Commun. Math. Phys. 211, 615 – 628 (2000)

Communications in
Mathematical Physics

Quadratic Bosonic and Free White Noises

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Received: 23 August 1999 / Accepted: 8 December 1999

Abstract: We discuss the meaning of renormalization used for deriving quadratic bosonic commutation relations introduced by Accardi [ALV] and find a representation of these relations on an interacting Fock space. Also, we investigate classical stochastic processes which can be constructed from noncommutative quadratic white noise. We postulate quadratic free white noise commutation relations and find their representation on an interacting Fock space.

1. Introduction

Hudson and Parthasarathy [HP] showed that a Brownian motion $B(T)$ can be represented as a sum of two noncommuting operators: annihilation $a_{(0,T)}$ and creation $a^{\star}_{(0,T)}$,

$$B(T) = a_{(0,T)} + a^{\star}_{(0,T)} = \int_0^T (a_t + a^{\star}_t),$$

where a_t and $a^{\star}_t$ stand for the infinitesimal annihilation and creation operators respectively.

Accardi [ALV], in order to study some physical problems, introduced quadratic white noise operators, which informally can be written as $n_t = a^{\star}_t a_t$, $b_t = (a_t)^2$ and $b^{\star}_t = (a^{\star}_t)^2$. The first one, called the number operator has been already considered in the white noise calculus and it does not cause serious difficulties. The other two, called quadratic annihilation and quadratic creation operators, in fact represent infinite quantities and therefore have to be redefined. Indeed, it can be shown that because of $[a_t, a^{\star}_s] = \delta(t-s)$ we have

$$[a_t^2, a_s^{\star 2}] = 2\delta^2(t-s) + 4\delta(t-s)a_t a^{\star}_s,$$

⋆ Research partially supported by the Scientific Research Committee in Warsaw under grant number P03A05415.

where δ denotes the Dirac distribution. Since the square of the delta function is not well defined, this relation is meaningless. Furthermore it is too singular to apply the subtraction renormalization [S]. By renormalization $\delta^2(x) = \gamma_0 \delta(x)$ Accardi postulates that the renormalized quadratic white noise operators should fulfill the following commutation relation:

$$[b_t, b_s^\star] = 2\gamma_0 \delta(t-s) + 4\delta(t-s) n_s, \tag{1}$$

which for smeared operators $b_\phi = \int \overline{\phi_t} b_t$, $b_\psi^\star = \int \psi_s b_s$ takes the form

$$[b_\phi, b_\psi^\star] = 2\gamma_0 \langle \phi, \psi \rangle + 4 n_{\bar{\phi}\psi}. \tag{2}$$

In Sect. 2 we present another discussion of this relation in a more general context of q-deformed commutation relations.

In Sect. 3 we show that from this discussion follows for the bosonic case the meaning of renormalization constant γ_0 as the inverse of the lengthscale taken for quadratic variation of a (noncommutative) Brownian motion and we discuss other commutation relations. Furthermore, from quadratic white noise operators we construct some classical stochastic processes.

Accardi and Skeide [ALV,AS] have constructed a Fock representation of quadratic white noise relations. The construction presented in the paper [ALV] uses the Kolmogorov decomposition for a certain positive kernel. Another approach is presented in the paper [AS] where the construction of quadratic white noise operators is based on the theory of Hilbert modules. In Sect. 4 and 5 we present a direct construction of such a representation on an interacting Fock space. Our method is based on defining explicitly a scalar product on a symmetric Fock space.

In Sect. 6 we discuss the existence of a Fock representation of an algebra containing both quadratic and usual linear white noise operators. It turns out that it is in general not possible to find such a representation. The main reason is that under a certain lengthscale the renormalized quadratic operators lose their intuitive meaning as squares of creation and annihilation operators.

In Sect. 7 we introduce free quadratic white noise operators which should describe the squares of free creation and annihilation operators with small violation of freeness and construct their representation.

Both standard and quadratic white noises are weak processes, i.e. mappings from some linear space S to operators on a Hilbert space. Contrary to white noise commutation relations, the quadratic relation (2) involves not only a scalar product in S, but a product of two elements of S as well. From the noncommutative geometry viewpoint [C] it would be interesting to consider noncommutative spacetime algebras S as well and quadratic white noise relations provide appropriate examples. Unfortunately, for the bosonic white noise there seems to be some limitations on the choice of S but for the free case the construction works for all associative algebras.

2. General Renormalized Quadratic White Noise

For a Hilbert space $\mathcal{H}$ and a real number q, $-1 < q \leq 1$ let us consider q-deformed white noise operators [FB,BKS]: the creation $a_\phi^\star$ and its adjoint annihilation a_ϕ indexed by $\phi \in \mathcal{H}$. These operators fulfill the following commutation relation:

$$a_\phi a_\psi^\star - q a_\psi^\star a_\phi = \langle \phi, \psi \rangle. \tag{3}$$

For the case $\mathcal{H} = \mathcal{L}^2(M, d\mu)$ we can write informally

$$a_\phi = \int_M \overline{\phi(t)} a_t, \qquad a_\phi^\star = \int_M \phi(t) a_t^\star,$$

where $a_t, a_t^\star$ denote white noise annihilation and creation operators.

Our goal is to introduce operators b_ϕ and $b_\phi^\star$ which would be informally treated as integrals of squares of white noise operators

$$b_\phi = \int_M \overline{\phi(t)} (a_t)^2, \qquad b_\phi^\star = \int_M \phi(t)(a_t^\star)^2.$$

In order to give meaning to these expressions let us consider a sequence (I_i) of disjoint measurable subsets of M, each of the same measure l and a sequence (χ_i) of orthogonal functions

$$\chi_i(x) = \begin{cases} 1 : x \in I_i \\ 0 : x \notin I_i \end{cases}.$$

Furthermore let us consider piecewise constant functions ϕ, ψ,

$$\phi(x) = \sum_i \phi(x_i)\chi_i(x), \qquad \psi(x) = \sum_i \psi(x_i)\chi_i(x),$$

for a sequence (x_i) such that $x_i \in I_i$. Now let us define

$$b_\phi = \sum_i \overline{\phi(x_i)} (a_{\frac{1}{\sqrt{l}}\chi_i})^2, \qquad b_\psi^\star = \sum_i \psi(x_i)(a^\star_{\frac{1}{\sqrt{l}}\chi_i})^2.$$

A simple computation shows that for squares of creation and annihilation operators

$$a_\zeta^2 a_\xi^{\star 2} - q^4 a_\xi^{\star 2} a_\zeta^2 = (1+q)\langle \zeta, \xi\rangle^2 + q(1+q)^2 \langle \zeta, \xi\rangle a_\xi^\star a_\zeta$$

hold. For this reason we have

$$\begin{aligned} &b_\phi b_\psi^\star - q^4 b_\psi^\star b_\phi \\ &\quad = (1+q)\sum_i \psi(x_i)\overline{\phi(x_i)} + q(1+q)^2 \sum_i \psi(x_i)\overline{\phi(x_i)}\; a^\star_{\frac{1}{\sqrt{l}}\chi_i}\; a_{\frac{1}{\sqrt{l}}\chi_i}. \end{aligned}$$

