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Exchange Dynamical Quantum Groups

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Abstract: For any simple Lie algebra $\mathfrak{g}$ and any complex number q which is not zero or a nontrivial root of unity, we construct a dynamical quantum group (Hopf algebroid), whose representation theory is essentially the same as the representation theory of the quantum group $U_q(\mathfrak{g})$. This dynamical quantum group is obtained from the fusion and exchange relations between intertwining operators in representation theory of $U_q(\mathfrak{g})$, and is an algebraic structure standing behind these relations.

1. Introduction

One of the most important equations in statistical mechanics is the so-called Star-Triangle relation, introduced by Baxter. In 1994, G. Felder [F] suggested to write this relation in the form of the quantum dynamical Yang–Baxter equation (QDYB) (which previously appeared in some form in physical literature), and proposed the concept of a quantum group associated to a solution of this equation. He also considered the quasi-classical limit of this equation, and showed that a solution of the classical dynamical Yang–Baxter equation (CDYB) appears naturally on the right-hand side of the Knizhnik–Zamolodchikov–Bernard equations for conformal blocks on an elliptic curve. Since then, this theory has found many applications in the theory of integrable systems.

In [EV1], we proposed a geometric interpretation of the CDYB equation without spectral parameter. Namely, we assigned to any solution of this equation, whose symmetric part is invariant, a certain Poisson groupoid. This construction is a generalization of Drinfeld's construction which assigns a Poisson–Lie group to any solution of the usual classical Yang–Baxter equation, with invariant symmetric part. We also classified such solutions for simple Lie algebras and showed that there are two classes of solutions (without spectral parameter) – rational and trigonometric.

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In [EV2], we quantized the results of [EV1] and presented a "noncommutative geometric" interpretation of solutions of the QDYB equation without spectral parameter. Namely, we assigned to any solution of the QDYB equation, satisfying a special "Hecke type" condition, a certain dynamical quantum group (Hopf algebroid). This construction is a generalization of the Faddeev–Reshetikhin–Takhtajan–Sklyanin construction which assigns to any solution of the usual quantum Yang–Baxter equation of Hecke type a quantum group (Hopf algebra), defined by the so-called $RTT = TTR$ relations. We also classified the Hecke type solutions of the QDYB equation and showed that, like in the classical case, there are two classes of solutions (without spectral parameter) – rational and trigonometric.

The solutions of the QDYB equation and corresponding dynamical quantum groups from [EV2] provide quantizations of the solutions of the CDYB equation and Poisson groupoids from [EV1], but only for the Lie algebra $\mathfrak{g} = sl_N$. For other simple Lie algebras, especially for exceptional ones, one needs to use a different method to quantize the Poisson groupoids from [EV1]; this method has to be applicable to any simple Lie algebra $\mathfrak{g}$ and should not use its particular matrix realization.

Such a method is suggested in the present paper. Namely, it turns out that with any simple complex Lie group G (with the Lie algebra $\mathfrak{g}$) and a nonzero complex number q (which is not a nontrivial root of unity but may be equal to 1) one can associate a Hopf algebroid $F_q(G)$, which is a quantization of the Poisson groupoid associated with the simple Lie algebra $\mathfrak{g}$ in [EV1]. More precisely, the case $q = 1$ corresponds to the Poisson groupoid defined by the rational solution, and $q \neq 1$ corresponds to the Poisson groupoid defined by the trigonometric solution.

The Hopf algebroid $F_q(G)$ is constructed from the representation theory of G. Namely, the structure constants of the multiplication in $F_q(G)$ are obtained from the structure constants of the multiplication (fusion) of intertwining operators between a Verma module over $U_q(\mathfrak{g})$ and a tensor product of a Verma module with a finite dimensional module over $U_q(\mathfrak{g})$. These structure constants have been known for a long time under various names (Racah coefficients, Wigner 6j symbols) and play an important role in quantum physics.

The commutation relations between generators of $F_q(G)$ are defined by certain dynamical R-matrices, which satisfy the quantum dynamical Yang–Baxter equations. These R-matrices are exactly the matrices which arise in commutation (=exchange) relations between intertwining operators and are therefore called the exchange matrices. This makes it natural to call the Hopf algebroids $F_q(G)$ **the exchange dynamical quantum groups.**

The results of this paper demonstrate how to use representation theory to construct new quantum groups, and conversely, how the multiplication of intertwining operators, one of the main structures in representation theory, is controlled by a dynamical quantum group.

We note that the main idea of this paper (to use commutation relations between intertwining operators to obtain new quantum groups) was inspired by the pioneering paper [FR].

Let us briefly describe the contents of the paper.

In Sect. 2 we introduce, for any polarized Hopf algebra, the fusion and exchange matrices $J(\lambda)$, $R(\lambda)$ and consider their main properties.

In Sect. 3 we recall the notion of an H-Hopf algebroid and its dynamical representations introduced in [EV2] (where H is a commutative, cocommutative Hopf algebra).

In Sect. 4 we construct Hopf algebriods defined by the fusion and exchange matrices.

In Sect. 5 we specialize our construction to the case of simple Lie groups and quantum groups, and construct the Hopf algebroids $F_q(G)$.

In Sect. 6 we compute the exchange R-matrix for the vector representation of $G = GL(N)$, and show that $F_q(GL(N))$ is isomorphic to the Hopf algebroid A_R defined by the trigonometric solution R of the quantum dynamical Yang–Baxter equation in [EV2].

In Sect. 7 we consider the representation theory of $F_q(G)$ for a simple complex group G and show that its category of rational finite dimensional dynamical representations contains the category of finite dimensional representations of $U_q(\mathfrak{g})$ as a full subcategory. In the next paper we plan to show that these categories are actually the same.

In Sect. 8, we describe the precise connection between fusion and exchange matrices (for sl_2) and the classical and quantum 6j-symbols.

In Sect. 9, we show that the universal fusion matrix $J(l)$ satisfies the defining property of the quasi-Hopf twist discovered in [A]. In particular, this shows that our fusion matrices are the same as the quasi-Hopf twists introduced in [A].

In a subsequent paper, we plan to consider the analogue of this theory for affine and quantum affine algebras. This will help one to understand better the monodromy of classical and quantum Knizhnik–Zamolodchikov equations following the ideas of [FR] and [TV1-2,FTV].

In conclusion we would like to mention the paper [BBB], in which another algebraic interpretation of the QDYB equation was given (via quasi-Hopf algebras), and a version of our main construction for the Lie algebra sl_2 was presented. See also [JKOS] where the approach of [BBB] was generalized to an arbitrary Kac–Moody algebra. We would also like to point out the recent paper [Xu], where the relationship between the quasi-Hopf algebra and Hopf algebroid interpretation of the quantum dynamical Yang–Baxter equation is explained.

2. Exchange Matrices

2.1. Polarized Hopf algebras. A polarized Hopf algebra is a Hopf algebra A over $\mathbb{C}$ with the following properties:

I. The algebra is $\mathbb{Z}$-graded, $A = \oplus_{k=-\infty}^{\infty} A[k]$.
II. The algebra is polarized. Namely, there exist graded subalgebras A_0, A_+, A_- such that the multiplication maps $A_+ \otimes A_0 \otimes A_- \to A$ and $A_- \otimes A_0 \otimes A_+ \to A$ are isomorphisms of vector spaces. We also assume that the graded components of A_+ and A_- (not of A_0) are finite dimensional.
III. Let $\epsilon : A \to \mathbb{C}$ be the counit, then Ker $\epsilon \cap A_+$ has only elements of positive degree, Ker $\epsilon \cap A_-$ has only elements of negative degree, A_0 has only elements of zero degree.
IV. The algebra A_0 is a commutative cocommutative finitely generated Hopf algebra.
V. $A_0 A_+$ and $A_0 A_-$ are Hopf subalgebras of A.

Let $T = \mathrm{Spec} A_0$. Since A_0 is finitely generated, commutative and cocommutative, T is a commutative affine algebraic group [M].

The main examples of polarized Hopf algebras are the universal enveloping algebra $U(\mathfrak{g})$ of a simple Lie algebra $\mathfrak{g}$ and the corresponding quantum group $U_q(\mathfrak{g})$.

If $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra, $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$ a polarization and $A = U(\mathfrak{g})$, then $A_- = U(\mathfrak{g}_-)$, $A_0 = U(\mathfrak{h})$, $A_+ = U(\mathfrak{g}_+)$, and $T = \mathbb{C}^n$, where n =dim $\mathfrak{h}$. If $A = U_q(\mathfrak{g})$, then $A_- = U_q(\mathfrak{g}_-)$, $A_0 = U_q(\mathfrak{h})$, $A_+ = U_q(\mathfrak{g}_+)$, and $T = (\mathbb{C}^*)^n$.

2.2. Verma modules. Let $A_+^0 = \mathrm{Ker}\,\epsilon \cap A_+$. Then $A_+ = \mathbb{C}\cdot 1 \oplus A_+^0$.

Lemma 1. *$A_0A_+^0$ is the set of elements of A_0A_+ of positive degree. Moreover, $A_0A_+^0$ is a two-sided ideal in A_0A_+.*

Proof. $A_0A_+^0$ obviously lies among the elements of positive degree. We prove the converse statement. Let $\alpha = \sum a_0^i a_+^i$ be a homogeneous element of degree $N > 0$ in A_0A_+, where $a_0^i \in A_0$ and $a_+^i \in A_+$ has degree N. Since $A_+ = \mathbb{C}\cdot 1 \oplus A_+^0$, we have $a_+^i \in A_+^0$. The second statement of the Lemma obviously follows from the first. $\square$

Corollary 2. $A_0A_+/A_0A_+^0 \cong A_0$.

In fact, $A_0 \to A_0A_+ \to A_0A_+/A_0A_+^0$ is an isomorphism.

Let $\varphi_+ : A_0A_+ \to A_0$ be the induced homomorphism, φ_+ is defined by $\varphi_+(a_0) = a_0$, $\varphi_+(a_+) = \epsilon(a_+)$, where $a_0 \in A_0$, $a_+ \in A_+$.

Let $\lambda : A_0 \to \mathbb{C}$ be a homomorphism, hence $\lambda \in T$. Let χ_λ be the corresponding one dimensional A_0-module. Define a homomorphism $\chi_\lambda^+ : A_0A_+ \to \mathbb{C}$ by $\chi_\lambda^+ = \lambda\varphi_+$. We also denote by$\chi_\lambda^+$ the corresponding one-dimensional A_0A_+-module. Define *a Verma module* M_λ^+ over A by $M_\lambda^+ = A \otimes_{A_0A_+} \chi_\lambda^+$.

Analogously, consider the homomorphism $\varphi_- : A_0A_- \to A_0$ and the corresponding A_0A_--module χ_λ^-. Define *a Verma module* M_λ^- over A by $M_\lambda^- = A \otimes_{A_0A_-} \chi_\lambda^-$.

Let $\chi_\lambda^+ = \mathbb{C}v_\lambda^+$.

Lemma 3. *M_λ^+ is a free A_--module generated by v_λ^+.*

The lemma follows from Property II.

Similarly, M_λ^- is a free A_+-module generated by v_λ^-.

Using Lemma 3 induce a grading on $M_\lambda^\pm$ from $A_\pm$ such that the degree of $v_\lambda^\pm$ is equal to zero.

2.3. The Shapovalov form. A polarized Hopf algebra A is called *nondegenerate* if Verma modules M_λ^+ and M_λ^- are irreducible for generic $\lambda \in T$. (This means that the modules are irreducible for all λ except a countable union of algebraic sets of lower dimension.) For example, $U(\mathfrak{g})$ and $U_q(\mathfrak{g})$ are nondegenerate.

Let A be polarized and nondegenerate. Consider the vector space $(M_\lambda^+)^*$, the restricted dual of M_λ^+ with respect to the grading of A_-. Define an A-module structure on $(M_\lambda^+)^*$ by $\pi_{(M_\lambda^+)^*}(a) = \pi_{M_\lambda^+}(S(a))^*$, where S is the antipode in A.

Define $(v_\lambda^+)^* \in (M_\lambda^+)^*$ to be the only degree zero element such that $< (v_\lambda^+)^*, v_\lambda^+ >= 1$.

Lemma 4. *If $a_- \in A_-^0$, then $a_-(v_\lambda^+)^* = 0$. If $a_0 \in A_0$, then $a_0(v_\lambda^+)^* = (-\lambda)(a_0)(v_\lambda^+)^*$, where $-\lambda$ means the inverse element in the abelian group T.*

In fact, $a_-(v_\lambda^+)^*$ has degree which does not occur in $(M_\lambda^+)^*$. The second statement is obvious.

Corollary 5. *There exists a unique homomorphism of A-modules $\psi_- : M_{-\lambda}^- \to (M_\lambda^+)^*$ such that $v_{-\lambda}^- \mapsto (v_\lambda^+)^*$.*

Analogously, one can define a homomorphism $\psi_+ : M_{-\lambda}^+ \to (M_\lambda^-)^*$ such that $v_{-\lambda}^+ \mapsto (v_\lambda^-)^*$.

Lemma 6. *If A is nondegenerate, then ψ_+, ψ_- are isomorphisms for generic λ.*

Proof. For generic λ, the homomorphisms ψ_+ and ψ_- are injective since $M^+_{-\lambda}$ and $M^-_{-\lambda}$ are irreducible and ψ_+, ψ_- differ from zero. This implies that $\dim A_+[n] \leqslant \dim A_-[-n]$ and $\dim A_+[n] \geqslant \dim A_-[-n]$. □

Corollary 7. *For all n, $\dim A_+[n] = \dim A_-[-n]$.*

The homomorphisms ψ_+, ψ_- are called *the Shapovalov forms*. They can be considered as bilinear forms $\psi_+(\lambda) : A_+ \otimes A_- \to \mathbb{C}$, $\psi_-(\lambda) : A_- \otimes A_+ \to \mathbb{C}$ depending on λ. The bilinear forms can be defined also by $\psi_+(\lambda)(a_+, a_-) =< (v^+_\lambda)^*, S(a_+)a_-v^+_\lambda >$ and $\psi_-(\lambda)(a_-, a_+) =< (v^-_\lambda)^*, S(a_-)a_+v^-_\lambda >$.

Choose bases in $A_\pm[n]$ and compute the determinants of the Shapovalov forms,

$$D^+_n(\lambda) = \det \psi_+(\lambda)[n], \quad D^-_n(\lambda) = \det \psi_-(\lambda)[n].$$

The determinants of the Shapovalov forms are regular nonzero functions of $\lambda \in T$ defined up to multiplication by a nonzero number.

2.4. Intertwining operators. Let A be polarized and nondegenerate. Let V be a $\mathbb{Z}$-graded A-module such that V is a diagonalizable A_0-module, $V = \oplus_{\lambda \in T} V[\lambda]$, where $a_0 v = \lambda(a_0)v$ for all $v \in V[\lambda]$, $a_0 \in A_0$.

Theorem 8.

I. *Assume that $M^-_{-\mu}$ is irreducible and V is bounded from above, i.e. the graded component of V corresponding a number N is equal to zero if $N >> 0$. Then* $\mathrm{Hom}_A(M^+_\lambda, M^+_\mu \otimes V) \cong V[\lambda - \mu]$, *where $\lambda - \mu$ means the difference in the abelian group T.*

II. *Assume that $M^+_{-\mu}$ is irreducible and V is bounded from below. Then*

$$\mathrm{Hom}_A(M^-_\lambda, M^-_\mu \otimes V) \cong V[\lambda - \mu].$$

The isomorphism is given by

$$\Phi \mapsto < \Phi >:=< (v^\pm_\mu)^* \otimes 1, \ \Phi v^\pm_\lambda > .$$

Proof. First we prove a lemma. Let B be a $\mathbb{Z}$-graded Hopf algebra. Let U, W be $\mathbb{Z}$-graded B-modules bounded from above and such that all homogeneous components of U are finite dimensional. Define the space $\mathrm{Hom}_B(U^*, W)$ as $\mathrm{Hom}_B(U^*, W) = \oplus_n \mathrm{Hom}_B(U^*, W)[n]$. Let $(U \otimes W)^B$ denote the subspace of invariants with respect to B, i.e. the subspace of all elements w such that $bw = \epsilon(b)w$ for any $b \in B$. The space $(U \otimes W)^B$ is $\mathbb{Z}$-graded, $(U \otimes W)^B = \oplus_n (U \otimes W)^B[n]$.

Lemma 9. *Let $w \in U \otimes W$. Let $\tilde{w} : U^* \to W$ be defined as the composition of the two maps:* $1 \otimes w : U^* \to U^* \otimes U \otimes W$ *and* $< \cdot, \cdot > \otimes 1 : U^* \otimes U \otimes W \to W$. *Then $w \in (U \otimes W)^B[n]$ for some n if and only if $\tilde{w} \in \mathrm{Hom}_B(U^*, W)[n]$. Thus the assignment $w \to \bar{w}$ is an isomorphism of $(U \otimes W)^B$ and* $\mathrm{Hom}_B(U^*, W)$.

Proof. The counit of B defines a trivial one-dimensional B-module $\mathbb{C}_B$. If $w \in U \otimes W$ is B-invariant, then w defines a homomorphism $\mathbb{C}_B \to U \otimes W$ of B-modules, which defines a homomorphism $1 \otimes w : U^* \to U^* \otimes U \otimes W$, $u^* \mapsto u^* \otimes w$. Since $<,>: U^* \otimes U \to \mathbb{C}$ is a homomorphism of B-modules, so is the composition $\tilde{w} = (<,> \otimes 1)(1 \otimes w)$. This proves one of the two claims of the lemma.

