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Random Unitary Matrices, Permutations and Painlevé

Craig A. Tracy[1], **Harold Widom**[2]

[1] Department of Mathematics and Institute of Theoretical Dynamics, University of California, Davis, CA 95616, USA. E-mail: tracy@itd.ucdavis.edu
[2] Department of Mathematics, University of California, Santa Cruz, CA 95064, USA. E-mail: widom@math.ucsc.edu

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Abstract: This paper is concerned with certain connections between the ensemble of $n \times n$ unitary matrices – specifically the characteristic function of the random variable $\mathrm{tr}(U)$ – and combinatorics – specifically Ulam's problem concerning the distribution of the length of the longest increasing subsequence in permutation groups – and the appearance of Painlevé functions in the answers to apparently unrelated questions. Among the results is a representation in terms of a Painlevé V function for the characteristic function of $\mathrm{tr}(U)$ and (using recent results of Baik, Deift and Johansson) an expression in terms of a Painlevé II function for the limiting distribution of the length of the longest increasing subsequence in the hyperoctahedral groups.

1. Introduction

The characteristic function of the random variable $\mathrm{tr}\, U$, where U belongs to the ensemble $\mathcal{U}(n)$ of $n \times n$ unitary matrices with Haar measure, is the expected value

$$E_n(e^{r\,\mathrm{tr}\,U + s\,\overline{\mathrm{tr}\,U}}). \tag{1.1}$$

In $\mathcal{U}(n)$ we have for any function g with Fourier coefficients g_k,

$$E_n\left(\prod_{j=1}^{n} g(e^{i\theta_j})\right) = \det\, T_n(g), \tag{1.2}$$

where $T_n(g)$ is the associated $n \times n$ Toeplitz matrix defined by

$$T_n(g) = (g_{j-k}), \quad (j, k = 0, \cdots, n-1).$$

It follows that the distribution function (1.1) equals the determinant of the Toeplitz matrix associated with the function $e^{rz+sz^{-1}}$. The determinant, which we denote by D_n, is a

function of the product rs (see Sect. 2) and so it is completely determined by its values when $r = s = t$. This function $D_n(t)$ has connections with both integrable systems and combinatorial theory. To state our results, and these connections, we introduce some notation.

We set

$$f(z) = e^{t\,(z+z^{-1})},$$

so that $D_n(t) = \det\, T_n(f)$. Notice that $T_n(f)$ is symmetric since $f(z^{-1}) = f(z)$. We introduce the n-vectors

$$\delta^+ = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \delta^- = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad f^+ = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{n-1} \\ f_n \end{pmatrix}, \quad f^- = \begin{pmatrix} f_n \\ f_{n-1} \\ \vdots \\ f_2 \\ f_1 \end{pmatrix},$$

and define

$$U_n = \left(T_n(f)^{-1} f^+, \delta^-\right) = \left(T_n(f)^{-1} f^-, \delta^+\right).$$

If we set

$$\Phi_n = 1 - U_n^2,$$

then Φ_n as a function of t satisfies the equation

$$\Phi_n'' = \frac{1}{2}\Big(\frac{1}{\Phi_n - 1} + \frac{1}{\Phi_n}\Big)(\Phi_n')^2 - \frac{1}{t}\Phi_n' - 8\Phi_n(\Phi_n - 1) + 2\frac{n^2}{t^2}\frac{\Phi_n - 1}{\Phi_n}, \tag{1.3}$$

which is a variant of the Painlevé V equation[1], and in terms of it D_n has the representation

$$D_n(t) = \exp\left(4\int_0^t \log(t/\tau)\,\tau\,\Phi_n(\tau)\,d\tau\right). \tag{1.4}$$

This is reminiscent of the many representations now in the literature for Fredholm determinants in terms of Painlevé functions. We shall also show that

$$W_n = U_n/U_{n-1}$$

satisfies

$$W_n'' = \frac{1}{W_n}\,(W_n')^2 - \frac{1}{t}\,W_n' + 4\,\frac{n-1}{t}\,W_n^2 - \frac{4n}{t} + 4\,W_n^3 - \frac{4}{W_n}, \tag{1.5}$$

which is a special case of the Painlevé III equation.

An important ingredient in the proofs is the following recurrence relation satisfied by the U_n:

$$\frac{n}{t}\,U_n + (1 - U_n^2)\,(U_{n-1} + U_{n+1}) = 0. \tag{1.6}$$

[1] The substitution $t^2 = z$ and $\Phi_n = w/(w-1)$ transforms (1.3) to the standard form of P_V with parameters $\alpha = 0$, $\beta = -n^2/2$, $\gamma = 2$ and $\delta = 0$.

We shall see that this recurrence formula, sometimes known as the discrete Painlevé II equation (see, e.g. [6]), is equivalent to one first shown to hold for $\mathcal{U}(n)$ by V. Periwal and D. Shevitz [9]. It was rediscovered by M. Hisakado [4], who also derived an equation equivalent to (1.3) and observed that this was one of the Painlevé V equations which, by results of K. Okamoto [8], is reducible to Painlevé III. Carrying through the Okamoto procedure is what led to our W_n, although the proof we give is direct. Our derivations of (1.3) and (1.6) are different from those in [4] and perhaps more down-to-earth since we use only the simplest properties of Toeplitz matrices and some linear algebra. (They cannot be entirely unrelated, though, since the orthogonal polynomials which are central to the argument of [4] can be defined in terms of Toeplitz matrices.)

A remarkable connection between $\mathcal{U}(n)$ and combinatorics was discovered by Gessel [3]. Place the uniform measure on the symmetric group $\mathcal{S}_N$, denote by $\ell_N(\sigma)$ the length of the longest increasing subsequence in σ, and define f_{Nn} by

$$\text{Prob}\,(\ell_N(\sigma) \le n) = \frac{f_{Nn}}{N!}.$$

Then $D_n(t)$ is a generating function for the f_{Nn}.[2] In fact

$$D_n(t) = \sum_{N \ge 0} f_{Nn} \frac{t^{2N}}{(N!)^2}. \tag{1.7}$$

Recently, E. Rains [10] gave an elegant proof that

$$f_{Nn} = E_n\left(|\text{tr}(U)|^{2N}\right), \tag{1.8}$$

which can be shown to be equivalent to (1.7) by a simple argument. Using the relationship (1.7) a sharp asymptotic result for the distribution function of the random variable $\ell_N(\sigma)$ was recently obtained by J. Baik, P. Deift and K. Johansson [1]. And at the same time they discovered yet another connection with Painlevé.

Their main result, which was quite difficult, was an asymptotic formula for $D_n(t)$ which we now describe. Introduce another parameter s and suppose that n and t are related by $n = 2t + s\,t^{1/3}$. Then as $t \to \infty$ with fixed s one has

$$\lim_{t\to\infty} e^{-t^2}\, D_{2t+s\,t^{1/3}}(t) = F(s). \tag{1.9}$$

Here F is the distribution function defined by

$$F(s) = \exp\left(-\int_s^\infty (x-s)\,q(x)^2\,dx\right), \tag{1.10}$$

where q is the solution of the Painlevé II equation

$$q'' = sq + 2q^3 \tag{1.11}$$

[2] Gessel in [3] does not write down the symbol of the Toeplitz matrix, nor does he mention random matrices. But in light of the well-known formula (1.2) and the subsequent work of Odlyzko et al. [7] and Rains [10], we believe it is fair to say that the connection with random matrix theory begins with the discovery of (1.7).

satisfying $q(s) \sim \mathrm{Ai(s)}$ as $s \to \infty$. (For a proof that such a solution exists, see, e.g. [2].) Using a "de-Poissonization" lemma due to Johansson [5] these asymptotics for the generating function $D_n(t)$ led to the asymptotic formula

$$\lim_{N\to\infty} \mathrm{Prob}\left(\frac{\ell_N(\sigma) - 2\sqrt{N}}{N^{1/6}} \le s\right) = F(s) \tag{1.12}$$

for the distribution function of the normalized random variable $(\ell_N(\sigma) - 2\sqrt{N})/N^{1/6}$.

