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Nonlinear Stability of Weak Detonation Waves for a Combustion Model

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Abstract: We show that the weak detonation waves for a combustion model of Rosales–Majda are nonlinearly stable. Because of the strongly nonlinear nature of the wave, usual stability analysis of weakly nonlinear nature does not apply. The chemical switch on-off is the main feature of nonlinearity. In particular, the propagation of the wave depends sensitively on the tail behaviour of the flow in front of it. Unlike the strong detonation waves, a weak detonation is supersonic and there is the separation of the gas waves from the reacting front. As a consequence, the reacting front needs to be traced.

Contents

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1. Introduction

Consider the combustion model, Rosales–Majda [15],

$$\begin{cases} u_t + (f(u) - qz)_x = u_{xx}, \\ z_x = K\ \psi(u)\ z, \end{cases} \tag{1.1}$$

where u represents the lumped gas variable and z the density of the reactant. The unburnt state is $z = 1$ and the burnt state $z = 0$. The positive constants q and K are the released energy and the reaction rate, respectively. We assume that there is an ignition temperature T^* so that the reaction function $\psi(u)$ is given by

$$\psi(u) \equiv \begin{cases} 1 \text{ if } u > T^* \\ 0 \text{ if } u \leq T^*. \end{cases} \tag{1.2}$$

To model the detonation waves, the flux function $f(u)$ is assumed to satisfy

$$f''(u) \geq 0, \quad f'(u) > 0 \text{ for all } u \text{ under consideration.} \tag{1.3}$$

Our purpose is to study the nonlinear stability of weak detonation waves. The detonation waves are travelling waves $(u, z)(x, t) = (\bar{\mathbf{u}}, \bar{\mathbf{z}})(x - st)$ of the model. We consider the perturbation of the wave:

$$\begin{cases} \lim_{x\to\infty} z(x, t) = 1, \\ \lim_{x\to-\infty} z(x, t) = 0, \\ u(x, 0) = u_0(x), \text{ with} \\ \lim_{x\to\infty} u_0(x) = u_R, \quad \lim_{x\to-\infty} u_0(x) = u_L. \end{cases}$$

The flow is unburnt in front of the wave and burnt behind it, $u_R < T^* < u_L$. System (1.1) is derived from the reactive Navier-Stokes equations to model the acoustic mode of the flow under the limit of low Mach number. For combustion waves for the reactive Navier Stokes equations, see [2].

There are two types of detonation waves, the strong and weak detonations.A strong detonation satisfies the same entropy condition as a gas dynamic shock in that it is supersonic (or subsonic) with respect to the flow in front of (or behind) it. Its stability can be shown by the same technique as that for the viscous conservation laws, [7,9,5,6]. The weak detonation is supersonic with respect to both sides of the wave. This is the classical inviscid Chapman-Jouget theory, [1]. The weak detonation waves are not inviscid waves and depend on the dissipation parameters. This is the general phenomena for waves which are either overcompressive, such as intermediate MHD waves, or undercompressive, such weak detonation waves, cf. [8]. This has two basic implications. A perturbation produces a gas wave leaving the combustion wave. The one conservation law can not determine both the location of the detonation wave and the amount of gas wave. Thus the weak detonation wave needs to be traced. The situation is similar to the interaction of shock waves with either the boundary, [10,14], or with other nonlinear waves, [17], see also [11], and [12] on discrete waves. In the study of weak detonation waves in Sect. 2, we fix all physical variables except for the ignition temperature T^*, which is allowed to vary. This is done for convenience and is equivalent to the usual practice of varying the energy release q, cf. [15].

One thing distinguishes the weak detonation wave is that it generates strongly nonlinear effects. For instance, there is a sensitive dependence of the propagation of the wave on the perturbation in front of it. These factors demand new techniques for the stability analysis. The tracing of the wave requires exact analysis of the chemical nonlinearity, which is done using the Laplace–Fourier transform.

The paper by Szepessy, [16] also studies the problem of stability of weak detonation waves. We have adopted his definition of detonation wave front $\gamma(t)$ by $u(\gamma(t), t) = T^*$, [16]. The derivation of the integral equation for the wave front in Sect. 3 is also motivated by the paper. On the other hand, [16] does not require the second condition in (1.3) and, as a consequence, the strength of the detonation wave can be assumed to be small. The gas nonlinearity $f(u) = u^2$ is emphasized in [16] and is taken care of through the Hopf-Cole transformation.

A strong detonation, the so-called ZND wave, [1], is a gas dynamic shock followed by a reacting zone. The shock raises the gas temperature through compression and thereby sets up the chemical reaction. Thus its stability mechanism is similar to that of the gas shocks. A weak detonation, on the other hand, runs ahead of the gas waves and decouples from them. Thus the nonlinearity of a weak detonation wave is mainly the chemical nonlinearity. To focus on this, we consider in Sect. 4 the simplified model with linear flux, $f''(u) \equiv 0$. This is the main part of the present paper. The study of the general situation, $f''(u) \geq 0$, requires an iteration scheme to take care of the gas nonlinearity and is done in Sect. 5.

Consistent with the derivation of the model (1.1), we require the strong seperation of the detonation front and the gas wave, $\alpha \equiv s - f'(T^*)$ large. The other main assumption is that both the reaction rate K and α/K are large. The precise assumptions, Assumptions 2.1 to 2.7, are listed in Sect. 2. These assumption are verified either numerically for convex flux in Sect. 2, or analytically for linear flux in Sect. 4. Under these assumptions, we have the following main theorem:

Theorem 1.1. *Suppose that the perturbation of a weak detonation wave* $v_0(x) \equiv u(x, 0) - \bar{\mathbf{u}}(x)$ *is sufficiently small:*

$$v_0 \in C^1(\mathbf{R}) \cap Lip^2(\mathbf{R}),$$
$$|\partial_x^i v_0(x)| \leq \delta\, \alpha^i\, e^{-\alpha \frac{5|x|}{8}} \text{ for } i = 0, 1, 2$$

for a constant δ *satisfying* $\delta < \alpha^{-6}$. *Then the solution of (1.1) tends to a translation* $\gamma(t)$ *of the detonation wave as time* t *tends to infinity:*

$$u(x, t) - \bar{\mathbf{u}}(x - st - \gamma(t)) = O(1)\delta[e^{-Ct} e^{-C|x-st-\gamma(t)|} + \frac{1}{\sqrt{t+1}} e^{-\frac{(x-f'(u_-)t)^2}{A(t+1)}}],$$

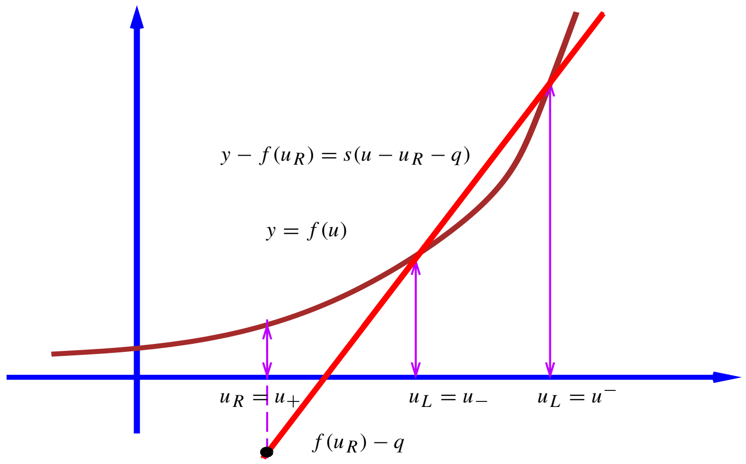
for some positive constants C *and* A*; and* $\gamma(t)$ *tends to its limit exponentially fast.*

Note that the convergence is at the exponential rate, except for the algebraic rate of $(t+1)^{-1/2}$ along the gas acoustic direction $x = f'(u)t$.

2. Structure of Weak Detonation Profiles

It follows from the system (1.1) that the end states and the speeds of a detonation wave satisfies the **Rankine–Hugoniot** condition:

$$s = \frac{f(u_L) - f(u_R) + q}{u_L - u_R}, \tag{2.1}$$



It is clear that we have

Lemma 2.1. *Suppose that u_R and $s > 0$ are given such that*

$$s > f'(u_R) > 0$$

as well as that

$$f'' > 0.$$

Then, u_L in (2.1) has two solutions $\{u_-, u^-\}$ with $u_- < u^-$.

Let $u_+ = u_R$ be given as in Lemma 2.1. The shock $((u_-, 0), (u_+, 1))$ is supersonic:

$$s > f'(u_-) > f'(u_+) > 0$$

and is called a weak detonation wave. The wave $((u^-, 0), (u_+, 1))$ satisfies the usual gas dynamics entropy condition:

$$f'(u^-) > s > f'(u_+),$$

and is called a strong detonation wave.

2.1. Construction of the profiles. So far we have studied the far field states $u_\pm$ of the combustion waves. For the actual existence and structure of these waves we need to study the ODE obtained from (1.1) when the solution is a travelling wave $(u, z)(x, t) = (\bar{\mathbf{u}}, \bar{\mathbf{z}})(x - st)$:

$$\begin{cases} -s(\bar{\mathbf{u}} - u_-) + f(u) - f(u_-) - q\bar{\mathbf{z}} = \bar{\mathbf{u}}_x, \\ K\psi(u)\bar{\mathbf{z}} = \bar{\mathbf{z}}_x, \\ \begin{cases} \lim_{x\to\infty}(\bar{\mathbf{u}}, \bar{\mathbf{z}})(x) = (u_+, 1), \\ \lim_{x\to-\infty}(\bar{\mathbf{u}}, \bar{\mathbf{z}})(x) = (u_-, 0), \end{cases} \\ \psi(u) \equiv \mathbf{H}(u - T^*), \end{cases} \tag{2.2}$$

where, for definiteness, we have made the normalization

$$\bar{\mathbf{u}}(0) = T^*,$$

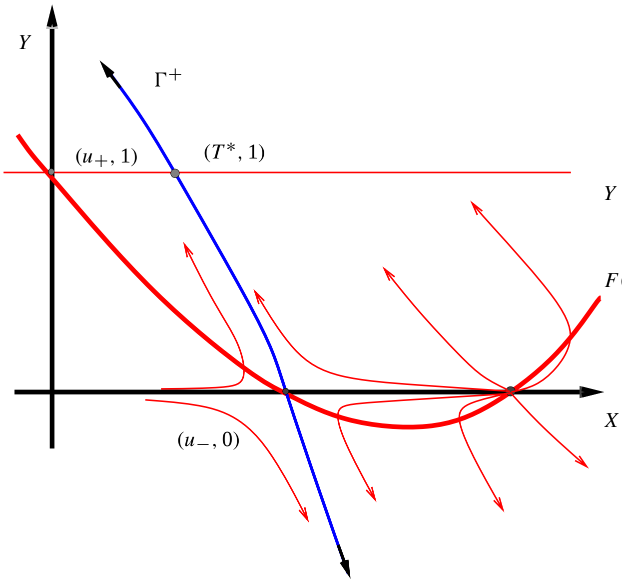
and $\mathbf{H}$ is the Heaveside function:

$$\mathbf{H}(u) = \begin{cases} 0 \text{ if } u \le 0, \\ 1 \text{ else.} \end{cases}$$

Lemma 2.2. *For any s and $u_+ \equiv u_R$ given in Lemma 2.1, there is a unique $T^* \in (u_+, u_-)$ such that (2.2) has a monotone solution.*

Proof. Consider the following dynamical system:

$$\begin{cases} \dot{X} = -s(X - u_-) + f(X) - f(u_-) - qY, \\ \dot{Y} = KY, \\ \lim_{t\to-\infty}(X, Y)(t) = (u_-, 0). \end{cases} \tag{2.3}$$



The state $(u_-, 0)$ is a fixed point of this dynamical system. At this point the dynamical system has a one-dimensional unstable manifold Γ. Let Γ^+ be the branch with positive Y component. Set

$$\Gamma^+ \equiv \{(\Gamma_1(t), \Gamma_2(t)) : t \in \mathbf{R}\}.$$

From the second equation of (2.3), Γ^+ can be normalized such that

$$\Gamma_2(t) = e^{Kt}.$$

Hence, Γ^+ will intersect $Y = 1$.

Set

$$F(X, Y) \equiv -s(X - u_-) + f(X) - f(u_-) - qY.$$

The set $F = 0$ contains $(u_+, 1)$ and $(u_-, 0)$. From the phase diagram of (X, Y), it follows that $F = 0$ never intersects Γ^+ in $Y > 0$. Thus, Γ^+ is to the right of $F = 0$ and

$\Gamma_1(t)$ is a monotone decreasing function. Hence, $\Gamma_1(0) < u_-$ and $(\Gamma_1(0), 1)$ is to the right of $F = 0$ and so

$$\Gamma_1(0) \in (u_+, u_-).$$

The profiles $(\bar{\mathbf{u}}, \bar{\mathbf{z}})(x)$ and $(\Gamma_1(x), e^{Kx})$ are identical for $x \leq 0$. Set

$$T^* \equiv \Gamma_1(0).$$

From this choice of T^*, the solution $(\bar{\mathbf{u}}, \bar{\mathbf{z}})(x)$ for $x > 0$ is $\bar{\mathbf{z}}(x) \equiv 1$ and $\bar{\mathbf{u}}(x) = X(x)$ solving

$$\begin{cases} -s(X - u_-) + f(X) - f(u_-) - q = \dot{X}, \\ X(0) = T^*. \end{cases}$$

Clearly, $X(t)$ is a strictly monotone decreasing function with

$$\lim_{t\to\infty} X(t) = u_+. \qquad \square$$

The ignition temperature T^* is a function depending on u_-, u_+, s, q, and K. With the additional constraint of the Rankine–Hugoniot condition, we have

$$T^* = T^*(u_+, s, q, K). \tag{2.4}$$

2.2. Hypothesis on the profiles. Let $(\bar{\mathbf{u}}, \bar{\mathbf{z}})(x - st)$ be the normalized weak detonation profile and α the separation of the combustion and the gas speeds:

$$\alpha \equiv s - f'(T^*).$$

For our stability analysis we need the following hypotheses on the combustion wave.

