Commun. Math. Phys. 217, 489 – 502 (2001)

Communications in
Mathematical Physics
© Springer-Verlag 2001

The Norm Convergence of the Trotter–Kato Product Formula with Error Bound

Takashi Ichinose[1,⋆], **Hideo Tamura**[2,⋆⋆]

[1] Department of Mathematics, Faculty of Science, Kanazawa University, Kanazawa, 920–1192, Japan.
E-mail: ichinose@kappa.s.kanazawa-u.ac.jp
[2] Department of Mathematics, Faculty of Science, Okayama University, Okayama, 700–8530, Japan.
E-mail: tamura@math.okayama-u.ac.jp

Received: 26 June 2000 / Accepted: 21 September 2000

Abstract: The norm convergence of the Trotter–Kato product formula with error bound is shown for the semigroup generated by that operator sum of two nonnegative selfadjoint operators A and B which is selfadjoint.

1. Introduction

If A and B are selfadjoint operators bounded below in a Hilbert space $\mathcal{H}$ with domains $D[A]$ and $D[B]$ and if their sum $A + B$ is essentially selfadjoint on $D[A] \cap D[B]$, then the exponential product formula

$$\lim_{n\to\infty} (e^{-tB/2n} e^{-tA/n} e^{-tB/2n})^n = \lim_{n\to\infty} (e^{-tA/n} e^{-tB/n})^n = e^{-tC} \tag{1.1}$$

holds in strong operator topology, where C is the closure of $A + B$. The convergence in (1.1) is uniform on each compact t-interval in the closed half line $[0, \infty)$. This is the celebrated result by Trotter [26]. It was extended by Kato [15] to the case for the form sum C of two arbitrary nonnegative selfadjoint operators A and B.

The aim of the present paper is to prove that (1.1) holds even in operator norm, uniformly on each compact t-interval in the open half line $(0, \infty)$, together with an error bound of order $O(n^{-1/2})$, when the sum $C := A + B$ is selfadjoint on $D[C] = D[A] \cap D[B]$.

To state our theorem, consider real-valued, Borel measurable functions f on $[0, \infty)$ satisfying

$$0 \le f(s) \le 1, \quad f(0) = 1, \quad f'(0) = -1. \tag{1.2}$$

⋆ Partially supported by the Grant-in-Aid for Scientific Research (B) No. 11440040, Japan Society for the Promotion of Science.

⋆⋆ Partially supported by the Grant-in-Aid for Scientific Research (B) No. 11440056, Japan Society for the Promotion of Science.

Some examples of functions satisfying (1.2) are

$$f(s) = e^{-s}, \quad f(s) = (1 + k^{-1}s)^{-k}, \quad k > 0. \tag{1.3}$$

In fact, it was also for $f(tA)$, $g(tB)$ in place of e^{-tA}, e^{-tB} with f and g being the functions satisfying (1.2) that Kato [15] proved the product formula (1.1) in strong operator topology.

We are interested in those functions f which satisfy (1.2) and further that for every small $\varepsilon > 0$ there exists a positive constant $\delta = \delta(\varepsilon) < 1$ such that

$$f(s) \le 1 - \delta(\varepsilon), \quad s \ge \varepsilon, \tag{1.4}$$

and that for some fixed constant κ with $1 < \kappa \le 2$,

$$[f]_\kappa := \sup_{s>0} \frac{|f(s) - 1 + s|}{s^\kappa} < \infty. \tag{1.5}$$

A function $f(s)$ satisfying (1.2) has property (1.4), if it is non-increasing. Of course, the functions in (1.3) have properties (1.4) and (1.5).

Theorem. *Let f and g be functions having properties (1.4) and (1.5) with $\kappa \ge 3/2$ as well as (1.2). If A and B are nonnegative selfadjoint operators in a Hilbert space $\mathcal{H}$ with domains $D[A]$ and $D[B]$ such that the operator sum $C := A + B$ is selfadjoint on $D[C] = D[A] \cap D[B]$, then it holds in operator norm that*

$$\begin{aligned} \left\| [g(tB/2n)f(tA/n)g(tB/2n)]^n - e^{-tC} \right\| &= O(n^{-1/2}), \\ \left\| [f(tA/n)g(tB/n)]^n - e^{-tC} \right\| &= O(n^{-1/2}), \quad n \to \infty. \end{aligned} \tag{1.6}$$

The convergence is uniform on each compact t-interval in the open half line $(0, \infty)$ and further, if C is strictly positive, i.e. $C \ge \eta$ for some constant $\eta > 0$, uniform on the closed half line $[T, \infty)$ for every fixed $T > 0$.

The first original result of such a norm convergence of the Trotter–Kato product formula (1.1) was proved by Rogava [21] under an additional condition that A is strictly positive and B is A-bounded, with error bound of order $O(n^{-1/2} \log n)$. The next is a result by Helffer [6] for the Schrödinger operators $H = H_0 + V \equiv -\frac{1}{2}\Delta + V(x)$ with C^∞ nonnegative potentials $V(x)$, roughly speaking, growing at most of order $O(|x|^2)$ for large $|x|$ with error bound of order $O(n^{-1})$. Each of these two results is independent of and does not cover the other. Then under some stronger or more general conditions, several further results are obtained.

As for the abstract case, a better error bound $O(n^{-1} \log n)$ than Rogava's is obtained by Ichinose–Tamura [13] (cf. [11]) when B is A^α-bounded for some $0 < \alpha < 1$, even though the $B = B(t)$ may be t-dependent, and by Neidhardt–Zagrebnov [16, 17] (cf. [18, 19]) when B is A-bounded with relative bound less than 1.

As for the Schrödinger operators, a different proof to Helffer's result was given by Dia–Schatzman [3]. Further, more general results were proved for continuous nonnegative potentials $V(x)$, roughly speaking, growing of order $O(|x|^\rho)$ for large $|x|$ with $\rho > 0$, together with error bounds dependent on the power ρ (for instance, of order $O(n^{-2/\rho})$, if $\rho \ge 2$), by Ichinose–Takanobu [7, 8], Doumeki–Ichinose–Tamura [4], Ichinose–Tamura [12], Takanobu [24] and Ichinose–Takanobu [9, 10]. It should be noted (see Guibourg [5], Shen [22, 23]) that in all these cases of the Schrödinger operators the sum $H = H_0 + V$ is selfadjoint on the domain $D[H] = D[H_0] \cap D[V]$.

Thus the present theorem not only extends Rogava's result, but also can include all the results mentioned above. It should be emphasized that the error bound $O(n^{-1/2})$ obtained, in fact, is even better than Rogava's, and than the error bounds (e.g. [8, 4, 12, 9]) known for the Schrödinger operators with potentials $V(x)$ growing of order $O(|x|^\rho)$ when $\rho > 4$.