It is a remarkable fact that this same distribution function F was first encountered in random matrix theory where it gives the limiting distribution for the normalized largest eigenvalue in the Gaussian Unitary Ensemble of Hermitian matrices. More precisely, we have for this ensemble [11],

$$\lim_{N\to\infty} \mathrm{Prob}\left(\left(\lambda_{\max}(N) - \sqrt{2N}\right)\sqrt{2}\,N^{1/6} \le s\right) = F(s).$$

In connection with these results just described, we shall do two things. We show, first, how one might have guessed the asymptotics (1.12). More precisely, we present a simple argument that if there is any limit theorem of this type, with $F(s)$ some distribution function and with some power N^α replacing $N^{1/6}$, then necessarily $\alpha = 1/6$ and F is given by (1.10) with q a solution to (1.11). (The boundary condition on q, however, cannot be anticipated.) This conclusion is arrived at by considering the implications of (1.9) with $t^{1/3}$ replaced by $t^{2\alpha}$ for the recurrence formula (1.6).

Secondly, we derive analogues of (1.8) and (1.7) for the subgroup $\mathcal{O}_N$ of "odd" permutations of $\mathcal{S}_N$.[3] These are described as follows: if $N = 2k$ think of $\mathcal{S}_N$ as acting on the integers from $-k$ to k excluding 0, and if $N = 2k+1$ think of $\mathcal{S}_N$ as acting on the integers from $-k$ to k including 0. In both cases $\sigma \in \mathcal{S}_N$ is called *odd* if $\sigma(-n) = -\sigma(n)$ for all n. The number of elements in the subgroup $\mathcal{O}_N$ of odd permutations equals $2^k\,k!$ in both cases. Therefore if b_{Nn} equals the number of permutations in $\mathcal{O}_N$ having no increasing subsequence of length greater than n,

$$\mathrm{Prob}(\ell_N(\sigma) \le n) = \frac{b_{Nn}}{2^k\,k!}, \tag{1.13}$$

where the uniform measure is placed on $\mathcal{O}_N$. Rains [10] proved identities analogous to (1.8) for these probabilities. Using these we are able to find representations for the two generating functions

$$G_n(t) = \sum_{k\ge 0} b_{2k\,n}\frac{t^{2k}}{(k!)^2}, \quad H_n(t) = \sum_{k\ge 0} b_{2k+1\,n}\frac{t^{2k}}{(k!)^2}, \tag{1.14}$$

analogous to the representation (1.7). (See Theorem 1 below.) The same determinants $D_n(t)$ arise as before but in the representation for $H_n(t)$, whose derivation uses the machinery developed in earlier sections, the quantities U_n also appear. Once the representations are established we can use (1.9) and Johansson's lemma to deduce the

[3] Our terminology for $\mathcal{O}_N$ is not standard. For $N = 2k$ one usually denotes $\mathcal{O}_N$ by $\mathcal{B}_k$, the hyperoctahedral group of order $2^k k!$ which is the centralizer of the reversal permutation in $\mathcal{S}_N$. Elements of $\mathcal{B}_k$ are commonly called *signed permutations*. Similar remarks hold for $N = 2k+1$.

Table 1. The mean (μ), standard deviation (σ), skewness (S) and kurtosis (K) of F and $F_{\mathcal{O}} := F^2$

Distr	μ	σ	S	K
F	−1.77109	0.9018	0.224	0.093
$F_{\mathcal{O}}$	−1.26332	0.7789	0.329	0.225

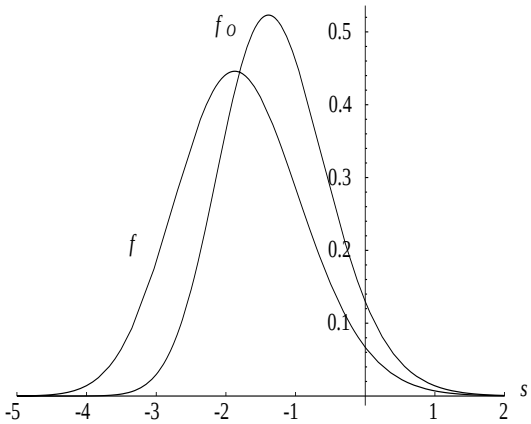

Fig. 1. The probability densities $f = dF/ds$ and $f_{\mathcal{O}} = dF_{\mathcal{O}}/ds$

asymptotics of (1.13). We show that as $N \to \infty$ we have for fixed s,

$$\text{Prob}\left(\frac{\ell_N(\sigma) - 2\sqrt{N}}{2^{2/3}N^{1/6}} \le s\right) \to F(s)^2, \tag{1.15}$$

where $F(s)$ is as in (1.12).

In Table 1 we give some statistics of the distribution functions F and $F_{\mathcal{O}} := F^2$. In Fig. 1 we graph their densities.

2. The Integral Representation for D_n

We write

$$\Lambda = T_n(z^{-1}), \quad \Lambda' = T_n(z).$$

Thus Λ is the backward shift and Λ' is the forward shift. It is easy to see that

$$T_n(z^{-1} f) = T_n(f)\,\Lambda + f^+ \otimes \delta^+ = \Lambda\, T_n(f) + \delta^- \otimes f^-, \tag{2.1}$$

$$T_n(z\, f) = T_n(f)\,\Lambda' + f^- \otimes \delta^- = \Lambda'\, T_n(f) + \delta^+ \otimes f^+, \tag{2.2}$$

where $\delta^{\pm}$ and $f^{\pm}$ were defined above and $a \otimes b$ denotes the matrix with j, k entry $a_j\, b_k$. Relation (2.1) holds for any f but (2.2) uses the fact that $f_{-k} = f_k$.

To derive (1.4) we temporarily reintroduce variables r and s and set $f(z) = e^{rz+sz^{-1}}$, so D_n and T_n are functions of r and s. Of course we are interested in $D_n(t, t)$. We shall compute

$$\partial_{r,s}^2 \log D_n(r, s)\Big|_{r=s=t}$$

in two different ways.

