \quad \text{and c.c.} \tag{40}$$

Consider now the discrete crossed product $\mathcal{H}_* = C_c^\infty(G_1) \rtimes G_2$, where G_2 acts on $C_c^\infty(G_1)$ by pullback. As a coalgebra, $\mathcal{H}$ is dual to the algebra $\mathcal{H}_*$. One has a natural action of $\mathcal{H}$ on $\mathcal{H}_*$:

$$\begin{aligned} &X.(fU_\psi^*) = X.fU_\psi^*, \quad f \in C_c^\infty(G_1), \psi \in G_2, \\ &\delta_n(fU_\psi^*) = y^n \partial_z^n \ln \psi' \, fU_\psi^*, \end{aligned} \tag{41}$$

and so on with $Y, \overline{X} \dots$. The operators $\delta_n, \overline{\delta_n}$ have in fact an interpretation in terms of coordinates on the group G_2: for $\psi \in G_2$, $\delta_n(\psi)$ is by definition the value of the function $\delta_n(U_\psi^*)U_\psi$ at $1 \in G_1$. For any $k \in G_1$, one has

$$[\delta_n(U_\psi^*)U_\psi](k) = \delta_n(\psi \triangleleft k). \tag{42}$$

Note that (40) rewrites

$$\omega_0 \circ \psi = \omega_0 + \delta_1(\psi \triangleleft k)\omega_{-1} \quad \text{at } k \in G_1. \tag{43}$$

The Hopf subalgebra of $\mathcal{H}$ generated by $\delta_n, \overline{\delta_n}$, $n \geq 1$, corresponds to the commutative Hopf algebra of functions on G_2 which are *polynomial* in these coordinates.

Let A be the complexification of the formal Lie algebra of G. It coincides with the jets of holomorphic and antiholomorphic vector fields of any order on $\mathbb{C}$:

$$\begin{aligned} &\partial_x, x\partial_x, \dots, \; x^n\partial_x, \dots, \quad x \in \mathbb{C}, \\ &\partial_{\overline{x}}, \overline{x}\partial_{\overline{x}}, \dots, \; \overline{x}^n\partial_{\overline{x}}, \dots. \end{aligned} \tag{44}$$

The Lie bracket between the elements of the above basis is thus

$$\begin{aligned} &[x^n\partial_x, x^m\partial_x] = (m-n)x^{n+m-1}\partial_x \quad \text{and c.c.}, \\ &[x^n\partial_x, \overline{x}^m\partial_{\overline{x}}] = 0. \end{aligned} \tag{45}$$

Define the generator of dilatations $H = x\partial_x + \overline{x}\partial_{\overline{x}}$ and of rotations $J = x\partial_x - \overline{x}\partial_{\overline{x}}$. They fulfill the properties

$$\begin{aligned} [H, x^n\partial_x] &= (n-1)x^n\partial_x; \quad [H, \overline{x}^n\partial_{\overline{x}}] = (n-1)\overline{x}^n\partial_{\overline{x}}, \\ [J, x^n\partial_x] &= (n-1)x^n\partial_x, \quad [J, \overline{x}^n\partial_{\overline{x}}] = -(n-1)\overline{x}^n\partial_{\overline{x}}. \end{aligned} \tag{46}$$

We are interested in the Lie algebra cohomology of A (see [7]). The complex $C^*(A)$ of cochains is the exterior algebra generated by the dual basis $\{\omega^n, \overline{\omega}^n\}_{n\geq -1}$:

$$\begin{aligned} &\omega^n(x^m\partial_x) = \delta^m_{n+1}, \quad \omega^n(\overline{x}^m\partial_{\overline{x}}) = 0, \\ &\overline{\omega}^n(x^m\partial_x) = 0, \qquad \overline{\omega}^n(\overline{x}^m\partial_{\overline{x}}) = \delta^m_{n+1}, \quad \forall n \geq -1, m \geq 0, \end{aligned} \tag{47}$$

and the coboundary operator is uniquely defined by its action on 1-cochains

$$d\omega(X, Y) = -\omega([X, Y]) \quad \forall X, Y \in A. \tag{48}$$

From [5] we know that the *periodic* cyclic cohomology $H^*(\mathcal{H}, SO(2))$ is isomorphic to the relative Lie algebra cohomology $H^*(A, SO(2))$, i.e. the cohomology of the basic subcomplex of cochains on A relative to the Cartan operation (L, i) of J:

$$L_J\omega = (i_J d + d i_J)\omega \quad \forall \omega \in C^*(A). \tag{49}$$

We say that a cochain $\omega \in C^*(A)$ is of weight r if $L_H\omega = -r\omega$. Remark that

$$L_H\omega^n = -n\omega^n, \quad L_H\overline{\omega}^n = -n\overline{\omega}^n \quad \forall n \geq -1, \tag{50}$$

so that $C^*(A)$ is the direct sum, for $r \geq -2$, of the spaces $C^*_r(A)$ of weight r. Since $[H, J] = 0$, $C^*_r(A)$ is stable under the Cartan operation of J and we note $C^*_r(A, SO(2))$ the complex of basic cochains of weight r. Then we have

$$C^*(A, SO(2)) = \bigoplus_{r=-2}^{\infty} C^*_r(A, SO(2)). \tag{51}$$

For any cocycle $\omega \in C^*_r(A, SO(2))$,

$$L_H\omega = d i_H\omega = -r\omega, \tag{52}$$

so that $C^*_r(A, SO(2))$ is acyclic whenever $r \neq 0$. Hence $H^*(A, SO(2))$ is equal to the cohomology of the finite-dimensional subcomplex $C^*_0(A, SO(2))$. The direct computation gives

$$\begin{aligned} &H^0(A, SO(2)) = \mathbb{C} \text{ with representative } 1, \\ &H^2(A, SO(2)) = \mathbb{C} \qquad '' \qquad \omega^{-1}\omega^1, \\ &H^3(A, SO(2)) = \mathbb{C} \qquad '' \qquad (\omega^{-1}\omega^1 - \overline{\omega}^{-1}\overline{\omega}^1)(\omega^0 + \overline{\omega}^0), \\ &H^5(A, SO(2)) = \mathbb{C} \qquad '' \qquad \omega^1\omega^{-1}\overline{\omega}^1\overline{\omega}^{-1}(\omega^0 + \overline{\omega}^0). \end{aligned} \tag{53}$$

The other cohomology groups vanish.

