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Nonlinear Stability of a Self-Similar 3-Dimensional Gas Flow

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Abstract: We show that the 3-dimensional supersonic gas flow past an infinite cone is nonlinear staple upon the perturbation of the obstacle. The perturbed flow exists globally in space and tends to the self-similar flow downstream. There is a thin layer of concentration of vorticities and entropy variation. Our analysis is based on an approximation scheme using local self-similar solutions as building blocks. This enables us to obtain global estimates of the nonlinear interactions of waves needed for the stability analysis.

1. Introduction

We are concerned with inviscid gas flow in three space dimension. The compressible Euler equations are:

$$\rho_t + \sum_{i=1}^{3} (\rho u_i)_{x_i} = 0 \qquad \text{(Conservation of mass)}, \tag{1.1}$$

$$(\rho u)_t + \sum_{j=1}^{3} (\rho u_i u_j)_{x_j} + P_{x_i} = 0,\ i = 1, 2, 3, \qquad \text{(Conservation of momentum)},$$

$$(\rho E)_t + \sum_{i=1}^{3} (\rho E u_i + P u_i)_{x_i} = 0, \qquad \text{(Conservation of energy)}, \tag{1.2}$$

$$P = f(\rho, S) \qquad \text{(Equation of state)}, \tag{1.3}$$

where (u_1, u_2, u_3) is the velocity, ρ the density, P the pressure, e the internal energy, $E = e + (u_1{}^2 + u_2{}^2 + u_3{}^2)/2$ the total energy, and S the specific entropy. The system is

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quasilinear hyperbolic and a general flow contains shock waves, which greatly complicate the analysis. However, in the presence of symmetries, the flow becomes self-similar and the system may be reduced to ordinary differential equations. In the present paper, we are inerested in steady flow past an infinite cone, with axis in the x_1 direction, and its stability with respect to perturbation of the cone. Such flow has cylindrical symmetry, the dependent variables are functions of $x \equiv x_1$ and the distance $y \equiv \sqrt{x_2{}^2 + x_3{}^2}$ from the axis. Let u and v represent the axial and radial components of velocity,

$$u_1(x_1, x_2, x_3) \equiv u(x, y), \ (u_2, u_3)(x_1, x_2, x_3) \equiv (\frac{x_2}{\sqrt{x_2{}^2 + x_3{}^2}}, \frac{x_3}{\sqrt{x_2{}^2 + x_3{}^2}})v(x, y).$$

With the additional simplification that the flow is isentropic, the Euler equations are reduced to:

$$(\rho u)_x + (\rho v)_y = \frac{-1}{y}(\rho v), \tag{1.4}$$

$$(\rho u^2 + P)_x + (\rho u v)_y = \frac{-1}{y}(\rho u v), \tag{1.5}$$

$$(\rho u v)_x + (\rho v^2 + P)_y = \frac{-1}{y}(\rho v^2), \tag{1.6}$$

$$P = P(\rho). \tag{1.7}$$

When a uniform supersonic flow

$$(u, v) = (q_0, 0)$$

hits the obstacle, which is an infinite cone

$$y/x = \theta_0,$$

and the angle θ_0 of opening at the vertex is sufficiently small, the conical flow can be constructed by studying self-similar solutions. The flow is deflected by an attached shock front beginning at the vertex and is continued so that the state of the air is constant on each concentric cone behind the shock cone and is parallel to the obstacle cone. The simplification is due to the fact that, between the shock and obstacle cones, the flow is isentropic $S = S_0$ and irrotational:

$$v_x = u_y, \tag{1.8}$$

and it follows from (1.4)–(1.7) that

$$(1 - \frac{u^2}{c^2})u_x - \frac{2uv}{c^2}v_x + (1 - \frac{v^2}{c^2})v_y + \frac{v}{y} = 0. \tag{1.9}$$

Here the sound speed c is a given function of u and v through Bernoulli's law. The flow depends on

$$\sigma \equiv x/y$$

and the equations are further reduced to a system of ordinary differential equations:

$$v_\sigma + \sigma u_\sigma = 0, \tag{1.10}$$

$$(1 - \frac{u^2}{c^2})u_\sigma - \frac{2uv}{c^2}v_\sigma - (1 - \frac{v^2}{c^2})\sigma v_\sigma + v = 0. \tag{1.11}$$

Such self-similar flow is suggested by Busemann([1]), who gave a graphical method for obtaining them, see Sect. 2.

We consider a more realistic case when the obstacle is a perturbation of the infinite cone. The shock front and the flow behind it are conical until the expansion wave and the shock wave coming from the bendings of the obstacle interact with the conical flow and the shock front. The flow then becomes rotational and, in general, contains infinitly many interacting shock waves. The main questions are: With all these wave interactions, does a solution exist globally? Is it stable with respect to the perturbation, for finite x and also asymptotically as $x \to \infty$? We answer these questions affirmatively and show that the long-range behavior of the flow is self-similar corresponding to an infinite cone with the asymptotic angle of the perturbation. In particular, the flow between the leading shock and the obstacle tends to be irrotational and isentropic. In fact, there is a boundary layer of high concentration of vorticity and entropy variation. The width of the layer tends to zero as $x \to \infty$. This can occur, of course, only for inviscid flow; for the viscous flow the vorticity would propagate into the flow. Nevertheless, experimental evidences show that the inviscid flow still accurately represents the actual flow, Courant–Friedrichs [3].

System (1.4)–(1.7) can be regarded as one-dimensional hyperbolic conservation laws with a source. There is a general existence and time-asymptotic theory, Liu [9, 10] and Lien [7], for a system of the form

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = \mathbf{g}(\mathbf{u}, x).$$

The main idea is to recognize that solutions, which are function of x only,

$$\mathbf{f}(\mathbf{u})_x = \mathbf{g}(\mathbf{u}, x),$$

are normal modes. The idea is then to approximate the solution by a piecewise smooth function consisting of these modes and then to study the nonlinear interaction of waves resulting form the resolution the discontinuities. This is done modifying the random choice method of Glimm [4] for hyperbolic conservation laws

$$u_t + f(u)_x = 0.$$

This approach applies when the source is finite, i.e.

$$\int_{-\infty}^{\infty} |\mathbf{g}(\mathbf{u}, x)| dx = O(1).$$

However, the source term here $(\frac{-1}{y}(\rho v), \frac{-1}{y}(\rho uv), \frac{-1}{y}(\rho v^2))$ is not finite both at the origin and for large y and the above theory does not apply. (To make the comparison, the independent variables are related by $(x, t) \to (y, x)$.) This simply means that solutions, which are functions of y only, play no natural role here. Nevertheless, the idea of identifying the normal modes for approximating the general solutions is shown to work in the present situation. We choose instead the self-similar solutions for (1.10) and (1.11) as the building blocks for the construction of solutions to (1.4)–(1.7). Besides the difference between the finite source and the present situation, one also notes that, unlike the finite source case, the system (1.10) and (1.11) is not automous. This reflects the fact that a self-similar solution is determined not only by its value at a location (x, y), but also by the origin of the self-similatity. For the perturbation of the infinite cone, the suitable origin $(x_0, 0)$ changes with the location (x, y), $x_0 = x_0(x, y)$, and so does the self-similar variable

$$\sigma = \sigma(x, y) = \frac{x - x_0}{y}.$$

For flow next to the obstacle, there is the clear choice of $\sigma = 1/\theta$, θ the slope of the obstacle. We allow this choice to propagate into the flow by the 3-waves. The numerical grids are moving along the constancy of the self-similar variable. The dominant shock next to the uniform upstream flow is traced, cf. Chern [2]. The construction of approximate solutions is done in Sect. 3.

Our analysis is based on the estimates of local wave interactions. Besides the interaction of elementary waves for the Riemann problem, one needs also to study the interaction of elementary waves and self-similar solutions, as well as the waves produced due to the changes in the origin $(x_0, 0)$ across a 3-wave. As with other studies of nonlinear waves, stability follows from the decoupling of waves, that is, wave interactions must decay. For hyperbolic conservation laws, this has been extensively studied, starting with Glimm-Lax [5]. A key observation here is that the angle between the self-similar rays, $\sigma =$ constant, and the shock and entropy waves decreases after interaction. These estimates are studied in Sect. 4.

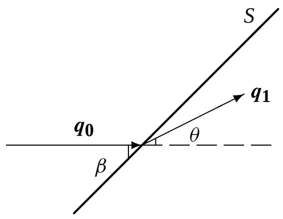
The local estimates allow us to introduce a global functional on nonlinear wave interactions to control the variation of the approximate solutions and thus prove the global existence of the solution in Sect. 5. The functional is defined also to take into account the fact that 1-waves are reflected by the obstacle to become 3-waves, and 3-waves propagate toward and then are combined with the dominant shock. These global wave estimates allow us in Sect. 6 to study the asymptotic behaviour of the flow as $x \to \infty$. The entropy waves, 2-waves, approach the obstacle and form the aforementioned boundary layer. The flow pattern eventually tends to a self-similar solution corresponding to the conical flow for the infinite cone without any deflections.

2. Self-Similar Solutions

In this section, we briefly review the quantitative analysis of shock polars and the construction of conical flow. We refer the readers to Courant and Friedrichs [3] and the references therein for more details. For simplicity, we assume that the flow is isentropic and consider the systems (1.4)–(1.7) for general flow, and (1.10) and (1.11) for self-similar flow. We consider the polytropic gases:

$$P = A\rho^{\gamma},\ \gamma > 1.$$

2.1. Shock polars. Consider a shock S in the (x, y)-plane with the upstream state of velocity $\boldsymbol{q}_0 = (q_0, 0)$ and the downstream state of velocity $\boldsymbol{q}_1$, which makes angle θ with the upstream flow. The angle the shock makes with the upstream flow is denoted by β.