Using the fact that

$$\partial_s \log D_n(r, s) = \operatorname{tr} T_n(f)^{-1} T_n(\partial_s f) = \operatorname{tr} T_n(f)^{-1} T_n(z^{-1} f),$$

then differentiating with respect to r, we find that

$$\begin{aligned}\partial_{r,s}^2 \log D_n(r, s) &= \operatorname{tr} [T_n(f)^{-1} T_n(f) - T_n(f)^{-1} T_n(z\, f)\, T_n(f)^{-1} T_n(z^{-1} f)] \\ &= \operatorname{tr} [I - T_n(f)^{-1} T_n(z\, f)\, T_n(f)^{-1} T_n(z^{-1} f)].\end{aligned}$$

We now set $r = s = t$. Since f is now as it was, we can use (2.1) and (2.2). If we multiply their first equalities on the left by $T_n(f)^{-1}$ and use the notation $u^{\pm} = T_n(f)^{-1} f^{\pm}$ we obtain

$$T_n(f)^{-1} T_n(z^{-1} f) = \Lambda + u^+ \otimes \delta^+, \quad T_n(f)^{-1} T_n(z\, f) = \Lambda' + u^- \otimes \delta^-.$$

Hence the last trace equals that of

$$\begin{aligned}&I - (\Lambda' + u^- \otimes \delta^-)(\Lambda + u^+ \otimes \delta^+) \\ &= I - \Lambda'\Lambda - u^- \otimes \Lambda'\delta^- - \Lambda' u^+ \otimes \delta^+ - (\delta^-, u^+)\, u^- \otimes \delta^+.\end{aligned}$$

The trace of $I - \Lambda'\Lambda$ equals 1, and

$$\Lambda'\delta^- = 0, \quad \operatorname{tr} \Lambda' u^+ \otimes \delta^+ = (\Lambda' u^+, \delta^+) = (u^+, \Lambda\delta^+) = 0,$$

so we have

$$\partial_{r,s}^2 \log\, D_n(r, s)\Big|_{r=s=t} = 1 - (\delta^-, u^+)\,(u^-, \delta^+).$$

But $(u^-, \delta^+) = (\delta^-, u^+) = (\delta^-, T_n(f)^{-1} f^+) = U_n$. Therefore

$$\partial_{r,s}^2 \log\, D_n(r, s)\Big|_{r=s=t} = 1 - U_n^2 = \Phi_n. \tag{2.3}$$

Now let us go back to general r and s. For any $p > 0$ Cauchy's theorem tells us that the j, k entry of $T_n(f)$ equals

$$\frac{1}{2\pi i} \int_{|z|=p} e^{t(rz+sz^{-1})}\, z^{-(j-k+1)} dz = p^{-j} \frac{1}{2\pi i} \int_{|z|=1} e^{t(prz+p^{-1}sz^{-1})}\, z^{-(j-k+1)} dz p^k.$$

It follows that $D_n(r, s) = D_n(pr, s/p)$, and by analytic continuation this holds for any (complex) p. Setting $p = \sqrt{s/r}$ we see that

$$D_n(r, s) = D_n(\sqrt{rs}, \sqrt{rs}).$$

It follows that $D_n(r, s)$ is a function of the product rs, as stated in the Introduction, and that

$$4\,\partial_{r,s}^2\,\log D_n(r,s)\Big|_{r=s=t} = \frac{d^2}{dt^2}\log D_n(t,t) + \frac{1}{t}\,\frac{d}{dt}\log D_n(t,t).$$

Comparing this with (2.3) we see that we have shown

$$\frac{d^2}{dt^2}\log D_n(t,t) + \frac{1}{t}\,\frac{d}{dt}\log D_n(t,t) = 4\,\Phi_n.$$

This gives the representation (1.4).

Of course it remains to show that this Φ_n satisfies (1.3). We do this by first finding a formula for dU_n/dt and then finding relations among the various quantities which occur for different values of n.

3. Differentiation

In addition to $u^{\pm} = T_n(f)^{-1} f^{\pm}$, we introduce $v^{\pm} = T_n(f)^{-1}\delta^{\pm}$ and we compute some derivatives with respect to t. First,

$$\frac{d}{dt}T_n(f) = T_n((z+z^{-1})f) = T_n(f)\,(\Lambda + \Lambda') + f^+ \otimes \delta^+ + f^- \otimes \delta^-$$

by the first equalities of (2.1) and (2.2), so

$$\begin{aligned} dt\,T_n(f)^{-1} &= -T_n(f)^{-1}\,\frac{dT_n(f)}{dt}\,T_n(f)^{-1} \\ &= -(\Lambda + \Lambda')\,T_n(f)^{-1} - u^+ \otimes v^+ - u^- \otimes v^-. \end{aligned} \tag{3.1}$$

Next,

$$\frac{df^+}{dt} = \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{n-2} \\ f_{n-1} \end{pmatrix} + \begin{pmatrix} f_2 \\ f_3 \\ \vdots \\ f_n \\ f_{n+1} \end{pmatrix} = T_n(f)\delta^+ + \Lambda f^+ + f_{n+1}\,\delta^-.$$

Hence

$$T_n(f)^{-1}\,\frac{df^+}{dt} = \delta^+ + T_n(f)^{-1}\Lambda f^+ + f_{n+1}\,v^-. \tag{3.2}$$

Multiplying the second equality of (2.1) left and right by $T_n(f)^{-1}$ gives

$$T_n(f)^{-1}\,\Lambda = \Lambda\,T_n(f)^{-1} + u^+ \otimes v^+ - v^- \otimes u^-. \tag{3.3}$$

Therefore

$$T_n(f)^{-1}\Lambda f^+ = \Lambda\,u^+ + u^+\,(v^+, f^+) - v^-\,(u^-, f^+),$$

and substituting this into (3.2) gives

$$T_n(f)^{-1}\frac{df^+}{dt} = \delta^+ + \Lambda u^+ + u^+(v^+, f^+) + v^-\left(f_{n+1} - (u^-, f^+)\right).$$

Adding this to (3.1) applied to f^+ gives

$$\frac{du^+}{dt} = \delta^+ - \Lambda' u^+ + v^-(f_{n+1} - (u^-, f^+)) - u^-(v^-, f^+).$$

Taking inner products with δ^- in the last displayed formula we obtain (recall the definition of U_n)

$$\frac{dU_n}{dt} = -(\Lambda' u^+, \delta^-) + (v^+, \delta^+)(f_{n+1} - (u^-, f^+)) - (u^+, \delta^+)(v^+, f^-).$$

We used the fact, which follows from the symmetry of $T_n(f)$, that all our inner products whose entries have signs as superscripts are unchanged if both signs are reversed.

To find $(\Lambda' u^+, \delta^-)$, which is the same as $(\Lambda u^-, \delta^+)$, we observe that

$$\Lambda f^- = \begin{pmatrix} f_n \\ f_{n-1} \\ \vdots \\ f_1 \\ 0 \end{pmatrix} = T_n(f)\,\delta^- - f_0\,\delta^-.$$

Applying (3.3) to f^- therefore gives

$$\delta^- - f_0 v^- = \Lambda u^- + u^+(v^+, f^-) - v^-(u^-, f^-).$$

Hence

$$-(\Lambda' u^+, \delta^-) = -(\Lambda u^-, \delta^+) = (u^+, \delta^+)(v^+, f^-) + (v^-, \delta^+)(f_0 - (u^+, f^+)).$$

Thus we have established the differentiation formula

$$\frac{dU_n}{dt} = (v^-, \delta^+)\left(f_0 - (u^+, f^+)\right) + (v^+, \delta^+)\left(f_{n+1} - (u^-, f^+)\right). \tag{3.4}$$

4. Relations

New quantities appearing in the differentiation formula are

$$V_n^{\pm} = (v^{\pm}, \delta^+).$$

There are others but we shall see that they may be expressed in terms of these (with different values of n), as indeed so will U_n. We obtain our relations through several applications of the following formula for the inverse of a 2×2 block matrix:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & \times \\ \times & \times \end{pmatrix}. \tag{4.1}$$

Here we assume A and D are square and the various inverses exist. Only one block of the inverse is displayed and the formula shows that $A - BD^{-1}C$ equals the inverse of this block of the inverse matrix. At first all that will be used about f is that $T_n(f)$ is symmetric. (There are modifications which hold in general.)

