Commun. Math. Phys. 210, 49 – 83 (2000)

Communications in
Mathematical Physics

Spectral Transformation Chains and Some New Biorthogonal Rational Functions

Vyacheslav Spiridonov[1], **Alexei Zhedanov**[2]

[1] Bogoliubov Laboratory of Theoretical Physics, JINR, Dubna, Moscow region 141980, Russia.
E-mail: svp@thsun1.jinr.ru
[2] Donetsk Institute for Physics and Technology, Donetsk 340114, Ukraine.
E-mail: zhedanov@host.dipt.donetsk.ua

Received: 14 May 1999 / Accepted: 5 September 1999

Abstract: A discrete-time chain, associated with the generalized eigenvalue problem for two Jacobi matrices, is derived. Various discrete and continuous symmetries of this integrable equation are revealed. A class of its rational, elementary and elliptic functions solutions, appearing from a similarity reduction, are constructed. The latter lead to large families of biorthogonal rational functions based upon the very-well-poised balanced hypergeometric series of three types: the standard hypergeometric series ${}_9F_8$, basic series ${}_{10}\varphi_9$ and its elliptic analogue ${}_{10}E_9$. For an important subclass of the elliptic biorthogonal rational functions the weight function and normalization constants are determined explicitly.

Contents

1. Introduction . . . 50
2. R_{II}-Polynomials and the R_{II}-Chain . . . 51
3. Companion Polynomials . . . 55
4. Symmetries of the R_{II}-Chain and a Similarity Reduction . . . 56
5. Rational and Elementary Functions Solutions . . . 61
6. Elliptic Solutions of the Basic Equation . . . 65
7. Elliptic Analogues of Hypergeometric Functions . . . 68
8. Finite-Dimensional Biorthogonality . . . 70
9. Duality Property and the Normalization Constants . . . 73
10. Möbius Transformations of the Grids and Some Other Similarity Reductions . . . 77
11. General Elliptic Biorthogonal Rational Functions . . . 79
12. Conclusions . . . 80
References . . . 82

1. Introduction

The theory of orthogonal polynomials is a well established subject [19]. Numerous and long history investigations of explicit examples culminated in the discovery of the Askey–Wilson polynomials – the most general set of the classical orthogonal polynomials [1]. The theory of biorthogonal rational functions is less developed, but it is actively pursued in many recent papers, see, e.g., [9,11,13–15,22–24]. A remarkable set of such functions related to a very-well-poised 2-balanced generalized hypergeometric series ${}_9F_8$ containing five free parameters was constructed by Wilson [22]. This class of biorthogonal rational functions was believed to be the most general one based upon the plain hypergeometric series. A q-analogue of this class, providing a six parameter family of functions expressed through a very-well-poised balanced basic hypergeometric series ${}_{10}\varphi_9$, has been built by Rahman and Wilson in [13,14,23] (see also [9,15]). These examples are intimately related to Ramanujan's entry 40 continued fraction and its basic analogue [8]. For a different interesting approach to the biorthogonality concept, see [10].

In the present work we extend known classes of biorthogonal rational functions in several respects. First, we describe an eight parameter family of functions expressed through the very-well-poised 2-balanced ${}_9F_8$ series with a more complicated parametrization of its arguments than in the Wilson case. These functions are orthogonal to a linear combination of three series of a similar form. A brief announcement of this result is given in our recent note [25]. Second, we present a q-analogue of this class – a nine parameter family of rational functions expressed through a very-well-poised balanced ${}_{10}\varphi_9$ series which are orthogonal to a linear combination of three similar functions.

The third generalization concerns the principally new type of series – the elliptic analogues of the very-well-poised balanced hypergeometric series. These series were introduced recently by Frenkel and Turaev under the name "modular hypergeometric functions" in the context of elliptic solutions of the Yang–Baxter equation [4]. As a natural generalization of the previous class of rational functions, we derive a ten parameter family of biorthogonal functions on the basis of an elliptic generalization of the mentioned ${}_{10}\varphi_9$ series.

The elliptic class of functions contains a subclass obeying a self-duality symmetry similar to the one of the Wilson's functions. For this special case we give explicit expressions for the discrete weight function and normalization constants in the biorthogonality relation. We conjecture that these self-dual functions define the most general set of classical biorthogonal rational functions in the spirit of the Askey–Wilson polynomials situation.

The key method of construction of new explicit examples of biorthogonal functions is based upon the analysis of solutions of a chain of spectral transformations for a specific three-term recurrence relation. This recurrence relation was introduced by Ismail and Masson in connection to the R_{II} type continued fractions [11]. Our spectral transformations generalize the ones investigated by Christoffel and Geronimus in the theory of orthogonal polynomials [6,7,19]. They may be considered also as discrete Darboux transformations for biorthogonal rational functions. From the point of view of the theory of integrable equations, we construct a specific discrete (1+1)-dimensional integrable chain and find its particular self-similar solutions associated with some generalized separation of variables. In the case of orthogonal polynomials an analogue of such a program leads to the discrete-time Toda chain (or the modified qd-algorithm) and its self-similar solutions comprising recurrence coefficients of the Askey–Wilson polynomials [17,18]. The biorthogonal functions mentioned above are derived in a systematic fashion as a result of a solution of a well-defined set of finite-difference equations. In particular, the modular hypergeometric series are emerging as special solutions of the R_{II} three-term recurrence relation with some elliptic functions coefficients.

2. R_{II}-Polynomials and the R_{II}-Chain

Denote as $P_n^j(z)$, $n, \ j \in \mathbf{Z}$, an infinite two-dimensional array of functions of the independent variable $z \in \mathbf{C}$. Let these functions satisfy the following relations:

$$P_n^{j+1}(z) = \frac{D_n^{j+1} P_{n+1}^j(z) + C_n^{j+1}(z - \alpha_n^{j+1}) P_n^j(z)}{z - \lambda_{j+1}}, \tag{2.1}$$

$$P_n^{j-1}(z) = B_n^j P_n^j(z) + A_n^j (z - \beta_n^j) P_{n-1}^j(z), \tag{2.2}$$

where the superpotentials $A_n^j, B_n^j, C_n^j, D_n^j$ and the spectral coefficients $\alpha_n^j, \beta_n^j, \lambda_j$ do not depend on z. Performing the shift $j \to j-1$ in (2.1) and removing $P_n^{j-1}(z)$ with the help of (2.2) we arrive at the three-term recurrence relation

$$P_{n+1}^j(z) + r_n^j (v_n^j - z) P_n^j(z) + u_n^j (z - \alpha_n^j)(z - \beta_n^j) P_{n-1}^j(z) = 0, \tag{2.3}$$

where the potentials u_n^j, r_n^j, v_n^j have the form

$$\begin{aligned} u_n^j &= \frac{A_n^j C_n^j}{D_n^j B_{n+1}^j}, \quad r_n^j = \frac{1 - D_n^j A_{n+1}^j - C_n^j B_n^j}{D_n^j B_{n+1}^j}, \\ r_n^j v_n^j &= \frac{\lambda_j - \beta_{n+1}^j D_n^j A_{n+1}^j - \alpha_n^j C_n^j B_n^j}{D_n^j B_{n+1}^j}. \end{aligned} \tag{2.4}$$

Analogously, shifting $j \to j+1$ in (2.2) and removing $P_n^{j+1}(z)$ with the help of (2.1) we come again to (2.3) but with different recurrence coefficients. The compatibility condition of these two recurrence relations yields the constraints

$$\beta_n^j = \beta_n, \quad \alpha_n^j = \alpha_{n+j},$$

and a set of three nonlinear finite-difference equations

$$\frac{A_n^j C_n^j}{B_{n+1}^j D_n^j} = \frac{A_n^{j+1} C_{n-1}^{j+1}}{B_n^{j+1} D_n^{j+1}}, \tag{2.5}$$

$$\frac{C_n^j B_n^j + A_{n+1}^j D_n^j - 1}{B_{n+1}^j D_n^j} = \frac{C_n^{j+1} B_n^{j+1} + A_n^{j+1} D_{n-1}^{j+1} - 1}{B_n^{j+1} D_n^{j+1}}, \tag{2.6}$$

$$\frac{\alpha_{n+j} C_n^j B_n^j + \beta_{n+1} A_{n+1}^j D_n^j - \lambda_j}{B_{n+1}^j D_n^j} = \frac{\alpha_{n+j+1} C_n^{j+1} B_n^{j+1} + \beta_n A_n^{j+1} D_{n-1}^{j+1} - \lambda_{j+1}}{B_n^{j+1} D_n^{j+1}}. \tag{2.7}$$

We say that this system of equations determines a (1+1)-dimensional discrete integrable chain, since it arises from the compatibility condition of two linear difference equations. It plays a crucial role in the following considerations. The variable j may be considered as a discrete time since the derived equations generalize the discrete time Toda chain appearing in a similar context for orthogonal polynomials [17].

If one takes in (2.3) the initial conditions

$$P_0^j(z) = p_j, \quad P_1^j(z) = r_0^j(z - v_0^j),$$

where p_j are some non-zero numbers, then $P_n^j(z)$, $n \geq 0$, represent n^{th} degree polynomials in z. In order to truncate the relation (2.2) at $n = 0$ we impose the constraints $A_0^j = A_0^j \beta_0 = 0$. Continued fractions associated with the three-term recurrence relation of the type (2.3) were named by Ismail and Masson as R_{II}-fractions [11]. Therefore we shall refer to $P_n^j(z)$ as the R_{II}-polynomials and to Eqs. (2.5)–(2.7) as the R_{II}-chain. Note that R_{II}-polynomials can be reduced to the so-called R_I and Laurent biorthogonal polynomials or to the standard orthogonal polynomials by removing in (2.3) the bilinear dependence on z in various ways. The transformations (2.1) and (2.2) are analogues of Christoffel's transformation of orthogonal polynomials to kernel polynomials [19] and of its inverse analyzed by Geronimus [6,7] respectively. Considered together they may also be called discrete Darboux transformations for the R_{II} recurrence relation (2.3).

As shown in [11] for a given set of R_{II}-polynomials such that $P_n^j(\alpha_{n+j}) \neq 0$, $P_n^j(\beta_n) \neq 0$ and $u_n^j \neq 0$ there always exists a linear functional $\mathcal{L}_j$ (the discrete variable j is considered as a dummy variable in (2.3)) such that

$$\mathcal{L}_j \left[\frac{z^m P_n^j(z)}{\prod_{k=1}^n (z - \alpha_{j+k})(z - \beta_k)} \right] = 0, \quad 0 \leq m < n. \tag{2.8}$$

On the basis of very simple linear algebra arguments this relation was rewritten in [24] as a biorthogonality condition of two rational functions built from $P_n^j(z)$:

$$\mathcal{L}_j \left[H_m^j(z)\, R_n^j(z) \right] = 0 \quad \text{for} \quad n \neq m, \tag{2.9}$$

where rational functions $R_n^j(z)$, $H_m^j(z)$ are defined below. Denote $R_0^j(z) = S_0^j(z) = 1$ and set

$$R_n^j(z) = \frac{P_n^j(z)}{\prod_{k=1}^n (z - \alpha_{j+k})}, \quad S_n^j(z) = \frac{P_n^j(z)}{\prod_{k=1}^n u_k^j (z - \beta_k)}, \tag{2.10}$$

for $n > 0$. These functions satisfy three-term recurrence relations with the linearized z-dependence:

$$(z - \alpha_{n+j+1}) R_{n+1}^j(z) + r_n^j(v_n^j - z) R_n^j(z) + u_n^j(z - \beta_n) R_{n-1}^j(z) = 0, \tag{2.11}$$

$$u_{n+1}^j(z - \beta_{n+1}) S_{n+1}^j(z) + r_n^j(v_n^j - z) S_n^j(z) + (z - \alpha_{n+j}) S_{n-1}^j(z) = 0. \tag{2.12}$$

Equations (2.11), (2.12) can be considered as generalized eigenvalue problems [21] of the form

$$L\psi(z) = zM\psi(z),$$

where the operators L, M are two general tri-diagonal Jacobi matrices.

The upper index j does not play an essential role in the derivation of the biorthogonality relations. Let us set temporarily $j = 0$ and suppress all the superscripts for a simplification. Then we may rewrite the relation (2.11) in the form $LR_n(z) = zMR_n(z)$, where

$$LR_n(z) \equiv \alpha_{n+1}R_{n+1}(z) - r_n v_n R_n(z) + \beta_n u_n R_{n-1}(z),$$
$$MR_n(z) \equiv R_{n+1}(z) - r_n R_n(z) + u_n R_{n-1}(z).$$

In these notations Eq. (2.12) takes the form $L^T S_n(x) = xM^T S_n(x)$, where L^T, M^T are the matrices L, M transposed with respect to the formal inner product

$$(S(x), R(z)) \equiv \sum_{k=0}^{\infty} S_n(x)R_n(z)$$

defined upon the space of rational functions. More precisely, one has

$$L^T S_n(z) = u_{n+1}\beta_{n+1}S_{n+1}(z) - r_n v_n S_n(z) + \alpha_n S_{n-1}(z),$$
$$M^T S_n(z) = u_{n+1}S_{n+1}(z) - r_n S_n(z) + S_{n-1}(z).$$

From the chain of relations

$$\begin{aligned} 0 &= (S(x), LR(z)) - z(S(x), MR(z)) \\ &= (L^T S(x), R(z)) - z(M^T S(x), R(z)) = (x - z)(M^T S(x), R(z)), \end{aligned} \tag{2.13}$$

one can conclude that the functions $H_n(x) \equiv M^T S_n(x)$ are orthogonal to $R_n(z)$ for different eigenvalues $x \neq z$. Restoring the superscript j one can find that the functions $H_n^j(z)$ are defined as follows:

$$H_n^j(z) \equiv u_{n+1}^j S_{n+1}^j(z) - r_n^j S_n^j(z) + S_{n-1}^j(z) \tag{2.14}$$

for $n = 1, 2, \ldots,$ and for $n = 0$ one has

$$H_0^j(z) = u_1^j S_1^j(z) - r_0^j = \frac{r_0^j(\beta_1 - v_0^j)}{z - \beta_1}.$$

Since we are dealing with matrices and their eigenvectors, the orthogonality for different eigenvalues (2.13) suggests that there is also a dual orthogonality relation for functions $H_m^j(z)$ and $R_n^j(z)$ with equal eigenvalues z. It is defined with the help of the functional $\mathcal{L}_j$ mapping rational functions of z onto the complex plane **C** (2.8). As a result, the biorthogonality of $H_m^j(z)$ to $R_n^j(z)$ for $m \neq n$ can be checked by direct substitution of the corresponding expressions into (2.9) and an application of the conditions (2.8).