Let $U \tilde{\otimes} U^* = \oplus_n U \tilde{\otimes} U^*[n]$, where $U \tilde{\otimes} U^*[n]$ is the space of all elements x of the form $x = \sum_{i=1}^{\infty} a_i \otimes b_i$ such that for any i the element $a_i \otimes b_i \in U \otimes U^*$ has degree n, the elements a_i, b_i are homogeneous, and $\deg a_i \to -\infty$ as $i \to \infty$. There is a canonical B-homomorphism $\mathbb{C}_B \to U \tilde{\otimes} U^*$, $1 \mapsto \sum_i a_i \otimes a_i^*$, where $\{a_i\}$ is a graded basis of U and $\{a_i^*\}$ is the dual basis of U^*.

Let $\tilde{w} : U^* \to W$ be a homogeneous B-homomorphism. Then $1 \otimes \tilde{w} : U \tilde{\otimes} U^* \to U \otimes W$ is a well defined B-homomorphism. The composition $\mathbb{C}_B \to U \tilde{\otimes} U^* \to U \otimes W$ gives a B invariant element w. $\square$

Now we prove the theorem. Introduce $A_{\geqslant 0} = A_0 A_+$ and $A_{\leqslant 0} = A_0 A_-$. We have $\mathrm{Hom}_A(M_\lambda^+, M_\mu^+ \otimes V) = \mathrm{Hom}_{A_{\geqslant 0}}(\chi_\lambda^+, M_\mu^+ \otimes V)$. The space $\mathrm{Hom}_{A_{\geqslant 0}}(\chi_\lambda^+, M_\mu^+ \otimes V)$ can be described as the space X of all $w \in M_\mu^+ \otimes V$ such that the $A_{\geqslant 0}$-submodule of $M_\mu^+ \otimes V$ generated by w is isomorphic to χ_λ^+. After tensoring with $\chi_{-\lambda}^+$ this submodule gives a trivial module. Thus the space X is isomorphic to the space $(M_\mu^+ \otimes V \otimes \chi_{-\lambda}^+)^{A_{\geqslant 0}}$. According to the lemma, the space X is isomorphic to $\mathrm{Hom}_{A_{\geqslant 0}}((M_\mu^+)^*, V \otimes \chi_{-\lambda}^+)$. This space is isomorphic to $\mathrm{Hom}_{A_{\geqslant 0}}(M_{-\mu}^-, V \otimes \chi_{-\lambda}^+)$ since $M_{-\mu}^-$ is irreducible. Now $\mathrm{Hom}_{A_{\geqslant 0}}(M_{-\mu}^-, V \otimes \chi_{-\lambda}^+) \cong \mathrm{Hom}_{A_0}(\chi_{-\mu}, V \otimes \chi_{-\lambda}) \cong \mathrm{Hom}_{A_0}(\chi_{\lambda-\mu}, V) \cong V[\lambda-\mu]$. The theorem is proved. $\square$

Let V be bounded from above. Let $v \in V[\lambda - \mu]$. Denote $\Phi_\lambda^v : M_\lambda^+ \to M_\mu^+ \otimes V$ the intertwining operator such that $< \Phi_\lambda^v >= v$.

Define $\Phi^v(\lambda) : A_- \to A_- \otimes V$ as the operator obtained from Φ_λ^v after identification of A_- with M_λ^+ and M_μ^+. Then $\Phi^v(\lambda)$ is a rational function, i.e. for any homogeneous $a_- \in A_-$, $a_-^* \in A_-^*$, and $f \in V^*$, the scalar function $(f^* \otimes a_-^*)\Phi^v(\lambda)a_-$ is a rational function.

2.5. A quasitriangular structure and dynamical R-matrices. Let $A_{\geqslant n} = \oplus_{j \geqslant n} A[j]$. Introduce a system of left A-ideals $I_n = A \cdot A_{\geqslant n}$.

Introduce a tensor product $A \hat{\otimes} A = \oplus_{i \in \mathbb{Z}} (A \hat{\otimes} A)[i]$ as follows. Let $(A \hat{\otimes} A)[i]$ be the projective limit of $(A/I_n \otimes A/I_n)[i]$ as $n \to \infty$, that is $(A \hat{\otimes} A)[i]$ consist of elements of the form $a = \sum_{k=1}^{\infty} a_k \otimes a_k'$ such that

I. For each k there is j such that $a_k \otimes a_k' \in A[j] \otimes A[i-j]$.

II. For each n there is only finitely many k such that $a_k \otimes a_k'$ does not belong to $A \otimes I_n + I_n \otimes A$.

Lemma 10. *$A \hat{\otimes} A$ is an algebra.* $\square$

Similarly we can define $A^{\hat{\otimes} n}$ for any n.

Consider the category $\mathcal{O}$ of graded A-modules bounded from above and diagonalizable over A_0. Let $V, W \in \mathcal{O}$, then $A \hat{\otimes} A$ acts on $V \otimes W$. Similarly, for any n one can define an action of the algebra $A^{\hat{\otimes} n}$ in a tensor product of n A-modules from the category $\mathcal{O}$.

An element $\mathcal{R} \in A \hat{\otimes} A$ is called *a quasitriangular structure* (QTS) if

I. $\mathcal{R}$ is invertible in $A \hat{\otimes} A$.
II. $\mathcal{R}\Delta(a) = \Delta^{op}(a)\mathcal{R}$.
III. $(\Delta \otimes 1)\mathcal{R} = \mathcal{R}^{13}\mathcal{R}^{23}$ and $(1 \otimes \Delta)\mathcal{R} = \mathcal{R}^{13}\mathcal{R}^{12}$.

Consider the category $\mathcal{O}$ of graded A-modules bounded from above and diagonalizable over A_0. The category $\mathcal{O}$ is a braided tensor category with the braiding equal to $P\mathcal{R}$.

Let A be a nondegenerate polarized Hopf algebra with a QTS $\mathcal{R}$. Let $V, W \in \mathcal{O}$. Let $v \in V[\lambda_v]$, $w \in W[\lambda_w]$. Assume that v, w are homogeneous with respect to the grading. Consider

$$M_\lambda^+ \xrightarrow{\Phi_\lambda^v} M_{\lambda-\lambda_v}^+ \otimes V \xrightarrow{\Phi_{\lambda-\lambda_v}^w \otimes 1} M_{\lambda-\lambda_v-\lambda_w}^+ \otimes W \otimes V$$

and denote this composition $\Phi_\lambda^{w,v}$.

Define the main object of this paper, a linear operator $J_{W,V}(\lambda) : W \otimes V \to W \otimes V$ as follows. Find $u \in W \otimes V[\lambda_v + \lambda_w]$ such that $\Phi_\lambda^{w,v} = \Phi_\lambda^u$ and set

$$J_{W,V}(\lambda)\, w \otimes v = u. \tag{1}$$

Lemma 11. *$J_{W,V}(\lambda)$ is strictly upper triangular, i.e. $J_{W,V}(\lambda)\, w\otimes v = w\otimes v + \sum w_i \otimes v_i$ where deg w_i < deg w.*

Corollary 12. *$J_{W,V}(\lambda) = 1 + N$, where N is locally nilpotent, hence $J_{W,V}(\lambda)$ is invertible and $J_{W,V}^{-1}(\lambda) = 1 - N + N^2 - \cdots$.*

We call the operators $J_{W,V}(\lambda)$ *fusion matrices*.

Define *a quantum dynamical R-matrix* $R_{V,W}(\lambda) : V \otimes W \to V \otimes W$ by

$$R_{V,W}(\lambda) = J_{V,W}^{-1}(\lambda)\mathcal{R}^{21}|_{V\otimes W} J_{W,V}^{21}(\lambda). \tag{2}$$

Theorem 13. *Let $v \in V$, $w \in W$ be homogeneous elements with respect to the grading and A_0. Let $R_{V,W}(\lambda)\, v \otimes w = \sum_i v_i \otimes w_i$, where v_i, w_i are homogeneous too. Then*

$$(1 \otimes P\mathcal{R}|_{W\otimes V})\, \Phi_\lambda^{w,v} = \sum_i \Phi_\lambda^{v_i,w_i}, \tag{3}$$

where P is the operator of permutation.

The proof follows from the definition of $R_{V,W}(\lambda)$.

Theorem 14. I. *J satisfies the 2-cocycle condition,*

$$J_{V\otimes W,U}(\lambda)\,(J_{V,W}(\lambda - h^{(3)}) \otimes 1) = J_{V,W\otimes U}(\lambda)\,(1 \otimes J_{W,U}(\lambda)). \tag{4}$$

II. *$R(\lambda)$ satisfies the quantum dynamical Yang–Baxter equation (QDYB),*

$$R^{12}(\lambda - h^{(3)})\, R^{13}(\lambda)\, R^{23}(\lambda - h^{(1)}) = R^{23}(\lambda)\, R^{13}(\lambda - h^{(2)})\, R^{12}(\lambda). \tag{5}$$

In these formulae $R^{12}(\lambda - h^{(3)})\, v \otimes w \otimes u = (R(\lambda - \lambda_u)\, v \otimes w) \otimes u$ if $u \in U[\lambda_u]$ and other symbols are defined analogously.

Proof. Define

$$\Phi_\lambda^{v_n,\dots,v_1} : M_\lambda^+ \to M^+_{\lambda - \sum_{i=1}^n \lambda_{v_i}} \otimes V_n \otimes \dots \otimes V_1$$

as the composition

$$M_\lambda^+ \xrightarrow{\Phi_\lambda^{v_1}} M^+_{\lambda-\lambda_{v_1}} \otimes V_1 \xrightarrow{\Phi^{v_2}_{\lambda-\lambda_{v_1}} \otimes 1} M^+_{\lambda-\lambda_{v_1}-\lambda_{v_2}} \otimes V_2 \otimes V_1 \to \dots .$$

In other words,

$$\Phi_\lambda^{v_n,\dots,v_1} = (\Phi^{v_n}_{\lambda - \sum_{i=1}^{n-1}\lambda_{v_i}} \otimes 1^{n-1}) \cdots (\Phi^{v_2}_{\lambda-\lambda_{v_1}} \otimes 1)\, \Phi_\lambda^{v_1}.$$

Lemma 15.

$$\Phi_\lambda^{v_n,\dots,v_1} = \Phi_\lambda^{v_n,\dots,v_{i+2}, J_{V_{i+1},V_i}(\lambda - \sum_{j=1}^{i-1}\lambda_{v_j})\, v_{i+1}\otimes v_i, v_{i-1},\dots,v_1}.$$

The proof is by definition of J.

Let $a \in V$, $b \in W$, $c \in U$. Then

$$\Phi_\lambda^{a,b,c} = \Phi_\lambda^{a, J_{W,U}(\lambda)\, b\otimes c} = \Phi_\lambda^{J_{V,W\otimes U}(\lambda)\,(1\otimes J_{W,U}(\lambda))\, a\otimes b\otimes c},$$
$$\Phi_\lambda^{a,b,c} = \Phi_\lambda^{J_{V,W}(\lambda-\lambda_c)\, a\otimes b, c} = \Phi_\lambda^{J_{V\otimes W,U}(\lambda)\,(J_{V,W}(\lambda-\lambda_c)\otimes 1)\, a\otimes b\otimes c}.$$

This proves the first statement of the theorem.

For $y \in V_n \otimes \dots \otimes V_1[\nu]$, $y = \sum_i v_n^i \otimes \dots \otimes v_1^i$, set $\Psi^y_{V_n,\dots,V_1}(\lambda) = \sum_i \Phi_\lambda^{v_n^i,\dots v_1^i}$.

Lemma 16.

$$P_{V_{i+1},V_i}\mathcal{R}_{V_{i+1},V_i}\Psi^{v_n\otimes\dots\otimes v_1}_{V_n,\dots,V_1}(\lambda) = \Psi^{R_{V_i,V_{i+1}}(\lambda - \sum_{j=1}^{i-1}\lambda_{v_j}) P_{V_{i+1},V_i}\, v_n\otimes\dots\otimes v_1}_{V_n,\dots,V_i,V_{i+1},\dots,V_1}(\lambda).$$

The proof is by definition of the quantum dynamical R-matrix.

In order to prove the second statement of the theorem we apply the lemma to the case $n = 3$. Namely, we consider the function $\Psi^{v_3\otimes v_2\otimes v_1}_{V_3,V_2,V_1}(\lambda)$ and express it via $\Psi^{w_1\otimes w_2\otimes w_3}_{V_1,V_2,V_3}(\lambda)$ in two different ways, using the two different reduced decompositions of the permutation $123 \to 321$. Comparing the two answers, we get the theorem. □

Remark. The explicit form of J_{VW} for $A = U_q(\mathfrak{g})$ has been recently computed in [A]. The fact that the twist in [A] coincides with our J_{VW} is proved in Sect. 9.

2.6. The tensor functor and exchange matrices. We recall (in a slightly generalized form) the setting of Sect. 3 of [EV2]. Let A_0 be a commutative, cocommutative finitely generated Hopf algebra such that the group $T = \mathrm{Spec} A_0$ is connected. Introduce a category $\mathcal{V}$ of A_0-vector spaces as follows.

The objects of $\mathcal{V}$ are diagonalizable A_0 modules, $V = \oplus_{\lambda \in T} V[\lambda]$, $V[\lambda] = \{v \in V \mid a_0 v = \lambda(a_0) v\}$.

Let M_T be the field of meromorphic functions on T and $V, W \in \mathcal{V}$. Define the space $\mathrm{Hom}_{\mathcal{V}}(V, W)$ as the space $\mathrm{Hom}_{A_0}(V, M_T \otimes W)$, thus a homomorphism of V to W (for finite dimensional $V, W \in \mathcal{V}$) is a meromorphic function on T with values in $\mathrm{Hom}_{A_0}(V, W)$.

Define a tensor structure on the category $\mathcal{V}$. Namely, let the tensor product of two objects $V \bar{\otimes} W$ be the standard tensor product of two diagonalizable A_0 modules. Define the tensor product $\bar{\otimes}$ of two morphisms $f : V \to V'$ and $g : W \to W'$ as

$$f \bar{\otimes} g(\lambda) = f^{(1)}(\lambda - h^{(2)})(1 \otimes g(\lambda)), \tag{6}$$

where $f^{(1)}(\lambda - h^{(2)})(1 \otimes g(\lambda))\, u \otimes v = (f(\lambda - \mu)u) \otimes g(\lambda)v$ if $g(\lambda)v \in W'[\mu]$.

Let A be a nondegenerate polarized Hopf algebra. Consider the category $\mathcal{O}$ of graded A-modules bounded from above and diagonalizable over A_0. We construct a tensor functor from the category $\mathcal{O}$ to the category $\mathcal{V}$.

By definition a tensor functor from $\mathcal{O}$ to $\mathcal{V}$ is a functor $F : \mathcal{O} \to \mathcal{V}$ and for any $V, W \in \mathcal{O}$ an isomorphism $J_{V,W} : F(V) \bar{\otimes} F(W) \to F(V \otimes W)$ such that $\{J_{V,W}\}$ is functorial and the two compositions $F(U) \bar{\otimes} F(V) \bar{\otimes} F(W) \to F(U \otimes V) \bar{\otimes} F(W) \to F(U \otimes V \otimes W)$ and $F(U) \bar{\otimes} F(V) \bar{\otimes} F(W) \to F(U) \bar{\otimes} F(V \otimes W) \to F(U \otimes V \otimes W)$ coincide. Then J is called *a tensor structure on* F.

Define a tensor functor $F : \mathcal{O} \to \mathcal{V}$ by sending an object $V \in \mathcal{O}$ to $F(V) = V$, considered as an A_0-module, and sending an A-homomorphism $\alpha : V \to W$ to $F(\alpha) = \alpha : V \to W$.

Define a tensor structure on F by

$$J_{V,W}(\lambda) : F(V) \bar{\otimes} F(W) \to F(V \otimes W), \tag{7}$$

where $J_{V,W}(\lambda)$ is defined by (1).

Lemma 17. *Formula (7) defines a tensor structure on F, i.e. the two compositions $F(U) \bar{\otimes} F(V) \bar{\otimes} F(W) \to F(U \otimes V) \bar{\otimes} F(W) \to F(U \otimes V \otimes W)$ and $F(U) \bar{\otimes} F(V) \bar{\otimes} F(W) \to F(U) \bar{\otimes} F(V \otimes W) \to F(U \otimes V \otimes W)$ coincide.*

Proof. The statement of the lemma is equivalent to formula (4). □

Define a braiding in $\mathcal{O}$ by $\beta = P\mathcal{R}$. Introduce

$$F(\beta) : F(V) \bar{\otimes} F(W) \to F(W) \bar{\otimes} F(V)$$

as the composition

$$F(V) \bar{\otimes} F(W) \xrightarrow{J_{V,W}(\lambda)} F(V \otimes W) \xrightarrow{P\mathcal{R}_{V,W}} F(W \otimes V) \xrightarrow{J^{-1}_{W,V}(\lambda)} F(W) \bar{\otimes} F(V). \tag{8}$$

Thus we have $F(\beta)_{V,W} = J^{-1}_{W,V}(\lambda) P_{V,W} \mathcal{R}|_{V \otimes W} J_{V,W}(\lambda)$. In particular, $F(\beta)_{V,W} P_{W,V} = J^{-1}_{W,V}(\lambda) \mathcal{R}^{21}|_{V \otimes W} J^{21}_{V,W}(\lambda) = R_{W,V}(\lambda)$, cf. (2). Notice that in Theorem 14 we showed that the R-matrix $R(\lambda)$ satisfies the QDYB equation; now it also follows from this tensor category construction and Theorem 3.3 in [EV2].

The operators $R_{V,W}(\lambda)$ will be called *the exchange matrices*.