We apply (4.1) first to the $(n+1) \times (n+1)$ matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 1 \\ f_1 & f_0 & \cdots & f_{n-1} \\ \vdots & \vdots & \cdots & \vdots \\ f_n & f_{n-1} & \cdots & f_0 \end{pmatrix},$$

with $A = (0)$, $D = T_n(f)$, $B = (0 \ \cdots \ 0 \ 1)$, $C = f^+$. In this case $A - BD^{-1}C = -(T_n(f)^{-1} f^+, \delta^-) = -U_n$. This equals the reciprocal of the upper-left entry of the inverse matrix, which in turn equals $(-1)^n$ times the lower-left $n \times n$ subdeterminant divided by D_n. Replacing the first row by $(f_0 \ f_1 \ \cdots \ f_n)$ gives the matrix

$$\begin{pmatrix} f_0 & f_1 & \cdots & f_n \\ f_1 & f_0 & \cdots & f_{n-1} \\ \vdots & \vdots & \cdots & \vdots \\ f_n & f_{n-1} & \cdots & f_0 \end{pmatrix} = T_{n+1}(f).$$

The upper-right entry of its inverse equals on the one hand V^-_{n+1} and on the other hand $(-1)^n$ times the same subdeterminant as arose above divided by D_{n+1}. This gives the identity

$$-U_n = V^-_{n+1} \frac{D_{n+1}}{D_n} = \frac{V^-_{n+1}}{V^+_{n+1}}. \tag{4.2}$$

(If we consider the polynomials on the circle which are orthonormal with respect to the weight function f then the right side above is equal to the constant term divided by the highest coefficient in the polynomial of degree n. Therefore our $-U_n$ equals the S_{n-1} of [4].)

If we now take A to be the upper-left corner of $T_{n+1}(f)$ and D the complementary $T_n(f)$, then $C = f^+$ and B is its transpose, and we deduce that

$$f_0 - (u^+, f^+) = \frac{1}{V^+_{n+1}}. \tag{4.3}$$

To evaluate $f_{n+1} - (u^-, f^+)$, the other odd coefficient appearing in (3.4), we consider

$$\begin{pmatrix} f_0 & f_1 & \cdots & f_n & f_{n+1} \\ f_1 & f_0 & \cdots & f_{n-1} & f_n \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ f_n & f_{n-1} & \cdots & f_0 & f_1 \\ f_{n+1} & f_n & \cdots & f_1 & f_0 \end{pmatrix} = T_{n+2}(f).$$

We apply to this an obvious modification of (4.1), where A is the 2×2 matrix consisting of the four corners of the large matrix, D is the central $T_n(f)$, C consists of the two columns f^+ and f^- and B consists of the rows which are their transposes. Then

$$A - BD^{-1}C = \begin{pmatrix} f_0 - (u^+, f^+) & f_{n+1} - (u^-, f^+) \\ f_{n+1} - (u^-, f^+) & f_0 - (u^+, f^+) \end{pmatrix},$$

and our formula tells us that this is the inverse of

$$\begin{pmatrix} V_{n+2}^+ & V_{n+2}^- \\ V_{n+2}^- & V_{n+2}^+ \end{pmatrix}.$$

This gives the two formulas

$$f_0 - (u^+, f^+) = \frac{V_{n+2}^+}{{V_{n+2}^+}^2 - {V_{n+2}^-}^2}, \quad f_{n+1} - (u^-, f^+) = \frac{-V_{n+2}^-}{{V_{n+2}^+}^2 - {V_{n+2}^-}^2}.$$

Comparing the first with (4.3) we see that

$${V_{n+2}^+}^2 - {V_{n+2}^-}^2 = V_{n+1}^+ V_{n+2}^+, \tag{4.4}$$

and therefore that the preceding relations can be written

$$f_0 - (u^+, f^+) = \frac{1}{V_{n+1}^+}, \quad f_{n+1} - (u^-, f^+) = -\frac{1}{V_{n+1}^+} \frac{V_{n+2}^-}{V_{n+2}^+}. \tag{4.5}$$

Notice that (4.2) and (4.4) give

$$1 - U_n^2 = \frac{V_n^+}{V_{n+1}^+}. \tag{4.6}$$

This is our Φ_n.

The relations we obtained so far in this section are completely general. The recurrence (1.6), however, depends on our specific function f. Integration by parts gives

$$k\, f_k = \frac{t}{2\pi i} \int (z - z^{-1})\, e^{t\,(z+z^{-1})}\, z^{-k-1}\, dz = t\,(f_{k-1} - f_{k+1}). \tag{4.7}$$

Hence if $M = \text{diag}\,(1\ 2\ \cdots\ n)$ we have

$$M\, T_n(f) - T_n(f)\, M = t\, T_n((z - z^{-1})\, f),$$

and by the first identities of (2.1) and (2.2) this equals

$$t\left[T_n(f)\,(\Lambda' - \Lambda) + f^- \otimes \delta^- - f^+ \otimes \delta^+\right],$$

so

$$T_n(f)^{-1}\, M - M\, T_n(f)^{-1} = t\left[(\Lambda' - \Lambda)\, T_n(f)^{-1} + u^- \otimes v^- - u^+ \otimes v^+\right].$$

Applying this to δ^- gives

$$n\, v^- - M\, v^- = t\Big[(\Lambda' - \Lambda)\, v^- + u^-\,(v^-, \delta^-) - u^+\,(v^+, \delta^-)\Big]. \tag{4.8}$$

Now (4.7) says

$$M\, f^+ = t \begin{pmatrix} f_0 - f_2 \\ f_1 - f_3 \\ \vdots \\ f_{n-2} - f_n \\ f_{n-1} - f_{n+1} \end{pmatrix}$$

whereas (this is relevant since the transpose of $\Lambda' - \Lambda$ is $\Lambda - \Lambda'$)

$$(\Lambda - \Lambda')\, f^+ = \begin{pmatrix} f_2 \\ f_3 - f_1 \\ \vdots \\ f_n - f_{n-2} \\ -f_{n-1} \end{pmatrix}.$$

Therefore

$$M\, f^+ + t\,(\Lambda - \Lambda')\, f^+ = t\,(f_0\, \delta^+ - f_{n+1}\, \delta^-),$$

and so taking inner products with f^+ in (4.8) gives

$$\frac{n}{t}(v^-, f^+) = f_0(v^-, \delta^+) - f_{n+1}(v^-, \delta^-) + (u^-, f^+)(v^-, \delta^-) - (u^+, f^+)(v^+, \delta^-),$$

or equivalently, since $(v^-, f^+) = U_n$,

$$\frac{n}{t}\, U_n = \big(f_0 - (u^+, f^+)\big)\; V_n^- - \big(f_{n+1} - (u^-, f^+)\big)\; V_n^+.$$

Using (4.5) we rewrite this as

$$\frac{n}{t}\, U_n = \frac{V_n^-}{V_{n+1}^+} + \frac{V_n^+}{V_{n+1}^+}\,\frac{V_{n+2}^-}{V_{n+2}^+} = \frac{V_n^+}{V_{n+1}^+}\,\frac{V_n^-}{V_n^+} + \frac{V_n^+}{V_{n+1}^+}\,\frac{V_{n+2}^-}{V_{n+2}^+}.$$

Using this, (4.6) and (4.2) we arrive at (1.6).

