

Algebraic Entropy

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Received: 20 May 1998 / Accepted: 2 February 1999

Abstract: For any discrete time dynamical system with a rational evolution, we define an entropy, which is a global index of complexity for the evolution map. We analyze its basic properties and its relations to the singularities and the irreversibility of the map. We indicate how it can be exactly calculated.

1. Introduction

Exploring the behaviour of dynamical systems is an old subject of mechanics [1,2]. Turning to discrete systems has triggered a huge activity and the notions of sensitivity to initial conditions, numerical (in)stability, Lyapunov exponents and various entropies remained at the core of the subject (see for example [3]).

We describe here the construction of a characterizing number associated to discrete systems having a rational evolution (the state at time $t + 1$ is expressible rationally in terms of the state at time t): it is defined in an algebraic way and we call it the *algebraic entropy* of the map. It is linked to global properties of the evolution map, which usually is not everywhere invertible. It is not attached to any particular domain of initial conditions and reflects its asymptotic behaviour. Its definition moreover does not require the existence of any particular object like an ergodic measure.

In previous works [4–7], a link has been observed between the dynamical complexity [8] and the degree of the composed map. The naive composition of n degree- d maps is of degree d^n , but common factors can be eliminated without any change to the map on generic points. This lowers the degree of the iterates. For maps admitting invariants, the growth of the degree was observed to be polynomial, while the generic growth is exponential.

We first define the algebraic entropy of a map from the growth of the degrees of its iterates, and give some of its fundamental properties. From the enumeration of the degrees of the first iterates, it is possible to infer the generating function and extract the exact value of the algebraic entropy, *even for systems with a large number of degrees of*

freedom. The reason underlying this calculability is the existence of a finite recurrence relation between the degrees.

After reviewing basic properties of birational maps and of their singularities, we prove such recurrences for specific families. The proof relies on the analysis of the singularities.

We describe the relations between the factorization process governing the growth of the degrees of the iterates and the geometry of the singularities of the evolution. This put a new light on the analysis of [9–11]. See also [12].

2. Algebraic Entropy

2.1. Definition. The primary notion we use is the degree of a rational map. In order to assign a well defined degree to a map, we require that all the components of the map are reduced to a common denominator of the smallest possible degree. The maximum degree of the common denominator and the various numerators is called the degree of the rational maps and it is the common degree of the homogeneous polynomials describing the map in *projective space*. From this definition we obtain the two basic properties:

- The degree is invariant by projective transformations of the source and image spaces.
- The degree of the composition of two maps is bounded by the product of the degrees of the maps.

From now on, rational maps will always be defined by homogeneous polynomials acting on homogeneous coordinates of the projective completion of affine space. When calculating the composition of two maps, common factors may appear which lower the degree of the resulting map. We then define a reduced composition $\phi_2 \times \phi_1$ of ϕ_1 and ϕ_2 by:

$$\phi_2 \circ \phi_1 = m(\phi_2, \phi_1) \cdot (\phi_2 \times \phi_1). \quad (1)$$

We denote by $\phi^{[n]}$ the “true” n^{th} iterate of a map ϕ , once all factors have been removed.

For a transformation ϕ , we can define the sequence d_n of the degrees of the successive iterates $\phi^{[n]}$ of ϕ .

Defining Proposition. *The sequence $1/n \log d_n$ always admits a limit as $n \rightarrow \infty$. By definition, we call this limit the algebraic entropy of the map ϕ .*

The proof is straightforward and is a consequence of the inequality $d_{n_1+n_2} \leq d_{n_1} d_{n_2}$.

The algebraic entropy is independent of the particular representation of the rational map ϕ . Indeed, if we take the conjugation of ϕ by some birational transformation ψ , $\phi' = \psi^{-1} \times \phi \times \psi$, the degree d'_n of $\phi'^{[n]}$ will satisfy $d'_n \leq k d_n$ for some constant k depending on the degree of ψ . A similar inequality can be obtained when writing $\phi = \psi \phi' \psi^{-1}$. In other words, the entropy is a *birational invariant* associated to ϕ .

This quantity can be rather easily computed by taking the images of an arbitrary line. The convergence to the asymptotic behavior is quite fast and can be obtained from the first iterates for which the degree can be exactly calculated.

The growth of d_n measures the complexity of the evolution, since d_n is the number of intersections of the n^{th} image of a generic line with a fixed hyperplane. It is related to the complexity introduced by Arnol'd [8], with the difference that we are not dealing with homeomorphisms.

The algebraic entropy also has an analytic interpretation. An invariant Kähler metric exists on the complex projective space \mathbb{P}^n and the volume of a k -dimensional algebraic variety is given by the integral of the k^{th} power of the Kähler form. This volume is proportional to the degree of the variety [13]. The area of the image by $\phi^{[n]}$ of a complex line can be expressed as the integral of the squared modulus of the differential of $\phi^{[n]}$. It is proportional to d_n by the above argument. The algebraic entropy can then be viewed as an averaged exponent: it does not depend on the choice of a starting point and it has the advantage of being of a global nature.

