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On the Virial Theorem in Quantum Mechanics

V. Georgescu[1], **C. Gérard**[2]

[1] CNRS, Département de Mathématiques, Université de Cergy-Pontoise, 2 avenue Adolphe Chauvin, 95302 Cergy-Pontoise Cedex, France
[2] Centre de Mathématiques, UMR 7640 CNRS, Ecole Polytechnique, 91128 Palaiseau Cedex, France

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Abstract: We review the various assumptions under which abstract versions of the quantum mechanical virial theorem have been proved. We point out a relationship between the virial theorem for a pair of operators H, A and the regularity properties of the map $\mathbb{R} \ni s \mapsto \mathrm{e}^{\mathrm{i}sA}(z-H)^{-1}\mathrm{e}^{\mathrm{i}sA}$. We give an example showing that the statement of the virial theorem in [CFKS] is incorrect.

The Virial Theorem in Quantum Mechanics

The virial relation is the statement that if H, A are two selfadjoint operators on a Hilbert space $\mathcal{H}$, the expectation value of the commutator $[H, \mathrm{i}A]$ vanishes on eigenvectors of H :

$$\mathbb{1}_{\{\lambda\}}(H)[H, \mathrm{i}A]\mathbb{1}_{\{\lambda\}}(H) = 0. \tag{1}$$

The virial relation is a very important part of Mourre's positive commutator method. In fact, combined with a positive commutator estimate, one can use the virial relation to obtain the local finiteness of point spectrum (or even the absence of point spectrum). Moreover, for Hamiltonians having a multiparticle structure, it is an essential tool to prove the positive commutator estimate itself (see eg [Mo, PSS, FH]).

If H, A are both unbounded operators, some care has to be taken with the definition of the commutator $[H, \mathrm{i}A]$ which a priori is only defined as a quadratic form on $\mathcal{D}(H) \cap \mathcal{D}(A)$. A rather weak assumption under which (1) can be formulated without ambiguity is the following one:

There exists a subspace $\mathcal{S} \subset \mathcal{D}(H) \cap \mathcal{D}(A)$ dense in $\mathcal{D}(H^n)$ for some $n \in \mathbb{N}$, such that

$$|(Hu, Au) - (Au, Hu)| \leq C(\|H^n u\|^2 + \|u\|^2),\ u \in \mathcal{S}. \tag{2}$$

The quadratic form $[H, \mathrm{i}A]$ extends then uniquely from $\mathcal{S}$ to $\mathcal{D}(H^n)$ which means that the left-hand side of (1) has an unambiguous meaning.

The obstacle to a direct proof of (1) is of course that an eigenvector of H needs not be in $\mathcal{D}(A)$. Actually the counterexample that we will construct below shows that the virial relation does not hold under assumption (2).

To overcome this, additional assumptions on H and A are needed. To our knowledge, three different types of assumptions have been used in the literature to prove the virial theorem in an abstract setting.

• In [Mo, Prop. II.4], (1) is proved under the following assumptions:

(M)
i) $\mathcal{D}(H) \cap \mathcal{D}(A)$ is dense in $\mathcal{D}(H)$,
ii) $\mathrm{e}^{\mathrm{i}sA}$ preserves $\mathcal{D}(H)$ and for each $u \in \mathcal{D}(H)$ $\sup_{|s|\leq 1} \|H\mathrm{e}^{\mathrm{i}sA}u\| < \infty$,
iii) the quadratic form $[H, \mathrm{i}A]$ on $\mathcal{D}(H) \cap \mathcal{D}(A)$ is bounded below, closeable, and it extends as a bounded operator from $\mathcal{D}(H)$ to $\mathcal{H}$.

In fact the condition " $\mathrm{e}^{\mathrm{i}sA}$ preserves $\mathcal{D}(H)$" implies i) and the second part of ii), see [ABG, Prop. 3.2.5]. Moreover, it was noticed in [PSS] that Mourre's proof works without change under a condition weaker than iii). So the assumptions which are really needed for the validity of Mourre's proof are:

(M′)
i) $\mathrm{e}^{\mathrm{i}sA}$ preserves $\mathcal{D}(H)$,
ii) $|(Hu, Au) - (Au, Hu)| \leq C(\|Hu\|^2 + \|u\|^2),\ u \in \mathcal{D}(H) \cap \mathcal{D}(A)$.