5. Painlevé V and Painlevé III

We first show that Φ_n satisfies (1.3). Our formula (3.4) for dU_n/dt can now be written

$$\begin{aligned}\frac{dU_n}{dt} &= \frac{V_n^-}{V_{n+1}^+} - \frac{V_n^+}{V_{n+1}^+}\,\frac{V_{n+2}^-}{V_{n+2}^+} = \frac{V_n^+}{V_{n+1}^+}\left(\frac{V_n^-}{V_n^+} - \frac{V_{n+2}^-}{V_{n+2}^+}\right) \\ &= -(1-U_n^2)\,(U_{n-1}-U_{n+1}),\end{aligned} \tag{5.1}$$

by (4.5), (4.6) and (4.2). Adding and subtracting (1.6) gives us the two formulas

$$\frac{dU_n}{dt} = \frac{n}{t}\,U_n + 2\,U_{n+1}\,(1-U_n^2), \tag{5.2}$$

$$\frac{dU_n}{dt} = -\frac{n}{t}\,U_n - 2\,U_{n-1}\,(1-U_n^2). \tag{5.3}$$

These are Eqs. (4.5) and (4.6) of [4]. As was done there, we solve (5.2) for U_{n+1} in terms of U_n and dU_n/dt and substitute this into (5.3) with n replaced by $n+1$. We get a second-order differential equation for U_n which is equivalent to Eq. (1.3) for $\Phi_n = 1-U_n^2$.

Next we show that $W_n = U_n/U_{n-1}$ satisfies (1.5). In computing the derivative of W_n we use (5.3) to compute the derivative of U_n and (5.2) with n replaced by $n-1$ to compute the derivative of U_{n-1}. We get

$$W_n' = -\frac{2n-1}{t}\,W_n - 2 + 4\,U_n^2 - 2\,W_n^2. \tag{5.4}$$

Using (5.3) once again we compute

$$(U_n^2)' = 2\,U_n\,\left(-\frac{n}{t}\,U_n - 2\,U_{n-1}\,(1-U_n^2)\right) = -2\,\frac{n}{t}\,U_n^2 - 4\,\frac{U_n^2\,(1-U_n^2)}{W_n}.$$

Differentiating (5.4) and using this expression for $(U_n^2)'$ we obtain a formula for W_n'' in terms of W_n, W_n' and U_n^2. Then we solve (5.4) for U_n^2 in terms of W_n and W_n'. Substituting this into the formula for W_n'' gives (1.5).

In order to specify the solutions of the Eqs. (1.3) and (1.5) we must determine the initial conditions at $t=0$. Clearly $\Phi_n(0)=1$, but this does not determine Φ_n uniquely. One can see that $\Phi_n^{(k)}(0) = 0$ for $k < 2n$ and that what determines Φ_n uniquely is $\Phi_n^{(2n)}(0)$. We shall show that

$$\Phi_n^{(2n)}(0) = -\frac{(2n)!}{n!^2}. \tag{5.5}$$

By (4.2), $U_n = V_{n+1}^-/V_{n+1}^+$. Now V_{n+1}^+ is the upper-left corner of $T_{n+1}(f)^{-1}$ and so tends to 1 as $t \to 0$. So let us see how V_{n+1}^-, which is the upper-right corner of $T_{n+1}(f)^{-1}$, behaves. More exactly, let us find the term in its expansion with the lowest power of t.

We have

$$e^{2t\cos\theta} = \sum_k \frac{t^k}{k!}(e^{i\theta}+e^{-i\theta})^k$$
$$= \sum_{0\le j\le k} \frac{t^k}{k!} C(k,j)\, e^{-i(2j-k)\theta} = \sum_{|j|\le k} \frac{t^k}{k!} C(k,(j+k)/2)\, e^{-ij\theta}.$$

This gives

$$T_n(f) = \sum_{|j|\le k} \frac{t^k}{k!} C(k,(j+k)/2)\,\Lambda^j = I + \sum_{\substack{k>0\\ |j|\le k}} \frac{t^k}{k!} C(k,(j+k)/2)\,\Lambda^j. \tag{5.6}$$

(Here Λ^j denotes the usual power when $j \ge 0$, but when $j < 0$ it denotes $\Lambda'^{|j|}$.)

We use the Neumann expansion

$$T_n(f)^{-1} = I + \sum_{l\ge 1}(-1)^l \left(\sum_{\substack{k>0\\ |j|\le k}} \frac{t^k}{k!} C(k,(j+k)/2)\,\Lambda^j \right)^l.$$

If we expand this out we get a sum of terms of the form coefficient times

$$t^{k_1+\cdots+k_l}\,\Lambda^{j_1}\cdots\Lambda^{j_l}.$$

Now the product $\Lambda^{j_1}\cdots\Lambda^{j_l}$ can only have a nonzero upper-right entry when $j_1+\cdots+j_l \ge n$. Since each $|j_i| \le k$, the power of t must be at least n, and this power occurs only when each $j_i = k$. That means that we get the same lowest power of t term for the upper-right entry if in (5.6) we only take the terms with $j = k$, in other words of we replace $T_{n+1}(f)$ by

$$\sum_{k\ge 0} \frac{t^k}{k!}\Lambda^k = e^{t\Lambda}.$$

The inverse of this operator is $e^{-t\Lambda}$ and the upper-right corner of this matrix is exactly $(-1)^n t^n/n!$. This shows that

$$U_n = -V_{n+1}^-/V_{n+1}^+ = (-1)^{n+1} t^n/n! + O(t^{n+1}),$$

and so

$$\Phi_n = 1 - U_n^2 = 1 - \frac{t^{2n}}{(n!)^2} + O(t^{2n+1}),$$

which gives (5.5). We also see that $W_n = U_n/U_{n-1}$ satisfies the initial condition

$$W_n(t) = -\frac{t}{n} + O(t^2).$$

Using the differential equation (1.5) together with this initial condition we find

$$W_n(t) = -\frac{t}{n} - \frac{t^3}{n^2(n+1)} - \frac{2t^5}{n^3(n+1)(n+2)}$$
$$- \frac{(5n+6)t^7}{n^4(n+1)^2(n+2)(n+3)} + O(t^9).$$

6. Painlevé II

We present a heuristic argument that if there is any limit theorem of the type (1.12), with some distribution function $F(s)$ and some power N^α replacing $N^{1/6}$, then necessarily $\alpha = 1/6$ and F is given by (1.10). First we note that Johansson's lemma (which we shall state in the next section) leads from (1.12) to (1.9) with the power $t^{1/3}$ replaced by $t^{2\alpha}$. We assume that F is smooth and that the limit in (1.9) commutes with d/ds, so that taking the second logarithmic derivative gives

$$\lim_{t\to\infty} \frac{d^2}{ds^2} \log\, D_{2t+s\,t^{2\alpha}} = -q(s)^2,$$

where q^2 is now *defined* by $-q^2 = (\log\, F)''$ and q is defined to be the positive square root of q^2 (for large s).