Assumption 2.3. *Assume that the following holds for* $\bar{\mathbf{u}}$

$$Q \equiv \frac{qK}{|\bar{\mathbf{u}}_x(0)|\alpha} = O(1),$$

where the bound $O(1)$ *is independent of the parameters involved.*

Assumption 2.4. *Assume that the gradient of* $-\bar{\mathbf{u}}_x(0)$ *is sufficiently large.*

Note. When $q, \alpha/K \gg 1$, the quantity $|\bar{\mathbf{u}}_x(0)|$ is proportional to $|u_- - u_+|K$. Thus, this hypothesis is a consequence of assuming $K \gg 1$. About Assumption 2.3 the quantity Q can not be made arbitrarily small by arranging the values of u_+, s, q, and K. This will be illustrated by a simplified model in the next subsection. For now we show by numerics that both Assumption 2.3 and Assumption 2.4 can be satisfied, by calculating the value Q for the flux function

$$f(u) = u^2.$$

The normalized profile $\bar{\mathbf{u}}$ satisfies

$$\bar{\mathbf{u}}_x = 2(\bar{\mathbf{u}} - u_-) + (\bar{\mathbf{u}} - u_-)^2 - qe^{Kx} \text{ for } x \leq 0. \tag{2.5}$$

By repeating Picard iteration eight times, we have the following approximate values for T^*, $\bar{\mathbf{u}}_x$, and Q with given (u_+, u_-) and varying (q, K):

u_+	u_-	s	q	K	T^*	$\bar{\mathbf{u}}_x(0)$	α	Q
1	2	103	100	1	1.0097	−0.98	100.98	1.0097
1	2	53	50	1	1.0189	−0.96	50.96	1.0189
1	2	28	25	1	1.0358	−0.931	25.92	1.036
1	2	15.5	12.5	1	1.0653	−0.88	13.37	1.0661
1	2	103	100	2	1.0192	−1.94	100.96	1.0194
1	2	103	100	4	1.0378	−3.81	100.92	1.04
1	2	103	100	8	1.0729	−7.36	100.85	1.08
1	2	103	100	16	1.1361	−13.73	100.72	1.16

The variables (u_+, s, q, K) in the above table are the basic variables which determine uniquely the other variables in the table. With K, q, and u_+ fixed one can vary s such that $K/\alpha \ll 1$. One has the following analytic properties of the resulted viscous shock profile $\bar{\mathbf{u}}$.

Lemma 2.5. *With $K > 0$, $q > 0$, and u_+ given, there exists $S(K, q, u_+) > 0$ such that for any $s > S(K, q, u_+)$ the following holds for $x > 0$:*

$$0 < \frac{T^* - u_+(x)}{u_- - u_+} < \frac{S(K, q, u_+)}{\alpha},$$
$$\frac{|\bar{\mathbf{u}}_x(0)|e^{-(s-f'(u_+))|x|}}{2} \leq -\bar{\mathbf{u}}_x(x) \leq 2\,|\bar{\mathbf{u}}_x(0)|e^{-\alpha|x|};$$

and for $x < 0$

$$0 \leq -\bar{\mathbf{u}}_x(x).$$

See the Appendix for the proof.

To handle the chemical nonlinearity, we pose the following hypothesis:

Assumption 2.6. *Assume that*

$$\frac{\alpha^{\frac{1}{16}}}{|\bar{\mathbf{u}}_x(0)|} \gg 1,$$
$$\frac{\alpha}{K} \gg 1.$$

On interaction between the fluid nonlinearity and chemical nonlinearity, the following assumption is required.

Assumption 2.7. *Assume that*

$$K \gg 1.$$

2.3. Explicit profiles for model with linear flux. Consider a simplified system with linear flux

$$\begin{cases} u_t + m\, u_x - q z_x = u_{xx}, \\ z_x = K\psi(u) z, \end{cases} \tag{2.6}$$

$$u_- > u_+, \quad s > m > 0,$$
$$q \equiv (s-m)(u_- - u_+).$$

From the analysis of the profiles in the last section it is easy to see that

$$T^* \equiv u_- - \frac{q}{K+s-m},$$
$$T^* - u_+ = \frac{K(u_- - u_+)}{K+s-m}.$$

Let $(\bar{\mathbf{u}}, \bar{\mathbf{z}})(x - st)$ be the travelling wave solution of (2.6), which connects $((u_-, 0), (u_+, 1))$:

$$\begin{cases} \bar{\mathbf{u}}_x = (m-s)(\bar{\mathbf{u}} - u_-) - q\bar{\mathbf{z}}, \\ \bar{\mathbf{z}}_x = K\psi(\bar{\mathbf{u}})\bar{\mathbf{z}}, \end{cases} \tag{2.7}$$

$$\begin{cases} \lim_{x\to\infty}(\bar{\mathbf{u}}, \bar{\mathbf{z}})(x) = (u_+, 1), \\ \lim_{x\to-\infty}(\bar{\mathbf{u}}, \bar{\mathbf{z}})(x) = (u_-, 0). \end{cases}$$

As before, we make the normalization:

$$\begin{cases} \bar{\mathbf{u}}(0) = T^*, \\ \bar{\mathbf{z}}(0) = 1. \end{cases} \tag{2.8}$$

From the monotonicity of the profile we have

$$(\bar{\mathbf{u}}(x) - T^*)\, x < 0 \text{ for } x \neq 0. \tag{2.9}$$

From (2.8) and (2.9),

$$\bar{\mathbf{z}}(x) = \begin{cases} 1 \text{ for } x > 0, \\ e^{Kx} \text{ for } x \leq 0. \end{cases} \tag{2.10}$$

Substitute (2.10) into (2.7), we obtain

$$\bar{\mathbf{u}}_x = (m-s)(\bar{\mathbf{u}} - u_-) - qe^{Kx} \text{ for } x \leq 0. \tag{2.11}$$

From (2.8) and (2.11) we have

$$\bar{\mathbf{u}}(x) = u_- - \frac{qe^{Kx}}{s-m+K} \text{ for } x \leq 0.$$

The profile for $x > 0$ is trivial as in the last section. For this simplified model

$$\alpha = s - m.$$

Furthermore, from (2.11) we have

$$\bar{\mathbf{u}}_x(0) = \frac{K\ (m-s)\ (u_- - u_+)}{s-m+K} < 0.$$

$$Q \equiv \frac{qK}{|\bar{\mathbf{u}}_x(0)|\alpha} = \frac{q(s-m+K)}{(s-m)^2(u_- - u_+)} = \frac{\alpha + K}{\alpha}.$$

Therefore Assumption 2.3 is satisfied under a weak version of Assumption 2.6: $\alpha = O(1)K$.

3. Evolution Equations and Wave Fronts

For a small perturbation of a weak detonation wave, the solution $u(x,t)$ assumes the value of the ignition temperature only at one location for each given time and the wave front $\gamma(t) + st$ is well-defined by:

$$u(\gamma(t) + st, t) \equiv T^*.$$

Without loss of generality, we assume that

$$\gamma(0) = 0.$$

With the change of coordinates:

$$\begin{cases} x \to x + st, \\ t \to t, \end{cases}$$

the system (1.1) becomes

$$\begin{cases} u_t - su_x + f(u)_x = u_{xx} + qz_x, \\ z_x = K\psi(u)z. \end{cases} \tag{3.1}$$

Expand $f(u)$ at T^*:

$$N_2(u) \equiv f(u) - f(T^*) - f'(T^*)(u - T^*),$$
$$\alpha \equiv s - f'(T^*).$$

Substitute this into (3.1):

$$\begin{cases} u_t - \alpha u_x - u_{xx} = qz_x - N_2(u)_x, \\ z_x = \psi(u)Kz. \end{cases} \tag{3.2}$$

The perturbation

$$v(x,t) \equiv u(x,t) - \bar{\mathbf{u}}(x)$$

of the weak detonation satisfies

$$v_t - \alpha\, v_x - v_{xx} = q(z - \bar{\mathbf{z}})_x - N_1(v)_x, \qquad N_1(v) \equiv N_2(v + \bar{\mathbf{u}}) - N_2(\bar{\mathbf{u}}). \tag{3.3}$$

The Green function for the left-hand side of (3.3) is the heat kernel

$$k(x - y + \alpha(t-\sigma), t - \sigma) = \frac{e^{-\frac{(x-y+\alpha(t-\sigma))^2}{4(t-\sigma)}}}{\sqrt{4\pi(t-\sigma)}}.$$

By Duhamel's principle,

$$\begin{aligned} v(x,t) = &\int_R k(x-y+\alpha t, t) v(y,0) dy \\ &+q \int_0^t \int_R k(x-y+\alpha(t-\sigma), t-\sigma)(z-\bar{\mathbf{z}})_y dy d\sigma \\ &- \int_0^t \int_R k(x-y+\alpha(t-\sigma), t-\sigma) N_{1y}(v)(y,\sigma) dy d\sigma. \end{aligned} \tag{3.4}$$

Set

$$\mathbf{W}(x) \equiv (1-\mathbf{H}(x))\, e^{Kx}.$$

From the reaction equation of z in (3.2), both $z(y,\sigma)$ and $\bar{\mathbf{z}}(y)$ can be represented in terms of the detonation wave locations:

$$\begin{cases} z_y(y,\sigma) = K\mathbf{W}(y-\gamma(\sigma)), \\ \bar{\mathbf{z}}_y(y) = K\mathbf{W}(y). \end{cases} \tag{3.5}$$

Since $u(\gamma(t),t) = T^*$:

$$v(\gamma(t),t) = u(\gamma(t),t) - \bar{\mathbf{u}}(\gamma(t)) = \bar{\mathbf{u}}(0) - \bar{\mathbf{u}}(\gamma(t)) = T^* - \bar{\mathbf{u}}(\gamma(t)),$$

we obtain, from (3.4) and (3.5) , the equation for $\gamma(t)$,

$$\begin{aligned} \bar{\mathbf{u}}(0) - \bar{\mathbf{u}}(\gamma(t)) = &\int_R k(\gamma(t)-y+\alpha t, t) v(y,0) dy \\ &+ qK \int_0^t \int_{y<0} (k(\gamma(t)-\gamma(\sigma)-y+\alpha(t-\sigma), t-\sigma) \\ &\qquad - k(\gamma(t)-y+\alpha(t-\sigma), t-\sigma))\mathbf{W}(y) dy d\sigma \\ &- \int_0^t \int_R k(\gamma(t)-y+\alpha(t-\sigma), t-\sigma) N_1(v)_y(y,\sigma) dy d\sigma. \end{aligned} \tag{3.6}$$

The same Duhamel's principle applies for the special solution $(\bar{\mathbf{u}}, \bar{\mathbf{z}})$ of (3.2):

$$\begin{aligned} \bar{\mathbf{u}}(x-\rho) - \bar{\mathbf{u}}(x) = &\int_R k(x-y+\alpha t, t)(\bar{\mathbf{u}}(y-\rho) - \bar{\mathbf{u}}(y)) dy \\ &+ qK \int_0^t \int_{y<0} (k(x-\rho-y+\alpha(t-\sigma), t-\sigma) \\ &- k(x-y+\alpha(t-\sigma), t-\sigma))\mathbf{W}(y)\, dy d\sigma \\ &- \int_0^t \int_R k(x-y+\alpha(t-\sigma), t-\sigma) N_1(\bar{\mathbf{u}}(y-\rho) - \bar{\mathbf{u}}(y))_y dy d\sigma. \end{aligned}$$

Set both x and ρ above equal to $\gamma(t)$, and subtract (3.6) from the resulting identity. We obtain a refined equation for $\gamma(t)$:

$$
\begin{aligned}
0 = & \int_R k(\gamma(t) - y + \alpha t, t) \{[\bar{\mathbf{u}}(y - \gamma(t)) - \bar{\mathbf{u}}(y)] - v(y, 0)\}\, dy \\
& + qK \int_0^t \int_{y<0} (k(-y + \alpha(t - \sigma), t - \sigma) \\
& - k(\gamma(t) - \gamma(\sigma) - y + \alpha(t - \sigma), t - \sigma))\mathbf{W}(y)\, dy d\sigma \\
& - \int_0^t \int_R k(\gamma(t) - y + \alpha(t - \sigma), t - \sigma) \{N_1(\bar{\mathbf{u}}(y - \gamma(t)) - \bar{\mathbf{u}}(y)) - N_1(v)\}_y \, dy d\sigma.
\end{aligned} \tag{3.7}
$$

Equation (3.7) contains not only the front $\gamma(t)$ but also the gradient of the fluid variable $\bar{\mathbf{u}}$. Thus, the study of the qualitative behavior of the wave front requires the global stability of the fluid. The idea is to utilize the local stability to trace the wave fronts in each small time interval. This is then used to show the local stability of the fluid. The time asymptotic stability is studied by repeating the local analysis. In the next section we carry out the analysis for the model with linear flux.

4. Stability Analysis I: Linear Flux Model

To concentrate on the analysis of the switch on-off reaction nonlinearity, we consider the simplified model with linear flux, $f'(u) = m$.