There is a one-parametric family of possible states, with velocity $\boldsymbol{q}_1$, which can be reached through a shock. These possible states are given by the Rankine–Hugoniot conditions of the conservation of mass, momentum and energy. On the phase space of the velocity $\boldsymbol{q} = (u, v)$, the states $\boldsymbol{q}_1$ lie on a curve, called the shock polar. Let N and L be the components of the velocity $\boldsymbol{q}$ normal and tangential to the shock line S respectively. We have

$$\begin{aligned} &u_0 = q_0, \quad v_0 = 0, \\ &u_1 = L_1 \cos\beta + N_1 \sin\beta, \quad v_1 = L_1 \sin\beta - N_1 \cos\beta, \\ &L_1 = L_0 = q_0 \cos\beta \quad \text{(Continuity of tangential component)}, \\ &N_0 = q_0 \sin\beta. \end{aligned} \tag{2.1}$$

From the conservation laws, Bernoulli's law for steady flow holds across a shock front:

$$\frac{1}{2}{q_0}^2 + i_0 = \frac{1}{2}{q_1}^2 + i_1 = \frac{1}{2}\hat{q}^2, \tag{2.2}$$

where i is the specific enthalpy. For a polytropic gas, $P(\rho) = A\rho^\gamma$, $\gamma > 1$, we have

$$i = \frac{c^2}{\gamma - 1}, \tag{2.3}$$

where $c = \sqrt{P'(\rho)}$ is the sound speed. We set $\mu^2 = \dfrac{\gamma - 1}{\gamma + 1}$ and $c_* = \mu\hat{q}$. The above identities yield the Prandtl's relation:

$$N_1 = \frac{(c_*)^2 - \mu^2 L_0^2}{N_0}. \tag{2.4}$$

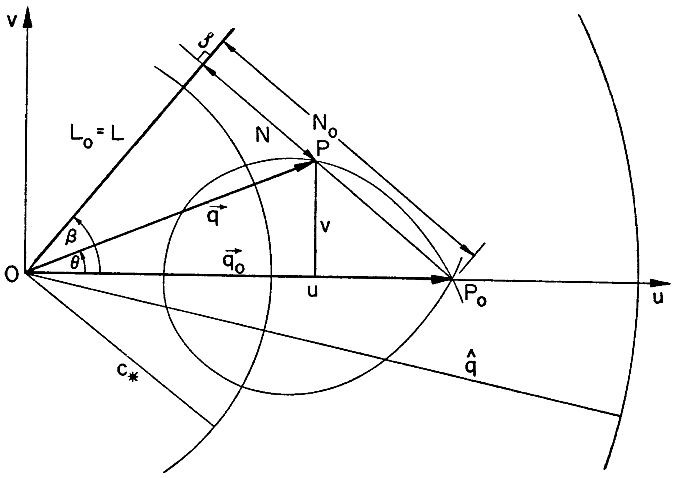
By (2.1)–(2.4), we obtain the relations

$$\begin{aligned} u_1 &= (1 - \mu^2) q_0 \cos^2\beta + \frac{c_*^2}{q_0}, \\ v_1 &= (q_0 - u_1)\cot\beta. \end{aligned}$$

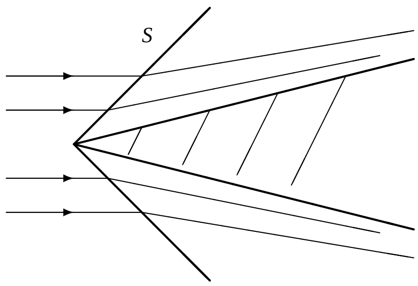
Eliminating the angle β, we find

$$v^2 = (q_0 - u)^2 \frac{u - \tilde{u}}{U - u}, \tag{2.5}$$

where $\tilde{u} = \dfrac{c_*^2}{q_0}$ and $U = (1-\mu^2)q_0 + \tilde{u}$. The curve in the (u, v)-plane given by Eq. (2.5), the shock polar, is the Folium of Descartes as shown in the following figure:



2.2. Conical flow. Now consider a conical body facing a supersonic stream of air at a uniform velocity $\boldsymbol{q_0} = (u_0, 0)$. Assume that the obstacle is an infinite cone with its vertex located at the origin in the (x, y)-plane. A shock wave S is formed and situated on a concentric cone where an abrupt change in density and velocity occurs. Between the shock and the obstacle cones, the flow is self-similar.



The self-similar flow satisfies the ordinary differential equations (1.10) and (1.11). There are two boundary requirements for the solution: The first requirement is that the flow velocity next to the obstacle is parallel to the obstacle. This is the natural condition for the inviscid flow. The second requirement is that the self-similar variable $\sigma = x/y$ next to the shock equals $1/\theta_S$, θ_S the slope of the shock. This is needed because the flow variables next to the shock are unchanged and the Rankine–Hugoniot condition is always satisfied. Such a solution is constructed by the shooting method. Given a state $\boldsymbol{q}_1$ on the shock polar through the given upstream state $\boldsymbol{q}_0$, we continue it by solving (1.10) and (1.11) with the initial condition $\boldsymbol{q}_1$ at $\sigma = 1/\theta_S$ so that the second requirement is satisfied. In other words, the initial value of (1.10) and (1.11) satisfies, with v regarded as a function of u,

$$v_u = -\sigma. \tag{2.6}$$

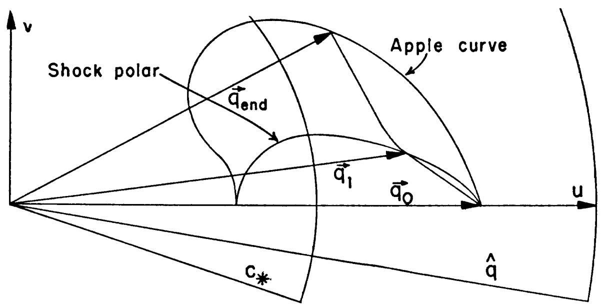
Since the shock line S is perpendicular to the straight line joining $(u_0, 0)$ and (u_1, v_1), the initial slope of the curve is given by

$$v_u = \frac{v_1 - v_0}{u_1 - u_0}. \tag{2.7}$$

The solution to (1.10) and (1.11) is continued so that $\sigma = \frac{x}{y}$ increases till an end state $\boldsymbol{q}_{\text{end}} \equiv (u_e, v_e)$ with the property that $u_e/v_e = \sigma_e$ there, or by (1.10),

$$v_u = -\frac{u}{v}. \tag{2.8}$$

In other words, the first requirement would be satisfied if the obstacle is $x = \sigma_e y$. The collection of the end states $\boldsymbol{q}_e$, for varying the state $\boldsymbol{q}_1$ on the shock polar, forms an *apple curve*, so called because of its shape. Note from (2.6) that the solution to (1.10) and (1.11) at an end point on the apple curve is normal to the line through the origin.



To construct the conical flow with the slope θ_0 of the obstacle given, we locate the point on the apple curve which intersects with the ray $x/y = 1/\theta_0$ through the origin. In general, there are two intersections of which the one corresponding to the weaker shock is more likely to occur in reality and is our main concern in the present paper.

3. Construction of Approximate Solutions

The approximate solutions to the system (1.4)–(1.7) will be constructed based on the self-similar solutions and the elementary waves for the homogeneous system

$$(\rho u)_x + (\rho v)_y = 0, \tag{3.1}$$
$$(\rho u^2 + P)_x + (\rho u v)_y = 0, \tag{3.2}$$
$$(\rho u v)_x + (\rho v^2 + P)_y = 0, \tag{3.3}$$
$$P = P(\rho). \tag{3.4}$$

The self-similar solutions have been considered in the first two sections. We now consider the elementary waves for (3.1). The system (3.1) is strictly hyperbolic and its characteristic speeds are:

$$\lambda_1 = \frac{uv}{u^2 - c^2} - \frac{c(q^2 - c^2)^{1/2}}{u^2 - c^2} < 0, \tag{3.5}$$

$$\lambda_2 = \frac{v}{u} > 0, \tag{3.6}$$

$$\lambda_3 = \frac{uv}{u^2 - c^2} + \frac{c(q^2 - c^2)^{1/2}}{u^2 - c^2} > 0. \tag{3.7}$$

Its first and third characteristic fields are genuinely nonlinear and the second characteristic field is linearly degenerate in the sense of Lax [6]. (For the non-isentropic flow, we need to consider the energy equation and the system is not strictly hyperbolic with double linearly degenerate eigenvalues v/u. Nevertheless, the system is completely hyperbolic and our analysis can be easily generalized for it.)