Since changing $n = 2t + s\,t^{2\alpha}$ by 1 is the same as changing s by $t^{-2\alpha}$, we have the large t asymptotics

$$\log\, D_{n+1} + \log\, D_{n-1} - 2\,\log\, D_n \sim t^{-4\alpha}\,\frac{d^2}{dt^2}\log\, D_{2t+s\,t^{2\alpha}} \sim -t^{-4\alpha}\,q(s)^2.$$

On the other hand $V_n^+ = D_{n-1}/D_n$ and so

$$\frac{D_{n+1}\,D_{n-1}}{D_n^2} = \frac{V_n^+}{V_{n+1}^+} = 1 - U_n^2, \tag{6.1}$$

by (4.6). We deduce

$$\log\,(1 - U_n^2) \sim -t^{-4\alpha}\,q(s)^2, \quad U_n^2 \sim t^{-4\alpha}\,q(s)^2. \tag{6.2}$$

Now the U_n are of variable sign, as is clear from (1.6). Let us consider those n going to infinity such that

$$U_{n-1} \ge 0, \quad U_n \le 0, \quad U_{n+1} \ge 0, \tag{6.3}$$

and write (1.6) as

$$t\,(U_{n+1} + U_{n-1} + 2\,U_n)\,(1 - U_n^2) = -(n - 2t)\,U_n - 2t\,U_n^3. \tag{6.4}$$

Because of (6.3), (6.2) and the fact that changing $n = 2t + s\,t^{2\alpha}$ by 1 is the same as changing s by $t^{-2\alpha}$, we have when t is large,

$$U_{n+1} + U_{n-1} + 2\,U_n \sim t^{-6\alpha}\,q''(s).$$

Since also

$$n - 2t \sim s\,t^{2\alpha}, \quad U_n \sim -t^{-2\alpha}\,q(s),$$

(6.4) becomes the approximation

$$t^{1-6\alpha}\,q''(s) \approx s\,q(s) + 2\,t^{1-6\alpha}\,q(s)^3. \tag{6.5}$$

Let us show that $\alpha = 1/6$. If $\alpha > 1/6$ then letting $t \to \infty$ in (6.5) gives $q(s) = 0$ and so F is the exponential of a linear function and therefore not a distribution function. If

$\alpha < 1/6$ then dividing by $t^{1-6\alpha}$ and letting $t \to \infty$ in (6.5) gives $q''(s) = 2q^3$. Solving this gives two sets of solutions

$$s = \pm \int_q^{q_0} \frac{dq}{\sqrt{q^4 + c_1}} + c_2,$$

where q_0, c_1 and c_2 can be arbitrary. Now q is small when s is large and positive, so s is large and positive when q is small. Therefore we have to have the + sign and we must have $c_1 = 0$. Then $F(s)$ is of the form $|s - c|^{-1}$ times the exponential of a linear function and therefore is not a distribution function. The only remaining case is $\alpha = 1/6$, and then (6.5) becomes (1.11). It follows that $F(s)$ must be given by (1.10) times the exponential of a linear function. This extra factor must be 1 since (1.10) is already a distribution function.

Now to derive this we assumed that the n under consideration were such that (6.3) held. We would have reached the same conclusion if all the inequalities were reversed. If $n \to \infty$ in such a way that, say, U_{n-1} and U_n have one sign and U_{n+1} the other, then (the reader can check this) we would have reached the conclusion $q = 0$. Thus the only possibility for q to give a distribution function occurs when $\alpha = 1/6$ and q satisfies (1.11).

7. Odd Permutations

Recall that b_{Nn} equals the number of permutations in $\mathcal{O}_N$ having no increasing subsequence of length greater than n. The representations of Rains [10] for these quantities are

$$b_{2k\,n} = E_n\left(|\mathrm{tr}\,(U^2)^k|^2\right), \tag{7.1}$$

$$b_{2k+1\,n} = E_n\left(|\mathrm{tr}\,(U^2)^k\,\mathrm{tr}\,(U)|^2\right). \tag{7.2}$$

Theorem 1. *Let $G_n(t)$ and $H_n(t)$ be the generating functions defined in (1.14). Then*

$$G_n(t) = \begin{cases} D_{\frac{n}{2}}(t)^2, & n \text{ even}, \\ D_{\frac{n-1}{2}}(t) D_{\frac{n+1}{2}}(t), & n \text{ odd}, \end{cases} \tag{7.3}$$

$$H_n(t) = \begin{cases} D_{\frac{n}{2}-1}(t) D_{\frac{n}{2}+1}(t), & n \text{ even}, \\ D_{\frac{n-1}{2}}(t) D_{\frac{n+1}{2}}(t), & n \text{ odd}. \end{cases} \tag{7.4}$$

We prove a lemma which gives a preliminary representation for the generating functions in terms of other Toeplitz determinants. Let

$$g(z, t_1, t_2) = g(z) = e^{t_1(z+z^{-1}) + t_2(z^2+z^{-2})}$$

and define

$$\hat{D}_n(t_1, t_2) = \hat{D}_n = \det T_n(g).$$

Lemma. *We have*

$$G_n(t) = \hat{D}_n(0,t), \tag{7.5}$$

$$H_n(t) = \frac{1}{4}\frac{\partial^2 \hat{D}_n}{\partial t_1^2}(0,t) + \frac{1}{4}\frac{\partial^2 \hat{D}_n}{\partial t_1^2}(0,-t). \tag{7.6}$$

The proof of (7.5) is essentially the same as the proof that (1.7) and (1.8) are equivalent. First observe that

$$E_n\left(\left(\mathrm{tr}(U^2) + \overline{\mathrm{tr}(U^2)}\right)^{2k}\right) = \sum_{m=0}^{2k}\binom{2k}{m} E_n\left(\mathrm{tr}(U^2)^m \overline{\mathrm{tr}(U^2)}^{2k-m}\right).$$

Each summand with $m \neq k$ vanishes since by the invariance of the Haar measure replacing each U by ζU, with ζ a complex number of absolute value 1, does not change the summand but at the same time multiplies it by ζ^{4m-4k}. Thus,

$$E_n\left(\left(\mathrm{tr}(U^2) + \overline{\mathrm{tr}(U^2)}\right)^{2k}\right) = \binom{2k}{k} E_n\left(\mathrm{tr}(U^2)^k \overline{\mathrm{tr}(U^2)}^k\right).$$

Hence (7.1) is equivalent to

$$b_{2k\,n} = \frac{(k!)^2}{(2k)!}\, E_n\left(\left(\mathrm{tr}(U^2) + \overline{\mathrm{tr}(U^2)}\right)^{2k}\right).$$

Therefore if the eigenvalues of U are $e^{i\theta_1}, \cdots, e^{i\theta_n}$, we have

$$\begin{aligned} G_n(t) &= \sum_{k\geq 0} b_{2k\,n}\frac{t^{2k}}{(k!)^2} = \sum_{k\geq 0} E_n\left(\left(\sum \cos 2\theta_j\right)^{2k}\right)\frac{(2t)^{2k}}{(2k)!} \\ &= E_n\left(\prod_{j=1}^{n} e^{2t\cos 2\theta_j}\right) = \hat{D}_n(0,t). \end{aligned}$$

The last step follows from (1.2).

This gives (7.5). To prove (7.6) we use (7.2):

$$\begin{aligned} b_{2k+1\,n} &= E_n\left(\mathrm{tr}(U)\mathrm{tr}(U^2)^k \overline{\mathrm{tr}(U)}\,\overline{\mathrm{tr}(U^2)}^k\right) \\ &= \frac{1}{2}\frac{(k!)^2}{(2k)!}\, E_n\left(\left(\mathrm{tr}(U) + \overline{\mathrm{tr}(U)}\right)^2\left(\mathrm{tr}(U^2) + \overline{\mathrm{tr}(U^2)}\right)^{2k}\right), \end{aligned}$$

by expanding the right side as before. Hence

$$\begin{aligned} H_n(t) &= \frac{1}{2}\sum_{k\geq 0}\frac{t^{2k}}{(2k)!} E_n\left(\left(\mathrm{tr}(U) + \overline{\mathrm{tr}(U)}\right)^2\left(\mathrm{tr}(U^2) + \overline{\mathrm{tr}(U^2)}\right)^{2k}\right) \\ &= 2\,E_n\left(\left(\sum \cos\theta_j\right)^2 \cosh\left(2t\sum\cos\theta_j\right)\right) \\ &= E_n\left(\left(\sum\cos\theta_j\right)^2 \prod_{j=1}^{n} e^{2t\cos 2\theta_j}\right) + E_n\left(\left(\sum\cos\theta_j\right)^2 \prod_{j=1}^{n} e^{-2t\cos 2\theta_j}\right) \\ &= \frac{1}{4}\frac{\partial^2}{\partial t_1^2} E_n\left(\prod_{j=1}^{n} g(e^{i\theta_j}, t_1, t_2)\right)(0,t) + \frac{1}{4}\frac{\partial^2}{\partial t_1^2} E_n\left(\prod_{j=1}^{n} g(e^{i\theta_j}, t_1, t_2)\right)(0,-t). \end{aligned}$$

From the last equality (7.6) follows.