The definition of the algebraic entropy can be generalized to sequences of maps $(\phi_k)_k$ such that the degree of ϕ_k is bounded. We define $\phi^{[n]}$ to be the regularized map $\phi_n \times \dots \times \phi_2 \times \phi_1$. This allows the extension to non-autonomous iterations and to maps which are the product of elementary steps, in which case the sequence $(\phi_k)_k$ is periodic.

In the cases where d_n grows polynomially with n , the algebraic entropy is zero, but we can make use of a new invariant, the degree of this polynomial in n . As the algebraic entropy, it is a birational invariant.

2.2. Entropy of the Hénon map. For a simple confrontation of the algebraic entropy with more usual approaches, let us consider the much studied Hénon map [14]. Since it is a polynomial map, it is usually considered as having no singularities. This is a misconception: using projective space shows that singularities exist and are located on the line at infinity.

$$t \rightarrow t^2, \tag{2}$$

$$x \rightarrow t^2 + ty - ax^2, \tag{3}$$

$$y \rightarrow btx. \tag{4}$$

Here t is the homogenizing coordinate. We immediately see on this expression that the line at infinity $t = 0$ is sent to the point with homogeneous coordinates $(t, x, y) = (0, 1, 0)$. This point is still on the line $t = 0$, so it is a fixed point of the transformation. It will therefore never be mapped to $(0, 0, 0)$ and there cannot be any factorization. The n^{th} iterate of this map is of degree 2^n and the algebraic entropy is $\log(2)$. The remarkable thing is that this number is independent of the parameters a and b , contrary to usual dynamical exponents.

3. Birational Maps

Among rational maps, we mainly use birational ones. They are almost everywhere invertible and are therefore quite appropriate for modeling systems possessing a certain amount of reversibility.

3.1. A little bit of algebraic geometry. Rational relations between two algebraic sets X and Y are relations with a graph Z which is an algebraic subset of $X \times Y$. It would be too restrictive to impose that this defines a map from X to Y . In fact, the only rational maps which are defined on the whole space \mathbb{P}^n are linear. One therefore only requires that a rational map is one to one on the complement of an algebraic variety, that is a Zariski open set. A birational map defines a bijection from an open subset X_0 of X to an open subset Y_0 of Y .

If we call p_1 and p_2 the projection on the components of the Cartesian products restricted to Z , the point x will correspond to $p_2(p_1^{-1}(x))$. When this subset of Y is not reduced to a point, x is by definition in the singular locus of Z . If we solve for the homogeneous coordinates of the image point, we get homogeneous polynomials in the coordinates of x . We therefore get a map defined in \mathbb{C}^{n+1} . Homogeneity makes it compatible with the scale relation defining projective space. Some vector lines in \mathbb{C}^{n+1} however are identically mapped to zero: they are projective points without definite images. The set of these points is exactly the singular locus.

If ϕ is the homogeneous polynomial representation of a rational map and P is a homogeneous polynomial in $(n + 1)$ variables, we denote $\phi^* P$ the pull-back of P by ϕ . It is simply obtained by the composition $P \circ \phi$. The hypersurface of equation $\phi^* P = 0$ is the image by ϕ^{-1} of the hypersurface $P = 0$. If x_j is one of the homogeneous coordinates of \mathbb{P}^n , $\phi^* x_j$ is simply the index j component of the polynomial function ϕ . Homogeneous polynomials do not define functions on \mathbb{P}^n , but sections of a line bundle which only depends on the homogeneity degree.

3.2. *Two examples.* Let us describe two examples of birational maps. The first one is the generalized Hadamard inverse in \mathbb{P}^n . Take two copies of \mathbb{P}^n with homogeneous coordinates (x_0, x_1, \dots, x_n) and (y_0, y_1, \dots, y_n) and define Z by the n equations in $\mathbb{P}^n \times \mathbb{P}^n$:

$$x_i y_i = x_0 y_0, \quad i = 1, \dots, n. \tag{5}$$

On the subset where all the x_i are different from zero, we can use affine coordinates by fixing $x_0 = 1$ and Z defines the map

$$(x_1, \dots, x_n) \rightarrow (1/x_1, \dots, 1/x_n). \tag{6}$$

If any of the x_i is zero, then all the products $x_j y_j$ must be zero. Let J be the set of indices for which x_i is zero. Z induces a correspondence between $\{x\}$ and the linear space $\bigcap_{i \in [0, \dots, n] - J} H_i$ (H_i is the hyperplane $y_i = 0$). Instead of Eqs. (5), it is often more convenient to give a functional definition of this correspondence. A polynomial definition is:

$$y_i = \prod_{j \neq i} x_j. \tag{7}$$

The y_i 's are polynomial functions of the x_i 's of degree $n - 1$ and they satisfy Eqs. (5). But no formula can give the proper relationship for singular points. For these, at least two of the x_i 's are zero and therefore all the y_i 's vanish.

