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Spectral Analysis and Feller Property for Quantum Ornstein–Uhlenbeck Semigroups

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Abstract: A class of dynamical semigroups arising in quantum optics models of masers and lasers is investigated. The semigroups are constructed, by means of noncommutative Dirichlet forms, on the full algebra of bounded operators on a separable Hilbert space. The explicit action of their generators on a core in the domain is used to demonstrate the Feller property of the semigroups, with respect to the C^*-subalgebra of compact operators. The Dirichlet forms are analysed and the L^2-spectrum together with eigenspaces are found. When reduced to certain maximal abelian subalgebras, the semigroups give rise to the Markov semigroups of classical Ornstein–Uhlenbeck processes on the one hand, and of classical birth-and-death processes on the other.

1. Introduction

The object of this paper is the investigation of the evolution equation

$$\frac{d}{dt} P_t x = \mathcal{L} P_t x; \quad P_0 = \mathrm{id} \tag{1.1}$$

determined by the Lindblad-type operator

$$\mathcal{L}x = -\frac{\mu^2}{2}(A^*Ax - 2A^*xA + xA^*A) - \frac{\lambda^2}{2}(AA^*x - 2AxA^* + xAA^*). \tag{1.2}$$

The equation is for an evolution of bounded linear operators on the complex Hilbert space $\mathfrak{h} = l^2(\mathbb{Z}_+)$; the operators A and A^* are the annihilation and creation operators of the usual representation of the canonical commutation relations associated with the quantum harmonic oscillator (defined in Sect. 4) and the constants λ and μ satisfy:

$$\mu > \lambda > 0 \quad \text{so that} \quad \nu := \lambda^2/\mu^2 \in]0, 1[.$$

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We may write $\mathcal{L}$ in the alternative form

$$\mathcal{L}x = L_\lambda^* x L_\lambda + L_\mu x L_\mu^* + Gx + xG, \tag{1.2'}$$

where

$$L_\lambda = \lambda A^*, \quad L_\mu = \mu A^*, \quad G = -\tfrac{1}{2}\left(\lambda^2(N+1) + \mu^2 N\right),$$

and N denotes the number operator A^*A. The relevance of such evolution equations for mathematical models of masers and lasers is discussed in [FRS]. Such equations also arise in weak-coupling models of open quantum systems (see [Da 1], Chap. 10, [AAFL]).

The first mathematical problem that must be faced with an equation of the form (1.1) is that the Lindbladian involves unbounded operators. One must therefore decide on the space in which one seeks solutions, and also the sense in which $\mathcal{L}$ is to be considered the generator of a one parameter semigroup $P = (P_t)_{t\geq 0}$ solving (1.1). Spaces on which it is natural to consider this problem include: the von Neumann algebra of all bounded operators on $\mathfrak{h}$; its predual, the Banach space of trace class operators on $\mathfrak{h}$; and the C^*-algebra of compact operators on $\mathfrak{h}$. We shall denote these L^∞, L^1 and $\mathcal{K}$ respectively, and also write L^p for the Schatten p-class, so that L^2 denotes the Hilbert–Schmidt class. As far as the continuity of the semigroup is concerned, due to the non-separability of L^∞ the appropriate topologies are respectively the weak*-topology on L^∞, the strong topology on $\mathcal{K}$ and the weak or strong topology on L^1.

In [FRS] weak *-continuous solutions on L^∞ are obtained from unitary solutions of associated quantum stochastic differential equations, thereby extending the method of [Hu P] to unbounded coefficient quantum SDE's. Ergodicity of these weak *-continuous solutions is also proved in [FRS].

In the present work a different approach is taken, based on the recent theory of noncommutative Dirichlet forms ([Ci 1,2, GL 1,2]). Instead of attacking the problem of closability and dissipativeness of $\mathcal{L}$ on the nonseparable Banach space L^∞, and seeking to apply the Lumer–Phillips Theorem, we consider the equivalent but more tractable problems of establishing closability and a Markov property for an associated nonnegative quadratic form $\mathcal{E}_\nu$, on the separable Hilbert space L^2. We exploit the fact that L^2, together with the cone of nonnegative Hilbert–Schmidt operators L^2_+, and the adjoint operation on L^2, comprise a standard form for L^∞. The equivalence of the L^∞ and L^2 problems (and solutions) is due to the existence of an invariant state for the dynamics. This state provides the means of moving back and forth between algebra and Hilbert space. Moreover it is the *KMS-symmetry* of the problem, with respect to this state, which permits a quadratic form description of the generator of the dynamics.

The first advantage of this approach is that symmetry, semiboundedness and Hilbert space domain consideration, make closability and the Markov property of $\mathcal{E}$ much easier to prove than closability and dissipativeness of $\mathcal{L}$. The second advantage is that studying the domain of the nonnegative self-adjoint operator corresponding to $\mathcal{E}$, one is able to characterize the action of $\mathcal{L}$ on an explicit core, and also to prove the strong continuity of the semigroup on $\mathcal{K}$ (weak Feller property) and the invariance of $\mathcal{K}$ under the L^∞-semigroup (Feller property). The third advantage is that we are able to obtain a complete description of the L^2-spectrum with associated eigenspaces.

It should be mentioned that the semigroups on L^∞ constructed here are quasi-free, having an explicit representation on Weyl operators (see [AlL], p. 63); in particular they leave invariant the C*-algebra of the Weyl relations ([Sla]). However, since the distance between distinct Weyl operators is $\sqrt{2}$, the semigroups fail to be strongly continuous on the Weyl algebra. These semigroups are also discussed in the (pre-quantum stochastic calculus) paper [HIP].

In Sect. 2 we recall some of the properties of the standard form $(\mathcal{L}^\infty, L^2, L^2_+, *)$, where $\mathcal{L}^\infty$ is the faithful normal representation of L^∞ on the Hilbert space L^2 obtained by left multiplication: $L_x\xi = x\xi$ for $x \in L^\infty, \xi \in L^2$. We then establish basic properties of left and right multiplication operators L_X, R_X, and bimodule derivations d_X on L^2, in which X is an unbounded operator on the original Hilbert space $\mathfrak{h}$.

In Sect. 3 we recall the correspondence between symmetric (noncommutative) Dirichlet forms on L^2, symmetric Markov semigroups on L^2 and KMS-symmetric Markov semigroups on L^∞ (specialized to the present setting). In Sect. 4, using the unbounded operators L_N, R_N and d_A we construct (for each $\mu > \lambda > 0$) a closed nonnegative form $\mathcal{E}$ on L^2 which is Markov with respect to a certain cyclic vector $\xi_\nu \in L^2_+$ (where $\nu = \lambda^2/\mu^2$). We shall refer to the associated Markov semigroups on L^2 and L^∞ as quantum Ornstein–Uhlenbeck semigroups.

In Sect. 5, motivated by Phillips' theory of dual semigroups ([Phi]) we define the weak Feller and Feller properties for a weak *-continuous semigroup on L^∞, and prove that the quantum O–U semigroups are Feller semigroups. We also show that, on an explicitly given core, the action of the L^∞-generator indeed coincides with the Lindblad-type operator (1.2). This amounts to solutions of (1.1) on both L^∞ and $\mathcal{K}$.

In Sect. 6 we prove ergodicity of the quantum O–U semigroups, and, by comparison of $\mathcal{E}$ with certain other related forms, and application of the minimax principle, we derive the discreteness of the L^2-spectra of each quantum O–U generator. Although both these results follow from the spectral analysis in the following section, they are included here for the purpose of illustrating techniques that may be applicable when a complete spectral analysis is not available. We also show how these semigroups provide a realisation of the Markov semigroups of classical birth and death processes by restriction to the maximal abelian subalgebra generated by the number operator. This nicely illustrates an important feature of quantum Markov semigroups, namely that they may contain widely varying classical Markov semigroups through restriction to different abelian subalgebras.

Section 7 contains a complete L^2-spectral analysis, and reveals why we have chosen to use the name *quantum Ornstein-Uhlenbeck* rather than *quantum birth and death*. In the limiting case $\lambda = \mu$ we are dealing with a quantum Brownian motion semigroup. This is treated in the final section by means of the tracial theory of noncommutative Dirichlet forms ([AH-K, DaL]).

2. Unbounded Multiplication Operators and Derivations

Here we describe the standard form convenient for our present purposes, and introduce the unbounded multiplication operators and derivations which will be used (in the following section) to construct the noncommutative Dirichlet forms we wish to investigate. In future sections $\mathfrak{h}$ will always be the sequence space $l^2(\mathbb{Z}_+)$, but here it may be any complex separable Hilbert space. The inner product is *linear in its second argument*. The von Neumann algebra of all bounded operators on $\mathfrak{h}$ is denoted L^∞; its elements by $x, y, z, \dots$; and the faithful, normal, semifinite trace on L^∞ (normalized so that on projections it gives their dimension) is denoted Tr.

Let L^p $(1 \le p < \infty)$ denote the Schatten classes, whose elements will be denoted by Greek letters $\xi, \eta, \rho, \dots$. Thus L^2 is the Hilbert–Schmidt class, and its inner product is given by $\langle \xi, \eta \rangle = \mathrm{Tr}\,(\xi^* \eta)$. Consider the maps

$$\pi_L : L^\infty \to \mathcal{B}(L^2), \qquad \pi_L(x) = L_x,$$
$$\pi_R : L^\infty \to \mathcal{B}(L^2), \qquad \pi_R(x) = R_x,$$

where L_x and R_x are the left and right multiplication operators, $\xi \mapsto x\xi$ and $\xi \mapsto \xi x$, respectively. π_L defines a faithful, normal representation of L^∞, while π_R defines a faithful, normal representation of the opposite algebra $(L^\infty)^0$. Putting $\mathcal{L}^\infty = \pi_L(L^\infty)$ and $\mathcal{R}^\infty = \pi_R(L^\infty)$ we have the commutant relations

$$(\mathcal{L}^\infty)' = \mathcal{R}^\infty \quad \text{and} \quad (\mathcal{R}^\infty)' = \mathcal{L}^\infty$$

in $\mathcal{B}(L^2)$.