To prove the theorem we consider (7.3) first. Observe that $\hat{D}_n(0,t)$ is the determinant of $T_n(h)$, where $h(z) = f(z^2)$. It has Fourier coefficients

$$h_{2k} = f_k, \quad h_{2k+1} = 0.$$

Let us rearrange the basis vectors $e_0,\ e_1, \cdots, e_{n-1}$ of our underlying n-dimensional space as

$$e_0,\ e_2, \cdots;\ \ e_1,\ e_3, \cdots. \tag{7.7}$$

Then we see from the above that $T_n(h)$ becomes the direct sum of two Toeplitz matrices associated with f, the orders of these matrices being the sizes of the two groups of basis vectors in (7.7). If n is even both groups have size $n/2$ whereas if n is odd the sizes are $(n \pm 1)/2$. Since $\hat{D}_n(0,t) = \det T_n(h)$ is the product of the corresponding Toeplitz determinants associated with f, we have (7.3).

The proof of (7.4) is not so simple. We have

$$\frac{1}{\hat{D}_n}\frac{\partial^2 \hat{D}_n}{\partial t_1^2} = \partial_{t_1}^2 \log \hat{D}_n + \left(\partial_{t_1} \log \hat{D}_n\right)^2.$$

Now, as in the computation leading to (2.3),

$$\partial_{t_1}^2 \log \hat{D}_n(t_1,t_2) = \operatorname{tr}\left(T_n(g)^{-1}\, T_n(\partial_{t_1}^2 g) - T_n(g)^{-1}\, T_n(\partial_{t_1} g)\, T_n(g)^{-1}\, T_n(\partial_{t_1} g)\right)$$
$$= \operatorname{tr}\left[T_n(g)^{-1}\, T_n((z+z^{-1})^2 g) - T_n(g)^{-1}\, T_n((z+z^{-1})g)\, T_n(g)^{-1}\, T_n((z+z^{-1})g)\right].$$

This is to be evaluated first at $t_1 = 0,\ t_2 = t$. Since $g(z,0,t) = f(z^2) = h(z)$ we must compute

$$\operatorname{tr}\left[T_n(h)^{-1}\, T_n((z+z^{-1})^2 h) - T_n(h)^{-1}\, T_n((z+z^{-1})h)\, T_n(h)^{-1}\, T_n((z+z^{-1})h)\right].$$

We write $w^{\pm} = T_n(h)^{-1} h^{\pm}$, so that the $w^{\pm}$ are associated with h just as $u^{\pm}$ are associated with f. Using (2.1) and (2.2) we find that

$$T_n(h)^{-1}\, T_n((z+z^{-1})h)\, T_n(h)^{-1}\, T_n((z+z^{-1})h)$$
$$= (\Lambda' + w^- \otimes \delta^- + \Lambda + w^+ \otimes \delta^+)^2,$$

and from this that

$$\operatorname{tr}\left(T_n(h)^{-1}\, T_n((z+z^{-1})h)\, T_n(h)^{-1}\, T_n((z+z^{-1})h)\right) =$$
$$2n - 2 + 2(w^-, \Lambda\delta^-)$$
$$+2(w^+, \Lambda'\delta^+) + 2(\delta^-, w^+)\,(w^-, \delta^+) + (\delta^-, w^-)^2 + (\delta^+, w^+)^2.$$

Consequently

$$\operatorname{tr}\left[T_n(h)^{-1}\, T_n((z+z^{-1})^2 h) - T_n(h)^{-1}\, T_n((z+z^{-1})h)\, T_n(h)^{-1}\, T_n((z+z^{-1})h)\right]$$
$$= \operatorname{tr}\left(T_n(h)^{-1}\, T_n((z^2+z^{-2})h)\right)$$

$$+2-2(w^-,\Lambda\delta^-)-2(w^+,\Lambda'\delta^+)-2(\delta^-,w^+)\,(w^-,\delta^+)-(\delta^-,w^-)^2-(\delta^+,w^+)^2.$$

This is $\partial_{t_1}^2 \log \hat{D}_n(0,t)$. Similarly, we find

$$\partial_{t_1} \log \hat{D}_n(0,t) = (w^+,\delta^+)+(w^-,\delta^-),$$

so that when $t_1 = 0,\ t_2 = t$,

$$\frac{1}{\hat{D}_n}\frac{\partial^2 \hat{D}_n}{\partial t_1^2} = \partial_{t_1}^2 \log \hat{D}_n + \Big(\partial_{t_1} \log \hat{D}_n\Big)^2 = \operatorname{tr}\Big(T_n(h)^{-1}\,T_n((z^2+z^{-2})h)\Big)$$
$$+2-2(w^-,\Lambda\delta^-)-2(w^+,\Lambda'\delta^+)-2(w^+,\delta^-)\,(w^-,\delta^+)+2(w^+,\delta^+)\,(w^-,\delta^-).$$

Since $T_n(h)$ is symmetric all superscripts in the symbols in the inner product may be reversed as long as we interchange Λ and Λ'. We therefore have shown

$$\frac{1}{\hat{D}_n(0,t)}\frac{\partial^2 \hat{D}_n(0,t)}{\partial t_1^2} = \operatorname{tr}\Big(T_n(h)^{-1}\,T_n((z^2+z^{-2})h)\Big)$$
$$+2-4\,(w^+,\Lambda'\delta^+)-2\,(w^+,\delta^-)^2+2\,(w^+,\delta^+)^2.$$

Let us rearrange our basis elements as in (7.7) and suppose the first group has n_1 vectors and the second group has n_2. Then $T_n(h)^{-1}$ becomes the matrix direct sum

$$\begin{pmatrix} T_{n_1}(f)^{-1} & 0 \\ 0 & T_{n_2}(f)^{-1} \end{pmatrix},$$

and $T_n(h)^{-1}T_n\,((z^2+z^{-2})h)$ becomes

$$\begin{pmatrix} T_{n_1}(f)^{-1}\,T_{n_1}((z+z^{-1})f) & 0 \\ 0 & T_{n_2}(f)^{-1}\,T_{n_2}((z+z^{-1})f) \end{pmatrix}.$$

By a now familiar computation we find from this that

$$\operatorname{tr}\Big(T_n(h)^{-1}\,T_n((z^2+z^{-2})h)\Big) = 2\,(u_{n_1}^+,\delta_{n_1}^+)+2\,(u_{n_2}^+,\delta_{n_2}^+),$$

where u_m^+ and δ_m^+ denote the quantities u^+ and δ^+ associated with the index m. We use similar notation below. To continue, after rearranging our basis we have the replacements

$$h^+ \to \begin{pmatrix} 0 \\ f_{n_2}^+ \end{pmatrix},\quad w^+ \to \begin{pmatrix} 0 \\ u_{n_2}^+ \end{pmatrix},\quad \delta^+ \to \begin{pmatrix} \delta_{n_1}^+ \\ 0 \end{pmatrix},\quad \Lambda'\delta^+ \to \begin{pmatrix} 0 \\ \delta_{n_2}^+ \end{pmatrix}$$

and

$$\delta^- \to \begin{pmatrix} 0 \\ \delta_{n_2}^- \end{pmatrix} \text{ if } n \text{ is even},\quad \delta^- \to \begin{pmatrix} \delta_{n_1}^- \\ 0 \end{pmatrix} \text{ if } n \text{ is odd}.$$

It follows from these that

$$(w^+,\delta^+)=0,\quad (w^+,\Lambda'\delta^+)=(u_{n_2}^+,\delta_{n_2}^+),$$

and that $(w^+, \delta^-) = (u^+_{n_2}, \delta^-_{n_2})$ if n is even and $(\delta^-, w^+) = 0$ if n is odd.