The second example is given in two dimensions by

$$x \rightarrow x, \tag{8}$$

$$y \rightarrow f(x) - y, \tag{9}$$

with $f(x)$ any rational function of x . Here again, we can give a homogeneous polynomial formulation. We will have a third variable t which will be multiplied by the denominator of $f(x)$.

These two involutions give rise to interesting evolution maps when combined with simple linear transformations. The exchange of the two variables x and y combined

with (8) gives a family of transformation which contains for suitable f discrete versions of some Painlevé equations [11]. The transformation (5) and its conjugation by the Fourier transformation yields a birational transformation which appears naturally as a symmetry of the $(n + 1)$ -state chiral Potts model [15].

3.3. Singularities. The singular points of a birational map are the vector lines of \mathbb{C}^{n+1} which are sent to the origin $(0, 0, \dots, 0)$. This singular set is of codimension at least 2. In fact, if there was an algebraic set of codimension 1 sent to the origin, the equation of this set could be factored out of all the components of the image, allowing a reduced description of the map without this singularity.

There is a bigger set where the map is not bijective. Let ϕ be a birational map and ψ be its inverse. Then the composition $\psi \circ \phi$ of their representations as polynomial maps in \mathbb{C}^{n+1} is a map of degree d^2 . It is however equivalent to the identity, so that each of the components of the image are of the form $K_\phi x_i$, where K_ϕ is a homogeneous polynomial of degree $d^2 - 1$. The set of zeroes of K_ϕ , $V(K_\phi)$, is a set where the composition $\psi \circ \phi$ is a priori not defined and it plays a fundamental role.

K_ϕ is an example of a multiplier. When composing two birational maps ϕ_1 and ϕ_2 , a common factor $m(\phi_2, \phi_1)$ may appear in the components of $\phi_2 \circ \phi_1$. In the case of inverse birational transformations, $\psi \times \phi$ is the identity and $m(\psi, \phi)$ is K_ϕ .

A fundamental property of $m(\phi_2, \phi_1)$ is that it cannot vanish out of $V(K_{\phi_1})$. Otherwise ϕ_1 would map an open subset of the set of zeros of $m(\phi_2, \phi_1)$ to a codimension 1 set where ϕ_2 is singular, since ϕ_1 is a diffeomorphism outside of $V(K_{\phi_1})$. This gives us a contradiction since the singular set of rational maps are of codimension at least 2. Determining the multiplying factor amount to determining the exponents of the different irreducible components of K_{ϕ_1} in $m(\phi_2, \phi_1)$.

In fact we obtain a definition of the map on a number of apparently singular hyper-surfaces, which is a natural continuous extension of the map.

3.4. The meaning of factorization. Consider the successive iterates $\phi^{[n]}$ of a birational map ϕ . Suppose we have the following pattern of factorization:

$$\phi \circ \phi = \phi \times \phi = \phi^{[2]}, \tag{10}$$

$$\phi \circ \phi^{[2]} = \phi \times \phi^{[2]} = \phi^{[3]}, \tag{11}$$

$$\phi \circ \phi^{[3]} = \kappa \cdot \phi^{[4]}, \tag{12}$$

with κ different from 1. Equation (12) means that the variety $\kappa = 0$ is sent to singular points of ϕ by $\phi^{[3]}$. In other words, $\kappa = 0$ is blown down to some variety of codimension higher than one by ϕ . The latter is non singular for the action of ϕ and $\phi^{[2]}$ but is eventually blown up by $\phi^{[3]}$.

Two situations may occur: it may happen that the image by $\phi^{[4]}$ of the variety $\kappa = 0$ is again of codimension 1 and we have a self-regularization of the map. Such a situation was called singularity confinement in [9, 10]. We would rather call it *resolution of singularities*. Reversibility is recovered on the singular set of ϕ after a finite number of time steps. The other possibility is that the image of the variety $\kappa = 0$ by $\phi^{[4]}$ remains of codimension larger than one, a situation depicted in Fig. 1.

In the scheme of Fig. 1, the equation of Σ is $\kappa = 0$, and the factor κ appears anew in $\phi \circ \phi^{[4]}$. The fifth iterate $\phi^{[5]}$ is regular on Σ .