The closed convex cone in L^2 consisting of nonnegative Hilbert–Schmidt operators is a self-dual cone in the sense that

$$L^2_+ = \left\{ \xi \in L^2 : \langle \xi, \eta \rangle \geq 0 \; \forall \eta \in L^2_+ \right\}.$$

The associated antiunitary conjugation J on L^2 is simply the adjoint map on L^2 : $J\xi \equiv \xi^*$. We therefore have $JL_xJ = R_{x^*}$, so that $J\mathcal{L}^\infty J = \mathcal{R}^\infty = (\mathcal{L}^\infty)'$, and $L_x J L_x J \xi = x\xi x^* \geq 0$ whenever $\xi \in L^2_+$.

In summary

$$\left(\mathcal{L}^\infty, L^2, L^2_+, * \right)$$

is a standard form for L^∞. We refer to [Haa] for the definition of standard forms and the proof of their uniqueness. The Dirichlet forms and Markov semigroups will be constructed on this standard form, in the framework of [Ci 1,2]. We shall use the fact that L^2 is the complexification of the real Hilbert space of self-adjoint Hilbert–Schmidt operators which itself is characterized by

$$L^2_{\mathbb{R}} = \left\{ \xi \in L^2 : \langle \xi, \eta \rangle \in \mathbb{R} \; \forall \eta \in L^2_+ \right\}.$$

Also note that each element $\xi \in L^2_{\mathbb{R}}$ may be uniquely expressed as a difference $\xi = \xi_+ - \xi_-$ in which $\xi_\pm \in L^2_+$ and the support projections of ξ_+ and ξ_- in $\mathcal{L}^\infty$ (as well as in $\mathcal{R}^\infty$) are orthogonal.

The following notation (of Dirac) remains highly convenient. For vectors e, f in $\mathfrak{h}$, let $|e\rangle \langle f|$ denote the operator on $\mathfrak{h}$ given by

$$|e\rangle \langle f| \, v = \langle f, v \rangle \, e.$$

Thus, when e and f are unit vectors, $|e\rangle \langle f|$ is a partial isometry with initial space $\mathbb{C}f$ and final space $\mathbb{C}e$.

Now let $\rho_0 \in L^1$, $\varphi_0 \in \mathcal{L}^\infty_*$ and $\xi_0 \in L^2$ be respectively a strictly positive density matrix, the corresponding (vector) state, and the corresponding vector. Thus, in terms of a Hilbert basis (e_n) consisting of eigenvectors of ρ,

$$\rho_0 = \textstyle\sum_{n \geq 1} \gamma_n \, |e_n\rangle \langle e_n| \,, \text{ with } \gamma_n > 0 \text{ and } \sum \gamma_n = 1,$$
$$\varphi_0(L_x) = \mathrm{Tr}(\rho_0 x) = \langle \xi_0, x\xi_0 \rangle \,,$$
$$\xi_0 = \rho_0^{1/2} = \textstyle\sum_{n \geq 1} \gamma_n^{1/2} \, |e_n\rangle \langle e_n| \,.$$

The action of the associated modular operator and modular group are given by:

$$\Delta^{\frac{1}{2}}\xi = \left[\rho_0^{1/2}\xi\rho_0^{-1/2}\right], \quad \text{at least for } \xi \in L^\infty\xi_0 = L^\infty\rho_0^{1/2};$$
$$\sigma_{it}(L_x) = L_{\rho^{it}x\rho^{-it}}.$$

The *symmetric embedding* of algebra into Hilbert space, determined by the faithful normal state φ, takes the simple form:

$$\iota^{(2)} : \mathcal{L}^\infty \to L^2, \quad \iota^{(2)}(L_x) = \rho^{1/4}x\rho^{1/4} = \Delta^{1/4}L_x\xi_0.$$

More generally $\mathcal{L}^\infty$ is symmetrically embedded into L^p by $\iota^{(p)}(L_x) = \rho_0^{1/2p}x\rho_0^{1/2p}$.

We next consider unbounded multiplication operators and derivations on L^2. Let X be a closed and densely defined operator on $\mathfrak{h}$, with domain $\mathrm{Dom}(X)$. Its adjoint X^* is then also closed and densely defined. For each $\xi \in L^2$, viewed as an operator on $\mathfrak{h}$, $X\xi$ is closed, but not necessarily densely defined, whereas ξX is densely defined but not necessarily closed. We define left and right multiplication operators, and (unbounded) derivations, on L^2 as follows:

$$\mathrm{Dom}(L_X) = \left\{\xi \in L^2 : \mathrm{Dom}(X\xi) = \mathfrak{h} \text{ and } X\xi \in L^2\right\}; \quad L_X\xi = X\xi,$$
$$\mathrm{Dom}(R_X) = \left\{\xi \in L^2 : \xi X \text{ is bounded and } [\xi X] \in L^2\right\}; \quad R_X\xi = [\xi X],$$
$$\mathrm{Dom}(\delta_X) = \mathrm{Dom}(L_X) \cap \mathrm{Dom}(R_X); \quad \delta_X = L_X - R_X,$$

where [ ] denotes the closure of a (closable) operator. Notice that $\mathrm{Dom}(X\xi) = \mathfrak{h}$ already implies that $X\xi \in L^\infty$ and also that if ξX is bounded then $[\xi X] \in L^\infty$. Thus our definitions involve a natural progression of restrictions on ξ.

For a pair of Hilbert bases $[e] = (e_n)$ and $[f] = (f_n)$ for $\mathfrak{h}$, let

$$C_{00}\left([e, f]\right) = \mathrm{Lin}\left\{|e_n\rangle\langle f_m|\right\}$$

and let $C_{00}([e]) = C_{00}([e, e])$. Thus $C_{00}([e, f])$ is a dense subspace of L^2 consisting of finite rank operators, and moreover $C_{00}([e])$ is a weak *-dense *-subalgebra of L^∞ whose norm closure is $\mathcal{K}$.

Lemma 2.1. *Let X be a closed densely defined operator on $\mathfrak{h}$.*

(i) $J\mathrm{Dom}(L_X) = \mathrm{Dom}(R_{X^*})$*;* $JL_XJ = R_{X^*}$*;*
(ii) L_X *is a closed densely defined operator on* L^2 *affiliated to* $\mathcal{L}^\infty$*, and satisfying:*

$$L_{X^*} \subset (L_X)^*; \quad \mathrm{Dom}(L_X) = \mathrm{Dom}(L_{|X|});$$

(iii) R_X *is a closed densely defined operator on* L^2 *affiliated to* $\mathcal{R}^\infty$*, and satisfying*

$$R_{X^*} \subset (R_X)^*; \quad \mathrm{Dom}(R_X) = \mathrm{Dom}(R_{|X^*|}).$$

Proof. If $\xi \in \mathrm{Dom}(L_X)$, then $\mathrm{Dom}(X\xi) = \mathfrak{h}$ and $X\xi \in L^2$ so $\xi^*X^* \subset (X\xi)^* \in L^2$, which implies that $\xi^* \in \mathrm{Dom}(R_{X^*})$ and $(R_{X^*}\xi^*)^* = L_X\xi$. Thus $J\mathrm{Dom}(L_X) \subset \mathrm{Dom}(R_{X^*})$ and $JR_{X^*}J \supset L_X$. Conversely, if $\eta \in \mathrm{Dom}(R_{X^*})$ then ηX^* is bounded and $\left[\eta X^*\right] \in L^2$, so $X\eta^* = \left[\eta X^*\right]^* \in L^2$, thus $\eta^* \in \mathrm{Dom}(L_X)$. Therefore $\mathrm{Dom}(R_{X^*}) \subset J\mathrm{Dom}(L_X)$, and (i) follows.

If (e_n) and (f_n) are Hilbert bases for $\mathfrak{h}$ contained in $\mathrm{Dom}(X)$ and $\mathrm{Dom}(X^*)$ respectively, then $C_{00}\left([e, f]\right)$ is a dense subspace of L^2 contained in $\mathrm{Dom}(L_X) \cap \mathrm{Dom}(R_X)$. Hence L_X, R_X and δ_X are all densely defined.

Let $v|X|$ be the polar decomposition of X so that $|X| = v^*X$. If $\xi \in \mathrm{Dom}(L_X)$ then $\mathrm{Dom}(|X|\xi) = \mathrm{Dom}(v^*X\xi) = \mathrm{Dom}(X\xi) = \mathfrak{h}$ and $|X|\xi = (v^*X)\xi = v^*(X\xi) \in L^2$, since L^2 is an ideal of L^∞, so $\xi \in \mathrm{Dom}(L_{|X|})$. Hence $\mathrm{Dom}(L_X) \subset \mathrm{Dom}(L_{|X|})$. The reverse inclusion follows similarly, so $\mathrm{Dom}(L_X) = \mathrm{Dom}(L_{|X|})$.

If $\xi \in \mathrm{Dom}(L_{X^*})$ and $\eta \in \mathrm{Dom}(L_X)$, then $X\eta \in L^2$, so $\xi^*X\eta \in L^1$ and $X^*\xi \in L^2$ so $(X^*\xi)^* \in L^2$. But $(X^*\xi)^*\eta$ extends the everywhere defined operator $\xi^*X\eta$, so the two operators must coincide, and we have $\langle \xi, L_X\eta\rangle = \mathrm{Tr}\,(\xi^*X\eta) = \mathrm{Tr}\left((X^*\xi)^*\,\eta\right) = \langle L_{X^*}\xi, \eta\rangle$. Thus $(L_X)^* \supset L_{X^*}$.

The fact that L_X is closed follows easily from the closure of X: if (ξ_n) is a sequence in $\mathrm{Dom}(L_X)$ such that $\xi_n \to \xi$ and $X\xi_n \to \eta$ in L^2, then for each $u \in \mathfrak{h}$, $\xi_n u \to \xi u$ and $X\xi_n u \to \eta u$ in $\mathfrak{h}$, so $\xi u \in \mathrm{Dom}(X)$ and $X\xi u = \eta u$, therefore $\xi \in \mathrm{Dom}\,(L_X)$ and $L_X\xi = \eta$.