• In [ABG, Prop. 7.2.10], (1) is proved if H is of class $C^1(A)$ i.e., if

(AGB) $\exists z \in \mathbb{C}\backslash\sigma(H)$ such that $\mathbb{R} \ni s \mapsto \mathrm{e}^{\mathrm{i}sA}R_z\mathrm{e}^{-\mathrm{i}sA}$ is C^1 for the strong topology of $B(\mathcal{H})$.

We have used the notation $R_z = (z - H)^{-1}$. Two equivalent characterizations of the $C^1(A)$ property in terms of commutators are:

(AGB′) $\exists z \in \mathbb{C}\backslash\sigma(H)$ such that $|(Au, R_z u) - (R_z^* u, Au)| \leq C\|u\|^2,\ u \in \mathcal{D}(A)$,

and:

(AGB″)
i) $\exists z \in \mathbb{C}\backslash\sigma(H)$ such that $R_z\mathcal{D}(A) \subset \mathcal{D}(A)$, $R_z^*\mathcal{D}(A) \subset \mathcal{D}(A)$,
ii) $|(Hu, Au) - (Au, Hu)| \leq C(\|Hu\|^2 + \|u\|^2),\ u \in \mathcal{D}(H) \cap \mathcal{D}(A)$.

• Finally in [CFKS, Theorem 4.6], (1) is proved under the following assumptions:

(CKFS)
i) $\mathcal{D}(H) \cap \mathcal{D}(A)$ is dense in $\mathcal{D}(H)$,
ii) $|(Hu, Au) - (Au, Hu)| \leq C(\|Hu\|^2 + \|u\|^2),\ u \in \mathcal{D}(H) \cap \mathcal{D}(A)$,
iii) $\exists\ H_0$, selfadjoint such that $\mathcal{D}(H) = \mathcal{D}(H_0)$, $[H_0, \mathrm{i}A]$ extends as a bounded operator from $\mathcal{D}(H_0)$ to $\mathcal{H}$, and $\mathcal{D}(A) \cap \mathcal{D}(H_0 A)$ is a core for H_0.

Since $\mathcal{D}(H_0 A) = \{u \in \mathcal{D}(A) | Au \in \mathcal{D}(H_0)\} \subset \mathcal{D}(A)$ one can suspect that there is a misprint in the last condition and that it should be replaced by the stronger version: $\mathcal{D}(H_0) \cap \mathcal{D}(H_0 A)$ is a core for H_0. Anyway, such a change does not invalidate the discussion below.

It is easy to verify that (M) implies that $\mathrm{e}^{\mathrm{i}sA}R_z\mathrm{e}^{-\mathrm{i}sA}$ is in $B(\mathcal{H}, \mathcal{D}(H))$ and that

$$\mathbb{R} \ni s \mapsto \mathrm{e}^{\mathrm{i}sA}R_z\mathrm{e}^{-\mathrm{i}sA} \text{ is } C^1 \text{ for the strong topology of } B(\mathcal{H}, \mathcal{D}(H)),$$

and hence (M) implies (ABG). The relation between (M′) and (ABG) is even more straightforward: if e^{isA} preserves $\mathcal{D}(H)$ then (M′) is equivalent to (ABG) (see Theorem 6.3.4 in [ABG]).

If $H \in C^1(A)$ then $(Au, R_z u) - (R_z^* u, Au)$ is the quadratic form of a bounded operator $[A, R_z]_0 \in B(\mathcal{H})$ (cf. (ABG′)). From (ABG″) it follows then that $\mathcal{D}(H) \cap \mathcal{D}(A)$ is a core of H and that the quadratic form $(Hu, Au) - (Au, Hu)$ is continuous for the topology of $\mathcal{D}(H)$, hence it extends uniquely to a continuous quadratic form $[H, A]_0$ on $\mathcal{D}(H)$. Identifying $\mathcal{D}(H) \subset \mathcal{H} \subset \mathcal{D}(H)^*$ in the usual way $[H, A]_0$ becomes a continuous operator $\mathcal{D}(H) \longrightarrow \mathcal{D}(H)^*$ and then one has (see [ABG, Theorem 6.2.10])

$$[A, R_z]_0 = R_z [H, A]_0 R_z. \tag{3}$$

We shall prove in an appendix that $\mathcal{D}(H)$ is preserved by e^{isA} if $[H, A]_0 \mathcal{D}(H) \subset \mathcal{H}$. In other terms, if (ABG) holds and $[H, A]_0 \mathcal{D}(H) \subset \mathcal{H}$, then (M) is satisfied.