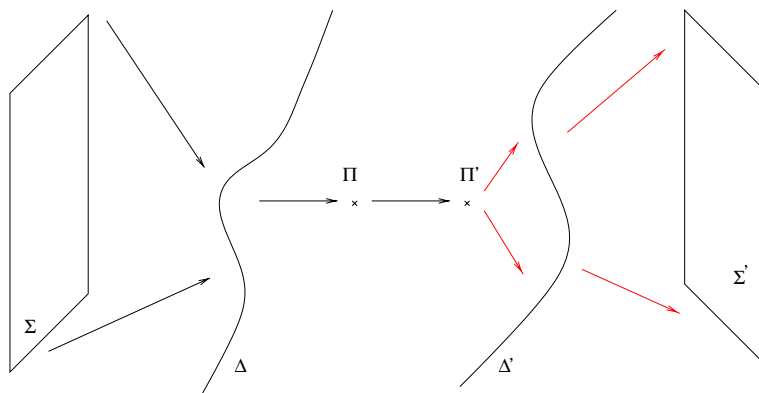


Fig. 1. A possible blow-down blow-up scheme in \mathbb{P}^3

The drop of the degree of the iterates is due to the presence of singularities on the successive images of a generic surface under the repeated action of ϕ . In other words, these images are less and less generic.

4. Recurrence Relations for the Degree

One of the basic properties of the sequence of degrees is that it seemingly always verifies a finite linear recurrence relation with integer coefficients. If this is true, the algebraic entropy is the logarithm of an algebraic number.

4.1. A simple case in \mathbb{P}^2 . Consider the map $\phi = \phi_2\phi_1$ with ϕ_1 and ϕ_2 given by:

$$\phi_1 : \begin{cases} t' = xy + 3ty + 3tx \\ x' = xy + \alpha tx + \beta ty \\ y' = xy + \beta tx + \alpha ty \end{cases}$$

$$\phi_2 : \begin{cases} t' = xy + 3ty + 3tx \\ x' = xy + \beta tx + \alpha ty \\ y' = xy + \alpha tx + \beta ty \end{cases}$$

with α and β the roots $\frac{1}{2}(-1 \pm i\sqrt{7})$ of $z^2 + z + 2 = 0$. It was used as an example of chaotic behavior in [15] and its singularities have been studied in [4]. The first few elements of the sequence d_n are:

$$1, 2, 4, 7, 12, 20, 33, 54, \dots \tag{13}$$

This sequence can be coded in the generating function:

$$g(z) = \frac{1}{1 - 2z + z^3}. \tag{14}$$

The rationality of the generating function is equivalent to the existence of a finite linear recurrence relation for the degrees, at least after a finite number of steps. The determination of the entropy is straightforward once the recurrence relation is known.

The iterated map is a product $\phi_2 \circ \phi_1$ of two linearly related transformations ϕ_1 and ϕ_2 of degree 2. It is useful in this case to look at the sequence of iterates of ϕ as the sequence built from $(\phi_1, \phi_2, \phi_1, \phi_2, \dots)$.

When calculating $\phi_1 \times \phi_2 \times \phi_1$, t will appear as a factor, since $\phi_2 \times \phi_1$ send the line $t = 0$ to a singular point of ϕ_1 .

We want to know the degree of the factor $m(\phi^{[n]}, \phi)$ or $m(\phi, \phi^{[n]})$. The former factor can only contain the factor t , but the exponent is not readily known, so we rather examine $m(\phi, \phi^{[n]})$. We have to determine the curve which is sent by $\phi^{[n-2]}$ to the line $t = 0$. This is just the first component of the polynomial expression of $\phi^{[n-2]}$ and can be written $(\phi^{[n-2]})^* t$. It has degree d_{n-2} . In two steps, the line $t = 0$ is mapped to a singular point of the following ϕ_i . The curve with equation $(\phi^{[n]})^* t$ is therefore mapped to a singular point by $\phi^{[n+2]}$ and its equation can be factorized in the calculation of $\phi^{[n+3]}$. This gives:

$$(\phi^{[n-2]})^* t \cdot \phi^{[n+1]} = \phi \circ \phi^{[n]}, \tag{15}$$

and consequently the following recurrence relation for d_n :

$$d_{n+1} = 2d_n - d_{n-2}. \tag{16}$$

This relation proves formula (14) and yields an exponential growth of the degrees and the value $\log \frac{1}{2}(1 + \sqrt{5})$ of the entropy.

4.2. Factors in the factors. The previous analysis is simple because the image of t remains an irreducible polynomial. This cannot be true in general, since the factors in K_ϕ generally break into pieces under further transformation by ϕ [5].

Let us take two birational transformations ϕ_1 and ϕ_2 with respective inverses ψ_1 and ψ_2 and calculate $\psi_1 \circ \phi_1 \circ \phi_2$ in two different ways:

$$\begin{aligned} \psi_1 \circ \phi_1 \circ \phi_2 &= (K_{\phi_1} \cdot Id) \circ \phi_2 = \phi_2^* K_{\phi_1} \cdot \phi_2 \\ &= m(\phi_1, \phi_2)^{d_{\psi_1}} \cdot \psi_1 \circ (\phi_1 \times \phi_2). \end{aligned} \tag{17}$$

Since the components of ϕ_2 cannot have any common factor, we deduce that $m(\phi_1, \phi_2)^{d_{\psi_1}}$ divides $\phi_2^* K_{\phi_1}$.