We note here that unless the sum $A + B$ is selfadjoint on $D[A] \cap D[B]$, the norm convergence of (1.1) does not always hold, even though the sum is essentially selfadjoint there and B is A-form-bounded with relative bound less than 1. This fact has recently been pointed out by Hiroshi Tamura [25] with a counterexample.

To prove our theorem, in Sect. 2, we establish an operator-norm version of Chernoff's theorem (cf. [1, 2]) with error bounds. The theorem is proved in Sect. 3. Section 4 remarks on conditions (1.4) and (1.5).

2. Operator-norm Version of Chernoff's Theorem

To prove the theorem, we shall use the following operator-norm version of Chernoff's theorem (cf. [1, 2]) with error bounds. The case without error bounds was noted by Neidhardt–Zagrebnov [18].

Lemma 2.1. *Let C be a nonnegative selfadjoint operator in a Hilbert space $\mathcal{H}$ and let $\{F(t)\}_{t\geq 0}$ be a family of selfadjoint operators with $0 \leq F(t) \leq 1$. Define $S_t = t^{-1}(1 - F(t))$. Then in the following two assertions, for $0 < \alpha \leq 1$,* (a) *implies* (b) .

(a)

$$\|(1 + S_t)^{-1} - (1 + C)^{-1}\| = O(t^\alpha), \quad t \downarrow 0. \tag{2.1}$$

(b) *For any $\delta > 0$ with $0 < \delta \leq 1$,*

$$\|F(t/n)^n - e^{-tC}\| = \delta^{-2} t^{-1+\alpha} e^{\delta t} O(n^{-\alpha}), \quad n \to \infty, \tag{2.2}$$

for all $t > 0$.

Therefore, for $0 < \alpha < 1$ (resp. $\alpha = 1$), the convergence in (2.2) is uniform on each compact t-interval in the open half line $(0, \infty)$ (resp. in the closed half line $[0, \infty)$).

Moreover, if C is strictly positive, i.e. $C \geq \eta$ for some constant $\eta > 0$, the error bound on the right-hand side of (2.2) can also be replaced by $(1 + 2/\eta)^2 t^{-1+\alpha} O(n^{-\alpha})$, so that, for $0 < \alpha < 1$ (resp. $\alpha = 1$), the convergence in (2.2) is uniform on the closed half line $[T, \infty)$ for every fixed $T > 0$ (resp. on the whole closed half line $[0, \infty)$).

Proof. Assume (a). Let $t > 0$. We have

$$F(t/n)^n - e^{-tC} = (F(t/n)^n - e^{-tS_{t/n}}) + (e^{-tS_{t/n}} - e^{-tC}). \tag{2.3}$$

To estimate the first term on the right-hand side of (2.3), let us note

$$0 \leq e^{-n(1-\lambda)} - \lambda^n \leq e^{-1}/n, \quad \text{for } 0 \leq \lambda \leq 1. \tag{2.4}$$

Though this can be in fact shown with the upper bound $2e^{-2}/n$ in place of e^{-1}/n, we shall content ourselves with it. To see (2.4) is easy. Since the function $\xi(\lambda) := e^{-n(1-\lambda)} - \lambda^n$ attains its maximum at λ_0 satisfying $e^{-n(1-\lambda_0)} = \lambda_0^{n-1}$, we obtain $0 \leq \xi(\lambda) \leq \xi(\lambda_0) = \lambda_0^{n-1} - \lambda_0^n = (1/n)n(1 - \lambda_0)e^{-n(1-\lambda_0)} \leq e^{-1}/n$.

Then by (2.4), we have by the spectral theorem for every $t > 0$,

$$\|F(t/n)^n - e^{-tS_{t/n}}\| = \|F(t/n)^n - e^{-n(1-F(t/n))}\| \le e^{-1}n^{-1}. \tag{2.5}$$

To estimate the second term, we use a formula in Kato [14, IX.4, (2.27)]

$$\begin{aligned}(1+S_\varepsilon)^{-1}&[e^{-t(\delta+S_\varepsilon)} - e^{-t(\delta+C)}](1+C)^{-1} \\ &= \int_0^t e^{-(t-s)(\delta+S_\varepsilon)}[(1+S_\varepsilon)^{-1} - (1+C)^{-1}]e^{-s(\delta+C)}ds \\ &= \int_0^{t/2} + \int_{t/2}^t \equiv S_1 + S_2,\end{aligned} \tag{2.6}$$

where $\delta > 0$ and $\varepsilon > 0$. Putting $D(\varepsilon) = (1+S_\varepsilon)^{-1} - (1+C)^{-1}$ in the following, we are assuming $\|D(\varepsilon)\| = O(\varepsilon^\alpha)$ by (2.1). For S_1 we have by integration by parts

$$\begin{aligned}S_1 =& \Big[-e^{-(t-s)(\delta+S_\varepsilon)}D(\varepsilon)e^{-s(\delta+C)}(\delta+C)^{-1}\Big]_{s=0}^{s=t/2} \\ &+ \int_0^{t/2}(\delta+S_\varepsilon)e^{-(t-s)(\delta+S_\varepsilon)}D(\varepsilon)e^{-s(\delta+C)}(\delta+C)^{-1}ds \\ =& -e^{-(t/2)(\delta+S_\varepsilon)}D(\varepsilon)e^{-(t/2)(\delta+C)}(\delta+C)^{-1} + e^{-t(\delta+S_\varepsilon)}D(\varepsilon)(\delta+C)^{-1} \\ &+ \int_0^{t/2}(\delta+S_\varepsilon)e^{-(t-s)(\delta+S_\varepsilon)}D(\varepsilon)e^{-s(\delta+C)}(\delta+C)^{-1}ds.\end{aligned}$$

Then

$$\begin{aligned}(1+&S_\varepsilon)S_1(1+C) \\ =& -(1+S_\varepsilon)e^{-(t/2)(\delta+S_\varepsilon)}D(\varepsilon)e^{-(t/2)(\delta+C)}(\delta+C)^{-1}(1+C) \\ &+ (1+S_\varepsilon)e^{-t(\delta+S_\varepsilon)}D(\varepsilon)(\delta+C)^{-1}(1+C) \\ &+ \int_0^{t/2}(1+S_\varepsilon)(\delta+S_\varepsilon)e^{-(t-s)(\delta+S_\varepsilon)}D(\varepsilon)e^{-s(\delta+C)}(\delta+C)^{-1}(1+C)ds,\end{aligned}$$

and similarly for S_2,

$$\begin{aligned}(1+&S_\varepsilon)S_2(1+C) \\ =& (1+S_\varepsilon)(\delta+S_\varepsilon)^{-1}D(\varepsilon)e^{-t(\delta+C)}(1+C) \\ &- (1+S_\varepsilon)(\delta+S_\varepsilon)^{-1}e^{-(t/2)(\delta+S_\varepsilon)}D(\varepsilon)e^{-(t/2)(\delta+C)}(1+C) \\ &+ \int_{t/2}^t(1+S_\varepsilon)(\delta+S_\varepsilon)^{-1}e^{-(t-s)(\delta+S_\varepsilon)}D(\varepsilon)e^{-s(\delta+C)}(\delta+C)(1+C)ds.\end{aligned}$$