In the following, (ρ, u, v) is denoted by ω. Let $S_i(\omega_-)$ and $R_i(\omega_-)$ denote the Rankine–Hugoniot curve and the rarefaction curve for the i-characteristic field, respectively. Set

$$\begin{aligned}
R_i^+(\omega_-) &= \{\omega : \omega \in R_i(\omega_-), \text{ and } \lambda_i(\omega) \geq \lambda_i(\omega_-)\},\\
S_i^-(\omega_-) &= \{\omega : \omega \in S_i(\omega_-), \text{ and } \lambda_i(\omega) < \mathbf{s}(\omega_-, \omega) < \lambda_i(\omega_-), \mathbf{s} \text{ is the shock speed}\},\\
T_i(\omega_-) &= R_i^+(\omega_-) \cup S_i^-(\omega_-), \quad \text{for } i = 1, 3,\\
T_2(\omega_-) &= R_2(\omega_-)(= S_2(\omega_-)).
\end{aligned}$$

By straightforward computations, we obtain that

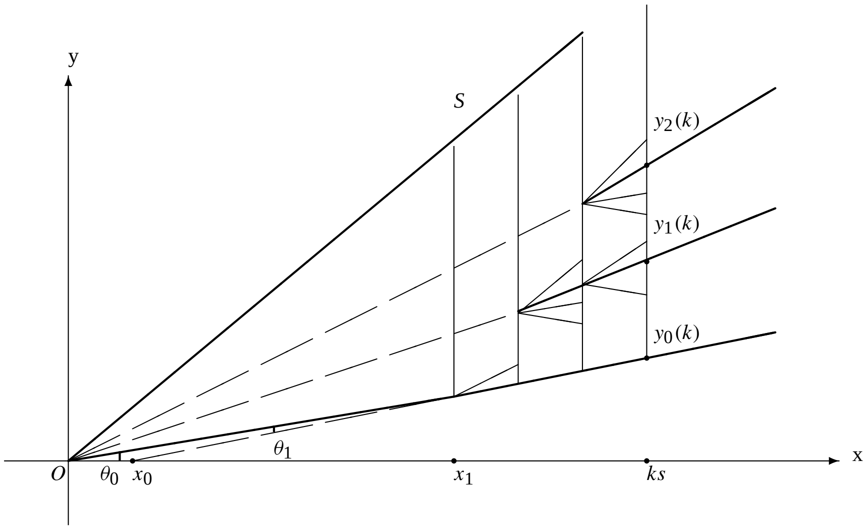
$$R_2(\omega_-) = \{\omega : P = P_-, \ \ \frac{v}{u} = \frac{v_-}{u_-}\} \tag{3.8}$$

and that $R_i, i = 1, 3,$ are the integral curve of

$$\begin{cases} \frac{du}{dv} = -\lambda_i, \\ \frac{dP}{dv} = \rho(\lambda_i u - v). \end{cases}$$

The Riemann problem for (3.1)–(3.4) with initial data a single jump can be solved by the elementary waves taking values along the wave curves $T_i, \ i = 1, 2, 3,$ just described, Lax [6].

To construct approximate solutions to (1.4)–(1.7), we adopt a generalization of the Glimm scheme [4]. The obstacle is a perturbation of the infinite cone $y/x = \theta_0$. In the following figure, we exhibit the construction of the numerical grids to be described below for the simplified case when the cone is perturbed only at one location $x = x_1$ and with the change of angle θ_1.



We now define the difference scheme. Choose the grid size $s = \Delta x$ for the variable x. Suppose that the obstacle is unperturbed before $x = N_0 \Delta x$. For $0 \leq k \leq N_0$, the grid points are the intersection of $x = k\Delta x$ with the self-similar rays centered at $(0, 0)$,

$$y = x\frac{1}{1/\theta_0 + h\Delta\sigma}, \ h = 0, -1, -2, \dots .$$

In this region, the approximate solution is the unperturbed conical flow centered at $(0, 0)$. We choose the initial numerical grid on $x = N_0 s$ to satisfy the usual C-F-L condition. The approximate solutions $\omega(x, y) = \omega_\Delta(x, y)$ and the numerical grids are defined inductively in k, $x = ks$, $k = N_0, N_0 + 1, \dots$, as follows:

Choose an equidistributed sequence $a_1, a_2, \dots$ in the unit interval $(0, 1)$. Approximate the obstacle by piecewise linear cones with changes in angle at $x = ks$, $k = N_0, N_0+1, \dots$. Suppose that the approximate solution and the grid points have been defined for $x \leq ks$. Let the grid points on $x = ks$ be denoted by $y = y_0(k) < y_1(k) < \dots$, with $y = y_0(k)$ the location of the obstacle. The approximate solution $\omega(ls + 0, y)$ is a piecewise smooth solution of the self-similar system (1.10) and (1.11) on each vertical grid line $x = ls + 0$. As part of the induction hypothesis, we assume that the center $(x_0, 0) = (x_0(l, h + 1/2), 0)$ of the self-similar variable $\sigma = (x - x_0)/y$ for $y_h(l) < y < y_{h+1}(l)$ have also been defined for $l \leq k$. We now define the approximate solution for the region $ks < x \leq (k + 1)s$. For $y_h(k) < y < y_{h+1}(k)$, $\omega(ks + 0, y)$ is the solution of (1.10) and (1.11) with

$$\begin{aligned}\omega(ks + 0, y_h + a_k(y_{h+1}(k) - y_h(k))) =& \omega_\Delta(ks - 0, y_h(k) + a_k(y_{h+1}(k) - y_h(k))),\\ & h = 0, 1, \dots, . \qquad (3.9)\end{aligned}$$

As noted before, the initial value above does not uniquely determine the solution of the non-autonomous system (1.10) and (1.11) and the center of the self-similar variable needs to be specified. We specify the center to be $(x_0, 0) = (x_0(k, h + 1/2), 0)$, which has been defined inductively, and this yields the self-similar variable $\sigma = (x - x_0)/y$. The discontinuities at the grid points $(k\Delta x, y_h)$, $h = 1, 2, \dots$ are resolved by solving the Riemann problem for (3.1)–(3.4) with initial data: $(\omega(ks + 0, y_h(k) - 0), \omega(ks + 0, y_h(k) + 0)$. The solution of the Riemann problem is a function of $(x - ks)/(y - y_h(k))$ and consists of rarefaction waves, shock waves or contact discontinuities.

The approximate solution $\omega(x, y),\ ks < x \le (k+1)s,\ y_{h-1/2}(k) < y < y_{h+1/2}(k)$ is defined according to (1.10) and (1.11) along the ray $(y - y_h(k))/(x - ks) = \xi$ with the initial value at $x = ks + 0$ given by the solution of the above Riemann problem. As before, we need to specify the center $x_0(\xi)$ of the self-similar variable. We do it according to the principle that the center propagates away from the obstacle and toward the leading shock. Let the upper edge of the 3-wave of the solution of the Riemann problem at $(ks, y_h(k))$ be $(y - y_h(k))/(x - ks) = a$. Since the 3-wave moves toward the leading shock, we set the center to be $x_0(\xi) = x_0(k, h - 1/2)$ (or $x_0(\xi) = x_0(k, h + 1/2)$) for the region below (or above) the upper edge of the 3-wave, $\xi < a$ (or $\xi > a$.)

The numerical grids on $x = (k+1)s$ are defined to be on the self-similar rays through the grids on $x = ks$. The new center on $x = (k+1)s$ inherits those $x_0(\xi)$ on $x = ks+0$ through the random choice (3.9). The choice of the centers is natural. The choice of the grid points is motivated by the study of moving sources in that the grids move along the constancy of the underlining self-similar flow.

On the obstacle, $(x - ks)/(y - y_0(k)) = \sigma_0(k)$, a 3-shock (or 3-rarefaction) wave emerges when the obstacle changes angle toward (or away from) the flow. For this, we solve the initial-boundary Riemann problem for (3.1)–(3.4) with initial data:

$$\omega_\Delta(ks + 0, \sigma) = \omega_\Delta(ks, \sigma_0(k)),\quad \sigma < \sigma_0(k)$$

and with a boundary condition posed at $\sigma = \sigma_0(k)$:

$$\frac{u}{v} = \sigma_0(k).$$

The approximate solution is extended to $(x, y),\ ks < x \le (k+1)s,\ y_0(k) + (x - ks)/\sigma_0(k) < y < y_0(k) + 1/2(y_1(k) - y_0(k))$ as before with center $x_0(k, 0) \equiv ks - \sigma_0(k)y_0(k)$.

The leading strong shock cone next to the uniform upstream flow is traced continuously, instead of the above random scheme. Suppose that the approximate solution is constructed for $0 \le x < ks$, $k \ge N_0$. Let $(x, y_f(x))$ denote the locus of the front of the 3-shock cone S. Suppose that $y_{j_f}(k) < y_f(ks) < y_{j_f+1}(k)$. We call the interval $y_{j_f-1}(k) < y < y_{j_f+1}(k)$ the front region at $x = ks$. Inside the front region, we first solve the self-similar solution to (1.10) and (1.11) with the initial value:

$$\begin{aligned}\omega(ks + 0, y_{j_f-1}(k) + a_k(y_{j_f}(k) - y_{j_f-1}(k))) = \omega_\Delta(ks - 0,\ & y_{j_f-1}(k) \\ & + a_k(y_{j_f}(k) - y_{j_f-1}(k))),\end{aligned}$$

and with the same center as the initial value. Denote the solution by $\omega(y)$. Next we solve the Riemann problem for (3.1)–(3.4) so that

$$\omega(ks, y) = \begin{cases} (\rho_0, u_0, v_0) = \omega_+, & \text{for } y > y_f(x) \\ \omega(y_f(x) - 0), & \text{for } y_f(x) > y > y_{j_f-1}(k). \end{cases}$$

The solution $\omega(x, y)$ thus contains a relatively strong 3-shock wave, (ω_+, ω_-), with speed **s**. Solve again Eqs. (1.10) and (1.11) in the interval $y_{j_f-1}(k) < y < y_f(x)$ with the initial value

$$\omega(y_f(x) - 0) = \omega_-.$$

Denote the solution by $\omega_-(y)$. Now, we can define the approximate solution in the front region as follows:

$$\omega_\Delta(x,y) = \begin{cases} \omega_+, & \text{for } y > y_f(x) \\ \omega_-(y), & \text{for } y_f(x) > y > y_{j_f-1}(k), \end{cases}$$
$$y_f(x) = \mathbf{s}(x - ks) + y_f(ks)$$

for $ks \le x < (k+1)s$. And the discontinuity at $y = y_{j_f-1}(k)$ is resolved by the same construction as before.