Geometrically, $m(\phi_1, \phi_2)$ is the equation of a hypersurface which ϕ_2 sends to singular points of ϕ_1 . Since K_{ϕ_1} vanishes on the singular points of ϕ_1 , its image $\phi_2^* K_{\phi_1}$ vanishes on the zero locus of $m(\phi_1, \phi_2)$.

In the example of the previous section, each new factor appearing in $\phi \circ \phi^{[n]}$ is the equation of a hypersurface which $\phi^{[n]}$ sends to the point $(1, 0, 0)$. The x and y components of $\phi^{[n]}$ therefore have a common factor $m(\phi, \phi^{[n]})$. Consequently, the image $\phi^{[n]*} K_\phi$ of $K_\phi = t x y$ by $\phi^{[n]}$ contains the expected factor $m(\phi, \phi^{[n]})^2$, while $\phi^{[n]*} t$ does not contain this factor.

4.3. An example in \mathbb{P}^{N-1} . Consider the algebra of the finite group \mathbb{Z}_N , its generic element $a(x) = \sum_{q=0}^{N-1} x_q \sigma^q$, with $x = (x_0, x_1, \dots, x_{N-1})$ and σ the generator of \mathbb{Z}_N .

The algebra has two homomorphic products:

$$a(x \circ y) = a(x) \circ a(y), \tag{18}$$

$$a(x \cdot y) = \sum_{q=0}^{N-1} x_q y_q \sigma^q. \tag{19}$$

The product \circ just comes from the product in \mathbb{Z}_N , and verifies $\sigma^p \circ \sigma^q = \sigma^{(p+q)}$, while $\sigma^p \cdot \sigma^q = \delta_q^p \sigma^p$. In terms of cyclic matrices, these two products respectively correspond to the matrix product and the element by element (Hadamard) product. The homomorphism between these two products is realized by the Fourier transform.

ϕ_1 and ϕ_2 will be the two inverses constructed from these products. The components $(x_0, x_1, \dots, x_{N-1})$ of x are the natural coordinates of projective space and ϕ_1 and ϕ_2 are involutions of degree $N - 1$. K_{ϕ_1} and K_{ϕ_2} are products of linear factors. These linear factors are the equations of hyperplanes which are sent by the corresponding ϕ_i into points which we call maximally singular.

The important fact is that these maximally singular points are permuted by the other involution. As an example, the maximally singular points of the Hadamard inverse are of the form σ^q , i.e., with only one non zero component. The matrix inverse permutes such points by $\sigma^q \rightarrow \sigma^{N-q}$.

If p has one vanishing component, say x_i , then $\phi^{[2]}(p)$ will have all its coordinates vanishing except x_{N-i} . It follows that x_i is a common factor to all these coordinates. The j^{th} coordinate of $\phi^{[2]}(p)$ can be written¹:

$$\phi^2(p)_j = x_j^{[2]} \prod_{i \neq N-j} x_i. \tag{20}$$

The Hadamard inverse is easily calculated on such an expression. The coordinates of $\phi^3(p)$ are given by:

$$\phi^3(p)_j = \left(\prod_{i \neq j} x_i^{[2]} \right)_{x_{N-j}} \left(\prod_{i=0 \dots N-1} x_i \right)^{N-2}. \tag{21}$$

The common factor is simply K_{ϕ_1} and this suggests that $\phi^{[3]}$ is a local diffeomorphism on the zeroes of K_{ϕ_1} .

We now want to determine the structure of the components of $\phi^{[n]}$ for any n . The situation for n odd and n even will be similar, since the conjugation by the Fourier transform exchanges the two inverses. From the expression (20) of $\phi^2(p)$, we see that this point is a generic element of the plane $x_j = 0$ if and only if $x_j^{[2]} = 0$. We define polynomials $x_j^{[n+2]}$ generalizing the $x_j^{[2]}$'s appearing in (20) such that²:

$$\phi^{[n+2]}(p)_j = x_j^{[n+2]} \prod_{i \neq N-j} x_i^{[n]}. \tag{22}$$

If g_n is the degree of the $x_j^{[n]}$'s, then Eq. (22) yields:

$$d_n = g_n + (N - 1)g_{n-2}. \tag{23}$$

The generalization of (21) gives:

$$m(\phi, \phi^{[n]}) = \left(\prod_j x_i^{[n-2]} \right)^{N-2}. \tag{24}$$

¹ As in Sect. 4.1, we write ϕ^n (resp. $\phi^{[n]}$) for the composition of alternatively the two inverses (resp. the reduced composition).

² In this formula, the coordinates are different according to the parity of n . They are always such that the following ϕ is the Hadamard inverse in those coordinates.