In the moving coordinate, the system (3.1) is

$$
\begin{cases} u_t - \alpha\, u_x - u_{xx} = q z_x, \\ z_x = K\psi(u) z, \end{cases}
$$

where $\alpha = m - s > 0$. The integral equation (3.7) of the detonation wave front $\gamma(t)$ is self-contained for this simplified model:

$$
\begin{aligned}
0 = & \int_R k(\alpha t - y, t)\, ([\bar{\mathbf{u}}(y) - \bar{\mathbf{u}}(y + \gamma(t))] - v(y + \gamma(t), 0))\, dy \\
& + qK \int_0^t \int_{y<0} \{k(\alpha(t - \sigma) - y, t - \sigma) \\
& - k(\alpha(t - \sigma) + \gamma(t) - \gamma(\sigma) - y, t - \sigma)\} \cdot \mathbf{W}(y)\, dy d\sigma.
\end{aligned} \tag{4.1}
$$

4.1. Initial step: upper bounds of the wave fronts. Let δ be a small positive number:

$$
\delta < \alpha^{-6}, \tag{4.2}
$$

and assume that the initial perturbation $v_0(x) \equiv u(x, 0) - \bar{\mathbf{u}}(x)$ satisfies:

$$
\begin{cases} v_0(0) = 0, \quad v_0 \in C^1(\mathbf{R}) \cap Lip^2(\mathbf{R}), \\ |\partial_x^i v_0(x)| \ \le\ \delta\, \alpha^i\, e^{-\alpha \frac{5|x|}{8}} \text{ for } i = 0, 1, 2. \end{cases} \tag{4.3}
$$

Rewrite (4.1) as

$$\gamma(t)\left\{-\int_R k(\alpha t-y,t)\left(\int_0^1 \bar{\mathbf{u}}_y(y+\gamma(t)\theta)\,d\theta\right)dy\right\}$$
$$+\int_0^t \gamma(t)\left\{qK\int_{y<0}\left(\int_0^1 k(\theta(\gamma(t)-\gamma(\sigma))+\alpha(t-\sigma)-y,t-\sigma)d\theta\right)_y e^{Ky}\,dy\right\}d\sigma$$
$$=\int_0^t \gamma(\sigma)\left\{qK\int_{y<0}\left(\int_0^1 k(\theta(\gamma(t)-\gamma(\sigma))+\alpha(t-\sigma)-y,t-\sigma)\,d\theta\right)_y e^{Ky}\,dy\right\}d\sigma$$
$$-\int_R k(\alpha t-y,t)v_0(y+\gamma(t))\,dy. \tag{4.4}$$

Lemma 4.1. *Suppose that α is sufficiently large. Then, there exists a constant $L_0>0$ such that for $|\rho|<\delta$,*

$$-L_0^{-1}\bar{\mathbf{u}}_x(0)\,e^{-\frac{\alpha^2 t}{4}}\min\left(1,\frac{1}{\alpha\sqrt{t}}\right)\le -\int_R k(\rho+\alpha t-y,t)\,\bar{\mathbf{u}}_y(y)\,dy,$$
$$\int_R k(\rho+\alpha t-y,t)e^{-\frac{5\alpha|y|}{8}}dy\le L_0 e^{-\frac{\alpha^2 t}{4}}\min\left(1,\frac{1}{\alpha\sqrt{t}}\right), \tag{4.5}$$
$$\int_{y<0} k(\rho+\alpha t-y,t)e^{\frac{\alpha|y|}{3}}dy\le L_0 e^{-\frac{\alpha^2 t}{4}}\min\left(1,\frac{1}{\alpha\sqrt{t}}\right). \tag{4.6}$$

Proof. Expand $k(\rho+\alpha t-y,t)$ as follows:

$$k(\rho+\alpha t-y,t)=\frac{e^{\frac{1}{2}\alpha y}}{e^{\frac{1}{2}\alpha\rho}}\,k(\rho-y,t)\,e^{-\frac{\alpha^2 t}{4}}.$$

The lemma follows from plugging Lemma 2.6 and the above expansion into the integrals. □

Lemma 4.2. *Suppose that α is sufficiently large and that (4.2) holds. Then, for $0<t<2\log\alpha/\alpha^2$,*

$$\left|qK\int_{t-\frac{8\delta}{\alpha}}^t\int_{y<0}\left(\int_0^1 k(\theta(\gamma(t)-\gamma(\sigma))+\alpha(t-\sigma)-y,t-\sigma)d\theta\right)_y e^{Ky}\,dy\,d\sigma\right|$$
$$\le\frac{1}{\alpha^{\frac{1}{2}}}\left|\int_R k(\alpha t-y,t)\left(\int_0^1 \bar{\mathbf{u}}_y(y+\gamma(t)\theta)\,d\theta\right)dy\right|.$$

Proof. Exchange the order of the last integrations and apply Lemma 4.1 to the resulting integral to yield

$$\frac{|\bar{\mathbf{u}}_x(0)|}{4L_0\alpha^{\frac{3}{2}}}\le\int_R k(\alpha t-y,t)\left(\int_0^1 |\bar{\mathbf{u}}_y(y+\gamma(t)\theta)|\,d\theta\right)dy \text{ for } 0\le t\le\frac{2\log\alpha}{\alpha^2}. \tag{4.7}$$

Set

$$\mathbf{II}(t,\sigma)\equiv\int_{y<0}\left(\int_0^1 k(\theta(\gamma(t)-\gamma(\sigma))+\alpha(t-\sigma)-y,t-\sigma)\,d\theta\right)_y e^{Ky}\,dy.$$

Using integration by parts we have

$$\begin{aligned}\mathbf{II}(t,\sigma) &= \int_0^1 k(\theta(\gamma(t)-\gamma(\sigma))+\alpha(t-\sigma),t-\sigma)\,d\theta\\ &\quad -\int_{y<0}\int_0^1 k(\theta(\gamma(t)-\gamma(\sigma))+\alpha(t-\sigma)-y,t-\sigma)\;Ke^{Ky}\,d\theta\,dy\\ &< \frac{2}{\sqrt{(t-\sigma)}}.\end{aligned}$$

This and (4.2) yield

$$qK\int_{t-\frac{8\delta}{\alpha}}^{t}\mathbf{II}(t,\sigma)\,d\sigma \le qK4\sqrt{\frac{8\delta}{\alpha}} = 8\sqrt{2}Q|\bar{\mathbf{u}}_x(0)|\sqrt{\delta\alpha} \le 8\sqrt{2}Q|\bar{\mathbf{u}}_x(0)|\alpha^{-\frac{5}{2}}. \tag{4.8}$$

The lemma follows from combining Assumption 2.3, (4.7), and (4.8) and the assumption that α is sufficiently large. □

Proposition 4.3. *Suppose that $\alpha > 0$ is sufficiently large. Then, for $t \in \left(0, \frac{2\ \log\alpha}{\alpha^2}\right)$,*

$$|\gamma(t)| < \frac{4L_0^2\delta}{|\bar{\mathbf{u}}_x(0)|}, \tag{4.9}$$

where L_0 is the constant given in Lemma 4.1.

Proof. We first relax (4.9) and make the a priori assumption

$$|\gamma(t)| < \frac{8L_0^2\delta}{|\bar{\mathbf{u}}_x(0)|} \text{ for } t \in \left[0, \frac{2\log\alpha}{\alpha^2}\right]. \tag{4.10}$$

Set

$$|||\gamma|||_t \equiv \sup_{\sigma\in(0,t]}|\gamma(\sigma)|.$$

Case 1. $0 \le t \le 8\delta/\alpha$. By applying Lemma 4.1, Lemma 4.2, (4.3) and (4.4), one obtains

$$\frac{\gamma(t)}{L_0\min(\sqrt{t}\alpha,1)}|\bar{\mathbf{u}}_x(0)|e^{-2\alpha\delta} - \frac{|\gamma(t)|}{\alpha^{\frac{3}{2}}|\bar{\mathbf{u}}_x(0)|} \le \frac{1}{\alpha^{\frac{3}{2}}}|\bar{\mathbf{u}}_x(0)|\cdot|||\gamma|||_t + \frac{L_0\delta}{\min(\sqrt{t}\alpha,1)}. \tag{4.11}$$

Due to Assumption 2.4, (4.11), and largeness of α, we have

$$|||\gamma|||_t \le \frac{4L_0^2\delta}{|\bar{\mathbf{u}}_x(0)|} \text{ for } 0 \le t \le 8\delta/\alpha,$$

and the proposition holds in this case.

Case 2. $8\,\delta\,\alpha^{-1} \le t \le 2\alpha^{-2}\log\alpha$. When $t-\sigma > 8\delta/\alpha$, due to (4.10) the function $\partial_y k(\theta(\gamma(t)-\gamma(\sigma))+\alpha(t-\sigma)-y,t-\sigma)$ is a positive function for $y < 0$ and $\theta \in [0,1]$. Thus, $\mathbf{II}(t,\sigma)$ is a positive function for $(t-\sigma) > 8\delta/\alpha$. It yields that

$$\left|\int_0^t \mathbf{II}(t,\sigma)\gamma(\sigma)\,d\sigma\right| \le \left(\int_0^{t-8\delta/\alpha}\mathbf{II}(t,\sigma)\,d\sigma + \int_{t-8\delta/\alpha}^{t}|\mathbf{II}(t,\sigma)|d\sigma\right)|||\gamma|||_t.$$

Applying this, Lemma 4.1, Lemma 4.2, (4.3) and (4.4), we conclude that

$$\begin{aligned}
&|\gamma(t)| \left(\frac{1}{L_0 \min(\sqrt{t}\alpha, 1)} |\bar{\mathbf{u}}_x(0)|\, e^{-\frac{\alpha^2 t}{4}} + qK \int_0^{t-8\delta/\alpha} \mathbf{II}(t,\sigma)\, d\sigma \right) \\
&- \frac{1}{\alpha^{\frac{5}{2}}} |\bar{\mathbf{u}}_x(0)| \cdot |\gamma(t)| \\
\leq &\left(\frac{1}{\alpha^{\frac{5}{2}}} |\bar{\mathbf{u}}_x(0)| + qK \int_0^{t-8\delta/\alpha} \mathbf{II}(t,\sigma)\, d\sigma \right) \cdot |||\gamma|||_t + \frac{L_0 e^{-\frac{\alpha^2 t}{4}} \delta}{\min(\sqrt{t}\alpha, 1)}.
\end{aligned} \tag{4.12}$$

Suppose that $|||\gamma|||_t = |\gamma(\tau_0)|$ for $\tau_0 \in [0, t]$, then from (4.12), Assumption 2.4, and largeness of α:

$$\begin{aligned}
&|\gamma(\tau_0)| \left(\frac{1}{L_0 \min(\sqrt{t}\alpha, 1)} |\bar{\mathbf{u}}_x(0)|\, e^{-\frac{\alpha^2 \tau_0}{4}} + qK \int_0^{t-8\delta/\alpha} \mathbf{II}(t,\sigma)\, d\sigma - \frac{|\bar{\mathbf{u}}_x(0)|}{\alpha^{\frac{5}{2}}} \right) \\
< &\left(\frac{1}{\alpha^{\frac{5}{2}}} |\bar{\mathbf{u}}_x(0)| + qK \int_0^{t-8\delta/\alpha} \mathbf{II}(t,\sigma)\, d\sigma \right) |\gamma(\tau_0)| + L_0 \frac{e^{-\frac{\alpha^2 \tau_0}{4}} \delta}{\min(\sqrt{t}\alpha, 1)}.
\end{aligned}$$

By canceling the integrals with qK coefficients in both sides, it yields the uniform bound for $|\gamma(\tau_0)|$. Thus, the estimate of $|||\gamma|||_t$ follows in this case. $\square$

Differentiate (4.1) with respect to t to result in the equation for $\gamma'(t)$:

$$\begin{aligned}
0 = &-\gamma'(t) \left\{ \int_R k(\alpha t, t) \left(\bar{\mathbf{u}}_y(y + \gamma(t)) + v_y(y + \gamma(t), 0) \right) dy \right\} \\
&+ \int_R \frac{d}{dt} \{k(\alpha t - y, t)\} \{\bar{\mathbf{u}}(y) - \bar{\mathbf{u}}(y + \gamma(t)) - v(y + \gamma(t), 0)\}\, dy \\
&+ qK \int_0^t \int_y \partial_y \left(k(\gamma(t) - \gamma(t-\tau) + \alpha\tau - y, \tau)\right) (\gamma'(t) - \gamma'(t-\tau)) \mathbf{W}(y) dy.
\end{aligned} \tag{4.13}$$

Applying the same arguments for obtaining the uniform bound of $\gamma(t)$ to (4.13), one obtains the following proposition about the uniform bound of $\gamma'(t)$.

Proposition 4.4. *Suppose that α is sufficiently large. Then, there exists a constant $L_1 > 0$ such that for $t \in [0, \frac{2\log\alpha}{\alpha^2}]$,*

$$|\gamma'(t)| \leq \frac{\delta L_1 L_0^2 \alpha^2}{|\bar{\mathbf{u}}_x(0)|}. \tag{4.14}$$

4.2. Initial step: rate of the wave fronts. In the above we have obtained the uniform bound of the detonation wave location $\gamma(t)$ in a finite time interval $[0, 2\log\alpha/\alpha^2]$. The analysis tries to minimize the chemical effect, the terms in (4.4) with coefficient qK. This uniform bound is not refined enough to trace the wave front. We need to obtain sharper estimates of γ' by using the Proposition 4.4 to obtain a refined wave front tracing. This is done by using the Laplace–Fourier transformation, [3], to make a full account of the chemical nonlinearity.

Let $G(t)$ be a function satisfying

$$|G(t)| < e^{Bt}.$$

Let $\widehat{G}(s)$ be the Laplace–Fourier transformation of G:

$$\widehat{G}(s) \equiv \int_0^\infty e^{-st} G(t)dt \text{ for } s \in \mathbf{C} \text{ and } \Re\, s > B. \tag{4.15}$$

One has the following Parseval's relation.

Lemma 4.5. *For* $\eta > B$,

$$\int_0^\infty e^{-2\eta t} |G(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^\infty |\widehat{G}(\eta + i\xi)|^2 \, d\xi.$$

See Appendix A.2 of [3].