We know

$$e^{-\delta t}(e^{-tS_\varepsilon} - e^{-tC}) = (1+S_\varepsilon)S_1(1+C) + (1+S_\varepsilon)S_2(1+C). \tag{2.7}$$

Since $\lambda^\gamma e^{-\lambda} \le (\gamma/e)^\gamma$ for $\lambda \ge 0$ and $\gamma \ge 0$, we can estimate (2.7) with assumption (2.1) by the spectral theorem as

$$\begin{aligned}\|(1+S_\varepsilon)S_1(1+C)\| &\le \Big(3e^{-1}/t + 4e^{-2}\int_0^{t/2}(t-s)^{-2}ds\Big)O(\varepsilon^\alpha)/\delta^2 \\ &\le 2O(\varepsilon^\alpha)/(\delta^2 t), \\ \|(1+S_\varepsilon)S_2(1+C)\| &\le \Big(3e^{-1}/t + 4e^{-2}\int_{t/2}^{t} s^{-2}ds\Big)O(\varepsilon^\alpha)/\delta^2 \le 2O(\varepsilon^\alpha)/(\delta^2 t).\end{aligned}$$

Here we have needed that $\delta \le 1$. Hence with (2.7),

$$\|e^{-\delta t}(e^{-tS_\varepsilon} - e^{-tC})\| = \|(1+S_\varepsilon)(S_1+S_2)(1+C)\| \le 4O(\varepsilon^\alpha)/(\delta^2 t). \tag{2.8}$$

It follows that with $\varepsilon = t/n$ the second term of (2.3) obeys

$$\|e^{-tS_{t/n}} - e^{-tC}\| \le (\delta^2 t)^{-1}e^{\delta t}O((t/n)^\alpha) = \delta^{-2}t^{-1+\alpha}e^{\delta t}O(n^{-\alpha}). \tag{2.9}$$

Thus, combining (2.5) and (2.9) with (2.3), we have the assertion (b) or (2.2).

In case C is strictly positive, that is, $C \ge \eta$ for some constant $\eta > 0$, we can show $S_\varepsilon \ge \eta/2$ or $\|S_\varepsilon^{-1}\| \le 2/\eta$ for sufficiently small $\varepsilon > 0$. In fact, by (2.1),

$$\|(1+S_\varepsilon)^{-1}\| = \|(1+C)^{-1}\| + O(\varepsilon^\alpha) \le (1+\eta)^{-1} + O(\varepsilon^\alpha) \le (1+\eta/2)^{-1} < 1,$$

so that S_ε has bounded inverse

$$S_\varepsilon^{-1} = [(1+S_\varepsilon)-1]^{-1} = (1+S_\varepsilon)^{-1}[1-(1+S_\varepsilon)^{-1}]^{-1}$$

with bound $\|S_\varepsilon^{-1}\| \le 2/\eta$. Therefore in the above argument around (2.8), though $(1+S_\varepsilon)(\delta+S_\varepsilon)^{-1}$ is bounded as well as $(1+C)(\delta+C)^{-1}$, with bound $(1+\eta/2)(\delta+\eta/2)^{-1}$ in place of $1/\delta$, one can in turn use the formula (2.5) with $\delta = 0$, and show, since both $(1+S_\varepsilon)S_\varepsilon^{-1}$ and $(1+C)C^{-1}$ are bounded with bound $(1+2/\eta)$ for small $\varepsilon > 0$, that the right-hand side of (2.9) simply becomes of order $(1+2/\eta)^2 t^{-1+\alpha}O(n^{-\alpha})$. In particular, it is of order $O(n^{-\alpha})$ uniformly on the closed half line $[T,\infty)$ for every $T > 0$ for $0 < \alpha < 1$, and on the whole closed half line $[0,\infty)$ for $\alpha = 1$. This proves Lemma 2.1. □

3. Proof of Theorem

We are now in a position to prove the theorem.

First note that since $C = A + B$ is itself selfadjoint and so a closed operator, by the closed graph theorem there exist constants a_1 and a_2 such that

$$\|Au\| + \|Bu\| \le a_1\|Cu\| + a_2\|u\|, \quad u \in D[C] = D[A]\cap D[B].$$

Therefore we may assume for some constant $a > 0$ that

$$\|(1+A)u\| + \|(1+B)u\| \le a\|(1+C)u\|, \quad u \in D[C] = D[A]\cap D[B]. \tag{3.1}$$

For $t > 0$ define positive bounded operators

$$A_t = t^{-1}[1-f(tA)], \quad B_t = t^{-1}[1-g(tB)], \quad C_t = t^{-1}[1-e^{-tC}]. \tag{3.2}$$

Note that

$$\|tA_t\| = \|1 - f(tA)\| \le 1, \quad \|tB_t\| = \|1 - g(tB)\| \le 1. \tag{3.3}$$

The proof of the theorem is now divided into two cases, (a) the symmetric product case concerning

$$F(t) = g(tB/2)f(tA)g(tB/2), \tag{3.4}$$

and (b) the non-symmetric product case concerning

$$G(t) = f(tA)g(tB). \tag{3.5}$$

In the former case we shall use Lemma 3.1. The latter case will follow from the former case.

(a) *The symmetric product case.* To prove the symmetric product case of the theorem, by Lemma 2.1 it suffices to show in operator norm that with $S_t = t^{-1}(1 - F(t))$,

$$\|(1 + S_t)^{-1} - (1 + C)^{-1}\| = O(t^{1/2}), \quad t \downarrow 0. \tag{3.6}$$

We should already know (cf. Chernoff [1, 2], Kato [15] and Reed–Simon [20]) that $(1 + S_t)^{-1} \to (1 + C)^{-1}$ in strong operator topology.