Any non-trivial solution of the R_{II}-chain with appropriate boundary conditions at $n = 0$ provides a system of biorthogonal rational functions. Let us sketch briefly a procedure of building $P_n^j(z)$ out of the given coefficients $A_n^j, \ldots, \lambda_j$. Introduce first two auxiliary polynomials of the n^{th} degree:

$$Y_n^j = \prod_{k=1}^{n}(z - \lambda_{j+k}), \quad Z_n^j = \prod_{k=1}^{n}(z - \alpha_{j+k}), \quad n > 0, \tag{2.15}$$

and $Y_0^j = Z_0^j = 1$. Then the polynomials $P_n^j(z)$ admit the following representation:

$$P_n^j(z) = Z_n^j(z) \sum_{k=0}^{n} \zeta_n^j(k) \frac{Y_k^j(z)}{Z_k^j(z)} \tag{2.16}$$

with some unknown coefficients $\zeta_n^j(k), k \leq n$. Substituting this expression into (2.1) we arrive at the system of equations

$$\zeta_{n+1}^j(0)\, D_n^{j+1} + \zeta_n^j(0)\, C_n^{j+1} = 0, \quad \zeta_n^{j+1}(n) = D_n^{j+1}\, \zeta_{n+1}^j(n+1),$$
$$D_n^{j+1}\, \zeta_{n+1}^j(k) + C_n^{j+1}\, \zeta_n^j(k) = \zeta_n^{j+1}(k-1), \quad k = 1, 2, \ldots, n.$$

From the first two equations one finds $\zeta_n^j(0)$ and $\zeta_n^j(n)$:

$$\zeta_n^j(0) = (-1)^n \prod_{m=0}^{n-1} \frac{C_m^{j+1}}{D_m^{j+1}}, \quad \zeta_n^j(n) = \prod_{m=0}^{n-1} \frac{1}{D_m^{j+n-m}}. \tag{2.17}$$

Introducing the normalized coefficients $\eta_n^j(k) = \zeta_n^j(k)/\zeta_n^j(0)$, we rewrite the remaining part of the equations as follows:

$$\eta_{n+1}^j(k) = \eta_n^j(k) - \frac{\zeta_n^{j+1}(0)}{\zeta_n^j(0) C_n^{j+1}} \eta_n^{j+1}(k-1), \quad k = 1, 2, \ldots, n. \tag{2.18}$$

Since $\eta_n^j(0) = 1$ and $\eta_k^j(k)$ are known already, this recurrence relation allows one to find all the coefficients $\eta_n^j(k)$ uniquely in an iterative manner.

Closing this section let us show that the R_{II}-chain allows one to generate from a given three term recurrence relation (2.3) another recurrence relation of the same nature. Indeed, from the relation (2.1) one can find

$$P_{n+1}^j(z) = \frac{z - \lambda_{j+1}}{D_n^{j+1}} P_n^{j+1}(z) - \frac{C_n^{j+1}(z - \alpha_{n+j+1})}{D_n^{j+1}} P_n^j(z). \tag{2.19}$$

In a similar way, from (2.2) one may express $P_{n-1}^j(z)$ in terms of $P_n^{j-1}(z)$ and $P_n^j(z)$:

$$P_{n-1}^j(z) = \frac{P_n^{j-1}(z) - B_n^j P_n^j(z)}{A_n^j(z - \beta_n)}, \quad n > 0. \tag{2.20}$$

Substituting (2.19) and (2.20) into (2.3) we get the three-term recurrence relation in the discrete time variable j:

$$\frac{z - \lambda_{j+1}}{D_n^{j+1}} P_n^{j+1}(z) + \left(r_n^j(v_n^j - z) + \frac{C_n^{j+1}(\alpha_{n+j+1} - z)}{D_n^{j+1}} + \frac{u_n^j B_n^j(\alpha_{n+j} - z)}{A_n^j} \right) P_n^j(z)$$
$$+ \frac{u_n^j(z - \alpha_{n+j})}{A_n^j} P_n^{j-1}(z) = 0. \tag{2.21}$$

This is again a representative of the generalized eigenvalue problems. Replacing $P_n^j(z)$ by $S_n^j(z)$ in (2.21) and comparing the result with (2.12) one can see that $S_n^j(z)$ satisfy R_{II}-type recurrence relations in both discrete variables n and j. Note however, that in the context of the R_{II}-polynomials one has $n \geq 0$, while the values of j are not limited. Suppose that the dependence on j enters $P_n^j(z)$ via some continuous parameters. Then (2.21) defines some contiguous relation for the corresponding system of polynomials.

3. Companion Polynomials

Consider the functions $H_n^j(z)$ in more detail. It is convenient to represent them in the form

$$H_n^j(z) = \frac{Q_n^j(z)}{(z-\beta_{n+1})\prod_{k=1}^n u_k^j(z-\beta_k)}, \tag{3.1}$$

where $Q_n^j(z)$ are some polynomials of the n^{th} degree which will be called the companion polynomials. Their explicit form is found from the definition (2.14):

$$Q_n^j(z) = P_{n+1}^j(z) - r_n^j(z-\beta_{n+1})P_n^j(z) + u_n^j(z-\beta_n)(z-\beta_{n+1})P_{n-1}^j(z), \tag{3.2}$$

for $n > 0$ and $Q_0^j(z) = r_0^j(\beta_1 - v_0^j)$. Using the recurrence relation (2.3) we can represent $Q_n^j(z)$ in one of the two forms

$$Q_n^j(z) = r_n^j(\beta_{n+1} - v_n^j)P_n^j(z) + u_n^j(z-\beta_n)(\alpha_{n+j} - \beta_{n+1})P_{n-1}^j(z) \tag{3.3}$$

or

$$Q_n^j(z) = \frac{(\beta_{n+1}-\alpha_{n+j})P_{n+1}^j(z) + r_n^j(\alpha_{n+j} - v_n^j)(z-\beta_{n+1})P_n^j(z)}{z-\alpha_{n+j}}. \tag{3.4}$$

From (3.3) it is clear that $Q_n^j(z)$ are indeed polynomials of the n^{th} degree. With the help of the formulas (3.3) and (3.4) it is possible to express $P_n^j(z)$ through $Q_n^j(z)$:

$$P_n^j(z) = \gamma_n^j Q_n^j(z) + \delta_n^j(z-\alpha_{n+j-1})Q_{n-1}^j(z) \tag{3.5}$$

or

$$P_n^j(z) = \frac{\sigma_n^j Q_{n+1}^j(z) + \tau_n^j(z-\alpha_{n+j})Q_n^j}{z-\beta_{n+1}}, \tag{3.6}$$

where

$$\gamma_n^j = \frac{r_{n-1}^j(\alpha_{n+j-1} - v_{n-1}^j)}{\epsilon_n^j}, \quad \delta_n^j = \frac{u_n^j(\beta_{n+1}-\alpha_{n+j})}{\epsilon_n^j},$$
$$\sigma_n^j = \frac{\alpha_{n+j}-\beta_{n+1}}{\epsilon_{n+1}^j}, \quad \tau_n^j = \frac{r_{n+1}^j(\beta_{n+2} - v_{n+1}^j)}{\epsilon_{n+1}^j},$$
$$\epsilon_n^j = r_n^j r_{n-1}^j(\alpha_{n+j-1} - v_{n-1}^j)(\beta_{n+1} - v_n^j) - u_n^j(\alpha_{n+j} - \beta_{n+1})(\beta_n - \alpha_{n+j-1}), \tag{3.7}$$

and it is assumed that $\epsilon_n^j \neq 0$. Substituting (3.5) and (3.6) into (2.1) and (2.2) we find that the polynomials $Q_n^j(z)$ satisfy the relations

$$Q_n^{j+1} = \frac{\tilde{D}_n^{j+1} Q_{n+1}^j + \tilde{C}_n^{j+1}(z - \tilde{\alpha}_{n+j+1}) Q_n^j}{z - \tilde{\lambda}_{j+1}},$$
$$Q_n^{j-1} = \tilde{B}_n^j Q_n^j + \tilde{A}_n^j (z - \tilde{\beta}_n) Q_{n-1}^j$$

with the following entries:

$$\tilde{\alpha}_{n+j} = \alpha_{n+j-1}, \qquad \tilde{\beta}_n = \beta_{n+1}, \qquad \tilde{\lambda}_j = \lambda_j, \tag{3.8}$$

$$\tilde{A}_n^j = \frac{D_{n-1}^j}{D_n^j} A_n^j, \qquad \tilde{B}_n^j = B_{n+1}^j. \tag{3.9}$$

The rest two superpotentials $\tilde{D}_n^j$ and $\tilde{C}_n^j$ have much more complicated form:

$$\tilde{D}_n^j = \frac{\epsilon_n^j}{\epsilon_{n+1}^{j-1}} D_{n-1}^j, \quad \tilde{C}_n^j = \frac{\epsilon_n^j}{\alpha_{n+j-1} - \beta_{n+1}} \left(D_{n-1}^j \tau_n^{j-1} + \frac{\lambda_j - \beta_{n+1}}{\epsilon_n^j B_{n+1}^j} \right). \tag{3.10}$$

Evidently, the compatibility condition of the $j \to j \pm 1$ transformations for $Q_n^j(z)$ polynomials generates the R_{II}-recurrence relation and the R_{II}-chain with new entries determined by $\tilde{A}_n^j, \ldots, \tilde{\lambda}_j$. We can formulate thus the following statement.

Theorem 1. *The transformations* (3.8)–(3.10) *define a particular symmetry of the R_{II}-chain* (2.5)–(2.7) *generated by the transition from a given set of R_{II}-polynomials $P_n^j(z)$ to the set of their companion polynomials $Q_n^j(z)$.*

4. Symmetries of the R_{II}-Chain and a Similarity Reduction

Let us describe some other symmetries of the R_{II}-chain. Let us start from a brief consideration of the normalization (gauge) freedom. Although this analysis is simple enough it is instructive to give it here.

We can transform recurrence coefficients in (2.3) by the multiplication of polynomials by an arbitrary gauge factor ξ_n^j independent on z, $P_n^j(z) = \xi_n^j \tilde{P}_n^j(z)$. This leads to recursions (2.1), (2.2) with the renormalized entries

$$\tilde{A}_n^j = \frac{A_n^j}{t_{n-1}^j}, \quad \tilde{B}_n^j = B_n^j w_n^j, \quad \tilde{C}_n^j = \frac{C_n^j}{w_n^j}, \quad \tilde{D}_n^j = D_n^j t_n^j, \tag{4.1}$$

where $t_n^j = \xi_{n+1}^{j-1}/\xi_n^j$, $w_n^j = \xi_n^j/\xi_n^{j-1}$. The coefficients t_n^j, w_n^j satisfy the relation $t_n^j w_{n+1}^j = t_n^{j+1} w_n^{j+1}$. The transformed recurrence coefficients have the form

$$\tilde{r}_n^j = \frac{\xi_n^j}{\xi_{n+1}^j} r_n^j, \quad \tilde{u}_n^j = \frac{\xi_{n-1}^j}{\xi_{n+1}^j} u_n^j, \tag{4.2}$$

with other entries in (2.3) being unchanged. There is thus a large freedom in the form of presentation of the recurrence coefficients of polynomials $P_n^j(z)$.

In [11] the gauge $r_n^j = 1$ was chosen. It is possible also to choose the gauge $r_n^j - u_n^j = 1, r_0^j = 1$, leading to monic polynomials $P_n^j(z) = z^n + O(z^{n-1})$, which may be convenient for some reasons. From (2.1) and (2.2) it is seen that the monicity condition implies the following constraints upon the superpotentials: $A_n^j + B_n^j = C_n^j + D_n^j = 1$. In this normalization one has actually only two independent equations (2.5) and (2.7), because Eq. (2.6) is fulfilled automatically.

There is an essential technical drawback with the monic gauge – it is not convenient for construction of explicit solutions of the R_{II}-chain $A_n^j, \ldots, D_n^j$. For the latter purpose it is necessary to reduce the number of superpotentials and another gauge will be chosen below: $B_n^j = 1$. As seen from (4.1) this choice leaves a freedom in the transformation of superpotentials:

$$A_n^j \to A_n^j / t_{n-1}, \quad D_n^j \to D_n^j t_n, \tag{4.3}$$

where the factor t_n does not depend on j. This freedom will be used in the following.

Describe now some more involved properties of the R_{II}-chain. Let $A_n^j, B_n^j, C_n^j, D_n^j$ satisfy Eqs. (2.5), (2.6). These functions would provide a solution of the whole R_{II}-chain if $\lambda_j = \alpha_{n+j} = \beta_n = \text{const.}$, because then Eq. (2.7) coincides with (2.6). Shifting the argument of polynomials and $z \to z - \text{const.}$ one can convert the latter constraints to $\lambda_j = \alpha_{n+j} = \beta_n = 0$. The solutions generated under these constraints are too trivial since the polynomials have the form $P_n^j(z) = \gamma_n^j z^n$, where γ_n^j do not depend on z. Indeed, using the initial condition $P_0^j(z) = 1$ and setting $n = 0$ in (2.1) one finds that $P_1^j(z) = \gamma_1^j z$ and the statement follows by induction. Below we shall assume that this trivial situation does not take place.

It is not difficult to see that the affine transformation of the argument z, $z \to \xi z + \eta$ can be compensated by the appropriate affine transformation of the parameters $\alpha_{n+j}, \beta_n, \lambda_j$ and recurrence coefficients, similar to the orthogonal polynomials case. However, the biorthogonal rational functions are associated with the generalized eigenvalue problem $L\psi(z) = zM\psi(z)$ which admits also the inversion symmetry $z \to 1/z$, since it amounts to the permutation of the operators L and M. As a result, rational transformations of the argument of R_{II}-polynomials accompanied by an appropriate gauge transformation

$$\tilde{P}_n^j(z) = (\zeta z + \sigma)^n P_n^j\left(\frac{\xi z + \eta}{\zeta z + \sigma}\right), \tag{4.4}$$

where ξ, η, ζ, σ are arbitrary parameters independent on j, leaves invariant the space of these polynomials.

Theorem 2. *The polynomials* (4.4) *satisfy* (2.3) *with the following recurrence coefficients:*

$$\begin{gathered} \tilde{r}_n^j = r_n^j(\xi - \zeta v_n^j), \quad \tilde{v}_n^j = \frac{\sigma v_n^j - \eta}{\xi - \zeta v_n^j}, \quad \tilde{u}_n^j = u_n^j(\xi - \zeta\alpha_{n+j})(\xi - \zeta\beta_n), \\ \tilde{\alpha}_{n+j} = \frac{\sigma\alpha_{n+j} - \eta}{\xi - \zeta\alpha_{n+j}}, \quad \tilde{\beta}_n = \frac{\sigma\beta_n - \eta}{\xi - \zeta\beta_n}. \end{gathered} \tag{4.5}$$

Consequently, the R_{II}*-chain is invariant with respect to the transformations* (4.5) *and*

$$\tilde{\lambda}_j = \frac{\sigma\lambda_j - \eta}{\xi - \zeta\lambda_j}, \quad \tilde{A}_n^j = A_n^j(\xi - \zeta\beta_n), \quad \tilde{B}_n^j = B_n^j,$$
$$\tilde{C}_n^j = \frac{C_n^j(\xi - \zeta\alpha_{n+j})}{\xi - \zeta\lambda_j}, \quad \tilde{D}_n^j = \frac{D_n^j}{\xi - \zeta\lambda_j}. \tag{4.6}$$

The proof is skipped being simple enough.

