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# Characters of Cycles, Equivariant Characteristic Classes and Fredholm Modules

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**Abstract:** We derive simple explicit formula for the character of a cycle in the Connes' $(b, B)$-bicomplex of cyclic cohomology and apply it to write formulas for the equivariant Chern character and characters of finitely-summable bounded Fredholm modules.

## 1. Introduction

The notion of a cycle, introduced by Connes in [4], plays an important role in his development of the cyclic cohomology and its applications. Many questions of the differential geometry and noncommutative geometry can be reformulated as questions about geometrically defined cycles. Associated with a cycle is its *character*, which is a characteristic class in cyclic cohomology, described by an explicit formula (see [4]).

Some natural constructs, like the transverse fundamental cycle of a foliation [6] or the superconnection in [15] require however consideration of more general objects, which we call "generalized cycles" (we recall the definition in Sect. 2). The simplest geometric example of generalized cycle is provided by the algebra of forms with values in the endomorphisms of some vector bundle, together with a connection. More interesting examples arise from vector bundles equivariant with respect to the action of the discrete group, or, more generally, holonomy equivariant vector bundles on foliated manifolds.

The original definition of the character of a cycle does not apply directly to generalized cycles. To overcome this, Connes ([4], cf. also [6]) has devised a canonical procedure allowing to associate a cycle with a generalized cycle. This allows to extend the definition of the character to the generalized cycles.

In this paper we show that the character of a generalized cycle can be defined by the explicit formula in the $(b, B)$-bicomplex, resembling the JLO formula for the Chern character [10]. In the geometric examples above this leads to formulas for Bott's Chern character [2] in cyclic cohomology. As another example we derive the formula for the character of the Fredholm module.

The paper is organized as follows. In Sect. 2 we define the character of a generalized cycle and, more generally, generalized chain. Closely related formulas also appear and play an important role in Nest and Tsygan's work on the algebraic index theorems [12, 13]. We then establish some basic properties of this character and prove that our definition of the character coincides with the original one given by Connes in [4]. In Sect. 3 we construct the cyclic cocycle, representing the equivariant Chern character in the cyclic cohomology, and discuss relation of this construction with the multidimensional version of the Connes construction of the Godbillon-Vey cocycle [5], and the transverse fundamental class of the foliation. In Sect. 4 we write explicit formulas for the character of a bounded finitely-summable Fredholm module, where $F^2 - 1$ is not necessarily $0$ (such objects are called pre-Fredholm modules in [4]). The idea is to associate with such a Fredholm module a generalized cycle, by the construction similar to [4]. We thus obtain finitely summable analogues of the formulas from [10] and [9].

## 2. Characters of Cycles

In this section we start by stating definitions of generalized chains and cycles, and writing the JLO-type formula for the character. We then show that this definition of character coincides with the original one from [4].

In what follows we require the algebra $\mathcal{A}$ to be unital. This condition will be later removed by adjoining the unit to $\mathcal{A}$.

One defines a *generalized chain* over an algebra $\mathcal{A}$ by specifying the following data:

1. Graded unital algebras $\Omega$ and $\partial\Omega$ and a surjective homomorphism $r : \Omega \to \partial\Omega$ of degree $0$ , and a homomorphism $\rho : \mathcal{A} \to \Omega^0$. We require that $\rho$ and $r$ be unital.
2. Graded derivations of degree $1$ $\nabla$ on $\Omega$ and $\nabla'$ on $\partial\Omega$ such that $r \circ \nabla = \nabla' \circ r$ and $\theta \in \Omega^2$ such that
$$\nabla^2(\xi) = \theta\xi - \xi\theta$$
$\forall \xi \in \Omega$. We require that $\nabla(\theta) = 0$ .
3. A graded trace $⨍$ on $\Omega^n$ for some $n$ (called the degree of the chain) such that
$$⨍\nabla(\xi) = 0$$
$\forall \xi \in \Omega^{n-1}$ such that $r(\xi) = 0$.

If one requires $\partial\Omega = 0$ one obtains the definition of the *generalized cycle*. The generalized cycle for which $\theta = 0$ is called *cycle*.

One defines the boundary of the generalized chain to be a generalized cycle ($\partial\Omega$, $\nabla'$, $\theta'$, $⨍'$) of degree $n - 1$ over an algebra $\mathcal{A}$, where the $⨍'$ is the graded trace defined by the identity
$$⨍'\xi' = ⨍\nabla(\xi), \tag{2.1}$$
where $\xi' \in (\partial\Omega)^{n-1}$ and $\xi \in \Omega^n$ such that $r(\xi) = \xi'$. Homomorphism $\rho' : \mathcal{A} \to \partial\Omega^0$ is given by
$$\rho' = r \circ \rho. \tag{2.2}$$

Notice that for $\xi' \in \partial\Omega$ $(\nabla')^2(\xi') = \theta'\xi' - \xi'\theta'$, where $\theta'$ is defined by

$$\theta' = r(\theta). \tag{2.3}$$

With every generalized chain $\mathcal{C}^n$ of degree $n$ one can associate by a JLO-type formula a canonical $n$-cochain $\mathrm{Ch}(\mathcal{C}^n)$ in the $(b, B)$-bicomplex of the algebra $\mathcal{A}$, which we call a character of the generalized chain,

$$\mathrm{Ch}^k(\mathcal{C}^n)(a_0, a_1, \dots a_k) = \frac{(-1)^{\frac{n-k}{2}}}{(\frac{n+k}{2})!} \sum_{i_0+i_1+\dots+i_k=\frac{n-k}{2}} ⨍ \rho(a_0)\theta^{i_0}\nabla(\rho(a_1))\theta^{i_1}\dots\nabla(\rho(a_k))\theta^{i_k}. \tag{2.4}$$

Note that if $\mathcal{C}^n$ is a (non-generalized) cycle $\mathrm{Ch}(\mathcal{C}^n)$ coincides with the character of $\mathcal{C}^n$ as defined by Connes.

For the generalized chain $\mathcal{C}$ let $\partial\mathcal{C}$ denote the boundary of $\mathcal{C}$.

**Theorem 2.1.** *Let $\mathcal{C}^n$ be a chain, and $\partial(\mathcal{C}^n)$ be its boundary. Then*

$$(B + b)\,\mathrm{Ch}(\mathcal{C}^n) = S\ \mathrm{Ch}(\partial(\mathcal{C}^n)). \tag{2.5}$$

*Here $S$ is the usual periodicity shift in the cyclic bicomplex.*

*Proof.* By direct computation. □

*Remark 2.1.* A natural framework for such identities in cyclic cohomology is provided by the theory of operations on cyclic cohomology of Nest and Tsygan, cf. [12,13].

**Corollary 2.2.** *If $\mathcal{C}^n$ is a generalized cycle then* $\mathrm{Ch}(\mathcal{C}^n)$ *is an $n$-cocycle in the cyclic bicomplex of an algebra $\mathcal{A}$.*

**Corollary 2.3.** *For two cobordant generalized cycles $\mathcal{C}_1^n$ and $\mathcal{C}_2^n$,*

$$[S\ \mathrm{Ch}(\mathcal{C}_1^n)] = [S\ \mathrm{Ch}(\mathcal{C}_2^n)]$$

*in $HC^{n+2}(\mathcal{A})$.*

Formula (2.4) can also be written in a different form. We will use the following notations. First, $⨍$ can be extended to the whole algebra $\Omega$ by setting $⨍\xi = 0$ if $\deg\xi \neq n$. For $\xi \in \Omega$ $e^\xi$ is defined as $\sum_{j=0}^{\infty}\frac{\xi^j}{j!}$. Then denote $\Delta^k$ the $k$-simplex $\{(t_0, t_1, \dots, t_k) | t_0 + t_1 + \dots + t_k = 1, t_j \geq 0\}$ with the measure $dt_1 dt_2 \dots dt_k$. Finally, $\alpha$ is an arbitrary nonzero real parameter. Then

$$\mathrm{Ch}^k(\mathcal{C}^n)(a_0, a_1, \dots a_k) = \alpha^{\frac{k-n}{2}} \int_{\Delta^k} \left(⨍ \rho(a_0)e^{-\alpha t_0\theta}\nabla(\rho(a_1))e^{-\alpha t_1\theta}\dots\nabla(\rho(a_k))e^{-\alpha t_k\theta}\right) dt_1 dt_2 \dots dt_k, \tag{2.6}$$

where $k$ is of the same parity as $n$. Indeed,

$$\begin{aligned}&⨍\rho(a_0)e^{-\alpha t_0\theta}\nabla(\rho(a_1))e^{-\alpha t_1\theta}\dots\nabla(\rho(a_k))e^{-\alpha t_k\theta} = \\ &(-\alpha)^{\frac{n-k}{2}}\sum_{i_0+i_1+\dots+i_k=\frac{n-k}{2}}\frac{t_0^{i_0}t_1^{i_1}\dots t_k^{i_k}}{i_0!i_1!\dots i_k!}⨍\rho(a_0)\theta^{i_0}\nabla(\rho(a_1))\theta^{i_1}\dots\nabla(\rho(a_k))\theta^{i_k} \quad (2.7)\end{aligned}$$

and our assertion follows from the equality

$$\int_{\Delta^n} t_0^{i_0}t_1^{i_1}\dots t_k^{i_k}\,dt_1dt_2\dots dt_k = \frac{i_0!i_1!\dots i_k!}{(i_0+i_1+\dots+i_k+k)!}.$$

*Remark 2.2.* We worked above only in the context of unital algebras and maps. The case of general algebras and maps can be treated by adjoining a unit. We follow [15]. The definition of the generalized chain in the nonunital case differ from the definition in the unital case only in two aspects: first, we do not require algebras and morphisms to be unital; second, we do not require any more that the curvature $\theta$ is an element of $\Omega^2$; rather we require it to be a multiplier of the algebra $\Omega$ which satisfies the following: for $\omega \in \Omega^k$, $\theta\omega$ and $\omega\theta$ are in $\Omega^{k+2}$, $\nabla(\theta\omega) = \theta\nabla(\omega)$, $\nabla(\omega\theta) = \nabla(\omega)\theta$ and $⨍\theta\omega = ⨍\omega\theta$ if $\omega \in \Omega^{n-2}$. We also need to require existence of the $\theta'$ – multiplier of $\partial\Omega$ such that $r(\theta\omega) = \theta' r(\omega)$, $r(\omega\theta) = r(\omega)\theta'$, and include it in the defining data of the chain.