The factor is therefore of degree $N(N - 2)g_n$. Finally:

$$\begin{aligned} g_{n+1} &= d_{n+1} - (N - 1)g_{n-1} = (N - 1)d_n - N(N - 2)g_{n-2} - (N - 1)g_{n-1} \\ &= (N - 1)g_n - (N - 1)g_{n-1} + g_{n-2}. \end{aligned} \tag{25}$$

It is easy from this recurrence relation to determine that for $N = 3$, (g_n) and therefore (d_n) are periodic sequences of period 6. The sequence of the $\phi^{[n]}$'s is known to have this periodicity. For $N = 4$, g_n is a polynomial of degree 2 in n , and for bigger N , the sequences are growing like β^n , with β the larger root of $x^2 - (N - 2)x + 1$.

4.4. Another proof. There is another way to prove the previous result, relating directly to the study of the singularities and the blow-down blow-up process.

We first need to introduce some notations, using a homogeneous coordinates system for \mathbb{P}^{N-1} . The Hadamard inverse ϕ_1 sends $(x_0, x_1, \dots, x_{N-1})$ into $(x'_0, x'_1, \dots, x'_{N-1})$, where $x'_k = \prod_{\alpha \neq k} x_\alpha$. The square of ϕ_1 is the multiplication by K_{ϕ_1} .

Define C to be the projective linear transformation constructed from the matrix

$$C = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \dots & \omega^{(N-1)^2} \end{pmatrix}, \tag{26}$$

with $\omega = \exp(2i\pi/N)$. The inverse of C is its complex conjugate \bar{C} . The involution ϕ_2 is linearly related to ϕ_1 by

$$\phi_2 = C \phi_1 \bar{C}. \tag{27}$$

The product $\phi_2 \circ \phi_1$ may thus be rewritten $\rho_1 \circ \rho_2$, with $\rho_1 = C \circ \phi_1$ and $\rho_2 = \bar{C} \circ \phi_1$. Denote by $\psi_1 = \phi_1 \circ \bar{C}$ and $\psi_2 = \phi_1 \circ C$ the inverses of ρ_1 and ρ_2 respectively.

The maximally singular points of the Hadamard inverse are the points P_i with x_i the only non vanishing component. They are the blow down by ϕ_1 of the planes $\Pi_i : \{x_i = 0\}$ for $i = 0, \dots, (N - 1)$. They are singular points of ρ_1 and ρ_2 . Denote by $Q_i, i = 0..(N - 1)$ the points $Q_i = (1, \omega^i, \omega^{2i}, \dots, \omega^{(N-1)i})$. The Q_i 's are singular points of ψ_1 and ψ_2 .

We have the following straightforward relations:

$$C P_i = Q_i, \quad \bar{C} P_i = Q_{-i}, \quad \bar{C} Q_i = P_i = C Q_{-i}, \tag{28}$$

$$\phi_1(\Pi_i) = P_i. \tag{29}$$

The relevant singularity structure is entirely described by the two sequences:

$$\Pi_i \xrightarrow{\phi_1} P_i \xrightarrow{C} Q_i \xrightarrow{\phi_1} Q_{-i} \xrightarrow{\bar{C}} P_{-i} \xrightarrow{\phi_1} \Pi_{-i}, \tag{30}$$

$$\Pi_i \xrightarrow{\phi_1} P_i \xrightarrow{\bar{C}} Q_{-i} \xrightarrow{\phi_1} Q_i \xrightarrow{C} P_{-i} \xrightarrow{\phi_1} \Pi_{-i}. \tag{31}$$

The first squiggly line indicates blow down from hyperplane to point and the last one indicates blow up from point to hyperplane.

Consider now a sequence $\{S_k\}$ of varieties of codimension one, constructed by the successive action of ρ_1, ρ_2, ρ_1 , and so on. Suppose the ordering is such that ρ_1 acts on

the S 's with even index and ρ_2 on the S 's with odd index. The successive images in the sequence are supposed to be regularized by continuity. We denote by $d_n = d(S_n)$ the degree of the equation of S_n .

Denote by $\alpha_k(n)$ (resp. $\beta_k(n)$) the order of P_k (resp. Q_k) on S_n . If a is the running point of \mathbb{P}^{N-1} , then we have the defining relations

$$S_{2n}(\rho_2(a)) = S_{2n-1}(a) \cdot \prod_{u=0}^{N-1} x_{-u}^{\beta_u(2n)}(a), \tag{32}$$

$$S_{2n-1}(\psi_2(a)) = S_{2n}(a) \cdot \prod_{v=0}^{N-1} x_v^{\alpha_v(2n-1)}(C a). \tag{33}$$