A different type of symmetries is induced by discrete transformations of the underlying two-dimensional grid formed by the variables n, j. Namely, the reflections 1) $j \to -j, n \to -n$; 2) $n \to j, j \to n$; 3) $j \to -j-n$; 4) $n \to -n-j$ induce peculiar involutions of the R_{II}-chain.

Theorem 3. *The following four involutions describe particular discrete symmetries of the* R_{II}*-chain:*

$$\begin{aligned}
&1.\ \tilde{A}_n^j = D_{-n}^{-j},\ \tilde{D}_n^j = A_{-n}^{-j},\ \tilde{B}_n^j = C_{-n}^{-j},\ \tilde{C}_n^j = B_{-n}^{-j},\\
&\quad \tilde{\beta}_n = \beta_{1-n},\ \tilde{\alpha}_{n+j} = \alpha_{-n-j},\ \tilde{\lambda}_j = \lambda_{-j};\\
&2.\ \tilde{A}_n^j = \frac{1}{A_j^n},\ \tilde{B}_n^j = \frac{B_j^n}{A_j^n},\ \tilde{C}_n^j = \frac{C_{j-1}^{n+1}}{D_{j-1}^{n+1}},\ \tilde{D}_n^j = \frac{1}{D_{j-1}^{n+1}},\\
&\quad \tilde{\alpha}_{n+j} = \alpha_{n+j},\ \tilde{\beta}_n = \lambda_n,\ \tilde{\lambda}_j = \beta_j;\\
&3.\ \tilde{A}_n^j = \frac{A_n^{1-j-n}}{B_n^{1-j-n}},\ \tilde{B}_n^j = \frac{1}{B_n^{1-j-n}},\ \tilde{C}_n^j = \frac{1}{C_n^{1-j-n}},\ \tilde{D}_n^j = \frac{D_n^{1-j-n}}{C_n^{1-j-n}},\\
&\quad \tilde{\lambda}_j = \alpha_{1-j},\ \tilde{\alpha}_{j+n} = \lambda_{1-j-n},\ \tilde{\beta}_n = \beta_n;\\
&4.\ \tilde{A}_n^j = B_{1-n-j}^j,\ \tilde{B}_n^j = A_{1-n-j}^j,\ \tilde{C}_n^j = D_{-n-j}^j,\ \tilde{D}_n^j = C_{-n-j}^j,\\
&\quad \tilde{\alpha}_{n+j} = \beta_{1-n-j},\ \tilde{\beta}_n = \alpha_{1-n},\ \tilde{\lambda}_j = \lambda_j.
\end{aligned} \tag{4.7}$$

The proof consists in the verification that after substitution of the tilded variables into (2.5)–(2.7) one gets the R_{II}-chain with the reflected grid points as indicated above. In a sense, this theorem shows an equivalence of the spectral coefficients $\lambda_j, \alpha_{n+j}, \beta_n$ despite their non-symmetric entrance into the original formulas (2.1), (2.2).

These four transformations do not cover all possible types of involutions of the R_{II}-chain. E.g., there should exist involutions generated by a freedom in the intermediate steps of double spectral transformation generalizing the corresponding symmetry for the standard orthogonal polynomials [16].

Suppose that the superpotentials A_n^j, B_n^j, C_n^j, D_n^j and spectral coefficients $\alpha_n, \beta_n, \lambda_n$ are described by some meromorphic functions of the continuous variables n and j. Such solutions of integrable chains appear usually from similarity reductions of the corresponding equations. In general the involutions (4.7) change essentially the form of a given solution. However, there is a special class of solutions for which only a change of parameters occurs.

First, note that there are specific combinations of the discrete variables n and j, namely,

$$u_1 = n, \quad u_2 = j, \quad u_3 = n + j, \quad u_4 = n - j, \ u_5 = 2n + j, \quad u_6 = 2j + n,$$

which are expressed through each other under the taken four grid reflections up to a change of the signs. Therefore symmetric products of some functions of these variables will not change their form under the grid reflections. This observation allows one to impose the constraint that the superpotentials $A_n^j, B_n^j, C_n^j, D_n^j$ split into products of functions each depending only on one of these six variables:

$$A_n^j = \prod_{k=1}^{6} A^{(k)}(u_k), \quad B_n^j = \prod_{k=1}^{6} B^{(k)}(u_k), \quad C_n^j = \prod_{k=1}^{6} C^{(k)}(u_k), \quad D_n^j = \prod_{k=1}^{6} D^{(k)}(u_k).$$

It is not guaranteed à priori that these restrictions are compatible with Eqs. (2.5)–(2.7). Before substituting them into the R_{II}-chain it is convenient to simplify superpotentials as much as possible using the gauge freedom. So, we impose the condition $B_n^j = 1$, which allows us to normalize the polynomials $P_0^j = 1$. Assume also that D_n^j does not depend on the variable $u_1 = n$, i.e. $D^{(1)}(u) = 1$, which can be always achieved by the transformation (4.3).

Then the first equation (2.5) can be resolved completely. It leads to the following relations between the functions $A^{(k)}, C^{(k)}, D^{(k)}$:

$$C^{(1)}(u) = 1, \quad C^{(6)}(u) = \frac{D^{(6)}(u)D^{(6)}(u+1)}{A^{(6)}(u)A^{(6)}(u+1)}, \quad D^{(2)}(u) = A^{(2)}(u)C^{(2)}(u),$$
$$D^{(3)}(u) = A^{(3)}(u), \quad D^{(4)}(u) = A^{(4)}(u)C^{(4)}(u)C^{(4)}(u-1),$$
$$D^{(5)}(u) = \frac{A^{(5)}(u)}{C^{(5)}(u-1)}.$$

Still, there remains eleven unknown functions giving too large a freedom. After a thorough analysis of different possibilities we have limited ourselves in this paper to the following restricted Ansatz of generalized separation of variables (some hints upon such a choice came from our analysis of the similar situation for orthogonal polynomials [17, 18]):

$$A_n^j = \frac{d(n)\rho(2j+n)}{g(2n+j)g(2n+j-1)\phi(n-j)\phi(n-j-1)}, \quad B_n^j = 1,$$
$$C_n^j = \frac{c(n+j)\phi(n-j)\phi(n-j+1)}{\sigma(j)g(2n+j)g(2n+j+1)}, \quad D_n^j = \frac{\rho(2j+n)\phi(n-j)\phi(n-j+1)}{\sigma(j)}, \tag{4.8}$$

where $d(0) = 0$. Equation (2.5) is satisfied automatically for arbitrary functions $d(x), \dots, \sigma(x)$. Note that the first, second and fourth involutions break the condition $B_n^j = 1$ and one should perform a gauge transformation (4.1) in order to restore it. Then it can be seen that the involutions being applied to (4.8) just permute the functions $d(x), c(x), \sigma(x)$ up to a simple transformation of their arguments. A similar situation takes place for $g(x), \rho(x), \phi(x)$. Therefore one may expect that the corresponding functions shall have identical forms.

It remains now to solve Eqs. (2.6), (2.7). In some particular cases a trick helps to reduce (2.7) to (2.6).

Proposition 1. *Suppose that the superpotentials* (4.8) *determine a solution of the equations* (2.5), (2.6) *such that the functions* $d(x), \sigma(x), c(x)$ *contain a number of free parameters which do not enter the functions* $g(x), \rho(x), \phi(x)$. *Then Eq.* (2.7) *is satisfied for the following choice of the spectral data coefficients:*

$$\lambda_j = \tilde{\sigma}(j)/\sigma(j), \quad \beta_n = \tilde{d}(n)/d(n), \quad \alpha_{n+j} = \tilde{c}(n+j)/c(n+j), \tag{4.9}$$

where the tilded functions differ from the untilded ones only by the choice of free parameters.

Substituting (4.8) into (2.6) we rewrite this equation in the form

$$\begin{aligned} &\frac{c(n+j)}{g(2n+j+1)g(2n+j)\rho(2j+n)\rho(2j+n+1)} \\ &+\frac{d(n+1)}{g(2n+2+j)g(2n+j+1)\phi(n+1-j)\phi(n-j)} \\ &-\frac{\sigma(j)}{\rho(2j+n)\,\phi(n+1-j)\,\phi(n-j)\,\rho(2j+n+1)} \\ =\ &\frac{c(n+j+1)}{g(2n+2+j)g(2n+j+1)\rho(2j+2+n)\rho(2j+n+1)} \\ &+\frac{d(n)}{g(2n+j+1)g(2n+j)\phi(n-j-1)\phi(n-j)} \\ &-\frac{\sigma(j+1)}{\rho(2j+2+n)\phi(n-j)\phi(n-j-1)\rho(2j+n+1)}. \end{aligned} \tag{4.10}$$

We were not able to find all solutions of this equation. However, a rich class of them has been derived from a set of natural additional constraints. Namely, suppose that the functions $g(x), \rho(x), \phi(x)$ have simple zeros at $x = x_2, x_1, x_0$ respectively, where x_2, x_1, x_0 are some constants. Let us demand that $g(x) \neq 0$ for $x = x_2 - 1, x_2 - 2$, $\rho(x) \neq 0$ for $x = x_1 - 1, x_1 - 2$ and $\phi(x) \neq 0$ for $x = x_0 \pm 1$. Now the condition of cancellation of poles in (4.10) leads to the equations

$$\begin{aligned} \frac{c(x_1-x)}{\sigma(x)} &= \frac{g(2x_1-3x)\,g(2x_1-3x+1)}{\phi(x_1-3x)\,\phi(x_1-3x+1)}, \\ \frac{c(x_2-x)}{d(x)} &= \frac{\rho(2x_2-3x)\,\rho(2x_2-3x+1)}{\phi(3x-x_2-1)\,\phi(3x-x_2)}, \\ \frac{\sigma(x-x_0)}{d(x)} &= \frac{\rho(3x-2x_0-1)\,\rho(3x-2x_0)}{g(3x-x_0-1)\,g(3x-x_0)}. \end{aligned}$$

These conditions are resolved if we set

$$\phi(x) = \psi(x-x_0), \quad g(x) = \psi(x-x_2), \quad \rho(x) = \psi(x-x_1) \tag{4.11}$$

and

$$\sigma(x) = d(x+x_0), \quad c(x) = d(x_2-x), \tag{4.12}$$

where $\psi(x)$ is an arbitrary odd function $\psi(x) = -\psi(-x)$ (there are minor restrictions upon the position of zeros of $\psi(x)$ mentioned above) and the parameters x_0, x_1, x_2 satisfy the constraint

$$x_2 = x_0 + x_1. \tag{4.13}$$

In the following we stick to this particular choice of the functions entering (4.8).

Evidently, there are now only two unknown functions $d(x)$, $\psi(x)$ and Eq. (4.10) takes the form:

$$\begin{aligned}
&\frac{d(x_2 - n - j)}{\psi(2n + j - x_2)\psi(2n + j + 1 - x_2)\psi(2j + n - x_1)\psi(2j + n + 1 - x_1)} \\
&+ \frac{d(n+1)}{\psi(2n + j + 1 - x_2)\psi(2n + j + 2 - x_2)\psi(n - j - x_0)\psi(n - j + 1 - x_0)} \\
&- \frac{d(j + x_0)}{\psi(2j + n - x_1)\psi(2j + n + 1 - x_1)\psi(n - j - x_0)\psi(n - j + 1 - x_0)} \\
= &\frac{d(x_2 - n - j - 1)}{\psi(2n + j + 1 - x_2)\psi(2n + j + 2 - x_2)\psi(2j + n + 1 - x_1)\psi(2j + n + 2 - x_1)} \\
&+ \frac{d(n)}{\psi(2n + j - x_2)\psi(2n + j + 1 - x_2)\psi(n - j - 1 - x_0)\psi(n - j - x_0)} \\
&- \frac{d(j + 1 + x_0)}{\psi(2j + n + 1 - x_1)\psi(2j + n + 2 - x_1)\psi(n - j - 1 - x_1)\psi(n - j - x_1)}.
\end{aligned} \tag{4.14}$$

We shall call (4.14) the basic equation. Assume that the functions $\psi(x)$ and $d(x)$ are entire, i.e. they do not have singularities for finite values of the argument x. Then it is clear from our considerations that there are no poles at finite values of n and j in (4.14) for arbitrary $\psi(x)$, $d(x)$, provided $\psi(x)$ has only simple zeroes.

5. Rational and Elementary Functions Solutions

Let us start from the analysis of a class of rational and elementary functions solutions of the basic equation (4.14).

If one limits consideration to rational functions, then it is possible to proceed further by giving to $\psi(x)$ the simplest possible forms and analyzing the resulting equation for $d(x)$. So, we have fixed $\psi(x) = x$ and looked for a polynomial solution for $d(x)$. Using the MAPLE software it was found that $d(x)$ can be a polynomial of the 6$^{\text{th}}$ degree

$$d(x) = x \prod_{k=1}^{5} (x - d_k) \tag{5.1}$$

with the curious restriction upon its roots:

$$\sum_{k=1}^{5} d_k = 1 + 2(x_0 + x_2). \tag{5.2}$$

There is a trivial freedom in the multiplication of $d(x)$ by an arbitrary factor, which we did not indicate, and one of the roots of $d(x)$ was chosen to be equal to zero in order to have $d(0) = 0$. As a result, there remains only four free parameters in $d(x)$. Taking in the formulation of Proposition 1 as $\tilde{d}(x)$ a polynomial of the same structure as $d(x)$:

$$\tilde{d}(x) = x \prod_{k=1}^{k} (x - e_k), \quad \sum_{k=1}^{5} e_k = 1 + 2(x_0 + x_2),$$

containing other 4 free parameters, we find the spectral coefficients

$$\lambda_j = \prod_{k=1}^{5} \frac{j + x_0 - e_k}{j + x_0 - d_k}, \quad \beta_n = \prod_{k=1}^{5} \frac{n - e_k}{n - d_k}, \quad \alpha_n = \prod_{k=1}^{5} \frac{n - x_2 + e_k}{n - x_2 + d_k}. \tag{5.3}$$

It is convenient to denote $s \equiv j + 2 - x_2$, $a \equiv 2j + 1 + x_0 - x_2$. The following result was announced in [25].