The affiliation properties easily follow using the fact that L^2 is an ideal of L^∞, and the remaining properties follow by similar arguments. □

Lemma 2.2. *Let X be a closed densely defined operator on $\mathfrak{h}$. Then δ_X is a closable densely defined operator satisfying*

$$\delta_X \subset (\delta_{X^*})^*\,; \quad J\delta_X J = -\delta_{X^*}.$$

Moreover, if $\mathrm{Dom}(L_{|X|}) = \mathrm{Dom}(L_{|X^*|})$*, then* $\mathrm{Dom}(\delta_X)$ *is J-invariant.*

Proof. We have already seen (in the proof of Lemma 2.1) that $\mathrm{Dom}(\delta_X) \supset C_{00}\left([e, f]\right)$ whenever (e_n) and (f_n) are Hilbert bases contained in $\mathrm{Dom}(X)$ and $\mathrm{Dom}(X^*)$ respectively. Since

$$(\delta_X)^* = (L_X - R_X)^* \supset L_X^* - R_X^* \supset L_{X^*} - R_{X^*} = \delta_{X^*},$$

replacing X by X^* we have $\delta_X \subset (\delta_{X^*})^*$, in particular δ_X is closable. Since J is antiunitary,

$$J\delta_X J = JL_X J - JR_X J = R_{X^*} - L_{X^*} = -\delta_{X^*}.$$

If $\mathrm{Dom}(L_{|X|}) = \mathrm{Dom}\left(L_{|X^*|}\right)$ then $\mathrm{Dom}\,(L_X) = \mathrm{Dom}\,(L_{X^*})$ and so $\mathrm{Dom}\,(R_X) = J\mathrm{Dom}\,(L_{X^*}) = J\mathrm{Dom}\,(L_X) = \mathrm{Dom}\,(R_{X^*})$. Thus

$$J\,(\mathrm{Dom}(L_X) \cap \mathrm{Dom}\,(R_X)) = \mathrm{Dom}\,(R_{X^*}) \cap \mathrm{Dom}\,(L_{X^*}) = \mathrm{Dom}\,(R_X) \cap \mathrm{Dom}\,(L_X)\,,$$

in other words $J\mathrm{Dom}\,(\delta_X) = \mathrm{Dom}\,(\delta_X)$. □

In view of the previous lemma we make the following definition:

$$d_X := [\delta_X],$$

for X closed and densely defined on $\mathfrak{h}$.

Lemma 2.3. *Let X be a closed densely defined operator on $\mathfrak{h}$. The real parts of the domains* $\mathrm{Dom}(L_X)$, $\mathrm{Dom}(R_X)$ *and* $\mathrm{Dom}(\delta_X)$ *are invariant under the modulus map* $\xi \mapsto |\xi| = (\xi^*\xi)^{\frac{1}{2}}$ *in L^2. In fact we have the following characterisations of domains:*
$\mathrm{Dom}(L_X) =$
$\{\xi \in L^2 : X\xi\xi^*X^*$ *is bounded, densely defined and has trace class closure*$\}$
and, for $\xi \in \mathrm{Dom}(L_X)$, $\|X\xi\|_2^2 = \mathrm{Tr}\left([X\xi\xi^*X^*]\right)$;
$\mathrm{Dom}(R_X) =$
$\{\xi \in L^2 : X^*\xi^*\xi X$ *is bounded, densely defined and has trace class closure*$\}$
and, for $\xi \in \mathrm{Dom}(R_X)$, $\|[\xi X]\|_2^2 = \mathrm{Tr}\left([X^*\xi^*\xi X]\right)$.

Proof. Let $\xi \in L^2$ and let $T_0 = X\xi\xi^*X^*$. If $\xi \in \mathrm{Dom}(L_X)$ then $\mathrm{Dom}\,(X\xi) = \mathfrak{h}$ and $X\xi \in L^2$, so $\mathrm{Dom}(T_0) = \mathrm{Dom}(X^*)$ which is dense and $T_0 \subset X\xi(X\xi)^* \in L^1$. Thus the closure of T_0 coincides with $X\xi(X\xi)^*$ and, by Lemma 2.1,

$$\|X\xi\|_2^2 = \left\|[\xi^*X^*]\right\|_2^2 = \mathrm{Tr}\left(X\xi\,(X\xi)^*\right) = \mathrm{Tr}(T) = \|T\|_1\,.$$

Conversely, if T_0 is densely defined and bounded, and its closure T is trace class, then $T = X\xi\xi^*X^* \subset X\xi(X\xi)^* = \left|(X\xi)^*\right|$. Since a densely defined bounded operator has only one closed extension, $T = \left|(X\xi)^*\right|^2$. Thus $(X\xi)^*$ is everywhere defined and Hilbert–Schmidt, so $\xi^* \in \mathrm{Dom}\,(R_{X^*}) = J\mathrm{Dom}\,(L_X)$, so $\xi \in \mathrm{Dom}\,(L_X)$. This establishes the first characterisation, and the second now follows from Lemma 2.1(i). The invariance properties are now immediate too. □

3. Noncommutative Dirichlet Forms

In this section we first summarize the general results on Dirichlet forms and Markov semigroups, specialized to the standard form $\left(\mathcal{L}^\infty, L^2, L^2_+, *\right)$ described in Sect. 2. The full theory is developed in [Ci 1,2] and [GL 1,2]. We also recall the definition and basic properties of the unbounded annihilation, creation and number operators on $l^2(\mathbb{Z}_+)$. Let ρ_0, φ_0 and ξ_0 be corresponding strictly positive density matrix, faithful normal state and positive cyclic vector, as in Sect. 2. The order intervals $\{\eta \in L^2 : 0 \le \eta \le \xi_0\}$ and $\{\eta \in L^2 : \eta \le \xi_0\}$ will be denoted $[0, \xi_0]$ and $]-\infty, \xi_0]$ respectively. These are closed convex subsets of L^2, and we shall denote the nearest point projection onto $[0, \xi_0]$ and $]-\infty, \xi_0]$ by $\eta \mapsto \eta_I$ and $\eta \mapsto \eta_\wedge$ respectively. For $\eta \in L^2_{\mathbb{R}}$ we have

$$\eta_\wedge = \eta - (\eta - \xi_0)_+ = \xi_0 - (\eta - \xi_0)_-\,.$$

Corresponding to any self-adjoint contraction semigroup $\left(P_t^{(2)}\right)$ on L^2 its form generator is the unique closed nonnegative quadratic form $\mathcal{E}$, given by

$$\mathcal{E}[\eta] = \lim_{t\to 0} t^{-1}\left\langle \eta, (I - P_t^{(2)})\eta\right\rangle,$$

and conversely such a form determines the semigroup through $P_t^{(2)} = \mathrm{e}^{-tH_{(2)}}$, where $H_{(2)} = H_{(2)}^* \ge 0$ is determined by

$$\left\|(H_{(2)})^{1/2}\eta\right\|^2 = \mathcal{E}[\eta], \qquad \mathrm{Dom}\left((H_{(2)})^{1/2}\right) = \left\{\eta \in L^2 : \mathcal{E}[\eta] < \infty\right\}.$$

A self-adjoint contraction semigroup on L^2 is called *(sub-)Markov* with respect to ξ_0 if

$$P_t^{(2)}\left([0,\xi_0]\right) \subset [0,\xi_0] \ \ \forall t,$$

and *conservative* if also

$$P_t^{(2)}\xi_0 = \xi_0 \ \forall t.$$

A densely defined nonnegative quadratic form $\mathcal{E}$ is called *Markov* with respect to ξ_0 if

$$\eta \in \mathrm{Dom}\,(\mathcal{E}) \Rightarrow \eta^* \in \mathrm{Dom}\,(\mathcal{E}) \text{ and } \mathcal{E}\left[\eta^*\right] = \mathcal{E}\left[\eta\right], \tag{3.1a}$$

$$\eta = \eta^* \in \mathrm{Dom}\,(\mathcal{E}) \Rightarrow \eta_I \in \mathrm{Dom}\,(\mathcal{E}) \text{ and } \mathcal{E}[\eta_I] \le \mathcal{E}[\eta], \tag{3.1b}$$

and is called *Dirichlet* if it is also closed.

The Markov conditions (a) and (b) on a closed densely defined nonnegative quadratic form on L^2 are equivalent to the (sub-)Markov property of the corresponding semigroup ([Ci 1] Theorem 4.11, [GL 1] Theorem 5.7). This amounts to a noncommutative generalisation of the Beurling–Deny characterisation of the form generators of classical symmetric Markov semigroups. As in the commutative case, there is a bijective correspondence between such semigroups and symmetric weak *-continuous positive contraction semigroups on L^∞. In the noncommutative case the correspondence arises not simply through common restriction, but through intertwining with the symmetric embedding of the algebra into Hilbert space: $\iota^{(2)} \circ P_t^{(\infty)} = P_t^{(2)} \circ \iota^{(2)}$; explicitly

$$\rho_0^{1/4} P_t^{(\infty)}(x)\rho_0^{1/4} = P_t^{(2)}\left(\rho_0^{1/4} x \rho_0^{1/4}\right).$$

Symmetry of the L^∞-semigroups involves the modular automorphism group of the state, and also arises through the symmetric embedding:

$$\varphi\left(\sigma_{i/2}(a) P_t^{(\infty)}(b)\right) = \varphi\left(P_t^{(\infty)}(a)\sigma_{-i/2}(b)\right), \tag{3.2a}$$

$$\mathrm{Tr}\left(\iota^{(1)}(a) P_t^{(\infty)}(b)\right) = \mathrm{Tr}\left(P_t^{(\infty)}(a)\iota^{(1)}(b)\right), \tag{3.2b}$$

where, in the first identity a and b are restricted to the algebra of analytic elements of (σ_t). To emphasize this involvement of the state, the condition (3.2) is called *KMS-symmetry*. In the present standard form it takes the explicit form

$$\mathrm{Tr}\left(\rho_0^{1/2} a \rho_0^{1/2} P_t^{(\infty)}(b)\right) = \mathrm{Tr}\left(P_t^{(\infty)}(a)\rho_0^{1/2} b \rho_0^{1/2}\right). \tag{3.2$'$}$$

This kind of symmetry was discussed by several authors in the eighties (see [Pet, GrK]).