3. *H*-Hopf Algebroids

3.1. Definitions. In this section we recall the definition of an H-Hopf algebroid, cf. [EV2]. Let H be a commutative and cocommutative Hopf algebra over $\mathbb{C}$, $T = \mathrm{Spec} H$ a commutative affine algebraic group. Let M_T denote the field of meromorphic functions on T. *An H-algebra* is an associative algebra A over $\mathbb{C}$ with 1, endowed with an T-bigrading $A = \oplus_{\alpha,\beta\in T} A_{\alpha\beta}$ (called the weight decomposition), and two algebra embeddings $\mu_l, \mu_r : M_T \to A_{00}$ (the left and the right moment maps), such that for any $a \in A_{\alpha\beta}$ and $f \in M_T$, we have

$$\mu_l(f(\lambda))a = a\mu_l(f(\lambda+\alpha)), \quad \mu_r(f(\lambda))a = a\mu_r(f(\lambda+\beta)). \tag{9}$$

Here $0 \in T$ denotes the unit element and $\lambda + \alpha$ denotes the sum in T.

A morphism $\varphi : A \to B$ of two H-algebras is an algebra homomorphism, preserving the moment maps.

Let A, B be two H-algebras and $\mu_l^A, \mu_r^A, \mu_l^B, \mu_r^B$ their moment maps. Define their *matrix tensor product*, $A\tilde{\otimes}B$, which is also an H-algebra. Let

$$(A\tilde{\otimes}B)_{\alpha\delta} := \oplus_\beta A_{\alpha\beta} \otimes_{M_T} B_{\beta\delta}, \tag{10}$$

where $\otimes_{M_T}$ means the usual tensor product modulo the relation $\mu_r^A(f)\, a \otimes b = a \otimes \mu_l^B(f)b$, for any $a \in A, b \in B, f \in M_T$. Introduce a multiplication in $A\tilde{\otimes}B$ by the rule $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$. Define the moment maps for $A\tilde{\otimes}B$ by $\mu_l^{A\tilde{\otimes}B}(f) = \mu_l^A(f) \otimes 1$, $\mu_r^{A\tilde{\otimes}B}(f) = 1 \otimes \mu_r^B(f)$.

A coproduct on an H-algebra A is a homomorphism of H-algebras $\Delta : A \to A\tilde{\otimes}A$.

Let D_T be the algebra of difference operators $M_T \to M_T$, i.e. the operators of the form $\sum_{i=1}^n f_i(\lambda)\mathcal{T}_{\beta_i}$, where $f_i \in M_T$, and for $\beta \in T$ we denote by $\mathcal{T}_\beta$ the field automorphism of M_T given by $(\mathcal{T}_\beta f)(\lambda) = f(\lambda+\beta)$.

The algebra D_T is an example of an H-algebra if we define the weight decomposition by $D_T = \oplus(D_T)_{\alpha\beta}$, where $(D_T)_{\alpha\beta} = 0$ if $\alpha \neq \beta$, and $(D_T)_{\alpha\alpha} = \{f(\lambda)\mathcal{T}_\alpha^{-1} : f \in M_T\}$, and the moment maps $\mu_l = \mu_r : M_T \to (D_T)_{00}$ to be the tautological isomorphism.

For any H-algebra A, the algebras $A\tilde{\otimes}D_T$ and $D_T\tilde{\otimes}A$ are canonically isomorphic to A. In particular, D_T is canonically isomorphic to $D_T\tilde{\otimes}D_T$. Thus the category of H-algebras equipped with the product $\tilde{\otimes}$ is a monoidal category, where the unit object is D_T.

A counit on an H-algebra A is a homomorphism of H-algebras $\epsilon : A \to D_T$.

An H-bialgebroid is a H-algebra A equipped with a coassociative coproduct Δ (i.e. such that $(\Delta \otimes \mathrm{Id}_A) \circ \Delta = (\mathrm{Id}_A \otimes \Delta) \circ \Delta$, and a counit ϵ such that $(\epsilon \otimes \mathrm{Id}_A) \circ \Delta = (\mathrm{Id}_A \otimes \epsilon) \circ \Delta = \mathrm{Id}_A$.

For example, D_T is an H-bialgebroid where $\Delta : D_T \to D_T \tilde{\otimes} D_T$ is the canonical isomorphism and $\epsilon = \mathrm{Id}$.

Let A be an H-algebra. A linear map $S : A \to A$ is called *an antiautomorphism* of an H-algebra if it is an antiautomorphism of algebras and $\mu_r \circ S = \mu_l$, $\mu_l \circ S = \mu_r$. From these conditions it follows that $S(A_{\alpha\beta}) = A_{-\beta,-\alpha}$.

Let A be an H-bialgebroid, and let Δ, ϵ be the coproduct and counit of A. For $a \in A$, let

$$\Delta(a) = \sum_i a_i^1 \otimes a_i^2. \tag{11}$$

An antipode on the H-bialgebroid A is an antiautomorphism of H-algebras $S : A \to A$ such that for any $a \in A$ and any presentation (11) of $\Delta(a)$, one has

$$\sum_i a_i^1 S(a_i^2) = \mu_l(\epsilon(a)1), \ \sum_i S(a_i^1)a_i^2 = \mu_r(\epsilon(a)1),$$

where $\epsilon(a)1 \in M_T$ is the result of the application of the difference operator $\epsilon(a)$ to the constant function 1.

An H-bialgebroid with an antipode is called *an H-Hopf algebroid.*

Let W be a diagonalizable H-module, $W = \oplus_{\lambda \in T} W[\lambda]$, $W[\lambda] = \{w \in W \mid aw = \lambda(a)w$, for all $a \in H\}$, and let $D^{\alpha}_{T,W} \subset \mathrm{Hom}_{\mathbb{C}}(W, W \otimes D_T)$ be the space of all difference operators on T with coefficients in $\mathrm{End}_{\mathbb{C}}(W)$, which have weight $\alpha \in T$ with respect to the action of H in W.

Consider the algebra $D_{T,W} = \oplus_{\alpha} D^{\alpha}_{T,W}$. This algebra has a weight decomposition $D_{T,W} = \oplus_{\alpha,\beta}(D_{T,W})_{\alpha\beta}$ defined as follows: if $g \in \mathrm{Hom}_{\mathbb{C}}(W, W \otimes M_T)$ is an operator of weight $\beta - \alpha$, then $g\mathcal{T}_{\beta}^{-1} \in (D_{T,W})_{\alpha\beta}$.

Define the moment maps $\mu_l, \mu_r : M_T \to (D_{T,W})_{00}$ by the formulas $\mu_r(f(\lambda)) = f(\lambda)$, $\mu_l(f(\lambda)) = f(\lambda - h)$, where $f(\lambda - h)w = f(\lambda - \mu)w$ if $w \in W[\mu]$, $\mu \in T$. The algebra $D_{T,W}$ equipped with this weight decomposition and these moment maps is an H-algebra.

Let $f \in \mathrm{Hom}(W, W \otimes M_T)$ and $g \in \mathrm{Hom}(U, U \otimes M_T)$. Define $f \bar{\otimes} g \in \mathrm{Hom}\,(W \otimes U, W \otimes U \otimes M_T)$ as

$$f \bar{\otimes} g(\lambda) = f^{(1)}(\lambda - h^{(2)})(1 \otimes g(\lambda)), \tag{12}$$

where $f^{(1)}(\lambda - h^{(2)})(1 \otimes g(\lambda))\, w \otimes u = (f(\lambda - \mu)w) \otimes g(\lambda)u$ if $g(\lambda)u \in U[\mu]$, cf. (6).

Lemma 18 **(**[EV2]**).** *There is a natural embedding of H-algebras $\theta_{W,U} : D_{T,W} \widetilde{\otimes} D_{T,U} \to D_{T,W\otimes U}$, given by the formula $f\mathcal{T}_{\beta} \otimes g\,\mathcal{T}_{\delta} \to (f \bar{\otimes} g)\mathcal{T}_{\delta}$. This embedding is an isomorphism if W, U are finite-dimensional.*

A dynamical representation of an H-algebra A is a diagonalizable H-module W endowed with a homomorphism of H-algebras $\pi_W : A \to D_{T,W}$. *A homomorphism* of dynamical representations $\varphi : W_1 \to W_2$ is an element of $\mathrm{Hom}_{\mathbb{C}}(W_1, W_2 \otimes M_T)$ such that $\varphi \circ \pi_{W_1}(x) = \pi_{W_2}(x) \circ \varphi$ for all $x \in A$.

Example. If A has a counit, then A has *the trivial representation*: $W = \mathbb{C}$, $\pi = \epsilon$.

If A is an H-bialgebroid, W and U are two dynamical representations of A, then the H-module $W \otimes U$ is a dynamical representation, $\pi_{W\otimes U}(x) = \theta_{WU} \circ (\pi_W \otimes \pi_U) \circ \Delta(x)$. If $f : W_1 \to W_2$ and $g : U_1 \to U_2$ are homomorphisms of dynamical representations, then so is $f \bar{\otimes} g : W_1 \otimes U_1 \to W_2 \otimes U_2$. Thus, dynamical representations of A form a monoidal category $\mathrm{Rep}(A)$, whose identity object is the trivial representation.

If A is an H-Hopf algebroid and V is a dynamical representation, then one can define the left and right dual dynamical representations ${}^*W$ and W^* as follows, see [EV2].

If (W, π_W) is a dynamical representation of an H-algebra A, we denote $\pi^0_W : A \to \mathrm{Hom}(W, W \otimes M_T)$ the map defined by $\pi^0_W(x)w = \pi_W(x)w$, $w \in W$ (the difference operator $\pi_W(x)$ restricted to the constant functions). It is clear that π_W is completely determined by π^0_W.

Let (W, π_W) be a dynamical representation of A. Then *the right dual representation* to W is (W^*, π_{W^*}), where W^* is the H-graded dual to W, and

$$\pi^0_{W^*}(x)(\lambda) = \pi^0_W(S(x))(\lambda + h - \alpha)^t \tag{13}$$

for $x \in A_{\alpha\beta}$, where t denotes dualization. The left dual representation to W is $({}^*W, \pi_{{}^*W})$, where ${}^*W = W^*$, and

$$\pi^0_{{}^*W}(x)(\lambda) = \pi^0_W(S^{-1}(x))(\lambda + h - \alpha)^t \tag{14}$$

for $x \in A_{\alpha\beta}$. Here $\pi^0_W(S(x))(\lambda + h - \alpha)^t$ denotes the result of two operations applied successively to $\pi^0_W(S(x))$: shifting of the argument, and dualization.

Formulas (13) and (14) define dynamical representations of A. Moreover, if $A(\lambda) : W_1 \to W_2$ is a morphism of dynamical representations, then $A^*(\lambda) := A(\lambda+h)^t$ defines morphisms $W_2^* \to W_1^*$ and ${}^*W_2 \to {}^*W_1$.

3.2. An H-bialgebroid associated to a function $R : T \to$ End $(V \otimes V)$. In this Section we recall a construction from [EV2] of an H-bialgebroid A_R associated to a meromorphic function $R : T \to \text{End}\,(V \otimes V)$, where V is a finite dimensional diagonalizable H-module and $R(\lambda)$ is invertible for generic λ.

By definition the algebra A_R is generated by two copies of M_T (embedded as subalgebras) and matrix elements of the operators $L^{\pm 1} \in \text{End}(V) \otimes A_R$. We denote the elements of the first copy of M_T by $f(\lambda^1)$ and of the second copy by $f(\lambda^2)$, where $f \in M_T$. We denote $(L^{\pm 1})_{\alpha\beta}$ the weight components of $L^{\pm 1}$ with respect to the natural T-bigrading on End (V), so that $L^{\pm 1} = (L^{\pm 1}_{\alpha\beta})$, where $L^{\pm 1}_{\alpha\beta} \in \text{Hom}_{\mathbb{C}}(V[\beta], V[\alpha]) \otimes A_R$.

Introduce the moment maps for A_R by $\mu_l(f) = f(\lambda^1)$, $\mu_r(f) = f(\lambda^2)$, and the weight decomposition by $f(\lambda^1), f(\lambda^2) \in (A_R)_{00}$, $L_{\alpha\beta} \in \text{Hom}_{\mathbb{C}}(V[\beta], V[\alpha]) \otimes (A_R)_{\alpha\beta}$.

The defining relations for A_R are:

$$f(\lambda^1)L_{\alpha\beta} = L_{\alpha\beta} f(\lambda^1 + \alpha); \qquad f(\lambda^2)L_{\alpha\beta} = L_{\alpha\beta} f(\lambda^2 + \beta); \tag{15}$$

$$LL^{-1} = L^{-1}L = 1; \qquad [f(\lambda^1), g(\lambda^2)] = 0; \tag{16}$$

and the dynamical Yang–Baxter relation

$$R^{12}(\lambda^1)L^{13}L^{23} =: L^{23}L^{13}R^{12}(\lambda^2) : . \tag{17}$$

Here the :: sign means that the matrix elements of L should be put on the right of the matrix elements of R. Thus, if $\{v_a\}$ is a homogeneous basis of V, and $L = \sum E_{ab} \otimes L_{ab}$, $R(\lambda)(v_a \otimes v_b) = \sum R^{ab}_{cd}(\lambda) v_c \otimes v_d$, then (17) has the form

$$\sum R^{xy}_{ac}(\lambda^1) L_{xb} L_{yd} = \sum R^{bd}_{xy}(\lambda^2) L_{cy} L_{ax},$$

where we sum over repeated indices.

Define the coproduct on A_R, $\Delta : A_R \to A_R \tilde{\otimes} A_R$, by

$$\Delta(L) = L^{12}L^{13}, \Delta(L^{-1}) = (L^{-1})^{13}(L^{-1})^{12}.$$

Define the counit by

$$\epsilon(L_{\alpha\beta}) = \delta_{\alpha\beta}\text{Id}_{V[\alpha]} \otimes \mathcal{T}_\alpha^{-1}, \epsilon((L^{-1})_{\alpha\beta}) = \delta_{\alpha\beta}\text{Id}_{V[\alpha]} \otimes \mathcal{T}_\alpha,$$

where $\text{Id}_{V[\alpha]} : V[\alpha] \to V[\alpha]$ is the identity operator. On an antipode for A_R see Sect. 4.5 in [EV2].

3.3. A rational H-bialgebroid associated to a rational function $R : T \to \mathrm{End}\,(V \otimes V)$. Assume that a function $R : T \to \mathrm{End}\,(V \otimes V)$ is a rational function of λ, where V is a finite dimensional diagonalizable H-module and $R(\lambda)$ is invertible for generic λ.

The H-bialgebroid A_R is defined over the field of meromorphic functions M_T. We replace the field of meromorphic functions M_T by the field of rational functions $\mathbb{C}(T)$ and define in the same way *the rational H-bialgebroid $A_{rat,R}$* associated to a rational function R.

4. The Exchange Dynamical Quantum Groups

4.1. The definition of an exchange dynamical quantum group. Let A be a polarized and nondegenerate Hopf algebra as in Sect. 1. Assume that T =Spec A_0 is connected.

Let $\mathcal{R} \in A \hat{\otimes} A$ be a quasitriangular structure on A. We always assume that $\mathcal{R} \in A_{\geqslant 0} \hat{\otimes} A_{\leqslant 0}$.

Let $\mathcal{O}_0 \subset \mathcal{O}$ be a full abelian tensor subcategory which is semisimple and such that all modules in $\mathcal{O}_0$ are finite dimensional. (Recall that a full subcategory $\mathcal{O}_0$ consists of some objects of $\mathcal{O}$ and for any $V, W \in \mathcal{O}_0$ we have $\mathrm{Hom}_{\mathcal{O}_0}(V, W) = \mathrm{Hom}_{\mathcal{O}}(V, W)$.) Let $Ir \subset \mathcal{O}_0$ be the set of all irreducible modules.

Examples of such categories $\mathcal{O}_0$ are provided by the categories of finite dimensional representations of semisimple Lie algebras and corresponding quantum groups (not at roots of unity).

The goal of this section is to define an A_0-Hopf algebroid $E = E(\mathcal{O}_0)$ called *an exchange dynamical quantum group*.

Define E as a vector space to be

$$M_T \otimes_{\mathbb{C}} M_T \otimes_{\mathbb{C}} \bar{E},$$

where $\bar{E} = \oplus_{U \in Ir} U \otimes U^*$ and U^* is the dual module to U. A T-bigrading on E is defined by $E = \oplus_{\alpha,\beta \in T} E_{\alpha,\beta}$, where $E_{\alpha,\beta} = M_T \otimes_{\mathbb{C}} M_T \otimes_{\mathbb{C}} \bar{E}_{\alpha,\beta}$ and $\bar{E}_{\alpha,\beta} \subset \oplus_{U \in Ir} U \otimes U^*$ is the subspace generated by all elements of the form $u \otimes v \in U[\alpha] \otimes (U[\beta])^*$, $U \in Ir$.

Let $\mathbb{C}_A$ be the trivial A-module, $\mathbb{C}_A = \mathbb{C}e$. The subspace $E_{0,0}$ has a component coming from the trivial module, $M_T \otimes M_T \otimes \mathbb{C}_A \otimes \mathbb{C}_A^*$. For a meromorphic function $f(\lambda) \in M_T$, the elements $f(\lambda) \otimes 1 \otimes e \otimes e^*$ and $1 \otimes f(\lambda) \otimes e \otimes e^*$ will be denoted $f(\lambda^1)$ and $f(\lambda^2)$, respectively.

Let v_i^U be a basis in $U \in Ir$, which is homogeneous with respect to T and the $\mathbb{Z}$-grading. Then $v_i^U \otimes (v_j^U)^*$ form a basis in $\bar{E}$. Let $\omega_i^U \in T$ be the weight of v_i^U.