Next we construct a map C from $C^*(A)$ to the bicomplex $(C^{n,m}, d_1, d_2)_{n,m\in\mathbb{Z}}$ of [3] chap. III.2.δ. Let $\Omega^m(G_1)$ be the space m-forms on G_1. $C^{n,m}$ is the space of totally antisymmetric maps $\gamma : {G_2}^{n+1} \to \Omega^m(G_1)$ such that

$$\gamma(g_0 g, \ldots, g_n g) = \gamma(g_0, \ldots, g_n) \circ g, \quad g_i \in G_2, g \in G\ , \tag{54}$$

where $g_i g$ is given by the right action of G on G_2, and G acts on $\Omega^*(G_1)$ by pullback (left action of G on G_1).

The first differential $d_1 : C^{n,m} \to C^{n+1,m}$ is

$$(d_1\gamma)(g_0, \dots, g_{n+1}) = (-)^m \sum_{i=0}^{n+1} (-)^i \gamma(g_0, \dots, \overset{\vee}{g_i}, \dots, g_{n+1}), \tag{55}$$

and $d_2 : C^{n,m} \to C^{n,m+1}$ is just the de Rham coboundary on $\Omega^*(G_1)$:

$$(d_2\gamma)(g_0, \dots, g_n) = d(\gamma(g_o, \dots, g_n)). \tag{56}$$

Of course ${d_1}^2 = {d_2}^2 = d_1 d_2 + d_2 d_1 = 0$. We remark that for $\gamma \in C^{n,m}$, the invariance property (54) implies

$$\gamma(g_0, \dots, g_n) \circ k = \gamma(g_0 \triangleleft k, \dots, g_n \triangleleft k) \quad \forall k \in G_1, \tag{57}$$

in other words the value of $\gamma(g_0, \dots, g_n) \in \Omega^m(G_1)$ at k is deduced from its value at 1.

Let us describe now the construction of C. As a vector space, the Lie algebra A is just the direct sum $\mathbf{G}_1 \oplus \mathbf{G}_2$, $\mathbf{G}_i$ being the (complexified) Lie algebra of G_i. The cochain complex $C^*(A)$ is then the exterior product $\Lambda A^* = \Lambda {\mathbf{G}_1}^* \otimes \Lambda {\mathbf{G}_2}^*$. One identifies ${\mathbf{G}_1}^*$ with the cotangent space $T_1^*(G_1)$ of G_1 at the identity. Since G_2 fixes $1 \in G_1$, there is a right action of G_2 on $\Lambda {\mathbf{G}_1}^*$ by pullback. The basis $\{\omega^{-1}, \omega^0, \overline{\omega}^{-1}, \overline{\omega}^0\}$ of ${\mathbf{G}_1}^*$ is represented by left-invariant one-forms on G_1 through the identification

$$\begin{aligned} \omega^{-1} &\to -\omega_{-1} = -y^{-1}dz, & \overline{\omega}^{-1} &\to -\overline{\omega}_{-1} = -\overline{y}^{-1}d\overline{z}, \\ \omega^0 &\to -\omega^0 = -y^{-1}dy, & \overline{\omega}^0 &\to -\overline{\omega}^0 = -\overline{y}^{-1}d\overline{y}, \end{aligned} \tag{58}$$

and the right action of $\psi \in G_2$ reads (cf. (40))

$$\omega^{-1} \cdot \psi = \omega^{-1}, \quad \omega^0 \cdot \psi = \omega^0 + \delta_1(\psi)\omega^{-1}. \tag{59}$$

Next, we view a cochain $\omega \in C^*(A)$ as a cochain of the Lie algebra of G_2 with coefficients in the right G_2-module $\Lambda {\mathbf{G}_1}^*$. It is represented by a $\Lambda {\mathbf{G}_1}^*$-valued right-invariant form μ on G_2. Then $C(\omega) \in C^{*,*}$ evaluated on $(g_0, \dots, g_n) \in {G_2}^{n+1}$ is a differential form on G_1 whose value at $1 \in G_1$ is

$$C(\omega)(g_0, \dots, g_n) = \int_{\Delta(g_0, \dots, g_n)} \mu \quad \in \Lambda T_1^*(G_1), \tag{60}$$

where $\Delta(g_0, \dots, g_n)$ is the affine simplex in the coordinates $\delta_i, \overline{\delta_i}$, with vertices $(g_0, \dots, g_n)$. Let $\{\rho_j\}$ be a basis of left-invariant forms on G_1. Then

$$C(\omega)(g_0, \dots, g_n) = \sum_j p_j(g_0, \dots, g_n)\rho_j \quad \text{at } 1 \in G_1, \tag{61}$$

where $p_j(g_0, \ldots, g_n)$ are polynomials in the coordinates $\delta_i, \overline{\delta_i}$. The invariance property (54) enables us to compute the value of $C(\omega)(g_0, \ldots, g_n)$ at any $k \in G_1$,

$$C(\omega)(g_0, \ldots, g_n)(k) = \sum_j p_j(g_0 \triangleleft k, \ldots, g_n \triangleleft k)\rho_j \tag{62}$$

because $\rho_j \circ k = \rho_j$.