Rewrite (4.13) as follows:

$$\begin{aligned}
&\gamma'(t) \int_0^\infty \left\{ qK \int_{y<0} \partial_y \left(k(\alpha\tau - y, \tau) e^{Ky} \right) dy \right\} d\tau \\
&\quad - \int_0^t \gamma'(t-\tau) \left\{ qK \int_{y<0} \partial_y \left(k(\alpha\tau - y, \tau) e^{Ky} \right) dy \right\} d\tau \\
&= \gamma'(t) \left\{ \int_R \left(-\bar{\mathbf{u}}_y(y+\gamma(t)) + v_y(y+\gamma(t), 0) \right) \, dy \right\} \\
&\quad + \int_R \frac{d}{dt} \{k(\alpha t - y, t)\} \, \{\bar{\mathbf{u}}(y) - \bar{\mathbf{u}}(y+\gamma(t)) - v(y+\gamma(t), 0)\} \, dy \\
&\quad + \left\{ qK \int_0^t \int_y \partial_y ((k(\gamma(t) - \gamma(t-\tau) + \alpha\tau - y, \tau) - k(\gamma(t) - \gamma(t-\tau) + \alpha\tau - y, \tau)) \right. \\
&\quad \cdot (\gamma'(t) - \gamma'(t-\tau)) \mathbf{W}(y)) dy d\tau \\
&\quad \left. + \gamma'(t) qK \int_t^\infty \int_{y<0} \partial_y \left(k(\alpha\tau - y, \tau) e^{Ky} \right) dy d\tau \right\} \\
&\equiv j_1 + j_2 + j_3.
\end{aligned} \tag{4.16}$$

From the uniform bounds of γ and γ' in Propositions 4.3 and 4.4, one can see that j_1 and j_2 in (4.16) satisfy, for $t \in (0, 2\log\alpha\,/\alpha^2)$,

$$\begin{aligned}
|j_1| &\le e^{-\frac{\alpha^2 t}{4}} \frac{\alpha^2 \delta L_1 L_0^2}{|\bar{\mathbf{u}}_x(0)|} \left(|\bar{\mathbf{u}}_x(0)| + \alpha^2 \delta \right), \\
|j_2| &\le 8 e^{-\frac{\alpha^2 t}{4}} \alpha^2 \delta (L_0^2 + 2), \\
|j_3| &= \frac{qK\alpha^2}{|\bar{\mathbf{u}}_x(0)|} \left(O(1) \frac{\delta e^{-\frac{\alpha^2 t}{4}}}{\alpha} + O(1) \frac{\delta^2 \alpha^2}{|\bar{\mathbf{u}}_x(0)|} \right) \\
&= O(1) \alpha^2 \delta e^{-\alpha^2 t/4} \text{ for } t \in \left[0, \frac{2\log\alpha}{\alpha^2} \right].
\end{aligned} \tag{4.17}$$

Here we have noticed from (4.14) and (4.9) that

$$
\begin{aligned}
&\gamma'(t)qK\int_t^\infty \mathbf{III}(\tau)d\tau = O(1)\frac{qK\alpha^2}{|\bar{\mathbf{u}}_x(0)|}\int_t^\infty \mathbf{III}(\tau)d\tau\\
&= O(1)\frac{qK\alpha}{|\bar{\mathbf{u}}_x(0)|}e^{-\frac{\alpha^2 t}{4}} = O(1)\alpha^2 e^{-\frac{\alpha^2 t}{4}},\\
&\quad qK\int_0^t\int_{y<0}(\gamma'(t)-\gamma'(t-\tau))(\gamma(t)-\gamma(t-\tau))k_{yy}(\cdot,\cdot)\mathbf{W}(y)dyd\tau\\
&= O(1)\frac{qK\delta^2\alpha^3}{|\bar{\mathbf{u}}_x(0)|^2} = O(1)\frac{\delta^2\alpha^4}{|\bar{\mathbf{u}}_x(0)|}.
\end{aligned}
$$

Set

$$
\begin{aligned}
\mathbf{III}(\tau) &\equiv \int_{y<0}\partial_y k(\alpha\tau-y,\tau)e^{Ky}\,dy,\\
\mathbf{E}(\tau) &\equiv \mathbf{III}(\tau)-k(\alpha\tau,\tau), \qquad (4.18)\\
|\mathbf{E}(\tau)| &= \left|\int_{y<0}k(\alpha\tau-y,\tau)Ke^{Ky}\,dy\right|\\
&\le O(1)\frac{K}{\alpha+K}\frac{e^{-\frac{\alpha^2\tau}{4}}}{\sqrt{\tau}},\\
V_0 &\equiv \int_0^\infty \mathbf{III}(\tau)\,d\tau\\
&= \int_0^\infty \mathbf{E}(\tau)d\tau + \int_0^\infty k(\alpha\tau,\tau)d\tau = \frac{1}{\alpha}\left(1+O(1)\frac{K}{\alpha}\right),\\
P(\tau) &\equiv \frac{k(\alpha\tau,\tau)}{V_0},\quad \mathbf{E}_0(\tau)\equiv\frac{\mathbf{E}}{V_0}, \qquad (4.19)\\
G(t) &\equiv \gamma'(t).
\end{aligned}
$$

From the above estimates about j_1, j_2, and j_3 , (4.16) can be rewritten as

$$
\begin{aligned}
G(t) &= (P*G)(t)+(\mathbf{E}_0*G)(t)+\mathbf{F}(t), \qquad (4.20)\\
\mathbf{F}(t) &\equiv O(1)\frac{Q}{|\bar{\mathbf{u}}_x(0)|}(j_1+j_2+j_3)\cdot char_{\left[0,\frac{2\log\alpha}{\alpha^2}\right]}(t),\\
\mathbf{F}(t) &= O(1)\frac{\delta e^{-\alpha^2 t/4}\alpha^2}{|\bar{\mathbf{u}}_x(0)|}.
\end{aligned}
$$

Let's rewrite (4.20) in terms of convolution operators

$$
\begin{aligned}
(1-\mathbf{P})\cdot G &= \mathbf{E}_0\cdot G+\mathbf{F}, \qquad (4.21)\\
\mathbf{P}\cdot G(\tau) &\equiv P*G,\\
\mathbf{E}_0\cdot G(\tau) &\equiv \mathbf{E}_0*G.
\end{aligned}
$$

We study (4.21) by formally expressing G as

$$
G = \sum_{i=0}^\infty[(1-\mathbf{P})^{-1}\mathbf{E}_0]^i(1-\mathbf{P})^{-1}\mathbf{F}. \qquad (4.22)
$$

One needs to construct a special functional space such that the operator $(1-\mathbf{P})^{-1}$ is a bounded linear operator and that $\{[(1-\mathbf{P})^{-1}\mathbf{E}_0]^i(1-\mathbf{P})^{-1}\mathbf{F}\}_{i\geq 0}$ is a Cauchy's sequence.

Consider the Fourier–Laplace transformation (4.15) of (4.20) with

$$s=-\alpha^2/8+i\xi,\quad \xi\in\mathbf{R}.$$

Both $\mathbf{P}$ and $\mathbf{E}_0$ are convolution operators, and so

$$\widehat{\mathbf{P}\circ\mathbf{E}}_0(s)=\widehat{\mathbf{P}}(s)\cdot\widehat{\mathbf{E}}_0(s),$$

and (4.22) yields

$$\widehat{G}(s)=\sum_{i=0}^{\infty}\frac{\widehat{\mathbf{E}_0}(s)^i}{(1-\widehat{\mathbf{P}}(s))^{i+1}}\,\widehat{\mathbf{F}}(s).$$

The function space is defined by the norm:

$$\||h\||_\alpha^2=\int_0^\infty e^{\frac{\alpha^2 t}{4}}\;h(t)\,dt.$$

By Lemma 4.5

$$\begin{aligned}\||G\||_\alpha&\equiv\left(\int_0^\infty G(t)^2e^{\alpha^2t/4}dt\right)^{1/2}\\&\leq\sum_{i=0}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{|\widehat{\mathbf{E}}_0(s)|^{2i}|\widehat{\mathbf{F}}(s)|^2}{|1-\widehat{\mathbf{P}}(s)|^{2(i+1)}}d\xi\right)^{1/2}\ \text{with}\ \Re\,s=-\frac{\alpha^2}{8}.\end{aligned}\tag{4.23}$$

From (4.20),

$$\||\mathbf{F}\||_\alpha=O(1)\frac{\delta\alpha}{|\bar{\mathbf{u}}_x(0)|}.\tag{4.24}$$

Thus, it is sufficient to show that $|1-\widehat{\mathbf{P}}(s)|^{-1}$ is bounded for $\Re\,s=-\alpha^2/8$.

Lemma 4.6. *For any $\xi\in\mathbf{R}$ there is positive constant C_0 such that*

$$|(1-\widehat{\mathbf{P}}(-\alpha^2/8+i\xi))|^{-1}<C_0,$$

$$|\widehat{\mathbf{E}}_0(-\alpha^2/8+i\xi)|<C_0\,\frac{K}{\alpha+K}.$$

Proof. Set $s=-\alpha^2/8+i\xi$ in (4.15),

$$\begin{aligned}\widehat{\mathbf{P}}(\xi)\equiv\widehat{\mathbf{P}}(s)&=\int_0^\infty e^{-(-\frac{\alpha^2}{8}+i\xi)t}\;P(t)\,dt\\&=\int_0^\infty e^{-(-\frac{\alpha^2}{8}+i\xi)t}\;\frac{k(\alpha t,t)}{V_0}\,dt,\end{aligned}$$

$$\widehat{\mathbf{P}}(0)=\alpha\left(1+O(1)\frac{K}{\alpha+K}\right)\int_0^\infty k(\alpha\tau,\tau)\,e^{\frac{\alpha^2}{8}}\,d\tau=\left(1+O(1)\frac{K}{\alpha+K}\right)\sqrt{2}.\tag{4.25}$$

Differentiate $\widehat{\mathbf{P}}(\xi)$ with respect to ξ to yield

$$\widehat{\mathbf{P}}_\xi(\xi) = -i \int_0^\infty \frac{\sqrt{\tau}}{\sqrt{4\pi}} e^{-\frac{\alpha^2\tau}{8} - i\xi\tau}\, d\tau$$
$$= -\frac{\widehat{\mathbf{P}}(\xi)}{2(\xi - i\frac{\alpha^2}{8})}.$$

From this,

$$\left(\widehat{\mathbf{P}}(\xi)\sqrt{\xi - \frac{i\alpha^2}{8}}\right)_\xi = 0.$$

Combine this with (4.25) to obtain

$$\widehat{\mathbf{P}}(\xi) = \frac{(1 + O\frac{K}{\alpha+K})\sqrt{2}}{\sqrt{1 + \frac{i8\xi}{\alpha^2}}}. \tag{4.26}$$

From (4.26), it follows that

$$|1 - \widehat{\mathbf{P}}(\xi)| > \frac{1}{16} \text{ for } \xi \in \mathbf{R}$$

and $|(1 - \widehat{\mathbf{P}}(\xi))^{-1}|$ is bounded.

From the definition of $\widehat{\mathbf{E}_0}$ in (4.20) and (4.21),

$$|\widehat{\mathbf{E}_0}(-\alpha^2/8 + i\xi)| = K\,\alpha \left| \int_0^\infty \int_{y<0} \frac{e^{-\frac{\alpha^2 t}{8} + \frac{\alpha y}{2} + Ky - \frac{y^2}{4t} - i\xi t}}{\sqrt{4\pi t}}\, dy dt \right|$$
$$\leq K \frac{2\alpha}{\alpha + 2K} \left| \int_0^\infty \frac{e^{-\frac{\alpha^2 t}{8}}}{\sqrt{4\pi t}}\, dt \right| = O(1)\frac{K}{\alpha + K}.$$

This proves the lemma. □

From Lemma 4.6, the series in (4.23) converges, and together with (4.24),

$$\left(\int_0^\infty \gamma'(t)^2 e^{\frac{\alpha^2 t}{4}}\, dt\right)^{\frac{1}{2}} \equiv |||G|||_\alpha \tag{4.27}$$
$$\leq O(1)\,\delta \left(1 - \frac{C_0^2 K}{\alpha + K}\right)^{-1} \frac{\alpha}{|\bar{\mathbf{u}}_x(0)|}.$$

With this we may improve Proposition 4.3:

Proposition 4.7. *For $t \in [t_0/2, t_0]$, $t_0 = \frac{2\log\alpha}{\alpha^2}$, there exists $C_2 > 0$, which is independent of α, such that*

$$|\gamma(t) - \gamma(t_0)| \leq C_2 \frac{\alpha^{-\frac{1}{8}}\delta\sqrt{|\log\alpha|}}{|\bar{\mathbf{u}}_x(0)|}.$$

Proof. For $t \in [t_0/2, t_0]$, (4.27) yields

$$
\begin{aligned}
|\gamma(t) - \gamma(t_0)| &\le \int_t^{t_0} |\gamma'(\rho)| d\rho \\
&\le \sqrt{t_0 - t}\; e^{-\frac{\alpha^2 t_0}{16}} \left(\int_t^{t_0} e^{\frac{\alpha^2 \rho}{4}} \gamma'(\rho)^2 d\rho \right)^{1/2} \\
&= O(1) \frac{\sqrt{|\log \alpha|}}{\alpha} \alpha^{-\frac{1}{8}} \; \frac{\alpha \delta}{|\bar{\mathbf{u}}_x(0)|} \\
&= O(1) \frac{\alpha^{-\frac{1}{8}} \delta \sqrt{|\log \alpha|}}{|\bar{\mathbf{u}}_x(0)|}. \qquad \square
\end{aligned}
$$

By introducing a parameter β_0,

$$
\beta_0 \equiv \frac{-\log \left(C_2 \frac{\sqrt{|\log \alpha|}\, \alpha^{-\frac{1}{8}}}{|\bar{\mathbf{u}}_x(0)|} \right)}{2 \log \alpha},
$$

Proposition 4.7 becomes

$$
|\gamma(t) - \gamma(t_0)| \le \delta e^{-\beta_0 \alpha^2 t_0} \text{ for } t \in \left[\frac{t_0}{2}, t_0 \right].
$$