Since the $\mathcal{L}^2(M, d\mu)$ norm of the function $\frac{1}{\sqrt{l}}\chi_i$ is equal to 1, the operator $a^\star_{\frac{1}{\sqrt{l}}\chi_i} a_{\frac{1}{\sqrt{l}}\chi_i}$ is a number operator. If we consider only the creation and annihilation operators $a_\theta^\star$, a_θ for functions θ that are piecewise constant on sets I_i then the operators $a^\star_{\frac{1}{\sqrt{l}}\chi_i} a_{\frac{1}{\sqrt{l}}\chi_i}$ and $\int_{I_i} a_t^\star a_t$ have the same commutation relations with others and therefore they are indistinguishable in the sense of vacuum expectation values. Under these assumptions we can write

$$\begin{aligned} &b_\phi b_\psi^\star - q^4 b_\psi^\star b_\phi \\ &\quad = \frac{1+q}{l}\int_M \psi(x)\overline{\phi(x)}\, d\mu(x) + q(1+q)^2 \int_M \psi(x)\overline{\phi(x)}\; a_t^\star a_t. \end{aligned} \tag{4}$$

The preceding calculations hold only for a very limited class of functions ϕ and ψ. However, we shall postulate the following commutation relation between quadratic creation and annihilation operators for all ϕ and ψ:

$$b_\phi b_\psi^\star - q^4 b_\psi^\star b_\phi = \gamma\langle \phi, \psi\rangle + c\, n_{\bar{\phi}\psi}, \tag{5}$$

for some constants γ, c and where n_f, called a number operator, should be understood as a generalization of the usual number operator $\int_M f(x) a_t^\star a_t$.

2.1. Fock representations. Hudson–Parthasarathy's operators a_ϕ, $a^\star_\phi$ ($\phi \in \mathcal{H}$) are usually represented as operators acting on some Hilbert space with a cyclic vector Ω with the property that $a_\phi \Omega = 0$ for all $\phi \in \mathcal{H}$. Since the operators b_ϕ, $b^\star_\phi$ are interpreted as smeared renormalised squares of white noise operators a_t, $a^\star_t$, therefore it is natural to ask if it is possible to find a representation of operators b_ϕ, $b^\star_\phi$, n_ϕ acting on some Hilbert space Γ^2 such that Γ^2 contains a cyclic vector Ω, called a vacuum, such that $b_\phi \Omega = 0$, $n_\phi \Omega = 0$ for all ϕ. In such a setup we will be able to define a state τ on the space of operators acting on Γ^2 defined by $\tau(X) = \langle \Omega, X\Omega \rangle$ which would play the role of a (noncommutative) expectation value.

3. Bosonic Quadratic White Noise

3.1. Bosonic commutation relations. For the bosonic case $q = 1$ Eq. (4) takes the form

$$[b_\phi, b^\star_\psi] = 2\gamma_0 \langle \phi, \psi \rangle + 4 n_{\bar{\phi}\psi}, \tag{6}$$

where $\gamma_0 = \frac{1}{l}$. Furthermore, we postulate that two creation, two annihilation and two number operators should commute:

$$[b_\phi, b_\psi] = 0, \qquad [b^\star_\phi, b^\star_\psi] = 0, \qquad [n_\phi, n_\psi] = 0. \tag{7}$$

A simple calculation for piecewise constant functions

$$\left[\sum_i \phi(x_i) a^\star_{\frac{1}{\sqrt{l}}\chi_i} a_{\frac{1}{\sqrt{l}}\chi_i}, \sum_j \psi(x_j) (a^\star_{\frac{1}{\sqrt{l}}\chi_j})^2 \right] = 2 \sum_k \psi(x_k) \phi(x_k) (a^\star_{\frac{1}{\sqrt{l}}\chi_k})^2$$

gives us a motivation for the following commutation relations:

$$[n_\phi, b^\star_\psi] = 2 b^\star_{\phi\psi}, \qquad [b_\psi, n_\phi] = 2 b_{\psi\phi^\star}. \tag{8}$$

3.2. Classical quadratic processes. By the spectral theorem a commuting family of normal operators has a common spectral measure. After applying a state the spectral measure becomes an ordinary measure which has a natural probabilistic interpretation as a joint distribution of random variables corresponding to operators from our family.

Let us define for $s \in \mathbb{R}$,

$$Q_s(\phi) = b_{\phi^\star} + b^\star_\phi + s n_\phi. \tag{9}$$

Similar to the white noise it is a weak process [S], i.e. an operator valued function on a linear space $\mathcal{L}^2(M, d\mu) \cap \mathcal{L}^\infty(M, d\mu)$. In the case $M = \mathbb{R}_+$ we can construct from it a stochastic process $Q_s(t) = Q_s(\chi_{(0,t)})$.

Theorem 1. *Let us fix $s \in \mathbb{R}$. Then $\{Q_s(\phi)\}$ forms a commuting family of normal operators and therefore it is a classical stochastic process. With respect to the expectation value τ, it is a Markovian process.*

Proof. The first part of the proof is a simple application of (6)–(8).

The property that Q_s is a process with independent increments means exactly that for all disjoint sets $M_1, M_2 \subset M$ and $f_i \in \mathrm{Alg}\{Q_s(\phi) : \phi \in \mathcal{L}^2(M_i)\}$ the equality $\tau(f_1 f_2) = \tau(f_1)\tau(f_2)$ holds. Note that every expression containing operators n_ϕ, b_ϕ, $b^\star_\phi$ ($\phi \in \mathcal{H}$) can be written according to the relations (6), (8) in the normal form, a linear combination of products of type

$$b^\star_{\phi_1} \cdots b^\star_{\phi_k} n_{\chi_1} \ldots n_{\chi_m} b_{\psi_1} \cdots b_{\psi_l}. \tag{10}$$

Each of the operators n_{ϕ_1}, b_{ϕ_1}, $b^\star_{\phi_1}$ commutes with each of the operators n_{ϕ_2}, b_{ϕ_2}, $b^\star_{\phi_2}$ for $\phi_i \in \mathcal{L}^2(M_i, d\mu)$, therefore a product of two expressions of the form (10), one being an element of $\mathrm{Alg}\{n_\phi, b_\phi, b^\star_\phi : \phi \in \mathcal{L}^2(M_1)\}$ and the other an element of $\mathrm{Alg}\{n_\phi, b_\phi, b^\star_\phi : \phi \in \mathcal{L}^2(M_2)\}$ is–up to a permutation of factors–in a normally ordered form. The state τ has a property that on normally ordered products it takes nonzero values only on multiples of identity and $\tau(f_1 f_2) = \tau(f_1)\tau(f_2)$ follows.