Connes and Moscovici showed in [5] that C is a morphism from $C^*(A, d)$ to the bicomplex $(C^{n,m}, d_1, d_2)_{n,m\in\mathbb{Z}}$. In the relative case, it restricts to a morphism from $C^*(A, SO(2), d)$ to the subcomplex $(C^{n,m}_{\text{bas.}}, d_1, d_2)$ of antisymmetric cochains on G_2 with values in the *basic* de Rham cohomology $\Omega^*(P) = \Omega^*(G_1/SO(2))$.

It remains to compute the image of $H^*(A, SO(2))$ by C. We restrict ourselves to even cocycles, i.e. the unit $1 \in H^0(A, SO(2))$ and the first Chern class $c_1 \in H^2(A, SO(2))$, defined as the class

$$c_1 = [2\omega^{-1}\omega^1]. \tag{63}$$

One has $C(1) \in C^{0,0}_{\text{bas.}}$. The immediate result is

$$C(1)(g_0) = 1, \quad g_0 \in G_2. \tag{64}$$

For the first Chern class, we must transform c_1 into a right-invariant form on G_2 with values in $\Lambda T_1^*(G_1)$. We already know that ω^{-1} is represented by $-\omega_{-1} = -y^{-1}dz$, which satisfies $\omega_{-1} \circ \psi = \omega_{-1}$, $\forall\, \psi \in G_2$. Next, the Taylor expansion of an element $\psi \in G_2$ can be expressed in the coordinates δ_n thanks to the obvious formula

$$\ln \psi'(x) = \sum_{n=1}^{\infty} \frac{1}{n!}\delta_n(\psi)x^n, \quad \forall\, x \in \mathbb{C}. \tag{65}$$

One finds:

$$\psi(x) = x + \frac{1}{2}\delta_1(\psi)x^2 + \frac{1}{3!}(\delta_2(\psi) + \delta_1(\psi)^2)x^3 + O(x^4). \tag{66}$$

It shows that the cochain $\omega^1 \in C^*(A)$ is represented by the right-invariant 1-form $\frac{1}{2}d\delta_1$ on G_2. Thus at $1 \in G_1$, $C(c_1) \in C^{1,1}_{\text{bas.}}$ is given by

$$\begin{aligned} C(c_1)(g_0, g_1) &= \int_{\Delta(g_0,g_1)} -\omega_{-1}d\delta_1 \\ &= -\omega_{-1}(\delta_1(g_1) - \delta_1(g_0)) \quad g_i \in G_2, \end{aligned} \tag{67}$$

and at $k \in G_1$, the 1-form $C(c_1)(g_0, g_1)$ is

$$C(c_1)(g_0, g_1) = -\omega_{-1}(\delta_1(g_1 \triangleleft k) - \delta_1(g_0 \triangleleft k)). \tag{68}$$

Since $\omega_{-1} = y^{-1}dz$ and $\delta_1(g \triangleleft k) = y\partial_z \ln g'(z)$, z and y being the coordinates of k, one has explicitly

$$C(c_1)(g_0, g_1) = -dz(\partial_z \ln g_1{}'(z) - \partial_z \ln g_0{}'(z)). \tag{69}$$

It is a basic form on G_1 relative to $SO(2)$, which then descends to a form on $P = G_1/SO(2)$ as expected.

The last step is to use the map Φ of [3, Theorem 14, p. 220] from $(C^{n,m}, d_1, d_2)$ to the (b, B) bicomplex of the discrete crossed product $C_c^\infty(P) \rtimes G_2$. Define the algebra

$$\mathcal{B} = \Omega^*(P) \hat{\otimes} \Lambda\mathbb{C}(G_2'), \tag{70}$$

where $\Lambda\mathbb{C}(G_2')$ is the exterior algebra generated by the elements δ_ψ, $\psi \in G_2$, with $\delta_e = 0$ for the identity e of G_2. With the de Rham coboundary d of $\Omega^*(P)$, $\mathcal{B}$ is a differential algebra. Now form the crossed product $\mathcal{B} \rtimes G_2$, with multiplication rules

$$\begin{aligned} U_\psi^* \alpha U_\psi &= \alpha \circ \psi, & \alpha \in \Omega^*(P), \psi \in G_2, \\ U_{\psi_1}^* \delta_{\psi_2} U_{\psi_1} &= \delta_{\psi_2 \circ \psi_1} - \delta_{\psi_1}, & \psi_i \in G_2. \end{aligned} \tag{71}$$

Endow $\mathcal{B} \rtimes G_2$ with the differential $\tilde{d}$ acting on an element bU_ψ^* as

$$\tilde{d}(bU_\psi^*) = dbU_\psi^* - (-)^{\partial b} b\delta_\psi U_\psi^*, \tag{72}$$

where db comes from the de Rham coboundary of $\Omega^*(P)$. The map

$$\Phi : (C^{*,*}, d_1, d_2) \to (C_c^\infty(P) \rtimes G_2, b, B) \tag{73}$$

is constructed as follows. Let $\gamma \in C_{bas.}^{n,m}$. It yields a linear form $\tilde{\gamma}$ on $\mathcal{B} \rtimes G_2$:

$$\begin{aligned} \tilde{\gamma}(\alpha \otimes \delta_{g_1} \ldots \delta_{g_n}) &= \int_P \alpha \wedge \gamma(1, g_1, \ldots, g_n), \quad \alpha \in \Omega^*(P), g_i \in G_2, \\ \tilde{\gamma}(bU_\psi^*) &= 0 \quad \text{if } \psi \neq 1. \end{aligned} \tag{74}$$