4.3. Initial step: waves carried by detonation wave fronts. The analytic properties given in Propositions 4.3 and 4.7 are sufficient to obtain a fine wave structure at $t_0 = 2 \log \alpha / \alpha^2$. We first update the initial data at time $t = t_0$ as a perturbation of $\bar{\mathbf{u}}(x - \gamma(t_0))$,

$$
\bar{v}(x, t) \equiv u(x, t) - \bar{\mathbf{u}}(x - \gamma(t_0)).
$$

With the front updated, the estimate of the perturbation can be improved when compared with the initial data in (4.3):

Proposition 4.8. *There exists a positive constant C_0 such that, for $x > 0$,*

$$
|\bar{v}(x, t_0)| < C_0 \alpha^{-\frac{1}{8}} \sqrt{\log \alpha}\; \delta e^{-\frac{5\alpha|x|}{8}}.
$$

Proof. Similar to (3.4) for $v(x, t)$, one gets

$$
\begin{aligned}
\bar{v}(x, t) = & \int_R k(x - y + \alpha t, t) \left[v(y, 0) + (\bar{\mathbf{u}}(y) - \bar{\mathbf{u}}(y - (\gamma(t_0)))\right] dy \qquad (4.28) \\
& + qK \int_0^t \int_{y<0} (k(x + \alpha\tau - \gamma(t_0 - \tau) - y, \tau) \\
& \qquad - k(x + \alpha\tau - \gamma(t_0) - y, \tau)) e^{Ky}\, dy d\tau.
\end{aligned}
$$

The proof uses Proposition 4.7 for the integral over $[t_0/2, t_0]$ and the separation of the fluid and the combustion wave speed s, $\alpha \gg 1$, for the time interval $[0, t_0/2]$. By integration by parts and by mean value theorem,

$$
\begin{aligned}
&qK\int_0^{t_0}\int_{y<0}(k(x+\alpha\tau-\gamma(t_0-\tau)-y,\tau)-k(x+\alpha\tau-\gamma(t_0)-y,\tau))\,e^{Ky}\,dyd\tau\\
&\quad=-qK\int_0^{t_0}\int_0^1 k(x+\alpha\tau+\theta(\gamma(t_0-\tau)-\gamma(t)),\tau)(\gamma(t_0)-\gamma(t_0-\tau))\,d\theta d\tau\\
&\quad+qK^2\int_0^{t_0}\int_{y<0}\int_0^1 k(x+\alpha\tau+\theta(\gamma(t_0-\tau)-\gamma(t_0))-y,\tau)\\
&\qquad\qquad\qquad\qquad\cdot(\gamma(t_0)-\gamma(t_0-\tau))e^{Ky}\,d\theta dyd\tau.
\end{aligned}
$$

Expand the kernel function $k(x-y+\alpha\tau,\tau)$ as follows:

$$
\begin{aligned}
k(x+\alpha\tau-y,\tau)&=\frac{1}{\sqrt{4\pi\tau}}e^{-\frac{(x-y)^2}{4\tau}-\frac{\alpha^2\tau}{4}-\frac{\alpha(x-y)}{2}}\\
&\le\frac{1}{\sqrt{4\pi\tau}}e^{-\frac{3(x-y)^2}{16\tau}-\frac{3\alpha^2\tau}{16}-\frac{\alpha(x-y)}{2}-\frac{\alpha|x-y|}{8}}.
\end{aligned}
$$

Substitute this into the last integral of (4.29) and apply (4.9), together with Assumption 2.6, to yield, for $x>0$,

$$
\begin{aligned}
&qK\int_0^{t_0}\int_{y<0}(k(x+\alpha\tau-\gamma(\tau)-y,\tau)-k(x+\alpha\tau-\gamma(t_0)-y,\tau))\,e^{Ky}\,dyd\tau\\
&\le O(1)e^{-\frac{5\alpha|x|}{8}}qK\left(1+O(1)\frac{K}{\alpha+K}\right)\left\{\int_0^{t_0/2}\frac{\delta\sqrt{\log\alpha}\,\alpha^{-\frac{1}{8}}e^{-\frac{3\alpha^2\tau}{16}}}{|\bar{\mathbf{u}}_x(0)|\sqrt{\tau}}d\tau\right.\\
&\quad\left.+\int_{t_0/2}^{t_0}\frac{\delta e^{-\frac{3\alpha^2\tau}{16}}}{\sqrt{\tau}}\,d\tau\right\}\\
&=O(1)\ \left(1+O(1)\frac{K}{\alpha+K}\right)\delta\,qK\sqrt{\log\alpha}\left(\frac{\alpha^{-1-\frac{1}{8}}}{|\bar{\mathbf{u}}_x(0)|}+\alpha^{-1-\frac{3}{16}}\right)\ e^{-\frac{5\alpha|x|}{8}}\ \log\alpha\\
&=O(1)\sqrt{\log\alpha}\ \alpha^{-\frac{1}{8}}\left(1+|\bar{\mathbf{u}}_x(0)|\alpha^{-\frac{1}{16}}\right)\delta e^{-\frac{5\alpha|x|}{8}}\\
&=O(1)\alpha^{-\frac{1}{8}}\log\alpha\ \delta e^{-\frac{5\alpha|x|}{8}}.
\end{aligned}
\tag{4.29}
$$

The last two integrals are obtained by using Proposition 4.7 and the largeness of α. Similarly, for $x>0$,

$$
\begin{aligned}
&\int_R k(x-y+\alpha t_0,t_0)\,(\bar{\mathbf{u}}(y)-\bar{\mathbf{u}}(y-\gamma(t_0))+v(y,0)\ )\ dy\\
&=O(1)\ (|\bar{\mathbf{u}}_x|(0)+1)\delta e^{-\frac{3\alpha^2t_0}{16}}\ e^{-\frac{5\alpha|x|}{8}}.
\end{aligned}
\tag{4.30}
$$

Equations (4.30) and (4.30) imply the proposition. □

As before, the above proposition can be rewritten as

$$|\bar{v}(x, t_0)| \leq \delta e^{-\beta_1 \alpha^2 t_0} e^{-\frac{5\alpha|x|}{8}} \text{ for } x > 0,$$
$$\beta_1 \equiv \frac{-\log\left(C_0\sqrt{|\log\alpha|}\,\alpha^{-\frac{1}{8}}\right)}{2\log\alpha}.$$

From the condition $\alpha \gg 1$,

$$\beta_1 \in (0, \beta_0).$$

Before investigating the situation $x < 0$, we derive an estimate from (4.28). The following lemma will be used to show that, due to the supersonic speed of the combustion wave, the information to the left of the combustion does not influence the propagation of the combustion wave.

Lemma 4.9. *For $x < 0$ and $\alpha > 0$ it holds*

$$\int_R k(x+\alpha t - y, t)\, e^{-\frac{5\alpha|y|}{8}}\, dy = O(1)\left(\frac{e^{-\frac{(x+\alpha t)^2}{16t}}}{\alpha\sqrt{4\pi t}} + \frac{e^{-\frac{\alpha|x+\alpha t|}{4}}}{\alpha\sqrt{t}}\right), \tag{4.31}$$

$$\int_{y<0} k(x+\alpha t - y, t)\, e^{-K|y|}\, dy \tag{4.32}$$

$$= O(1)\begin{cases} \dfrac{e^{-\frac{(x+\alpha t)^2}{4t}}}{K\sqrt{4\pi t}} \text{ if } x+\alpha t \geq 0, \\ \dfrac{e^{-\frac{(x+\alpha t)^2}{16t}}}{K\sqrt{4\pi t}} + \dfrac{e^{-\frac{K|x+\alpha t|}{2}}}{K\sqrt{t}} \text{ else } x+\alpha t < 0, \end{cases}$$

$$\int_{y<0} k(x+\alpha t - y, t)\, e^{-K|y|}\, dy \leq 2\frac{e^{-\frac{\alpha x}{2}}\, e^{-\frac{\alpha^2 t}{4}}}{(\alpha+K)\sqrt{4\pi t}}. \tag{4.33}$$

Proof. The proofs of (4.31) and (4.32) are identical. So, one just needs to prove (4.32) and (4.33). When $x + \alpha t > 0$,

$$\int_{y<0} k(x+\alpha t - y, t)\, e^{-K|y|} dy \leq \int_{y<0} k(x+\alpha t, t)\, e^{-K|y|}\; dy \leq \frac{1}{K}\, k(x+\alpha t, t).$$

When $x + \alpha t < 0$,

$$\int_{y<0} k(x+\alpha t - y, t)\, e^{-K|y|} dy$$
$$= \left(\int_{y<\frac{x+\alpha t}{2}} + \int_{\frac{x+\alpha t}{2}<y<0}\right) k(x+\alpha t - y, t)\, e^{-K|y|}\, dy$$
$$\leq \frac{e^{-\frac{K|x+\alpha t|}{2}}}{K\sqrt{4\pi t}} + \frac{2}{K}\, k(x+\alpha t, 4t).$$

Hence, (4.32) follows:

$$\begin{aligned}&\int_{y<0} k(x+\alpha t-y,t)\, e^{-K|y|}dy\\&=\int_{y<0} k(x-y,t)\, e^{-\frac{\alpha(x-y)}{2}-K|y|-\frac{\alpha^2 t}{4}}\, dy\\&\le \frac{2}{\alpha+K}\,\frac{e^{-\frac{\alpha x}{2}-\frac{\alpha^2 t}{4}}}{\sqrt{4\pi t}},\end{aligned}$$

and (4.33) follows. □

If δ is sufficiently small, then

$$\begin{aligned}|\bar{v}(x,t_0)| \le &\int_R k(x-y+\alpha t_0,t_0)\left[|v(y,0)|+|\bar{\mathbf{u}}(y)-\bar{\mathbf{u}}(y-\gamma(t_0))|\right]\, dy\\&+\delta\,\omega_1(x),\end{aligned}$$

where

$$\begin{aligned}\omega_1(x)\equiv L_0Kq&\left(\int_{\frac{t_0}{2}}^{t_0}+\int_0^{\frac{t_0}{2}} e^{-\beta_0\alpha^2 t_0}\right)\\&\left(k(x+\alpha\tau,\tau)+K\int_{y<0} k(x+\alpha\tau-y,\tau)e^{Ky}\, dy\right)d\tau.\end{aligned}$$

From (4.32) and (4.33) we have

Lemma 4.10. *The function $\omega_1(x)$ satisfies that*

$$\omega_1(x)=O(1)\,|\bar{\mathbf{u}}_x(0)| \text{ for } x<0, \tag{4.34}$$

$$\omega_1(x)\le e^{-\beta_1\alpha^2 t_0}\, e^{\frac{\alpha|x|}{2}} \text{ for } x<0. \tag{4.35}$$

From Lemma 4.10 and Proposition 4.8, we have the following proposition.

Proposition 4.11. *The wave structure of $\bar{v}(x,t_0)$ is*

$$|\bar{v}(x,t_0)|\le\delta\begin{cases}O(1)\left(\omega_1(x)+\dfrac{e^{-\frac{(x+\alpha t_0)^2}{16 t_0}}+e^{-\frac{5\alpha|x+\alpha t_0|}{16}}}{\alpha\sqrt{t_0}}\right) \text{ for } x<0,\\ e^{-\beta_1\alpha^2 t_0}\, e^{-\frac{5|x|}{8}} \text{ for } x>0.\end{cases}$$

4.4. Pointwise convergence to viscous profiles. Propositions 4.11 and 4.7 provide a decaying structure of a perturbation in front of the detonation wave at time $t = t_0$ as well as the analytic property of the detonation wave location. In fact, the factor δ can be realized as the strength of the perturbation in front the detonation wave, and the updated perturbation in front of the detonation wave is of order $e^{-\beta_1\alpha^2 t_0}\delta$. Finally, it results in the convergence of the wave front to a fixed location exponentially fast as well as time asymptotic stability of the weak detonation wave. In this subsection, we will generalize both Propositions 4.7 and 4.11 to show this.

The above reasoning suggests the a priori assumption on the wave front $\gamma(nt_0)$:

$$
\begin{aligned}
&|\gamma(nt_0) - \gamma(t)| = O(1)\,\delta\,\mathbf{b}(t) \text{ for } t \in [0, nt_0],\\
&\mathbf{b}(t) \equiv \begin{cases} e^{-\left[\frac{t}{t_0}\right]\beta_1\alpha^2 t_0} \text{ for } t \in \left[\left[\frac{t}{t_0}\right] t_0, \left(\left[\frac{t}{t_0}\right] + \frac{1}{2}\right) t_0\right), \\ e^{-\left(\left[\frac{t}{t_0}\right]\beta_1+\beta_0\right)\alpha^2 t_0} \text{ for } t \in \left[\left(\left[\frac{t}{t_0}\right] + \frac{1}{2}\right) t_0, \left(\left[\frac{t}{t_0}\right] + 1\right) t_0, \right), \end{cases}
\end{aligned}
$$

where $[x]$ is the largest integer less than or equal to x. The solution to the left of the detonation wave front is analyzed by resolving an initial boundary value problem with boundary value bounded by the a priori bound $O(1)\delta\mathbf{b}$. The consideration of such an initial boundary value problem is motivated by the device in [4] for studying the stability of a viscous shock profile.