Now it is enough to notice that the expectation value of $Q_s(\phi)$ is equal to 0 for any ϕ. □

3.3. Quadratic variation of a Brownian motion. Let $M = \mathbb{R}_+$ and let us consider an arithmetic series (t_i), $t_i = li$. For the sum of squares of increments of a standard Brownian motion the following operator equality holds:

$$\begin{aligned}\sum_i \phi(t_i)[B(t_{i+1}) - B(t_i)]^2 &= \sum_i \phi(t_i)[a_{\chi_i} + a^\star_{\chi_i}]^2 \\ &= \sum_i \phi(t_i)[a^2_{\chi_i} + 2a^\star_{\chi_i} a_{\chi_i} + a^{\star\,2}_{\chi_i} + (t_{i+1} - t_i)],\end{aligned}$$

where χ_i is the characteristic function of an interval (t_i, t_{i+1}). In the preceding discussion we have chosen the commutation relations between operators lb_{χ_i}, $lb^\star_{\chi_j}$ and ln_{χ_k} to coincide with commutation relations between $a^2_{\chi_i}$, $a^{\star 2}_{\chi_j}$ and $a^\star_{\chi_k} a_{\chi_k}$ whenever the length of intervals is equal to $l = \frac{1}{\gamma_0}$. Therefore, for any function ϕ which is piecewise constant on intervals (t_i, t_{i+1}) we can write

$$\begin{aligned}&\sum_i \phi(t_i)[B(t_{i+1}) - B(t_i)]^2 \\ &\quad = \sum_i \phi(t_i)\left\{\frac{1}{l}[b_{\chi_i} + b^\star_{\chi_i} + 2n_{\chi_i}] + (t_{i+1} - t_i)\right\} = \frac{1}{l} Q_2(\phi) + \int_{\mathbb{R}_+} \phi(x)\, dx.\end{aligned}$$

This equation can be viewed as follows. Just like a_ϕ, $a^\star_\phi$ are quantum components of the Brownian motion, for functions ϕ which are piecewise constant on intervals which length is a multiplicity of $\frac{1}{\gamma_0}$ operators b_ϕ, $b^\star_\phi$, $2n_\phi$ are quantum components of the quadratic variation of Brownian motion. The constant $\frac{1}{\gamma_0}$ describes the lengthscale under which such interpretation is no longer valid.

The measures corresponding to $\gamma_0 Q_2(t) + t = \gamma_0 Q_2(\chi_{(0,t)}) + t$ for t being the multiplicity of $\frac{1}{\gamma_0}$ are therefore the χ^2 distributions. From this it follows that for arbitrary t these are gamma distributions and $\gamma_0 Q_2(t) + t$ is a gamma process.

4. Quadratic Bosonic White Noise on an Interacting Fock Space

Let $\mathcal{A}$ be a commutative $C^\star$-algebra of continuous functions on some set M with a measure μ and let the state on $\mathcal{A}$ induced by μ be denoted also by μ.

Definition 1. *A partition of a finite set A is a collection $\pi = \{\pi_1, \dots, \pi_m\}$ of nonempty sets π_p, which are pairwise disjoint and their union is equal to A.*

An ordered partition of a finite set A is a set $\pi = \{\pi_1, \dots, \pi_m\}$ of nonempty sequences $\pi_p = (\pi_{p1}, \dots, \pi_{p,n_p})$, such that the family of sets $\{\pi_{p1}, \dots, \pi_{p,n_p}\}$, $1 \le p \le m$ forms a partition of A.

For a fixed positive constant γ_0 let us consider a vector space $\widetilde{\Gamma_{\mathrm{b}}^2}(\mathcal{A}) = \bigoplus_{n\ge 0} \mathcal{A}^{\widehat{\otimes} n}$ (where $\mathcal{A}^{\widehat{\otimes} n}$ denotes the symmetric tensor power) with a sesquilinear form defined by

$$\langle \chi_1 \otimes \cdots \otimes \chi_k, \psi_1 \otimes \cdots \otimes \psi_l \rangle = \delta_{kl} \frac{2^k}{k!} \sum_{\{\pi_1,\dots,\pi_m\}} \prod_{1\le p\le m} \frac{\gamma_0}{n_p} \mu(\chi^\star_{\pi_{p1}} \psi_{\pi_{p1}} \cdots \chi^\star_{\pi_{pn_p}} \psi_{\pi_{pn_p}}), \tag{11}$$

where the sum is taken over all ordered partitions π of the set $\{1, \dots, n\}$.

Please note that this sesquilinear form is well-defined on the full tensor power $\mathcal{A}^{\otimes n}$, however we shall usually use it on the symmetric tensor power $\mathcal{A}^{\widehat{\otimes} n}$.

In the sum $\widetilde{\Gamma_{\mathrm{b}}^2} = \bigoplus_{n\ge 0} \mathcal{A}^{\widehat{\otimes} n}$ appears a summand $\mathcal{A}^{\widehat{\otimes} 0}$ which should be understood as a one-dimensional Hilbert space $\mathbb{C}\Omega$ where Ω is a unital vector called vacuum.

The Hilbert space Γ_{b}^2, a completion of $\widetilde{\Gamma_{\mathrm{b}}^2}$ will be called bosonic quadratic Fock space. In the following by $\mathcal{A}^{\widehat{\otimes} k}$ we shall mean the completion of the symmetric tensor power $\mathcal{A}^{\widehat{\otimes} k}$ with respect to the scalar product (11).

Question 1. For the sesqilinear form (11) all algebraic considerations of this section hold even if the algebra $\mathcal{A}$ is not commutative. If this case we only have to assume that the state μ is tracial and we have to replace the number operator (14) by a pair of left and right number operators. Unfortunately, in this general situation the form (11) is not always positively definite. Is it possible to find some nontrivial examples of noncommutative algebras $\mathcal{A}$ with tracial states μ such that (11) is positively definite?