4. Local Interaction Estimates

We first study the interaction among the weak waves between the shock cone S and the obstacle cone. In order to obtain the desired estimates, we consider space-like curves, which are piecewise linear curves consisting of line segments joining a_{kh} to $a_{k+1,h+1}$ or to $a_{k-1,h+1}$, where $a_{kh} = (ks, y_h(k) + a_k(y_{h+1}(k) - y_h(k))$. The shock cone in the first quadrant is covered by "diamonds," the corners of which are the mesh points, a_{kh}. Let Δ denote a diamond centered at $(ks, y_h(k))$. We consider the following case. Suppose that the waves entering Δ are denoted by α and β, which are centered at $((k-1)s, y_{h-1}(k-1))$ and $((k-1)s, y_h(k-1))$ respectively. Let δ denote the set of waves issuing from $(ks, y_h(k))$ and δ_i the strength of the i-wave in δ. Let $\omega_1(\sigma)$, $\omega_2(\bar\sigma)$ and $\omega_3(\bar\sigma)$ represent the self-similar solutions centered at O_1, O_2 and O_2 respectively such that

$$\begin{aligned} \alpha &= (\omega_2(\bar\sigma_1), \omega_1(\sigma_1)), \\ \beta &= (\omega_3(\bar\sigma_2), \omega_2(\bar\sigma_2)), \\ \delta &= (\omega_3(\bar\sigma_2), \omega_1(\sigma_2)). \end{aligned}$$

σ and $\bar\sigma$ are the self-similar variables with the corresponding centers O_1 and O_2 respectively. To measure the potential nonlinear wave interaction, we use the following notations:

$$\begin{aligned} Q^0(\Delta) &= Q^0(\alpha,\beta) \\ &\equiv \sum \left\{ |\alpha_i||\beta_j| : \alpha_i \text{ and } \beta_j \text{ are approaching} \right\}, \\ Q^1(\Delta) &\equiv |\alpha_1|\Delta\sigma + |\alpha_3|\Delta\sigma, \\ Q^2(\Delta) &\equiv |\alpha_2|\Delta\sigma, \\ Q^c(\Delta) &\equiv \begin{cases} x_0\Delta\sigma, & \text{if } O_1 \ne O_2, \\ 0, & \text{if } O_1 = O_2, \end{cases} \\ Q(\Delta) &\equiv Q^0(\Delta) + Q^1(\Delta) + Q^2(\Delta) + Q^c(\Delta), \end{aligned}$$

where $\Delta\sigma = |\sigma_2 - \sigma_1|$ and x_0 denotes the change of the location for different centers. Here, Q^0 measures the wave interaction between elementary waves, Q^1 and Q^2 measure the wave interaction between self-similar solutions and elementary waves, and Q^c measures the effect of the change of centers for self-similar solutions. Our interaction estimate is as follows:

Lemma 4.1. *For some constant $O(1)$ depending only on system (1.4)–(1.7),* $1 \le i \le 3$,

$$\delta_i = \alpha_i + \beta_i + O(1)Q(\Delta). \tag{4.1}$$

Proof. By the interaction estimates of elementary waves for conservation laws [12],

$$\begin{aligned}\delta_i &= (\omega_3(\bar{\sigma_2}), \omega_1(\sigma_2))_i \\ &= (\omega_3(\bar{\sigma_2}), \omega_2(\bar{\sigma_2}))_i + (\omega_2(\bar{\sigma_2}), \omega_1(\sigma_2))_i \\ &\quad + O(1)Q^0((\omega_3(\bar{\sigma_2}), \omega_2(\bar{\sigma_2})), (\omega_2(\bar{\sigma_2}), \omega_1(\sigma_2))).\end{aligned} \tag{4.2}$$

It follows from the elementary theory of ordinary differential equations that

$$\omega_2(\bar{\sigma_2}) - \omega_1(\sigma_2) = \omega_2(\bar{\sigma_1}) - \omega_1(\sigma_1) + O(1)(|\alpha| + x_0)(|\sigma_2 - \sigma_1| + |\bar{\sigma_2} - \bar{\sigma_1}|). \tag{4.3}$$

Note that $|\bar{\sigma_2} - \bar{\sigma_1}|$ is equivalent to $|\sigma_2 - \sigma_1|$ when x_0 is sufficiently small. Since the solution of the Riemann problem depends continuously on its end states, (4.2) and (4.3) yield

$$\delta_i = \alpha_i + \beta_i + O(1)Q(\Delta).$$

This completes the proof. □

Remark 4.1. For the other cases, such as when α issues from $((k-1)s, y_{h+1}(k-1))$ or when ω_2 and ω_3 have different centers, the Q^j's can be defined with the same meaning and the interaction estimate (4.1) holds by the same argument.

Remark 4.2. When $h = 0$, that is, Δ covers a part of the boundary of the obstacle cone, we need to solve the boundary Riemann problem. Let α and β denote the waves issuing from $((k-1)s, y_1(k-1))$ and $((k-1)s, y_0(k-1))$ respectively. By an analogous argument, we have the interaction estimate:

$$\delta = \delta_3 = \beta + C_0\alpha + O(1)Q(\Delta), \tag{4.4}$$

where C_0 depends only on system (3.1)–(3.4).

As for the case involving the relatively strong 3-shock wave S, the estimate is similar to the above lemma except that instead of advancing one diamond, we need to advance three diamonds in the front region simultaneously. We still denote these three diamonds by Δ. Let $\Delta_{k,h}$ represent the diamond whose center is $(ks, y_h(k))$. Assume $a_{k+1} \in (0, \frac{1}{2})$. Then,

$$\Delta = \Delta_{k+1,j_f-1} \cup \Delta_{k+1,j_f} \cup \Delta_{k+1,j_f+1}.$$

The case for $a_{k+1} \in [\frac{1}{2}, 1)$ can be treated by the same analysis. Let β_k stand for the relatively strong 3-shock wave issuing from $(ks, y_f(ks))$. We denote by α the set of waves issuing from $(ks, y_{j_f-1}(k))$. The waves in α entering Δ_{k+1,j_f} are denoted by α^l and α^r are the waves entering Δ_{k+1,j_f+1}. Let γ be the set of waves issuing from $(ks, y_{j_f-2}(k))$ and entering $\Delta_{k+1,j_f(k)+1}$. Set $\omega_1(y)$ (or $\omega_1(\bar{\sigma})$) and $\omega_2(y)$ (or $\omega_2(\sigma)$) to be the self-similar solutions such that β_k connects $\omega_0 = (\rho_0, u_0, v_0)$ and $\omega_1(y)$, and α^l connects $\omega_1(y)$ and $\omega_2(y)$ at $x = ks$. Let β_{k+1} denote the strong 3-shock issuing from

$((k+1)s, y_f((k+1)s))$. δ denotes the wave issuing from $((k+1)s, y_{j_f-1}(k+1))$. In this case, we set

$$
\begin{aligned}
&Q^0(\Delta) = Q^0(\beta_k, \alpha^l) + Q^0(\alpha^r, \gamma),\\
&Q^1(\Delta) = |\alpha^l|\Delta\sigma_\alpha + |\gamma_1|\Delta\sigma_\gamma + |\gamma_3|\Delta\sigma_\gamma,\\
&Q^2(\Delta) = |\gamma_2|\Delta\sigma_\gamma,\\
&Q^c(\Delta) = \begin{cases} x_0\Delta\sigma_\alpha, & \text{if } \omega_1(y) \text{ and } \omega_2(y) \text{ have different centers,}\\ 0, & \text{if } \omega_1(y) \text{ and } \omega_2(y) \text{ have the same center.}\end{cases}
\end{aligned}
$$

Here, $\Delta\sigma$ simply means the change of the self-similar variable for the corresponding wave as it propagates through the self-similar solution. In this case, $\Delta\sigma_\alpha = |\sigma(y_{j_f}(k+1)) - \sigma(y_{j_f-1}(k))|$ and $\Delta\sigma_\gamma = |\sigma(y_{j_f-1}(k+1)) - \sigma(y_{j_f-2}(k))|$. $\Delta\sigma_{\beta_k} = |\sigma_f(k+1) - \sigma_f(k)|$, where $\sigma_f(k)$ represents the value of the self-similar variable σ for the shock β_k . In the following, $O(1)$ always represents a constant depending only on system (1.4)–(1.7).

Lemma 4.2. *Suppose that β_k is sufficiently small. Then there exists a small constant $c_0 = O(1)|\beta_k|$ such that*

$$
\begin{aligned}
\beta_{k+1} &= \beta_k + \alpha_1^l + O(1)Q^0(\beta_k, \alpha^l)\\
&\quad + O(1)\Delta\sigma_{\beta_k} + O(1)|\alpha^l|\Delta\sigma_\alpha + O(1)Q^c(\Delta),\\
\delta_j &= \alpha_j^r + \gamma_j + O(1)\left\{Q^0(\beta_k, \alpha^l) + Q^0(\alpha^r, \gamma)\right\} + O(1)c_0\Delta\sigma_{\beta_k}\\
&\quad + O(1)|\alpha^l|\Delta\sigma_\alpha + O(1)|\gamma|\Delta\sigma_\gamma + O(1)Q^c(\Delta), \quad \textit{for} \quad 1 \le j \le 3.
\end{aligned}
$$

Proof. Owing to the interaction estimates of the elementary waves for conservation laws, we have

$$
\begin{aligned}
&(\omega_0, \omega_2(\sigma_f(k+1)))_j = (\omega_0, \omega_1(\sigma_f(k+1))_j + (w_1(\sigma_f(k+1)), w_2(\sigma_f(k+1)))_j\\
&\quad + O(1)Q^0((\omega_0, \omega_1(\sigma_f(k+1)), (w_1(\sigma_f(k+1)), w_2(\sigma_f(k+1))). \qquad (4.5)
\end{aligned}
$$

And by (2.7), there exists a small constant $c_0 = O(1)|\beta_k|$ such that

$$
(\omega_0, \omega_1(\sigma_f(k+1)))_j = (\omega_0, \omega_1(\sigma_f(k)))_j + \begin{cases} O(1)c_0\Delta\sigma_{\beta_k}, & j = 1, 2,\\ O(1)\Delta\sigma_{\beta_k}, & j = 3,\end{cases} \qquad (4.6)
$$

when β_k is sufficiently small. Also,

$$
\begin{aligned}
(w_1(\sigma_f(k+1)), (w_2(\sigma_f(k+1)))_j &= (\omega_1(y_{j_f-1}(k)), \omega_2(y_{j_f-1}(k)))_j\\
&\quad + O(1)|\alpha^l|\Delta\sigma_\alpha + O(1)Q^c(\Delta). \qquad (4.7)
\end{aligned}
$$