For this we consider the Green function G_*,

$$
\begin{aligned}
G_*(x, t; y, \sigma) \equiv\; & k(x - y + \alpha(t - \sigma), t - \sigma)\\
& - e^{\alpha x}\, k(x + y - \alpha(t - \sigma), t - \sigma),
\end{aligned} \tag{4.36}
$$

for the initial-boundary value problems:

$$u_t - \alpha u_x - u_{xx} = 0 \text{ for } x < 0, t > 0$$

with homogeneous boundary values $u(0, t) = 0$ for $t \geq 0$, and the solution Ω_* of the initial-boundary value problem:

$$
\begin{cases} u_t - \alpha u_x - u_{xx} = qK\mathbf{W}(x), \\ u(0, t) = L\mathbf{b}(t), \quad u(x, 0) = 0, \end{cases}
$$

$$
\begin{aligned}
\Omega^*(x, t; L) \equiv qK \int_0^t \mathbf{b}(\sigma) \cdot \left(G_*(x, t; 0, \sigma) + K \int_{y<0} G_*(x, t; y, \sigma) e^{Ky}\, dy \right) d\sigma \\
- L \int_0^t \mathbf{b}(\sigma) \cdot \partial_y G_*(x, t; 0, \sigma)\, d\sigma.
\end{aligned}
$$

We introduce comparison functions $\omega_n(x)$ for the estimate of $\Omega^*(x, t; L)$:

$$
\begin{aligned}
&\omega_n(x) \equiv qK \int_0^{nt_0} \mathbf{b}(\sigma)\cdot \\
&\left(k(x + \alpha(nt_0 - \sigma), nt_0 - \sigma) + K \int_{y<0} k(x - y + \alpha(nt_0 - \sigma), nt_0 - \sigma) e^{Ky} dy \right) d\sigma.
\end{aligned}
$$

Lemma 4.12. *For $x < 0$,*

$$|\omega_n(x)| \le O(1)\ Q\ |\bar{\mathbf{u}}_x(0)| \left(\frac{e^{-\frac{(x+\alpha n t_0)^2}{A n t_0}}}{\sqrt{n t_0}} + \frac{1}{K\sqrt{n t_0}}\, e^{-\frac{|x+\alpha n t_0| K}{2}} \right), \tag{4.37}$$

$$|\omega_n(x)| \le O(1)\ Q|\bar{\mathbf{u}}_x(0)| e^{\frac{|x|\alpha}{3}} \mathbf{b}(n t_0), \tag{4.38}$$

for any $A > \max(16, 32/\beta_1)$.

Proof. From Lemma 4.9, the double integral defining $\omega_n(x)$ can be bounded by a single integral:

$$\omega_n(x) \le O(1) q K \int_0^{n t_0} \mathbf{b}(\sigma) \cdot \tag{4.39}$$
$$\left[k(x + \alpha(n t_0 - \sigma), n t_0 - \sigma) + \frac{e^{-\frac{K|x+\alpha(n t_0-\sigma)|}{2}}}{\sqrt{n t_0 - \sigma}} \mathbf{H}(-x - \alpha(n t_0 - \sigma)) \right] d\sigma.$$

Break the integration into two parts: $\sigma \in [0, n t_0/2)$ and $\sigma \in [n t_0/2, n t_0]$. For the first part $\sigma \in [0, n t_0/2)$,

$$\begin{aligned} &\int_0^{n t_0/2} \mathbf{b}(\sigma) \cdot k(x + \alpha(n t_0 - \sigma), n t_0 - \sigma)\, d\sigma \\ &\le 4 \int_0^{n t_0/2} \mathbf{b}(\sigma) \cdot k(x + \alpha(n t_0 - \sigma), n t_0)\, d\sigma \\ &= \frac{O(1)}{\alpha^2} \left(k(x + \alpha n t_0, 4 n t_0) + \frac{e^{-\frac{\alpha\beta|x+\alpha n t_0|}{2}}}{\sqrt{n t_0}} \right). \end{aligned} \tag{4.40}$$

When $\sigma \in [n t_0/2, n t_0]$, break x in two cases: $x + n t_0 < 0$ and $0 < x + n t_0 < n t_0$. When $x + n t_0 < 0$, since

$$k(x + \alpha(t_0 n - \sigma), t_0 n - \sigma) < O(1) k(x + \alpha t_0 n, t_0 n - \sigma),$$

we have

$$\begin{aligned} &\int_{n t_0/2}^{n t_0} \mathbf{b}(\sigma) \cdot k(x + \alpha(n t_0 - \sigma), n t_0 - \sigma)\, d\sigma \\ &\le 4 \int_{n t_0/2}^{n t_0} \mathbf{b}(\sigma) \cdot k(x + \alpha n t_0, n t_0 - \sigma)\, d\sigma \\ &= O(1) n t_0 e^{-\frac{n}{2}\beta_1 \alpha^2 t_0} k(x + \alpha n t_0, n t_0) = \frac{O(1)}{\alpha^2} e^{-\frac{n}{4}\beta_1 \alpha^2 t_0} k(x + \alpha n t_0, n t_0). \end{aligned} \tag{4.41}$$

When $0 < x + n t_0 < n t_0$, due to choice of A, we have

$$e^{-\frac{\beta_1 n}{8}\alpha^2 t_0} \ll O(1) k(x + \alpha n t_0, A n t_0).$$

Therefore,

$$\int_{nt_0/2}^{nt_0} \mathbf{b}(\sigma)\cdot k(x+\alpha(nt_0-\sigma), nt_0-\sigma)\, d\sigma \le 4\int_{nt_0/2}^{nt_0} \mathbf{b}(\sigma)\cdot \frac{1}{\sqrt{nt_0-\sigma}}\, d\sigma$$

$$\le O(1)\, e^{-\frac{n}{4}\beta_1\alpha^2 t_0}\sqrt{nt_0} \le O(1)\frac{e^{-\frac{n}{8}\beta_1\alpha^2 t_0}}{\alpha} \le O(1)\frac{1}{\alpha}k(x+\alpha nt_0, Ant_0). \tag{4.42}$$

Equations (4.40), (4.41), and (4.42) yield that

$$\int_{nt_0/2}^{nt_0} \mathbf{b}(\sigma)\cdot k(x+\alpha(nt_0-\sigma), nt_0-\sigma)\, d\sigma$$

$$= O(1)\frac{1}{\alpha}\left(k(x+\alpha nt_0, Ant_0) + \frac{e^{-\frac{\alpha\beta_1|x+\alpha nt_0|}{2}}}{\sqrt{nt_0}}\right). \tag{4.43}$$

From a direct calculation one can have

$$\int_0^{nt_0} \mathbf{b}(\sigma)\cdot \frac{e^{-\frac{-K|x+\alpha(nt_0-\sigma)|}{2}}}{\sqrt{nt_0-\sigma}}\, \mathbf{H}(-x-\alpha(nt_0-\sigma))\, d\sigma \le \frac{O(1)}{\alpha}\frac{e^{-K|x+\alpha nt_0|}}{\sqrt{nt_0}}. \tag{4.44}$$

Combining $qK/\alpha = Q|\bar{\mathbf{u}}_x(0)|$ with (4.43) and (4.44), one has (4.37).
The estimate (4.38) follows by plugging the inequality

$$k(x-y+\alpha(t-\sigma), t-\sigma)$$

$$\le \sqrt{\frac{3}{2}}\, k(x-y+\alpha(t-\sigma), \frac{3}{2}(t-\sigma)) \le O(1)\frac{e^{-\frac{\alpha x}{3}+\frac{\alpha y}{3}-\frac{\alpha^2(t-\sigma)}{8}}}{\sqrt{t-\sigma}}$$

into (4.4) with $\beta_1 < 1/8$. □

Lemma 4.13. *It holds for $x < 0$ and $L > 1$ that*

$$\Omega^*(x,t;L) = O(L)\, |\bar{\mathbf{u}}_x(0)| \left(k(x+\alpha t, At) + \frac{e^{-\frac{K|x+\alpha t|}{2}}}{\sqrt{t}}\right), \tag{4.45}$$

$$\Omega^*(x, nt_0; L) \le L\mathbf{b}(nt_0+)e^{\frac{\alpha|x|}{3}}, \tag{4.46}$$

where $A > \max(16, 32/\beta_1)$.

Proof. The function $\Omega^*(x,t;L)$ can be identified with the solution $U(x,t)$ of

$$\begin{aligned} U_t - \alpha U_x - U_{xx} &= qK(e^{Kx}+\delta(x))\mathbf{b}(t) \text{ for } x<0,\ t>0,\\ U(0,t) &= L\,\mathbf{b}(t),\\ U(x,0) &\equiv 0, \end{aligned}$$

where $\delta(x)$ is a delta function.

Consider

$$\begin{aligned} V_t - \alpha V_x - V_{xx} &= l_0\, qK(\mathbf{W}(x)+\delta(x))\mathbf{b}(t) \text{ for } x\in\mathbf{R},\ t>0,\\ V(x,0) &\equiv 0. \end{aligned}$$

Duhamel's representation of $V(x,t)$ is identical to the representation $\omega_n(x)$ in (4.4). Hence, the estimate of $\omega_n(x)$ in Lemma 4.12 is applicable to $V(x,t)$ with nt_0 replaced by t. According to (4.38) in Lemma 4.12, one can find an $l_0 > L$ such that the solution $V(x,t)$ satisfies

$$V(0,t) \geq L\mathbf{b}(t).$$

By maximal principle, one has

$$U(x,t) \leq V(x,t) \text{ for } x < 0,\ t > 0.$$

So, (4.45) follows. Equation (4.46) is rather straightforward. Its proof is omitted. □

The perturbation at each updated detonation wave front $\gamma = \gamma(nt_0)$ is:

$$\bar{v}_n(x,t) \equiv u(x,t) - \bar{\mathbf{u}}(x - \gamma(nt_0)).$$

The following proposition yields both the convergence of the wave locations and the time asymptotic pointwise convergence to the viscous weak detonation profile.

Proposition 4.14. *There is an $L > 1$ such that for all $n \in \mathbf{N}$,*

$$|\gamma(nt_0) - \gamma((n-1)t_0)| \leq \frac{4L_0^2}{|\bar{\mathbf{u}}_x(0)|} e^{-(n-1)\beta_1\alpha^2 t_0}\delta, \tag{4.47}$$

$$|\bar{v}_n(x, nt_0)| \leq \delta \begin{cases} 2\Omega^*(x - \gamma(nt_0), nt_0; L) \\ +k(x + \alpha nt_0 - \gamma(nt_0), 4nt_0) \text{ for } x < \gamma(nt_0), \\ O(1)e^{-n\beta_1\alpha^2 t_0}\, e^{-\frac{5|x-\gamma(nt_0)|}{8}} \text{ for } x > \gamma(nt_0). \end{cases} \tag{4.48}$$

Proof. We will prove Proposition 4.14 by induction. Due to (4.9) and Proposition 4.11, the proposition holds for $n = 1$. Assume that (4.47) and (4.48) hold for $n \leq j$. For simplicity in notation, we take $\gamma(jt_0) = 0$. Set

$$\begin{cases} \tau \equiv t - jt_0, \\ v(x,\tau) \equiv \bar{v}_j(x, \tau + jt_0), \\ v_0(x) \equiv \bar{v}_j(x, jt_0), \\ \bar{\gamma}(\tau) \equiv \gamma(jt_0 + \tau). \end{cases} \tag{4.49}$$

The equation for $\bar{\gamma}(\tau)$ is identical to (4.4) for $\gamma(t)$ with $v_0(x)$ provided by (4.49) instead of (4.3):

$$\begin{cases} v_0(0) = 0, \\ |\partial_x^i v_0(x)| \leq \alpha^i\, e^{-j\beta_1\alpha^2 t_0} e^{-\alpha\frac{5|x|}{8}} \text{ for } x > 0,\ i = 0, 1, 2, \\ |\partial_x^i v_0(x)| \leq \alpha^i e^{-j\beta_1\alpha^2 t_0} e^{\alpha\frac{|x|}{3}} \text{ for } x < 0,\ i = 0, 1, 2. \end{cases} \tag{4.50}$$

The conditions involve derivatives up to second order. However, the induction hypotheses of the proposition yield only the information about zero$^{\text{th}}$ order. Nevertheless, since the equation is parabolic, the zero-th order information is enough to recover the higher order derivatives in any positive time. The recovery of the higher order derivatives is routine and is omitted. Here, the third condition is due to Lemma 4.13.

Due to (4.5) and (4.6), the consequences of (4.11) and (4.12) remain valid for $\bar{\gamma}(\tau)$ with δ replaced by $\delta e^{-j\beta_1\alpha^2 t_0}$ for $\tau \in [0, t_0]$:

$$|\gamma((j+1)t_0) - \gamma(jt_0)| = |\bar{\gamma}(t_0)| < \delta \frac{4L_0^2}{|\bar{\mathbf{u}}_x(0)|}\, e^{-j\beta_1\alpha^2 t_0}.$$

Thus, (4.47) holds for $n = j + 1$.

All the conditions for Proposition 4.7 are still valid for $\bar{\gamma}(\tau)$ and one has, for $\tau \in [\frac{t_0}{2}, t_0]$,

$$|\bar{\gamma}(\tau) - \bar{\gamma}(t_0)| \le e^{-(j\beta_1+\beta_0)\alpha^2 t_0} \le \delta e^{-(j+1)\beta_1\alpha^2 t_0}. \tag{4.51}$$

In deriving this estimate, one also has the following estimate as a by-product:

$$\begin{aligned} &|v(0,t)| \le O(1)\ \delta\ e^{-j\beta_1\alpha^2 t_0} \text{ for } t \in [0, t_0/2], \\ &|v(0,t)| \le O(1)\ \delta\ e^{-(j\beta_1+\beta_0)\alpha^2 t_0} \text{ for } t \in [t_0/2, t_0]. \end{aligned} \tag{4.52}$$

Now, we change the x coordinate to $x-\gamma((j+1)t_0)$. Therefore, we take $\gamma((j+1)t_0) = 0$, again.