With $\mathcal{C}^n = (\Omega, \partial\Omega, r, \nabla, \nabla', \theta, ⨍)$ – nonunital generalized chain over a (possibly nonunital) algebra $\mathcal{A}$ we associate canonically a unital chain

$$\widetilde{\mathcal{C}}^n = (\widetilde{\Omega}, \partial\widetilde{\Omega}, \widetilde{r}, \widetilde{\nabla}, \widetilde{\nabla}'\widetilde{\theta}, \widetilde{⨍})$$

over the algebra $\widetilde{\mathcal{A}} - \mathcal{A}$ with unit adjoined. The construction is the following: the algebra $\widetilde{\Omega}$ is obtained from the algebra $\Omega$ by adjoining a unit 1 , (of degree 0 ) and an element $\widetilde{\theta}$ of degree 2 with the relations $\widetilde{\theta}\omega = \theta\omega$ and $\omega\theta = \omega\widetilde{\theta}$ for $\omega \in \Omega$, and similarly for the algebra $\partial\widetilde{\Omega}$. The derivation $\widetilde{\nabla}$ coincides with $\nabla$ on the elements of $\Omega$ and satisfies equalities $\widetilde{\nabla}(\widetilde{\theta}) = 0$ and $\widetilde{\nabla}(1) = 0$, and $\widetilde{\nabla}'$ is defined similarly. The graded trace $\widetilde{⨍}$ on $\widetilde{\Omega}$ is defined to coincide with $⨍$ on the elements of $\Omega$ and, if $n$ is even, is required to satisfy the relation $\widetilde{⨍}\widetilde{\theta}^{\frac{n}{2}} = 0$.

Now if $\mathcal{C}^n$ is a (nonunital) generalized cycle over $\mathcal{A}$, formula (2.4), applied to $\widetilde{\mathcal{C}}^n$ defines a (reduced) cyclic cocycle over an algebra $\widetilde{\mathcal{A}}$ and hence a class in the reduced cyclic cohomology $\overline{HC}^n(\widetilde{\mathcal{A}}) = HC^n(\mathcal{A})$. Corollary 2.3 implies that this class is invariant under the (nonunital) cobordism. Note also that in the unital case the class defined after adjoining the unit agrees with the one defined before.

Alternatively, one can work from the beginning with the Loday–Quillen–Tsygan bicomplex, see e.g. [11], where the corresponding formulas can be easily written.

We now will show equivalence of the previous construction with Connes' original construction.

With every generalized cycle $\mathcal{C} = (\Omega, \nabla, \theta, \fint)$ over an algebra $\mathcal{A}$ Connes shows how to associate canonically a cycle $\mathcal{C}_X$.

One starts with a graded algebra $\Omega_\theta$, which as a vector space can be identified with the space of $2$ by $2$ matrices over an algebra $\Omega$, with the grading given by the following:

$$\begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{bmatrix} \in \Omega_\theta^k \text{ if } \omega_{11} \in \Omega^k \ \omega_{12}, \omega_{21} \in \Omega^{k-1} \text{ and } \omega_{22} \in \Omega^{k-2}.$$

The product of the two elements in $\Omega_\theta$ $\omega = \begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{bmatrix}$ and $\omega' = \begin{bmatrix} \omega'_{11} & \omega'_{12} \\ \omega'_{21} & \omega'_{22} \end{bmatrix}$ is given by

$$\omega * \omega' = \begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \theta \end{bmatrix} \begin{bmatrix} \omega'_{11} & \omega'_{12} \\ \omega'_{21} & \omega'_{22} \end{bmatrix}. \tag{2.8}$$

The homomorphism $\rho_\theta : \mathcal{A} \to \Omega_\theta$ is given by

$$\rho_\theta(a) = \begin{bmatrix} \rho(a) & 0 \\ 0 & 0 \end{bmatrix}. \tag{2.9}$$

On this algebra one can define a graded derivation $\nabla_\theta$ of degree $1$ by the formula (here $\omega = \begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{bmatrix}$)

$$\nabla_\theta(\omega) = \begin{bmatrix} \nabla(\omega_{11}) & \nabla(\omega_{12}) \\ -\nabla(\omega_{21}) & -\nabla(\omega_{22}) \end{bmatrix}. \tag{2.10}$$

One checks that

$$\nabla_\theta^2(\omega) = \begin{bmatrix} \theta & 0 \\ 0 & 1 \end{bmatrix} * \omega - \omega * \begin{bmatrix} \theta & 0 \\ 0 & 1 \end{bmatrix}. \tag{2.11}$$

More generally, one can define on this algebra a family of connections $\nabla_\theta^t$, $0 \le t \le 1$ by the equation

$$\nabla_\theta^t(\omega) = \nabla_\theta(\omega) + t(\mathcal{X} * \omega - (-1)^{\deg \omega} \omega * \mathcal{X}), \tag{2.12}$$

where $\mathcal{X}$ is degree $1$ element of $\Omega_\theta$ given by the matrix

$$\mathcal{X} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{2.13}$$

**Lemma 2.4.** $(\nabla_\theta^t)^2(\omega) = (1 - t^2) \left( \begin{bmatrix} \theta & 0 \\ 0 & 1 \end{bmatrix} * \omega - \omega * \begin{bmatrix} \theta & 0 \\ 0 & 1 \end{bmatrix} \right).$

*Proof.* Follows from an easy computation. □

Hence for $t = 1$ we obtain a graded derivation $\nabla_\theta^1$ whose square is 0 .

Finally, the graded trace $\displaystyle\fint_\theta$ is defined by

$$\fint_\theta \omega = \fint \omega_{11} - (-1)^{\deg \omega} \fint \omega_{22}\theta. \tag{2.14}$$

It is closed with respect to $\nabla_\theta$, and hence, being a graded trace, it is closed with respect to $\nabla_\theta^t$ for any $t$.

**Corollary 2.5.** *$\mathcal{C}_X = (\Omega_\theta, \nabla_\theta^1, \fint_\theta)$ is a (nongeneralized) cycle.*

The cycle $\mathcal{C}_X$ is Connes' canonical cycle, associated with the generalized cycle $\mathcal{C}$ . With every (nongeneralized) cycle of degree $n$ Connes associated a cyclic $n$-cocycle on the algebra $\mathcal{A}$ by the following procedure: let the cycle consist of a graded algebra $\Omega$, degree 1 graded derivation $d$ and a closed trace $\fint$. Then the character of the cycle is the cyclic cocycle $\tau$ in the *cyclic complex* given by the formula

$$\tau(a_0, a_1, \ldots, a_n) = \fint \rho(a_0) d\rho(a_1) \ldots d\rho(a_n). \tag{2.15}$$

To it corresponds a cocycle in the $(b, B)$-bicomplex with only one nonzero component of degree $n$, which equals $\frac{1}{n!} \fint \rho(a_0) d\rho(a_1) \ldots d\rho(a_n)$.

**Theorem 2.6.** *Let $\mathcal{C}^n$ be a generalized cycle of degree $n$ over an algebra $\mathcal{A}$, and $\mathcal{C}_X^n$ be the canonical cycle over $\mathcal{A}$, associated with $\mathcal{C}^n$ (see above). Then $[\mathrm{Ch}(\mathcal{C}^n)] = [\tau(\mathcal{C}_X^n)]$ in $HC^n(\mathcal{A})$.*

Note that equality here is in the cyclic cohomology, not only in the periodic cyclic cohomology. The theorem will follow easily from the above considerations and the following lemma.

**Lemma 2.7.** *Let $\mathcal{C}_0 = (\Omega, \nabla_0, \theta_0, \fint)$ , be a generalized cycle of degree $n$ over an algebra $\mathcal{A}$, and let $\eta$ be an element of $\Omega^1$. Consider the generalized cycle $\mathcal{C}_1 = (\Omega, \nabla_1, \theta_1, \fint)$, where*

$$\nabla_1 = \nabla_0 + \operatorname{ad} \eta,$$
$$\theta_1 = \theta_0 + \nabla_0 \eta + \eta^2.$$

*Then* $[\mathrm{Ch}(\mathcal{C}_0)] = [\mathrm{Ch}(\mathcal{C}_1)]$.

*Proof of Lemma 2.7.* First, we can suppose that the cycle is unital – in the other case one can perform a construction, explained in Remark 2.2.

We start by constructing a cobordism between cycles $\mathcal{C}_0$ and $\mathcal{C}_1$. This is analogous to a construction from [15]. The cobordism is provided by the chain $\mathcal{C}^c = (\Omega^c, \partial\Omega^c, r^c, \nabla^c, (\nabla^c)', \theta^c, \fint^c)$ with $\partial \mathcal{C}^c = -\mathcal{C}_0 \sqcup \mathcal{C}_1$ defined as follows.

The graded algebra $\Omega^c$ is defined as $\Omega^*([0,1])\widehat{\otimes}\Omega$, where $\widehat{\otimes}$ denotes the graded tensor product, and $\Omega^*([0,1])$ is the algebra of the differential forms on the segment $[0,1]$. The map $\rho^c : \mathcal{A} \to \Omega^c$ is given by

$$\rho^c(a) = 1\widehat{\otimes}\rho(a). \tag{2.16}$$

We denote by $t$ the variable on the segment $[0,1]$.

The graded derivation $\nabla^c$ is defined by

$$\nabla^c(\alpha\widehat{\otimes}\omega) = d\alpha\widehat{\otimes}\omega + (-1)^{\deg\alpha}\alpha\widehat{\otimes}\nabla_0\omega + (-1)^{\deg\alpha}t\alpha\widehat{\otimes}[\eta,\omega]. \tag{2.17}$$

Here $d$ is the de Rham differential on $[0,1]$.

The curvature $\theta^c$ is defined to be

$$1\widehat{\otimes}\theta_0 + t\widehat{\otimes}\nabla\eta + t^2\widehat{\otimes}\eta^2 + dt\widehat{\otimes}\eta. \tag{2.18}$$

As expected, the algebra $\partial\Omega^c$ is defined to be $\Omega \oplus \Omega$. The restriction map $r^c : \Omega^c \to \Omega \oplus \Omega$ is defined by

$$r^c(\alpha\widehat{\otimes}\omega) = \begin{cases} \alpha(0)\omega \oplus \alpha(1)\omega & \text{if } \deg\alpha = 0 \\ 0 & \text{otherwise} \end{cases}. \tag{2.19}$$

The connection $\nabla'$ on $\Omega \oplus \Omega$ is given by $\nabla_0 \oplus \nabla_1$.

The graded trace $\displaystyle{⨍}^c$ on $(\Omega^c)^{n+1}$ is given by the formula

$$⨍^c \alpha\widehat{\otimes}\omega = \begin{cases} \int_{[0,1]} \alpha ⨍\omega & \text{if } \deg\omega = n \text{ and } \deg\alpha = 1 \\ 0 & \text{otherwise} \end{cases}. \tag{2.20}$$

It is easy to see that

$$⨍^c \nabla^c(\alpha\widehat{\otimes}\omega) = \begin{cases} (\alpha(1) - \alpha(0))⨍\omega & \text{if } \deg\omega = n \text{ and } \deg\alpha = 0 \\ 0 & \text{otherwise} \end{cases}. \tag{2.21}$$

Hence the "boundary" trace $(⨍^c)'$ induced on $\Omega \oplus \Omega$ equals $-⨍ \oplus ⨍$.