Define a positive bounded operator

$$\begin{aligned} K_t &= 1 + A_t + B_{t/2} - \tfrac{t}{4}B_{t/2}^2 \\ &= 1 + A_t + \tfrac{1}{2}B_{t/2} + \tfrac{1}{2}B_{t/2}^{1/2}(1 - \tfrac{t}{2}B_{t/2})B_{t/2}^{1/2} \\ &= 1 + A_t + B_{t/2}^{1/2}\left(\tfrac{1+g(tB/2)}{2}\right)B_{t/2}^{1/2} \ge 1. \end{aligned} \tag{3.7}$$

Rewrite $1 + S_t$, by introducing Q_t, as

$$\begin{aligned} 1 + S_t &= 1 + A_t + B_{t/2} - \tfrac{t}{4}B_{t/2}^2 + \tfrac{t^2}{4}B_{t/2}A_tB_{t/2} - \tfrac{t}{2}(A_tB_{t/2} + B_{t/2}A_t) \\ &= K_t^{1/2}(1 + Q_t)K_t^{1/2}, \\ Q_t &= \tfrac{t^2}{4}K_t^{-1/2}B_{t/2}A_tB_{t/2}K_t^{-1/2} - \tfrac{t}{2}K_t^{-1/2}(A_tB_{t/2} + B_{t/2}A_t)K_t^{-1/2}. \end{aligned} \tag{3.8}$$

Then we need that $1 + Q_t$ has bounded inverse uniformly for $t > 0$. The proof of this fact in Reed–Simon [20] seems to contain a small flaw. So we prove it in the following lemma. At this stage note that differing from their proof, ours is exchanging the roles of A and B.

Lemma 3.1. *For $t > 0$,*

$$\|(1 + Q_t)^{-1}\| \le 2/(3 - \sqrt{5}). \tag{3.9}$$

If (3.9) is proved, then we can obtain that for $t > 0$,

$$\|(1 + S_t)^{-1}K_t^{1/2}\| = \|K_t^{-1/2}(1 + Q_t)^{-1}\| \le 2/(3 - \sqrt{5}). \tag{3.10}$$

Proof of Lemma 3.1. We shall use (3.3) and

$$\begin{aligned} &\|A_t^{1/2}K_t^{-1/2}\| \le \|(1 + A_t)^{1/2}K_t^{-1/2}\| \le 1, \\ &2^{-1/2}\|B_{t/2}^{1/2}K_t^{-1/2}\| \le \|(1 + \tfrac{1}{2}B_{t/2})^{1/2}K_t^{-1/2}\| \le 1. \end{aligned} \tag{3.11}$$

We see from the definition of Q_t in (3.8),

$$\begin{aligned}Q_t &= K_t^{-1/2}(\tfrac{t}{2}B_{t/2}-r)A_t(\tfrac{t}{2}B_{t/2}-r)K_t^{-1/2}\\ &\quad -\tfrac{(1-r)t}{2}K_t^{-1/2}(A_tB_{t/2}+B_{t/2}A_t)K_t^{-1/2}-r^2K_t^{-1/2}A_tK_t^{-1/2}\\ &\ge -\tfrac{(1-r)t}{2}K_t^{-1/2}(A_tB_{t/2}+B_{t/2}A_t)K_t^{-1/2}-r^2K_t^{-1/2}A_tK_t^{-1/2},\end{aligned}$$

where r is a constant with $0<r<1$ to be determined later. Hence we have for $u\in\mathcal{H}$,

$$\begin{aligned}&(Q_tu,u)\\ &\ge -2(1-r)\mathrm{Re}\Big((tA_t)^{1/2}\big(\tfrac{1-g(tB/2)}{1+g(tB/2)}\big)^{1/2}\big(\tfrac{1+g(tB/2)}{2}\big)^{1/2}B_{t/2}^{1/2}K_t^{-1/2}u,\ A_t^{1/2}K_t^{-1/2}u\Big)\\ &\quad -r^2\big\|A_t^{1/2}K_t^{-1/2}u\big\|^2\\ &\ge -2(1-r)\big\|A_t^{1/2}K_t^{-1/2}u\big\|\big\|\big(\tfrac{1+g(tB/2)}{2}\big)^{1/2}B_{t/2}^{1/2}K_t^{-1/2}u\big\|-r^2\big\|A_t^{1/2}K_t^{-1/2}u\big\|^2\\ &\ge -(1-r)\Big[p\big\|A_t^{1/2}K_t^{-1/2}u\big\|^2+(1/p)\big\|\big(\tfrac{1+g(tB/2)}{2}\big)^{1/2}B_{t/2}^{1/2}K_t^{-1/2}u\big\|^2\Big]\\ &\quad -r^2\big\|A_t^{1/2}K_t^{-1/2}u\big\|^2.\end{aligned}$$

Here p is an aribitray positive constant. Choose p such that $(1-r)p+r^2=(1-r)/p$, namely, $p=\frac{-r^2+\sqrt{r^4+4(1-r)^2}}{2(1-r)}$. Then with $\beta(r)=\frac{r^2+\sqrt{r^4+4(1-r)^2}}{2}$, we have

$$\begin{aligned}(Q_tu,u) &\ge -\beta(r)\Big[\big\|A_t^{1/2}K_t^{-1/2}u\big\|^2+\big\|\big(\tfrac{1+g(tB/2)}{2}\big)^{1/2}B_{t/2}^{1/2}K_t^{-1/2}u\big\|^2\Big]\\ &= -\beta(r)\Big(\Big[A_t+\big(\tfrac{1+g(tB/2)}{2}\big)B_{t/2}\Big]K_t^{-1/2}u,\ K_t^{-1/2}u\Big)\\ &\ge -\beta(r)\|u\|^2.\end{aligned}$$

We can see $\beta(r)$ attains its minimum at $r=\frac{\sqrt{5}-1}{2}$:

$$\begin{aligned}\beta(\tfrac{\sqrt{5}-1}{2}) &= \tfrac{1}{2}\Bigg(\big(\tfrac{\sqrt{5}-1}{2}\big)^2+\sqrt{\big(\tfrac{\sqrt{5}-1}{2}\big)^4+4\big(\tfrac{3-\sqrt{5}}{2}\big)^2}\Bigg)\\ &= \tfrac{1}{4}\Big(3-\sqrt{5}+\big(70-30\sqrt{5}\big)^{1/2}\Big)=\tfrac{1}{4}\big(3-\sqrt{5}+3\sqrt{5}-5\big)=\tfrac{\sqrt{5}-1}{2}.\end{aligned}$$

It follows that $(Q_tu,u)\ge -\frac{\sqrt{5}-1}{2}\|u\|^2$, so that $((1+Q_t)u,u)\ge\big(1-\frac{\sqrt{5}-1}{2}\big)\|u\|^2=\frac{3-\sqrt{5}}{2}\|u\|^2$. This yields (3.9), showing Lemma 3.1. □

Now we have

$$\begin{aligned}&(1+S_t)^{-1}-(1+C)^{-1}\\ &=(1+S_t)^{-1}\big[A+B-(A_t+B_{t/2}-\tfrac{t}{4}B_{t/2}(1-tA_t)B_{t/2}\\ &\qquad -\tfrac{t}{2}(A_tB_{t/2}+B_{t/2}A_t))\big](1+C)^{-1}\\ &=(1+S_t)^{-1}(A-A_t)(1+C)^{-1}+(1+S_t)^{-1}(B-B_{t/2})(1+C)^{-1}\\ &\quad +(1+S_t)^{-1}[\tfrac{t}{4}B_{t/2}(1-tA_t)B_{t/2}+\tfrac{t}{2}(A_tB_{t/2}+B_{t/2}A_t)](1+C)^{-1}\\ &\equiv R_1(t)+R_2(t)+R_3(t).\end{aligned}\tag{3.12}$$

We are going to show in the following lemma that all the three $R_i(t)$ in the last member of (3.12) converge to zero in operator norm of order $O(t^{1/2})$ as $t\downarrow 0$.