Theorem 4. *The recurrence relation (2.3) for the derived rational solution of the R_{II}-chain* (5.1)-(5.3) *leads to R_{II}-polynomials $P_n^j(z)$ which are expressed through a very-well-poised 2-balanced generalized hypergeometric series ${}_9F_8$:*

$$\begin{aligned} P_n^j(z) &= f_n^j(z) \\ &\quad {}_9F_8\left(\begin{matrix} a, a/2+1, -n, s+n-1, a+2-s-y_1, \ldots, a+2-s-y_5 \\ a/2, a+n+1, a+2-s-n, s-1+y_1, \ldots, s-1+y_5 \end{matrix}; 1\right), \\ f_n^j(z) &= \frac{(1-z)^n \prod_{k=1}^{5}(s-1+y_k)_n}{(n+s-1)_n (a+1)_n}, \end{aligned} \tag{5.4}$$

where $y_1(z), \ldots, y_5(z)$ are the roots of the following algebraic equation of the fifth degree:

$$z \prod_{k=1}^{5}(y - d_k) = \prod_{k=1}^{5}(y - e_k).$$

Let us recall that the generalized hypergeometric function

$${}_{r+1}F_r\left(\begin{matrix} a_1, \ldots, a_{r+1} \\ b_1, \ldots, b_r \end{matrix}; z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_{r+1})_n}{n!(b_1)_n \ldots (b_r)_n} z^n$$

is called well-poised if $1 + a_1 = a_{k+1} + b_k,\ k = 1, \ldots, r$. It becomes very-well-poised if, additionally, $a_2 = a_1/2 + 1$. And it is called k-balanced if $k + a_1 + \cdots + a_{r+1} = b_1 + \cdots + b_r$ and $z = 1$. Such types of series have some special properties, see e.g. [5].

The Wilson family of rational functions [22] corresponds to the case when $\beta_n, \alpha_n, \lambda_n$ are reduced to the polynomials of the second degree. This can be achieved if one takes $d(x)$ as a polynomial of the fourth degree and demands that it divides $\tilde{d}(x)$. The key new properties of the polynomials (5.4) consist in the facts that they contain eight independent free parameters (in [22] there were only five of them) and that it is necessary to solve an algebraic equation of the degree higher than two for presentation of the polynomials in the form of hypergeometric series. Actually, there are ten free parameters in (5.4) in addition to the degree of polynomials n and their argument z. However, two of them may be absorbed into the definition of the argument z with the help of the linear fractional transformation (4.4) which preserves the fixed leading $j, n \to \infty$ asymptotics $\lambda_j, \alpha_n, \beta_n \to 1$.

We are not giving the proof of the above theorem but consider instead in detail its q-generalization.

It is natural to replace the $\psi(x) = x$ choice in (4.14) by the following odd function defining well-known q-numbers

$$\psi(x) = \frac{q^{x/2} - q^{-x/2}}{q^{1/2} - q^{-1/2}},$$

where q is an arbitrary (complex) deformation parameter. With the help of the MAPLE software we have found that the q-analogue of the polynomial $d(x)$ has the form (up to a common multiplicative factor)

$$d(x) = \psi(x) \prod_{k=1}^{5} \psi(x - d_k), \tag{5.5}$$

where the same restriction (5.2) needs to be imposed upon the roots of $d(x)$. Note that taking various limits of parameters d_k one can reduce the number of entries in the product (5.5) from 6 down to 4, 3, 2, 1.

Taking $\tilde{d}(x) = \psi(x) \prod_{k=1}^{5} \psi(x - e_k)$, where e_k satisfy the same constraints as in the rational case, and substituting it into (4.9), we find

$$\lambda_j = \prod_{k=1}^{5} \frac{1 - q^{j+x_0-e_k}}{1 - q^{j+x_0-d_k}}, \quad \beta_n = \prod_{k=1}^{5} \frac{1 - q^{n-e_k}}{1 - q^{n-d_k}}, \quad \alpha_n = \prod_{k=1}^{5} \frac{1 - q^{x_2-n-e_k}}{1 - q^{x_2-n-d_k}}. \tag{5.6}$$

For completeness we give also the explicit form of superpotentials

$$A_n^j = -\frac{(q^{1/2} - q^{-1/2})^{-3}(1 - q^n)(1 - aq^{n-1}) \prod_{k=1}^{5}(1 - q^{n-d_k})}{a^{1/2} q^{n/2}(1 - sq^{2n-2})(1 - sq^{2n-3})(1 - sq^{n-1}/a)(1 - sq^{n-2}/a)}, \tag{5.7}$$

$$C_n^j = \frac{a^2(1 - sq^{n-2})(1 - sq^n/a)(1 - sq^{n-1}/a) \prod_{k=1}^{5}(1 - sq^{n+d_k-2})}{s^2 q^{2n-1}(1 - sq^{2n-2})(1 - sq^{2n-1})(1 - aq/s) \prod_{k=1}^{5}(1 - aq^{1-d_k}/s)}, \tag{5.8}$$

$$D_n^j = -(q^{1/2} - q^{-1/2})^3 \frac{a^{5/2}(1 - aq^{n-1})(1 - sq^n/a)(1 - sq^{n-1}/a)}{s^2 q^{(3n-1)/2}(1 - aq/s) \prod_{k=1}^{5}(1 - aq^{1-d_k}/s)}, \tag{5.9}$$

where we have introduced the convenient notations

$$a \equiv q^{2j+1-x_1}, \quad s \equiv q^{j+2-x_2}.$$

Let us recall the definition of q-hypergeometric series ${}_{r+1}\varphi_r$ [5]:

$${}_{r+1}\varphi_r \left(\begin{matrix} a_1, \ldots, a_{r+1} \\ b_1, \ldots, b_r \end{matrix} ; q, z \right) = \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_{r+1}; q)_k}{(q, b_1, \ldots, b_r; q)_k} z^k,$$

where the q-shifted factorial is defined as

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{k=1}^{n} (1 - aq^{k-1}), \quad (a_1, \ldots, a_r; q)_n = (a_1; q)_n \ldots (a_r; q)_n.$$

This series is called well-poised if $qa_1 = a_2 b_1 = \cdots = a_{r+1} b_r$ and very-well-poised if, additionally, $a_2 = qa_1^{1/2}$, $a_3 = -qa_1^{1/2}$. Analogously to the ${}_{r+1}F_r$ case, ${}_{r+1}\varphi_r$ is called balanced if $qa_1 \ldots a_{r+1} = b_1 \ldots b_r$ and $z = q$.

Theorem 5. *The three-term recurrence relation for R_{II}-polynomials* (2.3) *with the recurrence coefficients determined from* (5.6)–(5.9) *has the following explicit solution:*

$$P_n^j(z) = f_n^j(z)\,{}_{10}\varphi_9\left(\begin{matrix} a, qa^{1/2}, -qa^{1/2}, q^{-n}, sq^{n-1}, aq^2/sy_1, \ldots, aq^2/sy_5 \\ a^{1/2}, -a^{1/2}, aq^{n+1}, aq^{2-n}/s, sy_1/q, \ldots, sy_5/q \end{matrix}; q, q\right), \tag{5.10}$$

where

$$f_n^j(z) = \frac{(q^{1/2}-q^{-1/2})^{-3n}(z-1)^n \prod_{k=1}^5 (sy_k/q;q)_n}{a^{n/2}q^{n(n+1)/4}(sq^{n-1}, aq; q)_n},$$

and $y_1(z), \ldots, y_5(z)$ are solutions of the algebraic equation of the fifth degree

$$(z-1)\prod_{k=1}^5 (y_k(z)-y) = z\prod_{k=1}^5 (q^{d_k}-y) - \prod_{k=1}^5 (q^{e_k}-y). \tag{5.11}$$

Proof. In order to find the explicit form of $P_n^j(z)$ we use the representation (2.16). First we find the $\zeta_n^j(0)$ coefficients in new notations:

$$\zeta_n^j(0) = \frac{(q^{1/2}-q^{-1/2})^{-3n}\prod_{k=1}^5 (sq^{d_k-1};q)_n}{a^{n/2}q^{n(n+1)/4}(sq^{n-1}, aq; q)_n}.$$

Then, it is necessary to calculate q-shifted factorial forms of Y_n^j and Z_n^j for which the algebraic equation (5.11) is needed:

$$Y_n^j(z) = (z-1)^n \prod_{k=1}^5 \frac{(aq^2/sy_k;q)_n}{(aq^{2-d_k}/s;q)_n}, \quad Z_n^j(z) = (z-1)^n \prod_{k=1}^5 \frac{(sy_k/q;q)_n}{(sq^{d_k-1};q)_n}.$$

Finally, solving the recurrence relation (2.18), which is the most difficult part of the derivation, we find

$$\eta_n^j(k) = \frac{(a, qa^{1/2}, -qa^{1/2}, q^{-n}, sq^{n-1}, aq^{2-d_1}/s, \ldots, aq^{2-d_5}/s; q)_k q^k}{(q, a^{1/2}, -a^{1/2}, aq^{n+1}, aq^{2-n}/s, sq^{d_1-1}, \ldots, sq^{d_5-1}; q)_k}.$$

Now it is a matter of simple substitution into the initial formula (2.16) that leads to the representation of $P_n^j(z)$ in terms of the very-well-poised balanced ${}_{10}\varphi_9$ basic hypergeometric series given above. The theorem is proved. □

A particular subclass of the derived set of R_{II}-polynomials corresponds to the Rahman-Wilson biorthogonal rational functions considered earlier in [9,13–15,23]. It appears when $d(x)$ degenerates into a polynomial of the fourth degree with the roots $d_3 = e_3, d_4 = e_4, d_5 = e_5$. E.g., take $d_1 \to \infty, d_2 \to -\infty$ in such a way that $d_1 + d_2 = e_1 + e_2$ is a finite constant. The divergences in $d(x)$ appear only as a prefactor which can be removed by a scaling transformation. Then, $d(x)$ divides $\tilde{d}(x)$ and one may write

$$\lambda_j = q^{n+x_0-t} + q^{-n-x_0+t} - v,$$
$$\beta_n = q^{n-t} + q^{-n+t} - v, \quad \alpha_n = q^{n-x_2+t} + q^{-n+x_2-t} - v,$$

where

$$t = \frac{e_1 + e_2}{2}, \quad v = q^{(e_2-e_1)/2} + q^{(e_1-e_2)/2}.$$

In this situation it can be shown that the companion polynomials $Q_n^j(z)$ differ from $P_n^j(z)$ only by the replacement of parameters x_0, e_1, e_2 by $x_0 - 1, e_1 - 1, e_2 - 1$ respectively. As a result, one has the biorthogonality relation between two ${}_{10}\varphi_9$-functions differing from each other only by a choice of parameters. From the relation (2.9) it follows that our general ${}_{10}\varphi_9$-series (5.10) is biorthogonal to a linear combination of three similar ${}_{10}\varphi_9$-functions. It is not clear at the moment whether this combination can be reduced to one basic hypergeometric series.

In the general case, the superpotentials for companion polynomials $Q_n^j(z)$ depend on the parameters e_k (which was not so for $P_n^j(z)$) and have much more complicated form than (5.7)–(5.9). Note that we can build companion polynomials for $Q_n^j(z)$ in the same way as we did for $P_n^j(z)$ and they will not coincide with $P_n^j(z)$ or $Q_n^j(z)$. This follows from the fact that in the general case the change of spectral variables $\beta_n \to \beta_{n+1}, \alpha_n \to \alpha_{n-1}$ caused by the transition to companion polynomials (3.8) cannot be compensated by a redefinition of parameters of the system. Evidently these transitions to companion polynomials may be iterated to infinity. At each step we would deal with a new elementary function solution of the R_{II}-chain and a specific biorthogonality condition between linear combinations of the ${}_{10}\varphi_9$-series.

6. Elliptic Solutions of the Basic Equation

We were able to find a further generalization of the solutions of the basic relation (4.14) described in the previous section. This extension uses the elliptic theta functions.

Recall that the Jacobi theta function $\theta_1(u)$ is defined as [5]

$$\begin{aligned}\theta_1(u) &= 2\sum_{n=0}^{\infty}(-1)^n p^{(n+1/2)^2} \sin(2n+1)u \\ &= 2p^{1/4}\sin u \prod_{n=1}^{\infty}\left(1 - 2p^{2n}\cos 2u + p^{4n}\right)(1-p^{2n}),\end{aligned} \tag{6.1}$$

where p is a complex parameter, $|p| < 1$. The modular parameter τ is introduced in the standard way $p = \exp(\pi i \tau)$. This function possesses many useful properties. The most important from them are

(i) $\theta_1(u)$ is an odd function, $\theta_1(-u) = -\theta(u)$;
(ii) $\theta_1(u)$ is quasiperiodic with respect to the shifts by π and $\pi\tau$

$$\theta_1(u+\pi) = -\theta(u), \quad \theta(u+\pi\tau) = -p^{-1}\exp(-2iu)\,\theta_1(u); \tag{6.2}$$

(iii) an algebraic relation (the Riemann identity)

$$\begin{aligned}&\theta_1(x+z)\theta_1(x-z)\theta_1(y+w)\theta_1(y-w) - \theta_1(x+w)\theta_1(x-w)\theta_1(y+z)\theta_1(y-z) \\ &= \theta_1(x+y)\theta_1(x-y)\theta_1(z+w)\theta_1(z-w)\end{aligned} \tag{6.3}$$

holds for any variables x, y, z, w (see, e.g. [2], where a rescaled form of the θ_1 function $H(u) = \theta_1(\pi u/2I)$ is used).

Following [4], let us introduce the “elliptic numbers” (or, simply, e-numbers) through the definition

$$[x; h, \tau] = \frac{\theta_1(\pi h x)}{\theta_1(\pi h)}, \tag{6.4}$$

where h is an arbitrary constant. Clearly, e-numbers depend on three variables x, h and τ. In what follows the dependence on h, τ will be omitted in the notations, i.e. we shall write $[x] \equiv [x; h, \tau]$.

The e-numbers possess the following properties:

(i) $[-x] = -[x]$;

(ii) $[x + 1/h] = -[x], \quad [x + \tau/h] = -\exp(-i\pi\tau - 2\pi i h x)\,[x]$; (6.5)

(iii) $[x + z][x - z][y + w][y - w] - [x + w][x - w][y + z][y - z]$
$= [x + y][x - y][z + w][z - w]$; (6.6)

(iv) $\lim_{Im(\tau)\to+\infty} [x; h, \tau] = \frac{\sin(\pi h x)}{\sin(\pi h)}$; (6.7)

(v) $\lim_{h\to 0} [x; h, \tau] = x$. (6.8)

The property (iv) means that in the limit $Im(\tau) \to +\infty$ e-numbers become q-numbers mentioned in the previous section for $q = e^{2\pi i h}$. The property (v) relates e-numbers with the usual numbers.