Combining (4.48) for $n \le j$, (4.52), $\gamma((j+1)t_0) = 0$, and (4.51) together, we have obtained the information of $\gamma(t)$ and $\bar{v}_{j+1}(t)$ for $t \in [0, (j+1)t_0]$.
This leads to the boundary value problem:

$$\begin{cases} \partial_t \bar{v}_{j+1} - \alpha\partial_x \bar{v}_{j+1} - \partial_x^2 \bar{v}_{j+1} = qK\left(\mathbf{W}(x-\gamma(t)) - \mathbf{W}(x)\right), \\ |\bar{v}_{j+1}(t)| \le O(1)\ \delta\ \mathbf{b}(t), \\ |\gamma(t)| \le O(1)\ \delta\ \mathbf{b}(t)/|\bar{\mathbf{u}}_x(0)|. \end{cases} \tag{4.53}$$

By the Duhamel principle, the representation of $\bar{v}_{j+1}(x, (j+1)t_0)$ is

$$\begin{aligned} &\bar{v}_{j+1}(x,(j+1)t_0) = \int_R G_*(x,(j+1)t_0; y, 0)\ \bar{v}_{j+1}(y,0)\ dy \\ &+ qK \int_0^{(j+1)t_0} G_*(x,(j+1)t_0; y,\sigma)\left(\mathbf{W}(y-\gamma(\sigma)) - \mathbf{W}(y)\right)\ dy d\sigma \\ &- \int_0^{(j+1)t_0} \partial_y G_*(x,(n+1)t_0; 0,\sigma)\bar{v}_{j+1}(0,\sigma) d\sigma. \end{aligned}$$

Substitute the conditions about $\gamma(t)$ and $\bar{v}_{j+1}(0,t)$ in (4.53) into the above representation. Then, (4.48) is verified for $n = j+1$ and the proposition follows. □

5. Stability Analysis II: Nonlinear Flux

When the flux is nonlinear, one needs to linearize the problem at the left end state of the detonation wave profile as well as at the ignition point. The first is for studying waves propagating to the left far field, the other is for the purpose of tracing the wave fronts. About the wave travelling to the left far field, we need consider it as a boundary value problem with the boundary values provided by the front tracing.

Similar to the setting of an initial boundary value problem in the previous section, we introduce

$$B(t) \equiv \begin{cases} e^{-\left[\frac{t}{t_0}\right]\frac{\beta_1}{2}\alpha^2 t_0} \text{ if } t \in \left(\left[\frac{t}{t_0}\right]t_0, \left(\left[\frac{t}{t_0}\right]+\frac{1}{2}\right)t_0\right], \\ e^{-\left(\left[\frac{t}{t_0}\right]\frac{\beta_1}{2}+\frac{\beta_0}{2}\right)\alpha^2 t_0} \text{ if } t \in \left(\left(\left[\frac{t}{t_0}\right]+\frac{1}{2}\right)t_0, \left(\left[\frac{t}{t_0}\right]+1\right)t_0\right], \end{cases}$$

$$
\begin{aligned}
\alpha_- &\equiv s - f'(u_-),\\
G_-(x,t;y,\sigma) &\equiv k(x-y+\alpha_-(t-\sigma),t-\sigma) - e^{\alpha x}\, k(x+y-\alpha_-(t-\sigma),t-\sigma),\\
\Omega^-(x,t;L) &\equiv qK\int_0^t B(\sigma)\cdot\left(G_-(x,t;0,\sigma)+K\int_{y<0} G_-(x,t;y,\sigma)e^{Ky}\,dy\right)d\sigma\\
&\quad - L\int_0^t B(\sigma)\cdot\partial_y G_-(x,t;0,\sigma)\,d\sigma.
\end{aligned}\tag{5.1}
$$

Note. The estimate for $\Omega^*(x,t;L)$ in Lemma 4.13 can be applied to Ω^- by replacing α and $\mathbf{b}(nt_0)$ in the lemma by α_- and $B(nt_0))$ respectively.

Let $v(x,t)$ be the solution of (3.3) with initial values which satisfy (4.3) and (4.2); and $\bar{v}_n(x,t)$ stands for the same meaning as that in Proposition 4.14.

Proposition 5.1. *There is a constant $L>10$ such that it holds for all $n\in\mathbf{N}$,*

$$
|\gamma(nt_0)-\gamma((n-1)t_0)| \le \frac{8L_0^2}{|\bar{\mathbf{u}}_x(0)|}e^{-(n-1)\frac{\beta_1}{2}\alpha^2 t_0}\delta, \tag{5.2}
$$

$$
|\bar{v}_n(x,nt_0)| \le \delta\begin{cases} 2\Omega^-(x-\gamma(nt_0),nt_0;L)\\ +k(x+\alpha nt_0-\gamma(nt_0),4nt_0) \text{ for } x<\gamma(nt_0),\\ O(1)e^{-n\frac{\beta_1}{2}\alpha^2 t_0}\, e^{-\frac{5|x-\gamma(nt_0)|}{8}} \text{ for } x>\gamma(nt_0).\end{cases} \tag{5.3}
$$

We will also prove this proposition by mathematical induction. The procedure is similar to those in obtaining Proposition 4.14. However, one still needs to modify the wave front tracing and the stability analysis regarding the presence of the fluid nonlinearity. It should be mentioned that for weak detonation *the chemical nonlinearities* and *fluid nonlinearities* are *decoupled* in terms of *wave front tracing*, because *weak detonation wave is faster than any other non-chemical waves.*

5.1. Nonlinear front tracking. In order to proceed with the wave front tracing for nonlinear flux, one makes the a priori assumption that

$$
|\partial_x^i \bar{v}_n(x,t)| < 2\alpha^i|\bar{v}_n(x)|,\ i=1,2, \quad \text{for } t\in(nt_0,(n+1)t_0). \tag{5.4}
$$

Proposition 5.2. *Under the hypothesis of Proposition 4.3 and under (5.4), it holds for the nonlinear problem that*

$$
|\gamma(t)-\gamma(nt_0)| \le \frac{8L_0^2 e^{-\frac{n\beta_1}{2}\alpha^2 t_0}\delta}{|\bar{\mathbf{u}}_x(0)|} \text{ for } t\in(nt_0,(n+1)t_0)\,, \tag{5.5}
$$

and

$$
|\gamma(t)-\gamma((n+\frac{1}{2})t_0)| \le \frac{8L_0^2 e^{-\frac{(n\beta_1+\beta_0)}{2}\alpha^2 t_0}\delta}{|\bar{\mathbf{u}}_x(0)|} \text{ for } t\in\left((n+\frac{1}{2})t_0,(n+1)t_0\right). \tag{5.6}
$$

Proof. The equation of the front $\gamma(t)$ is given by (3.7). The difference between (3.7) and (4.4) is the fluid nonlinearity, which shows up in the last double integral in R.H.S. of (3.7). However, the influence of this nonlinearity can be ignored in deriving the uniformly bound estimate of the wave front, provided that this nonlinearity satisfies, for $t \in (0, t_0)$,

$$\left|\int_0^t \int_R k(\alpha(t-\sigma)-y, t-\sigma)\left\{-N_1(\bar{v}_n(y, \gamma(nt_0+\sigma)))\right\}_y \, dy d\sigma\right| \tag{5.7}$$
$$\ll \delta\left(\int_{y<0} k(\alpha t-y, t)\Omega^-(y, nt_0; L)\, dy + \frac{e^{-\frac{n\beta_1}{2}\alpha^2 t_0}}{|\bar{\mathbf{u}}_x(0)|}\int_{y>0} k(\alpha t-y, t)e^{-5\alpha y/8}\, dy\right).$$

With this, (4.11) and (4.12) are still valid by letting L_0 twice the value of that in Proposition 4.3. This results in (5.5), the uniform bound of the detonation wave location.

Similarly, if

$$\left|\frac{d}{dt}\int_0^t \int_R k(\alpha(t-\sigma)-y, t-\sigma)N_1(\bar{v}_n(y, t+nt_0))_y \, dy d\sigma\right| \tag{5.8}$$
$$\ll \delta\alpha^2\left(\int_{y<0} k(\alpha t-y, t)\Omega^-(y, nt_0; L)\, dy + \frac{e^{-\frac{n\beta_1}{2}\alpha^2 t_0}}{|\bar{\mathbf{u}}_x(0)|}\int_{y>0} k(\alpha t-y, t)e^{-5\alpha y/8}\, dy\right),$$

then (4.24) remains valid. Thus, Proposition 4.4 and Proposition 4.7 hold for this nonlinear flux, too. This proves (5.6). It remains to prove (5.7) and (5.8).

From the definition $N_1(\bar{v}_n)$ in (3.3), one can write

$$N_1(\bar{v}_n)(y, nt_0+t) \equiv f(\bar{v}_n(y,t) + \bar{\mathbf{u}}(y-\gamma(nt_0))) - f(\bar{\mathbf{u}}(y-\gamma(nt_0))) - \bar{v}_n f'(\bar{\mathbf{u}}(0))$$
$$= \bar{v}_n \int_0^1 f'(\theta\bar{v}_n + \bar{\mathbf{u}}) - f'(\bar{\mathbf{u}}(0))\, d\theta$$
$$= \bar{v}_n \int_0^1\int_0^1 f''\left(\{\phi(\theta\bar{v}_n + \bar{\mathbf{u}} - \bar{\mathbf{u}}(0)) + \bar{\mathbf{u}}(0)\}\right)\ [\theta\bar{v}_n + \{\bar{\mathbf{u}} - \bar{\mathbf{u}}(0)\}]\, d\phi\, d\theta.$$

By this identity and (5.4), one has that

$$|N_{1y}(v_n(y, nt_0+t))| \tag{5.9}$$
$$= O(1)\ \delta e^{-\frac{n\beta_1}{2}\alpha^2 t_0}|\bar{\mathbf{u}}_x(0) + \delta\alpha|e^{-\frac{5\alpha|y-\gamma(nt_0)|}{8}}$$
$$= O(1)\ \delta e^{-\frac{n\beta_1}{2}\alpha^2 t_0}|\bar{\mathbf{u}}_x(0)|e^{-\frac{5\alpha|y-\gamma(nt_0)|}{8}} \text{ for } y > \gamma(nt_0),$$
$$|N_{1y}(v_n(y, nt_0+t))|e^{-\frac{\alpha|y-\gamma(nt_0)|}{2}}$$
$$= O(1)\ \delta e^{-\frac{n\beta_1}{2}\alpha^2 t_0}|\bar{\mathbf{u}}_x(0)| \text{ for } y < \gamma(nt_0),$$

$$
\begin{aligned}
&|N_{1yy}(v_n(y, nt_0+t))| \\
&\quad = O(1)\ \delta\alpha e^{-\frac{n\beta_1}{2}\alpha^2 t_0}|\bar{\mathbf{u}}_x(0)|e^{-\frac{5\alpha|y-\gamma(nt_0)|}{8}} \text{ for } y > \gamma(nt_0), \\
&|N_{1yy}(v_n(y, nt_0+t))|e^{-\frac{\alpha|y-\gamma(nt_0)|}{2}} \\
&\quad = O(1)\ \delta\alpha e^{-\frac{n\beta_1}{2}\alpha^2 t_0}|\bar{\mathbf{u}}_x(0)| \text{ for } y < \gamma(nt_0).
\end{aligned} \tag{5.10}
$$

Substitute (5.9) into the double integral in the L.H.S. of (5.7), then by a straight calculation (5.7) follows.

Similarly, by (5.10) and by applying integration by parts, (5.8) follows. This completes the proof of the proposition. □

5.2. Update wave front. Let $\overline{\mathbf{w}} = w(x,t) + \bar{\mathbf{u}}(x)$ be the solution of

$$
\begin{cases}
\overline{\mathbf{w}}_t - s\overline{\mathbf{w}}_x + f(\overline{\mathbf{w}})_x - \overline{\mathbf{w}}_{xx} = qK\mathbf{W}(x - loc(t)) \text{ for } -x, t > 0, \\
\max\left(|\bar{\mathbf{u}}_x(0)\cdot loc(t)|, \frac{|w(0,t)|}{L}\right) \le \delta B(t), \\
|w(x,0)| \ \le\ \delta e^{-K|x|}.
\end{cases} \tag{5.11}
$$

The equation of $w(x,t)$ is

$$
\begin{aligned}
w_t - \alpha_- w_x - w_{xx} = {} & qK(\mathbf{W}(x - loc(t)) - \mathbf{W}(x)) \\
& + \left((s - f'(\bar{\mathbf{u}}(x)) - \alpha_-)w\right)_x - \mathbf{N}(w)_x,
\end{aligned} \tag{5.12}
$$

where

$$
\begin{aligned}
\mathbf{N}(w) &\equiv f(\bar{\mathbf{u}}(x) + w) - f(\bar{\mathbf{u}}(x)) - f'(\bar{\mathbf{u}}(x))\, w, \\
s - f'(\bar{\mathbf{u}}(x)) - \alpha_- &= O(1)e^{Kx} \text{ for } x < 0.
\end{aligned}
$$

Lemma 5.3. *Suppose that Assumption 2.7 holds. Then, there exists constant C_3 such that*

$$
|w(x,t)| \le 2C_3\delta\left(\Omega^-(x, nt_0; L) + k(x + \alpha_- nt_0, 2nt_0)\right) \textit{ for } t \in (nt_0, (n+1)t_0).
$$

Proof. By Duhamel's principle,

$$
\begin{aligned}
w(x,t) = {} & \int_{y<0} G_-(x,t;y,0)w(y,0)dy \\
& + qK\int_0^t \int_{y<0} G_-(x,t;y,\sigma)\left(\mathbf{W}(y - loc(\sigma)) - \mathbf{W}(y)\right)\, dyd\sigma \\
& - \int_0^t \int_{y<0} \partial_y G_-(x,t;y,\sigma)\left(O(1)e^{Ky}w + \mathbf{N}(w)\right)(y,\sigma)dyd\sigma \\
& - \int_0^t \partial_y G_-(x,t;0,\sigma)w(0,\sigma)\, d\sigma.
\end{aligned}
$$

We introduce a standard iteration scheme to construct the solution $w(x,t)$,

$$\begin{aligned} w^1(x,t) = &\int_{y<0} G_-(x,t;y,0)w(y,0)dy \\ &+ qK\int_0^t\int_{y<0} G_-(x,t;y,\sigma)\left(\mathbf{W}(y-loc(\sigma))-\mathbf{W}(y)\right)\,dyd\sigma \\ &-\int_0^t \partial_y G_-(x,t;0,\sigma)w(0,\sigma)\,d\sigma. \end{aligned} \tag{5.13}$$