Set $L_{ij}^U = 1 \otimes 1 \otimes v_i^U \otimes (v_j^U)^*$. Define a linear map $E_{ij}^U : U \to U$ by $E_{ij}^U v_k^U = \delta_{jk} v_i^U$. Introduce $L^U \in \mathrm{End}\,(U) \otimes \bar{E}$ by

$$L^U = \sum_{ij} E_{ij}^U \otimes L_{ij}^U.$$

The relations in E between $f(\lambda^1)$, $f(\lambda^2)$, and L_{ij}^U are defined by

$$f(\lambda^1) f(\lambda^2) = f(\lambda^2) f(\lambda^1), \tag{18}$$

$$f(\lambda^1) L_{ij}^U = L_{ij}^U f(\lambda^1 + \omega_i^U), \qquad f(\lambda^2) L_{ij}^U = L_{ij}^U f(\lambda^2 + \omega_j^U). \tag{19}$$

In order to define the product of two elements L_{ij}^V and $L_{i'j'}^W$ we will consider

$$(L^V)^{23}(L^W)^{13} \in \text{End }(V) \otimes \text{End }(W) \otimes E.$$

Let $U \in Ir$, $V, W \in \mathcal{O}_0$ and $H_{V,W}^U = \text{Hom}_A(U, V \otimes W)$. Then we have an isomorphism $\tau_{V,W} : \oplus_{U \in Ir} H_{V,W}^U \otimes U \to V \otimes W$ given by $\tau_{V,W}(h \otimes u) = h(u)$. Let $\bar\tau_{V,W}^U : V \otimes W \to H_{V,W}^U \otimes U$ be the projection along the other summands, $\tau_{V,W}^U : H_{V,W}^U \otimes U \to V \otimes W$ the restriction of $\tau_{V,W}$ to the isotypic component $H_{V,W}^U \otimes U$. We have

$$\bar\tau_{V,W}^U \tau_{V,W}^U = \text{Id}, \quad \tau_{V,W}^U \bar\tau_{V,W}^U = p_U,$$

where p_U is the projection on the U-isotypical component.

Define the product of elements in the exchange quantum group by a formula analagous to the formula for the product of matrix elements of representations of a group considered as functions on the group. Namely, define the product $L_{ij}^V L_{i'j'}^W$ by

$$(L^V)^{23}(L^W)^{13} = :(J_{W,V}^{12}(\lambda^1))^{-1} \sum_{U \in Ir} (\tau_{W,V}^U)^{12} \, (\text{Id}_{H_{W,V}^U} \otimes L^U) \, (\bar\tau_{W,V}^U)^{12} \, J_{W,V}^{12}(\lambda^2) : . \tag{20}$$

This is an identity in End $(W) \otimes$ End $(V) \otimes E$. Here the :: sign ("normal ordering") means that the matrix elements of L^U should be put on the right of the matrix elements of $J_{V,W}(\lambda^1)$, $J_{V,W}(\lambda^2)$. Thus, if

$$(J_{V,W}^{12}(\lambda^1))^{-1} = \sum E_{ij}^V \otimes E_{kl}^W \otimes a_{ijkl}(\lambda^1),$$

$$\sum_{U \in Ir} (\tau_{V,W}^U)^{12} \, (\text{Id}_{H_{V,W}^U} \otimes L^U) \, (\bar\tau_{V,W}^U)^{12} = \sum E_{i"j"}^V \otimes E_{k"l"}^W \otimes a"_{i"j"k"l"},$$

$$J_{V,W}^{12}(\lambda^2) = \sum E_{i'j'}^V \otimes E_{k'l'}^W \otimes a'_{i'j'k'l'}(\lambda^2),$$

then (20) has the form

$$L_{kl'}^W L_{ij'}^V = \sum a_{ijkl}(\lambda^1) a'_{j"j'l"l'}(\lambda^2) a"_{jj"ll"}.$$

More generally, let $a = a_1 \dots a_n$ be a monomial in generators of E; so each of the factors has the form $f(\lambda^1)$, $f(\lambda^2)$, or L_{ij}^V. Define the normal ordering $: a :$ as the product of the same elements $a_1, \dots, a_n$ in which all elements of the form $f(\lambda^1)$, $f(\lambda^2)$ are put on the left and the remaining elements of the form L_{ij}^V are put on the right in the same order as in a. Extend by linearity the normal ordering operation to all polynomials in generators in E. If $v = (v_1, \dots, v_l)$ is a vector whose coefficients are polynomials in generators of E, then define the normal ordering $: v :$ as $: v := (: v_1 :, \dots, : v_l :)$.

Let $\mathbb{C}_A$ be the trivial module. Since $J_{\mathbb{C}_A,V} = J_{V,\mathbb{C}_A} = \text{Id}_V$ we have

$$(L^{\mathbb{C}_A})^{23}(L^W)^{13} = (L^W)^{13}, \quad (L^V)^{23}(L^{\mathbb{C}_A})^{13} = (L^V)^{23}. \tag{21}$$

Corollary 19. *The element* $1 \otimes 1 \otimes e \otimes e^*$ *of* E *corresponding to the trivial module is the unit element of the algebra* E.

Theorem 20. *E is an associative algebra.*

Proof. We start with preliminary lemmas.

Lemma 21. I. *Let $V' \subset V$ be objects in $\mathcal{O}_0$, then $J_{V,W}|_{V'\otimes W} = J_{V',W}$. Let $W' \subset W$ be A-modules, then $J_{V,W}|_{V\otimes W'} = J_{V,W'}$.*

II. *Let $V = V_1 \oplus V_2$, then $J_{V,W} = J_{V_1,W} \oplus J_{V_2,W}$. Let $W = W_1 \oplus W_2$, then $J_{V,W} = J_{V,W_1} \oplus J_{V,W_2}$.*

III. *For $U \in Ir$, $V, W, Z \in \mathcal{O}_0$, the maps $Z \otimes W \otimes V \to H^U_{Z,W} \otimes U \otimes V$ given by $(\bar\tau^{\,U}_{Z,W} \otimes \mathrm{Id}_V)\, J_{Z\otimes W,V}$ and $(\mathrm{Id}_{H^U_{Z,W}} \otimes J_{U,V})\,(\bar\tau^{\,U}_{Z,W} \otimes \mathrm{Id}_V)$ coincide.*

IV. *The maps $Z \otimes H^U_{W,V} \otimes U \to Z \otimes W \otimes V$ given by $J_{Z,W\otimes V}\,(\mathrm{Id}_Z \otimes \tau^U_{W,V})$ and $(\mathrm{Id}_Z \otimes \tau^U_{W,V})\,(J_{Z,U})^{13}$ coincide. In particular, $J^{-1}_{Z,W\otimes V}\,(\mathrm{Id}_Z \otimes \tau^U_{W,V}) = (\mathrm{Id}_Z \otimes \tau^U_{W,V})(J^{-1}_{Z,U})^{13}$*

The lemma follows from functorial properties of J. Now we prove the theorem. We want to show that

$$(L^V)^{34}((L^W)^{24}(L^Z)^{14}) = ((L^V)^{34}(L^W)^{24})(L^Z)^{14}. \tag{22}$$

We have

$$\begin{aligned}&\mathrm{RHS} =:\\ &(J^{-1}_{W,V}(\lambda^1))^{23} \sum_{U\in Ir} (\tau^U_{W,V})^{23}(\mathrm{Id}_{H^U_{W,V}} \otimes L^U)^{234}(\bar\tau^{\,U}_{W,V})^{23} J^{23}_{W,V}(\lambda^2)(L^Z)^{14} :.\end{aligned} \tag{23}$$

First we replace $(\bar\tau^{\,U}_{W,V})^{23} J^{23}_{W,V}(\lambda^2)(L^Z)^{14}$ with $(L^Z)^{14}(\bar\tau^{\,U}_{W,V})^{23} J^{23}_{W,V}(\lambda^2)$. Consider $(\mathrm{Id}_{H^U_{W,V}} \otimes L^U)^{234}(L^Z)^{14}$ as an element of the tensor product $\mathrm{End}\,(Z) \otimes \mathrm{End}\,(H^U_{W,V}) \otimes \mathrm{End}\,(U) \otimes \bar E$, then the element $(\mathrm{Id}_{H^U_{W,V}} \otimes L^U)^{234}(L^Z)^{14}$ takes the form $(\mathrm{Id}_{H^U_{W,V}})^{2'}$ $(L^U)^{3'4}$ $(L^Z)^{14}$, where $^{2'}$, $^{3'}$ label these new tensor factors. Applying formula (20) to the first, third and fourth factors, we get

$$(L^U)^{3'4}(L^Z)^{14} =: (J^{13'}_{Z,U}(\lambda^1))^{-1} \sum_{Y\in Ir} (\tau^Y_{Z,U})^{13'}\,(\mathrm{Id}_{H^Y_{Z,U}} \otimes L^Y)\,(\bar\tau^{\,Y}_{Z,U})^{13'}\, J^{13'}_{Z,U}(\lambda^2) :.$$

Returning to (23) we get

$$\begin{aligned}&\mathrm{RHS} =: (J^{-1}_{W,V}(\lambda^1))^{23} \sum_{U\in Ir} (\tau^U_{W,V})^{23}\\ &(J^{13}_{Z,U}(\lambda^1))^{-1} \sum_{Y\in Ir} (\tau^Y_{Z,U})^{13}\,(\mathrm{Id}_{H^Y_{Z,U}} \otimes L^Y)\,(\bar\tau^{\,Y}_{Z,U})^{13}\, J^{13}_{Z,U}(\lambda^2)(\bar\tau^{\,U}_{W,V})^{23} J^{23}_{W,V}(\lambda^2) :.\end{aligned}$$

Applying Lemma 21 we get

$$\begin{aligned}\mathrm{RHS} =\,&: (J^{-1}_{W,V}(\lambda^1))^{23}(J^{-1}_{Z,W\otimes V}(\lambda^1))^{1,23} \sum_{U\in Ir} (\tau^U_{W,V})^{23} \sum_{Y\in Ir} (\tau^Y_{Z,U})^{13}\,(\mathrm{Id}_{H^Y_{Z,U}} \otimes L^Y)\times\\ &(\bar\tau^{\,Y}_{Z,U})^{13}\,(\bar\tau^{\,U}_{W,V})^{23} J^{1,23}_{Z,W\otimes V}(\lambda^2) J^{23}_{W,V}(\lambda^2) :.\end{aligned} \tag{24}$$

Now we compute the left-hand side of (22),

$$\begin{aligned}
&\text{LHS} = \sum_{ij} (E^V_{ij})^3 (L^V_{ij})^4 : (J^{12}_{Z,W}(\lambda^1))^{-1} \sum_{U\in Ir} (\tau^U_{Z,W})^{12} \,(\mathrm{Id}_{H^U_{Z,W}} \otimes L^U) \times \\
&(\bar\tau^{\,U}_{\ Z,W})^{12} J^{12}_{Z,W}(\lambda^2) := \sum_{ij} (E^V_{ij})^3 : (J^{12}_{Z,W}(\lambda^1 - \omega_i))^{-1} (L^V_{ij})^4 \sum_{U\in Ir} (\tau^U_{Z,W})^{12} \times \\
&(\mathrm{Id}_{H^U_{Z,W}} \otimes L^U)\, (\bar\tau^{\,U}_{\ Z,W})^{12} J^{12}_{Z,W}(\lambda^2 - \omega_j) := \\
&: (J^{12}_{Z,W}(\lambda^1 - h^{(3)}))^{-1} \sum_{U\in Ir} (\tau^U_{Z,W})^{12} (L^V)^{34} (\mathrm{Id}_{H^U_{Z,W}} \otimes L^U) (\bar\tau^{\,U}_{\ Z,W})^{12} J^{12}_{Z,W}(\lambda^2 - h^{(3)}) := \\
&: (J^{12}_{Z,W}(\lambda^1 - h^{(3)}))^{-1} \sum_{U\in Ir} (\tau^U_{Z,W})^{12} (J^{23}_{U,V}(\lambda^1))^{-1} \\
&\sum_{Y\in Ir} (\tau^Y_{U,V})^{23} (\mathrm{Id}_{H^Y_{U,V}} \otimes L^Y) (\bar\tau^{\,Y}_{\ U,V})^{23} J^{23}_{U,W}(\lambda^2) (\bar\tau^{\,U}_{\ Z,W})^{12} J^{12}_{Z,W}(\lambda^2 - h^{(3)}) := \\
&: (J^{23}_{Z,W}(\lambda^1 - h^{(3)}))^{-1} (J^{12,3}_{Z\otimes W,V}(\lambda^1))^{-1} \sum_{U\in Ir} (\tau^U_{Z,W})^{12} \\
&\sum_{Y\in Ir} (\tau^Y_{U,V})^{23} (\mathrm{Id}_{H^Y_{U,V}} \otimes L^Y) (\bar\tau^{\,Y}_{\ U,V})^{23} (\bar\tau^{\,U}_{\ Z,W})^{12} J^{12,3}_{Z\otimes W,V}(\lambda^2) J^{12}_{Z,W}(\lambda^2 - h^{(3)}) : .
\end{aligned} \tag{25}$$

Formulas (24) and (25) and Theorem 14 imply the theorem. □

Theorem 22. *For $V, W \in Ir$, we have*

$$R^{12}_{V,W}(\lambda^1)(L^V)^{13}(L^W)^{23} =: (L^W)^{23}(L^V)^{13} R^{12}_{V,W}(\lambda^2) :, \tag{26}$$

where the normal ordering sign :: as before means that the matrix elements of L should be put on the right of the matrix elements of R. Thus, if $L^V = \sum E_{ij} \otimes L^V_{ij}$, $L^W = \sum E_{kl} \otimes L^W_{kl}$, $R(\lambda) = \sum E^V_{ij} \otimes E^W_{kl} \otimes R_{ijkl}(\lambda)$, then (26) has the form

$$\sum_{j,l} R_{ijkl}(\lambda^1) L^V_{jj'} L^W_{ll'} = \sum_{j,l} R_{jj'll'}(\lambda^2) L^W_{kl} L^V_{ij}.$$

Proof.

$$\begin{aligned}
&\text{RHS} =: (L^W)^{23}(L^V)^{13} R^{12}_{V,W}(\lambda^2) := \\
&: (J^{12}_{V,W}(\lambda^1))^{-1} \sum_{U\in Ir} (\tau^U_{V,W})^{12} (\mathrm{Id}_{H^U_{V,W}} \otimes L^U) (\bar\tau^{\,U}_{\ V,W})^{12} J^{12}_{V,W}(\lambda^2) R^{12}_{V,W}(\lambda^2) := \\
&: (J^{12}_{V,W}(\lambda^1))^{-1} \sum_{U\in Ir} (\tau^U_{V,W})^{12} (\mathrm{Id}_{H^U_{V,W}} \otimes L^U) (\bar\tau^{\,U}_{\ V,W})^{12} \mathcal{R}^{21}|_{V\otimes W} P_{W,V} P_{V,W} J^{21}_{W,V}(\lambda^2) :
\end{aligned}$$

Since $\mathcal{R}^{21}|_{V\otimes W} P_{W,V}$ is an intertwiner, the last expression is equal to

$$\begin{aligned}
&: (J^{12}_{V,W}(\lambda^1))^{-1} \mathcal{R}^{21}|_{V\otimes W} P_{W,V} \sum_{U\in Ir} (\tau^U_{W,V})^{12} (\mathrm{Id}_{H^U_{W,V}} \otimes L^U) (\bar\tau^{\,U}_{\ W,V})^{12} P_{V,W} J^{21}_{W,V}(\lambda^2) := \\
&: (J^{12}_{V,W}(\lambda^1))^{-1} \mathcal{R}^{21}|_{V\otimes W} P_{W,V} J^{12}_{W,V}(\lambda^1) (L^V)^{23} (L^W)^{13} P_{V,W} := \\
&: (J^{12}_{V,W}(\lambda^1))^{-1} \mathcal{R}^{21}|_{V\otimes W} J^{21}_{W,V}(\lambda^1) (L^V)^{13} (L^W)^{23} := \text{LHS}. \qquad \square
\end{aligned}$$

We proved that E is an A_0-algebra. Hence $E\tilde{\otimes} E$ is an A_0-algebra. Define *a comultiplication* $\Delta : E \to E\tilde{\otimes} E$ by

$$\Delta f(\lambda^1) = f(\lambda^1), \quad \Delta f(\lambda^2) = f(\lambda^2), \quad \Delta(L^V) = (L^V)^{12}\,(L^V)^{13},$$

where $\Delta(L^V)$ means that Δ acts in the second factor.