For $\psi \in \mathcal{A}$ we define the action of the quadratic creation, annihilation and number operators on simple tensors by

$$b^\star_\psi(\chi_1 \otimes \cdots \otimes \chi_k) = \sum_{0\le i\le k} \chi_1 \otimes \cdots \otimes \chi_i \otimes \psi \otimes \chi_{i+1} \otimes \cdots \otimes \chi_k, \tag{12}$$

$$\begin{aligned} b_\psi(\chi_1 \otimes \cdots \otimes \chi_k) = {} & 2\gamma_0\, \mu(\psi^\star \chi_1)\, \chi_2 \otimes \cdots \otimes \chi_k \\ & + 2 \sum_{2\le i\le k} \chi_2 \otimes \cdots \otimes \chi_{i-1} \otimes (\chi_i \psi^\star \chi_1) \otimes \chi_{i+1} \otimes \cdots \otimes \chi_k, \end{aligned} \tag{13}$$

$$n_\psi(\chi_1 \otimes \cdots \otimes \chi_k) = \sum_{1\le i\le k} \chi_1 \otimes \cdots \chi_{i-1} \otimes (\psi \chi_i) \otimes \chi_{i+1} \otimes \cdots \otimes \chi_k, \tag{14}$$

for $k \ge 1$ and their action on the vacuum by

$$b^\star_\psi(\Omega) = \psi, \qquad b_\psi(\Omega) = 0, \qquad n_\psi(\Omega) = 0. \tag{15}$$

Please note that simple tensors are in general not elements of the symmetric tensor power of $\mathcal{A}$. However, by linearity these definitions extend to a dense subspace of the symmetric tensor power $\mathcal{A}^{\widehat{\otimes} n}$. What is important, the range of operators $b_\phi : \mathcal{A}^{\widehat{\otimes} n} \to \mathcal{A}^{\widehat{\otimes}(n-1)}$, $b^\star_\phi : \mathcal{A}^{\widehat{\otimes} n} \to \mathcal{A}^{\widehat{\otimes}(n+1)}$, $n_\phi : \mathcal{A}^{\widehat{\otimes} n} \to \mathcal{A}^{\widehat{\otimes} n}$ is again a symmetric power of $\mathcal{A}$.

A difficulty arises from the fact that such defined operators are not bounded. For example we shall not claim that $b^\star_\phi$ is an adjoint of b_ϕ because such a statement is not easy to prove since it demands careful discussion of domains of operators. It seems that in order to do this we would have to define these operators on some analogue of exponential domain of Hudson and Parthasarathy [HP] in a less intuitive way. Similarly commutation relations will hold only in a restricted sense.

Theorem 2. *Operators b_ϕ, $b^\star_\phi$, n_ϕ fulfill the following operator norm estimates with respect to the scalar product (11):*

$$\left\| b_\phi : \mathcal{A}^{\widehat{\otimes} k} \to \mathcal{A}^{\widehat{\otimes}(k-1)} \right\| \le \sqrt{2k} \left(\sqrt{\gamma_0}\, \|\phi\|_{\mathcal{L}^2} + (k-1)\|\phi\|_{\mathcal{L}^\infty} \right), \tag{16}$$

$$\left\| b^\star_\phi : \mathcal{A}^{\widehat{\otimes}(k-1)} \to \mathcal{A}^{\widehat{\otimes} k} \right\| \le \sqrt{2k} \left(\sqrt{\gamma_0}\, \|\phi\|_{\mathcal{L}^2} + (k-1)\|\phi\|_{\mathcal{L}^\infty} \right), \tag{17}$$

$$\left\| n_\phi : \mathcal{A}^{\widehat{\otimes} k} \to \mathcal{A}^{\widehat{\otimes} k} \right\| \le k\, \|\phi\|_{\mathcal{L}^\infty}. \tag{18}$$

Proof. Let us consider a map $\mathcal{A}^{\otimes k} \to \mathcal{A}^{\otimes(k-1)}$ defined on simple tensors by $\psi_1 \otimes \cdots \otimes \psi_k \mapsto 2\gamma_0 \langle \phi, \psi_1 \rangle \psi_2 \otimes \cdots \otimes \psi_k$. It is easy to see that the operator norm of this map does not exceed $\sqrt{2k\gamma_0}\, \|\phi\|_{\mathcal{L}^2}$.

And now, for any i let us consider a map $\mathcal{A}^{\otimes k} \to \mathcal{A}^{\otimes(k-1)}$ defined on simple tensors by $\psi_1 \otimes \cdots \otimes \psi_k \mapsto 2\psi_2 \otimes \cdots \otimes \psi_{i-1} \otimes (\psi_i \phi^\star \psi_1) \otimes \psi_{i+1} \otimes \cdots \otimes \psi_k$. It is easy to see that the operator norm of this map does not exceed $\sqrt{2k}\, \|\phi\|_{\mathcal{L}^\infty}$.

The sum of these maps is equal to b_ϕ, which shows the estimate (16).

The estimation (17) follows from (16) because b_ϕ is an adjoint of $b^\star_\phi$ what will be proven in Theorem 3 and therefore their norms are equal.

The inequality (18) is obvious. □

This theorem allows us to define the action on $\mathcal{A}^{\widehat{\otimes} k}$ of operators a_ϕ, $a^\star_\phi$ for all $\phi \in \mathcal{L}^2(M, d\mu) \cap \mathcal{L}^\infty(M, d\mu)$ and of operator n_ϕ for all $\phi \in \mathcal{L}^\infty(M, d\mu)$.

Theorem 3. *For any $\zeta \in \mathcal{L}^2(M, d\mu) \cap \mathcal{L}^\infty(M, d\mu)$ operators b_ζ and $b^\star_\zeta$ are adjoint in the sense that*

$$\langle b_\zeta \Psi, \Phi \rangle = \langle \Psi, b^\star_\zeta \Phi \rangle$$

for all $\Psi \in \mathcal{A}^{\widehat{\otimes} k}$, $\Phi \in \mathcal{A}^{\widehat{\otimes} l}$. For any $\phi \in \mathcal{L}^\infty(M, d\mu)$ the adjoint of n_ϕ is equal to $n_{\phi^\star}$ in the sense that

$$\langle n_\zeta \Psi, \Phi \rangle = \langle \Psi, n_{\zeta^\star} \Phi \rangle,$$

for all $\Psi \in \mathcal{A}^{\widehat{\otimes} k}$, $\Phi \in \mathcal{A}^{\widehat{\otimes} l}$.

Proof. Let us consider $\Psi = \sum_M \psi^M_0 \otimes \cdots \otimes \psi^M_{k-1} \in \mathcal{A}^{\widehat{\otimes} k}$ and $\Phi = \sum_N \phi^N_1 \otimes \cdots \otimes \phi^N_l \in \mathcal{A}^{\widehat{\otimes} l}$. Since Ψ is a symmetric tensor the value of a scalar product $\left\langle \Psi, \sum_N \phi^N_1 \otimes \cdots \otimes \phi^N_{i-1} \otimes \zeta \otimes \phi^N_i \otimes \cdots \otimes \phi^N_l \right\rangle$ does not depend on i. This implies that

$$\langle \Psi, b^\star(\zeta)\Phi \rangle = (l+1)\langle \Psi, \zeta \otimes \Phi \rangle.$$