Thus, (4.5)–(4.7) imply that

$$
\begin{aligned}
\beta_{k+1} &= (\omega_0, \omega_2(\sigma_f(k+1)))_3 = (\omega_0, \omega_*)\\
&= \beta_k + \alpha_1^l + O(1)Q^0(\beta_k, \alpha^l)\\
&\quad + O(1)\Delta\sigma_{\beta_k} + O(1)|\alpha^l|\Delta\sigma_\alpha + O(1)Q^c(\Delta). \qquad (4.8)
\end{aligned}
$$

Denote the end states of δ by (δ_-, δ_+). To estimate the strength of δ, we first apply Lemma 4.1 to obtain

$$(\omega_2(y_{j_f-1}(k)), \delta_+)_j = \alpha_j^r + \gamma_j + O(1)Q^0(\alpha^r, \gamma) + O(1)|\gamma|\Delta\sigma_\gamma. \tag{4.9}$$

By the elementary theory of ordinary differential equations, we have

$$(\delta_-, \omega_2(y_{j_f-1}(k)))_j = (\omega_*, \omega_2(\sigma_f(k+1)))_j + O(1)|\omega_* - \omega_2(\sigma_f(k+1))|\Delta\sigma_\alpha. \tag{4.10}$$

Hence, (4.5)–(4.10) yield

$$\begin{aligned}\delta_j &= (\delta_-, \delta_+)_j \\ &= \alpha_j^r + \gamma_j + O(1)\left\{Q^0(\beta, \alpha^l) + Q^0(\alpha^r, \gamma)\right\} \\ &\quad + O(1)c_0\Delta\sigma_{\beta_k} + O(1)|\alpha^l|\Delta\sigma_\alpha + O(1)|\gamma|\Delta\sigma_\gamma + O(1)Q^c(\Delta).\end{aligned}$$

This completes the proof. □

We now establish the basic estimates on the change of speed of 3-waves and 2-waves. As 3-waves (or 2-waves) propagate along self-similar solutions, the characteristic speed λ_3 (or λ_2) is monotonely increasing with respect to σ.

Lemma 4.3. *Suppose that $\omega(\sigma) = (\rho(\sigma), u(\sigma), v(\sigma))$ is a self-similar solution to (1.10) and (1.11). Then,*

$$\begin{aligned}(i)\quad & \frac{d}{d\sigma}\lambda_2(\sigma) > 0, \\ (ii)\quad & \frac{d}{d\sigma}\lambda_3(\sigma) > 0.\end{aligned}$$

Proof. (i) By (1.10),

$$\frac{d}{d\sigma}\lambda_2(\sigma) = \frac{d}{d\sigma}\left\{\frac{v(\sigma)}{u(\sigma)}\right\} = \frac{v_\sigma u - v u_\sigma}{u^2} = \frac{-(\sigma u + v)u_\sigma}{u^2} > 0.$$

(ii) To compute $\frac{d}{d\sigma}\lambda_3(\sigma)$, we need to know $\frac{d\rho}{d\sigma}$ and $\frac{dc}{d\sigma}$. Applying Bernoulli's law, we obtain

$$udu + vdv = -c^2 d\rho/\rho.$$

It thus follows from (1.10) and (i) that

$$\begin{aligned}\frac{d\rho}{d\sigma} &= (uu_\sigma + vv_\sigma)\frac{-\rho}{c^2} \\ &= (u - \sigma v)u_\sigma\frac{-\rho}{c^2} > 0.\end{aligned}$$

Hence,

$$\frac{dc}{d\sigma} = \frac{1}{2c}P''(\rho)\frac{d\rho}{d\sigma} > 0.$$

Since λ_3 satisfies

$$(v - \lambda_3 u)^2 - c^2(1 + \lambda_3^2) = 0,$$

differentiating this equation with respect to σ yields

$$(c^2\lambda_3 + u(v - \lambda_3 u))\frac{d}{d\sigma}\lambda_3(\sigma) = (v - \lambda_3 u)(v_\sigma - \lambda_3 u_\sigma) - cc_\sigma(1 + \lambda_3^2). \tag{4.11}$$

Substituting λ_3, we have

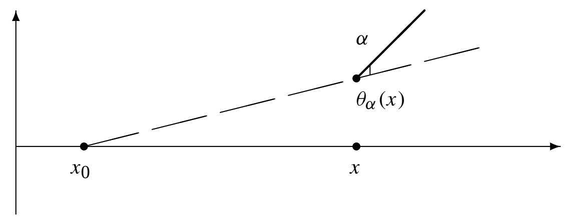
$$\begin{aligned} c^2\lambda_3 + u(v - \lambda_3 u) &= (c^2 - u^2)\lambda_3 + uv \\ &= -uv - c\sqrt{q^2 - c^2} + uv \\ &= -c\sqrt{q^2 - c^2} < 0. \end{aligned}$$

Applying (1.10) again, it is easy to check that the RHD of (4.11) is negative. Hence, it follows that

$$\frac{d}{d\sigma}\lambda_3(\sigma) > 0.$$

This completes the proof. □

We now turn to the interaction between self-similar solutions and elementary waves. To quantitatively measure how the elementary waves weave through self-similar solutions, we estimate the change of the angle between the elementary wave and the ray through the center of the wave itself, as shown in the following figure.



$\theta_\alpha(x)$ denotes the angle associated with the wave α issuing from x. The self-similar variable σ is employed in place of y to describe the coordinate in the (x, y)-plane. The following lemmas show that $\theta(x)$ is decreasing with respect to x for 2-waves and the relatively strong 3-shock S.

Lemma 4.4. *Suppose that $S = (\omega_0, \omega_1)$ at (x, σ_1). At the next step, $S = (\omega_0, \omega_2)$ at $(x + \Delta x, \sigma_2)$ by the construction described in Sect. 3. Then we have*

$$\theta_S(x) - \theta_S(x + \Delta x) = |O(1)||\sigma_1 - \sigma_2|.$$

Proof. Let $\mathbf{s}_i$ denote the shock speed $\mathbf{s}(\omega_0, \omega_i)$, $i = 1, 2$. Assume that $\mathbf{s}_i > \frac{1}{\sigma_1}$. The other case can be proved by analogous arguments.

Let $\omega(\sigma)$ denote the self-similar solution with the initial data

$$\omega(\sigma_1) = \omega_1.$$

Due to the construction of approximate solutions in Sect. 3, we need to solve the Riemann problem $(\omega_0, \omega(\sigma_2))$. Hence,

$$(\omega_0, \omega_2) = (\omega_0, \omega(\sigma_2))_3.$$

As σ_1 decreases to σ_2, $\omega(\sigma)$ moves along the integral curve $(u, v(u))$ to (1.10) and (1.11) and below the shock curve $S_3(\omega_0)$ on the (u, v)-plane. By (1.10), $v_u < 0$. This property together with the fact that T_1 and T_2 have positive slopes on the (u, v)-plane implies that $\mathbf{s}_2 < \mathbf{s}_1$ and thus

$$\mathbf{s}_1 - \mathbf{s}_2 = |O(1)||\sigma_1 - \sigma_2|.$$

Therefore, we have

$$\theta_S(x) - \theta_S(x + \Delta x) = |O(1)||\sigma_1 - \sigma_2|. \qquad \square$$

Lemma 4.5. *Suppose that* $\alpha = (\omega_l, \omega_r)$ *is a contact discontinuity at* (x, σ_1). *At the next step,* $\alpha = (\hat{\omega_l}, \hat{\omega_r})$ *at* $(x + \Delta x, \sigma_2)$ *by the construction described in Sect. 3. Then we have*

$$\theta_\alpha(x) - \theta_\alpha(x + \Delta x) = |O(1)||\sigma_1 - \sigma_2|.$$

Proof. Set $\mathbf{s}_1 = \mathbf{s}(\omega_l, \omega_r)$ and $\mathbf{s}_2 = \mathbf{s}(\hat{\omega_l}, \hat{\omega_r})$. Assume that the wave speed $\mathbf{s}_1 > \frac{1}{\sigma_1}$ and $\sigma_1 > \sigma_2$. The other cases can be proved by analogous arguments. Let $\omega_l(\sigma)$ and $\omega_r(\sigma)$ denote the self-similar solutions to (1.10) and (1.11) with the initial data

$$\omega_l(\sigma_1) = \omega_l,$$
$$\omega_r(\sigma_1) = \omega_r,$$

respectively. Due to the construction of approximate solutions, we obtain

$$(\hat{\omega_l}, \hat{\omega_r}) = (\omega_l(\sigma_2), \omega_r(\sigma_2))_2.$$

It follows from Lemma 4.3 that

$$\lambda_2(\omega_l(\sigma_2)) < \lambda_2(\omega_l), \tag{4.12}$$
$$\lambda_2(\omega_r(\sigma_2)) < \lambda_2(\omega_r). \tag{4.13}$$

And by Lemma 4.1, we have

$$(\omega_l(\sigma_2), \omega_r(\sigma_2))_j = O(1)|\omega_l - \omega_r||\sigma_1 - \sigma_2| \tag{4.14}$$

for $j = 1, 3$. Hence, (4.12)–(4.14) yield that $\mathbf{s}_2 < \mathbf{s}_1$ and

$$\mathbf{s}_2 - \mathbf{s}_1 = O(1)|\sigma_1 - \sigma_2|.$$

It thus implies that

$$\theta_\alpha(x) - \theta_\alpha(x + \Delta x) = |O(1)||\sigma_1 - \sigma_2|.$$

This completes the proof. $\square$

5. Global Existence

In this section, we adopt the difference scheme described in Sect. 3 to prove the global existence of the solution to (1.4)–(1.7). The obstacle is approximated by piecewise linear cones with the change of angle θ_i , $i = 1, \cdots, n$, at $x = N_0 s, \cdots, (N_0 + n - 1)s$ respectively, and the corresponding centers for these linear cones are $x_0^1, x_0^2, \cdots, x_0^n$ respectively. For convenience, we prove the simplified case when the cone is perturbed only at one location $x = x_1$ and after the perturbation, the obstacle is the infinite cone $\frac{x - x_0}{y} = \sigma_0$ with its corresponding center located at $x = x_0^1 = x_0$; hence, the self-similar variable σ equals $\frac{x - x_0}{y}$. Nevertheless, the functionals to be constructed below are also true for general situations.