Theorem 23. *The map Δ preserves the defining relations in E.*

Proof. Relations (9) are obviously preserved. We check that relation (20) is preserved. Compute the image under $1 \otimes 1 \otimes \Delta$ of the LHS and RHS of (20). The elements $(1 \otimes 1 \otimes \Delta)\,\mathrm{LHS}$, $(1 \otimes 1 \otimes \Delta)\,\mathrm{LHS}$, lie in $W \otimes V \otimes E\tilde{\otimes} E$. Denote λ_1^1, λ_1^2 the λ-variables of the third factor, and λ_2^1, λ_2^2 the λ-variables of the fourth. We have

$$\begin{aligned}&(1 \otimes 1 \otimes \Delta)\,\mathrm{LHS} = (L^V)^{23}(L^V)^{24}(L^W)^{13}(L^W)^{14} = (L^V)^{23}(L^W)^{13}(L^V)^{24}(L^W)^{14} = \\ &: (J^{12}_{W,V}(\lambda_1^1))^{-1} \sum_{U\in Ir} (\tau^U_{W,V})^{12}\,(\mathrm{Id}_{H^U_{W,V}} \otimes L^U)^{123}\,(\bar{\tau}^{\,U}_{W,V})^{12}\,J^{12}_{W,V}(\lambda_1^2) : \times \\ &: (J^{12}_{W,V}(\lambda_2^1))^{-1} \sum_{Y\in Ir} (\tau^Y_{W,V})^{12}\,(\mathrm{Id}_{H^Y_{W,V}} \otimes L^Y)^{124}\,(\bar{\tau}^{\,Y}_{W,V})^{12}\,J^{12}_{W,V}(\lambda_2^2) : . \end{aligned} \tag{27}$$

We cancel $J^{12}_{W,V}(\lambda_1^2)$ and $(J^{12}_{W,V}(\lambda_2^1))^{-1}$ since in $E\tilde{\otimes} E$ we have a relation $f(\lambda_1^2)a\tilde{\otimes} b = a\tilde{\otimes} f(\lambda_2^1)b$. We replace $\bar{\tau}^{\,U}_{W,V} \sum_{Y\in Ir} \tau^Y_{W,V}$ with $\mathrm{Id}_{H^U_{W,V}} \otimes \mathrm{Id}_U$ and use the relation $f(\lambda^2)(a\tilde{\otimes} b) = a\tilde{\otimes} f(\lambda_2^2)b$ in $E\tilde{\otimes} E$. Thus,

$$\begin{aligned}&(1 \otimes 1 \otimes \Delta)\,\mathrm{LHS} =: (J^{12}_{W,V}(\lambda_1^1))^{-1} \sum_{U\in Ir} (\tau^U_{W,V})^{12}\,(\mathrm{Id}_{H^U_{W,V}} \otimes L^U)^{123} \\ &(\mathrm{Id}_{H^U_{W,V}} \otimes L^U)^{124}\,(\bar{\tau}^{\,U}_{W,V})^{12}\,J^{12}_{W,V}(\lambda_2^2) := (1 \otimes 1 \otimes \Delta)\,\mathrm{RHS}. \qquad \square\end{aligned}$$

For $V \in \mathcal{O}_0$, define $\mathrm{Id}_{V[\mu]} : V \to V$ by $\mathrm{Id}_{V[\mu]}|_{V[\mu]} = \mathrm{Id}$ and $\mathrm{Id}_{V[\mu]}|_{V[\nu]} = 0$ for $\nu \neq \mu$.

Define *a counit* $\epsilon : E \to D_T$, where D_T is the A_0-algebra of scalar difference operators on T. Set

$$\epsilon(L^V) = \oplus_\mu \mathrm{Id}_{V[\mu]} \otimes \mathcal{T}_\mu^{-1}, \quad \epsilon(f(\lambda^j)) = f(\lambda). \tag{28}$$

Theorem 24. *ϵ is a counit in E.*

Proof. The relation

$$(\epsilon \otimes 1)\,\Delta = (1 \otimes \epsilon)\Delta = \mathrm{Id}$$

is obviously true.

We check that the counit ϵ preserves the relation (20). We have

$$\epsilon(LHS) = \epsilon((L^V)^{23}(L^W)^{13}) = \oplus_{\mu,\nu} \mathrm{Id}_{W[\mu]} \otimes \mathrm{Id}_{V[\nu]} \otimes \mathcal{T}^{-1}_{\mu+\nu},$$

$$\begin{aligned}\epsilon(RHS) =: (J^{12}_{W,V}(\lambda^1))^{-1} &\sum_{U\in Ir} (\tau^U_{W,V})^{12} \\ &\sum_\theta (\mathrm{Id}_{H^U_{W,V}} \otimes \mathrm{Id}_{U[\theta]} \otimes \mathcal{T}_\theta^{-1})\,(\bar{\tau}^{\,U}_{W,V})^{12}\,J^{12}_{W,V}(\lambda^2) : .\end{aligned} \tag{29}$$

Notice that

$$\oplus_{U\in Ir}\tau^{U}_{W,V}(\mathrm{Id}_{H^U_{W,V}}\otimes \mathrm{Id}_{U[\theta]})\bar\tau^{\;U}_{W,V}=\mathrm{Id}_{(W\otimes V)[\theta]}.$$

Returning to (29) we get

$$\epsilon(RHS)=:(J^{12}_{W,V}(\lambda^1))^{-1}\sum_\theta(\mathrm{Id}_{(W\otimes V)[\theta]}\otimes\mathcal{T}_\theta^{-1})\,J^{12}_{W,V}(\lambda^2):$$
$$=\sum_\theta \mathrm{Id}_{(W\otimes V)[\theta]}\otimes\mathcal{T}_\theta^{-1}=\epsilon(LHS).$$

The theorem is proved. □

4.2. The antipode in E.

Lemma 25. *If $S:E\to E$ is an antipode, then $S(L^V)=(L^V)^{-1}$, where $(L^V)^{-1}\in$ End $(V)\otimes E$ is such that*
$L^V(L^V)^{-1}=\mathrm{Id}_V\otimes 1$ *and* $(L^V)^{-1}L^V=\mathrm{Id}_V\otimes 1$.

Proof. The axioms of the antipode are

$$m\circ(\mathrm{Id}\otimes S)\circ\Delta(x)=\mu_l(\epsilon(x)\cdot 1),\quad m\circ(S\otimes \mathrm{Id})\circ\Delta(x)=\mu_r(\epsilon(x)\cdot 1).$$

Applying the first axiom to L^V we get

$$\text{LHS}\;:\;L^V\to(L^V)^{12}(L^V)^{13}\xrightarrow{1\otimes S}(L^V)^{12}\,S(L^V)^{13}\xrightarrow{m}L^V\,S(L^V),$$
$$\text{RHS}\;:\;L^V\to\sum_\theta \mathrm{Id}_{V[\theta]}\otimes\mathcal{T}_\theta^{-1}\to\mathrm{Id}_V\otimes 1.$$

Thus, $L^VS(L^V)=\mathrm{Id}_V\otimes 1$. Similarly, applying the second axiom, we get $S(L^V)L^V=\mathrm{Id}_V\otimes 1$. □

For $V\in\mathcal{O}_0$ define operators $\tilde K(\lambda):{}^*V\to{}^*V$ and $K'(\lambda):{}^*V\to{}^*V$ by

$$\tilde K(\lambda)=m(J^{t_2}_{{}^*V,V}(\lambda)),\quad K'(\lambda)=m(J^{t_1}_{V,{}^*V}(\lambda)),\tag{30}$$

where ^{t_j} means the dualization in the j^{th} component, $(\sum a_i\otimes b_i)^{t_1}=\sum a_i^*\otimes b_i$, and $m(a\otimes b)=ab$.

If $\tilde K(\lambda)$ is invertible, then denote $K(\lambda)=(\tilde K(\lambda-h))^{-1}$. Set

$$\bar L^V=(:K^{(1)}(\lambda^1)L^{{}^*V}(K^{(1)}(\lambda^2))^{-1}:)^{t_1},\tag{31}$$
$$\hat L^V=(:K'^{(1)}(\lambda^1)L^{{}^*V}(K'^{(1)}(\lambda^2))^{-1}:)^{t_1}.\tag{32}$$

Theorem 26. *Suppose that $\tilde K$ or K' is invertible for any module $V\in Ir$. Then $E=E(\mathcal{O}_0)$ is an A_0-Hopf algebroid with the antipode $S(f(\lambda^1))=f(\lambda^2)$, $S(f(\lambda^2))=f(\lambda^1)$ and $S(L^V)=(L^V)^{-1}=\bar L^{\,V}=\hat L^V$. Moreover, $K=K'$.*

The theorem is proved by direct verifications.

The A_0-Hopf algebroid $E(\mathcal{O}_0)$ will be called *the exchange dynamical quantum group associated to the category $\mathcal{O}_0$.*

4.3. The two point function and $K'(\lambda)$. Define a bilinear from $B_{\lambda,V} : V \otimes {}^*V \to \mathbb{C}$. For homogeneous $v \in V$, $v^* \in {}^*V$, with weights $\lambda_v + \lambda_{v^*} \neq 0$ set $B_{\lambda,V}(v, v^*) = 0$. If $\lambda_v + \lambda_{v^*} = 0$, then define $B_{\lambda,V}(v, v^*)$ by the property

$$(1\otimes <,>_{V\otimes {}^*V}) \circ \Phi_\lambda^{v,v^*} = B_{\lambda,V}(v, v^*)\,\mathrm{Id}_{M_\lambda}.$$

Notice that $(1\otimes <,>_{V\otimes {}^*V}) \circ \Phi_\lambda^{v,v^*}$ is an intertwiner, hence it has the form: Const Id_{M_λ}. The bilinear form $B_{\lambda,V}$ is called *the two point function*.

Lemma 27. *$B_{\lambda,V}(v, v^*) =< v, K'(\lambda)v^* >$, where $K'(\lambda)$ is defined in (30).*

Proof. Since $\Phi_\lambda^{v,v^*} = \Phi_\lambda^{J_{V,{}^*V}(\lambda)(v\otimes v^*)}$, we have $B_{\lambda,V}(v, v^*) = \sum < a_i v, b_i v^* >= < v, K'(\lambda)v^* >$. □

Remark. Let k, n be natural numbers, U the vector representation of the quantum group $U_q(sl_n)$. Let $V = S^{kn}U$ be the kn^{th} symmetric power of U. Then $V[0]$ is one dimensional and $B_{\lambda,V}|_{V[0]}$ is a scalar function of λ equal to the squared norm of a Macdonald polynomial, see Theorem 2.4 in [EK].

5. Exchange Quantum Groups Associated to Simple Lie Algebras

5.1. The exchange quantum groups $F(\mathfrak{g})$, $F_q(\mathfrak{g})$. In this section we consider the exchange dynamical quantum groups associated to the category of finite dimensional representations of simple Lie algebras and their quantum groups. We consider two types of polarized Hopf algebras.

I. Let $\mathfrak{g}$ be a simple Lie algebra, $\alpha_i, i = 1, \ldots, r$, simple roots, e_i, f_i, h_i the corresponding Chevalley generators, $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ the polar decomposition. Consider the polarized Hopf algebra $A = U(\mathfrak{g})$ with the $\mathbb{Z}$-grading and polarizations defined by $\deg(e_i) = 1$, $\deg(f_i) = -1$, $\deg(h_i) = 0$, $A_+ = U(\mathfrak{n}_+)$, $A_- = U(\mathfrak{n}_-)$, $A_0 = U(\mathfrak{h})$, $A_{\geqslant 0} = U(\mathfrak{b}_+)$, $A_{\leqslant 0} = U(\mathfrak{b}_-)$, where $\mathfrak{b}_\pm = \mathfrak{h} \oplus \mathfrak{n}_\pm$. In this case $T = \mathrm{Spec} A_0 = \mathfrak{h}^*$. Fix on A the quasitriangular structure $\mathcal{R} = 1 \in A \hat{\otimes} A$.

II. Fix $\varepsilon \in \mathbb{C}$ and set $q = e^\varepsilon$. Assume that q is not a root of unity. Let $\mathfrak{g}$ be a simple Lie algebra, $\alpha_i, i = 1, \ldots, r$, simple roots, $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ the polar decomposition. Consider the quantum group $A = U_q(\mathfrak{g})$ with the Chevalley generators $e_i, f_i, K_i^{\pm 1}$ as defined on p. 280 in [CP]. Fix in A a counit ϵ, a comultiplication Δ, and an antipode S as defined on p. 281 in [CP]. We consider A as a polarized Hopf algebra with the $\mathbb{Z}$-grading and polarizations defined by deg $(e_i) = 1$, deg $(f_i) = -1$, deg $(K_i^{\pm 1}) = 0$, $A_+ = U_q(\mathfrak{n}_+)$, $A_- = U_q(\mathfrak{n}_-)$, $A_0 = U_q(\mathfrak{h})$, $A_{\geqslant 0} = U_q(\mathfrak{b}_+)$, $A_{\leqslant 0} = U_q(\mathfrak{b}_-)$.

Remark. Let $a_{ij} = 2 < \alpha_i, \alpha_j > / < \alpha_i, \alpha_i >$ be the Cartan matrix. Let d_i be coprime positive integers such that the matrix $d_i a_{ij}$ is symmetric. Let $h_i \in \mathfrak{h}$ be the elements such that $\alpha_i(h_j) = a_{ij}$. Then one can think of the generators $K_i^{\pm 1}$ as of elements of the form $q^{\pm d_i h_i}$, see p. 281 in [CP].

For $A = U_q(\mathfrak{g})$, the spectrum $T =$ Spec A_0 is the spectrum of the algebra of Laurent polynomials $\mathbb{C}[K_1^{\pm 1}, \ldots, K_r^{\pm 1}]$. The spectrum T can be identified with $\mathfrak{h}^*/L$, where L is the lattice such that its dual lattice L^* is generated by elements $d_i h_i$, i.e. the lattice L consists of the points, where $q^{d_i h_i}$ are equal to 1.

Fix on A the quasitriangular structure $\mathcal{R} \in A \hat{\otimes} A$, where $\mathcal{R}$ is the universal R-matrix of the quantum group $U_q(\mathfrak{g})$.

Remark. If $q = 1$, then sometimes we shall use the notation $U_{q=1}(\mathfrak{g})$ for the universal enveloping algebra $U(\mathfrak{g})$ considered above.

If $A = U(\mathfrak{g})$, then let $\mathcal{O}_0$ be the category of finite dimensional modules over $U(\mathfrak{g})$. If $q \neq 1$ and $A = U_q(\mathfrak{g})$, then let $\mathcal{O}_0(q)$ be the category of finite dimensional modules over $U_q(\mathfrak{g})$ such that all of the eigenvalues of K_i are integer powers of q, i.e. $\mathcal{O}_0(q)$ is the category of finite dimensional modules over $U_q(\mathfrak{g})$ which are quantizations of finite dimensional modules of $U(\mathfrak{g})$ when q tends to 1.

Consider the exchange dynamical quantum group $E(\mathcal{O}_0)$ associated to the category $\mathcal{O}_0$ of modules over $U(\mathfrak{g})$ and denote it $\hat{F}(\mathfrak{g})$. The exchange dynamical quantum group $\hat{F}(\mathfrak{g})$ is defined over the field of meromorphic functions M_T, $T = \mathfrak{h}^*$. We replace the field of meromorphic functions M_T by the field of rational functions $\mathbb{C}(T)$ and define in the same way *the rational exchange dynamical quantum group* $E_{\text{rat}}(\mathcal{O}_0)$. We denote the rational exchange dynamical quantum group $F(\mathfrak{g})$.

If $q \neq 1$, then consider the exchange dynamical quantum group $E(\mathcal{O}_0(q))$ associated to the category $\mathcal{O}_0(q)$ of modules over $U_q(\mathfrak{g})$ and denote it $\hat{F}_q(\mathfrak{g})$. The exchange dynamical quantum group $\hat{F}(\mathfrak{g})$ is defined over the field of meromorphic functions M_T, where the torus T has the form $T = \mathfrak{h}^*/L$. We replace the field of meromorphic functions M_T by the field of rational functions $\mathbb{C}(T)$ and define in the same way *the rational exchange dynamical quantum group* $E_{\text{rat}}(\mathcal{O}_0(q))$. The field $\mathbb{C}(T)$ can be considered as the subfield $\mathbb{C}(T) \subset M_{\mathfrak{h}^*}$ of "trigonometric" functions with respect to the lattice $L \subset \mathfrak{h}^*$. We denote the rational exchange dynamical quantum group $F_q(\mathfrak{g})$.

According to Theorem 26, the exchange quantum group $F(\mathfrak{g})$ (resp. $F_q(\mathfrak{g})$) has a well defined antipode if for any $V \in Ir \subset \mathcal{O}_0$ (resp. $V \in Ir \subset \mathcal{O}_0(q)$) the operator $K'(\lambda) : {}^*V \to {}^*V$ is invertible for generic values of λ. By Lemma 4.3 this property holds if the two point function $B_{\lambda,V}$ is a nondegenerate bilinear form for generic values of λ.

Theorem 28. *For any $V \in \mathcal{O}_0$ (resp. $V \in \mathcal{O}_0(q)$ for generic q) the two point function $B_{\lambda,V} : V \otimes {}^*V \to \mathbb{C}$ is a nondegenerate bilinear form for generic values of λ.*

Proof. For $F(\mathfrak{g})$ the theorem follows from the next lemma.

Recall that $B_{\lambda,V}(v, v^*) = \sum < a_i v, b_i v^* >$ if $J_{V,{}^*V}(\lambda) = \sum a_i \otimes b_i$. Let $\rho \in \mathfrak{h}^*$ be the half sum of positive roots.

Lemma 29. *For $A = U(\mathfrak{g})$ and any $V, W \in \mathcal{O}_0$, we have $J_{V,W}(t\rho) \to 1$ when $t \in \mathbb{C}$ and t tends to infinity.*

Proof. In [ES1], the intertwining operator $\Phi^v(\lambda)$ was computed in terms of the Shapovalov form (formula (3-5) in [ES1]). From formula (3-5) in [ES1] it is easy to obtain the following asymptotic expansion of $\Phi^v(\lambda)$:

$$\Phi^w_\lambda v_\lambda = v_{\lambda - wt(w)} \otimes w + O(\frac{1}{|\lambda|}),$$

where $O(\frac{1}{|\lambda|})$ denotes terms of degree -1 and lower in λ. This implies the lemma. □

Corollary 30. $B_{t\rho,V}(\,,\,) \to <\,,\,>$ *as t tends to infinity.*

For $F_q(\mathfrak{g})$ and $|q| < 1$ or $|q| > 1$ the theorem follows in a similar way from [ES2], Sect. 2. However, in the q-case, the above lemma holds only for $t \to +\infty$ if $|q| < 1$ and for $t \to -\infty$ if $q > 1$. □

5.2. The exchange groups and A_0-bialgebroids associated with R-matrices. Let $V \in Ir \subset \mathcal{O}_0$ (resp. $V \in Ir \subset \mathcal{O}_0(q)$). Let $R(\lambda) = R_{V,V}(\lambda) : V \otimes V \to V \otimes V$ be the R-matrix defined in (2). R is a rational function of $\lambda \in T$. Consider the rational A_0-bialgebroid $A_{rat,R}$ constructed in Sect. 3.2. Recall that $A_{rat,R}$ is generated by matrix elements of operators $L^{\pm 1}$ and rational functions of $\lambda^1, \lambda^2 \in T$.