If the quadratic form of a self-adjoint contraction semigroup on L^2 satisfies (3.1a) and $\mathcal{E}\,[\xi_0] = 0$, then (3.1b) is equivalent to the weaker condition

$$\eta = \eta^* \in \mathrm{Dom}\,(\mathcal{E}) \Rightarrow \eta_\pm \in \mathrm{Dom}\,(\mathcal{E}) \quad \text{and } \mathcal{E}\,(\eta_+, \eta_-) \le 0 \tag{3.3}$$

which is also equivalent to

$$\eta = \eta^* \in \mathrm{Dom}\,(\mathcal{E}) \Rightarrow |\eta| \in \mathrm{Dom}\,(\mathcal{E}) \quad \text{and } \mathcal{E}\,[|\eta|] \le \mathcal{E}\,[\eta]. \tag{3.3$'$}$$

In general, under (3.1a), the condition (3.3) is equivalent only to *positivity* of the semigroup ([Ci 2], Theorem 4.10).

4. Quantum Ornstein–Uhlenbeck Semigroups

In this section we obtain the KMS-symmetric Markov semigroup on L^∞, which solves (1.1), by constructing a Dirichlet form on L^2 and using the theory outlined in Sect. 3. Thus, *from now on*, let $\mathfrak{h} = l^2(\mathbb{Z}_+)$, let $[e] = \{e_n : n \in \mathbb{Z}_+\}$ be the usual Hilbert basis, and with $\mu > \lambda > 0$ and $\nu = \lambda^2/\mu^2$ fixed, let $\rho_\nu \in L^1$, $\varphi_\nu \in \mathcal{L}_*^\infty$ and $\xi_\nu \in L^2$ be given by

$$\begin{aligned}\rho_\nu &= (1-\nu)\textstyle\sum_{n\geq 0} \nu^n \,|e_n\rangle\,\langle e_n|\,,\\ \varphi_\nu(L_x) &= \mathrm{Tr}\,(\rho_\nu x) = \langle \xi_\nu, x\xi_\nu\rangle\,,\\ \xi_\nu &= \rho_\nu^{1/2} = (1-\nu)^{1/2}\textstyle\sum_{n\geq 0}\nu^{n/2}\,|e_n\rangle\,\langle e_n|\,.\end{aligned}$$

Also let (σ_t) and $\iota^{(p)}$ denote the associated modular automorphism group and symmetric embeddings, and we shall abbreviate $C_{00}([e])$ to C_{00}.

For constructing the Dirichlet forms we shall apply the results of Sect. 2 to the number, annihilation and creation operators defined as follows. The *number operator* N is the self-adjoint multiplication operator $\alpha = (\alpha_n) \mapsto (n\alpha_n)$, with maximal domain $\left\{\alpha \in \mathfrak{h} : \sum_{n\geq 0} |n\alpha_n|^2 < \infty\right\}$. The *annihilation* and *creation operators* are given by $\mathrm{Dom}(A) = \mathrm{Dom}(A^*) = \mathrm{Dom}\left(\sqrt{N}\right)$ with

$$Ae_n = \begin{cases}\sqrt{n}e_{n-1} & \text{if } n>0,\\ 0 & \text{if } n=0;\end{cases}\quad A^*e_n = \sqrt{n+1}e_{n+1}.$$

The operators A and A^* are closed and mutually adjoint, $A^*A = N$, whereas $AA^* = N + I$, and in terms of the isometric right shift operator S given by $Se_n = e_{n+1}$, we have the relations

$$A^* = \sqrt{N}S = S\sqrt{N+I};\quad A = \sqrt{N+I}S^* = S^*\sqrt{N}, \tag{4.1}$$

which are not merely algebraic, but are also precise in terms of operator domains.

Proposition 4.1. *Let* $\xi \in D := \mathrm{Dom}\left(L_{\sqrt{N}}\right) \cap \mathrm{Dom}\left(R_{\sqrt{N}}\right)$. *Then the following expressions are all finite, and they coincide:*

$$\tfrac{1}{2}\left\{\|(\mu L_A - \lambda R_A)\,\xi\|^2 + \left\|(\mu L_A - \lambda R_A)\,\xi^*\right\|^2\right\}, \tag{4.2a}$$

$$\tfrac{1}{2}\left\{\|(\mu L_A - \lambda R_A)\,\xi\|^2 + \|(\mu R_{A^*} - \lambda L_{A^*})\,\xi\|^2\right\}, \tag{4.2b}$$

$$\lambda\mu\,\|d_A\xi\|^2 + \tfrac{1}{2}(\lambda-\mu)^2\left\{\left\|L_{\sqrt{N}}\xi\right\|^2 + \left\|R_{\sqrt{N}}\xi\right\|^2\right\} - \lambda(\mu-\lambda)\,\|\xi\|^2\,, \tag{4.2c}$$

$$\begin{aligned}&\tfrac{1}{2}\textstyle\sum_{n,m\geq 0}\left\{\left|\lambda\sqrt{m+1}\alpha_m^n - \mu\sqrt{n+1}\alpha_{m+1}^{n+1}\right|^2,\right.\\ &\quad\left. + \left|\lambda\sqrt{n+1}\alpha_m^n - \mu\sqrt{m+1}\alpha_{m+1}^{n+1}\right|^2\right\} + \tfrac{1}{2}\mu^2\sum_{n\geq 1} n\left\{\left|\alpha_n^0\right|^2 + \left|\alpha_0^n\right|^2\right\},\end{aligned} \tag{4.2d}$$

where $\xi = \sum_{m,n\geq 0}\alpha_m^n\,|e_m\rangle\,\langle e_n|$. *Moreover, if* $\xi \in \mathrm{Dom}(L_N) \cap \mathrm{Dom}(R_N)$, *then there is a fifth useful equal expression:*

$$\left\langle \xi, \tfrac{1}{2}\left(\lambda^2+\mu^2\right)(N\xi + [\xi N]) + \lambda^2\xi - \lambda\mu\left(A\left[\xi A^*\right] + A^*\left[\xi A\right]\right)\right\rangle. \tag{4.3}$$

Proof. By Lemma 2.1(ii) the domains of L_A, $L_{\sqrt{N}}$, $L_{\sqrt{N+1}}$ and L_{A^*} all coincide. By Lemma 2.1(i) the domains of R_A, $R_{\sqrt{N+1}}$, $R_{\sqrt{N}}$ and R_{A^*} all coincide too. It follows that each of the expressions (4.2a–d) is finite and, since $\mathrm{Dom}\,(L_N) \cap \mathrm{Dom}\,(R_N) \subset \mathrm{Dom}\,(L_{A^*} R_A)$, (4.3) is finite too. By another application of Lemma 2.1(i), (4.2a) and (4.2b) coincide. Straightforward calculations verify that (4.2b–d) also coincide, and that (4.3) equals (4.2b) under the given domain constraint on ξ. □

Theorem 4.2. *Let $D = \mathrm{Dom}\left(L_{\sqrt{N}}\right) \cap \mathrm{Dom}\left(R_{\sqrt{N}}\right)$, and let $\mathcal{E} : D \to \mathbb{R}_+$ be the map defined by any of the expressions (4.2a–d). Then $\mathcal{E}$ is a Dirichlet form with domain D satisfying $\mathcal{E}\,[\xi_\nu] = 0$. Moreover C_{00} is a core for $\mathcal{E}$.*

Proof. By (4.2a) we see that $\mathcal{E}$ is a nonnegative quadratic form satisfying the J-invariance condition (3.1a). Using (4.2c) write

$$\mathcal{E} = \lambda\mu\mathcal{E}^{[1]} + \tfrac{1}{2}\,(\lambda - \mu)^2\,\mathcal{E}^{[2]} - \lambda\,(\mu - \lambda)\,, \tag{4.4}$$

where $\mathcal{E}^{[1]}[\xi] = \|d_A\xi\|^2$ and $\mathcal{E}^{[2]}[\xi] = \left\|L_{\sqrt{N}}\xi\right\|^2 + \left\|R_{\sqrt{N}}\xi\right\|^2$. By Lemma 2.1, $\mathcal{E}^{[2]}$ is the sum of two closed quadratic forms, and is therefore closed. Therefore $\mathcal{E}$ itself is closed, being the sum of closed forms $\lambda\mu\mathcal{E}^{[1]}$, $\frac{1}{2}\,(\lambda-\mu)^2\,\mathcal{E}^{[2]}$ and $-\lambda\,(\mu-\lambda)\,I$.

By Lemma 2.1 if $\xi \in D$ then

$$\begin{aligned}\|d_A\xi\|^2 = \|L_A\xi - R_A\xi\|^2 &\le 2\left(\|L_A\xi\|^2 + \|R_A\xi\|^2\right)\\ &= 2\left(\left\|L_{\sqrt{N}}\xi\right\|^2 + \left\|R_{\sqrt{N+1}}\xi\right\|^2\right)\\ &= 2\left(\mathcal{E}^{[2]}[\xi] + \|\xi\|^2\right).\end{aligned}$$

Thus we have the comparison of forms:

$$\mathcal{E} \le (\lambda+\mu)^2\,\mathcal{E}^{[2]} + \lambda\,(\lambda+\mu)\,I. \tag{4.5}$$

In particular, since $\mathrm{Dom}\,(\mathcal{E}) = \mathrm{Dom}\left(\mathcal{E}^{[2]}\right)$, any form core for $\mathcal{E}^{[2]}$ is also a form core for $\mathcal{E}$. Putting $p_k = \sum_{n=0}^{k} |e_n\rangle\,\langle e_n|$, we have $p_k\xi p_k \in C_{00}$ for any $\xi \in L^2$ and it is easy to see that, for $\xi \in D$,

$$p_k\xi p_k \to \xi; \quad \sqrt{N}\,p_k\xi p_k \to \sqrt{N}\xi \quad \text{and} \quad \left[p_k\xi p_k\sqrt{N}\right] \to \left[\xi\sqrt{N}\right].$$

Hence C_{00} is a form core for $\mathcal{E}^{[2]}$ and thus by (4.5) it is also a form core for $\mathcal{E}$.

Since $\nu = \lambda^2/\mu^2$ and

$$\begin{aligned}L_A\xi_\nu &= (1-\nu)^{1/2}\textstyle\sum_{n\ge1}\nu^{n/2}\sqrt{n}\,|e_{n-1}\rangle\,\langle e_n|\\ &= (1-\nu)^{1/2}\textstyle\sum_{n\ge0}\nu^{(n+1)/2}\sqrt{n+1}\,|e_n\rangle\,\langle e_{n+1}| = \sqrt{\nu}R_A\xi_\nu,\end{aligned}$$

we have $(\mu L_A - \lambda R_A)\,\xi_\nu = 0$. But $\xi_\nu^* = \xi_\nu$ so by (4.2a) $\mathcal{E}\,[\xi_\nu] = 0$. It therefore remains only to establish that $\mathcal{E}$ satisfies (3.3).