If we modify our notation by writing

$$U_n^{\pm} = (u_n^+, \delta_n^{\pm}),$$

so that U_n^- is what we have been denoting by U_n, the above gives

$$\frac{1}{2}\frac{1}{\hat{D}_n(0,t)}\frac{\partial^2 \hat{D}_n(0,t)}{\partial t_1^2} = 1 + U^+_{n_1} - U^+_{n_2} - (w^+, \delta^-)^2$$
$$= \begin{cases} 1 - U^-_{\frac{n}{2}}(t)^2, & n \text{ even}, \\ 1 + U^+_{\frac{n+1}{2}}(t) - U^+_{\frac{n-1}{2}}(t), & n \text{ odd}. \end{cases}$$

To evaluate our quantities at $(0, -t)$ we observe that if C is the diagonal matrix with diagonal entries $1, -1, 1, \cdots, (-1)^n$, and we replace t by $-t$, then we have the replacements $T_n(f) \to CT_n(f)C$ and $f^+ \to -Cf^+$ and therefore $u^+ \to -Cu^+$. Therefore also $U_n^+ = (u^+, \delta^+) \to -U_n^+$ and $U_n^- = (u^+, \delta^-) \to (-1)^{n+1} U_n^-$. Hence in the last displayed formula ${U^-_{n/2}}^2$ is an even function of t whereas $U^+_{(n\pm1)/2}$ are odd functions of t. Also, $\hat{D}_n(0,t)$ is an even function of t. Thus

$$H_n(t) = G_n(t) \times \begin{cases} 1 - U^-_{\frac{n}{2}}(t)^2, & n \text{ even}, \\ 1, & n \text{ odd}. \end{cases}$$

Recalling (4.6) and the general fact $V_m^+(t) = D_{m-1}(t)/D_m(t)$ we obtain (7.4).

Theorem 2. *Let $\ell_N(\sigma)$ denote the length of the longest increasing subsequence of σ in the subgroup $\mathcal{O}_N$ of $\mathcal{S}_N$. Then*

$$\lim_{N\to\infty} \text{Prob}\left(\frac{\ell_N(\sigma) - 2\sqrt{N}}{2^{2/3}N^{1/6}} \le s\right) = F(s)^2,$$

where $F(s)$ is as in (1.12).

For the proof we shall apply Johansson's lemma [5], which we now state:

Lemma. *Let $\{P_k(n)\}_{k\ge0}$ be a family of distribution functions defined on the nonnegative integers n and $\varphi_n(\lambda)$ the generating function*

$$\varphi_n(\lambda) = e^{-\lambda}\sum_{k\ge0} P_k(n)\frac{\lambda^k}{k!}.$$

(Set $P_0(n) = 1$.) Suppose that for all $n, k \ge 1$,

$$P_{k+1}(n) \le P_k(n). \tag{7.8}$$

If we define $\mu_k = k + 4\sqrt{k\log k}$ and $\nu_k = k - 4\sqrt{k\log k}$, then there is a constant C such that

$$\varphi_n(\mu_k) - \frac{C}{k^2} \le P_k(n) \le \varphi_n(\nu_k) + \frac{C}{k^2}$$

for all sufficiently large k, $0 \le n \le k$.

This allows one to deduce that $P_k(n) \sim \varphi_n(k)$ as $n, k \to \infty$ under suitable conditions, and is how one obtains the equivalence of (1.9) and (1.12). We shall apply the lemma to the distributions functions

$$\varphi_n^e(\lambda) = e^{-\lambda} G_n\left(\sqrt{\lambda/2}\right) = e^{-\lambda} \sum_{k\geq 0} F_{2k}(n) \frac{\lambda^k}{k!},$$
$$\varphi_n^o(\lambda) = e^{-\lambda} H_n\left(\sqrt{\lambda/2}\right) = e^{-\lambda} \sum_{k\geq 0} F_{2k+1}(n) \frac{\lambda^k}{k!},$$

where

$$F_N(n) = \text{Prob}(\ell_N(\sigma) \leq n) = \frac{b_{Nn}}{2^k\, k!} \tag{7.9}$$

when $N = 2k$ or $2k+1$. To apply this lemma we must prove another

Lemma. *We have $F_{N+2} \leq F_N$ for all N.*

We show this simultaneously for $N = 2k$ and $N = 2k+1$. Take a $\sigma \in \mathcal{O}_{N+2}$ and remove the two-point set $\{-1, 1\}$ from its domain. Then σ maps the remaining N-point set one-one onto another N-point set. If we identify both of these sets with the integers from $-k$ to k (including or excluding 0 depending on the parity of N) by order-preserving maps, then under this identification the restriction of σ becomes an element of $\mathcal{S}_N$. In fact it becomes an odd permutation because the two identification maps are odd. Thus we have described a mapping $\sigma \to \mathcal{F}(\sigma)$ from $\mathcal{O}_{N+2}$ to $\mathcal{O}_N$. The mapping is $2k+2$ to 1 and is clearly onto. It is also clear that $\ell_N(\mathcal{F}(\sigma)) \leq \ell_{N+2}(\sigma)$, from which it follows that $b_{N+2\,n} \leq (2k+2)\, b_{N\,n}$ for all n. The assertion of the lemma follows upon using (7.9).

Theorem 1 tells us that

$$e^{2t^2} \varphi_n^e(2t^2) = \begin{cases} D_{\frac{n}{2}}(t)^2, & n \text{ even}, \\ D_{\frac{n-1}{2}}(t) D_{\frac{n+1}{2}}(t), & n \text{ odd}, \end{cases}$$
$$e^{2t^2} \varphi_n^o(2t^2) = \begin{cases} D_{\frac{n}{2}-1}(t) D_{\frac{n}{2}+1}(t), & n \text{ even}, \\ D_{\frac{n-1}{2}}(t) D_{\frac{n+1}{2}}(t), & n \text{ odd}. \end{cases}$$

From the asymptotics (1.9) we find that if we set

$$\frac{n}{2} = 2t + s t^{1/3},$$

then

$$\lim_{t\to\infty} \varphi_n^e(2t^2) = F(s)^2.$$

Setting $t = \sqrt{2k}/2$ gives

$$\lim_{k\to\infty} \varphi_n^e(k) = F(s)^2.$$

Johansson's lemma tell us that when N runs through the even integers $2k$,

$$\lim_{N\to\infty} F_N(2\sqrt{N} + 2^{2/3} s N^{1/6}) = \lim_{N\to\infty} F_N(n) = F(s)^2.$$

For N running through the odd integers we obtain the same relation. Thus the proof is complete.

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