Lemma 3.2. *For small $t > 0$,*

$$\|R_1(t)\| \le cat^{1/2}, \quad \|R_2(t)\| \le cat^{1/2}, \quad \|R_3(t)\| \le cat^{1/2}, \tag{3.13}$$

with a constant $c > 0$ independent of $t > 0$.

Proof. First note by the spectral theorem that

$$\begin{aligned} a_0 &:= \|A_t(1+A)^{-1}\| = \sup_{\lambda\ge 0} \frac{1-f(t\lambda)}{t(1+\lambda)} < \infty, \\ b_0 &:= \|B_{t/2}(1+B)^{-1}\| = \sup_{\lambda\ge 0} \frac{1-g(t\lambda/2)}{t(1+\lambda)/2} < \infty. \end{aligned} \tag{3.14}$$

I. For $R_1(t)$ we have

$$\begin{aligned} R_1(t) = {} & [(1+S_t)^{-1}K_t^{1/2}][K_t^{-1/2}(1+A_t)^{1/2}] \\ & \times [(1+A_t)^{-1/2} - (1+A_t)^{1/2}(1+A)^{-1}](1+A)(1+C)^{-1}. \end{aligned}$$

Hence by (3.1), (3.10) and (3.11),

$$\|R_1(t)\| \le \tfrac{2}{3-\sqrt{5}} a \|(1+A_t)^{-1/2} - (1+A_t)^{1/2}(1+A)^{-1}\|.$$

Then by the spectral theorem we have

$$\|(1+A_t)^{-1/2} - (1+A_t)^{1/2}(1+A)^{-1}\| = \sup_{\lambda\ge 0} |a_t(\lambda)|,$$

$$\begin{aligned} a_t(\lambda) &= \Big(1+\frac{1-f(t\lambda)}{t}\Big)^{-1/2} - \Big(1+\frac{1-f(t\lambda)}{t}\Big)^{1/2}(1+\lambda)^{-1} \\ &= \Big(\frac{t}{1+t-f(t\lambda)}\Big)^{1/2}\Big[1-\frac{1}{1+\lambda}\Big(1+\frac{1-f(t\lambda)}{t}\Big)\Big] \\ &= \Big(\frac{t}{1+t-f(t\lambda)}\Big)^{1/2}\frac{f(t\lambda)-1+t\lambda}{t(1+\lambda)}. \end{aligned}$$

Since f satisfies $f'(0) = -1$ by (1.2), there exists a small positive constant s_1 such that for $0 \le s \le s_1$,

$$-s/2 \le f(s) - 1 + s \le s/2, \quad \text{or} \quad s/2 \le 1 - f(s) \le 3s/2.$$

Then

$$\begin{aligned} \sup_{\lambda\ge 0}|a_t(\lambda)| &= \sup_{\mu\ge 0}|a_t(\mu/t)| \\ &= \sup_{\mu\ge 0}\Big(\frac{t}{1+t-f(\mu)}\Big)^{1/2}\frac{|f(\mu)-1+\mu|}{t+\mu} \\ &= \max\Big\{\sup_{0\le\mu\le s_1}|a_t(\mu/t)|, \sup_{\mu\ge s_1}|a_t(\mu/t)|\Big\}. \end{aligned}$$

As for the first component in the last member above, we have, since f satisfies (1.5) with $\kappa \ge 3/2$,

$$\sup_{0\le\mu\le s_1}|a_t(\mu/t)| \le \sup_{0\le\mu\le s_1}\Big(\frac{t}{t+\mu/2}\Big)^{1/2}\frac{[f]_\kappa \mu^\kappa}{t+\mu} \le \sqrt{2}[f]_\kappa s_1^{\kappa-3/2} t^{1/2}.$$

As for the second component, since, by (1.4), for the same s_1 as above there exists a positive constant $\delta = \delta(s_1) < 1$ such that if $s \geq s_1$ then $f(s) \leq 1 - \delta(s_1)$, we have

$$\sup_{\mu \geq s_1} |a_t(\mu/t)| \leq \Big(\frac{t}{t + \delta(s_1)}\Big)^{1/2}(1 + a_0) \leq \delta(s_1)^{-1/2}(1 + a_0)t^{1/2}.$$

This proves the estimate for $R_1(t)$.

II. The proof for $R_2(t)$ is the same as for $R_1(t)$. We have only to replace A_t, A and f by $B_{t/2}$, B and g, and only note that

$$\begin{aligned} R_2(t) = {} & \big[(1 + S_t)^{-1}K_t^{1/2}\big]\big[K_t^{-1/2}(1 + \tfrac{1}{2}B_{t/2})^{1/2}\big]\big[(1 + \tfrac{1}{2}B_{t/2})^{-1/2}(1 + B_{t/2})^{1/2}\big] \\ & \times \big[(1 + B_{t/2})^{-1/2} - (1 + B_{t/2})^{1/2}(1 + B)^{-1}\big](1 + B)(1 + C)^{-1}. \end{aligned}$$

III. For $R_3(t)$ we have

$$\begin{aligned} R_3(t) = {} & \tfrac{\sqrt{2}}{4}t^{1/2}\big[(1 + S_t)^{-1}K_t^{1/2}\big]\big[K_t^{-1/2}B_{t/2}^{1/2}\big]\big[(\tfrac{t}{2}B_{t/2})^{1/2}(1 - tA_t)\big] \\ & \times \big[B_{t/2}(1 + B)^{-1}\big](1 + B)(1 + C)^{-1} \\ & + \Big(\tfrac{1}{2}t^{1/2}\big[(1 + S_t)^{-1}K_t^{1/2}\big]\big[K_t^{-1/2}A_t^{1/2}\big] \\ & \times \big[(tA_t)^{1/2}B_{t/2}(1 + B)^{-1}\big](1 + B)(1 + C)^{-1} \\ & + \tfrac{\sqrt{2}}{2}t^{1/2}\big[(1 + S_t)^{-1}K_t^{1/2}\big]\big[K_t^{-1/2}B_{t/2}^{1/2}\big] \\ & \times \big[(\tfrac{t}{2}B_{t/2})^{1/2}A_t(1 + A)^{-1}\big](1 + A)(1 + C)^{-1}\Big). \end{aligned}$$

It follows by (3.1), (3.9), (3.10) and (3.14) that

$$\|R_3(t)\| \leq \Big[\tfrac{\sqrt{2}}{4}\tfrac{2\sqrt{2}}{3-\sqrt{5}}b_0 + \Big(\tfrac{1}{2}\tfrac{2}{3-\sqrt{5}}b_0 + \tfrac{\sqrt{2}}{2}\tfrac{2\sqrt{2}}{3-\sqrt{5}}\Big)a_0\Big]at^{1/2} \leq \tfrac{2}{3-\sqrt{5}}(a_0 + b_0)at^{1/2}.$$

This completes the proof of Lemma 3.2. Thus we have proved (3.6), so that by Lemma 3.1 with $F(t)$ in (3.4),

$$\big\|F(t/n)^n - e^{-tC}\big\| = \delta^{-2}t^{-1/2}e^{\delta t}O\big(n^{-1/2}\big), \quad n \to \infty, \tag{3.15}$$

and in particular, the symmetric product case of the theorem.