The proof requires estimates on the total variations of the approximate solutions $\omega_\Delta(x, y)$. Our strategy is to use induction on certain nonlinear functionals constructed to detect global wave interactions. Once this uniform bound is established, with the aid of Helly's theorem, we can extract a convergent subsequence of $\omega_\Delta(x, y)$ in $L^1_{loc}(R^2)$, and by the consistency theorem (Liu [10]), this subsequence converges to a weak solution $\omega(x, y)$ to the system (1.4)–(1.7). Let J be a space-like curve. To establish the uniform bound, we define a nonlinear functional $F(J)$ as follows:

$$
\begin{aligned}
F(J) &\equiv L(J) + KQ(J),\\
L(J) &\equiv L_0(J) + L_1(J),\\
L_0(J) &\equiv \sum \{c_\alpha|\alpha| : \alpha \text{ is the strength of any elementary waves crossing}\\
&\qquad J \text{ and } \alpha \neq S\},\\
L_1(J) &\equiv \theta_S(J) + \sum\{\theta_\alpha : \alpha \text{ is a contact discontinuity crossing } J\},\\
Q(J) &\equiv Q_0(J) + Q_1(J) + Q_3(J) + Q_c(J),\\
Q_0(J) &\equiv \sum \{|\alpha\beta| : \alpha \text{ and } \beta \text{ are strengths of elementary waves which are}\\
&\qquad \text{approaching, and cross } J\},\\
Q_1(J) &\equiv \sum \{|\alpha|(\sigma_0 - \sigma_\alpha) : \alpha \text{ is a 1-wave crossing } J\},\\
Q_3(J) &\equiv \sum \{|\alpha|(\sigma_\alpha - \sigma_\epsilon) : \alpha \text{ is a 3-wave crossing } J \text{ and } \alpha \neq S\},\\
Q_c(J) &\equiv \sum_{i=1}^{n} Q_c^i(J),\\
Q_c^i(J) &\equiv (x_0^i - x_0^{i-1})(\sigma_c^i(J) - \sigma_\epsilon), \quad (x_0^0 = 0).
\end{aligned}
$$

Here

$$
c_\alpha = \begin{cases} C_0, & \text{when } \alpha \text{ is a 1-wave or 2-wave.}\\ 1, & \text{when } \alpha \text{ is a 3-wave.} \end{cases}
$$

C_0 is the same constant as in Remark 4.2, which depends only on system (1.4)–(1.7). σ_α denotes the σ-coordinate of the center for the wave α. $\sigma_c^i(J)$ is the σ-coordinate of the grid point where the center of the self-similar solutions passing through J changes from x_0^{i-1} to x_0^i. If the centers do not change anymore, $Q_c(J) \equiv 0$. And $\sigma_S(J)$ is the

σ-coordinate of the 3-shock S when S crosses J. $\sigma_\epsilon \equiv \sigma_S(0) - \epsilon$, for some suitably chosen small constant ϵ. And K is some large number to be determined later.

The terms Q's are defined to detect the potential amount of wave interactions in the solution. Since 3-waves and 1-waves between S and the obstacle move upwards and downwards respectively with respect to the σ coordinate, $Q_3(J)$ and $Q_1(J)$ are so defined according to the domain of influence. $Q_0(J)$ is the amount of the usual waves interactions between elementary waves. And $Q_c(J)$ is defined to measure the effect of the change of centers for self-similar solutions, which also reflects the fact that this effect propagates upwards. As for the 2-waves nearby the obstacle boundary and the relatively strong shock S, we do not know a priori how they move ahead. Consequently, we cannot foresee their potential wave interactions. However, the local analysis gives us a decreasing quantity θ, which constitutes $L_1(J)$. We will show that the decrease in $L_1(J)$ is sufficient to dominate the increase in the remaining parts of $F(J)$.

We now give the global interaction estimates. Let 0 stand for the space-like curve in the strip $N_0 s \le x \le (N_0+1)s$. Λ represents the region between 0 and J. And $Q(\Lambda)$ is the sum over all $Q(\Delta)$, Δ any diamond in Λ.

Lemma 5.1. *Suppose that $L(0)$, $\sigma_0 - \sigma_\epsilon$ and $\sum_{i=1}^n \theta_i$ are sufficiently small. For sufficiently large K, we have*

$$F(J) \le F(0) - \frac{1}{2}Q(\Lambda) + C_1 \sum_{i=1}^n \theta_i - c_1 \left(\sum_k \Delta\sigma_S(J_k) + \sum_{\alpha_2} \Delta\sigma_{\alpha_2} \right), \quad (5.1)$$

$$Q(J) \le Q(0) - \frac{1}{2}Q(\Lambda) + Q^2(\Lambda) + \sum_{i=1}^n \theta_i + \frac{1}{2}\sum_k \Delta\sigma_S(J_k), \quad (5.2)$$

where C_1 and c_1 are positive constants depending only on system (1.4) -(1.7), and J_k's are all the space-like curves between 0 and J. $\sum_k \Delta\sigma_S(J_k)$ is the sum taken over the change of $\sigma_S(J_k)$. And $\sum_{\alpha_2} \Delta\sigma_{\alpha_2}$ is the sum over the change of σ for all the contact discontinuities in Λ.

Proof. We choose

$$\epsilon \equiv \left(F(0) + C_1 \sum_{i=1}^n \theta_i \right) c_1^{-1}.$$

K, C_1 and c_1 will be determined later. We will prove by induction. For $J = 0$, we can choose $L(0)$ and $Q(0)$ as small as needed. Suppose that (5.1) and (5.2) have been shown for $J = J_1$. It thus follows from (5.1) that $\sigma_S(J_1) > \sigma_\epsilon$. Let J_2 be an immediate successor and Δ denote the diamond between J_1 and J_2. To show that (5.1) and (5.2) hold for $J = J_2$, we divide the proof into three cases:

Case 1. Δ is between the shock cone S and the obstacle cone.

Let us consider the case when Δ is under the same setting as Lemma 4.1. The other cases can be proved similarly. With the help of Lemma 4.1 and 4.5, we obtain

$$L_0(J_2) - L_0(J_1) = O(1)Q(\Delta),$$
$$L_1(J_2) - L_1(J_1) = -|O(1)|\Delta\sigma + O(1)Q(\Delta),$$

where the term $-|O(1)|\Delta\sigma$ is due to the change of the angle θ_{α_2} if the contact discontinuity $\alpha_2 \neq 0$ in Lemma 4.1,

$$\begin{aligned} Q_0(J_2) - Q_0(J_1) &\le O(1)L_0(J_1)Q(\Delta) - Q^0(\Delta),\\ (Q_1+Q_3)(J_2) - (Q_1+Q_3)(J_1) &\le O(1)Q(\Delta)(\sigma_0-\sigma_\epsilon) - Q^1(\Delta),\\ Q_c(J_2) - Q_c(J_1) &= -Q^c(\Delta). \end{aligned}$$

It follows from the above inequalities that

$$Q(J_2) - Q(J_1) \le O(1)\left(L_0(J_1) + (\sigma_0-\sigma_\epsilon)\right)Q(\Delta) - \Big(Q(\Delta) - Q^2(\Delta)\Big),$$

and thus

$$Q(J_2) - Q(J_1) \le -\frac{3}{4}Q(\Delta) + Q^2(\Delta), \tag{5.3}$$

provided that $L(J_1)$ and $\sigma_0 - \sigma_\epsilon$ are sufficiently small. Therefore,

$$F(J_2) - F(J_1) \le (O(1) - \frac{K}{2})Q(\Delta) + KQ^2(\Delta) - |O(1)|\Delta\sigma. \tag{5.4}$$

Note that $Q^2(\Delta)$ is a quadratic term and $Q^2(\Delta) \le L_0(J_1)\Delta\sigma$. Hence, when $F(J_1)$ is sufficiently small, by choosing suitably large constant K, we have

$$F(J_2) - F(J_1) \le -\frac{1}{2}Q(\Delta) - c_1\Delta\sigma$$

for some positive constant c_1. By the induction hypothesis, it thus follows that

$$\begin{aligned} F(J_2) &\le F(0) - \frac{1}{2}Q(\Lambda_2) + C_1\sum_{i=1}^{n}\theta_i - c_1\left(\sum_k \Delta\sigma_S(J_k) + \sum_{\alpha_2}\Delta\sigma_{\alpha_2}\right),\\ Q(J_2) &\le Q(0) - \frac{1}{2}Q(\Lambda_2) + Q^2(\Lambda_2) + \sum_{i=1}^{n}\theta_i + \frac{1}{2}\sum_k \Delta\sigma_S(J_k), \end{aligned}$$

where Λ_2 is the region between 0 and J_2. Thus, (5.1) and (5.2) hold for $J = J_2$.

Case 2. Δ covers a part of the obstacle boundary.