By Lemma 2.3 both $\mathrm{Dom}\left(L_{\sqrt{N}}\right) \cap L^2_{\mathbb{R}}$ and $\mathrm{Dom}\left(R_{\sqrt{N}}\right) \cap L^2_{\mathbb{R}}$ are left invariant by the modulus map $\xi \to |\xi|$, hence $D \cap L^2_{\mathbb{R}}$ is invariant under this map, and therefore also under the maps $\xi \mapsto \xi_\pm$. Lemma 2.3 also implies that $\mathcal{E}^{[2]}[|\xi|] = \mathcal{E}^{[2]}[\xi]$ for $\xi \in D \cap L^2_{\mathbb{R}}$. Now for $\xi \in C_{00} \cap L^2_{\mathbb{R}}$, since $\left[A^*\xi_- A\right], \left[A\xi_- A^*\right] \in L^2_+$,

$$\begin{aligned}\mathcal{E}^{[1]}(\xi_+, \xi_-) &= \left\langle \xi_+, d_A^* d_A \xi_- \right\rangle \\ &= \left\langle \xi_+, A^*A\xi_- - \left[A^*\xi_- A\right] - \left[A\xi_- A^*\right] + \left[\xi_- AA^*\right]\right\rangle \\ &\le \left\langle \xi_+, A^*A\xi_- \right\rangle + \left\langle \xi_+, \left[\xi_- AA^*\right]\right\rangle \\ &= \langle A\xi_+, A\xi_- \rangle + \left\langle \left[\xi_+ A\right], \left[\xi_- A\right]\right\rangle \\ &= \mathrm{Tr}\left(\left[A\xi_+\xi_- A^*\right]\right) + \mathrm{Tr}\left(\left[A^*\xi_+\xi_- A\right]\right) = 0,\end{aligned}$$

using the polarised form of the identities in Lemma 2.3. Therefore, for $\xi \in C_{00} \cap L^2_{\mathbb{R}}$, $\mathcal{E}^{[1]}[|\xi|] \le \mathcal{E}^{[1]}[\xi]$ and so, by (4.4), $\mathcal{E}[|\xi|] \le \mathcal{E}[\xi]$ too. Now let $\xi \in D \cap L^2_{\mathbb{R}}$ and choose a sequence (ξ_n) in $C_{00} \cap L^2_{\mathbb{R}}$ converging to ξ in the quadratic form norm of $\mathcal{E}$. It is easily verified that $|\xi_n| \to |\xi|$ so, by the lower semicontinuity of $\mathcal{E}$,

$$\mathcal{E}[|\xi|] \le \liminf \mathcal{E}[|\xi_n|] \le \liminf \mathcal{E}[\xi_n] = \mathcal{E}[\xi].$$

This completes the proof. □

From the results of Sect. 3 we therefore have

Corollary 4.3. *There is a self-adjoint contraction semigroup $P^{(2)}$ on L^2, with form generator (4.2), which is Markov with respect to ξ_ν, and a weak *-continuous positive contraction semigroup $P^{(\infty)}$ on L^∞, determined by $\iota^{(2)} \circ P_t^{(\infty)} = P_t^{(2)} \circ \iota^{(2)}$, which is KMS-symmetric with respect to φ_ν, and also conservative.*

The generator $-H_{(2)}$ of the symmetric Markov semigroup $P^{(2)}$ satisfies

$$H_{(2)} \supset \tfrac{1}{2}\left(\lambda^2 + \mu^2\right)(L_N + R_N) + \lambda^2 - \lambda\mu\left(L_A R_{A^*} + L_{A^*} R_A\right), \tag{4.6}$$

as is clear from (4.3). In the next section we investigate $P^{(\infty)}$, and its weak *-generator $-H_{(\infty)}$.

5. The Feller Property

The R.S. Phillips theory of dual semigroups ([Phi])implies that there is a Banach subspace of L^∞, which we shall call the *Phillips subspace* and denote by $\mathcal{B}$, on which the semigroup $P^{(\infty)}$ is strongly continuous. Moreover $\mathcal{B}$ is the norm closure of the domain of the weak *-generator of $P^{(\infty)}$, and $\mathcal{B}$ is also the maximal subspace on which $P^{(\infty)}$ is strongly continuous. This justifies the following definition.

A weak *-continuous semigroup $T = (T_t)_{t\ge 0}$ on L^∞ satisfies a *weak Feller property* if there is a weak *-dense, separable C^*-subalgebra $\mathcal{A}$ of L^∞ on which T is strongly continuous:

$$\lim_{t\to 0} \|T_t a - a\| = 0 \quad \forall a \in \mathcal{A}.$$

The semigroup is *Feller* if moreover it leaves such a C^*-subalgebra invariant:

$$T_t(\mathcal{A}) \subset \mathcal{A} \quad \forall t \ge 0.$$

By the maximality of the Phillips subspace, any such algebra $\mathcal{A}$ satisfies $\mathcal{A} \subset \mathcal{B}$.

Theorem 5.1. *Each quantum Ornstein–Uhlenbeck semigroup $P^{(\infty)}$ is Feller with respect to the algebra of compact operators $\mathcal{K}$. The subalgebra C_{00} is both an L^∞-core for the generator of $\left(P_t^{(\infty)}|_{\mathcal{K}}\right)$ and also a weak *-core for $\mathcal{L}$, the weak *-generator of $P_t^{(\infty)}$. Moreover the action of $\mathcal{L}$ on $x \in C_{00}$ is given by each of the following expressions:*

$$-\tfrac{1}{2}\mu^2\left\{A^*Ax + \left[xA^*A\right] - 2\left[A^*xA\right]\right\} - \tfrac{1}{2}\lambda^2\left\{AA^*x + \left[xAA^*\right] - 2\left[AxA^*\right]\right\}, \tag{5.1a}$$

$$-\tfrac{1}{2}\mu^2\left\{Nx + [xN] - 2\left[\sqrt{N}SxS^*\sqrt{N}\right]\right\}$$
$$-\tfrac{1}{2}\lambda^2\left\{(N+1)x + [x(N+1)] - 2\left[\sqrt{N+1}S^*xS\sqrt{N+1}\right]\right\}, \tag{5.1b}$$

$$Gx + [xG] + \left[L_\lambda^* x L_\lambda\right] + \left[L_\mu x L_\mu^*\right], \tag{5.1c}$$

where $G = -\frac{1}{2}\left\{\left(\lambda^2 + \mu^2\right)N + \lambda^2\right\}$, $L_\lambda = \lambda A^*$ and $L_\mu = \mu A^*$.

The proof proceeds through a series of lemmas. Since the weak *-generator $\mathcal{L}$ is $-H_{(\infty)}$, we use both notations, according to convenience.

Lemma 5.2. *Each O–U semigroup is weakly Feller with respect to $\mathcal{K}$ and its L^∞-generator satisfies:*

$$\mathrm{Dom}\,(\mathcal{L}) \supset C_{00}, \tag{5.2}$$

$$\iota^{(2)}\left(H_{(\infty)}x\right) = H_{(2)}\iota^{(2)}(x) \quad \forall x \in C_{00}, \tag{5.3}$$

$$\mathcal{L}(C_{00}) \subset C_{00}. \tag{5.4}$$

Proof. Since $\iota^{(2)}\left(|e_n\rangle\langle e_m|\right) = (1-\nu)^{1/2}\nu^{(n+m)/4}|e_n\rangle\langle e_m|$, we have the identity $\iota^{(2)}\left(C_{00}\right) = C_{00}$. Since $H_{(2)}$ leaves C_{00} invariant, as is clear from (4.6),

$$K : x \mapsto \left(\iota^{(2)}\right)^{-1} H_{(2)}\iota^{(2)}(x)$$

defines an operator on L^∞ with domain C_{00}, which leaves this domain invariant. Let $x \in C_{00}$, $\xi \in L^2$ and $z = (\iota^{(2)})_*(\xi) = \rho_\nu^{1/4}\xi\rho_\nu^{1/4} \in L^1$, then $\iota^{(2)}(x) \in C_{00} \subset \mathrm{Dom}\left(H_{(2)}\right)$, so

$$\begin{aligned}\left\langle z, (x - P_t^{(\infty)}x)\right\rangle &= \left\langle \xi, \iota^{(2)}(x) - P_t^{(2)}\iota^{(2)}(x)\right\rangle \\ &= \left\langle \xi, \int_0^t ds\ P_s^{(2)}H_{(2)}\iota^{(2)}(x)\right\rangle \\ &= \int_0^t ds\ \left\langle \xi, \iota^{(2)} \circ P_s^{(\infty)}(Kx)\right\rangle \\ &= \int_0^t ds\ \left\langle z, P_s^{(\infty)}(Kx)\right\rangle.\end{aligned}$$

Since the semigroup $P^{(\infty)}$ is contractive, this identity for z extends from the dense subspace $\left(\iota^{(2)}\right)_*(L^2)$ to all of L^1 by the Dominated Convergence Theorem. Dividing by t and letting $t \searrow 0$ therefore gives

$$x \in \mathrm{Dom}\left(H_{(\infty)}\right) \quad \text{and} \quad H_{(\infty)}x = Kx.$$

This establishes (5.2–4) and also that the Phillips subspace $\mathcal{B}$ contains C_{00}. Since the closure of C_{00} in L^∞ is $\mathcal{K}$, and $\mathcal{B}$ is closed, $\mathcal{B}$ must contain $\mathcal{K}$ also. □

Lemma 5.3. *The expressions (5.1a–c) all coincide with $\mathcal{L}x$, for $x \in C_{00}$.*