Then $\Phi(\gamma)$ is the following l-cochain on $C_c^\infty(P) \rtimes G_2$, $l = \dim P - m + n$,

$$\begin{aligned} \Phi(\gamma)(x_0, \ldots, x_l) &= \frac{n!}{(l+1)!} \sum_{j=0}^{l} (-)^{j(l-j)} \tilde{\gamma}(\tilde{d}x_{j+1} \ldots \tilde{d}x_l x_0 \tilde{d}x_1 \ldots \tilde{d}x_j), \\ & x_i \in C_c^\infty(P) \rtimes G_2 \subset \mathcal{B} \rtimes G_2. \end{aligned} \tag{75}$$

The essential tool is that Φ is a morphism of bicomplexes:

$$\Phi(d_1\gamma) = b\Phi(\gamma), \quad \Phi(d_2\gamma) = B\Phi(\gamma). \tag{76}$$

Moreover, if $d_1\gamma = d_2\gamma = 0$, $\Phi(\gamma)$ is a cyclic cocycle. This happens in our case. Since P is a 3-dimensional manifold, the image of $C(1)$ under Φ is the cyclic 3-cocycle

$$\Phi(C(1))(x_0, \ldots, x_3) = \int_P x_0 dx_1 \ldots dx_3, \quad x_i \in C_c^\infty(P) \rtimes G_2, \tag{77}$$

where $d(fU_\psi^*) = df U_\psi^*$ for $f \in C_c^\infty(P)$, $\psi \in G_2$, and the integration is extended over $\Omega^*(P) \rtimes G_2$ by setting

$$\int_P \alpha U_\psi^* = 0 \quad \text{if } \psi \neq 1,\ \alpha \in \Omega^*(P). \tag{78}$$

The image of $\gamma = C(c_1)$ is more complicated to compute. One has

$$\tilde{\gamma}(\alpha \otimes \delta_g) = -\int_P \alpha \wedge y^{-1} dz \delta_1(g \triangleleft k), \quad \alpha \in \Omega^2(P), g \in G_2, \tag{79}$$

where $y^{-1}dz\delta_1(g \triangleleft k) = dz\partial_z \ln g'(z)$ is, of course, a 1-form on P. $\Phi(\gamma)$ is the cyclic 3-cocycle

$$\begin{aligned}
&\Phi(\gamma)(f_0U^*_{\psi_0}, \dots, f_3U^*_{\psi_3}) \\
&\qquad = -\tilde{\gamma}(f_0U^*_{\psi_0}df_1U^*_{\psi_1}df_2U^*_{\psi_2}f_3\delta_{\psi_3}U^*_{\psi_3} \\
&\qquad\quad + f_0U^*_{\psi_0}df_1U^*_{\psi_1}f_2\delta_{\psi_2}U^*_{\psi_2}df_3U^*_{\psi_3} \\
&\qquad\quad + f_0U^*_{\psi_0}f_1\delta_{\psi_1}U^*_{\psi_1}df_2U^*_{\psi_2}df_3U^*_{\psi_3}) \\
&\qquad = \tilde{\gamma}(f_0\,(df_1\circ\psi_0)\,(df_2\circ\psi_1\psi_0)\,(f_3\circ\psi_2\psi_1\psi_0)\delta_{\psi_2\psi_1\psi_0} \\
&\qquad\quad + f_0\,(df_1\circ\psi_0)\,(f_2\circ\psi_1\psi_0)\,(df_3\circ\psi_2\psi_1\psi_0)(\delta_{\psi_2\psi_1\psi_0}-\delta_{\psi_1\psi_0}) \\
&\qquad\quad - f_0\,(f_1\circ\psi_0)\,(df_2\circ\psi_1\psi_0)\,(df_3\circ\psi_2\psi_1\psi_0)(\delta_{\psi_1\psi_0}-\delta_{\psi_0})),
\end{aligned} \tag{80}$$

upon assuming that $\psi_3\psi_2\psi_1\psi_0 = \mathrm{Id}$. Using the relation

$$\delta_1(\psi \triangleleft k) = [\delta_1(U^*_\psi)U_\psi](k), \quad \forall\, k \in G_1, \psi \in G_2, \tag{81}$$

the computation gives

$$\Phi(\gamma)(x_0, \dots, x_3) = \int_P x_0(dx_1dx_2\delta_1(x_3) + dx_1\delta_1(x_2)dx_3 + \delta_1(x_1)dx_2dx_3)y^{-1}dz. \tag{82}$$

Now recall that P has an invariant volume form $dv_1 = e^{2r}dzd\bar{z}dr$. The differential df of a function on P makes use of the horizontal $X = y\partial_z$, $\overline{X} = \overline{y}\partial_{\bar{z}}$ and vertical $Y + \overline{Y} = -\partial_r$ vector fields:

$$df = y^{-1}dzX.f + \overline{y}^{-1}d\bar{z}\overline{X}.f - dr(Y + \overline{Y}).f. \tag{83}$$

Then using the relations (40) one sees that $\Phi(C(c_1))$ is a sum of terms involving the Hopf algebra

$$\Phi(C(c_1))(x_0, \dots, x_3) = \sum_i \int_P x_0h_1^i(x_1)\dots h_3^i(x_3)dv_1, \tag{84}$$

where the sum $\sum_i h_1^i \otimes h_2^i \otimes h_3^i$ is a cyclic 3-cocycle of $\mathcal{H}$ relative to $SO(2)$. This follows from the existence of a characteristic map

$$HC^*(\mathcal{H}, SO(2)) \to HC^*(C_c^\infty(P) \rtimes G_2) \tag{85}$$

and the duality between $\mathcal{H}$ and $\mathcal{H}_* = C_c^\infty(G_1) \rtimes G_2$ (cf. [5]).