We will use also the notations [4]

$$[x]_n = [x][x + 1] \ldots [x + n - 1], \quad [n]! = [1]_n,$$

which are natural elliptic generalizations of the Pochhammer symbol and factorial.

Now we are ready to construct elliptic solutions of the basic equation (4.14). Compare the properties of $[x]$ and $\psi(x)$. Both are odd functions and the limiting cases of $[x]$ (iv), (v) coincide with $\psi(x)$ for the elementary functions and rational solutions of (4.14) described in the previous section. Therefore, it is natural to identify them,

$$\psi(x) = [x]. \tag{6.9}$$

For $d(x)$ we choose the following Ansatz

$$d(x) = [x] \prod_{k=1}^{5} [x - d_k], \tag{6.10}$$

where parameters d_k are restricted by the condition (5.2). Then the limits to previous solutions are obvious. In order to prove that (6.10) is a solution of the R_{II}-chain (4.14) it is necessary to rewrite the resulting equation in such a form that it will define a doubly periodic function without singularities in the fundamental rectangle of the elliptic function $\theta_1(x)$. By the Liouville theorem such a function should be a constant the value of which is determined separately.

To this end let us consider the combination

$$
\begin{aligned}
R(n) = \Bigg(& \frac{d(x_2-n-j)}{[2n+j-x_2][2n+j-x_2+1][2j+n+x_0-x_2][2j+n+x_0-x_2+1]} \\
& + \frac{d(n+1)}{[2n+j-x_2+1][2n+j-x_2+2][n-j-x_0][n-j-x_0+1]} \\
& - \frac{d(x_2-n-j-1)}{[2n+j-x_2+1][2n+j-x_2+2][2j+n+x_0-x_2+1][2j+n+x_0-x_2+2]} \\
& - \frac{d(n)}{[2n+j-x_2][2n+j-x_2+1][n-j-x_0-1][n-j-x_0]} \\
& + \frac{d(j+x_0+1)}{[2j+n+x_0-x_2+1][2j+n+x_0-x_2+2][n-j-x_0-1][n-j-x_0]} \Bigg) \\
& \times [2j+n+x_0-x_2][2j+n+x_0-x_2+1][n-j-x_0][n-j-x_0+1],
\end{aligned}
$$

where $d(x)$ is given by (6.10). Let us treat n as a continuous variable. Then it is not difficult to see that $R(n+1/h) = R(n)$ and $R(n+\tau/h) = R(n)$ due to the special restriction (5.2). By the construction, all the poles of $R(n)$ as a function of n have been cancelled in advance by the special choice of the $c(x)$, $\sigma(x)$, $\phi(x)$ and $\rho(x)$ functions, i.e. $R(n)$ is entire and doubly periodic. By the Liouville theorem $R(n) = C_1$ is a constant not depending on n, which may depend, however, on other variables j, x_2, x_0. In order to prove that $C_1 = d(j+x_0)$, which would imply (4.14), it is necessary to consider the combination

$$
\begin{aligned}
S(j) = \Bigg(& \frac{d(x_2 - j)}{[j - x_2][j - x_2 + 1][2j + x_0 - x_2][2j + x_0 - x_2 + 1]} \\
& - \frac{d(j + x_0)}{[2j + x_0 - x_2][2j + x_0 - x_2 + 1][-j - x_0][-j - x_0 + 1]} \\
& - \frac{d(x_2 - j - 1)}{[j - x_2 + 1][j - x_2 + 2][2j + x_0 - x_2 + 1][2j + x_0 - x_2 + 2]} \\
& - \frac{d(0)}{[j - x_2][j - x_2 + 1][-j - x_0 - 1][-j - x_0]} \\
& + \frac{d(j + x_0 + 1)}{[2j + x_0 - x_2 + 1][2j + x_0 - x_2 + 2][-j - x_0 - 1][-j - x_0]} \Bigg) \\
& \times [j - x_2 + 1][j - x_2 + 2][-j - x_0][-j - x_0 + 1].
\end{aligned}
$$

Again, taking j as a continuous variable, it can be checked that $S(j + 1/h) = S(j)$ and $S(j + \tau/h) = S(j)$. By the construction $S(j)$ does not have poles in j. Therefore $S(j) = C_2$ is a constant not depending on j. Taking the limit $j \to -x_0$, one can see that $C_2 = -d(1)$, which implies that $C_1 = d(j + x_0)$.

We thus proved that the function $d(x)$ given by (6.10) satisfies the basic equation (4.14) provided the constraint (5.2) is satisfied. Note an important difference of the derived elliptic solution from the rational one and its q-generalization – in the latter cases one can take parameters of the system to infinity and lower the number of products of $\psi(x)$ in $d(x)$, whereas in the elliptic case this is not possible for finite τ due to the quasi-periodicity of theta functions.

We can construct another solution of (4.14),

$$\tilde{d}(x) = [x] \prod_{k=1}^{5} [x - e_k], \quad \sum_{k=1}^{5} e_k = 1 + 2(x_0 + x_2),$$

with the help of which the spectral variables $\alpha_n, \beta_n, \lambda_n$ are restored similarly to the previous cases.

Consider first in detail biorthogonal rational functions corresponding to the following special choice of parameters:

$$e_3 = d_3,\ e_4 = d_4,\ e_5 = d_5, \quad e_1 + e_2 = d_1 + d_2. \tag{6.11}$$

Taking into account formulas (4.9) we get the expressions

$$\beta_k = \frac{[k - e_1][k - e_2]}{[k - d_1][k - d_2]}, \tag{6.12}$$

$$\alpha_k = \frac{[k - x_2 + e_1][k - x_2 + e_2]}{[k - x_2 + d_1][k - x_2 + d_2]}, \tag{6.13}$$

$$\lambda_k = \frac{[k + x_0 - e_1][k + x_0 - e_2]}{[k + x_0 - d_1][k + x_0 - d_2]}. \tag{6.14}$$

7. Elliptic Analogues of Hypergeometric Functions

In this section we reconstruct an explicit form of the R_{II}-polynomials $P_n^j(z)$ corresponding to the restricted elliptic solution of the R_{II}-chain (6.11).

Consider the expression (2.16) for R_{II}-polynomials. First of all we choose the following parametrization of the argument z:

$$z(\xi) = \frac{[\xi][\xi + e_2 - e_1]}{[\xi + d_2 - e_1][\xi + d_1 - e_1]}. \tag{7.1}$$

Using the identity (6.6) we can write

$$z(\xi) - \lambda_k = \frac{[k + \xi + x_0 - e_1][k - \xi + x_0 - e_2][d_2 - e_1][e_1 - d_1]}{[\xi + d_2 - e_1][\xi + d_1 - e_1][k + x_0 - d_1][k + x_0 - d_2]}, \tag{7.2}$$

$$z(\xi) - \alpha_k = \frac{[k + \xi - x_2 + e_2][k - \xi - x_2 + e_1][d_2 - e_1][e_1 - d_1]}{[\xi + d_2 - e_1][\xi + d_1 - e_1][k - x_2 + d_1][k - x_2 + d_2]}. \tag{7.3}$$

Hence

$$Z_n^j(z) = \prod_{k=1}^{n} (z - \alpha_{j+k}) = \left(\frac{[d_2 - e_1][e_1 - d_1]}{[\xi + d_2 - e_1][\xi + d_1 - e_1]} \right)^n \times \frac{[1 + j + \xi + e_2 - x_2]_n [1 + j - \xi + e_1 - x_2]_n}{[1 + j - x_2 + d_1]_n [1 + j - x_2 + d_2]_n}, \tag{7.4}$$

$$Y_n^j(z) = \prod_{k=1}^{n} (z - \lambda_{j+k}) = \left(\frac{[d_2 - e_1][e_1 - d_1]}{[\xi + d_2 - e_1][\xi + d_1 - e_1]} \right)^n \times \frac{[1 + j + \xi - e_1 + x_0]_n [1 + j - \xi - e_2 + x_0]_n}{[1 + j + x_0 - d_1]_n [1 + j + x_0 - d_2]_n}. \tag{7.5}$$

Determine now the coefficients $\eta_n^j(k)$ from the difference equation (2.18). The coefficients $\zeta_n^j(0)$ have the form

$$\zeta_n^j(0) = (-1)^n \prod_{m=0}^{n-1} \frac{C_m^{j+1}}{D_m^{j+1}} = (-1)^n \frac{[j+1-x_2]_n \prod_{k=1}^5 [j+1-x_2+d_k]_n}{[j+1-x_2]_{2n}[2j+2-x_1]_n}. \quad (7.6)$$

Whence

$$\frac{\zeta_n^{j+1}(0)}{\zeta_n^j(0) C_n^{j+1}} = \frac{[j+x_0+1][2j+2-x_1][2j+3-x_1][2n+j+2-x_2]}{[n-j-x_0][n-j-x_0-1][2j+n+2-x_1][2j+n+3-x_1]} \times \prod_{k=1}^5 \frac{[j+1+x_0-d_k]}{[j+1-x_2+d_k]}. \quad (7.7)$$

Consider now the Ansatz

$$\eta_n^j(k) = G(k;j) \frac{[-n]_k [1-x_2+j+n]_k}{[x_0+1-n+j]_k \, [2-x_1+n+2j]_k}, \quad (7.8)$$

where $G(k,j)$ are coefficients to be determined.

Substituting (7.7), (7.8) into (2.18) and using the identity (6.6) we see that the part containing dependence on the argument n is cancelled. The remaining part yields the equation for $G(k;j)$:

$$\frac{G(k;j)}{G(k-1;j+1)} = \frac{[2j+2-x_1][2j+3-x_1]}{[k][k+2j-x_1+1]} \prod_{m=1}^5 \frac{[j+1+x_0-d_m]}{[j+1-x_2+d_m]} \quad (7.9)$$

with the initial condition $G(0;j)=1$. It is easily verified that the only solution of the equation (7.9) for the taken initial condition is

$$G(k;j) = \frac{[-x_1+1+2j]_k [1-x_1+2j+2k]}{[k]![1-x_1+2j]} \prod_{m=1}^5 \frac{[j+1+x_0-d_m]_k}{[j+1-x_2+d_m]_k}. \quad (7.10)$$

We thus have found $\eta_n^j(k)$ to be given by (7.8), where $G(k;j)$ is fixed in (7.10).

Substituting (7.5), (7.4), (7.8) into (2.16) we arrive at the following expression:

$$\begin{aligned} P_n^j(z) = \; & Z_n^j(z)\zeta_n^j(0)\, {}_{10}E_9(2j+1-x_1; -n, 1+j-x_2+n, j+1+x_0-d_3, \\ & j+1+x_0-d_4, j+1+x_0-d_5, j+1+\xi+x_0-e_1, \\ & j+1-\xi+x_0-e_2; h, \tau), \end{aligned} \quad (7.11)$$

where ${}_{10}E_9$ is the particular terminating very-well-poised balanced "elliptic" hypergeometric function. The general series of this type ${}_{r+1}E_r$ were defined in [4] as (we use slightly different notations)

$${}_{r+1}E_r(a_1;\ a_4, a_5, \ldots, a_{r+1}; h, \tau) = \sum_{k=0}^{\infty} \frac{[a_1]_k [a_1+2k]}{[k]![a_1]} \prod_{m=1}^{r-2} \frac{[a_{3+m}]_k}{[1+a_1-a_{3+m}]_k}, \quad (7.12)$$

where the parameters $a_1, a_4, \dots, a_{r+1}$ satisfy the balancing condition

$$\frac{r-5}{2} + \frac{r-3}{2} a_1 - \sum_{m=1}^{r-2} a_{3+m} = 0. \tag{7.13}$$

In order to avoid the convergence problems of (7.12), one has to assume that one of the parameters a_k is equal to a negative integer. In our case the condition (7.13) is fulfilled due the constraints (4.13) and (5.2). For $Im\,(\tau) \to +\infty$ this ${}_{r+1}E_r$ function is transformed into the very-well-poised balanced q-series ${}_{r+1}\varphi_r$. We suggest to use the capital letter "E" to denote this new type of series to make it similar to the plain hypergeometric series case (capital "F") and to keep a trace of the "E"lliptic functions.

We thus have the following statement.

Theorem 6. *The polynomials* $P_n^j(z)$ (7.11) *are polynomials of* R_{II}*-type satisfying the recurrences* (2.1), (2.2) *and the biorthogonality conditions* (2.8) *with respect to some functional* $\mathcal{L}_j$.

As clearly seen from (7.4), (7.5) and (7.11), the j-dependence enters only in the combinations $x_0 + j, x_1 - 2j, x_2 - j$, i.e. the shifts $j \to j \pm 1$ are equivalent to a simple redefinition of the parameters.

In the next section we describe explicitly the linear functional $\mathcal{L}_j$ and a pair of biorthogonal functions $R_n^j(z), T_n^j(z)$ corresponding to $P_n^j(z)$ in a finite-dimensional case.

8. Finite-Dimensional Biorthogonality

In this section we fix the value of discrete time $j = 0$ in all formulas and remove the superscript 0 in $P_n^0(z)$ and other functions. Then, the j-dependence of all expressions can be restored if one makes the shifts $x_0 \to x_0 + j, x_1 \to x_1 - 2j, x_2 \to x_2 - j$ keeping all the parameters $d_1, \dots, d_5, e_1, e_2$ fixed. The key recurrence relation (2.3) takes now the form

$$P_{n+1}(z) + r_n\,(v_n - z) P_n(z) + u_n\,(z - \alpha_n)(z - \beta_n)\,P_{n-1}(z) = 0. \tag{8.1}$$

Let us impose one more constraint upon the parameters in addition to (6.11),

$$d_3 - x_2 = -N = 1, 2, 3, \dots. \tag{8.2}$$

Then it is seen from (4.12) that $c(N) = 0$. In turn, this means that $u_N = 0$ in (8.1). Therefore the recurrence relation is truncated naturally if z is a solution of the equation $P_N(z) = 0$ and one gets a finite-dimensional system of polynomials $P_n(z), n = 0, \dots, N-1$. It is assumed that there are no other relevant zeroes or poles in u_n and r_n for $n = 1, \dots, N$. From our formalism it follows that $P_N(z)$ has the following N zeroes

$$z_s = \lambda_{s+1} = \frac{[s+1+x_0-e_1][s+1+x_0-e_2]}{[s+1+x_0-d_1][s+1+x_0-d_2]}, \quad s = 0, 1, \dots, N-1. \tag{8.3}$$

Indeed, if one restores j-dependence for a minute, then it is seen that $C_{N-j}^j = 0$, $j = 0, \dots, N-1$. Substituting these conditions into (2.1) one concludes that the parameters $\lambda_{j+1}, \dots, \lambda_N$ define zeroes of the polynomials $P_{N-j}^j(z)$, $j = 0, \dots, N-1$.