(b) *The non-symmetric product case.* What we have proved in the symmetric product case (a) of the theorem, namely, (3.15), is that $F(t/n)^n = e^{-tC} + O_p(t,n)$, where $O_p(t,n)$ is some bounded operator with norm of order $\delta^{-2}t^{-1/2}e^{\delta t}O(n^{-1/2})$ for n large and $t > 0$ with $0 < \delta \leq 1$. We are now going to show this implies that $G(t/n)^n = e^{-tC} + O_p(t,n)$; here it should be noted that the following proof is equally valid, even if $O_p(t,n)$ means some bounded operator with norm of such an order $\delta^{-2}t^{-1+\alpha}e^{\delta t}O(n^{-\alpha})$ for some $0 < \alpha \leq 1$ as we have had on the right-hand side of (2.2) in Lemma 2.1.

Given $g(t)$, put $g_1(t) = g(2t)^{1/2}$ or $g_1(t)^2 = g(2t)$. We can see that g_1 satisfies the same condition as f and g. Put $F_1(t) = g_1(tB/2)f(tA)g_1(tB/2)$, similarly to (3.4). Then by the symmetric product case (a), we have

$$F_1(t/n)^n = [g_1(tB/2n)f(tA/n)g_1(tB/2n)]^n = e^{-tC} + O_p(t,n). \tag{3.16}$$

Then we have by (3.5) and (3.16),

$$\begin{aligned} G(t/n)^n &= [f(tA/n)g(tB/n)]^n = \big[f(tA/n)g_1(tB/2n)^2\big]^n \\ &= f(tA/n)g_1(tB/2n)F_1(t/n)^{n-1}g_1(tB/2n) \\ &= f(tA/n)g_1(tB/2n)\big[e^{-(n-1)tC/n} + O_p(t,n)\big]g_1(tB/2n) \\ &= f(tA/n)g_1(tB/2n)e^{-(n-1)tC/n}g_1(tB/2n) + O_p(t,n). \end{aligned} \tag{3.17}$$

In the following lemma we are denoting by $[U, V] = UV - VU$ the commutator of bounded linear operators U and V.

Lemma 3.3. *For $\tau = t/n$ or $\tau = t/2n$,*

$$\begin{aligned} &\|[f(\tau A), e^{-tC}]\| = \delta^{-1}e^{\delta t}O(n^{-1}), \\ &\|[g(\tau B), e^{-tC}]\| = \delta^{-1}e^{\delta t}O(n^{-1}), \quad \|[g_1(\tau B), e^{-tC}]\| = \delta^{-1}e^{\delta t}O(n^{-1}), \end{aligned}$$

with $0 < \delta \le 1$ a constant. Therefore the norm bounds on the right-hand side are of order $O(n^{-1})$ uniformly on each compact t-interval in the closed half line $[0, \infty)$.

If C is strictly positive, i.e. $C \ge \eta$ for some constant $\eta > 0$, then these norm bounds are of order $O(n^{-1})$ uniformly on the whole closed half line $[0, \infty)$.

Proof. We have only to prove the first one for $f(\tau A)$. We see by (3.2) for $\delta > 0$,

$$\begin{aligned} [f(\tau A), e^{-tC}] &= e^{\delta t}\big(f(\tau A)e^{-t(\delta+C)} - e^{-t(\delta+C)}f(\tau A)\big) \\ &= e^{\delta t}\big((1-\tau A_\tau)e^{-t(\delta+C)} - e^{-t(\delta+C)}(1-\tau A_\tau)\big) \\ &= -e^{\delta t}\tau(A_\tau e^{-t(\delta+C)} - e^{-t(\delta+C)}A_\tau). \end{aligned}$$

Since by (3.1) and (3.14) the norm of

$$A_\tau e^{-t(\delta+C)} = t^{-1}[A_\tau(1+A)^{-1}][(1+A)(1+C)^{-1}][(1+C)(\delta+C)^{-1}]t(\delta+C)e^{-t(\delta+C)}$$

is bounded by $a_0ae^{-1}/(\delta t)$ and similarly for $e^{-t(\delta+C)}A_\tau$, we have $\|[f(\tau A), e^{-tC}]\| = a_0ae^{-1}\delta^{-1}e^{\delta t}O(n^{-1})$.

In case C is strictly positive, i.e. $C \ge \eta$ for $\eta > 0$, we may begin the above argument with $\delta = 0$ to get the norm bound $\|A_\tau e^{-tC}\| \le a_0ae^{-1}/(\eta t)$, so that $\|[f(\tau A), e^{-tC}]\| = a_0ae^{-1}\eta^{-1}O(n^{-1})$.

This proves Lemma 3.3. □

By Lemma 3.3, we obtain from (3.17),

$$\begin{aligned} G(t/n)^n &= [f(tA/n)g(tB/n)]^n = f(tA/n)g(tB/n)e^{-(n-1)tC/n} + O_p(t,n) \\ &= f(tA/n)e^{-(n-1)tC/2n}g(tB/n)e^{-(n-1)tC/2n} + O_p(t,n) \\ &= e^{-(n-1)tC/2n}f(tA/n)g(tB/n)e^{-(n-1)tC/2n} + O_p(t,n). \end{aligned} \tag{3.18}$$

Lemma 3.4. *For $\tau = t/n$,*

$$\big\|(1+C)^{-1/2}[f(\tau A)g(\tau B) - e^{-\tau C}](1+C)^{-1/2}\big\| = O(\tau).$$

Proof. We have by (3.2),

$$\begin{aligned}
&(1+C)^{-1/2}\big[f(\tau A)g(\tau B)-e^{-\tau C}\big](1+C)^{-1/2}\\
&\quad=(1+C)^{-1/2}\big[(1-\tau A_\tau)(1-\tau B_\tau)-e^{-\tau C}\big](1+C)^{-1/2}\\
&\quad=\tau(1+C)^{-1/2}C_\tau(1+C)^{-1/2}-\tau(1+C)^{-1/2}(A_\tau+B_\tau)(1+C)^{-1/2}\\
&\qquad+\tau^2(1+C)^{-1/2}A_\tau B_\tau(1+C)^{-1/2}\\
&\quad\equiv E_1(\tau)+E_2(\tau)+E_3(\tau).
\end{aligned}$$