Proof. The expressions (5.1a) and (5.1b) coincide by the explicit polar decompositions (4.1). A simple computation on the basis elements $|e_n\rangle\langle e_m|$ of C_{00} shows that (5.1c) agrees with (5.1a). Let $K_0 : C_{00} \to C_{00}$ be the map given by these common expressions. The identities

$$\rho_\nu^{1/4} A^* e_n = \rho_\nu^{1/4}\sqrt{n+1}e_{n+1} = (1-\nu)^{1/4}\nu^{(n+1)/4}\sqrt{n+1}e_{n+1} = \nu^{1/4}A^*\rho_\nu^{1/4}e_n,$$

give the following commutation relations, for $x \in C_{00}$:

$$\rho_\nu^{1/4}A^*x = \nu^{1/4}A^*\rho_\nu^{1/4}x;\quad \rho_\nu^{1/4}Ax = \nu^{-1/4}A\rho_\nu^{1/4}x;\quad \rho_\nu^{1/4}Nx = N\rho_\nu^{1/4}x.$$

Thus, since $\nu = \lambda^2/\mu^2$,

$$\begin{aligned}\rho_\nu^{1/4}\left(\mu^2\left[A^*xA\right] + \lambda^2\left[AxA^*\right]\right)\rho^{1/4} &= \mu^2\nu^{1/2}\left[A^*\xi A\right] + \lambda^2\nu^{-1/2}\left[A\xi A^*\right]\\ &= \lambda\mu\left\{\left[A^*\xi A\right] + \left[A\xi A^*\right]\right\},\end{aligned}$$

where $\xi = \iota^{(2)}(x)$. This gives the following identity for $x \in C_{00}$:

$$\begin{aligned}&\iota^{(2)}(K_0x)\\ &= -\tfrac{1}{2}\left\{\left(\lambda^2+\mu^2\right)\left(N\iota^{(2)}(x) + \left[\iota^{(2)}(x)N\right]\right) + \lambda^2\iota(x)\right\} + \lambda\mu\left\{\left[A^*\xi A\right] + \left[A\xi A^*\right]\right\}\\ &= -H_{(2)}\iota^{(2)}(x)\\ &= \iota^{(2)}\left(\mathcal{L}x\right),\end{aligned}$$

by (4.6) and (5.3). By the injectivity of $\iota^{(2)}$, this completes the proof. □

Lemma 5.4. *There is a time $T > 0$, depending only on λ and μ, such that*

$$\sum_{k\geq 0} t^k \left\|\mathcal{L}^k x\right\|/k! < \infty$$

for all $x \in C_{00}$ and $t \in [0, T[$. In particular each element of C_{00} is an analytic vector for $\mathcal{L}$. For $t \in [0, T[$ and $x \in C_{00}$, $P_t^{(\infty)}x = \sum_{k\geq 0}(k!)^{-1}t^k\mathcal{L}^k x$.

Proof. Put $\mathcal{L} = -H_{(\infty)}$. We know, by (5.4), that $\mathcal{L}$ leaves C_{00} invariant so that $C_{00} \subset \bigcap_{N\geq 1}\mathrm{Dom}\left(\mathcal{L}^k\right)$. If $x = |e_n\rangle\langle e_m|$ then

$$\mathcal{L}x = \alpha_{nm}\,|e_{n-1}\rangle\langle e_{m-1}| + \beta_{nm}\,|e_n\rangle\langle e_m| + \gamma_{nm}\,|e_{n+1}\rangle\langle e_{m+1}|, \tag{5.5}$$

where

$$\alpha_{nm} = \lambda^2\sqrt{nm};\quad \beta_{nm} = -\frac{1}{2}\left(\lambda^2+\mu^2\right)(n+m) - \lambda^2;$$
$$\gamma_{nm} = \mu^2\sqrt{(n+1)(m+1)}.$$

Since $|\alpha_{nm}| + |\beta_{nm}| + |\gamma_{nm}| \le \left(\lambda^2 + \mu^2\right)(n+m+1)$, iterating (5.5) leads to the estimate

$$\begin{aligned}\left\|\mathcal{L}^k x\right\| &\le \left(\lambda^2+\mu^2\right)^k (n+m+1)(n+m+3)\cdots(n+m+2k-1)\\ &\le 2^k\left(\lambda^2+\mu^2\right)^k (n+m+1)(n+m+2)\cdots(n+m+k)\\ &= 2^k\left(\lambda^2+\mu^2\right)^k k!\binom{n+m+k}{n+m}\\ &\le 2^{n+m}4^k\left(\lambda^2+\mu^2\right)^k k!\,,\end{aligned}$$

thus putting $T = \left[4(\lambda^2+\mu^2)\right]^{-1}$, $\sum_{k\ge 0} t^k \left\|\mathcal{L}^k x\right\| / k! < \infty$ for $t < T$. Since any element of C_{00} is a finite linear combination of elements of the form $|e_n\rangle\langle e_m|$, this finiteness holds for any x in C_{00}. We may therefore define maps $S_t : C_{00} \to L^\infty$, for $t \in [0, T[$, by

$$S_t x = \sum_{k\ge 0} (k!)^{-1} t^k \mathcal{L}^k x.$$

Since each $\mathcal{L}^k x \in C_{00}$, $S_t x \in \mathcal{K}$. By (5.4), for $x \in C_{00}$,

$$\frac{d}{dt} S_t x = \sum_{k\ge 0} (k!)^{-1} t^k \mathcal{L}^{k+1} x = \sum_{k\ge 0} \mathcal{L}\left((k!)^{-1}\, t^k \mathcal{L}^k x\right) \quad \forall t \in [0, T[.$$

Now the series $\sum (k!)^{-1} t^k \mathcal{L}^k x$ and $\sum \mathcal{L}\left((k!)^{-1}\, t^k \mathcal{L}^k x\right)$ are both norm convergent and so also weak *-convergent, and $\mathcal{L}$ is weak *-closed, so

$$S_t x \in \mathrm{Dom}\,(\mathcal{L}) \quad \text{and} \quad \frac{d}{dt} S_t x = \mathcal{L}\,(S_t x) \quad \forall t \in [0, T[.$$

By the uniqueness of the solution of the Cauchy problem,

$$P_t^{(\infty)} x = S_t x = \sum_{k\ge 0} (k!)^{-1} t^k \mathcal{L}^k x \tag{5.6}$$

for $x \in C_{00}$ and $t \in [0, T[$. □

Proof of the theorem. Since $\mathcal{K}$ is the norm closure of C_{00}, the inclusion (5.2) implies that the Phillips subspace includes $\mathcal{K}$, and so $P^{(\infty)}$ is weakly Feller with respect to $\mathcal{K}$. The contractivity of each $P_t^{(\infty)}$ and (5.6) together imply that $P_t^{(\infty)}(\mathcal{K}) \subset \mathcal{K}$ for $t \in [0, T[$. Invariance for all positive times now follows from the semigroup property, so $P^{(\infty)}$ is strongly Feller. The identification of (5.1) with $\mathcal{L}x$, for $x \in C_{00}$, is contained in Lemma 5.3. Finally, since C_{00} is weak *-dense and $H_{(\infty)}$-invariant, it is a weak *-core for $H_{(\infty)}$ ([Br R], Corollary 3.7), and since it is norm dense in $\mathcal{K}$, C_{00} is also a core for the generator of $\left(P_t^{(\infty)}|_{\mathcal{K}}\right)$. This completes the proof. □

6. Ergodicity and Discreteness of Spectrum

We saw in Theorem 4.2 that the quantum O–U semigroups have zero at the bottom of their L^2-spectrum, and that zero is an eigenvalue corresponding to the eigenvector ξ_ν. In the present section we shall strengthen this by proving ergodicity of the semigroups $P^{(2)}$ and $P_t^{(\infty)}$, and discreteness of the L^2-spectrum. The results contained in this section are subsumed by those of the final section, however they are included here since the arguments used may be applicable in cases where it is difficult to find the whole spectrum. L^∞-ergodicity has been demonstrated by different methods in [FRS].

A positivity preserving self-adjoint contraction semigroup (T_t) on L^2 is *ergodic* if

$$\forall \xi, \eta \in L^2_+ \setminus \{0\} \quad \exists t > 0 \text{ such that } \langle \xi, T_t \eta \rangle > 0.$$

We shall use the following result from [Ci 1]:

Theorem 6.1. *Let (T_t) be a positivity preserving self-adjoint contraction semigroup on L^2. If zero is an eigenvalue of the generator of the semigroup, then the following are equivalent:*

(i) *the multiplicity of the zero eigenvalue is one and it has a strictly positive eigenvector;*
(ii) *(T_t) is ergodic.*

Strict positivity for a vector in L^2 means that its support is the identity in $\mathcal{L}^\infty$, equivalently the vector is cyclic for $\mathcal{L}^\infty$. A semigroup (S_t) on L^∞ is *ergodic* if

$$S_t x = x \;\forall t \geq 0 \Rightarrow x = \alpha 1 \text{ for some } \alpha \in \mathbb{C}.$$

Theorem 6.2. *The quantum O–U semigroups $P^{(2)}$ and $P^{(\infty)}$ are ergodic.*

Proof. By Theorem 4.2, $\mathcal{E}[\xi_\nu] = 0$, so zero is an eigenvalue of the generator of $P^{(2)}$. The representation (4.2d) makes it clear that only multiples of ξ_ν satisfy $\mathcal{E}[\xi] = 0$. Since ξ_ν is strictly positive, L^2-ergodicity follows from Theorem 6.1. Now L^∞-ergodicity follows from the injectivity of the symmetric embedding $\iota^{(2)}$. □

Theorem 6.3. *The L^2-spectrum of the quantum O–U semigroups are discrete.*

Proof. As in the proof of Theorem 4.2 we represent the O–U Dirichlet form as

$$\mathcal{E} = \lambda\mu\mathcal{E}^{[1]} + \tfrac{1}{2}(\lambda - \mu)^2 \mathcal{E}^{[2]} - \lambda(\mu - \lambda) I,$$

where $\mathcal{E}^{[1]}[\xi] = \|d_A \xi\|^2$ and $\mathcal{E}^{[2]}[\xi] = \left\| L_{\sqrt{N}} \xi \right\|^2 + \left\| R_{\sqrt{N}} \xi \right\|^2$. Let $H^{[1]} = d_A^* d_A$ and $H^{[2]} = \sum_{k \geq 0} k R_k$, where R_k is the orthogonal projection onto the linear span of $\{|e_n\rangle \langle e_m| : n + m = k\}$. Clearly C_{00} is a core for $H^{[2]}$ so that (by the proof of Theorem 4.2) $H^{[2]}$ is the self-adjoint operator associated with $\mathcal{E}^{[2]}$. Now recall the comparison of forms obtained in the same proof (4.4) – this may be written $H_{(2)} \geq K$, where

$$K = \tfrac{1}{2} (\lambda - \mu)^2 H^{[2]} - \lambda(\mu - \lambda) I.$$

It follows from the minimax principle that the infimum of the essential spectrum of $H_{(2)}$ is greater than that of K ([Da 2], Lemma 1.2.2). Since K has empty essential spectrum, so does $H_{(2)}$; the spectrum is therefore discrete. □

As a final result of this section we consider the restriction of the quantum O–U semigroups to the abelian subalgebra consisting of bounded functions of the number operator, i.e. the weak *-closed linear span of $\{|e_n\rangle\langle e_n| : n \in \mathbb{Z}_+\}$, which we identify with $l^\infty(\mathbb{Z}_+)$.