Thus we constructed the generalized chain $\mathcal{C}^c$, providing the cobordism between $\mathcal{C}_0$ and $\mathcal{C}_1$. Corollary 2.3 implies that $[S\ \mathrm{Ch}(\mathcal{C}_0)] = [S\ \mathrm{Ch}(\mathcal{C}_1)]$. To obtain the more precise statement of the lemma and finish the proof of the theorem, we need to examine the character $\mathrm{Ch}(\mathcal{C}^c)$, since $S\ \mathrm{Ch}(\mathcal{C}_0) - S\ \mathrm{Ch}(\mathcal{C}_1) = (b+B)\,\mathrm{Ch}(\mathcal{C}^c)$.

$\mathrm{Ch}(\mathcal{C}^c)$ has components $\mathrm{Ch}^k(\mathcal{C}^c)$ for $k = n+1, n-1, \ldots$. Its top component $\mathrm{Ch}^{n+1}(\mathcal{C}^c)$ is given by the formula,

$$\mathrm{Ch}^{n+1}(\mathcal{C}^c)(a_0, a_1, \ldots, a_{n+1}) = \frac{1}{(n+1)!}⨍^c \rho^c(a_0)\nabla^c(\rho^c(a_1))\ldots\nabla^c(\rho^c(a_{n+1})), \tag{2.22}$$

where $a_i \in \mathcal{A}$. But the expression under $⨍^c$ is easily seen to be of the form $\alpha\widehat{\otimes}\omega$, with $\alpha$ of degree 0. Hence the expression (2.22) is identically 0, by the definition (2.20) of $⨍^c$. It follows that $\mathrm{Ch}(\mathcal{C}^c)$ is in the image of the map $S$, and this implies that $[\mathrm{Ch}(\mathcal{C}_0)] = [\mathrm{Ch}(\mathcal{C}_1)]$. □

*Remark 2.3.* The above lemma remains true if we relax its conditions to allow $\eta$ to be a multiplier of degree 1 , such that $⨍\eta\omega = (-1)^{(n-1)/2}⨍\omega\eta$ and $r(\eta\omega) = r(\omega\eta) = 0$ if $r(\omega) = 0$. Then $\nabla_0\eta$ is a multiplier, defined by $(\nabla_0\eta)\omega = \nabla_0(\eta\omega) + \eta\nabla_0\omega$. The same proof then goes through if we enlarge the algebra $\Omega$ to the subalgebra of the multiplier algebra of $\Omega$ obtained from $\Omega$ by adjoining 1 , $\theta_0$, $\eta$, $\nabla_0\eta$, and extending $⨍$ to this algebra by zero (i.e. we put $⨍P = 0$ for any $P$ – monomial in $\theta_0$ and $\eta$).

*Proof of Theorem 2.6.* The lemma above applies directly to the cycles

$$\mathcal{C}_1 = \left(\Omega_\theta, \nabla_\theta, \begin{bmatrix}\theta & 0\\ 0 & 1\end{bmatrix}, ⨍_\theta\right)$$

and $\mathcal{C}_1 = \mathcal{C}^n_X$ (with $\eta = \mathcal{X}$). This shows that $\mathrm{Ch}(\mathcal{C}^n) = \mathrm{Ch}(\mathcal{C}^n_X)$ in $HC^n(\mathcal{A})$. Since $\mathcal{C}^n_X$ is a (nongeneralized) cycle, comparison of the definitions shows that $\mathrm{Ch}(\mathcal{C}^n_X) = \tau(\mathcal{C}^n_X)$, even on the level of cocycles, and the theorem follows. □

**Corollary 2.8.** *For two generalized cycles*

$$\mathcal{C}^n_1 = (\Omega_1, \nabla_1, \theta_1, ⨍_1) \quad \textit{and}$$
$$\mathcal{C}^m_2 = (\Omega_2, \nabla_2, \theta_2, ⨍_2)$$

*define the product by* $\mathcal{C}_1 \times \mathcal{C}_2 = (\Omega_1\widehat{\otimes}\Omega_2, \nabla_1\widehat{\otimes}1 + 1\widehat{\otimes}\nabla_2, \theta_1\widehat{\otimes}1 + 1\widehat{\otimes}\theta_2, ⨍_1\widehat{\otimes}⨍_2)$. *Then* $\mathrm{Ch}(\mathcal{C}_1 \times \mathcal{C}_2) = \mathrm{Ch}(\mathcal{C}_1) \cup \mathrm{Ch}(\mathcal{C}_2)$.

*Proof.* For the non-generalized cycles this follows from Connes' definition of the cup-product. In the general case, the statement follows from the existence of the natural map of cycles (i.e. homomorphism of the corresponding algebras, preserving all the structure) $(\mathcal{C}_1 \times \mathcal{C}_2)_X \to (\mathcal{C}_1)_X \times (\mathcal{C}_2)_X$, which agrees with taking the character.

The simplest way to describe this map is by using another Connes' description of his construction. In this description matrix $\begin{bmatrix}\omega_{11} & \omega_{12}\\ \omega_{21} & \omega_{22}\end{bmatrix}$, $\omega_{ij} \in \Omega$ is identified with the element $\omega_{11} + \omega_{12}X + X\omega_{21} + X\omega_{22}X$, where $X$ is a formal symbol of degree 1. The multiplication law is formally defined by $\omega X\omega' = 0$, $X^2 = \theta$. This should be understood as a short way of writing identities like $\omega X * X\omega' = \omega\theta\omega'$ (note that $X$ is not an element of the algebra).

If we denote by $X_1$, $X_2$, $X_{12}$ formal elements, corresponding to $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_1 \times \mathcal{C}_2$ respectively, the homomorphism mentioned above is the unital extension of the identity map $\Omega_1\widehat{\otimes}\Omega_2 \to \Omega_1\widehat{\otimes}\Omega_2$ defined (again formally) by $X_{12} \mapsto (X_1\widehat{\otimes}1 + 1\widehat{\otimes}X_2)$. □

## 3. Equivariant Characteristic Classes

This section concerns vector bundles equivariant with respect to discrete group actions. We show that there is a generalized cycle associated naturally to such a bundle with (not necessarily invariant) connection. The character of this generalized cycle turns out to be related (see Theorem 3.1) to the equivariant Chern character.

Let $V$ be an orientable smooth manifold of dimension $n$, $E$ a complex vector bundle over $V$, and $\mathcal{A} = \mathrm{End}(E)$ – algebra of endomorphisms with compact support. One can construct a generalized cycle over an algebra $\mathcal{A}$ in the following way. The algebra $\Omega = \Omega^*(V, \mathrm{End}(E))$ – the algebra of endomorphism-valued differential forms. Any connection $\nabla$ on the bundle $E$ defines a connection for the generalized cycle, with the curvature $\theta \in \Omega^2(V, \mathrm{End}(E))$ – the usual curvature of the connection. On the $\Omega^n(V, \mathrm{End}(E))$ one defines a graded trace $⨍$ by the formula $⨍\omega = \int_V tr\omega$, where in the right-hand side we have a usual matrix trace and a usual integration over a manifold. Note that when $V$ is noncompact, this cycle is nonunital. The formula (2.6), define a cyclic $n$-cocycle $\{\mathrm{Ch}^k\}$ on the algebra $\mathcal{A}$, given by the formula

$$\mathrm{Ch}^k(a_0, a_1, \dots a_k) = \int_{\Delta^k} \left( \int_V tr\, a_0 e^{-t_0\theta} \nabla(a_1) e^{-t_1\theta} \dots \nabla(a_k) e^{-t_k\theta} \right) dt_1 dt_2 \dots dt_k. \quad (3.1)$$

Hence we recover the formula of Quillen from [16]. (Recall that for noncompact $V$ these expressions should be viewed as defining the reduced cocycle over the algebra $\mathcal{A}$ with unit adjoined, with $\mathrm{Ch}^0$ extended by $\mathrm{Ch}^0(1) = 0$.)

One can restrict this cocycle to the subalgebra of functions $C^\infty(V) \subset \mathrm{End}(E)$. As a result one obtains an $n$-cocycle on the algebra $C^\infty(V)$, which we still denote by $\{\mathrm{Ch}^k\}$, given by the formula

$$\mathrm{Ch}^k(a_0, a_1, \dots a_k) = \frac{1}{k!} \int_V a_0 da_1 \dots da_k\, tr\, e^{-\theta}. \quad (3.2)$$

To this cocycle corresponds a current on $V$, defined by the form $tr\, e^{-\theta}$. Hence in this case we recover the Chern character of the bundle $E$. Note that we use normaliztion of the Chern character from [1].

Let now an orientable manifold $V$ of dimension $n$ be equipped with an action of the discrete group $\Gamma$ of orientation preserving transformations, and $E$ be a $\Gamma$-invariant bundle. In this situation, one can again construct a cycle of degree $n$ over the algebra $\mathcal{A} = \mathrm{End}(E) \rtimes \Gamma$. Our notations are the following: the algebra $\mathcal{A}$ is generated by the elements of the form $aU_g$, $a \in \mathrm{End}(E)$, $g \in \Gamma$, and $U_g$ is a formal symbol. The product is $(a'U_{g'})(aU_g) = a'a^{g'}U_{gg'}$. The superscript here denotes the action of the group.

The graded algebra $\Omega$ is defined as $\Omega^*(V, \mathrm{End}(E)) \rtimes \Gamma$. Elements of $\Omega$ clearly act on the forms with values in $E$, and any connection $\nabla$ in the bundle $E$ defines a connection for the algebra $\Omega$, which we also denote by $\nabla$, by the identity (here $\omega \in \Omega$, and $s \in \Omega^*(V, E)$)

$$\nabla(\omega s) = \nabla(\omega)s + (-1)^{\deg \omega}\omega\nabla(s). \quad (3.3)$$

One checks that the above formula indeed defines a degree 1 derivation, which can be described by the action on the elements of the form $\alpha U_g$, where $\alpha \in \Omega^*(V, \mathrm{End}(E))$, $g \in \Gamma$, by the equation

$$\nabla(\alpha U_g) = (\nabla(\alpha) + \alpha \wedge \delta(g))\, U_g, \quad (3.4)$$

where $\delta$ is $\Omega^1(V, \mathrm{End}(E))$-valued group cocycle, defined by

$$\delta(g) = \nabla - g \circ \nabla \circ g^{-1}. \tag{3.5}$$

One defines a curvature as an element $\theta U_1$, where 1 is the unit of the group, and $\theta$ is the (usual) curvature of $\nabla$ . The graded trace $⨍$ on $\Omega^n$ is given by

$$⨍ \alpha U_g = \begin{cases} \int_V \alpha & \text{if } g = 1 \\ 0 & \text{otherwise} \end{cases}. \tag{3.6}$$