Let us consider the case when Δ is under the same setting as Remark 4.2. The other cases can be proved similarly. Using Remark 4.2 and Lemma 4.5, we have

$$\begin{aligned} L_0(J_2) - L_0(J_1) &= O(1)Q(\Delta),\\ L_1(J_2) - L_1(J_1) &= -|O(1)|\Delta\sigma + O(1)Q(\Delta), \end{aligned}$$

where the term $-|O(1)|\Delta\sigma$ is due to the change of the angle θ_{α_2} if the contact discontinuity $\alpha_2 \neq 0$ in Remark 4.1. Also,

$$\begin{aligned} Q_0(J_2) - Q_0(J_1) &\le O(1)L_0(J_1)Q(\Delta) - Q^0(\Delta),\\ (Q_1+Q_3)(J_2) - (Q_1+Q_3)(J_1) &\le O(1)(|\alpha| + Q(\Delta))(\sigma_0-\sigma_\epsilon) - Q^1(\Delta),\\ Q_c(J_2) - Q_c(J_1) &= -Q^c(\Delta). \end{aligned}$$

Thus,

$$Q(J_2) - Q(J_1) \le O(1)\left(L_0(J_1) + (\sigma_0 - \sigma_\epsilon)\right) Q(\Delta) - \left(Q(\Delta) - Q^2(\Delta)\right) + O(1)|\alpha||\sigma_0 - \sigma_\epsilon|.$$

When $L(J_1)$ and $\sigma_0 - \sigma_\epsilon$ are sufficiently small, the last inequality yields

$$Q(J_2) - Q(J_1) \le -\frac{3}{4} Q(\Delta) + Q^2(\Delta) + O(1)|\alpha||\sigma_0 - \sigma_\epsilon|. \tag{5.5}$$

Therefore,

$$F(J_2) - F(J_1) \le (O(1) - \frac{K}{2}) Q(\Delta) + K Q^2(\Delta) - |O(1)|\Delta\sigma + O(1) K |\alpha||\sigma_0 - \sigma_\epsilon|. \tag{5.6}$$

By telescoping the estimates of the three cases for every step between 0 and J_2, we obtain from (5.3)–(5.8) (see also Case 3)

$$\begin{aligned}
F(J_2) - F(0) \le\ & (O(1) - \frac{K}{2}) Q(\Lambda_2) \\
& + O(1) K \left(\sum_{i=1}^{n} \theta_i + Q(\Lambda_2) + c_0 \sum_k \Delta\sigma_S(J_k) \right) |\sigma_0 - \sigma_\epsilon| \\
& + \sum_k \left(K Q^2(\Delta_k) - |O(1)|\Delta\sigma_k \right) \\
& + |O(1)| \left(K(\sigma_0 - \sigma_\epsilon) + c_0 - 1 \right) \sum_k \Delta\sigma_S(J_k) \\
& + \left(\sum_{i=1}^{n} \theta_i + O(1) Q(\Lambda_2) + O(1) c_0 \sum_k \Delta\sigma_S(J_k) \right), \\
Q(J_2) - Q(0) \le\ & -\frac{3}{4} Q(\Lambda_2) + Q^2(\Lambda_2) + O(1) \\
& \cdot \left(\sum_{i=1}^{n} \theta_i + Q(\Lambda_2) + c_0 \sum_k \Delta\sigma_S(J_k) \right) |\sigma_0 - \sigma_\epsilon| + \frac{1}{4} \sum_k \Delta\sigma_S(J_k),
\end{aligned}$$

where Δ_k is any diamond between 0 and J_2. When $F(J_1)$ is sufficiently small and K sufficiently large, we obtain

$$\begin{aligned}
F(J_2) &\le F(0) - \frac{1}{2} Q(\Lambda) + C_1 \sum_{i=1}^{n} \theta_i - c_1 \left(\sum_k \Delta\sigma_S(J_k) + \sum_{\alpha_2} \Delta\sigma_{\alpha_2} \right), \\
Q(J_2) &\le Q(0) - \frac{1}{2} Q(\Lambda_2) + Q^2(\Lambda_2) + \sum_{i=1}^{n} \theta_i + \frac{1}{2} \sum_k \sigma_S(J_k),
\end{aligned}$$

where Λ_2 is the region between 0 and J_2. C_1 and c_1 are positive constants depending only on system (1.4)–(1.7). Thus, (5.1) and (5.2) hold for $J = J_2$.

Case 3. Δ is in the shock front region as in Lemma 4.2.

Let us consider the case when Δ is under the same setting as Lemma 4.2. The other cases can be proved similarly. By Lemma 4.2 and 4.4, we have

$$L_0(J_2) - L_0(J_1) = O(1)Q(\Delta) + O(1)c_0\Delta\sigma_S(J_1),$$

where $\Delta\sigma_S(J_1) = |\sigma_S(J_2) - \sigma_S(J_1)|$. And

$$L_1(J_2) - L_1(J_1) = -|O(1)|\Delta\sigma + O(1)Q(\Delta) + O(1)c_0\Delta\sigma_S(J_1) + (\theta_S(J_2) - \theta_S(J_1)),$$

where the sum $-|O(1)|\Delta\sigma$ is due to the change of the angle θ_{γ_2} if the contact discontinuity $\gamma_2 \neq 0$ in Lemma 4.2. And

$$\begin{aligned} Q_0(J_2) - Q_0(J_1) &\leq O(1)L_0(J_1)Q(\Delta) - Q^0(\Delta) \\ (Q_1 + Q_3)(J_2) - (Q_1 + Q_3)(J_1) &\leq O(1)Q(\Delta)(\sigma_0 - \sigma_\epsilon) \\ &\quad - Q^1(\Delta) + O(1)\Delta\sigma_\beta(J_1)(\sigma_0 - \sigma_\epsilon) \\ Q_c(J_2) - Q_c(J_1) &= -Q^c(\Delta). \end{aligned}$$

Hence,

$$\begin{aligned} Q(J_2) - Q(J_1) \leq\ & O(1)\left(L_0(J_1) + (\sigma_0 - \sigma_\epsilon)\right) Q(\Delta) \\ & - \left(Q(\Delta) - Q^2(\Delta)\right) + O(1)\Delta\sigma_\beta(J_1)|\sigma_0 - \sigma_\epsilon|. \end{aligned}$$

If $L(J_1)$ and $\sigma_0 - \sigma_\epsilon$ are sufficiently small, we have

$$Q(J_2) - Q(J_1) \leq -\frac{3}{4}Q(\Delta) + Q^2(\Delta) + \frac{1}{4}\Delta\sigma_\beta(J_1). \tag{5.7}$$

Thus,

$$\begin{aligned} F(J_2) - F(J_1) \leq\ & (O(1) - \frac{K}{2})Q(\Delta) + KQ^2(\Delta) - |O(1)|\Delta\sigma \\ & + O(1)(K(\sigma_0 - \sigma_\epsilon) + c_0)\Delta\sigma_S(J_1) - |O(1)|\Delta\sigma_S(J_1) + |\alpha^l|. \end{aligned} \tag{5.8}$$

By telescoping the estimates of the three cases for every step between 0 and J_2, we have

$$\begin{aligned} F(J_2) \leq\ & F(0) + (O(1) - \frac{K}{2})Q(\Lambda_2) + \sum_k \left(KQ^2(\Delta_k) - |O(1)|\Delta\sigma_k\right) \\ & + |O(1)|\left(K(\sigma_0 - \sigma_\epsilon) + c_0 - 1\right)\sum_k \Delta\sigma_S(J_k) \\ & + O(1)K\left(\sum_{i=1}^n \theta_i + Q(\Lambda_2) + c_0\sum_k \Delta\sigma_S(J_k)\right)|\sigma_0 - \sigma_\epsilon| \\ & + \left(\sum_{i=1}^n \theta_i + O(1)Q(\Lambda_2) + O(1)c_0\sum_k \Delta\sigma_S(J_k)\right), \\ Q(J_2) \leq\ & Q(0) - \frac{3}{4}Q(\Lambda_2) + Q^2(\Lambda_2) + \frac{1}{4}\sum_k \Delta\sigma_S(J_k) \\ & + O(1)\left(\sum_{i=1}^n \theta_i + Q(\Lambda_2) + c_0\sum_k \Delta\sigma_S(J_k)\right)|\sigma_0 - \sigma_\epsilon|. \end{aligned}$$

Thus,

$$F(J_2) \leq F(0) - \frac{1}{2}Q(\Lambda) + C_1 \sum_{i=1}^{n} \theta_i - c_1 \left(\sum_k \Delta\sigma_S(J_k) + \sum_{\alpha_2} \Delta\sigma_{\alpha_2} \right),$$
$$Q(J_2) \leq Q(0) - \frac{1}{2}Q(\Lambda_2) + Q^2(\Lambda_2) + \sum_{i=1}^{n} \theta_i + \frac{1}{2} \sum_k \sigma_S(J_k),$$

provided that K is large enough. Thus, (5.1) and (5.2) hold for $J = J_2$.

Furthermore, by Lemma 4.2, (5.1) and (5.2), we can establish the estimate of the strength of the relatively strong shock:

$$|S(x)| \leq O(1)|S_0|, \tag{5.9}$$

where $|S_0|$ denotes the initial strength.

This completes the proof. □

Remark 5.1. It is to be noted that the assumption in Lemma 5.1 can be achieved by choosing the Mach number $M = q_0/c_0$ sufficiently large. For simplicity in the presentation, we assume that the gas is polytropic. As the shock strength $|S_0|$ tends to zero, its corresponding σ_S tends to $\sqrt{\dfrac{q_0 - \tilde{u}}{U - q_0}}$ by (2.5). By direct computation, we obtain

$$\sqrt{\frac{q_0 - \tilde{u}}{U - q_0}} = \sqrt{\left(\frac{q_0}{c_0}\right)^2 - 1}.$$

Hence, we can choose $\sigma_0 - \sigma_\epsilon$ in Lemma 5.1 sufficiently small by simultaneously requiring $|S_0|$ sufficiently small and the Mach number sufficiently close to σ_0.