Theorem 31. *For any $V \in Ir \subset \mathcal{O}_0$ (resp. $V \in Ir \subset \mathcal{O}_0(q)$), there exists a unique homomorphism $\varphi : A_{rat,R} \to F(\mathfrak{g})$ (resp. $\varphi : A_{rat,R} \to F_q(\mathfrak{g})$) of rational A_0-bialgebroids such that $(1 \otimes \varphi)(L) = L^V$. Moreover, $(1 \otimes \varphi)(L^{-1}) = (L^V)^{-1}$, $\varphi(f(\lambda^1)) = f(\lambda^1)$, $\varphi(f(\lambda^2)) = f(\lambda^2)$.*

The theorem follows from definitions. □

Theorem 32. *For $V \in Ir \subset \mathcal{O}_0$ (resp. $V \in Ir \subset \mathcal{O}_0(q)$), let V and ${}^*V$ generate the tensor category $\mathcal{O}_0$ (resp. $\mathcal{O}_0(q)$) in the sense that any object in Ir is a sub-object in $V^{\otimes n} \otimes ({}^*V)^{\otimes m}$ for suitable n, m. Then the homomorphism φ is surjective.*

Proof. Clearly the matrix components of L^V and $L^{{}^*V}$ belong to the image of φ, since $(L^V)^{-1}$ is $L^{{}^*V}$ up to some invertible factors in λ^1, λ^2.

Let $U \in Ir$ and U is a sub-object in $V^{\otimes n} \otimes ({}^*V)^{\otimes m}$ for suitable n, m. Consider the product

$$(L^{{}^*V})^{m+n,m+n+1} \quad \dots \quad (L^{{}^*V})^{n+1,m+n+1} \quad (L^V)^{n,m+n+1} \quad \dots \quad (L^V)^{1,m+n+1}.$$

It is clear that the matrix components of L^U are linear combinations of the matrix components of this product with coefficients in rational functions of λ^1, λ^2. □

5.3. The exchange groups corresponding to classical Lie groups $GL(N)$, $SL(N)$, $O(N)$, $SP(2N)$. In this section we modify the construction of Sect. 5.1.

Consider the Lie algebra $gl(N)$. Let e_i, f_i, $i = 1, \dots, N-1$, and h_i, $i = 1, \dots, N$, be its standard Chevalley generators. Let $\mathcal{O}_0(GL(N))$ be the category of all finite dimensional modules over $gl(N)$ which can be integrated to a representation of the Lie group $GL(N)$. Consider the rational exchange dynamical quantum group $E_{rat}(\mathcal{O}_0(GL(N)))$ associated to the category $\mathcal{O}_0(GL(N))$ and denote it $F(GL(N))$.

Fix $\varepsilon \in \mathbb{C}$ and set $q = e^{\varepsilon}$. Assume that q is not a root of unity. Consider the quantum group $A = U_q(gl(N))$ with the standard Chevalley generators e_i, f_i, $i = 1, \dots, N-1$, and $k_i^{\pm 1}$, $i = 1, \dots, N$. Let $\mathcal{O}_0(GL(N), q)$ be the category of all finite dimensional modules over $U_q(gl(N))$ which are q-deformations of finite dimensional modules over $GL(N)$. Consider the rational exchange dynamical quantum group $E_{rat}(\mathcal{O}_0(GL(N), q))$ associated to the category $\mathcal{O}_0(GL(N), q)$ and denote it $F_q(GL(N))$.

Similarly, let G be a simple complex Lie group and $\mathfrak{g}$ its Lie algebra. Consider the category $\mathcal{O}_0(G)$ of all finite dimensional modules over $\mathfrak{g}$ which can be integrated to a module over G. Consider the rational exchange dynamical quantum group $E_{rat}(\mathcal{O}_0(G))$ associated to the category $\mathcal{O}_0(G)$ and denote it $F(G)$. If $\varepsilon \in \mathbb{C}$, $q = e^{\varepsilon}$, and q is not a root of unity, consider the quantum group $A = U_q(\mathfrak{g})$ and the category $\mathcal{O}_0(G, q)$ of all finite dimensional modules over $U_q(\mathfrak{g})$ which are q-deformations of finite dimensional modules over G. The rational exchange dynamical quantum group $E_{rat}(\mathcal{O}_0(G, q))$ associated to the category $\mathcal{O}_0(G, q)$ is denoted $F_q(G)$.

Let G be a Lie group of type $GL(N)$, $SL(N)$, $SO(N)$, $SP(2N)$ and $\mathfrak{g}$ its Lie algebra. Let V be the vector representation of $U(\mathfrak{g})$ (resp. $U_q(\mathfrak{g})$). We have $V \in Ir \subset \mathcal{O}_0(G)$ (resp. $V \in Ir \subset \mathcal{O}_0(G, q)$).

Lemma 33. *V and $^*V$ generate $\mathcal{O}_0(G)$ (resp. $\mathcal{O}_0(G, q)$).*

The Lemma follows from the fact that the vector representation is faithful as a representation of G.

Corollary 34. *Let V be the vector representation of $U(\mathfrak{g})$ (resp. $U_q(\mathfrak{g})$), $R(\lambda) = R_{V,V}(\lambda) : V \otimes V \to V \otimes V$ the R-matrix defined in (2), $A_{rat,R}$ the rational A_0-bialgebroid constructed in Sect. 3.2. Then the homomorphism $\varphi : A_{rat,R} \to F(G)$ (resp. $\varphi : A_{rat,R} \to F_q(G)$) of Theorem 31 is an epimorphism.*

Theorem 35. *Let $G = GL(N)$. Then*

I. *For $F(G)$, the homomorphism $\varphi : A_{rat,R} \to F(G)$ of Corollary 34 is injective.*
II. *For $F_q(G)$, the homomorphism $\varphi : A_{rat,R} \to F_q(G)$ of Corollary 34 is injective for all q except a countable set.*

Proof. To prove the theorem for $F(G)$ recall that in this case $\lambda \in T = \mathfrak{h}^*$. For $\gamma \in \mathbb{C}^*$ introduce a new variable $\tilde{\lambda} = \lambda/\gamma$. Then, by the results of Sect. 3 in [ES1], for any modules $V, W \in \mathcal{O}(GL(N))$, we have $J_{V,W}(\tilde{\lambda}) = \mathrm{Id} + \gamma J_1(\tilde{\lambda}) + \gamma^2 J_2(\tilde{\lambda}) + \dots$. Hence $J_{V,W}(\tilde{\lambda}) \to \mathrm{Id}$ as $\gamma \to 0$.

Let $A^\gamma_{rat,R}$, $F^\gamma(G)$ be the algebras defined by the same relations as $A_{rat,R}$, $F(G)$ with λ replaced by λ/γ and $\varphi_\gamma : A^\gamma_{rat,R} \to F^\gamma(G)$ the corresponding homomorphisms. It is easy to see that the algebras $A^0_{rat,R}$, $F^0(G)$ are well defined,

$$A^0_{rat,R} = F^0(G) = \mathbb{C}(\mathfrak{h}^*) \otimes \mathbb{C}(\mathfrak{h}^*) \otimes \mathbb{C}[G]$$

and $\varphi_\gamma \to \varphi_0 = \mathrm{Id}$ as $\gamma \to 0$. Here $\mathbb{C}[G]$ is the algebra of polynomials on G.

The algebras $A^\gamma_{rat,R}$, $F^\gamma(G)$ and the homomorphism φ_γ are deformations of the algebras $A^0_{rat,R}$, $F^0(G)$ and the homomorphism φ_0. Elementary reasonings of the deformation theory imply that the homomorphism φ_γ is an isomorphism.

The theorem for $F_q(G)$ is deduced from the theorem for $F(G)$ by taking the limit $q \to 1$. □

Now consider the case of $SL(N)$.

For $G = GL(N)$, consider the exchange group $F_q(G)$. Let $C \in \mathcal{O}_0(G, q)$ be a one dimensional module. Then L^C is a 1×1-matrix and can be considered as an element of $F_q(G)$.

Lemma 36. *L^C is a central element in $F_q(G)$ and L^C is invertible, $(L^C)^{-1} = L^{*C}$.*

Proof. For any $W \in \mathcal{O}_0(G, q)$, L^C and L^W satisfy the R-matrix relation (26). In this case the R-matrix $R_{C,W}(\lambda)$ is a scalar constant, hence L^C is central. □

For $F_q(GL(N))$, consider the one dimensional module $C = \wedge^N_q V$ over $U_q(gl(N))$, which is the N^{th} quantum exterior power of the vector representation V. For generic q, consider the isomorphism $\varphi_{GL(N)} : A^{GL(N)}_{rat,R} \to F_q(GL(N))$ of Theorem 35. Define $D \in A^{GL(N)}_{rat,R}$ by $D = \varphi^{-1}_{GL(N)}(L^C)$.

Consider the quantum group $U_q(sl(N))$. There is a natural embedding of $U_q(sl(N))$ to $U_q(gl(N))$ sending the Chevalley generators

$$e_i, f_i, K_i \in U_q(sl(N)) \text{ to } e_i, f_i, \quad k_{i+1}/k_i \in U_q(gl(N)).$$

Let V be the vector representation of $U_q(sl(N)) \subset U_q(gl(N))$. Consider the corresponding R-matrices $R^{GL(N)}(\lambda) = R^{GL(N)}_{V,V}(\lambda)$, $\lambda \in T_{GL(N)} = (\mathbb{C}^*)^N$ and $R^{SL(N)}(\lambda) = R^{SL(N)}_{V,V}(\lambda)$, $\lambda \in T_{SL(N)} = (\mathbb{C}^*)^N/\mathbb{C}^*(1,\ldots,1)$. Any rational function on $(\mathbb{C}^*)^N/\mathbb{C}^*(1,\ldots,1)$ can be considered as a rational function on $(\mathbb{C}^*)^N$ invariant with respect to the diagonal action of $\mathbb{C}^*$. It is easy to see that R-matrix $R^{SL(N)}(\lambda)$ considered as a function on $T_{GL(N)}$ coincides with the R-matrix $R^{GL(N)}(\lambda)$ up to a multiplicative scalar constant. This construction allows us to define a natural embedding $A^{SL(N)}_{rat,R} \to A^{GL(N)}_{rat,R}$. Clearly, the element D belongs to the image of the imbedding.

Theorem 37. I. *For $F(SL(N))$, the kernel of the epimorphism $\varphi : A^{SL(N)}_{rat,R} \to F(SL(N))$ of Corollary 34 is generated by the relation $D = 1$.*

II. *For $F_q(SL(N))$, the kernel of the epimorphism $\varphi : A^{SL(N)}_{rat,R} \to F_q(SL(N))$ of Corollary 34 contains the ideal generated by the relation $D = 1$. Moreover, the kernel is generated by this relation for all q except a countable set.*

Proof. For $F(SL(N))$, clearly the kernel contains the relation $D = 1$, since for $sl(N)$, the module $\wedge^N V$ is trivial.

Introduce (as before) the algebras $A^{SL(N),\gamma}_{rat,R}$, $F^{\gamma}(SL(N))$ and a homomorphism $\varphi_\gamma : A^{SL(N),\gamma}_{rat,R} \to F^{\gamma}(SL(N))$ depending on a parameter $\gamma \in \mathbb{C}^*$. It is easy to see that for $\gamma = 0$, the homomorphism $\bar{\varphi}_{\gamma=0} : A^{SL(N),\gamma=0}_{rat,R}/\{D = 1\} \to F^{\gamma=0}(SL(N))$ is an isomorphism. This statement (as before) implies the theorem. □

Now let G be a Lie group of type $SO(N)$ or $SP(2N)$. Let V be the vector representation of its Lie algebra $\mathfrak{g}$ (resp. $U_q(\mathfrak{g})$). In this case there is an isomorphism $T : {}^*V \to V$ of $\mathfrak{g}$-modules (resp. $U_q(\mathfrak{g})$-modules).

Theorem 38. *Let G be a Lie group of type $SO(N)$ or $SP(2N)$. Then*

I. *For $F(G)$ and $F_q(G)$ the kernel of the epimorphism $\varphi : A_{rat,R} \to F(G)$ of Corollary 34 contains the ideal generated by the relations*

$$L =: T^{(1)}(K^{(1)}(\lambda^1))^{-1}(L^{-1})^{t_1}K^{(1)}(\lambda^2)(T^{(1)})^{-1} :, \tag{33}$$

where K is defined in Sect. 4.2.

II. *For $F(G)$ and $F_q(G)$, the element D defined above equals 1 modulo (33) for $G = SP(2N)$, and is a central grouplike element of order 2 modulo (33) for $G = SO(N)$.*

III. *For $F(G)$ and $F_q(G)$ with q outside of a countable set, the kernel of φ is generated by relations (33) in the case of $G = SP(2N)$, and by (33) and $D = 1$ for $G = SO(N)$.*

Proof.

Lemma 39. *Relations (33) belong to the kernel.*

Proof. In fact, by Theorem 26 we have

$$L^{*V} =: (K^{(1)}(\lambda^1))^{-1}((L^V)^{-1})^{t_1}K^{(1)}(\lambda^2) : .$$

Since $T : {}^*V \to V$ is an isomorphism, we have

$$(T \otimes 1)L^{*V}(T^{-1} \otimes 1) = L^V. \qquad \square$$

Let $I \subset A_{rat,R}$ be the ideal generated by relations (33). Consider the quotient $A_{rat,R}/I$ and the homomorphism $\bar{\varphi} : A_{rat,R}/I \to F_q(G)$. One can prove as for $GL(N)$ that the homomorphism $\bar{\varphi}$ is an isomorphism for $q = 1$ and for generic q if $G = SP(2N)$, and has kernel generated by $D = 1$ if $G = SO(N)$.

Remark. If $G = SO(N)$, then it is natural to denote the quotient $A_{rat,R}/I$ by $F_q(O(N))$

Remark. If $q = 1$, then in the limit $\gamma \to 0$ we have $J = 1$. In this case relations (33) take the form

$$L = (T \otimes 1)(L^{-1})^{t_1}(T^{-1} \otimes 1),$$

which is the defining relation for the orthogonal and symplectic groups. □

6. The R-Matrix $R_{V,V}(\lambda)$ for the Vector Representation of $U_q(gl(N))$

6.1. Matrices $J_{V,V}(\lambda)$ and $R_{V,V}(\lambda)$. Let $V = \mathbb{C}^N$ be the vector representation of $A = U_q(gl(N))$. Let $v_j = (0, \dots, 0, 1_j, \dots, 0)$ be the standard basis in V. We have $f_i v_j = \delta_{i,j} v_{i+1},\ e_i v_j = \delta_{i+1,j} v_i$, where f_i, e_i are the Chevalley generators of $U_q((N))$. Introduce a basis E_{ij} in End (V) by $E_{ij} v_k = \delta_{jk} v_i$.

The $U_q(gl(N))$-module $V \otimes V$ has the weight decomposition,

$$V \otimes V = \oplus_{a=1}^N V_{aa} \ \oplus \oplus_{a<b} V_{ab}\,, \tag{34}$$

where $V_{aa} = \mathbb{C}\, v_a \otimes v_a$ and $V_{ab} = \mathbb{C}\, v_a \otimes v_b \oplus \mathbb{C}\, v_b \otimes v_a$.

The action of the quasi-triangular structure $\mathcal{R} \in A \hat{\otimes} A$ on $V \otimes V$ takes the form

$$\mathcal{R} = q \sum_{a=1}^N E_{aa} \otimes E_{aa} + \sum_{a \neq b} E_{aa} \otimes E_{bb} + \sum_{a<b} (q - q^{-1})\, E_{ab} \otimes E_{ba}.$$

Consider the maps

$$J(\lambda) = J_{V,V}(\lambda) : V \otimes V \to V \otimes V, \quad R(\lambda) = R_{V,V}(\lambda) : V \otimes V \to V \otimes V$$

defined in (1) and (2). Here $\lambda \in T = (\mathbb{C}^*)^N$, if $q \neq 1$, and $\lambda \in T = \mathbb{C}^N$, if $q = 1$. We shall use the coordinates $\lambda = (q^{\lambda_1}, \dots, q^{\lambda_N})$ on $(\mathbb{C}^*)^N$ and the coordinates $\lambda = (\lambda_1, \dots, \lambda_N)$ on $\mathbb{C}^N$.

Recall that $R(\lambda) = J^{-1} \mathcal{R}^{21} J^{21}$.

Theorem 40. I. *For $F_q(GL(N))$, we have*

$$J(\lambda) = \sum_{a,b} E_{aa} \otimes E_{bb} + \sum_{a<b} \frac{q^{-1} - q}{q^{2(\lambda_a - \lambda_b + b - a)} - 1}\, E_{ba} \otimes E_{ab},$$

$$\begin{aligned} R(\lambda) = q \sum_{a=1}^N E_{aa} \otimes E_{aa} &+ \sum_{a \neq b} \frac{q^{-1} - q}{q^{2(\lambda_b - \lambda_a + a - b)} - 1}\, E_{ba} \otimes E_{ab} \\ &+ \sum_{a<b} E_{aa} \otimes E_{bb} + \\ &\sum_{a>b} \frac{(q^{2(\lambda_b - \lambda_a + a - b)} - q^{-2})(q^{2(\lambda_b - \lambda_a + a - b)} - q^2)}{(q^{2(\lambda_b - \lambda_a + a - b)} - 1)^2}\, E_{aa} \otimes E_{bb}. \end{aligned} \tag{35}$$

II. *For $F(GL(N))$, we have*

$$J(\lambda) = \sum_{a,b} E_{aa} \otimes E_{bb} + \sum_{a<b} \frac{1}{\lambda_b - \lambda_a + a - b} E_{ba} \otimes E_{ab},$$

$$\begin{aligned} R(\lambda) = \sum_{a=1}^{N} E_{aa} \otimes E_{aa} &+ \sum_{a\neq b} \frac{1}{\lambda_a - \lambda_b + b - a} E_{ba} \otimes E_{ab} \\ &+ \sum_{a<b} E_{aa} \otimes E_{bb} \\ &+ \sum_{a>b} \frac{(\lambda_b - \lambda_a + a - b - 1)(\lambda_b - \lambda_a + a - b + 1)}{(\lambda_b - \lambda_a + a - b)^2} E_{aa} \otimes E_{bb}. \end{aligned} \tag{36}$$

The theorem is proved by direct calculations. More precisely, the coefficients of J corresponding to simple roots (i.e. with $b = a + 1$) are easily calculated explicitly, after which all other coefficients are found using the classification of dynamical quantum R-matrices of Hecke type given in [EK2].