One can associate with this cycle a cyclic $n$-cocycle over an algebra $\mathcal{A}$, by Eq. (2.6). By restricting it to the subalgebra $C_0^\infty(V) \rtimes \Gamma$ one obtains an $n$-cocycle $\{\chi^k\}$ on this algebra. Its $k^{\text{th}}$ component is given by the formula

$$\chi^k(a_0 U_{g_0}, a_1 U_{g_1}, \dots a_k U_{g_k}) = \sum_{1 \le i_1 < i_2 < \dots < i_l \le k} \int_V a_0 da_1^{\gamma_1} da_2^{\gamma_2} \dots da_{i_1-1}^{\gamma_{i_1-1}} a_{i_1}^{\gamma_{i_1}} da_{i_1+1}^{\gamma_{i_1+1}} \dots \Theta_{i_1, i_2, \dots, i_l}(\gamma_1, \dots, \gamma_k). \tag{3.7}$$

for $g_0 g_1 \dots g_k = 1$ and 0 otherwise. Here the summation is over all the subsets of $\{1, 2, \dots k\}$ and the following notations are used: $\gamma_j$ are group elements defined by $\gamma_j = g_0 g_1 \dots g_{j-1}$. $\Theta_{i_1, i_2, \dots, i_l}(\gamma_1, \dots, \gamma_k)$ is the form (depending on $g_0, g_1 \dots$ ) defined by the formula

$$\Theta_{i_1, i_2, \dots, i_l}(\gamma_1, \dots, \gamma_k) = \int_{\Delta^k} tr e^{-t_0 \theta^{\gamma_1}} e^{-t_1 \theta^{\gamma_2}} \dots e^{-t_{i_1-1} \theta^{\gamma_{i_1}}} \delta(g_{i_1})^{\gamma_{i_1}} e^{-t_{i_1} \theta^{\gamma_{i_1+1}}} \dots e^{-t_{i_2-1} \theta^{\gamma_{i_2}}} \delta(g_{i_2})^{\gamma_{i_2}} \dots e^{-t_k \theta} dt_1 \dots dt_k. \tag{3.8}$$

The change of connection does not change the class in the cyclic cohomology, as can be seen by constructing a cobordism between corresponding cycles. This formula is a cyclic cohomological analogue of the formula of Bott [2]. More precisely, the following theorem holds:

**Theorem 3.1.** *Let* $\mathrm{Ch}_\Gamma(E) \in H^*(V \times_\Gamma \mathrm{E}\Gamma)$ *be the equivariant Chern character. Let* $\Phi : H^*(V \times_\Gamma \mathrm{E}\Gamma) \to \mathrm{HP}^*(\mathrm{C}_0^\infty(\mathrm{V}) \rtimes \Gamma)$ *be the canonical imbedding, constructed by Connes, cf. [6]. Then*

$$\Phi(\mathrm{Ch}_\Gamma(E)) = [\chi].$$

Here $E$ pulls back to an equivariant bundle on $V \times \mathrm{E}\Gamma$, and then drops down to $V \times_\Gamma \mathrm{E}\Gamma$, and the equivariant Chern character $\mathrm{Ch}_\Gamma(E)$ is the Chern character of the resulting bundle. We recall that we use normalization from [1].

To prove the theorem we need some preliminary constructions and facts. For a $\Gamma$-manifold $Y$ by $Y_\Gamma$ we denote the homotopy quotient $Y \times_\Gamma \mathrm{E}\Gamma$.

Suppose we are given $\Gamma$-manifolds $V$ and $X$, $X$ oriented. We construct then a map $I : HC^j(C_0^\infty(V) \rtimes \Gamma) \to HC^{j+\dim X}(C_0^\infty(V \times X) \rtimes \Gamma)$. The construction is the following: in $HC^{\dim X}(C_0^\infty(X) \rtimes \Gamma)$ there is a class represented by the cocycle

$$\tau(f_0 U_{g_0}, f_1 U_{g_1}, \ldots, f_k U_{g_k}) = \begin{cases} \frac{1}{k!} \int_X f_0 df_1^{g_0} \ldots df_k^{g_0 g_1 \ldots g_{k-1}} & \text{if } g_0 g_1 \ldots g_k = 1 \\ 0 & \text{otherwise} \end{cases},$$

$k = \dim X$.

One then constructs the map $I$ from the following diagram:

$$HC^j(C_0^\infty(V) \rtimes \Gamma) \stackrel{\cup\tau}{\to} HC^{j+\dim X}\left(\left(C_0^\infty(V) \rtimes \Gamma\right) \otimes \left(C_0^\infty(X) \rtimes \Gamma\right)\right) \stackrel{\Delta^*}{\to} HC^{*+\dim X}(C_0^\infty(V \times X) \rtimes \Gamma). \quad (3.9)$$

Here the last arrow is induced by the natural map

$$\Delta : C_0^\infty(V \times X) \rtimes \Gamma = \left(C_0^\infty(V) \otimes C_0^\infty(X)\right) \rtimes \Gamma \to \left(C_0^\infty(V) \rtimes \Gamma\right) \otimes \left(C_0^\infty(X) \rtimes \Gamma\right) \quad (3.10)$$

defined by $\Delta\left((f \otimes f')U_g\right) = \left(fU_g\right) \otimes \left(f'U_g\right)$.

Suppose now that $V$ is also oriented.

**Proposition 3.2.** *The following diagram is commutative:*

$$\begin{array}{ccc} HP^*(C_0^\infty(V \times X) \rtimes \Gamma) & \stackrel{\Phi}{\longleftarrow} & H^*\left((V \times X)_\Gamma\right) \\ {\scriptstyle I}\uparrow & & \uparrow{\scriptstyle \pi^*} \\ HP^*(C_0^\infty(V) \rtimes \Gamma) & \stackrel{\Phi}{\longleftarrow} & H^*(V_\Gamma) \end{array} . \quad (3.11)$$

*Here $\pi : (V \times X)_\Gamma \to V_\Gamma$ is induced by the ($\Gamma$-equivariant) projection $V \times X \times \mathrm{E}\Gamma \to$* $\mathrm{V} \times \mathrm{E}\Gamma$.

*Proof.* We can consider $V \times X$ with action of $\Gamma \times \Gamma$. We start with showing that the following diagram is commutative:

$$\begin{array}{ccc} HP^*(C_0^\infty(V \times X) \rtimes (\Gamma \times \Gamma)) & \stackrel{\Phi}{\longleftarrow} & H^*\left((V \times X)_{(\Gamma\times\Gamma)}\right) \\ {\scriptstyle \cup\tau}\uparrow & & \uparrow{\scriptstyle \pi^*} \\ HP^*(C_0^\infty(V) \rtimes \Gamma) & \stackrel{\Phi}{\longleftarrow} & H^*(V_\Gamma) \end{array} . \quad (3.12)$$

Here we identify $C_0^\infty(V \times X) \rtimes (\Gamma \times \Gamma)$ with $(C_0^\infty(V) \rtimes \Gamma) \otimes (C_0^\infty(X) \rtimes \Gamma)$ and $(V \times X)_{(\Gamma\times\Gamma)}$ with $X_\Gamma \times V_\Gamma$. This is verified by the direct computation, using the Eilenberg-Silber theorem and shuffle map in cyclic cohomology, cf. [11].

Now we note that the commutativity of the following diagram is clear:

$$\begin{array}{ccc} HP^*(C_0^\infty(V \times X) \rtimes (\Gamma \times \Gamma)) & \stackrel{\Phi}{\longleftarrow} & H^*\left((V \times X)_{(\Gamma\times\Gamma)}\right) \\ \downarrow & & \downarrow \\ HP^*(C_0^\infty(V \times X) \rtimes \Gamma) & \stackrel{\Phi}{\longleftarrow} & H^*\left((V \times X)_\Gamma\right) \end{array} , \quad (3.13)$$

where both vertical arrows are induced by the diagonal maps $\Gamma \to \Gamma \times \Gamma$ and $\mathrm{E}\Gamma \to \mathrm{E}\Gamma \times \mathrm{E}\Gamma$. This ends the proof. $\square$

**Proposition 3.3.** *Let $E$ be an equivariant vector bundle on $V$ with connection $\nabla$. Let $\chi \in HC^n(C_0^\infty(V) \rtimes \Gamma)$ be the character of the associated cycle, and let $\chi' \in HC^{n+k}(C_0^\infty(V \times X) \rtimes \Gamma)$ be the character of the cycle constructed with the bundle $pr_V^* E$ and connection $pr_V^* \nabla$, where $pr_V : X \times V \to V$. Then $I(\chi) = \chi'$. (Here $n$ and $k$ are dimensions of $V$ and $X$ respectively.)*

*Proof.* Let $\mathcal{C}$ denote the corresponding cycle over $C_0^\infty(V) \rtimes \Gamma$, and $T$ -transverse fundamental cycle of $X$. Then $\mathcal{C} \times T$ is a cycle over $(C_0^\infty(V) \rtimes \Gamma) \otimes (C_0^\infty(X) \rtimes \Gamma)$, and

$$\mathrm{Ch}(\mathcal{C} \times T) = \mathrm{Ch}(\mathcal{C}) \cup \tau$$

by Corollary 2.8. If by $pr^*\mathcal{C}$ we denote the corresponding cycle over $C_0^\infty(V \times X) \rtimes \Gamma$, we have

$$\mathrm{Ch}(pr^*\mathcal{C}) = \Delta^* \left(\mathrm{Ch}(\mathcal{C} \otimes T)\right) = \Delta^* \left(\mathrm{Ch}(\mathcal{C}) \cup \tau\right) = I(\mathrm{Ch}(\mathcal{C})). \qquad \square$$

**Lemma 3.4.** *Suppose in addition to the conditions of Theorem 3.1 that $\Gamma$ acts freely and properly on $V$. Then the statement of the theorem holds.*

*Proof.* Since the group acts freely and properly, one can find a connection on $E$ which is $\Gamma$-invariant. For the class of the cocycle $\chi$ written with the invariant connection the result follows easily from the definition of the map $\Phi$. $\square$

*Proof of Theorem 3.1.* Comparison of the construction from [7], [14] with the definition of the map $\Phi$ implies that (class of) $\chi$ is in the image of $\Phi$, $[\chi] = \Phi(\xi)$ for some (necessarily unique) $\xi \in H^*(V \times_\Gamma \mathrm{E}\Gamma)$. We need to verify that $\xi = \mathrm{Ch}_\Gamma(E)$. We do this by showing that for any oriented manifold $W$ and any map continuous $f : W \to V_\Gamma$ $f^*\xi = f^* \mathrm{Ch}_\Gamma(E)$.

Let $\widetilde{W}$ be the principal $\Gamma$-bundle obtained by pullback of the bundle $V \times \mathrm{E}\Gamma \to \mathrm{V}_\Gamma$, so that the following diagram is commutative, and $\widetilde{f}$ is $\Gamma$-equivariant:

$$\begin{array}{ccc} \widetilde{W} & \xrightarrow{\widetilde{f}} & V \times \mathrm{E}\Gamma \\ \downarrow & & \downarrow \\ W & \xrightarrow{f} & V_\Gamma \end{array} . \qquad (3.14)$$

We can write $\widetilde{f}$ as a composition of two $\Gamma$-equivariant maps $\widetilde{f_1} : \widetilde{W} \to \widetilde{W} \times V \times \mathrm{E}\Gamma$, which embeds $\widetilde{W}$ as the graph of $\widetilde{f}$ and $pr : \widetilde{W} \times V \times \mathrm{E}\Gamma \to \mathrm{V} \times \mathrm{E}\Gamma$, projection. Let $\pi : (\widetilde{W} \times V)_\Gamma \to V_\Gamma$ and $f_1 : W \to (\widetilde{W} \times V)_\Gamma$ be the induced maps. We have $f = \pi f_1$.