Returning to the initial situation, where F is the frame bundle of a flat Riemann surface Σ, and $P = F/SO(2)$ the bundle of metrics, the above computation shows that the cyclic 3-cocycle on $\mathcal{A}_1 = C_c^\infty(P) \rtimes \Gamma$,

$$[c_1](a_0, \dots, a_3) = \sum_i \int_P a_0h_1^i(a_1)\dots h_3^i(a_3)dv_1, \quad a_i \in \mathcal{A}_1, \tag{86}$$

is the image of $C(c_1)$ by the characteristic map $HC^*(\mathcal{H}, SO(2)) \to HC^*(\mathcal{A}_1)$. Also the fundamental class

$$[P](a_0, \ldots, a_3) = \int_P a_0 da_1 da_2 da_3 \tag{87}$$

is in the range of the characteristic map.

Since Connes and Moscovici showed that the Gelfand–Fuchs cohomology $H^*(A, SO(2))$ is isomorphic to the periodic cyclic cohomology of $\mathcal{H}$, we have completely determined the odd part of the range of the characteristic map. We can summarize the result in the following

Proposition 1. *Under the characteristic map*

$$H^*(A, SO(2)) \simeq H^*(\mathcal{H}, SO(2)) \to H^*(\mathcal{A}_1), \tag{88}$$

the unit $1 \in H^0(A, SO(2))$ *maps to the fundamental class* $[P]$ *represented by the cyclic 3-cocycle,*

$$[P](a_0, \ldots, a_3) = \int_P a_0 da_1 da_2 da_3, \quad a_i \in \mathcal{A}_1, \tag{89}$$

and the first Chern class $c_1 \in H^2(A, SO(2))$ *gives the cocycle* $[c_1] \in HC^3(\mathcal{A}_1)$*:*

$$[c_1](a_0, \ldots, a_3) = \int_P a_0 (da_1 da_2 \delta_1(a_3) + da_1 \delta_1(a_2) da_3 + \delta_1(a_1) da_2 da_3) y^{-1} dz. \tag{90}$$

In Sect. 2 we considered an odd K-cycle on $C_0(P \times \mathbb{R}^2) \rtimes \Gamma$ represented by a differential operator Q', which is equivalent, up to Bott periodicity, to an odd K-cycle on $C_0(P) \rtimes \Gamma$. Q' is a matrix-valued polynomial in the vector fields $X, \overline{X}, Y + \overline{Y}$ and the partial derivatives along the two directions of $\mathbb{R}^2$. Its Chern character is the cup product

$$\mathrm{ch}_*(Q') = \varphi \# [\mathbb{R}^2] \tag{91}$$

of a cyclic cocycle $\varphi \in HC^{\mathrm{odd}}(C_c^\infty(P) \rtimes \Gamma)$ by the fundamental class of $\mathbb{R}^2$. The index theorem of Connes and Moscovici states that φ is in the range of the characteristic map (we have to assume that the action of Γ on Σ has no fixed point). Hence it is a linear combination of the characteristic classes $[P]$ and $[c_1]$. We shall determine the coefficients by using the classical Riemann–Roch theorem.

5. A Riemann–Roch Theorem for Crossed Products

We shall first use the Thom isomorphism in K-theory [1],

$$K_i(C_0(\Sigma) \rtimes \Gamma) \to K_{i+1}(C_0(P) \rtimes \Gamma) \tag{92}$$

to descend the characteristic classes $[P]$ and $[c_1]$ down to the cyclic cohomology of $C_c^\infty(\Sigma) \rtimes \Gamma$. Recall that $C_0(P) \rtimes \Gamma$ is just the crossed product of $C_0(\Sigma) \rtimes \Gamma$ by the modular automorphism group σ of the associated von Neumann algebra

$$C_0(P) \rtimes \Gamma = (C_0(\Sigma) \rtimes \Gamma) \rtimes_\sigma \mathbb{R}. \tag{93}$$

By homotopy we can deform σ continuously into the trivial action. For $\lambda \in [0,1]$, let $\sigma^\lambda_t = \sigma_{\lambda t}$, $\forall t \in \mathbb{R}$. Then $\sigma^1 = \sigma$, $\sigma^0 = \mathrm{Id}$ and

$$(C_0(\Sigma) \rtimes \Gamma) \rtimes_{\mathrm{Id}} \mathbb{R} = C_0(\Sigma) \rtimes \Gamma \otimes C_0(\mathbb{R}). \tag{94}$$

Next, the coordinate system $(z, \bar{z})$ of Σ gives a smooth volume form $\frac{dz \wedge d\bar{z}}{2i}$ together with a representative of σ, whose action on the subalgebra $C^\infty_c(\Sigma) \rtimes \Gamma$ is

$$\sigma_t(fU^*_\psi) = f|\psi'|^{2it} U^*_\psi, \quad f \in C^\infty_c(\Sigma), \psi \in \Gamma, \tag{95}$$

and accordingly

$$\sigma^\lambda_t(fU^*_\psi) = f|\psi'|^{2i\lambda t} U^*_\psi. \tag{96}$$

We remark that the algebra $(C_0(\Sigma) \rtimes \Gamma) \rtimes_{\sigma^\lambda} \mathbb{R}$ is equal to the crossed product $C_0(P) \rtimes_\lambda \Gamma$ obtained from the following deformed action of Γ on P:

$$\begin{aligned} z &\to \psi(z), \quad \bar{z} \to \overline{\psi(z)}, \\ r &\to r - \frac{1}{2}\lambda \ln |\psi'(z)|^2, \quad \psi \in \Gamma. \end{aligned} \tag{97}$$