Set $R^\vee(\lambda) = PR(\lambda)$, where $P : V \otimes V \to V \otimes V$ is the permutation of factors.

Lemma 41. I. *The operator $R^\vee(\lambda)$ preserves the weight decomposition (34).*
II. *For any $a = 1, \dots, N$, we have $R^\vee(\lambda) v_a \otimes v_a = q\, v_a \otimes v_a$.*
III. *For any $a \neq b$, the operator $R^\vee(\lambda)$ restricted to the two dimensional space V_{ab} has eigenvalues q and $-p$, where $p = q^{-1}$.*

A meromorphic function $R : T \to \mathrm{End}\,(V \otimes V)$ with these three properties and satisfying the dynamical Yang–Baxter equation (5) is called *an R-matrix of Hecke type with parameters q and p*. We classified such R-matrices in [EV2] up to gauge transformations.

6.2. Gauge transformations of R-matrices of Hecke type. Consider the torus $T = (\mathbb{C}^*)^N$ with coordinates $\lambda = (q^{\lambda_1}, \dots, q^{\lambda_N})$. *A multiplicative k-form* on T is a collection,

$$\varphi = \{\varphi_{a_1,\dots,a_k}(q^{\lambda_1}, \dots, q^{\lambda_N})\},$$

of meromorphic functions on T, where $a_1, \dots, a_k$ run through all k element subsets of $\{1, \dots, N\}$, such that for any subset $a_1, \dots, a_k$ and any i, $1 \leqslant i < k$, we have

$$\varphi_{a_1,\dots,a_{i+1},a_i,\dots,a_k}(q^{\lambda_1}, \dots, q^{\lambda_N})\, \varphi_{a_1,\dots,a_k}(q^{\lambda_1}, \dots, q^{\lambda_N}) = 1.$$

Let Ω^k be the set of all multiplicative k-forms.

If φ and ψ are multiplicative k-forms, then

$$\{\varphi_{a_1,\dots,a_k}(q^{\lambda_1}, \dots, q^{\lambda_N}) \cdot \psi_{a_1,\dots,a_k}(q^{\lambda_1}, \dots, q^{\lambda_N})\}$$

and

$$\{\varphi_{a_1,\dots,a_k}(q^{\lambda_1}, \dots, q^{\lambda_N}) \,/\, \psi_{a_1,\dots,a_k}(q^{\lambda_1}, \dots, q^{\lambda_N})\}$$

are multiplicative k-forms. This gives an abelian group structure on Ω^k. The zero element in Ω^k is the form $\{\varphi_{a_1,\dots,a_k}(q^{\lambda_1}, \dots, q^{\lambda_N}) \equiv 1\}$.

For any $a = 1, \ldots, N$, introduce an endomorphism δ_a of the multiplicative group of nonzero meromorphic functions $f(q^{\lambda_1}, \ldots, q^{\lambda_N})$ on T by

$$\delta_a : f(q^{\lambda_1}, \ldots, q^{\lambda_N}) \mapsto \frac{f(q^{\lambda_1}, \ldots, q^{\lambda_N})}{f(q^{\lambda_1}, \ldots, q^{\lambda_a}/q, \ldots, q^{\lambda_N})}$$

and a homomorphism $d : \Omega^k \to \Omega^{k+1}$, $\varphi \mapsto d\varphi$, by

$$(d\varphi)_{a_1,\ldots,a_{k+1}}(q^{\lambda_1}, \ldots, q^{\lambda_N}) = \prod_{i=1}^{k+1} (\delta_{a_i}\varphi_{a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_{k+1}}(q^{\lambda_1}, \ldots, q^{\lambda_N}))^{(-1)^{i+1}}.$$

We have $d^2 = 0$ (0 means the trivial homomorphism which maps everything to the zero element). A multiplicative form φ is called *closed* if $d\varphi = 0$.

Introduce gauge transformations of R-matrices, $R : T \to \mathrm{End}\,(V \otimes V)$, of the form

$$R(\lambda) = \sum_{a=1}^{N} \alpha_{aa}(\lambda)\, E_{aa} \otimes E_{aa} + \sum_{a\neq b} \alpha_{ab}(\lambda)\, E_{aa} \otimes E_{bb} + \sum_{a\neq b} \beta_{ab}(\lambda) E_{ba} \otimes E_{ab}, \tag{37}$$

where $\alpha_{ab}(\lambda)$, $\beta_{ab}(\lambda)$ are suitable functions.

I. Let $\{\varphi_{ab}\}$ be a meromorphic closed multiplicative 2-form on T. Set

$$R(\lambda) \mapsto \sum_{a=1}^{N} \alpha_{aa}(\lambda)\, E_{aa} \otimes E_{aa} + \sum_{a\neq b} \varphi_{ab}(\lambda)\, \alpha_{ab}(\lambda)\, E_{aa} \otimes E_{bb} + \sum_{a\neq b} \beta_{ab}(\lambda)\, E_{ba} \otimes E_{ab}.$$

II. Let the symmetric group S_N, the Weyl group of gl_N, act on T and V by permutation of coordinates. For any permutation $\sigma \in S_N$, set

$$R(\lambda) \mapsto (\sigma \otimes \sigma)\, R(\sigma^{-1} \cdot \lambda)\, (\sigma^{-1} \otimes \sigma^{-1}).$$

III. For a nonzero complex number c, set

$$R(\lambda) \mapsto c\, R(\lambda).$$

IV. For an element $\mu \in T$, set

$$R(\lambda) \mapsto R(\lambda + \mu),$$

(recall that we always use the additive notation for the standard group structure on T).

By Theorem 1.1 in [EV2] any gauge transformation transforms a matrix satisfying the QDYB (5) to a matrix satisfying the QDYB (5). In all cases, if the R-matrix is of Hecke type, then the transformed matrix is of Hecke type. If the transformation is of type III and the Hecke parameters of the R-matrix are q and p, then the Hecke parameters of the transformed matrix are cq and cp. For all other types of transformations the Hecke parameters do not change.

Two R-matrices will be called *rationally equivalent* if one of them can be transformed into another by a sequence of gauge transformations of types II-IV and of type I with only rational functions $\varphi_{ab}(\lambda)$.

The R-matrices of Hecke type were classified in [EV2] up to gauge transformations. Here are two main examples of that classification.

Examples. I. For $q \neq 1$, let the R-matrix $R^q : (\mathbb{C}^*)^N \to V \otimes V$ have the form (37), where

$$\beta_{ab}(\lambda) = \frac{q^{-2} - 1}{q^{2(\lambda_b - \lambda_a)} - 1},$$

$\alpha_{aa} = 1$ and $\alpha_{ab}(\lambda) = \beta_{ab}(\lambda) + q^{-2}$ for $a \neq b$. $R^q(\lambda)$ is an R-matrix of Hecke type with parameters q and $p = q^{-1}$.

II. For $q = 1$, let the R-matrix $R : \mathbb{C}^N \to V \otimes V$ have the form (37), where

$$\beta_{ab}(\lambda) = \frac{1}{\lambda_a - \lambda_b},$$

$\alpha_{aa} = 1$ and $\alpha_{ab}(\lambda) = \beta_{ab}(\lambda) + 1$ for $a \neq b$. $R(\lambda)$ is an R-matrix of Hecke type with parameters $q = p = 1$.

Lemma 42. I. *The R-matrix $R(\lambda)$ in (35) is rationally equivalent to the R-matrix $R^q(\lambda)$ of the first example.*

II. *The R-matrix $R(\lambda)$ in (36) is rationally equivalent to the R-matrix $R(\lambda)$ of the second example.*

Proof. To transform $R^q(\lambda)$ to the R-matrix in (35) one needs to make the gauge transformation of type IV with μ equal to the half sum of positive roots, then the gauge transformation of type II with $c = q$, and finally, the gauge transformation of type I corresponding to the closed multiplicative 2-form $\varphi_{ab}(\lambda)$, where

$$\varphi_{ab}(\lambda) = q\frac{q^{2(\lambda_a - \lambda_b + a - b - 1)} - 1}{q^{2(\lambda_a - \lambda_b + a - b)} - 1} \quad \text{for } a > b \tag{38}$$

and $\varphi_{ab}(\lambda)$ for $a < b$ are reconstructed from the multiplicative "skew symmetry" relation $\varphi_{ba}(\lambda)\varphi_{ab}(\lambda) = 1$. The second statement of the lemma is proved analogously. □

6.3. Gauge transformations of A_0-bialgebroids corresponding to R-matrices. For $q \neq 1$, let $R, \tilde{R} : (\mathbb{C}^*)^N \to V \otimes V$ be two R-matrices having the form (37). Consider the Lie algebra $gl(N)$ and its Cartan subalgebra $\mathfrak{h}$ generated by elements h_i. Consider the polarized algebra $A = U_q(gl(N))$ with the earlier distinguished polarization $A_\pm$, $A_0 = U_q(\mathfrak{h})$. Let A_R and $A_{\tilde{R}}$ be the $A_0 = U_q(\mathfrak{h})$-bialgebroids associated to R and $\tilde{R}$, respectively, and constructed in Sect. 3.2. Assume that the R-matrix $\tilde{R}$ is obtained from the R-matrix R by a gauge transformation of type II, III, or IV, then clearly the A_0-bialgebroids A_R and $A_{\tilde{R}}$ are isomorphic.

Theorem 43. *Assume that the R-matrix $\tilde{R}$ is obtained from the R-matrix R by a gauge transformation of type I associated to a multiplicative 2-form $\{\varphi_{ab}(\lambda)\}$ which is exact, $\{\varphi_{ab}(\lambda)\} = d\,\{\xi_a(\lambda)\}$, where $\{\xi_a(\lambda)\}$ is a multiplicative 1-form. Then the A_0-bialgebroids A_R and $A_{\tilde{R}}$ are isomorphic.*

Proof. Let $\xi = \sum \xi_a E_{aa}$. If L satisfies the QDYB equation (17), and $\tilde{L}$ is such that

$$L =: \xi^{(1)}(\lambda^1)\tilde{L}(\xi^{(1)}(\lambda^2))^{-1} :,$$

then $\tilde{L}$ satisfies

$$\tilde{R}^{12}(\lambda^1)\tilde{L}^{13}\tilde{L}^{23} =: \tilde{L}^{23}\tilde{L}^{13}\tilde{R}^{12}(\lambda^2) :,$$

where

$$\tilde{R}(\lambda) = (\xi^{(1)}(\lambda - h^{(2)}))^{-1}(\xi^{(2)}(\lambda))^{-1}R(\lambda)\xi^{(1)}(\lambda)\xi^{(2)}(\lambda - h^{(1)}).$$

This means that if $R(\lambda)$ has the form (37), then $\tilde{R}(\lambda)$ is obtained from $R(\lambda)$ by the gauge transformation of type I corresponding to the 2-form $\{\varphi_{ab}(\lambda)\} = d\,\{\xi_a(\lambda)\}$. □

Notice that the multiplicative 2-form given by (38) is exact, $\{\varphi_{ab}(\lambda)\} = d\,\{\xi_a(\lambda)\}$, where

$$\xi_a(\lambda) = \prod_{b<a} q^{-\lambda_b}\,(q^{2(\lambda_a - \lambda_b + a - b - 1)} - 1).$$

Hence the corresponding bialgebroids are isomorphic.

7. Elements of Representation Theory of Exchange Groups $F_q(G)$

7.1. A construction of representations. Let G be a simple group and $\mathfrak{g}$ its Lie algebra. Consider an exchange group $F_q = F_q(G)$.

A dynamical representation $\pi_W : F_q \to D_{T,W}$ is called *rational* if the image of π_W consists of difference operators with rational coefficients. A homomorphism of dynamical representations $\varphi : W_1 \to W_2$ is called *rational* if the matrix elements of φ are rational functions.

Denote $Rep_f(F_q)$ the tensor category of rational finite dimensional (dynamical) representations of F_q and rational morphisms between the representations.

Let $W \in \mathcal{O}_0(G, q)$. Define a rational dynamical representation of F_q on W. Recall that a rational dynamical representation is a diagonalizable $A_0 = U_q(\mathfrak{h})$-module W and a homomorphism of A_0-algebras $\pi_W : F_q \to D_{T,W}$ such that the image of the homomorphism consists of difference operators with rational coefficients. We consider $W \in \mathcal{O}_0(G, q)$ with the A_0-module structure induced by $U_q(\mathfrak{h}) \subset U_q(\mathfrak{g})$ and define π_W by

$$\pi_W(f(\lambda^1)) = f(\lambda), \quad \pi_W(f(\lambda^2)) = f(\lambda - h), \tag{39}$$

$$(1 \otimes \pi_W^0)(L^V)(\lambda) = R_{V,W}(\lambda), \tag{40}$$

for any $V \in Ir \subset \mathcal{O}_0(G, q)$.

Recall that for a dynamical representation $\pi_W : F_q \to D_{T,W}$, one defines a map $\pi_W^0 : F_q \to \mathrm{End}\,(W, W \otimes M_T)$ as explained in Sect. 3.1 and this map uniquely determines π_W.

Theorem 44. *Formulas* (*39*) *and* (*40*) *define a structure of a rational dynamical representation of F_q on W.*

The theorem follows from definitions.

Define a functor F from the category $\mathcal{O}_0(G,q)$ of finite dimensional modules over $U_q(\mathfrak{g})$ (defined in Sect. 5.3) to the category $Rep_f(F_q)$ sending an object $W \in \mathcal{O}_0(G,q)$ to (W,π_W) and sending a morphism $\alpha : W \to U$ to the same linear map $\alpha : W \to U$.

Theorem 45. I. *This construction defines a tensor functor* $F:\mathcal{O}_0(G,q) \to Rep_f(F_q)$ *with a tensor structure*

$$J_{W,U} : F(W)\tilde{\otimes}F(U) \to F(W \otimes U),$$

where $J_{W,U}$ *is defined in (1).*

II. *For generic* q *the map* $\mathrm{Hom}_{\mathcal{O}_0(G,q)}(W,U) \to \mathrm{Hom}_{Rep_f(F_q)}(F(W),F(U))$ *defined by* F *is an isomorphism. Thus,* F *defines a tensor equivalence of* $\mathcal{O}_0(G,q)$ *onto a full subcategory of* $Rep_f(F_q)$.

Remark 1. In the next paper we plan to show that F is an equivalence of categories, i.e. that any object of $Rep_f(F_q)$ is in the image of F.

Remark 2. We see that the representation category of $F_q(G)$ is essentially the same as for $U_q(\mathfrak{g})$. A similar result was obtained in [BBB], where it is shown that the quasi-Hopf algebra associated to the dynamical R-matrix is twist equivalent to $U_q(\mathfrak{g})$ (for $\mathfrak{g} = sl(2)$). These two results are closely related, because as follows from [Xu], representation theory of the Hopf algebroid $F_q(G)$ is tautologically equivalent to representation theory of the corresponding quasi-Hopf algebra.

Remark 3. Theorem 45 raises a question: why is it interesting to study dynamical quantum groups if they have the same representation theory as the usual ones? In our opinion, it is interesting to study not only tensor categories but also their realizations (e.g. tensor functors on them to other tensor categories), which contain extra structure. In particular, $F_q(G)$ and $U_q(G)$ correspond to two different realizations of the same tensor category. In other words, dynamical quantum groups do not provide new tensor categories, but do provide new realizations of already known tensor categories.

Now let us prove the theorem. The first part of the theorem is trivial. The second part of the theorem is proved in Sect. 7.4. In order to prove the second part we first prove that any rational morphism $b : F(W) \to F(U)$ does not depend on λ and then show that there exists $a \in \mathrm{Hom}_{\mathcal{O}_0(G,q)}(W,U)$ such that $b = F(a)$.

7.2. Rational morphisms. Let $W \in \mathcal{O}_0(G,q)$. Let π_W, π_W^0 be representation maps. By definition

$$(1 \otimes \pi_W^0)(L^V)(\lambda) = R_{V,W}(\lambda) = J_{V,W}^{-1}(\lambda)\mathcal{R}^{21}|_{V\otimes W} J_{W,V}^{21}(\lambda).$$

Let v_i^V be a homogeneous basis of V, and v_0^V be the highest weight vector. Let $(v_i^V)^*$ be the dual basis. Consider the matrix element $R_{V,W}^{00}(\lambda) \in \mathrm{End}\,(W)$ defined by

$$< y^*, R_{V,W}^{00}(\lambda)x >=< (v_0^V)^* \otimes y^*, R_{V,W}(\lambda)\, v_0^V \otimes x >,$$

where $x \in V$, $y^* \in V^*$.