Using the fact that ρ_i and ψ_i are inverse of each other, and relations (32, 33), we get by evaluating $S_{2n}(\rho_2\psi_2(a)) = K_{\psi_2}^{d_{2n}} \cdot S_{2n}(a)$, the following relation on the degrees:

$$(N - 1) d_{2n} = \alpha_v(2n - 1) + \sum_{b \neq -v} \beta_b(2n). \tag{34}$$

Similarly, calculating $S_{2n}(\psi_1\rho_1(a))$ produces

$$(N - 1) d_{2n} = \beta_u(2n + 1) + \sum_{k \neq u} \alpha_k(2n). \tag{35}$$

Let $\Theta_\alpha(n) = \sum_k \alpha_k(n)$ and $\Theta_\beta(n) = \sum_k \beta_k(n)$. Relations (34,35) yield

$$\begin{aligned} N(N - 1) d_{2n} &= \Theta_\alpha(2n - 1) + (N - 1) \Theta_\beta(2n), \\ N(N - 1) d_{2n} &= \Theta_\beta(2n + 1) + (N - 1) \Theta_\alpha(2n). \end{aligned} \tag{36}$$

From the singularity pattern (30,31), we see that $\alpha_i(2n) = \beta_{-i}(2n - 1)$ and $\alpha_i(2n + 1) = \beta_i(2n)$, so that $\Theta_\alpha(2n) = \Theta_\beta(2n - 1)$ and $\Theta_\alpha(2n + 1) = \Theta_\beta(2n)$. It follows that

$$\Theta_\alpha(k) = \Theta_\beta(k - 1). \tag{37}$$

This combined with (36) yields

$$d_{n+3} - (N - 1)d_{n+2} + (N - 1)d_{n+1} - d_n, \tag{38}$$

which is the recurrence relation on the degrees of the iterates, with generating function

$$f_q(z) = \frac{1 + z^2(N - 1)}{(1 - z)(z^2 - z(N - 2) + 1)}. \tag{39}$$

4.5. *Discrete Painlevé I.* The discrete Painlevé I system is given by the following transformations:

$$\begin{aligned} x &\rightarrow \frac{c_n}{x} + b - x - y, \\ y &\rightarrow x, \end{aligned} \tag{40}$$

where c_n depends on three parameters and is given by $c_n = c + an + d(-1)^n$. The transformation is just an involution of the form (8) followed by the exchange of x and y . The homogeneous form is ϕ_n given by:

$$(t, x, y) \rightarrow (xt, c_n t^2 + bxt + x^2 - yx, x^2). \tag{41}$$

It is easy to obtain that K_ϕ is simply x^3 , so that x is the only factor which can appear in $m(\psi, \phi)$. The line x is sent to the point of coordinates $(0, 1, 0)$, but it is not sufficient to characterize a possible blowing up. In fact, at leading order in x , the image of points approaching this line satisfy the equation $xy = c_n t^2$. We therefore have to follow the image up to second order. x remains a factor in the successive transformations of t . In $(\phi^3)^* x, x^2$ appears as a factor. This gives a factor of x^6 in the transformation of K_ϕ and is the signal according to Sect. 4.2 of the factor x^3 appearing in $m(\phi, \phi^3)$. The x^2 factor is however not sufficient to guarantee the factorization of x^3 in the next composition. The factorization of x^3 depends on the relation $c_{n+3} - c_{n+2} - c_{n+1} + c_n$ which characterizes the form of the c_n given in [11].

We may now establish the recurrence relation obeyed by the degrees. We introduce the polynomials $x^{[n]}$ of degree g_n such that the x component of $\phi^{[n]}$ is $x^{[n]}(x^{[n-3]})^2$. As in the preceding case, the factor which will replace x in the successive factorization is $x^{[n]}$. The factors $x^{[n-3]}$ have disappeared in $\phi^{[n+1]}$ and the images of $x^{[n-3]} = 0$ are not singular. Since $d_n = g_n + 2g_{n-3}$, we have:

$$d_{n+1} = 2d_n - 3g_{n-3} = 2g_n + g_{n-3}. \tag{42}$$

whose solution is:

$$g_n = (1 + \frac{1}{2}n)^2 - \frac{1}{8}(1 - (-1)^n), \tag{43}$$

$$d_n = \frac{3}{4}n^2 + \frac{9}{8} - \frac{1}{8}(-1)^n. \tag{44}$$

These results agree with the explicit calculations, producing the sequence of degrees:

$$1, 2, 4, 8, 13, 20, 28, 38, 49, 62, 76, \dots \tag{45}$$

We can also consider a slight generalization introduced in [6]. The pole part of the transformation of x is replaced by a double pole but we do not use variable coefficients,

$$\begin{aligned} x &\rightarrow \frac{c}{x^2} + b - x - y, \\ y &\rightarrow x. \end{aligned} \tag{46}$$

This is now a degree 3 birational map, with $K_\phi = x^3$. It was shown that we still have the same pattern, but with higher powers of x appearing. In $\phi^{[3]}$, the x component gets a x^3 factor and we can factorize x^8 from ϕ^4 . Defining similarly $x^{[n]}$ such that the x component of $\phi^{[n]}$ is $x^{[n]}(x^{[n-3]})^3$, we get the following recurrence relation for its degree g_n :

$$g_{n+1} - 3g_n + 3g_{n-2} - g_{n-3} = 0. \tag{47}$$

The solution of this equation allows to recover the results of [6]. The algebraic entropy is given by the logarithm of the largest solution of $x^4 - 3x^3 + 3x - 1$ which is $\frac{1}{2}(3 + \sqrt{5})$, the square of the golden ratio.