Hence for any $\lambda \in [0,1]$, one has a Thom isomorphism

$$\Phi^\lambda : K_0(C_0(\Sigma) \rtimes \Gamma) \to K_1(C_0(P) \rtimes_\lambda \Gamma), \tag{98}$$

and Φ^0 is just the connecting map $K_0(C_0(\Sigma) \rtimes \Gamma) \to K_1(S(C_0(\Sigma) \rtimes \Gamma))$. We introduce also the family $\{[P]^\lambda\}_{\lambda \in [0,1]}$ of cyclic cocycles

$$[P]^\lambda(a^\lambda_0, \ldots, a^\lambda_3) = \int_P a^\lambda_0 da^\lambda_1 \ldots da^\lambda_3, \quad \forall a^\lambda_i \in C^\infty_c(P) \rtimes_\lambda \Gamma. \tag{99}$$

One has $[P]^1 = [P]$ and $[P]^0 = [\Sigma]\#[\mathbb{R}] \in (C^\infty_c(\Sigma) \rtimes \Gamma) \otimes C^\infty_c(\mathbb{R})$, where

$$[\Sigma](a_0, a_1, a_2) = \int_\Sigma a_0 da_1 da_2 \quad \forall a_i \in C^\infty_c(\Sigma) \rtimes \Gamma. \tag{100}$$

Moreover for any element $[e] \in K_0(C_0(\Sigma) \rtimes \Gamma)$ such that $\Phi^\lambda([e])$ is in the domain of definition of $[P]^\lambda$, the pairing

$$\langle \Phi^\lambda([e]), [P]^\lambda \rangle \tag{101}$$

depends continuously upon λ. Next for any $\lambda \in]0,1]$, consider the vertical diffeomorphism of P whose action on the coordinates $(z, \bar{z}, r)$ reads

$$\tilde{\lambda}(z) = z, \quad \tilde{\lambda}(\bar{z}) = \bar{z}, \quad \tilde{\lambda}(r) = \lambda r. \tag{102}$$

Thus for $\lambda \neq 0$ one has an algebra isomorphism

$$\chi_\lambda : C^\infty_c(P) \rtimes_\lambda \Gamma \to C^\infty_c(P) \rtimes \Gamma \tag{103}$$

by setting

$$\chi_\lambda(fU^*_\psi) = f \circ \tilde{\lambda}\, U^*_\psi \quad \forall f \in C^\infty_c(P), \psi \in \Gamma. \tag{104}$$

For any $\lambda \neq 0$,

$$(\chi_\lambda)_* \circ \Phi^\lambda = \Phi^1, \tag{105}$$

$$(\chi_\lambda)^*[P]^1 = [P]^\lambda. \tag{106}$$

Equation (105) comes from the unicity of the Thom map (cf. [1]), and (106) is obvious. Thus $\langle \Phi^\lambda([e]), [P]^\lambda \rangle$ is constant for $\lambda \neq 0$, and by continuity at 0,

$$\langle \Phi^1([e]), [P] \rangle = \langle [e], [\Sigma] \rangle. \tag{107}$$

This shows that the image of $[P]$ by Thom isomorphism is the cyclic 2-cocycle $[\Sigma]$ corresponding to the fundamental class of Σ. In exactly the same way we show that the image of $[c_1]$ is the cyclic 2-cocycle τ defined, for $a_i = f_i U^*_{\psi_i} \in C^\infty_c(\Sigma) \rtimes \Gamma$, by

$$\tau(a_0, a_1, a_2) = \int_\Sigma a_0 (da_1 \partial \ln \psi_2' \, a_2 + \partial \ln \psi_1' \, a_1 da_2), \tag{108}$$

with $\partial = dz\partial_z$. Note that in the decomposition of the differential on Σ, $d = \partial + \overline{\partial}$, both ∂ and $\overline{\partial}$ commute with the pullbacks by the conformal transformations $\psi \in \Gamma$.

So far we have considered a *flat* Riemann surface and the constructions we made were relative to a coordinate system $(z, \overline{z})$. We shall now remove this unpleasant feature by using the Morita equivalence [5]. In order to understand the general situation, let us first treat the particular case of the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$. We consider an open covering of the sphere by two planes: $S^2 = U_1 \cup U_2$, $U_1 = \mathbb{C}$, $U_2 = \mathbb{C}$, together with the glueing function g:

$$\begin{aligned} g : U_1 \backslash \{0\} &\to U_2 \backslash \{0\}, \\ z &\mapsto \frac{1}{z}. \end{aligned} \tag{109}$$

The pseudogroup of conformal transformations Γ_0 generated by $\{U_g^*, U_g\}$ acts on the disjoint union $\Sigma = U_1 \amalg U_2$, which is flat. Then S^2 is described by the groupoid $\Sigma \rtimes \Gamma_0$. If Γ is a pseudogroup of local transformations of S^2, there exists a pseudogroup Γ' containing Γ_0, acting on Σ and such that the crossed product $C^\infty(S^2) \rtimes \Gamma$ is Morita equivalent to $C^\infty_c(\Sigma) \rtimes \Gamma'$. The latter splits into four parts: it is the direct sum, for $i, j = 1, 2$, of elements of the form $f_{ij} U^*_{\psi_{ij}}$ with

$$\psi_{ij} : U_i \to U_j \quad \text{and} \quad \operatorname{supp} f_{ij} \subset \operatorname{Dom} \psi_{ij}. \tag{110}$$

For convenience, we adopt a matricial notation for any generic element $b \in C^\infty_c(\Sigma) \rtimes \Gamma'$:

$$b = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \quad b_{ij} = f_{ij} U^*_{\psi_{ij}}. \tag{111}$$

