Commun. Math. Phys. 204, 269 – 312 (1999)

Communications in
Mathematical Physics

Interface, Surface Tension and Reentrant Pinning Transition in the 2D Ising Model

C.-E. Pfister[1], **Y. Velenik**[2]

[1] Département de Mathématiques, EPF-L, CH-1015 Lausanne, Switzerland.
E-mail: charles.pfister@dma.epfl.ch
[2] Fachbereich Mathematik, Sekr. MA 7-4, TU-Berlin, Strasse des 17. Juni 136, D-10623 Berlin, Germany.
E-mail: velenik@math.tu-berlin.de

Received: 28 January 1998 / Accepted: 22 December 1998

Abstract: We develop a new way to look at the high-temperature representation of the Ising model up to the critical temperature and obtain a number of interesting consequences. In the two-dimensional case, it is possible to use these tools to prove results on phase-separation lines in the whole phase-coexistence regime, by way of a duality transformation. We illustrate the power of these techniques by studying an Ising model with a boundary magnetic field, in which a reentrant pinning transition takes place; more precisely we show that the typical configurations of the model can be described, at the macroscopic level, by interfaces which are solutions of the corresponding thermodynamic variational problem; this variational problem is solved explicitly. There exist values of the boundary magnetic field and temperatures $0 < T_1 < T_2 < T_c$ such that the interface is not pinned for $T < T_1$ or $T > T_2$, but is pinned for $T_1 < T < T_2$; we can also find values of the boundary magnetic field and temperatures $0 < T_1 < T_2 < T_3 < T_c$ such that for $T < T_1$ or $T_2 < T < T_3$ the interface is pinned, while for $T_1 < T < T_2$ or $T > T_3$ it is not pinned. An important property of the surface tension which is used in this paper is the sharp triangle inequality about which we report some new results. The techniques used in this work are robust and can be used in a variety of different situations.

1. Introduction

Let us consider a 2D Ising model in some rectangular box with boundary conditions imposing the presence of a phase-separation line crossing the box from one fixed point of a vertical side to another fixed point of the other vertical side. We suppose that the model is in the phase-coexistence region; the boundary conditions are chosen so that above the phase-separation line we have the $+$ phase and below it the $-$ phase. The bottom horizontal side of the box, which we call the wall, is subject to a negative boundary magnetic field. By varying the temperature or the boundary magnetic field one can observe an interfacial pinning-depinning or critical wetting transition as established by Abraham

[A1]. In [A1], however, this surface phase transition was called a "roughening transition" (although the analysis demonstrated the depinning character); further comments are made in Sect. 2.3.2 in connection with the work of McCoy and Wu who observed a related "boundary hysteresis" (see Chapters VI and XIII in [MW]). We now describe the pinning-depinning transition at the macroscopic level. For values of these parameters for which the $+$ phase wets partially the wall, and under appropriate geometrical conditions, the equilibrium shape of the interface changes from a straight line crossing the box to a broken line touching a macroscopic part of the wall. Moreover, we show in this paper that there exist values of the boundary magnetic field and temperatures $0 < T_1 < T_2 < T_c$ such that the interface is not pinned for $T < T_1$ or $T > T_2$ and pinned for $T_1 < T < T_2$; we can also find values of the boundary magnetic field and temperatures $0 < T_1 < T_2 < T_3 < T_c$ such that for $T < T_1$ or $T_2 < T < T_3$ the interface is pinned, while for $T_1 < T < T_2$ or $T > T_3$ it is not pinned. These reentrant pinning-depinning transitions are predicted by a macroscopic variational problem for the interface, which is formulated in terms of the surface tension and wall free energies of the model. One of the main results of the paper is the derivation of this macroscopic theory starting from the Boltzmann formula defining the equilibrium states of the model at the microscopic level.

It is important to distinguish different length-scales. To do so we use two different words, "interface" and "phase-separation line". We use the word "interface" to denote the boundary at the macroscopic scale between the two phases. At this scale the boundary is fixed (nonfluctuating). The fundamental thermodynamical function associated with an interface is the surface tension, which is non-zero below the critical temperature. (In [ABCP] similar ideas are developed). By contrast, the "phase-separation line" is a stochastic line whose probability distribution is determined by the Gibbs measure; it describes the boundary between the two phases at the lattice spacing scale. In this respect it is very interesting to read the introduction of [T], where Talagrand develops a similar analysis of the Law of Large Numbers for independent random variables.