Comparing (8.3) with the continuous parametrization of the argument z (7.1), we find that $z_s = z(\xi_s)$ for

$$\xi_s = s + 1 + x_0 - e_2. \tag{8.4}$$

Assume that all the zeroes (8.3) are distinct, i.e. $z_j \neq z_k,\ \ j \neq k$, and do not coincide with the points $\alpha_i, \beta_i,\ i = 1, 2, \dots, N-1$. Then it can be shown (see, e.g. [24]) that the biorthogonality relation (2.8) can be rewritten explicitly in the form

$$\sum_{s=0}^{N-1} \frac{w_s\, z_s^m\, P_n(z_s)}{q_n(z_s)} = 0, \quad m = 0, 1, 2, \dots, n-1, \tag{8.5}$$

where

$$q_n(z) = \prod_{i=1}^{n} (z - \alpha_i)(z - \beta_i) \tag{8.6}$$

and the weight function w_s is

$$w_s = \frac{q_{N-1}(z_s)}{P_{N-1}(z_s)\, P_N'(z_s)}. \tag{8.7}$$

Let us calculate all the entries $P_N'(z_s), q_{N-1}(z_s), P_{N-1}(z_s)$ of the weight function w_s.

First, notice that

$$P_N'(z_s) = \rho_N (z_s - z_0) \dots (z_s - z_{s-1})(z_s - z_{s+1}) \dots (z_s - z_{N-1}), \tag{8.8}$$

where ρ_N is some constant. Substituting (8.3) into (8.8) and using the identity (6.6) we find

$$P_N'(z_s) = \rho_N (-1)^{N-s-1} [N-s-1]! \left(\frac{[d_2 - e_1][d_1 - e_1]}{[s + x_0 + 1 - d_1][s + x_0 + 1 - d_2]} \right)^{N-1} \\ \times \frac{[x_0 + 1 + s - d_1][x_0 + 1 + s - d_2]}{[2x_0 + 2 - d_1 - d_2 + 2s]} \, \frac{[s]![2x_0 + 2 - d_1 - d_2 + s]_N}{[x_0 + 1 - d_1]_N [x_0 + 1 - d_2]_N}. \tag{8.9}$$

In order to calculate $q_{N-1}(z_s)$ we need the expressions (6.12), (6.13) for α_k, β_k. Using the identity (6.6) we find

$$q_{N-1}(z_s) = \frac{\delta_N [1 - x_1 + N]_s [1 + x_0 + x_2 - d_1 - d_2]_s}{[2 - x_1]_s [2 + x_0 + x_2 - d_1 - d_2 - N]_s [2 + x_0 - N]_s} \\ \times \frac{[1 + x_0 - d_1 - d_2 + N]_s [x_0 + 1]_s}{[2 + x_0 - d_1 - d_2]_s [s + x_0 - d_1 + 1]^{2N-2} [s + x_0 - d_2 + 1]^{2N-2}}, \tag{8.10}$$

where δ_N is a factor independent on z_s.

In order to calculate $P_{N-1}(z_s)$ we substitute (8.4) into (7.11) for $n = N-1$ and find

$$P_{N-1}(z_s) = Z_{N-1}(z_s)\, \zeta_{N-1}(0) \times \\ {}_8E_7(1 - x_1; N - x_2, 1 + x_0 - d_4, 1 + x_0 - d_5, s + 2 + 2x_0 - d_1 - d_2, -s; h, \tau). \tag{8.11}$$

The elliptic analogue of the very-well-poised hypergeometric series ${}_8E_7$ in (8.11) can be summed up exactly using the generalized Jackson summation formula derived in [4]:

$$
\begin{aligned}
&{}_8E_7(a_1; a_4, \ldots, a_8; h, \tau) \\
&= \frac{[a_1+1]_n[a_1+1-a_4-a_5]_n[a_1+1-a_4-a_6]_n[a_1+1-a_5-a_6]_n}{[a_1+1-a_4]_n[a_1+1-a_5]_n[a_1+1-a_6]_n[a_1+1-a_4-a_5-a_6]_n},
\end{aligned}
\tag{8.12}
$$

where $n = -a_8$ is a nonnegative integer. This formula yields

$$
\begin{aligned}
P_{N-1}(z_s) = {} & Z_{N-1}(z_s)\, \zeta_{N-1}(0) \times \\
& \frac{[2-x_1]_s[d_4+d_5-x_0-x_2]_s[1-N+d_4]_s[1-N+d_5]_s}{[1+d_4-x_2]_s[1+d_5-x_2]_s[2+x_0-N]_s[-N-x_0+d_4+d_5]_s}.
\end{aligned}
\tag{8.13}
$$

Combining all the derived expressions together, we get

$$
w_s = \kappa_N\, (z_s - \beta_1)\, \omega_s, \tag{8.14}
$$

where κ_N is a factor not depending on s and

$$
\begin{aligned}
\omega_s = {} & \frac{[2x_0+2-d_1-d_2+2s][1-N]_s[2x_0+2-d_1-d_2]_s}{[2x_0+2-d_1-d_2][s]![2x_0+2-d_1-d_2+N]_s} \\
& \times \frac{[x_0]_s[1+d_4-x_2]_s[1+d_5-x_2]_s[1+x_0+x_2-d_1-d_2]_s}{[2-x_1]_s[3+x_0-d_1-d_2]_s[1-N+d_4]_s[1-N+d_5]_s}.
\end{aligned}
\tag{8.15}
$$

From (2.9) we have

$$
\sum_{s=0}^{N-1} \omega_s\, (z_s - \beta_1)\, R_n(z_s) H_m(z_s) = 0, \quad n \neq m, \tag{8.16}
$$

where

$$
\begin{aligned}
R_n(z) = P_n(z)/(Z_n(z)\, \zeta_n(0)) = {}_{10}E_9(& 1-x_1; -n, 1-x_2+n, 1+x_0-d_3, \\
& 1+x_0-d_4, 1+x_0-d_5, 1+\xi+x_0-e_1, 1-\xi+x_0-e_2; h, \tau).
\end{aligned}
\tag{8.17}
$$

Here we have divided for convenience $R_n(z)$ defined in (2.10) by $\zeta_n(0)$. The rational functions $H_n(z)$ have the structure $H_n(z) = Q_n(z)/(z-\beta_{n+1}) \prod_{k=1}^{n} u_k(z-\beta_k)$. Introduce the modified rational functions

$$
T_n(z) = (z-\beta_1) u_1 \ldots u_n\, H_n(z) = \frac{Q_n(z)}{\prod_{k=2}^{n+1}(z-\beta_k)}. \tag{8.18}
$$

Then the biorthogonality relation (8.16) is rewritten as

$$
\sum_{s=0}^{N-1} \omega_s\, R_n(z_s) T_m(z_s) = h_n\, \delta_{nm}, \tag{8.19}
$$

where h_n are the normalization constants to be determined in the next section. Note that the rational functions $R_n(z)$, $T_n(z)$ have the same structure $[n/n]$, i.e. both are the ratios of two n^{th} degree polynomials. Poles of the functions $R_n(z)$ and $T_n(z)$ are located at the points $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\beta_2, \beta_3, \ldots, \beta_{n+1}$ respectively, where α_k, β_k are given by (6.12), (6.13).

Consider the following transformation of the parameters:

$$
\begin{aligned}
&d_1^* = 1 + x_2 - d_1, \quad e_1^* = 1 + x_2 - e_1, \quad d_2^* = 1 + x_2 - d_2, \quad e_2^* = 1 + x_2 - e_2, \\
&d_3^* = d_3, \quad d_4^* = d_4, \quad d_5^* = d_5, \quad N^* = N, \\
&x_0^* = 1 + x_0 + x_2 - d_1 - d_2, \quad x_2^* = x_2, \quad \xi^* = -\xi.
\end{aligned} \tag{8.20}
$$

It can be checked that this transformation is an involution, i.e. $p^{**} = p$ for any parameter p. Note that $z^*(\xi) = z(\xi)$, i.e. the dependence of z on the parameter ξ remains unchanged under the involution $*$. Moreover, $\alpha_k^* = \beta_{k+1},\ \beta_k^* = \alpha_{k-1}$, i.e. the poles of $R_n(z)$ and $T_n(z)$ are interchanged under this involution. The weight function appears to be an invariant function $\omega_s^* = \omega_s$. This means, in particular, that $R_n^*(z) = T_n(z),\ T_n^*(z) = R_n(z)$, because the pair of biorthogonal rational functions $R_n(z)$, $T_n(z)$ is defined uniquely (up to a normalization factor) by their poles from the relation (8.19).

As a result, we recover an explicit expression for the rational function $T_n(z)$:

$$
\begin{aligned}
T_n(z) = R_n^*(z) = {}_{10}E_9(&2 + x_0 - d_1 - d_2; -n, 1 - x_2 + n, \\
&2 + x_0 + x_2 - d_1 - d_2 - d_3, 2 + x_0 + x_2 - d_1 - d_2 - d_4, \\
&2 + x_0 + x_2 - d_1 - d_2 - d_5, 1 + \xi + x_0 - e_1, 1 - \xi + x_0 - e_2; h, \tau).
\end{aligned} \tag{8.21}
$$

Note that this expression is valid even without the quantization condition (8.2).

To summarize, the biorthogonality relation (8.19) holds for the rational functions $R_n(z)$ and $T_n(z)$ given by (8.17) and (8.21) with the weight function ω_s (8.15).

In order to find the normalization constants h_n we need the duality property of the functions $R_n(z)$ and $T_n(z)$. This will be analyzed in the next section.

9. Duality Property and the Normalization Constants

Let us rewrite the expressions for $R_n(z_s)$ and $T_n(z_s)$ after the substitution of (8.4) into (8.17) and (8.21),

$$
\begin{aligned}
R_{ns} = {}_{10}E_9(&1 - x_1; -n, 1 - x_2 + n, 1 + x_0 - d_3, 1 + x_0 - d_4, \\
&1 + x_0 - d_5, 2 + 2x_0 - d_1 - d_2 + s, -s; h, \tau),
\end{aligned} \tag{9.1}
$$

$$
\begin{aligned}
T_{ns} = {}_{10}E_9(&2 + x_0 - d_1 - d_2; -n, 1 - x_2 + n, 2 + x_0 + x_2 - d_1 - d_2 - d_3, \\
&2 + x_0 + x_2 - d_1 - d_2 - d_4, 2 + x_0 + x_2 - d_1 - d_2 - d_5, \\
&2 + 2x_0 - d_1 - d_2 + s, -s; h, \tau),
\end{aligned} \tag{9.2}
$$

where we use the matrix notations $R_{ns} \equiv R_n(z_s),\ T_{ns} \equiv T_n(z_s)$ for brevity. Consider the following transformation of the parameters:

$$
\begin{aligned}
&\tilde{x}_0 = -1 - x_0 - x_2 + d_1 + d_2, \quad \tilde{x}_1 = x_1, \quad \tilde{x}_2 = -1 - 2x_0 + d_1 + d_2, \\
&\tilde{d}_3 = -1 - N - 2x_0 + d_1 + d_2, \quad \tilde{d}_4 = -1 - 2x_0 - x_2 + d_1 + d_2 + d_4, \\
&\tilde{d}_5 = -1 - 2x_0 - x_2 + d_1 + d_2 + d_5, \quad \tilde{e}_1 + \tilde{e}_2 = \tilde{d}_1 + \tilde{d}_2, \\
&\tilde{d}_1 + \tilde{d}_2 = 2d_1 + 2d_2 - 1 - 2x_0 - x_2.
\end{aligned} \tag{9.3}
$$

It is directly verified that

$$
\tilde{R}_{ns} = R_{sn}, \quad \tilde{T}_{ns} = T_{sn}, \quad 0 \le s, n \le N - 1, \tag{9.4}
$$

where by $\tilde{R}_{ns}$ we mean the matrix obtained from $R_n(z_s)$ by the replacement of all parameters $d_1, \ldots, x_2$ by $\tilde{d}_1, \ldots, \tilde{x}_2$. One may conclude that the transformation (9.3) is equivalent to the permutation of n and s or to the transposition of matrices R_{ns}, T_{ns}. Since $\tilde{\tilde{R}}_{ns} = R_{ns}, \tilde{\tilde{T}}_{ns} = T_{ns}$, we have an involution which will be called the duality transformation.

Return to the biorthogonality relation and observe that if $h_n\, \omega_s \neq 0, \quad n, s = 0, 1, \ldots, N-1$, then the relation (8.19) means the mutual orthogonality of two matrices with the entries R_{ns}/h_n and $T_{ms}\, \omega_s$. Hence there exists the dual orthogonality relation for the same matrices

$$\sum_{n=0}^{N-1} \frac{T_{ns} R_{ns'}}{h_n} = \sum_{n=0}^{N-1} \frac{\tilde{T}_{sn} \tilde{R}_{s'n}}{h_n} = \frac{\delta_{ss'}}{\omega_s}. \tag{9.5}$$

Applying the duality transformation to the original biorthogonality relation (8.19), we get

$$\sum_{s=0}^{N-1} \tilde{\omega}_s\, \tilde{R}_{ns} \tilde{T}_{ms} = \tilde{h}_n\, \delta_{nm}. \tag{9.6}$$

Comparing (9.5) and (9.6) we arrive at the equalities

$$h_n = \frac{\kappa}{\tilde{\omega}_n}, \quad \tilde{h}_n = \frac{\kappa}{\omega_s}, \quad \tilde{\kappa} = \kappa, \tag{9.7}$$

where κ is a normalization constant not depending on n and s. Since $\omega_0 = 1$, this constant can be found if one puts $s = s' = 0$ in (9.5):

$$\kappa = \sum_{n=0}^{N-1} \tilde{\omega}_n. \tag{9.8}$$

Applying the transformation (9.3) to ω_s expressed by (8.15) we get

$$\begin{aligned} \tilde{\omega}_s = & \frac{[1 - x_2 + 2s][1 - N]_s [1 - x_2]_s}{[s]![1 - x_2][1 - x_2 + N]_s} \\ & \times \frac{[-1 - x_0 - x_2 + d_1 + d_2]_s [1 + d_4 - x_2]_s [1 + d_5 - x_2]_s [-x_0]_s}{[2 - x_1]_s [3 + x_0 - d_1 - d_2]_s [1 - d_4]_s [1 - d_5]_s}. \end{aligned} \tag{9.9}$$

The sum (9.8) is reduced to the function ${}_8E_7$ and can be calculated using the formula (8.12):

$$\kappa = \frac{[2 - x_2]_{N-1} [x_2 - d_4 - d_5]_{N-1} [1 + x_0 - d_4]_{N-1} [1 + x_0 - d_5]_{N-1}}{[1 - d_4]_{N-1} [1 - d_5]_{N-1} [2 - x_1]_{N-1} [x_0 + x_2 - d_4 - d_5]_{N-1}}. \tag{9.10}$$

It can be checked that, indeed, $\tilde{\kappa} = \kappa$. So, the normalization constants have the explicit expression

$$\begin{aligned} h_n = & \kappa\, \frac{[1 - x_2][n]![1 - x_2 + N]_n}{[1 - x_2 + 2n][1 - N]_n [1 - x_2]_n} \\ & \times \frac{[2 - x_1]_n [3 + x_0 - d_1 - d_2]_n [1 - d_4]_n [1 - d_5]_n}{[-1 - x_0 - x_2 + d_1 + d_2]_n [1 + d_4 - x_2]_n [1 + d_5 - x_2]_n [-x_0]_n}. \end{aligned} \tag{9.11}$$

Gathering the results of the previous and this section we arrive at the following theorem.