Now the Morita equivalence is explicitly realized through the following idempotent $e \in C^\infty_c(\Sigma) \rtimes \Gamma'$:

$$e = \begin{pmatrix} {\rho_1}^2 & \rho_1 \rho_2 U_g^* \\ U_g \rho_2 \rho_1 & U_g {\rho_2}^2 U_g^* \end{pmatrix}, \quad e^2 = e, \tag{112}$$

where $\{\rho_i\}_{i=1,2}$ is a partition of unity relative to the covering $\{U_i\}$:

$$\rho_1 \in C_c^\infty(U_1), \quad {\rho_1}^2 + {\rho_2}^2 = 1 \text{ on } S^2 = U_1 \cup \{\infty\}. \tag{113}$$

The reduction of $C_c^\infty(\Sigma) \rtimes \Gamma'$ by e is the subalgebra

$$(C_c^\infty(\Sigma) \rtimes \Gamma')_e = \{b \in C_c^\infty(\Sigma) \rtimes \Gamma' / b = be = eb\}. \tag{114}$$

Its elements are of the form

$$ebe = \begin{pmatrix} \rho_1 c \rho_1 & \rho_1 c \rho_2 U_g^* \\ U_g \rho_2 c \rho_1 & U_g \rho_2 c \rho_2 U_g^* \end{pmatrix} \tag{115}$$

with $c = \rho_1 b_{11} \rho_1 + \rho_2 U_g^* b_{21} \rho_1 + \rho_1 b_{12} U_g \rho_2 + \rho_2 U_g^* b_{22} U_g \rho_2$. Then c can be considered as an element of $C^\infty(S^2) \rtimes \Gamma$ under the identification $S^2 = U_1 \cup \{\infty\}$. $(C_c^\infty(\Sigma) \rtimes \Gamma')_e$ and $C^\infty(S^2) \rtimes \Gamma$ are isomorphic through the map

$$\begin{aligned} \theta : C^\infty(S^2) \rtimes \Gamma &\longrightarrow (C_c^\infty(\Sigma) \rtimes \Gamma')_e \\ a &\longmapsto \begin{pmatrix} \rho_1 a \rho_1 & \rho_1 a \rho_2 U_g^* \\ U_g \rho_2 a \rho_1 & U_g \rho_2 a \rho_2 U_g^* \end{pmatrix}. \end{aligned} \tag{116}$$

We are ready to compute the pullbacks of $[\Sigma]$ and $\tau \in HC^2(C_c^\infty(\Sigma) \rtimes \Gamma')$ by θ. This yields the following cyclic 2-cocycles on $C^\infty(S^2) \rtimes \Gamma$:

$$\begin{aligned} \theta^*[\Sigma] &= [S^2], \\ (\theta^*\tau)(a_0, a_1, a_2) &= \int_{S^2} a_0 \Big(da_1 (\partial \ln \psi_2'\, a_2 + [a_2, {\rho_2}^2 \partial \ln g']) \\ &\quad + (\partial \ln \psi_1' a_1 + [a_1, {\rho_2}^2 \partial \ln g']) \Big) \\ &\quad - \int_{S^2} a_2 a_0 a_1 d({\rho_2}^2) \partial \ln g', \end{aligned} \tag{117}$$

with $a_i = f_i U_{\psi_i}^* \in C^\infty(S^2) \rtimes \Gamma$. In formula (117), $S^2 = U_1 \cup \{\infty\}$ is gifted with the coordinate chart $(z, \bar{z})$ of U_1, which makes sense to $\psi_i'(z) = \partial_z \psi_i(z)$ and $g'(z) = \partial_z g(z) = -1/z^2$, but gives singular expressions at 0 and ∞. We can overcome this difficulty by introducing a smooth volume form $\nu = \rho(z, \bar{z}) \frac{dz \wedge d\bar{z}}{2i}$ on S^2. The associated modular automorphism group σ^ν leaves $C^\infty(S^2) \rtimes \Gamma$ globally invariant and is expressed in the coordinates $(z, \bar{z})$ by

$$\sigma_t^\nu(f U_\psi^*) = \left(\frac{\nu \circ \psi}{\nu} \right)^{it} f U_\psi^* = \left(\frac{\rho \circ \psi}{\rho} |\partial_z \psi|^2 \right)^{it} f U_\psi^*, \quad \forall t \in \mathbb{R}. \tag{118}$$

Define the derivation δ^ν on $C^\infty(S^2) \rtimes \Gamma$,

$$\begin{aligned} \delta^\nu(f U_\psi^*) &\equiv -i [\partial, \frac{d}{dt} \sigma_t^\nu](f U_\psi^*)|_{t=0} \\ &= [\partial, \ln \left(\frac{\rho \circ \psi}{\rho} |\partial_z \psi|^2 \right)](f U_\psi^*) && (119) \\ &= \partial \ln \psi' \, f U_\psi^* - [\partial \ln \rho, \, f U_\psi^*]. && (120) \end{aligned}$$

One has

$$\partial \ln \psi' \, fU_\psi^* + [fU_\psi^*, {\rho_2}^2 \partial \ln g'] = \delta^\nu(fU_\psi^*) + [\partial \ln \rho - {\rho_2}^2 \partial \ln g', fU_\psi^*], \quad (121)$$

where the 1-form $\omega = \partial \ln \rho - {\rho_2}^2 \partial \ln g'$ is globally defined, nowhere singular on S^2. Let $R^\nu = \partial\bar{\partial} \ln \rho$ be the curvature 2-form associated to the Kähler metric $\rho dz \otimes d\bar{z}$. One has the commutation rule

$$(\bar{\partial}\delta^\nu + \delta^\nu \bar{\partial})a = [R^\nu, a] \quad \forall\, a \in C^\infty(S^2) \rtimes \Gamma. \quad (122)$$