Lemma 46. *$R^{00}_{V,W}(\lambda)$ is a nonzero scalar operator on each weight subspace $W[\alpha]$, $\alpha \in T$, of W. Moreover, the value of the scalar is determined by α and does not depend on W.*

Proof. We have

$$\begin{aligned} &< (v_0^V)^* \otimes y^*, J^{-1}_{V,W}(\lambda)\mathcal{R}^{21}|_{V\otimes W} J^{21}_{W,V}(\lambda)\, v_0^V \otimes x >= \\ &< (J^{-1}_{V,W}(\lambda))^* \,(v_0^V)^* \otimes y^*, \mathcal{R}^{21}|_{V\otimes W} J^{21}_{W,V}(\lambda)\, v_0^V \otimes x > . \end{aligned} \tag{41}$$

Since deg $J = 0$ and $J = 1 + \sum a_i \otimes b_i$, deg $a_i < 0$, we have $J^{21} = 1 + \sum b_i \otimes a_i$ and $(J^{-1})^* = 1 + \sum c_i \otimes d_i$, where deg $c_i < 0$. Continuing (41), we get

$$\begin{aligned} < y^*, R^{00}_{V,W}(\lambda)x > &= < (v_0^V)^* \otimes y^*, \mathcal{R}^{21}|_{V\otimes W}\, v_0^V \otimes x > \\ &= < (\mathcal{R}^{21}|_{V\otimes W})^* \,(v_0^V)^* \otimes y^*, v_0^V \otimes x > . \end{aligned}$$

It is well known that the operator $(\mathcal{R}^{21}|_{V\otimes W})^*$ has the form $\mathcal{R}_0 Q$, where $\mathcal{R}_0 = 1+$ (a strictly upper triangular element in $U_q(\mathfrak{n}_+)\hat{\otimes} U_q(\mathfrak{n}_-)$) and $Q \in U_q(\mathfrak{h})\hat{\otimes} U_q(\mathfrak{h})$. Hence,

$$< (\mathcal{R}^{21}|_{V\otimes W})^* \,(v_0^V)^* \otimes y^*, v_0^V \otimes x >=< Q\,(v_0^V)^* \otimes y^*, v_0^V \otimes x > .$$

This proves the lemma. □

Lemma 47. *Let $a(\lambda) : F(W) \to F(U)$ be an intertwining operator, then $a(\lambda)$ does not depend on λ.*

Proof. We have $R^{00}_{V,W} = \pi^0_W(L^V_{00})$, where L^V_{00} is the matrix component of L^V corresponding to the highest weight vector v_0^V. Hence, $\pi_W(L^V_{00}) = R^{00}_{V,W}\mathcal{T}^{-1}_{wt(v_0^V)}$, where $wt(v_0^V)$ is the weight of v_0^V.

The intertwining operator has to satisfy $a(\lambda)\circ \pi_W(L^V_{00}) = \pi_W(L^V_{00})\circ a(\lambda)$. Hence, $a(\lambda) = a(\lambda - wt(v_0^V))$ for any $V \in Ir \subset \mathcal{O}_0(G,q)$. Since $a(\lambda)$ is rational, this means that $a(\lambda)$ does not depend on λ. □

7.3. Asymptotics of $J_{V,W}(\lambda)$ and $R_{V,W}(\lambda)$. First assume that $q = 1$ and $A = U(\mathfrak{g})$. Consider $J_{V,W}(\lambda)$. Change variables $\lambda \to \lambda/\gamma$, where $\gamma \in \mathbb{C}^*$. Then $J_{V,W}(\lambda/\gamma)$ has the form

$$J_{V,W}(\lambda/\gamma) = 1 + \gamma j_{V,W}(\lambda) + O(\gamma^2).$$

To describe $j_{V,W}(\lambda)$ we fix notations. Namely, we fix an invariant nondegenerate bilinear form $(\cdot,\cdot)$ on $\mathfrak{g}$. The bilinear form identifies $\mathfrak{g}$ and $\mathfrak{g}^*$. For any positive root α, fix generators $e_\alpha \in \mathfrak{g}_\alpha$, $f_\alpha \in \mathfrak{g}_{-\alpha}$, such that $h_\alpha = [e_\alpha, f_\alpha]$ has the property $< h_\alpha, \lambda >= (\alpha,\lambda)$ for all $\lambda \in \mathfrak{h}^*$.

Theorem 48. *We have $j_{V,W}(\lambda) = j(\lambda)|_{V\otimes W}$, where $j(\lambda) \in \mathfrak{n}_- \otimes \mathfrak{n}_+$ and*

$$j(\lambda) = -\sum_{\alpha>0} \frac{f_\alpha \otimes e_\alpha}{(\lambda,\alpha)}.$$

Corollary 49. *For $q = 1$ and $A = U(\mathfrak{g})$, we have*

$$J_{V,W}(u \otimes w) = u \otimes w - \sum_{\alpha>0} \frac{f_\alpha \otimes e_\alpha}{(\lambda, \alpha)} u \otimes w + O(\frac{1}{|\lambda|^2}),$$

$$R_{V,W}(u \otimes w) = u \otimes w + \sum_{\alpha>0} \frac{f_\alpha \otimes e_\alpha - e_\alpha \otimes f_\alpha}{(\lambda, \alpha)} u \otimes w + O(\frac{1}{|\lambda|^2}).$$

Proof of the theorem. Let $w \in W$. Consider the intertwining operator $\Phi_\lambda^w : M_\lambda \to M_{\lambda-wt(w)} \otimes W$. Let $v_\lambda \in M_\lambda$ be the highest weight vector (we write M_λ for M_λ^+). It follows from [ES1], Sect. 3 that

$$\Phi_\lambda^w v_\lambda = v_{\lambda-wt(w)} \otimes w - \sum_{\alpha>0} \frac{1}{(\lambda, \alpha)} f_\alpha v_{\lambda-wt(w)} \otimes e_\alpha w + O(\frac{1}{|\lambda|^2}).$$

Now computing the leading term of the composition $\Phi^u_{\lambda-wt(w)} \Phi^w_\lambda v_\lambda$ we conclude that

$$J_{V,W}(u \otimes w) = u \otimes w - \sum_{\alpha>0} \frac{f_\alpha \otimes e_\alpha}{(\lambda, \alpha)} u \otimes w + O(\frac{1}{|\lambda|^2}).$$

This proves the theorem. □

Let $q \neq 1$, $A = U_q(\mathfrak{g})$ and $r =$dim $\mathfrak{h}$. It is well known that $\mathcal{R} \in A\hat{\otimes}A$ has the form $\mathcal{R} = \mathcal{R}_0 Q$, where $Q \in U_q(\mathfrak{h})\hat{\otimes}U_q(\mathfrak{h})$ is a suitable invertible element, and $\mathcal{R}_0 = 1+$ (a strictly upper triangular element in $U_q(\mathfrak{n}_+)\hat{\otimes}U_q(\mathfrak{n}_-)$).

Theorem 50. *For $|q| < 1$ and $A = U_q(\mathfrak{g})$,*

I. *$J_{V,W}(\lambda) \to 1$, when $\lambda \in T = (\mathbb{C}^*)^r$ tends to infinity along the positive alcove, and $J_{V,W}(\lambda) \to \mathcal{R}_0^{21}$, when λ tends to infinity along the negative alcove.*
II. *$R_{V,W}(\lambda) \to \mathcal{R}^{21}$, when λ tends to infinity along the positive alcove, and $R_{V,W}(\lambda) \to Q\mathcal{R}Q^{-1}$, when λ tends to infinity along the negative alcove.*

Proof. It is clear that statement II follows from I. The first statement of I follows from [ES2], Sect. 2, as explained in the proof of Theorem 28. So it remains to prove the second statement of I.

It follows from Proposition 19.3.7 in [L] that the asymptotics of the Shapovalov form on $M_\lambda = U_q(\mathfrak{n}_-)$ for λ tending to ∞ in the negative alcove equals to the Drinfeld form on $U_q(\mathfrak{n}_-)$ (i.e. the form which defines an injective map of Hopf algebras $U_q(\mathfrak{b}_+)$ to its dual). This fact together with the explicit formula for the intertwining operator via the Shapovalov form ([ES2], Sect. 2) implies the second statement of I. □

7.4. Proof of part II of Theorem 45. First assume that $q = 1$ and $A = U(\mathfrak{g})$. Let $W, U \in \mathcal{O}_0(G)$ and $b \in \mathrm{Hom}_{Rep_f(F_q)}(F(W), F(U))$. Recall that $b \in \mathrm{End}_{\mathbb{C}}(W, U)$ does not depend on λ by Lemma 47.

Lemma 51. *The linear operator b commutes with the action of elements e_α, f_α where α is any positive root.*

Corollary 52. $b \in \mathrm{Hom}_{\mathcal{O}_0(G,q)}(W, U)$.

The corollary implies part II of Theorem 45 for $q = 1$.

Proof of the lemma. We prove the lemma for $W = U$. For $W \neq U$, the proof is similar. For any $V \in Ir \subset \mathcal{O}_0(G)$, we have $[1 \otimes b, R_{V,W}(\lambda)] = 0$. Hence, $[1 \otimes b, (R_{V,W}(\lambda) - 1)|\lambda|] = 0$. Setting $\lambda = t\lambda_0$ and taking the limit $t \to \infty$, we get

$$\sum_{\alpha>0} \frac{f_\alpha \otimes [b, e_\alpha] - e_\alpha \otimes [b, f_\alpha]}{(\lambda, \alpha)} = 0.$$

Since there exists V such that the linear operators $e_\alpha|_V$, $f_\alpha|_V$ are linear independent in $\mathrm{End}_{\mathbb{C}}(V)$, we get the lemma. □

Part II of Theorem 45 for $|q| < 1$ follows similarly from Theorem 50. Namely, from Theorem 50 we get that any intertwining operator b must commute with all elements of the form $(f \otimes 1)(\mathcal{R}^{21})$ and $(f \otimes 1)(Q\mathcal{R}Q^{-1})$ $(f \in U_q(\mathfrak{g})^*)$, which obviously generate $U_q(\mathfrak{g})$. (Here if $X = \sum a_i \otimes b_i$ then $(f \otimes 1)(X)$ denotes $\sum f(a_i)b_i$.) Thus, b has to commute with $U_q(\mathfrak{g})$, Q.E.D.

For $|q| > 1$, the proof is analogous.

8. Appendix: Fusion Matrices and 6j-Symbols

In this appendix we discuss the relationship between fusion matrices introduced in Sect. 2, and 6j-symbols, for the Lie algebra $sl(2)$. For quantum $sl(2)$, the relationship is the same.

Recall the definition of 6j-symbols (see e.g. [CFS], p. 29). Let V_a, $a \in \mathbb{Z}_+/2$, be the irreducible representation of $sl(2)$ with spin a. Let v_a be the highest weight vector of V_a, and $v_{a,n} = f^n v_a$. Let $\varphi_a^{bc} : V_a \to V_b \otimes V_c$ be the intertwiner such that $\varphi_a^{bc} v_a = v_b \otimes v_{c,b+c-a} + l.o.t.$ (here l.o.t. is "lower order terms").The 6j-symbol is defined by the formula

$$(1 \otimes \varphi_j^{bc})\varphi_k^{aj} = \sum_n \begin{pmatrix} a & b & n \\ c & k & j \end{pmatrix} (\varphi_n^{ab} \otimes 1)\varphi_k^{nc}.$$

The 6j-symbols not defined in this way are defined to be zero.

Remark. Our definition coincides with the standard one only up to normalization. Namely, it is more common to use a different normalization of the operators φ_a^{bc}, which results in a different normalization of the 6j-symbols.

Now define $J_{bc}(\lambda) := J_{V_b V_c}(\lambda)$. The next proposition, which gives a connection between fusion matrices and 6j-symbols, follows easily from the definitions.

Proposition 53. *For any $k \in \mathbb{Z}_+/2$, one has*

$$\sum_n \begin{pmatrix} a & b & n \\ c & k & j \end{pmatrix} (v_{b,b-n+a} \otimes v_{c,c-k+n}) = J_{bc}^{-1}(k)\varphi_j^{bc} v_{j,j-k+a}.$$

Thus $J_{bc}(k)$ is the unique rational function of k which satisfies the above equation for $k \in \mathbb{Z}_+/2$.

It is easy to check that under this correspondence, the 2-cocycle condition for $J(\lambda)$ corresponds to the Elliott-Biedenharn identity for 6j-symbols [CFS] (known to mathematicians as the Maclane pentagon relation). The dynamical Yang–Baxter equation for $R(\lambda) = J^{-1}(\lambda)J^{21}(\lambda)$ corresponds to the star-triangle relation.

9. Appendix: Recursive Relations for Fusion Matrices

In [A], the authors defined fusion matrices as unique solutions of certain linear equations, and checked that they satisfy the 2-cocycle condition. In this appendix, we will show that our fusion matrices satisfy the same linear equations, which implies that they are the same fusion matrices as in [A].

We will use a finite-dimensional version of the quantum Knizhnik-Zamolodchikov equations, which were deduced by Frenkel and Reshetikhin for quantum affine algebras. Consider the function $\Psi_{wv}(\lambda) \in \mathrm{End}\,(W \otimes V)$ given by

$$\Psi_{wv}(\lambda) := J_{WV}(\lambda)(w \otimes v) =< (\Phi^w_{\lambda-\lambda_v} \otimes 1)\Phi^v_\lambda >,$$

where the notation $<, >$ was defined in Sect. 2.4. It follows from a finite dimensional degeneration of the Frenkel-Reshetikhin theorem (Theorem 10.3.1 in [EFK]) that this function satisfies the following version of the quantum Knizhnik-Zamolodchikov equations:

$$q^{2(\lambda_v,\lambda+\rho)-(\lambda_v,\lambda_v)}\Psi_{wv}(\lambda) = R^{21}_{VW}(1 \otimes q^{2\lambda-\lambda_v-\lambda_w+2\rho})\Psi_{wv}(\lambda).$$

This implies that

$$J_{WV}(\lambda)(1 \otimes q^{2(\lambda+\rho)-\sum x_i^2}) = R^{21}_{VW}q^{-\sum x_i\otimes x_i}(1 \otimes q^{2(\lambda+\rho)-\sum x_i^2})J_{WV}(\lambda).$$

It is easy to see that the last equation is (up to simple changes of variable) the same as relation (18) in [A].

A similar computation is valid for an arbitrary quantized Kac–Moody algebra. This computation yields the linear relation for J discussed in [JKOS].

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References

[A] Arnaudon, D., Buffenoir, E., Ragoucy, E. and Roche, Ph.: Universal Solutions of Quantum Dynamical Yang–Baxter Equations. Preprint q-alg/9712037

[BBB] Babelon, O., Bernard, D., Billey, E.: A quasi-Hopf algebra interpretation of quantum 3j and 6j symbols and difference equations. Phys. Lett. B **375**, 89–97 (1996)

[CFS] Carter, J., Flath, D., Saito, M.: *The classical and quantum 6j-symbols*. Math. Notes Princeton, NJ: Princeton University Press, 1995

[CP] Chari, V. and Pressley, A.: *A Guide to Quantum Groups* Cambridge: Cambridge University Press, 1994

[D] Drinfeld, V.: Quasi-Hopf Algebras. Leningrad Math. J. **1**, 1419–1457 (1990)

[EFK] Etingof, P., Frenkel, I. and Kirillov Jr, A.: *Lectures on representation theory and Knizhnik–Zamolodchikov equations*. Providence, RI: AMS, 1998

[EK] Etingof, P. and Kirillov Jr, A.: Representation-theoretic proof of Macdonald inner product and symmetry identities. Comp. Math. **102**, 179–202 (1996)

[ES1] Etingof, P. and Styrkas, K.: Algebraic integrability of Schrödinger operators and representations of Lie algebras. hep-th 9403135, Comp. Math. **98**, No. 1, 91–112 (1995)

[ES2] Etingof, P. and Styrkas, K.: Algebraic integrability of Macdonald operators and representations of quantum groups. q-alg 9603022, to appear in Comp. Math.

[EV1] Etingof, P. and Varchenko, A.: Geometry and classification of solutions of the classical dynamical Yang–Baxter equation. Commun. Math. Phys. (1998)

[EV2] Etingof, P. and Varchenko, A.: Solutions of the quantum dynamical Yang–Baxter equation and dynamical quantum groups. Preprint, 1997

[F] Felder, G.: *Conformal field theory and integrable systems associated to elliptic curves*. Proceedings of the International Congress of Mathematicians, Zürich 1994, Basel–Boston: Birkhäuser, 1994, pp. 1247–1255; Elliptic quantum groups. Preprint hep-th/9412207, to appear in the Proceedings of the ICMP, Paris 1994

[FR] Frenkel, I. and Reshetikhin, N.: Quantum affine algebras and holonomic difference equations. Commun. Math. Phys. **146**, 1–60 (1992)

[FTV] Felder, G., Tarasov, V. and Varchenko, A.: Solutions of the elliptic qKZB equations and Bethe ansatz I. Am. Math. Soc. Transl. (2) Vol. **180**, 45–75 (1997)

[FV1] Felder, G. and Varchenko, A.: On representations of the elliptic quantum group $E_{\tau,\eta}(sl_2)$. Commun. Math. Phys. **181**, 746–762 (1996)

[FV2] Felder, G. and Varchenko, A.: Algebraic Bethe ansatz for the elliptic quantum group $E_{\tau,\eta}(sl_2)$. Nuclear Physics **B 480**, 485–503 (1996)

[JKOS] Jimbo, M., Konno, H., Odake, S., Shiraishi, J.: Quasi-Hopf twistors for elliptic quantum groups. q-alg/9712029

[L] Lusztig, G.: *Introduction to quantum groups*. Boston: Birkhäuser, 1993

[M] Mumford, D.: *Abelian varieties*. Tata Studies in Math., Oxford: Oxford Univ. Press, 2nd ed., 1975

[TV1] Tarasov, V. and Varchenko, A.: Geometry of q-hypergeometric functions as a bridge between Yangians and quantum affine algebras. Invent. Math. **128**, 501–588 (1997)

[TV2] Tarasov, V. and Varchenko, A.: Geometry of q-Hypergeometric Functions, Quantum Affine Algebras and Elliptic Quantum Groups. preprint q-alg/9703044

[Xu] Xu, P.: Quantum groupoids and universal dynamical R-matrices. IHES preprint, June 1998

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