It is easy to see that $\|E_1(\tau)\|\le\tau$. We have also $\|E_2(\tau)\|\le(a_0+b_0)\tau$ and $\|E_3(\tau)\|\le(a_0b_0)^{1/2}\tau$, by (3.3), because

$$\begin{aligned}
E_2(\tau)=&-\tau(1+C)^{-1/2}(1+A)^{1/2}\big[(1+A)^{-1/2}A_\tau(1+A)^{-1/2}\big]\\
&\times(1+A)^{1/2}(1+C)^{-1/2}\\
&-\tau(1+C)^{-1/2}(1+B)^{1/2}\big[(1+B)^{-1/2}B_\tau(1+B)^{-1/2}\big]\\
&\times(1+B)^{1/2}(1+C)^{-1/2},\\
E_3(\tau)=&\,\tau(1+C)^{-1/2}(1+A)^{1/2}\big[(1+A)^{-1/2}A_\tau^{1/2}\big]\\
&\times(\tau A_\tau)^{1/2}(\tau B_\tau)^{1/2}\big[B_\tau^{1/2}(1+B)^{-1/2}\big](1+B)^{1/2}(1+C)^{-1/2}.
\end{aligned}$$

This proves Lemma 3.4. □

Finally, by Lemma 3.4 we obtain from (3.18),

$$\begin{aligned}
G(t/n)^n&=\big[f(tA/n)g(tB/n)\big]^n\\
&=e^{-(n-1)tC/2n}(1+C)^{1/2}\big((1+C)^{-1/2}f(tA/n)g(tB/n)(1+C)^{-1/2}\big)\\
&\quad\times(1+C)^{1/2}e^{-(n-1)tC/2n}+O_p(t,n)\\
&=e^{-(n-1)tC/2n}(1+C)^{1/2}\big[(1+C)^{-1/2}e^{-tC/n}(1+C)^{-1/2}+O_p(t/n)\big]\\
&\quad\times(1+C)^{1/2}e^{-(n-1)tC/2n}+O_p(t,n)\\
&=e^{-tC}+\big[e^{-(n-1)tC/2n}(1+C)^{1/2}\big]O_p(t/n)\big[(1+C)^{1/2}e^{-(n-1)tC/2n}\big]\\
&\quad+O_p(t,n)\\
&=e^{-tC}+\delta^{-1}e^{\delta t}O_p(n^{-1})+O_p(t,n)\\
&=e^{-tC}+O_p(t,n).
\end{aligned}\tag{3.19}$$

Here $O_p(t/n)$ and $O_p(n^{-1})$ also mean some bounded operators with norm of order $O(t/n)$ and $O(n^{-1})$, respectively, for n large and $t>0$. Therefore we can conclude from (3.19),

$$\|G(t/n)^n-e^{-tC}\|=O(n^{-1/2}),\quad n\to\infty,\tag{3.20}$$

uniformly on each compact t-interval in the open half line $(0,\infty)$.

If C is strictly positive, then we can see this norm bound $O(n^{-1/2})$ on the right-hand side of (3.20) is uniform on the closed half line $[T,\infty)$ for every $T>0$, taking this case of both Lemma 2.1 and Lemma 3.3 into consideration.

Thus we have proved the non-symmetric product case of the theorem.

4. Remarks on Conditions (1.4) and (1.5)

In this section, we note that condition (1.4) is necessary, and make a remark on what both conditions (1.4) and (1.5) imply.

First, let f and g be real-valued smooth functions satisfying (1.2) and (1.5) such that $f(s) = g(s) = 1$ for $s > 1$. Note that these f and g do not satisfy (1.4). Let H be a nonnegative selfadjoint operator in $\mathcal{H}$. Assume that H has only discrete eigenvalues divergent to infinity. Let $\{\lambda_j\}_{j=1}^\infty$ be the eigenvalues with $\{\psi_j\}_{j=1}^\infty$ the corresponding normalized eigenvectors.

Take three operators A, B and C as $A = B = \frac{1}{2}H$, $C = A + B = H$. Fix n sufficiently large, and take N so large that $\lambda_N > 2n$. Then

$$\begin{aligned}[f(A/n)g(B/n)]^n\psi_N &= [f(H/2n)g(H/2n)]^n\psi_N \\ &= [f(\lambda_N/2n)g(\lambda_N/2n)]^n\psi_N = \psi_N,\end{aligned}$$

which preserves the norm as vectors in the Hilbert space $\mathcal{H}$. On the other hand, we have $e^{-C}\psi_N = e^{-H}\psi_N = e^{-\lambda_N}\psi_N \to 0$, strongly as $N \to \infty$. This means that $[f(A/n)g(B/n)]^n$ never converges to e^{-C} in operator norm.

Next, in general, let f and g be real-valued smooth functions satisfying (1.2) and (1.5), but one of them, say, f not (1.4). We may suppose that $f(1) = 1$. Let H be a selfadjoint operator as above but with eigenvalues $\{\lambda_j = j\}_{j=1}^\infty$. Take $A = H$, $B = O$, $C = A + B = H$. Then

$$[f(A/n)g(B/n)]^n\psi_n = f(H/n)^n\psi_n = f(1)^n\psi_n = \psi_n,$$

which preserves the norm, while $e^{-C}\psi_n = e^{-H}\psi_n = e^{-n}\psi_n \to 0$, strongly as $n \to \infty$. This means again that $[f(A/n)g(B/n)]^n$ never converges to e^{-C} in operator norm.