5. Conclusion and Perspectives

We have not produced the general proof of the existence of a finite recurrence on the degrees. We have however shown that its origin lies in the singularity structure of the evolution and the possible recovering of reversibility.

In numerous examples which we will not enumerate, we have been able either to establish recurrence equations or to infer a generating function from the first degrees which successfully predict the following ones. This supports the following conjecture.

Conjecture. *The generating function of the sequence of the degrees is always a rational function with integer coefficients.*

This may even be the case for rational transformation which are not birational [16]. The algebraic entropy is in this case the logarithm of an algebraic number and in the case of vanishing entropy, the sequence of the degrees is of polynomial growth.

There is a keyword which we did not use yet: integrability. Proving integrability in our setting amounts to showing that the motion is a translation on a torus. From the numerous examples we have examined, we believe the algebraic entropy measures a deviation from this type of integrability. We can actually propose the following:

Conjecture 2. *If the birational transformation ϕ is equivalent to a bijection defined on an algebraic variety M deduced from \mathbb{P}^n by a finite sequence of blow-ups, then the sequence of degrees of $\phi^{[n]}$ has at most a polynomial growth.*

There also is the question of the relation of the algebraic entropy to other dynamical entropies [3]. The fact that the algebraic maps we study do not necessarily admit an ergodic measure precludes the definition of the Kolmogorov-Sinai entropy in many cases. The most natural correspondence would be with the topological entropy, but requires more work. We must also stress that in any case, the algebraic entropy is a property of the map in the complex domain.

The special properties of rational maps allow to characterize the complexity of the dynamics from the study of a single number, the degree, and to control it through the study of the behaviour of the map in a small number of singular points.

References

1. Liouville, J.: Sur l'intégration des équations différentielles de la Dynamique. *J. de Mathématiques Pures et Appliquées* **XX**, 137–138 (1855)
2. Poincaré, H.: *Les méthodes nouvelles de la mécanique céleste*. Paris: Gauthier-Villars, 1892
3. Eckmann, J.-P. and Ruelle, D.: Ergodic theory of chaos. *Rev. Mod. Phys.* **57**(3), 617–656 (1985)
4. Falqui, G. and Viallet, C.-M.: Singularity, complexity, and quasi-integrability of rational mappings. *Commun. Math. Phys.* **154**, 111–125 (1993)
5. Boukraa, S., Maillard, J.-M. and Rollet, G.: Integrable mappings and polynomial growth. *Physica A* **209**, 162–222 (1994)
6. Hietarinta, J. and Viallet, C.-M.: Singularity confinement and chaos in discrete systems. *Phys. Rev. Lett.* **81**, 325–328 (1998); solv-int/9711014
7. Veselov, A.P.: Growth and Integrability in the Dynamics of Mappings. *Commun. Math. Phys.* **145**, 181–193 (1992)
8. Arnold, V.I.: Dynamics of complexity of intersections. *Bol. Soc. Bras. Mat.* **21**, 1–10 (1990)
9. Grammaticos, B., Ramani, A. and Papageorgiou, V.: Do integrable mappings have the Painlevé property? *Phys. Rev. Lett.* **67**, 1825–1827 (1991)
10. Ramani, A., Grammaticos, B., and Hietarinta, J.: Discrete versions of the Painlevé equations. *Phys. Rev. Lett.* **67**, 1829–1832 (1991)
11. Grammaticos, B., Nijhoff, F.W. and Ramani, A.: Discrete painlevé equations. In: R. Conte, ed., *The Painlevé property, one century later*, 1996 Cargèse Proceedings, to appear in CRM Proceedings and lecture notes

12. Hietarinta, J. and Viallet, C.-M.: *Discrete Painlevé and singularity confinement in projective space*. Proceeding Bruxelles July 1997 meeting on "Integrability and chaos in discrete systems", 1997
13. Mumford, D.: *Algebraic geometry, Vol. 1: Complex projective varieties*. Berlin–Heidelberg–New York: Springer-Verlag, 1995
14. Hénon, M.: A two-dimensional mapping with a strange attractor. *Commun. Math. Phys.* **50**, 69–77 (1976)
15. Bellon, M.P., Maillard, J.-M. and Viallet, C.-M.: Infinite Discrete Symmetry Group for the Yang-Baxter Equations: Spin models. *Phys. Lett. A* **157**, 343–353 (1991)
16. Boukraa S., Abenkova, N., Anglès d'Auriac, J.-C. and Maillard, J.-M.: In preparation (1998)

Communicated by Ya. G. Sinai