Theorem 6.4. *The semigroup $P^{(\infty)}$ leaves $l^\infty(\mathbb{Z}_+)$ invariant. Its restriction to $l^\infty(\mathbb{Z}_+)$ is the Markov semigroup of the classical birth and death process with birth rates $\big(\lambda^2(k+1)\big)_{k\geq 0}$ and death rates $\big(\mu^2 k\big)_{k\geq 0}$.*

Proof. Let c_0 and c_{00} denote the subalgebras of l^∞ consisting of sequences which tend to zero, respectively vanish after a finite number of terms. Recall the proof of Lemma 5.4. By (5.5), $\mathcal{L}(|e_n\rangle\langle e_n|)$ is given by

$$\lambda^2 n\,|e_{n-1}\rangle\langle e_{n-1}| - \left\{\left(\mu^2+\lambda^2\right)n+\lambda^2\right\}|e_n\rangle\langle e_n| + \mu^2(n+1)\,|e_{n+1}\rangle\langle e_{n+1}|\,. \quad (6.1)$$

Moreover, for $t \in [0,T[$ and x in $l^\infty \cap C_{00} = c_{00}$, $P_t^{(\infty)}x \in l^\infty \cap \mathcal{K} = c_0$, by Lemma 5.4. By norm density of c_{00} in c_0 and weak *-density of c_{00} in l^∞, together with the norm and weak*-continuity of each $P_t^{(\infty)}$ and the semigroup property of $P^{(\infty)}$, both c_0 and l^∞ are left invariant, and semigroups are induced on these abelian subalgebras. Now one can recognise in (6.1) the action of the generator of a classical birth-and-death process (put $\varphi = \delta_n$, where $\delta_n(k) = 1$ if $k = n$ and 0 otherwise, in (7.8) below). □

7. L^2-Spectrum: Case $\lambda < \mu$

In this section we shall obtain a complete description of the L^2–spectrum of the quantum Ornstein–Uhlenbeck semigroups, together with multiplicities and eigenspaces. We shall also see how both classical Ornstein–Uhlenbeck semigroups and classical birth and death processes are embedded within each quantum semigroup.

The notation developed in the previous sections will be used, together with:

$$\Gamma(z) = z^{-N};\;\; Q_z = 2^{-1/2}[\overline{z}A + zA^*]$$

for $z \in \mathbb{C}$ of unit modulus. Thus, writing Q for Q_z when $z = 1$,

$$Q_z = \Gamma(z)^* Q \Gamma(z).$$

Lemma 7.1. *Let $\rho = \rho_\nu$. Then*

(a) $V := \mathrm{Lin}\{\rho^{1/4}A^{*i}A^j\rho^{1/4} : i, j \geq 0\}$ *is a dense subspace of* L^2.
(b) $U_n = W_n$ *for each* $n \geq 0$, *where* $U_n := \mathrm{Lin}\{\rho^{1/4}A^{*i}A^j\rho^{1/4} : i+j \leq n\}$ *and* $W_n := \mathrm{Lin}\{\rho^{1/4}Q_z^m\rho^{1/4} : |z| = 1,\; m \leq n\}$.

Proof. First note that, for $\gamma \in\,]0,1[$,

$$\mathrm{Ran}(\gamma^N) = \mathrm{Dom}(\gamma^{-N}) \subset \textstyle\bigcap_{k\geq 1}\mathrm{Dom}(N^k) = \bigcap_{l,m}\mathrm{Dom}(A^{*l}A^m).$$

Since $\rho = (1-\nu)\nu^N$, it follows that $A^{*l}A^m\rho^{1/4}$ is everywhere defined and closed, and therefore bounded, for each $l, m \geq 0$. In particular $V \subset \bigcap_{p\geq 1} L^p$.

(a) Let $\xi = \sum_{n,m\geq 0} \alpha_{nm} |e_n >< e_m| \in L^2$ be orthogonal to V. Since V is invariant under the adjoint operation it suffices to show that $\alpha_{nm} = 0$ for $m \geq n \geq 0$. Now

$$\begin{aligned}&\rho^{1/4} A^{*(i+k)} A^i \rho^{1/4}\\ &= (1-\nu)^{1/2} \nu^{k/4} \textstyle\sum_{n\geq 0} \nu^{n/2} \sqrt{(n+1)\cdots(n+k)}\, n(n-1)\cdots\\ &\qquad \cdots (n-i+1)|e_{n+k} >< e_n|,\end{aligned}$$

so, fixing $k \geq 0$, the orthogonality condition implies that, for each i,

$$\sum_{n\geq i} \alpha_{n,n+k} \sqrt{(n+1)\cdots(n+k)}\, \nu^{n/4} n(n-1)\cdots(n-i+1)\nu^{(n-i)/4} = 0. \qquad (7.1)$$

Now let $\beta_n = \alpha_{n,n+k}\sqrt{(n+1)\cdots(n+k)}\,\nu^{n/4}$. Then the sequence (β_n) is square–summable and so $f(z) = \sum_{n\geq 0} \beta_n z^n$ defines an analytic function on the open unit disc, and (7.1) says that f and all of its derivatives vanish at $z = \nu^{1/4}$. Thus f must be identically zero, and so $\alpha_{n,n+k} = 0\ \forall\, n$. Since $k \geq 0$ was fixed arbitrarily, this establishes (a).

(b) The inclusion $W_n \subset U_n$ is obvious from the definition and canonical commutation relations. For the opposite inclusion let $m \leq n$ and let ω be a unit modulus number whose square is a primitive $(m+1)^{\text{th}}$ root of unity, and note that for $r \in \mathbb{Z}$, $\sum_{j=0}^m \omega^{2jr} = 0$ unless $\omega^{2r} = 1$. Thus, for $l \in \{0, 1, \ldots, m\}$,

$$\begin{aligned}\textstyle\sum_{j=0}^m \omega^{j(m-2l)} (Q_{\omega^j})^m &= \textstyle\sum_{j=0}^m \omega^{-2lj} (A + \omega^{2j} A^*)^m\\ &= \textstyle\sum_{k=0}^m \binom{m}{k} \sum_{j=0}^m \omega^{2j(k-l)} A^{*k} A^{(m-k)} + \text{l.o.t.}\\ &= (m+1)\binom{m}{l} A^{*l} A^{(m-l)} + \text{l.o.t.},\end{aligned}$$

where l.o.t. is a linear combination of terms of the form $A^{*j}A^k$ with $j + k < m$. Since $W_0 = U_0 = \mathbb{C}\rho^{1/2}$, it follows inductively that $U_n \subset W_n$. □

The differential operator given by

$$(\mathcal{G}^{\text{OU}}\varphi)(t) = \Big(\frac{\mu^2+\lambda^2}{4}\Big)\varphi''(t) - \Big(\frac{\mu^2-\lambda^2}{2}\Big)t\varphi'(t) \qquad (7.2)$$

is the generator of a classical Ornstein–Uhlenbeck semigroup; its eigenpolynomials are $\{p_n : n \geq 0\}$, where

$$p_n(t) = \textstyle\sum_{2r\leq n} \Big\{ -\frac{1}{4}\cdot\frac{\mu^2+\lambda^2}{\mu^2-\lambda^2}\Big\}^r \frac{n!}{r!(n-2r)!} t^{n-2r}, \qquad (7.3)$$

and corresponding eigenvalues $-\{n(\mu^2-\lambda^2)/2\}$.

Theorem 7.2. *The L^2-generator of the quantum Ornstein–Uhlenbeck semigroup, with parameters $\lambda < \mu \in\,]0,\infty[$, has the form*

$$H_{(2)} = \Big(\frac{\mu^2-\lambda^2}{2}\Big) \textstyle\sum_{n\geq 0} n P_{E_n},$$

where P_{E_n} is the orthogonal projection onto

$$E_n := \text{Lin}\{\rho^{1/4} p_n(Q_z)\rho^{1/4} : |z| = 1\}$$

and p_n is given by (7.3). A basis for E_n is obtained by restricting z to the set

$$\Omega_n := \{\omega_n^j : j = 0, \ldots, n\},$$

where ω_n is a unit modulus number whose square is a primitive $(n+1)^{th}$ root of unity.