We can split the sum in the definition (11) of $\langle \Psi, \zeta \otimes \Phi \rangle$ into two parts: over ordered partitions π of the set $\{0, 1, \dots, k-1\}$ which contain a block consisting of a single element 0 and all the others ordered partitions. Since the state μ is tracial we have

$$\begin{aligned}\langle \Psi, \zeta \otimes \Phi \rangle = \delta_{k,l+1} \frac{2^k}{k!} \sum_M \sum_N \Big[& \gamma_0 \, \mu(\psi_0^{M\star} \zeta) \\ & \times \sum_{\{\pi_1,\dots,\pi_m\}} \prod_{1 \le p \le m} \frac{\gamma_0}{n_p} \mu\big(\psi^{M\star}_{\pi_{p1}} \phi^N_{\pi_{p1}} \cdots \psi^{M\star}_{\pi_{p,n_p}} \phi^N_{\pi_{p,n_p}}\big) \\ & + \sum_{\{\pi_1,\dots,\pi_m\}} \sum_{1 \le q \le m} \gamma_0 \, \mu\big(\psi_0^{M\star} \zeta \psi^{M\star}_{\pi_{q1}} \phi^N_{\pi_{q1}} \cdots \psi^{M\star}_{\pi_{q,n_q}} \phi^N_{\pi_{q,n_q}}\big) \\ & \times \prod_{1 \le p \le m, p \ne q} \frac{\gamma_0}{n_p} \mu\big(\psi^{M\star}_{\pi_{p1}} \cdots \psi^{M\star}_{\pi_{p,n_p}} \phi^N_{\pi_{p1}} \cdots \phi^N_{\pi_{p,n_p}}\big) \Big],\end{aligned}$$

where the sums over π are taken over all ordered partitions π of the set $\{1, \dots, k-1\}$.

Note that for any nonempty subset A of the set $\{1, \dots, k-1\}$ we have

$$\begin{aligned}&\sum_{\pi_q} \mu\big(\psi_0^{M\star} \zeta \psi^{M\star}_{\pi_{q1}} \phi^N_{\pi_{q1}} \cdots \psi^{M\star}_{\pi_{q,n_q}} \phi^N_{\pi_{q,n_q}}\big) = \sum_{\pi_q} \sum_{1 \le r \le n_q} \frac{1}{n_q} \\ &\times \mu\big(\psi^{M\star}_{\pi_{q1}} \phi^N_{\pi_{q1}} \cdots \psi^{M\star}_{\pi_{q,r-1}} \phi^N_{\pi_{q,r-1}} \big(\psi^M_{\pi_{q,r}} \zeta^\star \psi^M_0\big)^\star \phi^N_{\pi_{q,r}} \psi^{M\star}_{\pi_{q,r+1}} \phi^N_{\pi_{q,r+1}} \cdots \psi^{M\star}_{\pi_{q,n_q}} \phi^N_{\pi_{q,n_q}}\big),\end{aligned}$$

where the sums are taken over all sequences $\pi_q = (\pi_{q,1}, \dots, \pi_{q,n_q})$ such that each of the elements of A appears in π_q exactly once.

Now, it is easy to see that

$$\begin{aligned}\langle \Psi, b^\star_\zeta \Phi \rangle = 2\delta_{k,l+1} \Big[& \gamma_0 \sum_M \langle \psi_0^M, \zeta \rangle \langle \psi_1^M \otimes \cdots \otimes \psi_k^M, \Phi \rangle + \\ & + \sum_i \sum_M \langle \psi_1^M \otimes \cdots \otimes \psi_{i-1}^M \otimes \big(\psi_i^M \zeta^\star \psi_0^M\big) \otimes \psi_{i+1}^M \otimes \cdots \psi_k^M, \Phi \rangle \Big],\end{aligned}$$

which proves the first part of the theorem.

The proof of the fact that the adjoint of n_ϕ is equal to $n_{\phi^\star}$ is very simple and we shall omit it. □

Theorem 4. *For any $\phi, \psi \in \mathcal{L}^2(\mathcal{A}) \cap \mathcal{L}^\infty(\mathcal{A})$, $\zeta, \eta \in \mathcal{L}^\infty(\mathcal{A})$ and $\Phi \in \mathcal{A}^{\widehat{\otimes} k}$ we have*

$$[b^\star_\phi, b^\star_\psi]\Phi = 0, \qquad [b_\phi, b_\psi]\Phi = 0, \tag{19}$$

$$[b_\phi, b^\star_\psi]\Phi = (2\gamma_0 \langle \phi, \psi \rangle + 4 n_{\phi^\star \psi})\Phi, \tag{20}$$

$$[n_\zeta, b^\star_\psi]\Phi = 2 b^\star_{\zeta \psi} \Phi, \qquad [b_\psi, n_\zeta]\Phi = 2 b_{\zeta^\star \psi} \Phi. \tag{21}$$

Proof. Since the definitions of creation quadratic operators and standard creation operators coincide, two quadratic creation operators commute. Quadratic annihilation operators are their adjoints so they commute with each other as well.

Let us consider two auxiliary annihilation operators

$$\hat{b}_\psi(\chi_1 \otimes \cdots \otimes \chi_k) = 2\gamma_0\, \mu(\psi^\star \chi_1)\, \chi_2 \otimes \cdots \otimes \chi_k,$$
$$\tilde{b}_\psi(\chi_1 \otimes \cdots \otimes \chi_k) = 2 \sum_{2 \le i \le k} \chi_2 \otimes \cdots \otimes \chi_{i-1} \otimes (\chi_i \psi^\star \chi_1) \otimes \chi_{i+1} \otimes \cdots \otimes \chi_k.$$

We have $b_\psi = \hat{b}_\psi + \tilde{b}_\psi$.

The definition of $\hat{b}$ coincides up to a factor with the definition of the standard annihilation operator, therefore

$$[\hat{b}_\phi, b^\star_\psi] = 2\gamma_0 \langle \phi, \psi \rangle.$$

It is easy to see that there are exactly two terms in the commutator which do not cancel:

$$[\tilde{b}_\phi, b^\star_\psi](\chi_1 \otimes \cdots \otimes \chi_k) = 2\gamma_0(\psi\phi^\star\chi_1 + \chi_1\phi^\star\psi) \otimes \chi_2 \otimes \cdots \otimes \chi_k,$$

which is equal to the action of $4\gamma_0 n_{\psi\phi^\star}$. If we do not assume that $\mathcal{A}$ is commutative we have to replace n by an appropriate sum of left and right multiplication operators. □

5. Another Representation of the Quadratic Bosonic Fock Space

The construction from the previous subsection can be presented in a more direct way. Let us consider an isomorphism $C(M)\otimes\cdots\otimes C(M) = C_{\mathrm{alg}}(M\times\cdots\times M)$, where $C_{\mathrm{alg}}(M^n)$ denotes the space of continuous functions on $M^n = M \times \cdots \times M$ which are finite sums of simple tensors. The multiplication map $\mathcal{A}^{\otimes n} \ni x_1 \otimes \cdots \otimes x_n \mapsto x_1 \cdots x_n \in \mathcal{A}$ under this isomorphism is equal to the diagonal map $C_{\mathrm{alg}}(M^n) \ni f \mapsto \Delta f \in C(M)$, where $(\Delta f)(x) = f(x, x, \dots, x)$ for any $x \in M$.