On the conceptual level one point of our paper is to show that the theory of the Gibbs states for the infinite volume model is inadequate for discussing some macroscopic properties of the model. The famous theorem, which states that all Gibbs states are translation-invariant for the 2D Ising model [Az1,Hi1], is not pertinent when we study the model at scales L^{α}, $\alpha > 1/2$, L being the linear size of the box containing the system. There are non-translation invariant states at that scale, with well-defined (fixed) interfaces! Let us illustrate this point by considering the so-called $\pm$ boundary conditions, which corresponds to a special case of the present paper, where the phase-separation line goes from the middle of a vertical side of the box to the middle of the other vertical side. The definition of the phase separation line in [BLP1] coincides with the one of Gallavotti in his work [G] about the phase separation in the 2D Ising model; it differs slightly from the one used here, but in no essential way. (Notice that the terminology "interface" is sometimes used for "phase-separation line" in [BLP1].) There are three natural scales in the study of the phase-separation line, which have been first studied by Abraham and Reed [AR] in a non-perturbative manner.

At the scale of the lattice spacing the phase-separation line is a stochastic geometrical line, which has well-defined properties, which depend strongly on the microscopic interaction [BLP1]. Its middle point has fluctuations typically of the order $O(L^{1/2})$, L being the linear size of the box Λ_L containing the system [G,AR]. Because of these fluctuations the projection of the corresponding limiting Gibbs state, at the middle of the box, when $L \to \infty$, is translation invariant [G]; in particular the magnetization (at the middle of the box) is zero. If we scale the lengths vertically by $(1/L)^{1/2}$ and horizontally by $1/L$, then in the limit $L \to \infty$ the phase-separation line converges to a Brownian bridge, [Hi2, DH, D]. The magnetization profile on that scale has been computed by [AR]. At that intermediate scale the phase-separation line is still stochastic, but its properties show some universal features (Central Limit Theorem). However, at a scale of order $O(L^\alpha)$, $\alpha > 1/2$, the system has a well-defined fixed horizontal interface and a deterministic macroscopic magnetization profile [AR].

To describe the system at the scale $O(L^\alpha)$ we partition the box Λ_L into square boxes C_i of linear size $O(L^\alpha)$; the state of the system in each of these boxes is specified by the empirical magnetization $|C_i|^{-1}\sum_{t\in C_i}\sigma(t)$. Then we rescale all lengths by $1/L$ in order to get a measure for these normalized block-spins in the fixed (macroscopic box) Q. When $L \to \infty$ these measures converge to a deterministic macroscopic magnetization profile showing a well-defined horizontal interface separating the two phases of the model, characterized by a value $\pm m^*$ of the normalized block-spins, m^* being the spontaneous magnetization of the model. This coarsed-grained description of the equilibrium state at the thermodynamic limit is in sharp contrast with the above mentioned result implying that the equilibrium state converges to a translation invariant measure at the thermodynamical limit. These two limits are related to properties of the model at two different scales, the lattice spacing scale and the macroscopic one.

We outline the content of the paper. In Sect. 2 we recall the definitions and some properties of phase-separation line, duality, surface tension and wall free energy. We give here no proof. By duality the statistical properties of the phase-separation line at $\beta > \beta_c$ between two distant but fixed points, say t and t', are (essentially) the same as the statistical properties of the high-temperature contour λ in the random-line representation (1.1) of the two-point correlation function at $\beta^* < \beta_c$,

$$\langle\, \sigma(t)\sigma(t')\,\rangle(\beta^*) = \sum_{\lambda:t\to t'} q(\lambda). \tag{1.1}$$

In (1.1) λ is an open contour of the high-temperature representation with end-points t and t'; $q(\lambda)$ is the weight of the contour λ; $q(\lambda)$ depends of course on β^*. We can also interpret λ as the part of the phase-separation line going from t to t' and $q(\lambda)$ is the weight at β of that part of the phase-separation line. The sum over λ in (1.1) is the partition function of the ensemble of stochastic lines λ from t to t'. We exploit the fact that this partition function is equal to $\langle\, \sigma(t)\sigma(t')\,\rangle(\beta^*)$; consequently we have a good control of this sum since we can use information, either from explicit computations or from correlation inequalities, available for the two-point correlation function. Our (working) definition of the surface tension of an interface described at the macroscopic level by a line passing through t and t', perpendicular to the direction n, is the thermodynamical function corresponding to this ensemble of stochastic lines, that is

$$\hat{\tau}(n;\beta) := \lim_{\substack{k\in\mathbb{N}\\ k\to\infty}} -\frac{1}{k\|t-t'\|}\ln \sum_{\lambda:kt\to kt'} q(\lambda). \tag{1.2}$$