Theorem 7. *Let $d_1, d_2, d_4, d_5, e_1, e_2$ and x_0, x_1, x_2 be arbitrary parameters with the restrictions*

$$x_2 = x_0 + x_1, \quad e_1 + e_2 = d_1 + d_2,$$
$$d_1 + d_2 + d_4 + d_5 = 1 + 2x_0 + x_2 + N,$$

where N is a fixed positive integer. Then the rational functions

$$R_n(z) = {}_{10}E_9(1 - x_1; -n, 1 - x_2 + n, 1 - x_1 + N, 1 + x_0 - d_4, \\ 1 + x_0 - d_5, 1 + \xi + x_0 - e_1, 1 - \xi + x_0 - e_2; h, \tau) \tag{9.12}$$

and

$$T_n(z) = {}_{10}E_9(2 + x_0 - d_1 - d_2; -n, 1 - x_2 + n, 2 + x_0 - d_1 - d_2 + N, \\ 1 - x_0 + d_4 - N, 1 - x_0 + d_5 - N, 1 + \xi + x_0 - e_1, 1 - \xi + x_0 - e_2; h, \tau) \tag{9.13}$$

of the argument

$$z(\xi) = \frac{[\xi][\xi + e_2 - e_1]}{[\xi + d_2 - e_1][\xi + d_1 - e_1]}$$

are biorthogonal

$$\sum_{s=0}^{N-1} R_n(z_s)T_m(z_s)\omega_s = h_n\, \delta_{nm} \tag{9.14}$$

on the "elliptic grid"

$$z_s = \frac{[s + 1 + x_0 - e_1][s + 1 + x_0 - e_2]}{[s + 1 + x_0 - d_1][s + 1 + x_0 - d_2]}, \quad s = 0, 1, 2, \ldots, N - 1 \tag{9.15}$$

with the weight function ω_s and normalization constants h_n given by (8.15) *and* (9.10), (9.11) *respectively.*

We conjecture that the functions defined in this theorem represent the most general set of self-dual biorthogonal rational functions, i.e. they are the top level classical biorthogonal rational functions in the spirit of the Askey–Wilson polynomials status [1].

Consider some limiting cases of the functions $R_n(z)$. If $Im(\tau) \to +\infty$, then $[x; h, \tau] \to \sin(\pi hx)/\sin(\pi h)$ and, hence, we arrive at the biorthogonal rational functions expressed in terms of the very-well-poised balanced basic hypergeometric series ${}_{10}\varphi_9$ with the discrete measure [9,15,23]. In this case $\tilde{z}_s \equiv (1 - z_s)^{-1} \propto \sin(\pi h(s - a_1)) \sin(\pi h(s - a_2))$ with some constants a_1, a_2. Hence in this limit one can perform a rational transformation of the argument z such that the functions are parametrized with the help of the q-quadratic grid $\tilde{z}_s$ (in the terminology of [15]).

In the limit $h \to 0$ we have $[x; h, \tau] \to x$ and we arrive at Wilson's family of functions which are biorthogonal on the quadratic grid $\tilde{z}_s = (s - a_1)(s - a_2)$. Note that only in these limiting cases one can reduce parametrization of the argument of rational functions to the quadratic or q-quadratic grids. In the elliptic case the grid z_s becomes inevitably *rational* in a *quadratic* combination of the key elliptic theta function of s (9.15).

For a special choice of parameters one can make the functions $R_n(z)$ and $T_n(z)$ equal to each other. Indeed, consider the following restriction upon the parameters:

$$d_1 + d_2 = x_2 + 1. \tag{9.16}$$

Then from the explicit expressions (9.12) and (9.13) one can see that $T_n(z) = R_n(z)$ and, hence, in this case we have the pure orthogonality relation

$$\sum_{s=0}^{N-1} \omega_s R_n(z_s)\, R_m(z_s) = h_n \delta_{nm} \tag{9.17}$$

instead of (9.14). The condition (9.16) means, in particular, that $\alpha_n = \beta_{n+1}$, i.e. all poles of the function $R_n(z)$ coincide with the poles of the function $T_n(z)$. In [24] it was shown that this condition (coincidence of the poles of rational functions $R_n(z)$ and $T_n(z)$) is necessary and sufficient for the equality $R_n(z) = T_n(z)$ in the general case.

Let us restore now the discrete time dependence in the rational functions and discuss briefly the effects of the shifts $j \to j+1$ associated with the analogues of Christoffel transformations (2.1). As was mentioned, the j-dependence in all expressions is recovered by the shifts $x_0 \to x_0 + j, x_1 \to x_1 - 2j, x_2 \to x_2 - j$ without change of the parameters $d_1, \ldots, d_5, e_1, e_2$. Note that due to the condition (8.2) this leads to the shift of the integer parameter N, $N \to N - j$, i.e. each Christoffel transformation reduces the dimensionality of the system of rational functions N by 1. Denote as ω_s^j the weight function obtained after these substitutions into (8.15). Then it is seen that $\omega_0^j = 1$ and $\omega_{N-j}^j = 0$. Moreover, the following relation between ω_s^{j+1} and ω_s^j takes place

$$\omega_{s-1}^{j+1} = \gamma_j \frac{z_s - \lambda_{j+1}}{z_s - \alpha_{j+1}}\, \omega_s^j, \tag{9.18}$$

where γ_j is easily determined from the condition $\omega_0^{j+1} = 1$. As a result, one has the following relation between the functionals $\mathcal{L}_j$ at different j:

$$\mathcal{L}_{j+1} = \gamma_j \frac{z - \lambda_{j+1}}{z - \alpha_{j+1}}\, \mathcal{L}_j, \tag{9.19}$$

where the standard notation for the product of a functional $\mathcal{L}$ by a function $g(z)$ is used: $g(z)\mathcal{L}(f(z)) \equiv \mathcal{L}(g(z)f(z))$. Note that the functions $R_n^j(z)$ are orthogonal on the set $\{z_s^j\} = \lambda_{j+1}, \lambda_{j+2}, \ldots, \lambda_N$, whereas the functions $R_n^{j+1}(z)$ are orthogonal on the set $\{z_s^{j+1}\} = \lambda_{j+2}, \lambda_{j+2}, \ldots, \lambda_N$, which differs from the previous one by deletion of the first point λ_{j+1} .

Rational modifications of the functional were used already by Wilson in the construction of his ${}_9F_8$-family of biorthogonal functions [22,23]. Namely, he has built it from the requirement that multiplication of the weight function by particular rational factors is equivalent to simple shifts of the free parameters of some hypergeometric series. We have shown that a similar property holds for a much wider system of functions. Actually this is true for any explicit solution of the R_{II}-chain with the dependence on j entering through continuous parameters because the relation (9.19) is valid for arbitrary biorthogonal rational functions for an appropriate choice of the constants γ_j.

Indeed, suppose that $\mathcal{L}_j$ provides the biorthogonality condition (2.8) for some j. Then, using the definition (2.1), one easily verifies that $\mathcal{L}_{j+1}$ defined by (9.19) provides the biorthogonality functional for the polynomials $P_n^{j+1}(z)$:

$$\mathcal{L}_{j+1}\left[\frac{z^m P_n^{j+1}(z)}{\prod_{k=1}^n (z-\alpha_{j+k+1})(z-\beta_k)}\right] = \gamma_j D_n^{j+1}\mathcal{L}_j\left[\frac{z^m (z-\beta_{n+1})P_{n+1}^j(z)}{\prod_{k=1}^{n+1}(z-\alpha_{j+k})(z-\beta_k)}\right]$$
$$+\gamma_j C_n^{j+1}\mathcal{L}_j\left[\frac{z^m P_n^j(z)}{\prod_{k=1}^n (z-\alpha_{j+k})(z-\beta_k)}\right] = 0, \quad 0 \le m < n.$$

The transformations (2.1) are similar to Christoffel's transformations in the theory of orthogonal polynomials (transitions to kernel polynomials, see, e.g. [19]). However, instead of the linear transformation of the functional $\mathcal{L}_{j+1} \propto (z-\lambda_{j+1})\mathcal{L}_j$ characteristic for orthogonal polynomials, one arrives at the more involved rule (9.19). For a more detailed comparison of these two transformations, see [24].

Self-similarity of the functional $\mathcal{L}$, i.e. the requirement that there exist some non-trivial rational multiplication factors which lead only to a change of parameters of the underlying system of functions is a highly non-trivial constraint. Systematic search of systems obeying such a property requires an investigation of symmetries of the chains of spectral transformations (the R_{II}-chain in our case or the discrete-time Toda chain in the case of orthogonal polynomials [18]) with the subsequent analysis of solutions which are invariant under the corresponding symmetry transformations (self-similar solutions). The main problem consists now in the generation of all discrete and continuous symmetries of the taken nonlinear discrete time equation, which is a nontrivial problem. Some additional self-similar reductions of the R_{II}-chain, differing from (4.8), are discussed in the next section.

10. Möbius Transformations of the Grids and Some Other Similarity Reductions

As we know, R_{II}-polynomials $P_n(z)$, as well as the corresponding rational functions $R_n(z)$ and $T_n(z)$, are covariant with respect to the Möbius transformation $z \to (\xi z + \eta)/(\zeta z + \sigma)$. Consider what happens with the elliptic grid z_s (9.15) under this transformation. It is sufficient to consider two elementary transformations:

(i) $z \to z - C,\ C = \text{const};$
(ii) $z \to 1/z.$

For the linear transformation (i) one can write

$$z_s - C = \frac{[s+1+x_0-e_1][s+1+x_0-e_2]}{[s+1+x_0-d_1][s+1+x_0-d_2]} - \frac{[t+1+x_0-e_1][t+1+x_0-e_2]}{[t+1+x_0-d_1][t+1+x_0-d_2]}$$
$$= \rho\,\frac{[s-t][s+t+2x_0+2-d_1-d_2]}{[s+1+x_0-d_1][s+1+x_0-d_2]}, \tag{10.1}$$

where we have chosen a specific parametrization of the constant C via the variable t. The uniform scaling factor ρ has the form

$$\rho = \frac{[d_2-e_1][d_1-e_1]}{[t+x_0+1-d_1][t+x_0-d_2+1]}.$$

We see that for arbitrary C the points $z_s - C$ belong to the same set of elliptic grids with the changed parameters e_1, e_2. Moreover, obviously $1/z_s$ also belongs to this set of grids with the permutation $\{d_{1,2}\} \leftrightarrow \{e_{1,2}\}$. We thus arrive at the following proposition:

Proposition 2. *Möbius transforms of the elliptic grid points* $z_s \to (\xi z_s + \eta)/(\zeta z_s + \sigma)$ *belong again to an elliptic grid from the class* (9.15) *for a different choice of parameters* d_1, d_2, e_1, e_2 *and of the uniform scaling factor* ρ.

Let us outline another application of the Möbius transformations. For orthogonal polynomials, R_I and Laurent biorthogonal polynomials only affine transformations of z preserve the form of the corresponding three-term recurrence relation. Using this symmetry it is possible to define a class of polynomials whose discrete spectrum contains a number of independent geometric progressions which can be considered as generalizations of the corresponding Schrödinger equation situation [16]. Since symmetry transformations of R_{II}-polynomials comprise the full $SL(2, C)$ group, we may define a particular self-similar set of solutions of the R_{II}-chain from the requirement for the shift $j \to j + M$, M – integer, to be equivalent to the discrete shift of another grid variable $n \to n + k$, $k \in \mathbf{Z}$, combined with the transformation (4.5), (4.6) for some fixed ξ, ζ, σ, η. These conditions correspond to the following reduction of the R_{II}-chain:

$$\lambda_{j+M} = \frac{\sigma\lambda_j - \eta}{\xi - \zeta\lambda_j}, \quad \alpha_{j+M} = \frac{\sigma\alpha_{j+k} - \eta}{\xi - \zeta\alpha_{j+k}}, \quad \beta_n = \frac{\sigma\beta_{n+k} - \eta}{\xi - \zeta\beta_{n+k}},$$
$$A_n^{j+M} = A_{n+k}^j(\xi - \zeta b_{n+k}), \quad B_n^{j+M} = B_{n+k}^j,$$
$$C_n^{j+M} = C_{n+k}^j \frac{\xi - \zeta\alpha_{n+j+k}}{\xi - \zeta\lambda_j}, \quad D_n^{j+M} = \frac{D_{n+k}^j}{\xi - \zeta\lambda_j}.$$

Then formally the spectral coefficients λ_j are composed from up to M independent sequences of numbers having the form of the ratios of geometric progressions:

$$\lambda_{Mi+m} = \frac{a_m q^i + b_m}{c_m q^i + d_m}, \quad m = 1, 2, \dots M, \tag{10.2}$$

where a_m, b_m, c_m, d_m, q are some constants. The coefficients β_n and α_{n+j} are composed from up to k and $M - k$ sequences of numbers of a similar form.

Another type of reductions is associated with the companion polynomials $Q_n^j(z)$. Since they satisfy recurrence relation of the R_{II} type, it is possible to define a system of R_{II}-polynomials from the following constraint:

$$P_n^{j+M}(z) = (\zeta z + \sigma)^n Q_n^j\left(\frac{\xi z + \eta}{\zeta z + \sigma}\right) \tag{10.3}$$

or from a similar condition imposed after a number of transitions to companion polynomials. A generalization of such closures can be reached if one considers the associated R_{II}-polynomials. Corresponding constraints imposed upon the R_{II}-chain look cumbersome because of the complexity of transformations (3.9), (3.10).

Analysis of the structure of the last two types of closures lies beyond the scope of the present paper. The spectrum λ_j of the elliptic biorthogonal rational functions (9.12) is defined as a ratio of theta functions. Comparing this with (10.2), it is natural to conjecture that for some similarity closures λ_j will consist of superpositions of a number of "elliptic sequences" of points of the form (9.15).