Proof. In view of the relations $\mathrm{Ran}(\rho^{1/4}) \subset \mathrm{Dom}(N)$,

$$N\rho^{1/4}A^{*i}A^j\rho^{1/4} = \rho^{1/4}A^*AA^{*i}A^j\rho^{1/4},$$
$$J\rho^{1/4}A^{*i}A^j\rho^{1/4} = \rho^{1/4}A^{*j}A^i\rho^{1/4},$$

and Lemma 2.1, the subspace V is contained in the domain of L_N and R_N, and is left invariant by both operators. Therefore, by Proposition 4.1, $V \subset \mathrm{Dom}(H_{(2)})$ and for $\xi \in V$,

$$H_{(2)}\xi = \Big(\frac{\lambda^2+\mu^2}{2}\Big)(N\xi + [\xi N]) + \lambda^2\xi - \lambda\mu(A[\xi A^*] + A^*[\xi A]). \tag{7.4}$$

The gauge invariance

$$\xi \in V \Rightarrow \Gamma(z)^*\xi\Gamma(z) \in V \quad \text{and} \quad H_{(2)}(\Gamma(z)^*\xi\Gamma(z)) = \Gamma(z)^*(H_{(2)}\xi)\Gamma(z)$$

follows from the commutation relation $A^*\Gamma(z) = z[\Gamma(z)A^*]$. Switching for a moment to the Schrödinger representation in which $A = 2^{-1/2}[Q + iP]$, $Q = M_x$ and $P = -i\frac{d}{dx}$, commutation relations yield the identities ([FaR])

$$(A^*AM_\varphi - 2A^*M_\varphi A + M_\varphi A^*A)f = \Big(-\frac{1}{2}\varphi'' + x\varphi'\Big)f, \tag{7.5a}$$

$$(AA^*M_\varphi - 2AM_\varphi A^* + M_\varphi AA^*)f = \Big(-\frac{1}{2}\varphi'' - x\varphi'\Big)f \tag{7.5b}$$

for smooth φ, and $f \in L^2(\mathbb{R})$ for which both f and φf lie in the domain of $N = A^*A$. Putting $\xi = \rho^{1/4}p(Q)\rho^{1/4}$ in (7.4), where p is a polynomial, and using the commutation relations $N\rho^{1/4} = [\rho^{1/4}N]$, $A\rho^{1/4} = \nu^{1/4}[\rho^{1/4}A]$, gives

$$\begin{aligned} H_{(2)}(\rho^{1/4}p(Q)\rho^{1/4}) = \rho^{1/4}\Big\{&\frac{\mu^2}{2}(A^*Ap(Q) - 2A^*p(Q)A + p(Q)A^*A) \\ &+\frac{\lambda^2}{2}(AA^*p(Q) - 2Ap(Q)A^* + p(Q)AA^*)\Big\}\rho^{1/4}. \end{aligned}$$

Using (7.5) and the functional calculus, this gives

$$-H_{(2)}(\rho^{1/4}p(Q)\rho^{1/4}) = \rho^{1/4}(\mathcal{G}^{\mathrm{OU}}p)(Q)\rho^{1/4}, \tag{7.6}$$

where $\mathcal{G}^{\mathrm{OU}}$ is the classical OU–generator (7.2). Applying gauge invariance to (7.6) we obtain

$$H_{(2)}(\rho^{1/4}p(Q_z)\rho^{1/4}) = \rho^{1/4}(\mathcal{G}^{\mathrm{OU}}p)(Q_z)\rho^{1/4}. \tag{7.7}$$

Thus

$$H_{(2)}\xi = n\Big(\frac{\mu^2-\lambda^2}{2}\Big)\xi \quad \forall\, \xi \in E_n,$$

in particular the subspaces $\{E_n : n = 0, 1, \dots\}$ are mutually orthogonal.

Since p_n is a polynomial of degree n, it now follows from Lemma 7.1 that

$$E_n = W_n \ominus W_{n-1}\ (n \geq 1);\ \dim E_n = (n+1);\ \bigoplus_{n\geq 0} E_n = L^2;$$

and $H_{(2)}$ is of the form claimed. It remains only to prove that $S_n := \{\rho^{1/4}p_n(Q_z)\rho^{1/4} : z \in \Omega_n\}$ is a basis for E_n. From the proof of Lemma 7.1 if $\omega \in \mathbb{C}$ is such that ω^2 is a primitive $(n+1)^{\text{th}}$ root of unity then, for each $l = 0, 1, \dots, n$, there is $\{\alpha_{j,l,n} : j = 0, 1, \dots, n\} \subset \mathbb{C}$ such that

$$A^{*l}A^{(n-l)} = \textstyle\sum_{j=0}^{n} \alpha_{j,l,n}\, p_n(Q_{\omega^j}) + \text{l.o.t.},$$

in particular, for each unit modulus z, there are $\{\alpha_j(z) : j = 0, \dots, n\} \subset \mathbb{C}$ such that

$$p_n(Q_z) = \textstyle\sum_{j=0}^{n} \alpha_j(z)\, p_n(Q_{\omega^j}) + \text{l.o.t.}$$

By orthogonality the lower order terms (l.o.t.) must all vanish, thus S_n spans E_n. Since $\#S_n = (n+1) = \dim E_n$, S_n is a basis for E_n and the proof is complete. □

Let $\mathcal{G}^{\text{BD}}$ be the difference operator defined by

$$(\mathcal{G}^{\text{BD}}\varphi)(k) = \mu^2 k\{\varphi(k-1) - \varphi(k)\} + \lambda^2(k+1)\{\varphi(k+1) - \varphi(k)\}, \tag{7.8}$$

with the understanding $\varphi(-1) = 0$. Then $\mathcal{G}^{\text{BD}}$ is the generator of a birth and death process, and its eigenvalues are $\{n(\mu^2 - \lambda^2) : n \geq 0\}$, each having multiplicity 1. We may now give an L^2-view of Theorem 6.4.

Proposition 7.3. *For any polynomial q,*

$$-H_{(2)}(\rho^{1/4}q(N)\rho^{1/4}) = \rho^{1/4}(\mathcal{G}^{BD}q)(N)\rho^{1/4}. \tag{7.9}$$

Proof. In view of the commutation relations $A\varphi(N)e = \varphi(N+1)Ae$, valid for $e \in \text{Lin}\{e_n : n \geq 0\}$, if $\xi = \rho^{1/4}q(N)\rho^{1/4}$, then

$$A\xi A^*e = \nu^{1/2}\rho^{1/4}q(N+1)(N+1)\rho^{1/4}e;$$
$$A^*\xi Ae = \nu^{-1/4}\rho^{1/4}Nq(N-1)\rho^{1/4}e.$$

Equation (7.9) now follows easily from (7.4). □

Putting $m = 2k$ and $l = k$ in the computation in the proof of Lemma 7.1 (b), and using mutual orthogonality of the eigenspaces of $H_{(2)}$ as in the proof of Theorem 7.2, leads to the following interesting relationship between the respective eigenpolynomials of the Ornstein–Uhlenbeck and birth and death generators:

Proposition 7.4. *Let $\{p_n : n \geq 0\}$ and $\{q_m : m \geq 0\}$ be respectively the (monic) eigenpolynomials of the Ornstein–Uhlenbeck and birth and death generators, indexed by increasing eigenvalues, then for each k,*

$$q_k(N) = \left\{(2k+1)\binom{2k}{k}\right\}^{-1} \textstyle\sum_{j=0}^{2k} p_{2k}(Q_{\omega^j}),$$

where ω^2 is a primitive $(2k+1)^{th}$ root of unity.

Theorem 7.2 and Eqs. (7.7) and (7.9) together show how quantum theory can manufacture a discrete (classical) process by knitting together a one parameter family of classical continuous processes into a single quantum process.

8. Quantum Brownian Motion

When $\lambda = \mu$ there is no longer an invariant state for the dynamics. However the quadratic form (4.2) reduces to a multiple of $\|d_A\cdot\|^2$ which is a Dirichlet form with respect to the trace, and therefore generates a symmetric Markov semigroup on L^2, and also determines a semigroup on the algebra, by the theory developed in [AH–K] and [DaL]. The counterpart to (7.7) is

$$\mathcal{L}\varphi(Q_z) = (\mathcal{G}^{BM}\varphi)(Q_z),$$

where $\mathcal{G}^{BM}$ is the generator of a classical Brownian motion. We shall therefore refer to the *quantum Brownian motion semigroup*. The arguments of Sect. 5, in a simplified form, continue to apply when $\lambda = \mu$, and so Theorem 5.1 holds in this case too. Thus the quantum BM semigroup is a Feller semigroup with respect to the algebra of compact operators. Not unexpectedly the L^2-spectrum is now the whole of the positive half-line.

Theorem 8.1. *The L^2-generator of the quantum Brownian motion semigroup with equal parameters $\lambda = \mu \in]0,\infty[$ has spectrum* $[0,\infty[$.

Proof. Without loss we may suppose that $\lambda = \mu = 1$, thus let $H_{(2)} = d_A^* d_A$. For $t \in \mathbb{R}$ and $\varepsilon > 0$ let

$$\xi_{t,\varepsilon} = V_t R_\varepsilon,$$

where V_t is the Weyl operator $\exp iQ_{t\sqrt{2}} = \exp it[A + A^*]$ and R_ε is the Yosida approximation to the identity $\mathrm{e}^{-\varepsilon N}$, N being the number operator. Thus V_t is unitary, $p(A, A^*)R_\varepsilon$ is Hilbert–Schmidt for any polynomial p, and the following commutation relations are easily verified:

$$V_t^* A V_t = A + itI, \quad V_t^* N V_t = N + i[tA^* - tA] + t^2 I, \quad [R_\varepsilon A] = \mathrm{e}^\varepsilon A R_\varepsilon.$$

It follows that $\xi_{t,\varepsilon} \in \mathrm{Dom}(H_{(2)})$, and

$$\begin{aligned}
&H_{(2)}\xi_{t,\varepsilon}\\
&= NV_tR_\varepsilon + V_t[R_\varepsilon N] + V_tR_\varepsilon - AV_t[R_\varepsilon A^*] - A^*V_t[R_\varepsilon A]\\
&= V_t\{N + itA^* - itA + t^2 I + N + I\}R_\varepsilon\\
&\quad -V_t(A + itI)\mathrm{e}^{-\varepsilon}A^*R_\varepsilon - V_t(A^* - itI)\mathrm{e}^{\varepsilon}AR_\varepsilon\\
&= t^2\xi_{t,\varepsilon} + (1-\mathrm{e}^{\varepsilon})V_t\{N - itA\}R_\varepsilon + (1-\mathrm{e}^{-\varepsilon})V_t\left\{(N+1) + itA^*\right\}R_\varepsilon.
\end{aligned}$$

But

$$\left\|\xi_{t,\varepsilon}\right\|_2^2 = \left\|\mathrm{e}^{-\varepsilon N}\right\|_2^2 = \sum_{n\geq 0}\mathrm{e}^{-2\varepsilon n} = \left(1 - \mathrm{e}^{-2\varepsilon}\right)^{-1},$$

therefore

$$\left\|\xi_{t,\varepsilon}\right\|^{-1}\left\|H_{(2)}\xi_{t,\varepsilon} - t^2\xi_{t,\varepsilon}\right\|_2 \to 0 \text{ as } \varepsilon \to 0.$$

This shows that $[0,\infty[\subset \sigma(H_{(2)})$, but $H_{(2)}$ is nonnegative so the reverse inclusion holds too. □

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