Construct now the class $\chi' \in HP^n(C_0^\infty(\widetilde{W} \times V) \rtimes \Gamma)$ using the bundle $pr^*E$ with connection $pr^*\nabla$. By Proposition 3.3 $\chi' = I(\chi)$, where $I : HP^*(C_0^\infty(V) \rtimes \Gamma) \to HP^{*+\dim W}(C_0^\infty(\widetilde{W} \times V) \rtimes \Gamma)$. By Proposition 3.2 $\chi' = I(\chi) = I(\Phi(\xi)) = \Phi(\pi^*\xi)$. By Lemma 3.4, since $\widetilde{W} \times V$ is acted by $\Gamma$ freely and properly, $\chi' = \Phi(\mathrm{Ch}(pr^*E))$. But since $\mathrm{Ch}(pr^*(E)) = \pi^* \mathrm{Ch}(E)$, and using injectivity of $\Phi$ we conclude that

$$\pi^* \mathrm{Ch}(E) = \pi^*\xi.$$

Hence

$$f^* \operatorname{Ch}(E) = f_1^* \pi^* \operatorname{Ch}(E) = f_1^* \pi^* \xi = f^* \xi. \qquad \square$$

*Remark 3.1 (Relation with Connes' Godbillon-Vey cocycle).* In the paper [5] Connes considers (in particular) the case of the circle $S^1$ acted on by the group of its diffeomorphisms $\operatorname{Diff}(S^1)$ . Here we present the Connes construction in the multidimensional case and indicate some relations with our construction of cyclic cocycles representing equivariant classes.

In the situation of the previous example take the bundle $E$ to be $\bigwedge^n T^*X$. This is a 1-dimensional trivial bundle, naturally equipped with the action of the group $\Gamma = \operatorname{Diff}(X)$. Let $\phi$ be a nowhere 0 section of this bundle, i.e. a volume form. Define a flat connection $\nabla$ on $E$ by

$$\nabla(f\phi) = df\phi,\ \phi \in C^\infty(X). \tag{3.15}$$

We can thus define the cycle $\mathcal{C}$ over the algebra $C^\infty(X)$.

Let now $\delta(g)$ be defined as above, and put

$$\mu(g) = \frac{\phi^g}{\phi} \in C^\infty(X). \tag{3.16}$$

Then $\mu$ is a cocycle, i.e.

$$\mu(gh) = \mu(h)^g \mu(g). \tag{3.17}$$

We also have

$$\delta(g) = d \log \mu(g). \tag{3.18}$$

Indeed,

$$\delta(g)\phi^g = \nabla\phi^g - (\nabla(\phi))^g = \nabla(\mu(g)\phi) = d\mu(g)\phi$$

and

$$\delta(g) = \frac{d\mu(g)\phi}{\phi^g} = \frac{d\mu(g)}{\mu(g)} = dlog(\mu(g)).$$

For every $t$ we define a homomorphism $\rho_t : C^\infty(X) \rtimes \Gamma \to \operatorname{End}(E) \rtimes \Gamma$ by

$$\rho_t(aU_g) = a(\mu(g))^t U_g. \tag{3.19}$$

This is a homomorphism due to the cocycle property of $\mu$, which according to [5] is the Tomita-Takesaki flow associated with the state given by the volume form $\phi$.

Consider now the transverse fundamental cycle $\Phi$ over the algebra $\mathcal{A} = C^\infty(X)$ defined by the following data:

the differential graded algebra $\Omega^*(X) \rtimes \Gamma$ with the differential $d(\omega U_g) = (d\omega)U_g$, the graded trace $\fint$ on $\Omega^*(X) \rtimes \Gamma$ defined by

$$\fint \omega U_g = \begin{cases} \int\limits_X \omega & \text{if } g = 1 \\ 0 & \text{otherwise} \end{cases},$$

the homomorphism $\rho = \rho_0 = id$ from $\mathcal{A} = C^\infty(X) \rtimes \Gamma$ to $C^\infty(X) \rtimes \Gamma$.

The flow (3.19) acts on the cycle $\Phi$, by replacing $\rho_0$ by $\rho_t$. We call the cycle thus obtained $\Phi_t$. Using the identities

$$d(\rho_t(aU_g)) = \big(da + ta\; dlog\mu(g)\big)\mu(g)^t U_g = \big(da + ta\;\delta(g)\big)\mu(g)^t U_g \tag{3.20}$$

and

$$\mu(g_0)\mu(g_1)^{g_0}\mu(g_2)^{g_0 g_1}\dots\mu(g_k)^{g_0 g_1 \cdots g_{k-1}} = \mu(g_0 g_1 \dots g_k), \tag{3.21}$$

we can explicitly compute $\mathrm{Ch}(\Phi_t)$. This is the cyclic $n$-cocycle with the only component of degree $n$. The result is:

$$\mathrm{Ch}(\Phi_t) = \sum_{j=0}^{n} t^j p_j, \tag{3.22}$$

where $p_j$ is the cyclic cocycle given by

$$p_j(a_0 U_{g_0}, a_1 U_{g_1}, \dots, a_n U_{g_n}) = \frac{1}{n!} \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} \int\limits_X a_0 da_1^{\gamma_1} da_2^{\gamma_2} \dots da_{i_1-1}^{\gamma_{i_1-1}} a_{i_1}^{\gamma_{i_1}} da_{i_1+1}^{\gamma_{i_1+1}} \dots \Theta_{i_1, i_2, \dots, i_j}(\gamma_1, \dots, \gamma_k) \tag{3.23}$$

for $g_0 g_1 \dots g_k = 1$ and $0$ otherwise, where we define as before $\gamma_j = g_0 g_1 \dots g_{j-1}$, and the $j$-form $\Theta_{i_1, i_2, \dots, i_j}(\gamma_1, \dots, \gamma_k)$ is given by

$$\Theta_{i_1, i_2, \dots, i_j}(\gamma_1, \dots, \gamma_k) = \delta(g_{i_1})^{\gamma_{i_1}} \delta(g_{i_2})^{\gamma_{i_2}} \dots \delta(g_{i_j})^{\gamma_{i_j}}. \tag{3.24}$$

In particular, $p_0$ is the transverse fundamental class. Comparing these formulas from the formulas in the previous example we obtain

**Proposition 3.5.** *Let $\Phi_1$ be the image of the transverse fundamental cycle $\Phi$ under the action of the Tomita-Takesaki flow for the time 1. Let $\mathcal{C}$ be the cycle over $C^\infty(X) \rtimes \Gamma$ associated to the equivariant bundle $\bigwedge^n T^*X$ with the connection from* (3.15). *Then, on the level of cocycles* $\mathrm{Ch}(\Phi_1) = \mathrm{Ch}(\mathcal{C})$.

We now sketch a construction of a family of chains $\Psi_s$ providing the cobordism between $\Phi_0$ and $\Phi_s$, $s \in \mathbb{R}$. The algebra $\Omega^* = \Omega * ([0, s]) \widehat{\otimes} \Omega^*(X) \rtimes \Gamma$. The homomorphism from $\mathcal{A}$ to $\Omega^0$ maps $aU_g \in \Omega^*(X) \rtimes \Gamma$ to $a\mu(g)^t U_g$, where $t$ is the variable on $[0, s]$. The connection is given by $1 \widehat{\otimes} \nabla + d \widehat{\otimes} 1$, where $d$ is the de Rham differential, and the curvature is 0 . The restriction map is given by the restriction to the endpoints of the interval and the graded trace is given by

$$\fint \alpha \widehat{\otimes} (\omega U_g) = (-1)^{\deg \omega} \int\limits_{[0,s]} \alpha \int\limits_X \omega$$

if deg $\alpha = 1$ and $g = 1$ and 0 otherwise. This chain provides a cobordism between $\Phi_0$ and $\Phi_s$. Its character is given by the formula

$$\mathrm{Ch}(\Psi_s) = \sum_{j=1}^{n+1} s^j q_j, \tag{3.25}$$

where $q_j$ is the cyclic cochain given by

$$q_j(a_0 U_{g_0}, a_1 U_{g_1}, \ldots, a_n U_{g_n}) = \frac{1}{n!} \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} \int\limits_X a_0 da_1^{\gamma_1} da_2^{\gamma_2} \ldots da_{i_1-1}^{\gamma_{i_1-1}} a_{i_1}^{\gamma_{i_1}} da_{i_1+1}^{\gamma_{i_1+1}} \ldots \Xi_{i_1, i_2, \ldots, i_j}(\gamma_1, \ldots, \gamma_k) \tag{3.26}$$

for $g_0 g_1 \ldots g_k = 1$ and 0 otherwise, where we define as before $\gamma_j = g_0 g_1 \ldots g_{j-1}$, and the $j-1$-form $\Xi_{i_1,i_2,\ldots,i_j}(\gamma_1, \ldots, \gamma_k)$ is given by

$$\Xi_{i_1,i_2,\ldots,i_j}(\gamma_1, \ldots, \gamma_k) = \frac{1}{j} \sum_{l=1}^{j} (-1)^l \delta(g_{i_1})^{\gamma_{i_1}} \delta(g_{i_2})^{\gamma_{i_2}} \ldots \log \mu(g_{i_l})^{g_{i_l}} \ldots \delta(g_{i_j})^{\gamma_{i_j}}. \tag{3.27}$$

Comparing this formula with (3.22) we obtain:

**Proposition 3.6.** *Let $p_j$, $j = 1,\ldots, n$ be the chains, defined in* (3.22), (3.23)*, and $q_j$, $j = 1,\ldots, n+1$ be from* (3.25), (3.26)*. Then for $j = 1, \ldots, n$ we have*

$$B\, q_j = p_j \text{ and } b\, q_j = 0. \tag{3.28}$$

*Also*

$$B\, q_{n+1} = 0 \text{ and } b\, q_{n+1} = 0. \tag{3.29}$$

*In particular all $p_j$ define trivial classes in periodic cyclic cohomology, and $q_{n+1}$ is a cyclic cocycle.*

The cocycle $q_{n+1}$ should represent (up to a constant) the Godbillon-Vey class in the cyclic cohomology (i.e. class defined by $h_1 c_1^n$, while $p_j$ and $q_j$ represent forms $c_1^j$ and $h_1 c_1^j$, $j = 1,\ldots,n$, see [2]).

*Remark 3.2 (Transverse fundamental class).* The construction of the equivariant characteristic classes works equally well in the case of a foliation. The new ingredient required here is the Connes' construction of the transverse fundamental (generalized) cycle. We now will write a simple formula for the character of this cycle.