For any ordered partition $\pi = \{\pi_1, \dots, \pi_k\}$ of the set $\{1, \dots, n\}$ let $\Delta_\pi : M^k \to M^n$ be an embedding of M^k onto the diagonal of M^n defined by partition π:

$$\Delta_\pi(x_1, \dots, x_k) = (y_1, \dots, y_n),$$

where $y_r = x_s$ for $r \in \pi_s$. $\Delta^\star_\pi(\mu^{\otimes m})$ denotes the pull–back of the measure $\mu^{\otimes m}$ on M^m onto a multidiagonal of M^k defined by Δ_π, namely

$$\int_{M^k} f(x_1, \dots, x_k)\, d\Delta^\star_\pi(\mu^{\otimes m}) = \int_{M^m} f[\Delta_\pi(y_1, \dots, y_m)] d\mu(y_1) \cdots d\mu(y_m).$$

Note that however the function Δ_π depends on the choice of order of blocks of partition π, the pull–back measure $\Delta^\star_\pi(\mu^{\otimes m})$ does not depend on it.

Therefore the scalar product (11) can be represented as

$$\langle \Phi, \Psi \rangle = \delta_{kl} \int_{M^k} \overline{\Phi(x_1, \dots, x_k)} \Psi(x_1, \dots, x_k)\, d\mu_k(x_1, \dots, x_k)$$

for $\Phi \in C_{\mathrm{alg}}(M^k)$, $\Psi \in C_{\mathrm{alg}}(M^l)$, where the measure μ_k on M^k is given by

$$\mu_k = \frac{2^k}{k!} \sum_{\{\pi_1, \dots, \pi_m\}} \frac{\gamma_0^m}{|\pi_1| \cdots |\pi_m|} \Delta^\star_\pi(\mu^{\otimes m}).$$

In other words: the measure μ_k on M^k is a sum of the product measure on M^k and of product measures with supports on all multidiagonals of M^k.

The operators defined in the last section in this context are represented as follows:

$$(b^\star_\phi \Psi)(x_1, \dots, x_{n+1}) = \sum_i \phi(x_i)\Psi(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1}), \tag{22}$$

$$\begin{aligned}(b_\phi \Psi)(x_1, \dots, x_n) = {} & 2\gamma_0 \int_M \overline{\phi(x_{n+1})}\Psi(x_1, \dots, x_{n+1}) d\mu(x_{n+1}) \\ & + 2\sum_i \overline{\phi(x_i)}\Psi(x_1, x_2, \dots, x_{i-1}, x_i, x_i, x_{i+1}, \dots, x_n),\end{aligned} \tag{23}$$

$$(n_\phi \Psi)(x_1, \dots, x_n) = \Psi(x_1, \dots, x_n) \sum_i \phi(x_i). \tag{24}$$

6. Quadratic and Linear Bosonic White Noise

It is natural to ask if it is possible to incorporate both quadratic white noise operators b_ϕ, $b^\star_\phi$, n_ϕ and linear white noise operators a_ϕ, $a^\star_\phi$ to the same algebra. We postulate relations (6)–(8) of quadratic white noise, a relation of white noise

$$[a_\phi, a^\star_\psi] = \langle \phi, \psi \rangle,$$

and some relations linking quadratic and linear noises, among which we shall mention only

$$[a_\phi, b^\star_\psi] = 2a^\star_{\phi^\star\psi}, \qquad [b_\phi, a^\star_\psi] = 2a_{\phi\psi^\star}.$$

We shall prove now that in general it is impossible to find a Fock representation of these relations.

Let X be a measurable subset of M. Let $0 < \mu(X) = l < \infty$ and let $\chi(x) = 1$ for $x \in X$ and $\chi(x) = 0$ otherwise. By rewriting operators in the normal order we have for any $c \in \mathbb{R}$,

$$\begin{aligned}&\langle (ca^\star_\chi a^\star_\chi + b^\star_\chi)\Omega, (ca^\star_\chi a^\star_\chi + b^\star_\chi)\Omega \rangle = \langle \Omega, (ca_\chi a_\chi + b_\chi)(ca^\star_\chi a^\star_\chi + b^\star_\chi)\Omega \rangle \\ &\quad = 2c^2 \langle \chi, \chi \rangle^2 + 2\gamma_0 \langle \chi, \chi \rangle + 2c\langle \chi^2, \chi \rangle + 2c\langle \chi, \chi^2 \rangle = 2c^2 l^2 + 4cl + 2\gamma_0 l.\end{aligned}$$

It is easy to see that for $l < \frac{1}{\gamma_0}$ this expression takes negative values.

This is can be interpreted as another manifestation of the constant $\frac{1}{\gamma_0}$ which describes the lengthscale, under which a quadratic white noise loses its physical meaning.

7. Free Quadratic White Noise

7.1. Free commutation relations. For the free case $q = 0$ the coefficient standing at number operator in Eq. (4) is equal to 0 so this equation is equivalent to the commutation relations of free creation and annihilation operators. However if we redefine annihilation and creation operators by mulitiplying them by $\frac{1}{\sqrt{q}}$ and take the limit $q \to 0$ and $ql = \frac{1}{\gamma}$ we obtain

$$b_\phi b^\star_\psi = \gamma \langle \phi, \psi \rangle + n_{\bar{\phi}\psi}. \tag{25}$$

The simple calculation for $q = 0$,

$$(a^\star_{\frac{1}{\sqrt{l}}\chi_i} a_{\frac{1}{\sqrt{l}}\chi_i})(a^\star_{\frac{1}{\sqrt{l}}\chi_j})^2 = \delta_{ij}(a^\star_{\frac{1}{\sqrt{l}}\chi_j})^2,$$
$$(a^\star_{\frac{1}{\sqrt{l}}\chi_i} a_{\frac{1}{\sqrt{l}}\chi_i})(a^\star_{\frac{1}{\sqrt{l}}\chi_j} a_{\frac{1}{\sqrt{l}}\chi_j}) = \delta_{ij}(a^\star_{\frac{1}{\sqrt{l}}\chi_i} a_{\frac{1}{\sqrt{l}}\chi_i})$$

motivates us to postulate the other free commutation relations

$$n_\phi b^\star_\psi = b^\star_{\phi\psi}, \qquad b_\psi n_\phi = b_{\psi\phi^\star}, \qquad n_\phi n_\psi = n_{\phi\psi}. \tag{26}$$

Therefore, heuristically quadratic free white noise defined like this could be interpreted as a square of free white noise with small violations of freeness in the limit $q \to 0$. However, since we take the limit $l \to \infty$ it is impossible to repeat the arguments from Subsect. 3.3 and it should be stressed that this interpretation is very informal.