That (ABG) is more general than (M′) can be seen from the following example: consider in $L^2(\mathbb{R})$ the operator H of multiplication by a real rational function (which may have poles, e.g. take $H(x) = 1/x$) and let $A = -\mathrm{i}d/dx$; then clearly $H \in C^1(A)$ but e^{isA} and $(A + \mathrm{i}\lambda)^{-1}$ do not leave the domain of H invariant.

In conditions (M) and (ABG) assumptions either on the action of e^{isA} on $\mathcal{D}(H)$ or on the action of $(z - H)^{-1}$ on $\mathcal{D}(A)$ are made. No comparable assumptions are made in condition (CFKS). However reading the proof (in particular the proof of [CFKS, Lemma 4.5]) one can see that the assumption that $(z - H_0)^{-1}$ preserves $D(A)$ is implicitly used to justify the identity (3) (with H replaced by H_0). We give below an example showing that the virial relation does not hold if one only assumes (CFKS) (or a slightly stronger version of it). In particular, we show that the relation $(A + \mathrm{i}\lambda)^{-1}\mathcal{D}(H) \subset \mathcal{D}(H)$, which plays a crucial role in the argument from [CFKS], is not true under their conditions.

Finally let us mention that in concrete situations (e.g. $\mathcal{H}$ is an L^2 space and H, A are differential operators), the use of cutoff and regularization arguments can be an alternative to the abstract approach relying on (M) or (ABG) (see for example [W,K]).

Results

Let us introduce the following definition concerning multicommutators: we set $\mathrm{ad}_A^0 H = H$. For $k \geq 0$, if $\mathrm{ad}_A^k H$ is a bounded operator from $\mathcal{D}(H)$ to $\mathcal{H}$ and the quadratic form $[\mathrm{ad}_A^k H, A]$ on $\mathcal{D}(H) \cap \mathcal{D}(A)$ extends as a bounded operator from $\mathcal{D}(H)$ into $\mathcal{H}$ we denote it by $\mathrm{ad}_A^{k+1} H$.

Theorem 1. *There exists a pair H, A of selfadjoint operators on a Hilbert space $\mathcal{H}$ such that:*

i) *H, A satisfy (CFKS),*
ii) *the multicommutators $\mathrm{ad}_A^k H$ extend as bounded operators from $\mathcal{D}(H)$ to $\mathcal{H}$ for all $k \in \mathbb{N}$,*
iii) *the pair H, A satisfies a Mourre estimate away from 0: For each compact interval I in $\mathbb{R}\backslash\{0\}$ there exist $c > 0$, K compact such that*

$$\mathbb{1}_I(H)[H, \mathrm{i}A]\mathbb{1}_I(H) \geq c\mathbb{1}_I(H) + K,$$

iv) *the virial relation does not hold for* H, A*: there exists* $\lambda \in \sigma_{\mathrm{pp}}(H)$ *such that*

$$\mathbb{1}_{\{\lambda\}}(H)[H, \mathrm{i}A]\mathbb{1}_{\{\lambda\}}(H) \neq 0.$$

Theorem 1 is a consequence of Theorem 2 below, which establishes a link between the virial relation and the $C^1(A)$ property.

Let H_0 be a positive selfadjoint operator on a Hilbert space $\mathcal{H}$. For $\phi \in \mathcal{H}$ we consider the rank one perturbation of H_0,

$$H_\phi := H_0 - |\phi >< \phi|,$$

which is selfadjoint with $\mathcal{D}(H_\phi) = \mathcal{D}(H_0)$. Note that $\lambda < 0$ is an eigenvalue of H_ϕ if and only if $(\phi, (H_0 - \lambda)^{-1}\phi) = 1$ and $\mathrm{Ker}(H_\phi - \lambda)$ is generated by $(H_0 - \lambda)^{-1}\phi$.