11. General Elliptic Biorthogonal Rational Functions

In the discussion of elliptic solutions of the R_{II}-chain we have restricted ourselves to the special case (6.11). In this section we consider the general situation when *all* roots e_i of the polynomial $\tilde{d}(x) = [x]\prod_{i=1}^{5}[x - e_i]$ are different from d_i with the only restriction

$$\sum_{i=1}^{5} e_i = \sum_{i=1}^{5} d_i = 1 + 2(x_0 + x_2). \tag{11.1}$$

In this case we have

$$\alpha_k = \prod_{i=1}^{5} \frac{[k - x_2 + e_i]}{[k - x_2 + d_i]}, \quad \beta_k = \prod_{i=1}^{5} \frac{[k - e_i]}{[k - d_i]}, \quad \lambda_k = \prod_{i=1}^{5} \frac{[k + x_0 - e_i]}{[k + x_0 - d_i]}. \tag{11.2}$$

We need to find a convenient parametrization of the expressions $z - \lambda_k$ and $z - \alpha_k$. This can be done with the help of the following proposition.

Proposition 3. *Assume that* $d_i, e_i, i = 1, \ldots, 5,$ *are arbitrary numbers (pairwise distinct from each other) located inside the fundamental parallelogram of periods of the function* $[x]$ *with the restriction* (11.1). *Then the following identity*

$$z - \prod_{i=1}^{5} \frac{[x - e_i]}{[x - d_i]} = \kappa(z) \prod_{i=1}^{5} \frac{[x - \nu_i(z)]}{[x - d_i]} \tag{11.3}$$

holds, where the parameters $\kappa(z), \nu_i(z)$ *do not depend on* x.

Proof. It is easily verified that due to the condition (11.1) the function $\chi(x) = z - \prod_{i=1}^{5}[x - e_i]/[x - d_i]$ is double-periodic with the periods $1/h, \tau/h$. This function is meromorphic and has 5 simple poles at the points d_i inside the parallelogram of periods. Hence by the theorems concerning double-periodic meromorphic functions (see, e.g. [20, Ch.21.5]) the function $\chi(x)$ should have the expression (11.3) with zeroes $\nu_i(z)$ inside the fundamental parallelogram of periods. This proves the statement. □

Using the parametrization (11.3) and the property that $[-x] = -[x]$, we find

$$\begin{aligned} Z_n^j(z) &= \prod_{k=1}^{n}(z - \alpha_{j+k}) = \kappa^n(z) \prod_{i=1}^{5} \frac{[j + 1 - x_2 + \nu_i(z)]_n}{[j + 1 - x_2 + d_i]_n}, \\ Y_n^j(z) &= \prod_{k=1}^{n}(z - \lambda_{j+k}) = \kappa^n(z) \prod_{i=1}^{5} \frac{[j + 1 + x_0 - \nu_i(z)]_n}{[j + 1 + x_0 - d_i]_n}. \end{aligned} \tag{11.4}$$

The biorthogonal rational functions $R_n^j(z)$ have the form (differing from the functions entering (2.9) by the normalization factors $\zeta_n^j(0)$):

$$R_n^j(z) = \frac{P_n^j(z)}{Z_n^j(z)\zeta_n^j(0)} = \sum_{k=0}^{n} \eta_n^j(k) \frac{Y_k^j(z)}{Z_k^j(z)}, \tag{11.5}$$

where the coefficients $\eta_n^j(k)$ are given by (7.8), (7.10). Substituting (11.4) and (7.8) into (11.5) we arrive at the expression

$$R_n^j(z) = {}_{10}E_9(2j+1-x_1; -n, j+1-x_2+n, j+1+x_0-\nu_1(z), \\ j+1+x_0-\nu_2(z), \ldots, j+1+x_0-\nu_5(z); h, \tau). \quad (11.6)$$

We see that again $R_n^j(z)$ are expressed in terms of the elliptic analogue of the very-well-poised hypergeometric functions ${}_{10}E_9$. However, there are now more free parameters and the zeroes $\nu_i(z)$ have no simple expression. The solution of the R_{II} chain leading to (11.6) contains twelve natural parameters, say, $x_0, x_2, d_1, \ldots, d_4, e_1, \ldots, e_4, h, \tau$. One more free parameter appears as a ratio of the polynomials $d(x)$ and $\tilde{d}(x)$ for $e_i = d_i$; it was set equal to 1 in our considerations. Linear fractional transformations of z should allow one to fix three parameters, so that there remains only ten independent parameters. However, we did not consider explicitly how this minimization of the number of parameters takes place.

Set for simplicity $j = 0$ and remove the superscript 0 from the notations. Taking the constraint $d_1 - x_2 = N$, similar to (8.2), we arrive again at the finite-dimensional biorthogonality,

$$\sum_{s=0}^{N-1} R_n(z_s)\, T_m(z_s)\omega_s = h_n\, \delta_{nm}, \quad (11.7)$$

where for the spectral points z_s we have the expression

$$z_s = \lambda_{s+1} = \prod_{i=1}^{5} \frac{[s+1+x_0-e_i]}{[s+1+x_0-d_i]}. \quad (11.8)$$

We see that the grid (11.8) is again a double-periodic function (of the argument s) but it has now an essentially more complicated form than (8.3). Similar to the self-dual case, the linear fractional transformations of z_s do not change the general form of the grid (11.8) – this is a consequence of Proposition 3. The weight function ω_s is given again by the formula (8.7). However, in this case we were not able to find a simple expression for $P_{N-1}(z_s)$. Moreover, the companion rational function $T_n(z)$ has now much more complicated form than in the restricted case (6.11).

Similar to the rational and elementary functions solutions cases, transition to companion polynomials in the general elliptic case cannot be compensated by a redefinition of parameters. As a result, the corresponding superpotentials $\tilde{A}_n^j, \tilde{C}_n^j, \tilde{D}_n^j$ will not satisfy the similarity constraint (4.8) we have started from. One may thus conclude that actually we have an infinite sequence of elliptic solutions of the R_{II}-chain depending on ten free parameters.

12. Conclusions

In the literature on hypergeometric special functions satisfying three-term recurrence relations and some orthogonality conditions it was conjectured rather explicitly that Wilson's family of biorthogonal rational functions and their basic analogues of Rahman and Wilson provide "the most general model of its type" [9]. In this paper we have constructed a more general system of biorthogonal rational functions $R_n(z)$ which still possesses the main properties of these families:

(i) the functions $R_n(z)$ satisfy the three-term recurrence relation (2.11) (with the coefficients being given by elliptic functions);
(ii) a pair of rational functions $R_n(z)$ and $T_n(z)$ is biorthogonal on a finite number of points $z_s,\ s = 0, 1, \dots, N-1$ with respect to the explicitly found discrete weights ω_s (determined again by elliptic functions);
(iii) there is a self-duality property of the functions $R_n(z_s)$ (and $T_n(z_s)$) in the sense that the interchange of the number of rational function n and of the discrete variable s parametrizing its argument, $n \leftrightarrow s$, is equivalent to a change of parameters;
(iv) the functions $R_n(z)$ possess a self-similarity in the sense that there are Christoffel transformations which are equivalent to a redefinition of the free parameters.

The generalization with respect to previously known families of functions consists in the introduction of the new types of grids – the elliptic grids (8.3). Surprisingly these grids appeared in [2] in the study of exactly solvable models of statistical mechanics. Namely, the grid (8.3) is a solution of the following symmetric biquadratic difference equation [2]:

$$az_s^2 z_{s+1}^2 + bz_s z_{s+1}(z_s + z_{s+1}) + c(z_s^2 + z_{s+1}^2) + 2dz_s z_{s+1} + e(z_s + z_{s+1}) + f = 0. \tag{12.1}$$

For the special choice of parameters $a = b = 0$ one recovers the difference equation defining the quadratic and q-quadratic grids [12,15]:

(i) $z_s = A_1 q^s + A_2 q^{-s} + A_3$;
(ii) $z_s = A_1\, s^2 + A_2\, s + A_3$.

Existence of the elliptic grids for the case of biorthogonal rational functions could be guessed from the following considerations. Let us start from the hyperbolic grid (i) which is known to be associated with the Askey–Wilson polynomials [1] or the ${}_{10}\varphi_9$-family of biorthogonal functions considered in [9,13–15,23]. The grid (i) is determined from the difference equation

$$c(z_s^2 + z_{s+1}^2) + 2dz_s z_{s+1} + e(z_s + z_{s+1}) + f = 0. \tag{12.2}$$

We know, however, that the Möbius transformation $\tilde{z}_s = (\xi z_s + \eta)/(\zeta z_s + \sigma)$ is admissible: it transforms one set of biorthogonal rational functions to another. But the grid $\tilde{z}_s$ satisfies now Eq. (12.1) with some restriction upon the parameters a, b, c, d, e, f. It is natural to remove this restriction and consider the equation (12.1) as a starting point. Then, as shown in [2], one derives uniquely the elliptic grid (8.3). The additional free parameter, evidently, coincides with the modular parameter τ. Thus the elliptic grids appear quite naturally from the q-quadratic ones and this indicates the existence of the corresponding system of biorthogonal rational functions.

In the recent seminal paper [4] Frenkel and Turaev have introduced "elliptic" generalizations of the hypergeometric functions (more precisely, elliptic analogues of the very-well-poised balanced series). These new types of functions were overlooked in the previous works on special functions. The authors of [4] have offered many useful identities concerning these functions. They also identified "elliptic $6j$-symbols", appearing within some exactly solvable models of statistical mechanics [3], with the elliptic very-well-poised balanced hypergeometric function ${}_{10}E_9$ for some special choice of parameters. However, to the best of our knowledge, the relation of these functions to the three-term recurrence relation of the R_{II}-type, the corresponding generalized spectral problems and biorthogonal rational functions upon the elliptic grids constructed in this paper were not discussed in the literature.

It should be stressed that our approach is based upon self-similar reductions of the chains of spectral transformations for eigenvalue problems (see, e.g., our previous works [16–18], where this formalism was applied to the Schrödinger equation and ordinary orthogonal polynomials). In this general formalism the elliptic hypergeometric functions are derived in a completely regular way as solutions of the R_{II} recurrence relation for some elliptic recurrence coefficients, which correspond to some particular solutions of the R_{II}-chain. Considering other similarity solutions of the R_{II}-chain one can arrive at the biorthogonal rational functions determined in terms of the more complicated special functions.

Acknowledgements. The authors are supported in part through funds provided by the INTAS grant 96-0700 and RFBR (Russia) grant 97-01-00281. A part of this work was done when the second author was visiting the Centre de Recherches Mathématiques (Montréal).

References

1. Askey, R. and Wilson, J.A.: Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials. Mem. Am. Math. Soc. **54**, 1–55 (1985)
2. Baxter, R.: *Exactly Solvable Models in Statistical Mechanics*. London: Academic Press, 1982
3. Date, E., Jimbo, M., Kuniba, A., Miwa, T. and Okado, M.: Exactly solvable SOS models II: Proof of the star-triangle relation and combinatorial identities. In: *Conformal Field Theory and Lattice Models*, Advanced Studies in Pure Math. **16**, 17–122 (1988)
4. Frenkel, I.B. and Turaev, V.G.: Elliptic solutions of the Yang-Baxter equation and modular hypergeometric functions. In: *The Arnold-Gelfand Mathematical Seminars*, Boston, MA: Birkhäuser, 1997, pp. 171–204
5. Gasper, G. and Rahman, M.: *Basic Hypergeometric Series*. Cambridge: Cambridge University Press, 1990
6. Geronimus, Ya.L.: On the polynomials orthogonal with respect to a given number sequence. Zap. Mat. Otdel. Khar'kov Univ. i NII Mat. i Mehan. **17**, 3–18 (1940) (in Russian)
7. Geronimus, Ya.L.: On the polynomials orthogonal with respect to a given number sequence and a theorem by W. Hahn. Izv. Akad. Nauk **4**, 215–228 (1940)
8. Gupta, D.P. and Masson, D.R.: Watson's basic analogue of Ramanujan's entry 40 and its generalization. SIAM J. Math. Anal. **25**, 429–440 (1994)
9. Gupta, D.P. and Masson, D.R.: Contiguous relations, continued fractions and orthogonality. Trans. Am. Math. Soc. **350**, 769–808 (1998)
10. Iserles, A. and Nørsett, S.P.: On the theory of biorthogonal polynomials. Trans. Am. Math. Soc. **306**, 455–474 (1988)
11. Ismail, M.E.H. and Masson, D.R.: Generalized orthogonality and continued fractions. J. Approx. Theory **83**, 1–40 (1995)
12. Magnus, A.: Special non uniform lattice (SNUL) orthogonal polynomials on discrete dense sets of points. J. Comp. Appl. Math. **65**, 253–265 (1995)
13. Rahman, M.: An integral representation of a ${}_{10}\varphi_9$ and continuous biorthogonal ${}_{10}\varphi_9$ rational functions. Canad. J. Math. **38**, 605–618 (1986)
14. Rahman, M.: Biorthogonality of a system of rational functions with respect to a positive measure on [-1, 1]. SIAM J. Math. Anal. **22**, 1430–1441 (1991)
15. Rahman, M. and Suslov, S.K.: Classical biorthogonal rational functions. In: *Methods of Approximation Theory in Complex Analysis and Mathematical Physics IV*, Lecture Notes in Math. **1550**, Berlin: Springer-Verlag, 1993, pp. 131–146
16. Spiridonov, V.: Symmetries of factorization chains for the discrete Schrödinger equation: J. Phys. A: Math. & Gen. **30**, L15–L21, (1997)
17. Spiridonov, V. and Zhedanov, A.: Discrete Darboux transformations, discrete time Toda lattice and the Askey–Wilson polynomials. Methods and Appl. Anal. **2**, 369–398 (1995)
18. Spiridonov, V. and Zhedanov, A., Discrete-time Volterra chain and classical orthogonal polynomials. J. Phys. A: Math. & Gen. **30**, 8727–8737 (1997)
19. Szegő, G.: *Orthogonal Polynomials*. Providence, RI: Am. Math. Soc., 1959 (second edition)

20. Whittacker, E.T. and Watson, G.N.: *A Course of Modern Analysis*. Cambridge: Cambridge University Press, 1927 (fourth edition)
21. Wilkinson, J.H.: *The Algebraic Eigenvalue Problem*. Oxford: Clarendon Press, 1965
22. Wilson, J.A.: Hypergeometric series, recurrence relations and some new orthogonal functions. Ph.D. thesis, University of Wisconsin, Madison, WI, 1978
23. Wilson, J.A.: Orthogonal functions from Gram determinants. SIAM J. Math. Anal. **22**, 1147–1155 (1991)
24. Zhedanov, A.: Biorthogonal rational functions and generalized eigenvalue problem. J. Approx. Theory, to be published
25. Zhedanov, A. and Spiridonov, V.: Hypergeometric biorthogonal rational functions. Uspekhi Mat. Nauk (Russian Math. Surveys) **54** No. 2, 173–174 (1999) (in Russian)

Communicated by T. Miwa