7.2. Realization of quadratic free white noise. Let $\mathcal{A}$ be an associative $\star$-algebra and $\mu : \mathcal{A} \to \mathbb{C}$ be a state. Let us consider a pre–Hilbert space $\widetilde{\Gamma^2_{\mathrm{f}}}(\mathcal{A}) = \bigoplus_{n\geq 0} \mathcal{A}^{\otimes n}$ with a scalar product defined by

$$\begin{aligned}&\langle \psi_1 \otimes \cdots \otimes \psi_l, \chi_1 \otimes \cdots \otimes \chi_k\rangle \\ &\quad = \delta_{kl} \sum_{m\geq 1} \sum_{(n_0,\ldots,n_m)} \prod_{1\leq p\leq m} \gamma\mu(\psi^\star_{n_p}\psi^\star_{n_p-1}\cdots\psi^\star_{n_{p-1}+1}\chi_{n_{p-1}+1}\chi_{n_{p-1}+2}\cdots\chi_{n_p}),\end{aligned} \tag{27}$$

where the sum is taken over all increasing sequences of natural numbers $(n_0, n_1, \ldots, n_m)$ such that $n_0 = 0$ and $n_m = k$ what corresponds to all Boolean partitions of a set $\{1, \ldots, k\}$, i.e. partitions into blocks of consecutive elements $\{\{1, 2, \ldots, n_1\}, \{n_1 + 1, n_1 + 2, \ldots, n_2\}, \ldots, \{n_{m-1} + 1, n_{m-1} + 2, \ldots, n_m\}\}$.

The completion of $\widetilde{\Gamma^2_{\mathrm{f}}}(\mathcal{A})$ will be called the quadratic free Fock space and will be denoted by $\Gamma^2_{\mathrm{f}}(\mathcal{A})$.

For $\psi \in \mathcal{A}$ we define the action of the quadratic creation operators on $\widetilde{\Gamma^2_{\mathrm{f}}}$ by

$$\begin{aligned}b^\star_\psi(\chi_1 \otimes \cdots \otimes \chi_k) &= \psi \otimes \chi_1 \otimes \cdots \otimes \chi_k,\\ b_\psi(\chi_1 \otimes \cdots \otimes \chi_k) &= \gamma\mu(\psi^\star\chi_1)\, \chi_2 \otimes \cdots \otimes \chi_k + (\psi^\star\chi_1\chi_2) \otimes \chi_3 \otimes \cdots \otimes \chi_k,\\ n_\psi &= (\psi\chi_1) \otimes \chi_2 \otimes \cdots \otimes \chi_k.\end{aligned}$$

On the algebra $\mathcal{A}$ we shall introduce (noncommutative) $\mathcal{L}^2$ and $\mathcal{L}^\infty$ norms by

$$\begin{aligned}\|x\|_{\mathcal{L}^2} &= \sqrt{\mu(x^\star x)},\\ \|x\|_{\mathcal{L}^\infty} &= \sup_{y\in\mathcal{A},\ \|y\|_{\mathcal{L}^2}=1} |\mu(xy)|.\end{aligned}$$

Theorem 5. *For quadratic free operators the following estimations hold:*

$$\begin{aligned}&\left\|b^\star_\phi\right\| = \left\|b_\phi\right\| \leq \sqrt{\gamma}\|\phi\|_{\mathcal{L}^2} + \|\phi\|_{\mathcal{L}^\infty},\\ &\|n_\phi\| \leq \|\phi\|_{\mathcal{L}^\infty}.\end{aligned}$$

Theorem 6. *The operators b_ϕ and $b^\star_\phi$ are adjoint. The adjoint to n_ϕ is equal to $n_{\phi^\star}$.*

Proof. First note that the summands in (27) can be split into two groups: those for which the first component of the partition defined by (n_i) is and those for which is not a single element. Therefore for $\Psi = \psi_1 \otimes \cdots \otimes \psi_k$, $X = \chi_0 \otimes \cdots \otimes \chi_k$ we have

$$\begin{aligned}
\langle b^\star_\phi \Psi, X\rangle &= \langle \phi \otimes \psi_1 \otimes \cdots \otimes \psi_k, \chi_0 \otimes \cdots \otimes \chi_k\rangle = \gamma\mu(\phi^\star \chi_0) \\
&\quad \times \sum_{m\geq 1} \sum_{(n_0,\dots,n_m)} \prod_{1\leq p\leq m} \gamma\mu(\psi^\star_{n_p}\psi^\star_{n_p-1}\cdots\psi^\star_{n_{p-1}+1}\chi_{n_{p-1}+1}\chi_{n_{p-1}+2}\cdots\chi_{n_p}) \\
&\quad + \sum_{m\geq 1} \gamma^m \sum_{(n_0,\dots,n_m)} \mu[\psi^\star_{n_1}\psi^\star_{n_1-1}\cdots\psi^\star_1\phi^\star\chi_0\cdots\chi_{n_1}] \\
&\quad \times \prod_{2\leq p\leq m} \mu[\psi^\star_{n_p}\psi^\star_{n_p-1}\cdots\psi^\star_{n_{p-1}+1}\chi_{n_{p-1}+1}\chi_{n_{p-1}+2}\cdots\chi_{n_p}] \\
&= \langle \psi_1 \otimes \cdots \otimes \psi_k, \gamma\mu(\phi^\star\chi_0)\chi_1 \otimes \cdots \otimes \chi_k\rangle \\
&\quad + \langle \psi_1 \otimes \cdots \otimes \psi_k, (\phi^\star\chi_0\chi_1) \otimes \chi_2 \otimes \cdots \otimes \chi_k\rangle,
\end{aligned}$$

where the sums are taken over all increasing sequences of natural numbers $(n_0, \dots, n_m)$ such that $n_0 = 0$ and $n_m = k$.

The proof of the fact that n_ϕ is adjoint to $n^\star_\phi$ is straightforward and we shall omit it. □

Theorem 7. *The following operator equalities hold for all $\phi, \psi \in \mathcal{L}^2 \cap \mathcal{L}^\infty$ and $\zeta, \eta \in \mathcal{L}^\infty$:*

$$b_\psi b^\star_\phi = \gamma\mu(\psi^\star\phi) + n_{\psi^\star\phi}, \tag{28}$$

$$n_\zeta b^\star_\phi = b^\star_{\zeta\phi}, \qquad b_\psi n_\zeta = b_{\zeta^\star\psi}, \tag{29}$$

$$n_\zeta n_\eta = n_{\zeta\eta}. \tag{30}$$

7.3. Free quadratic Fock space and free probability. In this subsection we shall present some properties of the free quadratic Fock space related to the free probability of Voiculescu [V].