Let A be another selfadjoint operator on $\mathcal{H}$ such that

$$\begin{aligned}&\mathcal{D}(H_0) \cap \mathcal{D}(A) \text{ is dense in } \mathcal{D}(H_0),\\ &\text{the quadratic form } [H_0, A] \text{ on } \mathcal{D}(H_0) \cap \mathcal{D}(A) \text{ is bounded for the topology of } \mathcal{D}(H_0).\end{aligned} \tag{4}$$

Theorem 2. *Assume that* H_0 *is positive and* H_0, A *satisfy (4). Assume that the virial relation holds for* H_ϕ, A *for each* ϕ *in a core* S *of* A*. Then* H_0 *is of class* $C^1(A)$*.*

Proof. Let $\phi \in S$, $\lambda < 0$, $u = (H_0 - \lambda)^{-1}\phi$, $\alpha^2 = (\phi, u)^{-1}$, so that λ is an eigenvalue of $H_{\alpha\phi}$. Since $\alpha\phi \in S$ and by hypothesis the virial relation holds for $H_{\alpha\phi}, A$, we have:

$$\begin{aligned}0 &= (u, [H_0, A]_0 u) + \alpha^2(u, A\phi)(\phi, u) - \alpha^2(u, \phi)(A\phi, u)\\ &= ((H_0 - \lambda)^{-1}\phi, [H_0, A]_0(H_0 - \lambda)^{-1}\phi)\\ &\quad + ((H_0 - \lambda)^{-1}\phi, A\phi) - (A\phi, (H_0 - \lambda)^{-1}\phi).\end{aligned}$$

Using (4), this implies that

$$|((H_0 - \lambda)^{-1}\phi, A\phi) - (A\phi, (H_0 - \lambda)^{-1}\phi)| \leq C\|\phi\|^2, \ \forall \phi \in S.$$

Since S is dense in $\mathcal{D}(A)$, this implies (ABG') and hence that H_0 is of class $C^1(A)$. □

If we assume the following condition which is stronger than (4):

$$\begin{aligned}&\mathcal{D}(H_0) \cap \mathcal{D}(A) \text{ is dense in } \mathcal{D}(H_0),\\ &[H_0, A] \text{ extends to a bounded operator } [H_0, A]_0 : \mathcal{D}(H_0) \longrightarrow \mathcal{H},\\ &\mathcal{D}(H_0) \cap \mathcal{D}(H_0 A) \text{ is dense in } \mathcal{D}(H_0),\end{aligned} \tag{5}$$

then for $\phi \in \mathcal{D}(A)$ we have:

$$[H_\phi, A] = [H_0, A] - [|\phi >< \phi|, A] = [H_0, A]_0 + |A\phi >< \phi| - |\phi >< A\phi|,$$

and hence the pair H_ϕ, A satisfies then *(CFKS)*.

Note that if in addition to (5) we assume that the multicommutators $\mathrm{ad}_A^k H_0$ are bounded operators on $\mathcal{D}(H_0)$, then for $\phi \in \mathcal{D}(A^\infty) = \cap_{p \in \mathbb{N}} \mathcal{D}(A^p)$ the multicommutators $\mathrm{ad}_A^k H_\phi$ have the same property.

By Theorem 2 to construct the pair H, A in Theorem 1, it suffices to find a pair H_0, A satisfying (5) such that H_0 is not of class $C^1(A)$.

Let $\mathcal{H} = L^2(\mathbb{R}, dx)$, q the operator of multiplication by x in $\mathcal{H}$ and p the self-adjoint operator in $\mathcal{H}$ associated to $-\mathrm{i}d/dx$.