For $j \geq 1$,

$$\begin{aligned} w^{j+1}(x,t) = &\int_{y<0} G_-(x,t;y,0)w(y,0)dy \\ &+ qK\int_0^t\int_{y<0} G_-(x,t;y,\sigma)\left(\mathbf{W}(y-loc(\sigma))-\mathbf{W}(y)\right)\,dyd\sigma \\ &-\int_0^t \partial_y G_-(x,t;0,\sigma)w(0,\sigma)\,d\sigma \\ &-\int_0^t\int_{y<0} \partial_y G_-(x,t;y,\sigma)\left(O(1)e^{Ky}w^j+\mathbf{N}(w^j)\right)(y,\sigma)dyd\sigma, \end{aligned} \tag{5.14}$$

From the definition of $\Omega^-(x,t;L)$, there exists c_0 such that

$$\begin{aligned} &|w^1(x,t)| \leq A(x,t), \\ &A(x,t) \equiv c_0\delta\left(\Omega^-(x,t;L)+\frac{1}{\alpha}\,k(x+\alpha_-t,2t)\right). \end{aligned}$$

This leads to a priori assumption on $w^j(x,t)$ for $j \geq 1$:

$$|w^j(x,t)| \leq 2\;\delta\;A(x,t). \tag{5.15}$$

It is sufficient to show that

$$\int_0^t\int_{y<0} |\partial_y G_-(x,t;y,\sigma)\left(e^{Ky}A+\mathbf{N}(A)\right)(y,\sigma)\,|\,dy\,d\sigma \ll A(x,t).$$

Due to the quadratic nonlinearity of $\mathbf{N}$, one can easily show that

$$\int_0^t\int_{y<0} |\partial_y G_-(x,t;y,\sigma)\mathbf{N}(A)(y,\sigma)|\,dy\,d\sigma \ll A(x,t).$$

About $\int_0^t\int_{y<0}\partial_y G_-(x,t;y,\sigma)e^{Ky}A(y,\sigma)\,dyd\sigma$, it is sufficient to show that

$$\delta\int_0^t\int_{y<0} |\partial_y G_-(x,t;y,\sigma)|e^{Ky}\,k(y+\alpha_-\sigma,4\sigma)dyd\sigma \ll A(x,t).$$

For showing this, first we break the space integral into two parts,

$$\delta\int_0^t\left(\int_{y<-\frac{\alpha_-\sigma}{2}}+\int_{-\frac{\alpha_-\sigma}{2}}^0\right)|\partial_y G_-(x,t;y,\sigma)|e^{Ky}\,k(y+\alpha\sigma,4\sigma)dyd\sigma$$
$$\equiv r_1+r_2.$$

From the definition of G_-, (5.1), one has

$$|\partial_y G_-(x,t;y,\sigma)|\le O(1)\left(\frac{k(x-y+\alpha_-(t-\sigma),2(t-\sigma))}{\sqrt{t-\sigma}}\right.$$
$$\left.+\frac{|x+y-\alpha(t-\sigma)|}{t-\sigma}e^{-\alpha x}k(x+y-\alpha_-(t-\sigma),(t-\sigma))\right)$$
$$\le O(1)\left(\frac{k(x-y+\alpha_-(t-\sigma),2(t-\sigma))}{\sqrt{t-\sigma}}+\alpha k(x-y+\alpha_-(t-\sigma),(t-\sigma))\right).$$

From this,

$$r_1\le\delta O(1)\int_0^t\int_{y<-\frac{\alpha_-\sigma}{2}}$$
$$\left(\frac{k(x-y+\alpha_-(t-\sigma),2(t-\sigma))}{\sqrt{t-\sigma}}+\alpha_-k(x-y+\alpha_-(t-\sigma),t-\sigma)\right)e^{-K\frac{\alpha_-\sigma}{2}}$$
$$\cdot k(y+\alpha_-\sigma,4\sigma)dyd\sigma=O(1)\delta\left(\frac{1}{\sqrt{K\alpha_-}}+\frac{1}{K}\right)k(x+\alpha_-t,4t).$$

Therefore, when α, $K\gg 1$,

$$r_1\ll A(x,t). \tag{5.16}$$

When $y\in(-\frac{\alpha\sigma}{2},0)$, we have

$$e^{K(y+\alpha\sigma)}k(y+\alpha\sigma,4\sigma)\le\frac{e^{-(\frac{\alpha}{32}-K)|y+\alpha\sigma|}}{\sqrt{\sigma}}.$$

This yields

$$\int_{-\frac{\alpha\sigma}{2}}^0 k(x+\alpha t-(y+\alpha\sigma),t-\sigma)e^{Ky}k(y+\alpha\sigma,4\sigma)\,dy$$
$$\le O(1)e^{-K\alpha\sigma}\frac{1}{\alpha\sqrt{\sigma}}\left(k(x+\alpha t,4(t-\sigma))+e^{-(\frac{\alpha}{32}-K)|x+\alpha t|/2}\right).$$

This yields, when $\alpha\gg 1$,

$$r_2\ll A(x,t). \tag{5.17}$$

Thus, (5.16) and (5.17) imply (5.15). By a similar calculation as above one can show that the iteration scheme converges. □

Proof of Proposition 5.1. The analysis for the case $n = 1$ is similar to the following and is omitted.

Assume the proposition holds for $n \le j$ and suppose that (5.4) is satisfied for $n = j$. Then, Proposition 5.2 yields (5.5) and (5.6). We need to update the detonation wave front and consider $\bar{v}_{j+1}(x, t)$ in order to justify that (5.4) holds for $n = j$. The verification of the ansatz for $x > \gamma((j+1)t_0)$ is straightforward. We also omit it.

For convenience we assume $\gamma((j+1)t_0) = 0$. Thus, $\gamma(t)$ satisfies the same property of $loc(t)$, given in (5.11), for $t \in (0, (j+1)t_0)$. The value of $\bar{v}_{j+1}(0, t)$ also share the same property of $w(0, t)$, given in (5.11), for $t \in [0, jt_0]$. The property of $\bar{v}_{j+1}(0, t)$ for $t \in [jt_0, (j+1)t_0]$ can be obtained through verifying the ansatz $\bar{v}_{j+1}(x, t)$ in the region $x > 0$. This results in

$$\bar{v}_{j+1}(0, t) \le L\delta\, e^{-\frac{j\beta_1}{2}\alpha^2 t_0} \text{ for } t \in \left[jt_0, \left(j + \frac{1}{2}\right)\right),$$
$$\bar{v}_{j+1}(0, t) \le L\delta\, e^{-\frac{j\beta_1+\beta_0}{2}\alpha^2 t_0} \text{ for } t \in \left[\left(j + \frac{1}{2}\right), (j+1)t_0\right);$$

and also implies (5.4) for $n = j$, $x > 0$. It remains to show that (5.4) holds for $n = j$, $x < 0$. Since $\bar{v}_{j+1}(x, t)$ satisfies the criterion for $w(x, t)$ in (5.11) for $t \in [0, (j+1)t_0]$, one can apply Lemma 5.3 to $\bar{v}_{j+1}(x, t)$. So, (5.4) is true for $n = j$. Proposition 5.1 is true for $n = j + 1$. Thus, the proposition follows. □

From Proposition 5.1 we have proved Theorem 1.1.

6. Remarks

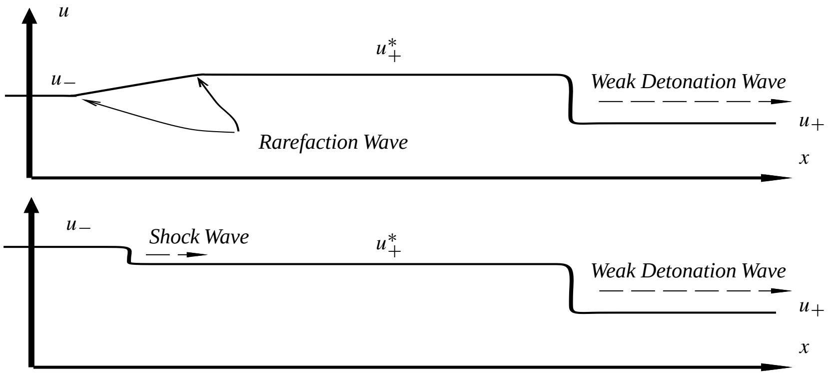
In a physical setting, the parameters q, K, and T^* are given, while the states s, u_-, and u_+ depend on the physical situation. In our setting of a weak detonation profile, (2.4), T^* is a function of u_+, s, q, and K. Keeping the ignition temperature T^*, q, and K as fixed constants, then (2.4) gives an implicit function for s. Thus, one can write s as a function $s(u_+)$. Then, by (2.1) the left state of the weak detonation wave is given uniquely. We note this state as u_+^*.

On the other hand, the weak detonation wave (u_+^*, u_+) seems so special compared to the usual fluid shock wave (u_-, u_+), for which both states can vary independently; for a weak detonation wave only one state can. Thus, it seems very difficult to produce such a weak detonation wave pattern. However, the weak detonation wave pattern (u_+^*, u_+) is generic and is a key wave pattern for a Riemann problem.

Consider the Riemann problem (u_-, u_+) with $u_+ \in \{\text{Domain of } s\}$. The decomposed wave pattern is a weak detonation wave followed either by a shock wave or by a rarefaction wave. The decomposed wave pattern is

$$(u_-, u_+) \Longrightarrow \underbrace{(u_-, u_+^*)}_{\text{Slow}} + \underbrace{(u_+^*, u_+)}_{\text{Fast}},$$

where (u_-, u_+^*) is a shock wave or a rarefaction wave depending on whether $u_- > u_+^*$ or $u_- < u_+^*$. See the following diagram.



Our analysis can be refined to support such wave patterns by replacing $\bar{\mathbf{u}}(x)$ in (5.11) either by a viscous shock profile or by a viscous rarefaction wave. Thus, a weak detonation profile is generic.

7. Appendix

Proof of Lemma 2.6. Let $\bar{\mathbf{u}}(x)$ the solution of (2.2). In order to obtain an analytic property of the profile, one needs an analytic property about s, q, K, (u_-, u_+), and T^* in term of the fluid nonlinearity $f(u)$. We assume $|u_- - u_+| = O(1)$ and $q \gg 1$. From (2.1),

$$s = \frac{q}{u_- - u_+} + O(1). \tag{7.1}$$

When $x < 0$, the normalized profile $\bar{\mathbf{u}} = \bar{\mathbf{v}} + u_-$ is given by

$$\bar{\mathbf{v}}_x = (f'(u_-) - s) \cdot \bar{\mathbf{v}}_x - qe^{Kx} + \mathbf{n}(\bar{\mathbf{v}}), \tag{7.2}$$

where

$$\begin{aligned} \mathbf{n}(\bar{\mathbf{v}}) &\equiv f(\bar{\mathbf{v}} + u_-) - f(u_-) - f'(u_-)\bar{\mathbf{v}}, \\ m_- &\equiv f'(u_-). \end{aligned}$$

Transform this into an integral equation and use (7.1),

$$\begin{aligned} \bar{\mathbf{v}}(x) &= \int_{-\infty}^{x} e^{(-s+m_-)(x-y)} \left(-qe^{Ky} + \mathbf{n}(\bar{\mathbf{v}}(y))\right) dy \\ &= -\frac{qe^{Kx}}{s - m_- + K} + \int_{-\infty}^{x} e^{(-s+m_-)(x-y)} \mathbf{n}(\bar{\mathbf{v}}(y)) dy \\ &= (u_+ - u_-) \left(1 + O(K + m_-) \frac{|u_- - u_+|}{q}\right) e^{Kx} \\ &\quad + \int_{-\infty}^{x} e^{(-s+m_-)(x-y)} \mathbf{n}(\bar{\mathbf{v}}(y))\, dy. \end{aligned}$$

By a priori assumption

$$|\bar{\mathbf{v}}(x)| < 2|u_- - u_+|\, e^{Kx}, \text{ for } x < 0$$

one obtains

$$\bar{\mathbf{v}}(0) = (u_+ - u_-) + \frac{O(1)|u_- - u_+|^2(K + m_-)}{q}.$$

This yields that

$$T^* - u_+ = \frac{O((K + m_-)|u_- - u_+|^2)}{q}. \tag{7.3}$$

By a straightforward calculation, one can verify this assumption and also (7.3).
Next we consider the profile in the unburnt zone. Denote

$$\bar{\mathbf{v}}^+ \equiv \bar{\mathbf{u}} - u_+.$$

The equation for $\bar{\mathbf{v}}^+$ is

$$\bar{\mathbf{v}}^+_x = (-s + f'(u_+))\bar{\mathbf{v}}^+ + \mathbf{n}^+(\bar{\mathbf{v}}^+),$$
$$\mathbf{n}^+(\bar{\mathbf{v}}^+) \equiv f(u_+ + \bar{\mathbf{v}}^+) - f(u_+) - f'(u_+)\bar{\mathbf{v}}^+.$$

By the smallness of $T^* - u_+$ in (7.3) and by Picard's iteration,

$$\bar{\mathbf{v}}^+_{n+1}(x) = \bar{\mathbf{v}}^+(0)e^{(-s+f'(u_+))x} + \int_0^x e^{(-s+f'(u_+))(x-y)}\mathbf{n}^+(\bar{\mathbf{v}}^+_n)(y)dy,$$

this yields

$$\bar{\mathbf{v}}^+(x) = \bar{\mathbf{v}}^+(0)\left(1 + \frac{O(1)}{q^2}\right)e^{(-s+f'(u_+))x}.$$

Roughly, this yields, for $x > 0$,

$$-\bar{\mathbf{v}}^+(x)_x > \frac{s - f'(u_+)}{2}\bar{\mathbf{v}}^+(0)e^{(-s+f'(u_+))x}$$
$$= -\frac{1}{2}\left(1 + O(\frac{1}{q})\right)\bar{\mathbf{u}}_x(0)e^{(-s+f'(u_+))x}.$$

The lemma follows. □

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