Simple algebraic manipulations show that the following 2-cochain:

$$\tau^\nu(a_0, a_1, a_2) = \int_{S^2} a_0(da_1\delta^\nu a_2 + \delta^\nu a_1 da_2) + \int_{S^2} a_2 a_0 a_1 R^\nu \quad (123)$$

is a cyclic cocycle. Moreover, τ^ν is cohomologous to $\theta^*\tau$. To see this, let φ be the cyclic 1-cochain

$$\varphi(a_0, a_1) = \int_{S^2} (a_0 da_1 - a_1 da_0)\omega. \quad (124)$$

Then for all $a_i \in C^\infty(S^2) \rtimes \Gamma$,

$$\begin{aligned}(\tau^\nu - \theta^*\tau)(a_0, a_1, a_2) &= -\int_{S^2} (a_0 da_1 a_2 + a_2 da_0 a_1 + a_1 da_2 a_0)\omega \\ &= b\varphi(a_0, a_1, a_2).\end{aligned} \quad (125)$$

It is clear now that the construction of characteristic classes for an arbitrary (non flat) Riemann surface Σ follows exactly the same steps as in the above example. Using an open cover with partition of unity, one gets the desired cyclic cocycles by pullback. Choose a smooth measure ν on Σ, then the associated modular group is

$$\sigma_t^\nu(fU_\psi^*) = \left(\frac{\nu \circ \psi}{\nu}\right)^{it} fU_\psi^*, \quad fU_\psi^* \in C_c^\infty(\Sigma) \rtimes \Gamma. \quad (126)$$

The corresponding derivation

$$D^\nu(fU_\psi^*) = \ln\left(\frac{\nu \circ \psi}{\nu}\right) fU_\psi^* \quad (127)$$

allows one to define the noncommutative differential

$$\delta^\nu = [\partial, D^\nu]. \quad (128)$$

Then the characteristic classes of the groupoid $\Sigma \rtimes \Gamma$ are given by $[\Sigma]$ and $[\tau^\nu] \in HC^2(C_c^\infty(\Sigma) \rtimes \Gamma)$, where τ^ν is given by Eq. (123) with S^2 replaced by Σ.

In the case $\Gamma = \mathrm{Id}$, the crossed product reduces to the commutative algebra $C_c^\infty(\Sigma)$ for which $(\delta^\nu = 0)$

$$\tau^\nu(a_0, a_1, a_2) = \int_\Sigma a_0 a_1 a_2 R^\nu \quad (129)$$

is just the image of the cyclic 0-cocycle

$$\tau_0^\nu(a) = \int_\Sigma a R^\nu \tag{130}$$

by the suspension map in cyclic cohomology

$$S : HC^*(C_c^\infty(\Sigma)) \to HC^{*+2}(C_c^\infty(\Sigma)).$$

Thus the periodic cyclic cohomology class of τ^ν corresponds in de Rham homology to the cap product

$$\frac{1}{2\pi i}[\tau^\nu] = c_1(\kappa) \cap [\Sigma] \quad \in H_0(\Sigma) \tag{131}$$

of the first Chern class of the holomorphic tangent bundle κ by the fundamental class. This motivates the following definition:

Definition 2. *Let Σ be a Riemann surface without boundary and Γ a discrete pseudogroup acting on Σ by local conformal transformations. Let ν be a smooth volume form on Σ, and σ^ν the associated modular automorphism group leaving $C_c^\infty(\Sigma) \rtimes \Gamma$ globally invariant. Then the Euler class $e(\Sigma \rtimes \Gamma)$ is the class of the following cyclic 2-cocycle on $C_c^\infty(\Sigma) \rtimes \Gamma$*

$$\frac{1}{2\pi i}\tau^\nu(a_0, a_1, a_2) = \frac{1}{2\pi i}\int_\Sigma (a_2 a_0 a_1 R^\nu + a_0(da_1\delta^\nu a_2 + \delta^\nu a_1 da_2)), \tag{132}$$

where δ^ν is the derivation $-i[\partial, \frac{d}{dt}\sigma_t^\nu|_{t=0}]$, and R^ν is the curvature of the Kähler metric determined by ν and the complex structure of Σ. Moreover, this cohomology class is independent of ν.

Now if $\Gamma = \mathrm{Id}$, the operator Q of Sect. 2 defines an element of the K-homology of $\Sigma \times \mathbb{R}^2$. It corresponds to the tensor product of the classical Dolbeault complex $[\bar\partial]$ of Σ by the signature complex $[\sigma]$ of the fiber $\mathbb{R}^2$, so that its Chern character in de Rham homology is the cup product

$$\begin{aligned} \mathrm{ch}_*(Q) &= \mathrm{ch}_*([\bar\partial])\#\mathrm{ch}_*([\sigma]) \\ &= ([\Sigma] + \frac{1}{2}c_1(\kappa) \cap [\Sigma])\#2[\mathbb{R}^2] \quad \in H_*(\Sigma \times \mathbb{R}^2) \end{aligned} \tag{133}$$

which yields, by Thom isomorphism, the homology class on Σ

$$2[\Sigma] + c_1(\kappa) \cap [\Sigma] \quad \in H_*(\Sigma). \tag{134}$$

Next for any Γ, we know from the last section that the Chern character of the Dolbeault K-cycle, expressed in the periodic cyclic cohomology of $C_c^\infty(\Sigma) \rtimes \Gamma$, is a linear combination of $[\Sigma]$ and $e(\Sigma \rtimes \Gamma)$. Thus we deduce immediately the following generalisation of the Riemann–Roch theorem:

Theorem 3. *Let Σ be a Riemann surface without boundary and Γ a discrete pseudogroup acting on Σ by local conformal mappings without fixed point. The Chern character of the Dolbeault K-cycle is represented by the following cyclic 2-cocycle on $C_c^\infty(\Sigma) \rtimes \Gamma$:*

$$ch_*(Q) = 2[\Sigma] + e(\Sigma \rtimes \Gamma). \tag{135}$$

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