We will consider the operators

$$H_0 = e^{\omega q}, \ A = e^{\omega p} - p, \tag{6}$$

which are selfadjoint operators on their natural domains given by the functional calculus. We note that $\mathcal{D}(A) = \mathcal{D}(p) \cap \mathcal{D}(e^{\omega p})$. Noting also that $\mathcal{D}(\mathrm{e}^{\alpha p}) \subset \mathcal{D}(\mathrm{e}^{\omega p})$ if $0 < \alpha < \omega$ and using Fatou's lemma we see that the domain of $\mathrm{e}^{\omega p}$ can be described as follows: a function $f \in L^2(\mathbb{R})$ belongs to $\mathcal{D}(\mathrm{e}^{\omega p})$ if and only if f has an analytic extension to the strip $\{x + \mathrm{i}y \mid -\omega < y < 0\}$ and $\|f(\cdot + \mathrm{i}y)\|_{L^2} \leq \text{const}$. Then $\lim_{y\to\omega} f(x + \mathrm{i}y) \equiv f(x + \mathrm{i}\omega)$ exists in L^2 and one has $(\mathrm{e}^{\omega p} f)(x) = f(x - \mathrm{i}\omega)$.

The operators $\mathrm{e}^{\omega p}, \mathrm{e}^{\omega q}$ were considered by Fuglede in [Fu] in order to show that the Heisenberg form of the canonical commutation relations is not equivalent to the Weyl form.

From the Weyl form of the canonical commutation relations $\mathrm{e}^{\mathrm{i}\alpha p}\mathrm{e}^{\mathrm{i}\beta q} = \mathrm{e}^{\mathrm{i}\alpha\beta}\mathrm{e}^{\mathrm{i}\beta q}\mathrm{e}^{\mathrm{i}\alpha p}$ it follows, by formally taking $\alpha = \beta = -\mathrm{i}\omega$ with $\omega = (2\pi)^{1/2}$, that $\mathrm{e}^{\omega p}\mathrm{e}^{\omega q} = \mathrm{e}^{\omega q}\mathrm{e}^{\omega p}$. This commutation property will certainly hold on a large domain (we give below the details of the proof) although the operators $\mathrm{e}^{\omega p}$ and $\mathrm{e}^{\omega q}$ do not commute, which is the reason why H_0 is not of class $C^1(A)$.

Lemma 1. *Let H_0, A be the pair defined in (6) for $\omega = (2\pi)^{\frac{1}{2}}$. Then*

i) *H_0, A satisfy (5),*
ii) *the multicommutators $\mathrm{ad}_A^k H_0$ are bounded operators from $\mathcal{D}(H_0)$ into $\mathcal{H}$ for all $k \in \mathbb{N}$,*
iii) *on $\mathcal{D}(H_0) \cap \mathcal{D}(A)$ we have $[H_0, \mathrm{i}A] = \omega H_0$,*
iv) *H_0 is not of class $C^1(A)$.*

Proof of Theorem 1. Applying Lemma 1 and Theorem 2 for $S = D(A^\infty)$, we see that there exists $\phi \in \mathcal{D}(A^\infty)$ such that for $H = H_\phi$ properties i), ii) and iv) of Theorem 1 are satisfied. Property iii) follows from Lemma 1 iii) and the fact that $H - H_0$, $[H, A] - [H_0, A]$ are compact operators. □

Proof of Lemma 1. Let us consider the sequence of operators $\mathrm{e}^{-q^2/n}$. Clearly $\mathrm{e}^{-q^2/n}$ tends strongly to $\mathbb{1}$ in the spaces $\mathcal{H}$ and $\mathcal{D}(\mathrm{e}^{\omega q})$. Let us verify that the same is true in $\mathcal{D}(\mathrm{e}^{\omega p})$. In fact using the Fourier transformation, we see that $\mathrm{e}^{\omega p}\mathrm{e}^{-q^2/n} = \mathrm{e}^{-(q-\mathrm{i}\omega)^2/n}\mathrm{e}^{\omega p}$, in particular $\mathrm{e}^{-q^2/n}$ preserves $\mathcal{D}(\mathrm{e}^{\omega p})$. This easily implies that $\mathrm{e}^{-q^2/n}$ tends strongly to $\mathbb{1}$ in $\mathcal{D}(\mathrm{e}^{\omega p})$. Similarly we have $p\mathrm{e}^{-q^2/n} = \mathrm{e}^{-q^2/n}p - 2\mathrm{i}\mathrm{e}^{-q^2/n}q/n$, which shows that $\mathrm{e}^{-q^2/n}$ tends strongly to $\mathbb{1}$ in $\mathcal{D}(p)$ and hence in $\mathcal{D}(e^{\omega p} - p)$.