We start by briefly recalling Connes' construction from [6]. Details can be found in [6]. Let $(V, F)$ be a transversely oriented foliated manifold, $F$ being an integrable subbundle of $TV$. The graph of the foliation $\mathcal{G}$ is a groupoid, whose objects are points of $V$ and morphisms are equivalence classes of paths in the leaves, with equivalence given by holonomy. Equipped with a suitable topology it becomes a smooth (possibly non-Hausdorff) manifold. By $r$ and $s$ we denote the range and source maps $\mathcal{G} \to V$. By $\Omega_F^{1/2}$ we denote the line bundle on $V$ of the half-densities in the direction of $F$. Let $\mathcal{A} = C_0^\infty\left(\mathcal{G}, s^*(\Omega_F^{1/2}) \otimes r^*(\Omega_F^{1/2})\right)$ be the convolution algebra of $\mathcal{G}$. We define a (nonunital) generalized cycle over the algebra $\mathcal{A}$ as follows. The $k^{\text{th}}$ component of the graded algebra $\Omega^*$ is given by $C_0^\infty\left(\mathcal{G}, s^*(\Omega_F^{1/2}) \otimes r^*(\Omega_F^{1/2}) \otimes r^*(\bigwedge^k \tau^*)\right)$. Here $\tau = TV/F$ is the normal bundle, and the product $\Omega^k \otimes \Omega^l \to \Omega^{k+l}$ is induced by the convolution and exterior product.

The definition of the transverse differentiation (connection) requires a choice of a subbundle $H \subset TV$, complementary to $F$. This choice allows one to identify $C^\infty(V, \bigwedge^* TV^*)$ with $C^\infty(V, \bigwedge^* F^* \otimes \bigwedge^* \tau^*)$. We say that form $\omega \in C^\infty(V, \bigwedge^* TV^*)$ is of the type $(r, s)$ if it is in $C^\infty(V, \bigwedge^r F^* \otimes \bigwedge^s \tau^*)$ under this identification. For such a form we have

$$d\omega = d_V\omega + d_H\omega + \sigma\omega, \tag{3.30}$$

where $d_V\omega$, $d_H\omega$, $\sigma\omega$ are defined to be components of $d\omega$ of the types $(r+1, s)$, $(r, s+1)$, $(r-1, s+2)$ respectively (our notations are slightly different from those of [6]). Now, writing locally $\rho \in C^\infty(V, \Omega_F^{1/2})$ as $\rho = f|\omega|^{1/2}$, $f \in C^\infty(V)$, $\omega \in C^\infty(V, \bigwedge^{\dim F} F^*)$ we define

$$d_H\rho = (d_H f)|\omega|^{1/2} + f|\omega|^{1/2}\frac{d_H\omega}{2\omega}. \tag{3.31}$$

Finally, $d_H$ can be extended uniquely as a graded derivation of the graded algebra $C_0^\infty\left(\mathcal{G}, s^*(\Omega_F^{1/2}) \otimes r^*(\Omega_F^{1/2}) \otimes r^*(\bigwedge^* \tau^*)\right)$ so that the following identities are satisfied:

$$\begin{aligned} &d_H\left(r^*(\rho_1) f s^*(\rho_2)\right) = \\ &\qquad r^*(d_H\rho_1) f s^*(\rho_2) + r^*(\rho_1) d_H f s^*(\rho_2) + r^*(\rho_1) f s^*(d_H\rho_2) \\ &\qquad\qquad \text{for } \rho_1, \rho_2 \in C^\infty(V, \Omega_F^{1/2}),\ f \in C_0^\infty(\mathcal{G}) \end{aligned} \tag{3.32}$$

and

$$\begin{aligned} &d_H(\phi r^*(\omega)) = d_H(\phi) r^*(\omega) + \phi r^*(d_H\omega) \\ &\qquad \text{for } \phi \in C_0^\infty\left(\mathcal{G}, s^*(\Omega_F^{1/2}) \otimes r^*(\Omega_F^{1/2})\right), \omega \in C^\infty(V, \bigwedge^* \tau^*). \end{aligned} \tag{3.33}$$

Now, for the form $\omega$ $d_H^2\omega = -(d_V\sigma + \sigma d_V)\omega$. The operator $\theta = -(d_V\sigma + \sigma d_V)$ contains only longitudinal Lie derivatives, and hence defines a multiplier (of degree 2) of the algebra $C_0^\infty\left(\mathcal{G}, s^*(\Omega_F^{1/2}) \otimes r^*(\Omega_F^{1/2}) \otimes r^*(\bigwedge^* \tau^*)\right)$.

Finally, the graded trace on $C_0^\infty\left(\mathcal{G}, s^*(\Omega_F^{1/2}) \otimes r^*(\Omega_F^{1/2}) \otimes r^*(\bigwedge^q \tau^*)\right)$, $q = \operatorname{codim} F$ is given by $⨍\omega = \int_V \omega$.

**Lemma 3.7 ([6]).** $\left(C_0^\infty\left(\mathcal{G}, s^*(\Omega_F^{1/2}) \otimes r^*(\Omega_F^{1/2}) \otimes r^*(\bigwedge^q \tau^*)\right), d_H, \theta, ⨍\right)$ *is a generalized cycle of degree $q$ over the algebra $\mathcal{A}$ .*

We can now write an explicit formula for the character of this cycle.

**Proposition 3.8.** *The following formula defines a (reduced) cyclic cocycle $\chi$ in the $(b, B)$-bicomplex of the algebra $\mathcal{A}$ (with adjoined unit),*

$$\chi^k(\phi_0, \phi_1, \dots, \phi_k) = \frac{(-1)^{\frac{q-k}{2}}}{\left(\frac{q+k}{2}\right)!} \sum_{i_0+\dots+i_k=\frac{q-k}{2}} \int_V \phi_0\theta^{i_0} d_H(\phi_1)\dots d_H(\phi_k)\theta^{i_k}. \quad (3.34)$$

*Here $k = q, q-2, ...,$ and $\phi_j$, $j \geq 1$ are elements of $\mathcal{A}$, while $\phi_0$ is an element of $\mathcal{A}$ with unit adjoined.*

Recall, that for $q$ even to define the cocycle over $\mathcal{A}$ with the unit adjoined we extend $⨍$ by requiring that $⨍\theta^{q/2} = 0$. The resulting class is independent of the choice of $H$. It follows from the fact that by varying the subbundle $H$ smoothly we obtain the cobordism between the corresponding cycles, satisfying the conditions of Lemma 2.7. Note that the equality here is in cyclic cohomology, not only periodic cyclic cohomology.

The results of Sect. 2 imply that the class of the cocycle $\chi$ is the transverse fundamental class of the foliation, as defined in [6].

## 4. Fredholm Modules

In this section we write formulas for the character of the generalized cycle associated with a finitely summable bounded Fredholm module (cf. [4]) . In other words we obtain a formula for the character of a Fredholm module. We show that this definition agrees with Connes' definition [4].

Let $(\mathcal{H}, F, \gamma)$ be an even finitely summable bounded Fredholm module over the algebra $\mathcal{A}$. Here $\mathcal{H}$ is a Hilbert space, on which the algebra $\mathcal{A}$ acts, $\gamma$ is a $\mathbb{Z}_2$-grading on $\mathcal{H}$, and $F$ is an odd selfadjoint operator on $\mathcal{H}$. We assume that $\mathcal{A}$ is represented by the even operators in $\mathcal{H}$, and since we almost always consider only one representation of $\mathcal{A}$, we drop this representation from our notations, and do not distinguish elements of the algebra and corresponding operators. We suppose that the algebra $\mathcal{A}$ is unital. Let $p$ be a number such that $[F, a] \in \mathcal{L}^p$ and $(F^2 - 1) \in \mathcal{L}^{\frac{p}{2}}$. We remark that for any $p$ summable Fredholm module one can achieve these summability conditions by altering the operator $F$ and keeping all the other data intact. We associate with the Fredholm module a generalized cycle similarly to [4] where it is done in the case when $F^2 = 1$. Consider a $\mathbb{Z}$-graded algebra $\Omega = \bigoplus_{m=0}^\infty \Omega^m$ generated by the symbols $a \in \mathcal{A}$ of degree 0 , $[F, a]$, $a \in \mathcal{A}$ of degree 1 and symbol $(F^2 - 1)$ of degree 2 , with a relation $[F, ab] = a[F, b] + [F, a]b$. This algebra can be naturally represented on the Hilbert space $\mathcal{H}$, and we will not distinguish in our notations between elements of the algebra and the corresponding operators. $\Omega$ is equipped with a natural connection $\nabla$, given by the formula $\nabla(\xi) = [F, \xi]$ (graded commutator) in terms of the representation of $\Omega$, or on generators by the formulas

$$\nabla(a) = [F, a], \tag{4.1}$$

$$\nabla([F, a]) = \Big((F^2 - 1)a - a(F^2 - 1)\Big) = [(F^2 - 1), a], \tag{4.2}$$

$$\nabla\Big((F^2 - 1)\Big) = 0. \tag{4.3}$$

Notice that $\nabla^2(\xi) = [(F^2 - 1), \xi]$ for $\xi \in \Omega$. Hence we define the curvature $\theta$ to be $(F^2 - 1)$. Clearly, $\xi \in \Omega^n$ is of trace class if $n \geq p$. Here we need to choose $n$ to be even, $n = 2m$. Hence we can define the graded trace on $\Omega^n$ by $⨍\xi = m! Tr\, \gamma \xi$. The equality $Tr\, \gamma \nabla(\xi) = 0$ for $\xi \in \Omega^{n-1}$ follows from the relation

$$Tr\, \gamma \omega = \frac{1}{2} Tr\, \gamma F \nabla(\omega) - Tr\, \gamma (F^2 - 1)\omega$$

which holds for $\omega$ of trace class). Indeed, for $\xi \in \Omega^{n-1}$ $\nabla(\xi)$ is of trace class and

$$\begin{aligned} Tr\, \gamma \nabla(\xi) &= \frac{1}{2} Tr\, \gamma F \nabla^2(\xi) + Tr\, \gamma \nabla(\xi) \\ &= \frac{1}{2} Tr\, \gamma F [(F^2 - 1), \xi] - Tr\, \gamma (F^2 - 1)[F, \xi] = 0. \end{aligned} \tag{4.4}$$

Now we can apply the formula (2.4) to obtain a cyclic cocycle $\mathrm{Ch}_{2m}(F)$ in the cyclic bicomplex of the algebra $\mathcal{A}$. Its components $\mathrm{Ch}^k_{2m}(F)$ $k = 0, 2, 4, \dots, 2m$ are given by the formula

$$\mathrm{Ch}^k(F)(a_0, a_1, \dots a_k) = \frac{m!}{(m + \frac{k}{2})!} \sum_{i_0 + i_1 + \dots + i_k = m - \frac{k}{2}} Tr\, \gamma a_0 (1 - F^2)^{i_0} [F, a_1] (1 - F^2)^{i_1} \dots [F, a_k] (1 - F^2)^{i_k}. \tag{4.5}$$

Note that for the case when $F^2 = 1$ we get the formula from [4], normalized as in [6].