Definition 2. *A noncrossing partition is a partition $\pi = \{\pi_1, \dots, \pi_k\}$ of a set $\{1, \dots, n\}$ such that there do not exist numbers $1 \leq a < b < c < d \leq n$ such that $a, c \in \pi_r$, $b, d \in \pi_s$ and $r \neq s$.*

Theorem 8. *Let $\mathcal{A}$ be an associative $\star$–algebra with a state μ. For $Q_s(\phi) = b^\star_\phi + b_{\phi^\star} + sn_\phi$ we have*

$$\begin{aligned}
&\tau[Q_s(\phi_1)\cdots Q_s(\phi_k)] \\
&\quad = \sum_{\pi=\{\pi_1,\dots,\pi_k\}} \prod_{1\leq i\leq k} \gamma\mu(\phi_{\pi_{i1}}\cdots\phi_{\pi_{i,n_i}}) \sum_{1\leq l\leq \frac{n_i-2}{2}} \frac{1}{l+1}\binom{2l}{l}\binom{n_i-2}{2l} s^{n_i-2l-2},
\end{aligned}$$

where the sum is taken over all noncrossing partitions $\{\pi_1, \dots, \pi_k\}$; $\pi_i = \{\pi_{i,1}, \dots, \pi_{i,n_i}\}$, $\pi_{i,1} < \cdots < \pi_{i,n_i}$.

The free cumulants [KS] are therefore

$$k_n(\phi_1, \dots, \phi_n) = \gamma\mu(\phi_1\cdots\phi_n) \sum_{1\leq l\leq \frac{n-2}{2}} \frac{1}{l+1}\binom{2l}{l}\binom{n-2}{2l} s^{n-2l-2}.$$

Proof. Let us consider two auxiliary annihilation operators

$$\hat{b}_\phi(\psi_1 \otimes \cdots \otimes \psi_k) = \gamma\mu(\phi^\star\psi_1)\,\psi_2 \otimes \cdots \otimes \psi_k,$$
$$\tilde{b}_\phi(\psi_1 \otimes \cdots \otimes \psi_k) = (\phi^\star\psi_1\psi_2) \otimes \psi_3 \otimes \cdots \otimes \psi_k.$$

Therefore $Q_s(\phi) = b^\star_\phi + sn_\phi + \tilde{b}_{\phi^\star} + \hat{b}_{\phi^\star}$ and $\tau[Q_s(\phi_1)\cdots Q_s(\phi_k)]$ is a sum of 4^k summands each equal to the state τ acting on a product of operators $b^\star$, sn, $\tilde{b}$ and $\hat{b}$. Each of these summands is of the form $\prod_{1\le i\le k}\gamma\mu(\phi_{\pi_{i1}}\cdots\phi_{\pi_{i,n_i}})$ times a power of s. Furthermore, only expressions coming from noncrossing partitions can appear. Our question is: with which coefficient such a term comes in the $\tau[Q_s(\phi_1)\cdots Q_s(\phi_k)]$. We shall discuss the 4^{n_i} ways of choosing one of four operators $b^\star$, sn, $\tilde{b}$ and $\hat{b}$ to be associated with each of the vectors $\phi_{\pi_{i1}}, \ldots, \phi_{\pi_{i,n_i}}$ forming a block of the partition π.

First note that there must be $n_i \ge 2$, and with the vector $\phi_{\pi_{i1}}$ must be associated the annihilator $\hat{b}$ and with the vector $\phi_{\pi_{i,n_i}}$ must be associated the creator $b^\star$ – otherwise such a summand does not contribute in the sum. There are remaining $n_i - 2$ places on which we have to choose operators $b^\star$, $\tilde{b}$ and sn. The number of creation operators l on these places must be equal to the number of annihilators, the other $n_i - 2 - 2l$ places must be occupied by number operators. There are $\binom{n_i-2}{2l}$ possibilites of choosing places on which number operators should be placed. The number of ways of choosing l places among $2l$ places on which creation operators should act is equal to the l Catalan number $\frac{1}{n+1}\binom{2l}{l}$ [GKP] which ends the proof. □

As a simple corollary we have the following

Theorem 9. *Let $\mathcal{A}$ be an associative $\star$-algebra with a tracial state μ and let $s \in \mathbb{R}$. Let $\mathcal{F}_s(\mathcal{A})$ be a $\star$-algebra generated by operators $Q_s(\phi)$ for $\phi \in \mathcal{A}$ and by the identity operator. Then a state ρ on $\mathcal{F}_s(\mathcal{A})$ given by $\rho(X) = \langle\Omega, X\Omega\rangle$ is tracial.*

Theorem 10. *Let $\mathcal{A}$ be an associative $\star$-algebra with a tracial state μ, $s \in \mathbb{R}$ and let $(\mathcal{A}_i)$ be a family of $\star$-subalgebras of $\mathcal{A}$ such that $xy = 0$ for any $x \in \mathcal{A}_i$, $y \in \mathcal{A}_j$ and $i \neq j$. Then subalgebras $\mathcal{F}_s(\mathcal{A}_i)$ of the algebra $\mathcal{F}_s(\mathcal{A})$ are free with respect to the state ρ.*

Proof. By the definition of freeness we have to prove that for any sequence $i_1, \ldots, i_n$ of indexes such that the consecutive indexes are not equal $i_k \neq i_{k+1}$ for $1 \le k \le n-1$ and a sequence (X_k) such that $X_k \in \mathcal{F}_s(A_{i_k})$ and $\rho(X_k) = 0$ for $1 \le k \le n$ we have $\rho(X_1\cdots X_n) = 0$.

Each of the operators X_k can be written as a sum of normally ordered operators, i.e. as a sum of products of the type $b^\star_{\phi_1}\cdots b^\star_{\phi_p} b_{\psi_1}\cdots b_{\psi_q}$ or $b^\star_{\phi_1}\cdots b^\star_{\phi_p} n_\zeta b_{\psi_1}\cdots b_{\psi_q}$. The assumption $\rho(X_k) = 0$ implies that in this sum the multiplicity of the identity operator does not appear. By distributing we see that the product $X_1\cdots X_n$ can be written as a sum of many summands, each being a product of normally ordered products.

The commutation relations (28)–(30) show that for each of these summands if one is not normally ordered then it is equal to zero. On the other hand, a vacuum expectation of any normally ordered product (containing no multiplicity of identity) is equal to 0. □

Acknowledgement. The author would like to thank Prof. L. Accardi and M. Skeide for a very inspiring presentation and Centro Internazionale per la Ricerca Mathematica, Consiglio Nazionale della Ricerche and University of Roma Tor Vergata for giving opportunity to participate in the Volterra International School *White Noise Approach to Quantum Stochastic Calculi and Quantum Probability* in Trento, July 1999.

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Communicated by H. Araki