Thus, finally we arrive at the following remark on both conditions (1.4) and (1.5). Since the Theorem should also hold in both the special and trivial cases $B = O$ or $C = A$, and $A = O$ or $C = B$, we expect (2.1) in Lemma 2.1 to hold with $F(t) = f(tA)$ and $\alpha = 1/2$:

$$\|(1 + A_t)^{-1} - (1 + A)^{-1}\| = O(t^{1/2}), \quad t \downarrow 0, \tag{4.1}$$

and similarly with $F(t) = g(tB/2)^2$. Here note that $g(s/2)^2$ also have the same properties (1.2), (1.4) and (1.5) as $g(s)$. The fact is, conditions (1.4) and (1.5) are giving sufficient conditions for (4.1) to hold. In fact, for $t > 0$ put

$$a_t'(\lambda) = \Big(1 + \frac{1 - f(t\lambda)}{t}\Big)^{-1} - (1 + \lambda)^{-1} = \frac{f(t\lambda) - 1 + t\lambda}{(1 + \lambda)(t + 1 - f(t\lambda))}.$$

Then the right-hand side of (4.1) is equal to

$$\sup_{\lambda \ge 0} |a_t'(\lambda)| = \sup_{\mu \ge 0} |a_t'(\mu/t)| = \sup_{\mu \ge 0} \frac{t|f(\mu) - 1 + \mu|}{(t + \mu)(1 + t - f(\mu))}. \tag{4.2}$$

Take the same $s_1 > 0$ as in proof I of Lemma 3.2. Then, dividing the supremum over $\mu \ge 0$ in (4.2) into those over two parts $0 \le \mu \le s_1$ and $\mu \ge s_1$, we have by (1.5),

$$\sup_{0 \le \mu \le s_1} |a_t'(\mu/t)| \le \sup_{0 \le \mu \le s_1} \frac{[f]_\kappa \mu^\kappa t}{(t + \mu)(t + \mu/2)} \le 2[f]_\kappa t^{\kappa - 1},$$

and by (1.4) with a_0 in (3.14),

$$\sup_{\mu \geq s_1} |a_t'(\mu/t)| \leq \sup_{\mu \geq s_1} \frac{t|f(\mu) - 1 + \mu|}{(t+\mu)(t+\delta(s_1))} \leq \sup_{\mu \geq s_1} \frac{(1+a_0)t}{(t+\delta(s_1))} \leq (1+a_0)\delta(s_1)^{-1}t.$$

Therefore, as for the bound of (4.1) we can conclude $O(t^{\kappa-1})$, which, for small $t > 0$, is less than or equal to $O(t^{1/2})$ because $3/2 \leq \kappa \leq 2$.

Acknowledgement. Thanks are due to Hiroshi Tamura for a careful reading of the manuscript and some useful comments. The authors are also very grateful to an anonymous kind referee for pointing out some errors in the original manuscript.

References

1. Chernoff, P.R.: Note on product formulas for operator semigroups. J. Funct. Anal. **2**, 238–242 (1968)
2. Chernoff, P.R.: *Product Formulas, Nonlinear semigroups and Addition of Unbounded Operators*. Mem. Amer. Math. Soc., **140**, 1–121 (1974)
3. Dia, B.O. and Schatzman, M.: An estimate on the Kac transfer operator. J. Funct. Anal. **145**, 108–135 (1997)
4. Doumeki, A., Ichinose, T. and Tamura, Hideo: Error bound on exponential product formulas for Schrödinger operators. J. Math. Soc. Japan **50**, 359–377 (1998)
5. Guibourg, D.: Inégalités maximales pour l'opérateur de Schrödinger. C. R. Acad. Sci. Paris **316**, Série I Math., 249–252 (1993)
6. Helffer, B.: Around the transfer operator and the Trotter–Kato formula. Operator Theory: Advances and Appl. **78**, 161–174 (1995)
7. Ichinose, T. and Takanobu, S.: On the norm estimate of the difference between the Kac operator and the Schrödinger semigroup. Differential Equations, Asymptotic Analysis, and Mathematical Physics. In: *(Proceedings of the International Conference on Partial Differential Equations, Potsdam, July 29 – Aug. 3, 1996)*, edited by M. Demuth and B.-W. Schulze, Berlin: Akademie-Verlag, 1997, pp. 165–173
8. Ichinose, T. and Takanobu, S.: Estimate of the difference between the Kac operator and the Schrödinger semigroup. Commun. Math. Phys. **186**, 167–197 (1997)
9. Ichinose, T. and Takanobu, S.: The norm estimate of the difference between the Kac operator and the Schrödinger semigroup: A unified approach to the nonrelativistic and relativistic cases. Nagoya Math. J. **149**, 53–81 (1998)
10. Ichinose, T. and Takanobu, S.: The norm estimate of the difference between the Kac operator and the Schrödinger semigroup II: The general case including the relativistic case. Electronic J. Probability **5**, Paper no. 5, 1–47 (2000); http://www.math.washington.edu/˜ejpecp/paper5.abs.html
11. Ichinose, T. and Tamura, Hideo: Error estimates in operator norm for Trotter–Kato product formula. Integr. Equat. Oper. Theory **27**, 195–207 (1997)
12. Ichinose, T. and Tamura, Hideo: Error bound in trace norm for Trotter–Kato product formula of Gibbs semigroups. Asymptotic Analysis **17**, 239–266 (1998)
13. Ichinose, T. and Tamura, Hideo: Error estimates in operator norm of exponential product formulas for propagators of parabolic evolution equations. Osaka J. Math. **35**, 751–770 (1998)
14. Kato, T.: *Perturbation Theory for Linear Operators*. 2nd edition, Berlin–Heidelberg–New York: Springer-Verlag, 1976
15. Kato, T.: Trotter's product formula for an arbitrary pair of self-adjoint contraction semigroups. In: *Topics in Functional Analysis* (G.C. Rota, ed.), New York: Academic Press, 1978, pp. 185–195
16. Neidhardt, H. and Zagrebnov, V.: On error estimates for the Trotter–Kato product formula. Lett. Math. Phys. **44**, 169–186 (1998)
17. Neidhardt, H. and Zagrebnov, V.: Fractional powers of selfadjoint operators and Trotter–Kato product formula. Integr. Equat. Oper. Theory **35**, 209–231 (1999)
18. Neidhardt, H. and Zagrebnov, V.: Trotter–Kato product formula and operator-norm convergence. Commun. Math. Phys. **205**, 129–159 (1999)
19. Neidhardt, H. and Zagrebnov, V.: Trotter–Kato product formula and symmetrically normed ideals. J. Funct. Anal. **167**, 113–147 (1999)
20. Reed, M. and Simon, B.: *Methods of Modern Mathematical Physics I: Functional Analysis*. Revised and enlarged ed., New York–London: Academic Press, 1980
21. Rogava, Dzh.L.: Error bounds for Trotter–type formulas for self-adjoint operators. Functional Analysis and Its Applications, **27**, 217–219 (1993)

22. Shen, Z.: L^p estimates for Schrödinger operators with certain potentials. Ann. Inst. Fourier, Grenoble **45**, 513–546 (1995)
23. Shen, Z.: Estimates in L^p for magnetic Schrödinger operators. Indiana Univ. Math. J. **45**, 817–841 (1996)
24. Takanobu, S.: On the error estimate of the integral kernel for the Trotter product formula for Schrödinger operators. Ann. Probab. **25**, 1895–1952 (1997)
25. Tamura, Hiroshi: A remark on operator-norm convergence of Trotter–Kato product formula. Integr. Equat. Oper. Theory **37**, 350–356 (2000)
26. Trotter, H.F.: On the product of semi-groups of operators. Proc. Amer. Math. Soc. **10**, 545–551 (1959)

Communicated by H. Araki