We will now associate the generalized chain with homotopy between Fredholm modules. If the two Fredholm modules $(\mathcal{H}, F_0, \gamma)$ and $(\mathcal{H}, F_1, \gamma)$ are connected by a smooth operator homotopy ( meaning that there exists a $C^1$ family $F_t$ of operators with $[F_t, a] \in \mathcal{L}^p$ and $(F_t^2 - 1) \in \mathcal{L}^{\frac{p}{2}}$, $t \in [0, 1]$ with $F_t|_{t=0} = F_0$, $F_t|_{t=1} = F_1$), this generalized chain will provide cobordism between cycles corresponding to the modules.

We start by constructing, exactly as before, an algebra $\Omega_t$ generated by the elements $a$, $[F_t, a]$, $(F_t^2 - 1)$, with the connection $\nabla_t$ and the curvature $\theta_t = (F_t^2 - 1)$. For each $t \in [0, 1]$ one constructs a natural representation $\pi_t$ of this algebra on the Hilbert space $\mathcal{H}$. Let $\Omega^*([0, 1])$ be the DGA of the differential forms on the interval $[0, 1]$ with the usual differential $d$. We can form a graded tensor product $\Omega^*([0, 1]) \widehat{\otimes} \Omega_t$. Choose an odd number $n = 2m + 1$ so that $n \geq p + 2$; if in addition we suppose that $\frac{dF_t}{dt} \in \mathcal{L}^p$, we can choose $n \geq p + 1$. In order to define the connection and the curvature we will have to adjoin to our algebra an element of degree 2 $dt \widehat{\otimes} \frac{dF_t}{dt}$ and an element of degree 3 $dt \widehat{\otimes} (F_t \frac{dF_t}{dt} + \frac{dF_t}{dt} F_t)$. The algebra with the adjoined elements will be denoted $\Omega_c$. The homomorphism $\rho_c : \mathcal{A} \to \Omega_c$ is given by $\rho_c(a) = 1 \widehat{\otimes} a$. We define the connection $\nabla_c$ as $\frac{d}{dt} \wedge dt + \nabla_t$, i.e. on the generators the definition is the following ($\beta \in \Omega^*([0, 1])$):

$$\nabla_c(\beta \widehat{\otimes} a) = d\beta \widehat{\otimes} a + (-1)^{\deg(\beta)} \beta \widehat{\otimes} [F_t, a], \tag{4.6}$$

$$\nabla_c(\beta \widehat{\otimes} [F_t, a]) = d\beta \widehat{\otimes} [F_t, a] + (-1)^{\deg(\beta)} \beta \widehat{\otimes} [(F_t^2 - 1), a] + \beta \wedge dt \widehat{\otimes} [\frac{dF_t}{dt}, a], \tag{4.7}$$

$$\nabla_c(\beta \widehat{\otimes} (F_t^2 - 1)) = d\beta \widehat{\otimes} (F_t^2 - 1) + \beta \wedge dt \widehat{\otimes} (F_t \frac{dF_t}{dt} + \frac{dF_t}{dt} F_t), \tag{4.8}$$

$$\nabla_c(dt \widehat{\otimes} \frac{dF_t}{dt}) = -dt \widehat{\otimes} (F_t \frac{dF_t}{dt} + \frac{dF_t}{dt} F_t), \tag{4.9}$$

$$\nabla_c(dt \widehat{\otimes} (F_t \frac{dF_t}{dt} + \frac{dF_t}{dt} F_t)) = dt \widehat{\otimes} [(F_t^2 - 1), \frac{dF_t}{dt}]. \tag{4.10}$$

The curvature $\theta_c$ of this connection is defined as

$$\theta_c = 1 \widehat{\otimes} (F_t^2 - 1) + dt \widehat{\otimes} \frac{dF_t}{dt}, \tag{4.11}$$

and the identity $(\nabla_c)^2 \cdot = [\theta_c, \cdot]$ is verified by computation. One then defines the graded trace $⨍_c$ on $(\Omega^*([0, 1]) \widehat{\otimes} \Omega_t)^n$ by the formula

$$⨍_c \beta \widehat{\otimes} \xi = \begin{cases} (-1)^{\deg(\xi)} m! \int_{[0,1]} \left(\beta \, Tr \, \gamma \pi_t(\xi)\right) & \text{if } \beta \in \Omega^1([0, 1]) \\ 0 & \text{if } \beta \in \Omega^0([0, 1]) \end{cases}.$$

The restriction maps $r_0 : \Omega_c \to \Omega_0$ and $r_1 : \Omega_c \to \Omega_1$ are defined as follows. $r_0(\beta \widehat{\otimes} \xi))$ is 0 if $\beta$ is of degree 1 , and $\beta(0)\xi_0$ where $\xi_0$ is obtained from $\xi$ by replacing $F_t$ by $F_0$ if $\beta$ is of degree 0 , and similarly for $r_1$. One can check that the map $r_1 \oplus r_0$ identifies $\partial \Omega_c$ with $\Omega^1 \oplus \widetilde{\Omega^0}$ and provides required cobordism.

Now we can use Theorem 2.1 to study the properties of $\mathrm{Ch}(F)$ with respect to the operator homotopy.

**Theorem 4.1.** *Suppose $(\mathcal{H}, F_0, \gamma)$ and $(\mathcal{H}, F_1, \gamma)$ are two finitely summable Fredholm modules over an algebra $\mathcal{A}$ which are connected by the smooth operator homotopy $F_t$ and $p$ is a number such that $[F_t] \in \mathcal{L}^p$ and $(F_t^2 - 1) \in \mathcal{L}^{\frac{p}{2}}$ for $0 \leq t \leq 1$. Choose $m$ such that $2m \geq p + 1$. Then $\mathrm{Ch}_{2m}(F_0) = \mathrm{Ch}_{2m}(F_1)$ in $HC^{2m}(\mathcal{A})$. If moreover $\frac{dF_t}{dt} \in \mathcal{L}^p$ one can choose $m$ such that $2m \geq p$.*

*Proof.* Let $Tch_{2m}^k$ denote the $k^{\text{th}}$ component of the character of the constructed above chain, providing the cobordism between the cycles associated with $(\mathcal{H}, F_0, \gamma)$ and $(\mathcal{H}, F_1, \gamma)$, $k = 1, 3, \dots, 2m+1$. It can be defined under the conditions on $m$ specified in the theorem. According to Theorem 2.1,

$$\mathrm{Ch}_{2m}(F_1) - \mathrm{Ch}_{2m}(F_0) = (b + B) \, Tch_{2m}.$$

Now,

$$Tch_{2m}^{2m+1}(a_0, a_1, \ldots a_{2m+1}) = \text{const} \fint_c \rho_c(a_0)\nabla_c(\rho_c(a_1)) \ldots \nabla_c(\rho_c(a_{2m+1})) = 0$$

(since the term under the $\fint_c$ does not contain $dt$). Hence $Tch_{2m}$ can be considered as the $2m-1$ chain (is in the image of $S$), and the result follows. $\square$

*Remark 4.1.* Suppose we have two Fredholm modules $(\mathcal{H}, F_0, \gamma)$ and $(\mathcal{H}, F_1, \gamma)$ such that $F_0 - F_1 \in \mathcal{L}^p$ and $F_i^2 - 1 \in \mathcal{L}^{\frac{p}{2}}$, $i = 0, 1$ . Then $\mathrm{Ch}_{2m}(F_0) = \mathrm{Ch}_{2m}(F_1)$, $2m \geq p$. Indeed, we can apply Theorem 4.1 to the linear homotopy $F_t = F_0 + t(F_1 - F_0)$, and need only to verify that $F_t^2 - 1 \in \mathcal{L}^{\frac{p}{2}}$. But

$$F_t^2 - 1 = (F_0^2 - 1) + t\big(F_0(F_1 - F_0) + (F_1 - F_0)F_0\big) + t^2(F_1 - F_0)^2.$$

The first and the last terms in the right-hand side are always in $\mathcal{L}^{\frac{p}{2}}$, and since the left-hand side is in $\mathcal{L}^{\frac{p}{2}}$ for $t = 1$, $\big(F_0(F_1 - F_0) + (F_1 - F_0)F_0\big) \in \mathcal{L}^{\frac{p}{2}}$.

**Corollary 4.2.** *Let $e$ be an idempotent in $M_N(\mathcal{A})$, and $(\mathcal{H}, F, \gamma)$ be an even Fredholm module over $\mathcal{A}$. Construct the Fredholm operator $F_e = e(F \otimes 1)e : \mathcal{H}^+ \otimes \mathbb{C}^N \to \mathcal{H}^- \otimes \mathbb{C}^N$ (where $\mathcal{H}^+$ and $\mathcal{H}^-$ are determined by the grading). Then*

$$\mathrm{index}(F_e) =< \mathrm{Ch}^*(F), \mathrm{Ch}_*(e) > .$$

*Here $\mathrm{Ch}_*(e)$ is the usual Chern character in the cyclic homology.*

*Proof.* By replacing $\mathcal{A}$ by $M_N(\mathcal{A})$ we reduce the situation to the case when $e \in \mathcal{A}$. Now we apply Connes' construction, which uses the homotopy $F_t = F + t(1-2e)[F, e]$ which connects $F$ (obtained when $t = 0$) with the operator $F_1 = eFe + (1-e)F(1-e)$, obtained when $t = 1$. Note that $1 - F_t^2 \in \mathcal{L}^{\frac{p}{2}}$. Indeed,

$$F_t^2 - 1 = \big(F^2 - 1\big) + \big(t(1-2e)[F, e]\big)^2 + t\big([F, (1-2e)[F, e]]\big).$$

The first two terms are clearly in $\mathcal{L}^{\frac{p}{2}}$. As for the third one, it can be rewritten as $-2[F, e][F, e] + (1-2e)[F, [F, e]] = -2[F, e]^2 + (1-2e)[(F^2 - 1), e] \in \mathcal{L}^{\frac{p}{2}}$.

The operator $F_1$ commutes with $e$, and homotopy does not change the pairing. Hence it is enough to prove the result in the case when $F$ and $e$ commute. In this case in the formula for the pairing all of the terms involving commutators are 0 , hence the only term with nonzero contribution is $\mathrm{Ch}^0(F)(e) = Tr\,\gamma e(1 - F^2)^m = Tr\,\gamma(e - (eFe)^2)^m = \mathrm{index}(F_e)$ by the well known formula. $\square$

In [4] Connes provides canonical construction, allowing one to associate with every $p$-summable Fredholm module such that $F^2 - 1 \neq 0$ another one for which $F^2 - 1 = 0$, and which defines the same $K$-homology class. This allows to reduce the definition of the character of a general Fredholm module to the case when $F^2 = 1$. The construction is the following. Given the Fredholm module $(\mathcal{H}, F, \gamma)$ one first constructs the Hilbert space $\widetilde{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}$ with the grading given by $\widetilde{\gamma} = \gamma \oplus (-\gamma)$. An element $a \in \mathcal{A}$ acts by $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. Then one constructs an operator $\widetilde{F}$, such that $\widetilde{F} - F' \in \mathcal{L}^p$ and $\widetilde{F}^2 = 1$; here by $F'$ we denote $\begin{pmatrix} F & 0 \\ 0 & -F \end{pmatrix}$. The character of the Fredholm module $(\mathcal{H}, F, \gamma)$ is then *defined* to be the character of the $(\widetilde{\mathcal{H}}, \widetilde{F}, \widetilde{\gamma})$.