After conjugation by Fourier transformation, we see that the same results hold for the operator $\mathrm{e}^{-p^2/n}$. Let now

$$T_n = \mathrm{e}^{-q^2/n}\mathrm{e}^{-p^2/n}.$$

We deduce from the above observations that

$$\mathrm{slim}_{n\to+\infty} T_n = \mathbb{1}, \ \text{in the spaces } \mathcal{D}(H_0), \ \mathcal{D}(A), \ \mathcal{D}(H_0) \cap \mathcal{D}(A), \tag{7}$$

where $\mathcal{D}(H_0) \cap \mathcal{D}(A)$ is equipped with the intersection topology. Since T_n maps $\mathcal{H}$ into $\mathcal{D}(H_0) \cap \mathcal{D}(H_0 A)$, we see that the first and third conditions of (5) are satisfied.

Let us now check the second condition of (5). We claim that

$$[H_0, \mathrm{i}A] = \omega H_0, \text{ on } \mathcal{D}(H_0) \cap \mathcal{D}(A). \tag{8}$$

In fact let $u \in \mathcal{D}(H_0) \cap \mathcal{D}(A)$, and $u_n = T_n u$. By (7) it suffices to check that $(Au_n, H_0 u_n) - (H_0 u_n, Au_n) = \mathrm{i}\omega(u_n, H_0 u_n)$ for each n. Since $Au_n \in \mathcal{D}(H_0)$ and $H_0 u_n \in \mathcal{D}(A)$, we have

$$(Au_n, H_0 u_n) - (H_0 u_n, Au_n) = (u_n, AH_0 u_n - H_0 A u_n).$$

But u_n is an entire function, decreasing faster than any exponential on each line $Imz = Cst$. Hence we have

$$\begin{aligned} AH_0 u_n(x) &= \mathrm{e}^{\omega(x-\mathrm{i}\omega)} u_n(x - \mathrm{i}\omega) + \mathrm{i}\tfrac{d}{dx}(\mathrm{e}^{\omega x} u_n(x)) \\ &= e^{\omega x}(u_n(x - \mathrm{i}\omega) + \mathrm{i}\tfrac{d}{dx} u_n(x)) + \mathrm{i}\omega e^{\omega x} u_n(x) \\ &= H_0 A u_n(x) + \mathrm{i}\omega H_0 u_n(x), \end{aligned}$$

since $\omega^2 = 2\pi$. This proves (8) and hence the second condition of (5). Moreover it follows from (8) that the multicommutators $\mathrm{ad}_A^k H_0$ are bounded on $\mathcal{D}(H_0)$.

Let us now prove that H_0 is not of class $C^1(A)$. Assume the contrary. Then $(H_0+1)^{-1}$ would send $\mathcal{D}(A)$ into itself. The function $u(x) = \mathrm{e}^{-x^2}$ belongs to $\mathcal{D}(A)$ and $(H_0+1)^{-1}u$ equals $(\mathrm{e}^{\omega x}+1)^{-1}\mathrm{e}^{-x^2}$. This function has a pole at $z = -\mathrm{i}\omega/2$ and hence is not in $\mathcal{D}(A)$. This gives a contradiction and hence H_0 is not of class $C^1(A)$. □

Appendix

The following result is of some independent interest.