The global existence theorem thus follows from Lemma 5.1 and the consistency theorem [8].

Theorem 5.1. *Suppose that the opening angle θ_0 of the obstacle cone and the initial strength $|S_0|$ of the relatively strong shock are sufficiently small and the Mach number $M = \frac{q_0}{c_0}$ is sufficiently close to σ_0. Then the initial boundary value problem (1.4)–(1.7) as stated in Sect. 3 has a global solution $\omega(x, y)$ satisfying*

$$Total\ Variation\ \{\omega(x, y) : 0 < y < \infty\} = O(1)|S_0|,$$

provided that the perturbation is small as compared to the shock strength $|S_0|$.

6. Decay of Solutions

In this section, we study the rate of the convergence of the solution $\omega(x, y)$ to a self-similar solution. We use the following notations. χ_i denotes an i-generalized characteristic curve [5], which is a Lipschitz continuous curve traveling either with i-shock speed or with i-characteristic speed. The one-sided limits of the weak solution exist along any such curves except possibly for a countable set of x and an i-wave may cross χ_i only due to interactions. We set

$$\begin{aligned}\chi_3^S &\equiv (x, y_S(x)), \quad x \ge x_1\\ &\equiv \text{ the 3-generalized characteristic curve issued from } (x_1, \sigma_S)\ ,\\ \chi_3^0(x) &\equiv \text{ the 3-generalized characteristic curve issued from } (x, \sigma_0)\ ,\\ \chi_j^1(x) &\equiv \text{ the } j\text{-generalized characteristic curve issued from } (x, y_S(x)).\end{aligned}$$

Suppose that $\chi_1^1(x)$ ends at $x = \hat{x}$ when $\sigma = \sigma_0$. We set

$$\chi_3^2(x) \equiv \text{ the 3-generalized characteristic curve issued from } (\hat{x}, \sigma_0).$$

The Lax entropy condition implies that $\chi_3^0(x)$ and $\chi_3^2(x)$ enter the relatively strong shock S before $O(1)|S|^{-1}x$.

To study the decay rate of the solution $\omega(x, y)$, we define the following functions:

$$\begin{aligned}X(x) &= \sum\{|\alpha| : \alpha \text{ is a 3-wave or a 1-wave at } x, \alpha \neq S\},\\ \bar{Y}(x) &= \sum\{|\alpha| : \alpha \text{ is a 2-wave at } x\},\\ Y(x) &= \sum\{|\alpha|\theta_\alpha(x) : \alpha \text{ is a 2-wave at } x\},\\ Z(x) &= |S(x)|\theta_S(x),\end{aligned}$$

where $S(x)$ is the strength of the relatively strong shock S at x. $Q(\tilde{x})$ denotes the limit of $Q(J)$ as the mesh lengths r, s tend to zero, where J is a space-like curve approaching $x = \tilde{x}$. We choose a sufficiently large number x_2 such that $Q_c(x) = 0$ for $x > x_2$.

Lemma 6.1. *There exist some constants $M > 1, k_1, k_2$ depending only on system (1.4)–(1.7), and $C = O(1)|S_0|^{-1}$ depending on system (1.4)–(1.7) and the shock strength $|S_0|$ such that for $x > x_2$,*

$$X(Cx) \le MI(x), \tag{6.1}$$

$$Y(Cx) \le C^{-k_1\sigma_\epsilon^2}Y(x) + M|S_0|^2(X(x) + I(x)), \tag{6.2}$$

$$Z(Cx) \le C^{-k_2\sigma_\epsilon^2}Z(x) + M|S_0|(X(x) + I(x)). \tag{6.3}$$

Here S_0 is the initial strength of the relatively strong shock, and $I(x)$ is due to wave interactions defined by

$$I(x) \equiv X(x)^2 + |S_0|X(x) + Y(x) + Z(x).$$

Proof. According to Lemma 5.1, there exists some constant C_2 depending only on system (1.4)–(1.7) such that

$$F(J) \le C_2|S_0|^2,$$

for any space-like curve J provided that the hypothesis of Theorem 5.1 holds. It thus implies that

$$\bar{Y}(x) \le C_2|S_0|^2, \tag{6.4}$$

$$\theta_S(x) + \sum\{\theta_\alpha(x) : \alpha \text{ is a 2-wave.}\} \le C_2|S_0|^2. \tag{6.5}$$

Since $\chi_3^0(x)$ and $\chi_3^2(x)$ enter χ_3^S before Cx, $C = O(1)|S_0|^{-1}$, 1-waves and 3-waves in $X(Cx)$ are those produced by wave interactions; hence, we have from (6.4), Lemma 4.1 and 4.2,

$$X(Cx) \le O(1)\left(X(x)^2 + |S_0|X(x) + Y(x) + Z(x)\right).$$

for $x > x_2$. Applying Lemma 4.5, we obtain the decay rate of θ_α:

$$\theta_\alpha(Cx) = \theta_\alpha(x)\left(\frac{x}{Cx}\right)^{k_1\sigma_\epsilon^2}, \tag{6.6}$$

provided that a contact discontinuity α interacts only with a self-similar solution. Here, k_1 is a constant depending only on system (1.4)–(1.7). It follows from (6.4)–(6.6) that

$$\begin{aligned} Y(Cx) &\le C_2|S_0|^2 I(x) + \sum_{\alpha:\ \text{2-wave}} \alpha(x)\theta_\alpha(x)C^{-k_1\sigma_\epsilon^2} + O(1)C_2|S_0|^2(X(x) + I(x)) \\ &\le C^{-k_1\sigma_\epsilon^2} Y(x) + O(1)|S_0|^2(X(x) + I(x)). \end{aligned}$$

By Lemma 4.4, we can derive the decay rate of θ_S:

$$\theta_S(Cx) = \theta_S(x)\left(\frac{x}{Cx}\right)^{k_2\sigma_\epsilon^2}, \tag{6.7}$$

provided that the relatively strong shock S interacts only with a self-similar solution. k_2 is a constant depending only on system (1.4)–(1.7). Therefore, (6.7), Lemma 4.1, 4.2, and 5.1 yield

$$\begin{aligned} Z(Cx) &\le |S(Cx)|\left(\theta_S(x)C^{-k_2\sigma_\epsilon^2} + O(1)X(x) + O(1)I(x)\right) \\ &\le C^{-k_2\sigma_\epsilon^2} Z(x) + O(1)|S_0|(X(x) + I(x)). \end{aligned}$$

Now, we can choose a sufficiently large number M such that (6.1)–(6.3) hold for $x > x_2$. This completes the proof. □

Theorem 6.1. *For given $\varepsilon > 0$, suppose that the hypothesis of Theorem 5.1 holds, the solution $\omega(x, y)$ to system (1.4)–(1.7) converges to a self-similar solution at the following rate:*

$$\begin{aligned} X(x) &\le M_1 x^{-\frac{1}{2+\varepsilon}}, \\ Y(x) &\le M_2 x^{-\frac{1}{2+\varepsilon}}, \\ Z(x) &\le M_3 x^{-\frac{1}{2+\varepsilon}}, \end{aligned} \tag{6.8}$$

where $M_i,\ i = 1, 2, 3$, are some constants depending on ε, $|S_0|$ and system (1.4)–(1.7).

Proof. We shall prove by induction. Set

$$M_1 = C_2|S_0|(Cx_2)^{\frac{1}{2+\varepsilon}}, \quad M_2 = M_3 = C_2|S_0|^{\frac{3}{2}}(Cx_2)^{\frac{1}{2+\varepsilon}},$$

so that (6.8) holds for $x \le Cx_2$. Suppose that (6.8) holds for $x \le C^p x_2$, $p \ge 1$. We want to establish (6.8) for $C^p x_2 < x \le C^{p+1} x_2$. By Lemma 6.1 and the induction hypothesis, for $x \le C^p x_2$,

$$\begin{aligned} X(Cx) &\le MI(x) \\ &\le M(X(x)^2 + |S_0|X(x) + Y(x) + Z(x)) \\ &\le M\left(M_1^2 x^{\frac{-2}{2+\varepsilon}} + |S_0|M_1 x^{\frac{-1}{2+\varepsilon}} + M_2 x^{\frac{-1}{2+\varepsilon}} + M_3 x^{\frac{-1}{2+\varepsilon}}\right) \\ &\le M_1(Cx)^{\frac{-1}{2+\varepsilon}} \end{aligned}$$

when $|S_0|$ is sufficiently small, which depends also on ε. Also by the same argument, we have

$$\begin{aligned} Y(Cx) &\le C^{-k_1\sigma_\epsilon^2} Y(x) + M|S_0|^2(X(x) + I(x)) \\ &\le C^{-k_1\sigma_\epsilon^2} M_2 x^{\frac{-1}{2+\varepsilon}} + M|S_0|^2\left(M_1 x^{\frac{-1}{2+\varepsilon}} + M_1(Cx)^{\frac{-1}{2+\varepsilon}}\right) \\ &\le M_2(Cx)^{\frac{-1}{2+\varepsilon}}, \end{aligned}$$

$$\begin{aligned} Z(Cx) &\le C^{-k_2\sigma_\epsilon^2} Z(x) + M|S_0|(X(x) + I(x)) \\ &\le C^{-k_2\sigma_\epsilon^2} M_3 x^{\frac{-1}{2+\varepsilon}} + M|S_0|\left(M_1 x^{\frac{-1}{2+\varepsilon}} + M_1(Cx)^{\frac{-1}{2+\varepsilon}}\right) \\ &\le M_3(Cx)^{\frac{-1}{2+\varepsilon}}. \end{aligned}$$

Therefore, (6.8) holds for $C^p x_2 < x \le C^{p+1} x_2$. The proof is complete. □

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