**Theorem 4.3.** *Let* $(\mathcal{H}, F, \gamma)$ *be an even finitely summable Fredholm module over the algebra* $\mathcal{A}$*, and let* $p$ *be a real number such that* $[F, a] \in \mathcal{L}^p$ *and* $(F^2 - 1) \in \mathcal{L}^{\frac{p}{2}}$*. Then the class of* $\mathrm{Ch}^*(F)$ *defined in* (4.5) *in the periodic cyclic cohomology coincides with the Chern character, as defined by Connes [4].*

*Proof.* First, let us consider the Fredholm module $(\widetilde{\mathcal{H}}, F', \widetilde{\gamma})$ over the algebra $\widetilde{\mathcal{A}}$ - the algebra $\mathcal{A}$ with adjoined unit (acting by the identity operator). Then $\mathrm{Ch}_{2m}(F')$ defines a class in the cyclic cohomology of $\widetilde{\mathcal{A}}$, where we choose $2m \geq p$. Since $Tr\, \widetilde{\gamma}(1 - (F')^2)^m = 0$, it defines a class in the reduced cyclic cohomology of $\widetilde{\mathcal{A}}$, and hence in the cyclic cohomology of $\mathcal{A}$. It coincides with the class defined by the Fredholm module $(\mathcal{H}, F, \gamma)$.

Theorem 4.1 and Remark 4.1 show that the classes defined by the Fredholm modules $\mathrm{Ch}(\widetilde{F})$ and $\mathrm{Ch}(F')$ coincide. To finish the proof we note that $\mathrm{Ch}(\widetilde{F})$ coincides with the Chern character as defined in [4]. □

The proof of Theorem 4.1 also provides an explicit transgression formula. We just need to compute explicitly formula for

$$Tch^k_{2m}(a_0, a_1, \dots a_k) = \frac{(-1)^{m-\frac{k-1}{2}}(m)!}{(m+\frac{k+1}{2})!} \sum_{i_0+i_1+\dots+i_k=m-\frac{k-1}{2}} \fint_c \rho_c(a_0)\theta_c^{i_0}\nabla_c(\rho_c(a_1))\theta_c^{i_1}\dots\nabla_c(\rho_c(a_k))\theta_c^{i_k}. \tag{4.12}$$

Since $\theta_c^{i_l} = \sum_{r+q=i_l-1} dt\widehat{\otimes}(F_t^2-1)^r \frac{dF_t}{dt}(F_t^2-1)^q$ one can rewrite this formula as

$$\frac{(-1)^{m-\frac{k-1}{2}}(m)!}{(m+\frac{k+1}{2})!}\int_0^1\Big(\sum_{i_0+i_1+\dots+i_k=m-\frac{k-1}{2}}\sum_{l=0}^k\sum_{r+q=i_l-1}(-1)^l Tr\,\gamma a_0(F_t^2-1)^{i_0}$$
$$[F_t,a_1](F_t^2-1)^{i_1}\dots[F_t,a_l](F_t^2-1)^r\frac{dF_t}{dt}(F_t^2-1)^q\dots[F_t,a_k](F_t^2-1)^{i_k}\Big)dt. \tag{4.13}$$

Finally we can write the answer as

$$Tch^k_{2m}(a_0, a_1, \dots a_k) = -\frac{(m)!}{(m+\frac{k+1}{2})!}\int_0^1\Big(\sum_{i_0+\dots+i_k+i_{k+1}=m-\frac{k+1}{2}}\sum_{l=0}^k(-1)^l Tr\,\gamma a_0(F_t^2-1)^{i_0}$$
$$[F_t,a_1](1-F_t^2)^{i_1}\dots[F_t,a_l](1-F_t^2)^{i_l}\frac{dF_t}{dt}(1-F_t^2)^{i_{l+1}}\dots[F_t,a_k](1-F_t^2)^{i_{k+1}}\Big)dt, \tag{4.14}$$

where $k$ is an odd number between 1 and $2m-1$.

All the considerations above can be repeated in the case of an odd finitely summable Fredholm module $(\mathcal{H}, F)$ over an algebra $\mathcal{A}$ . Here as before we suppose that $[F, a] \in$

$\mathcal{L}^p$, $(F^2-1) \in \mathcal{L}^{\frac{p}{2}}$. We choose the number $m$ such that $n = 2m+1 \geq p$. The trace now is given by $⨍\xi = \sqrt{2i}\,\Gamma(n/2+1) Tr\,\xi$.

The corresponding Chern character $\mathrm{Ch}_{2m+1}(F)$ has components $\mathrm{Ch}^k_{2m+1}$ for $k = 1, 3, \ldots, 2m+1$, given by the formula

$$\mathrm{Ch}^k_{2m+1}(a_0, a_1, \ldots, a_k) = \frac{\Gamma(m+\frac{3}{2})\sqrt{2i}}{(m+\frac{k+1}{2})!} \sum_{i_0+i_1+\cdots+i_k = m-\frac{k-1}{2}} Tr\, a_0(1-F^2)^{i_0}[F, a_1](1-F^2)^{i_1}\ldots[F, a_k](1-F^2)^{i_k}. \tag{4.15}$$

If the two Fredholm modules are connected via the operator homotopy $F_t$ one has the transgression formula

$$\mathrm{Ch}_{2m+1}(F_1) - \mathrm{Ch}_{2m+1}(F_0) = (b+B)\, Tch_{2m+1}, \tag{4.16}$$

where $Tch_{2m+1}$ is a $2m$ cyclic cochain having components $Tch^k_{2m}$ for $k$ even between 0 and $2m$, given by the formula:

$$Tch^k_{2m+1}(a_0, a_1, \ldots a_k) = -\frac{\Gamma(m+\frac{3}{2})\sqrt{2i}}{(m+\frac{k}{2}+1)!}\int_0^1 \Big( \sum_{i_0+\cdots+i_k+i_{k+1} = m-\frac{k}{2}} \sum_{l=0}^{k} (-1)^l Tr\, a_0(F_t^2-1)^{i_0}$$
$$[F_t, a_1](1-F_t^2)^{i_1}\ldots[F_t, a_l](1-F_t^2)^{i_l}\frac{dF_t}{dt}(1-F_t^2)^{i_{l+1}}\ldots[F_t, a_k](1-F_t^2)^{i_{k+1}}\Big)dt. \tag{4.17}$$

The proof of Theorem 4.3 works in the odd situation as well and shows that $\mathrm{Ch}^*(F)$ coincides with the Chern character as defined by Connes. In particular, this allows to recover the spectral flow via the pairing with $K^{\text{th}}$eory. More precisely, let $u \in M_N(\mathcal{A})$ be a unitary. Let $\mathrm{sf}(F\otimes 1, (F\otimes 1)^u)$ be the spectral flow of the operators $F\otimes 1$ and $(F\otimes 1)^u = u((F\otimes 1)u^*$ acting on the space $\mathcal{H}\otimes\mathbb{C}^N$. The Chern character of the class of $u$ in $K_1(\mathcal{A})$ is the periodic cyclic cycle defined by

$$Ch_*(u) = \frac{1}{2\sqrt{2\pi i}}\sum_{l=1}^{\infty}(-1)^l(l-1)!\,tr\left((u\otimes u^{-1})^l - (u^{-1}\otimes u)^l\right). \tag{4.18}$$

Then we have the following

**Corollary 4.4.** *Let $u \in \mathcal{A}$ be a unitary, and $(\mathcal{H}, F)$ be an odd Fredholm module over the algebra $\mathcal{A}$. Then* $< \mathrm{Ch}^*(F), \mathrm{Ch}_*(u) >= \mathrm{sf}(F\otimes 1, (F\otimes 1)^u)$.

*Remark 4.2.* This is a finitely summable analogue of the result of Getzler [8]. In the finitely summable case analytic formula for the spectral flow was derived in [3]; use of Theorem 4.3 allows to give a proof of Corollary 4.4 without using this formula.

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## References

1. Berline, N., Getzler, E. and Vergne, M.: *Heat kernels and Dirac operators*. Volume **298** of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Berlin: Springer-Verlag, 1992
2. Bott, R.: On some formulas for the characteristic classes of group-actions. In: *Differential topology, foliations and Gelfand–Fuks cohomology (Proc. Sympos., Pontifícia Univ. Católica, Rio de Janeiro, 1976)*. Volume **652** of *Lecture Notes in Math.*, Berlin: Springer, 1978, pp. 25–61
3. Carey, A. and Phillips, J.: Unbounded Fredholm modules and spectral flow. Canad. J. Math. **50** (4), 673–718 (1998)
4. Connes, A.: Non-commutative differential geometry. Publ. Math. IHES **62**, 257–360 (1985)
5. Connes, A.: Cyclic cohomology and the transverse fundamental class of a foliation. In: *Geometric methods in operator algebras (Kyoto, 1983)*. Volume **123** of *Pitman Res. Notes Math. Ser.* Harlow: Longman Sci. Tech., 1986, pp. 52–144
6. Connes, A.: *Noncommutative Geometry*. London–New York: Academic Press, 1994
7. Feĭgin, B.L. and Tsygan, B.L.: Cyclic homology of algebras with quadratic relations, universal enveloping algebras and group algebras. In: *$K$-theory, arithmetic and geometry (Moscow, 1984–1986)*, Berlin: Springer, 1987, pp. 210–239
8. Getzler, E.: The odd Chern character in cyclic homology and spectral flow. Topology **32** (3), 489–507 (1993)
9. Getzler, E. and Szenes, A.: On the Chern character of a theta-summable Fredholm module. J. Funct. Anal. **84** (2), 343–357 (1989)
10. Jaffe, A., Lesniewski, A. and Osterwalder, K.: Quantum $K$-theory. I. The Chern character. Commun. Math. Phys. **118** (1), 1–14 (1988)
11. Loday, J.-L.: *Cyclic homology*. Volume **301** of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Berlin: Springer-Verlag, 1992
12. Nest, R. and Tsygan, B.: Algebraic index theorem. Commun. Math. Phys. **172** (2), 223–262 (1995)
13. Nest, R. and Tsygan, B.: Algebraic index theorem for families. Adv. Math. **113** (2), 151–205 (1995)
14. Nistor, V.: Group cohomology and the cyclic cohomology of crossed products. Invent. Math. **99** (2), 411–424 (1990)
15. Nistor, V.: Super-connections and non-commutative geometry. In: *Cyclic cohomology and noncommutative geometry (Waterloo, ON, 1995)*. Volume 17 of Fields Inst. Commun. Providence, RI: Amer. Math. Soc., 1997, pp. 115–136
16. Quillen, D.: Algebra cochains and cyclic cohomology. Inst. Hautes Études Sci. Publ. Math. **68**, 139–174 (1989)

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