Lemma 2. *Let A, H be selfadjoint operators in a Hilbert space $\mathcal{H}$ such that $H \in C^1(A)$ and $[A, H]_0 \mathcal{D}(H) \subset \mathcal{H}$. Then $\mathrm{e}^{\mathrm{i}sA}\mathcal{D}(H) \subset \mathcal{D}(H)$ for all real s.*

Proof. For any bounded operator S of class $C^1(A)$ the commutator $[S, A]$ extends to a bounded operator in $\mathcal{H}$ denoted $[S, A]_0$, and one has

$$S\mathrm{e}^{\mathrm{i}tA} = \mathrm{e}^{\mathrm{i}tA} S + \int_0^t \mathrm{e}^{\mathrm{i}(t-s)A}[S, \mathrm{i}A]_0 \mathrm{e}^{\mathrm{i}sA} ds.$$

So if $t > 0$, $u \in \mathcal{H}$:

$$\|S\mathrm{e}^{\mathrm{i}tA}u\| \leq \|Su\| + \int_0^t \|[S, A]_0 \mathrm{e}^{\mathrm{i}sA}u\| ds.$$

We shall take

$$S = H_\varepsilon = H(1 + \mathrm{i}\varepsilon H)^{-1} = -\mathrm{i}/\varepsilon + (\mathrm{i}/\varepsilon)R^\varepsilon,$$

where $R^\varepsilon = (1 + \mathrm{i}\varepsilon H)^{-1}$. We set $T = [A, H]_0(H + \mathrm{i})^{-1} \in B(\mathcal{H})$ and we use [ABG, Theorem 6.2.10]; then

$$[A, H_\varepsilon]_0 = R^\varepsilon T(H + \mathrm{i})R^\varepsilon = R^\varepsilon T H_\varepsilon + \mathrm{i}R^\varepsilon T R^\varepsilon.$$

Since $\|R^\varepsilon\| \leq 1$ we obtain

$$\|H_\varepsilon e^{itA}u\| \leq \|H_\varepsilon u\| + t\|T\|\|u\| + \|T\| \int_0^t \|H_\varepsilon e^{isA}u\| ds.$$

From the Gronwall lemma it follows that for each $t_0 > 0$ there is a constant C such that $\|H_\varepsilon e^{itA}u\| \leq C(\|H_\varepsilon u\| + \|u\|)$ for all $\varepsilon > 0$, $0 \leq t \leq t_0$, $u \in \mathcal{H}$. Now it suffices to apply Fatou's lemma. □

As a final remark we shall prove a version of the virial theorem. Let A, H be self-adjoint operators on a Hilbert space $\mathcal{H}$ such that $e^{isA}\mathcal{D}(|H|^\sigma) \subset \mathcal{D}(|H|^\sigma)$ for some real number $\sigma \geq 1/2$ and all s (then the domain of $|H|^\tau$ will also be invariant if $0 \leq \tau \leq \sigma$). Set $\mathcal{K} = \mathcal{D}(|H|^\sigma)$ and identify $\mathcal{K} \subset \mathcal{H} \subset \mathcal{K}^*$. Then the group induced by e^{isA} in $\mathcal{K}$ is strongly continuous, hence the space $\mathcal{D}(A;\mathcal{K}) = \{u \in \mathcal{K} \cap \mathcal{D}(A) | Au \in \mathcal{K}\}$ is dense in $\mathcal{K}$. So the sesquilinear form $(Au, Hu) - (Hu, Au)$ is well defined on the dense linear subspace $\mathcal{D}(A;\mathcal{K})$ of $\mathcal{K}$ (one needs this restricted subspace only if $\sigma < 1$; e.g. if $\sigma = 1/2$ then one does not have anything better than $H\mathcal{K} \subset \mathcal{K}^*$).

Assume, moreover, that the preceding sesquilinear form is continuous for the topology of $\mathcal{K}$ and denote by $[A, H]_0$ the operator in $B(\mathcal{K}, \mathcal{K}^*)$ associated to it. If we set $A_\varepsilon = (e^{i\varepsilon A} - 1)(i\varepsilon)^{-1}$, then it is easily seen that

$$[H, A_\varepsilon] = \frac{1}{\varepsilon} \int_0^\varepsilon e^{i(\varepsilon - s)A}[H, iA]_0 e^{isA} ds$$

holds in the strong operator topology of $B(\mathcal{K}, \mathcal{K}^*)$. In particular we see that $[H, A_\varepsilon]$ converges strongly in $B(\mathcal{K}, \mathcal{K}^*)$ to $[H, iA]_0$. This clearly implies the virial theorem, because the eigenvectors of H belong to $\mathcal{K}$.

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