This is exactly the quantity, which enters in the macroscopic variational problem. (The same point of view is taken in [Pf2] and [PV1] in connection with the Wulff shape.) On the technical side this definition is much simpler to use in our problem than the previous definitions considered in the literature [A2]. The fact that this definition coincides with previous definitions considered in the literature is not trivial (Proposition 2.2). The "physical" reason, why Proposition 2.2 is true, is that the walls of the box are in the complete wetting regime; see Sect. 7 where the dual question, equality between the short and long correlation lengths, is considered. In Sect. 3 we define precisely the main problem, which we address. In this section we give some references to earlier works. We formulate the problem from the microscopic viewpoint, but then discuss it from the macroscopic viewpoint in Sects. 4 and 5. The main physical results are contained in Sect. 5, in which we prove that the physical situations described at the beginning of this introduction take place. These two sections dealing with the macroscopic theory are formulated in terms of surface tension and wall free energy. We use known results, mostly coming from explicit computations. In fact, we do not know how to predict the reentrant phenomena, which we display, without knowing explicitly the values of the surface tension and the wall free energy. In the second part of the paper we derive the macroscopic theory starting from the microscopic Hamiltonian by analysing the typical configurations. Our starting point is a new way of dealing with the high-temperature representation of the model, which has been developed recently in [PV1]. Although different, our approach is similar in some respect to the random current representation of the Ising model of Aizenman [Az2]. This method is exposed in Sect. 6; it is the core of the paper. The method is not restricted to dimension two. Except for two proofs, which can be read in [PV1], the method is developed from scratch, with new proofs and new results. This section has its own interest and can be read independently. The major results are concentration results for the random-line representation (1.1) of the two-point correlation function above the critical temperature when $D = 2$. Let

$$\mathcal{S}(t, t'; C \ln \|t - t'\|) := \{ x \in \mathbb{Z}^2 : \|t - x\| + \|t' - x\| \leq \|t - t'\| + C \ln \|t - t'\| \}. \tag{1.3}$$

There exists C, large enough, so that the stochastic lines contributing to the two-point function $\langle \sigma(t)\sigma(t') \rangle(\beta^*)$ are those contained inside the ellipse (1.3); more precisely, if C is large enough, by Lemma 6.10,

$$\lim_{\|t-t'\|\to\infty} \frac{\displaystyle\sum_{\substack{\lambda:t\to t' \\ \lambda\not\subset\mathcal{S}(t,t';C\ln\|t-t'\|)}} q(\lambda)}{\displaystyle\sum_{\lambda:t\to t'} q(\lambda)} = 0. \tag{1.4}$$

This result is sharp, since the width of the ellipse is $O\Big((\|t - t'\| \ln \|t - t'\|)^{1/2}\Big)$. Thus the lines λ contributing to (1.1) are concentrated in a region, whose size is essentially, that of the normal fluctuations of a random walk going from t to t'. When the model is defined on the half-infinite lattice $\mathbb{L} := \{ x \in \mathbb{Z}^2 : x_2 \geq 0 \}$ we have a random-line representation similar to (1.1) for the boundary two-point function (Lemma 6.13). There are two regimes, depending on the value of the boundary magnetic field. If the boundary coupling constant h^*, dual to the boundary magnetic field h, is not too large, then the concentration result is as above; in that case we know that the line λ undergoes an entropic repulsion from the boundary of $\mathbb{L}$. On the other hand, if the coupling constant h^* is high enough, then the line λ sticks to the boundary of $\mathbb{L}$. We show that the lines λ contributing to the boundary two-point correlation function, when $\|t - t'\| = |t_1 - t_1'|$ tends to infinity, are those contained in a rectangle ($t_1 < t_1'$)

$$\mathcal{B}(t, t'; \rho) := \{ x \in \mathbb{L} : x_1 \in [t_1 - \rho, t_1' + \rho],\ 0 \leq x_2 \leq \rho \}\ ,\ \ \rho = C \ln |t_1 - t_1'|. \tag{1.5}$$

Again, this result is optimal. We stress that the only condition about the temperature is $T > T_c$. We give a first application of the results of Sect. 6 in Sect. 7. This section also contains one of the main estimates, a lower bound for the two-point correlation function in a finite box in terms of the two-point correlation function of the infinite system. This bound is essential for Sect. 8. We show that the pinning transition below T_c has a dual interpretation above T_c; although there is a unique Gibbs state at the thermodynamical limit, we may have surface effects. Inspired by [SML] we introduce the notions of short correlation length and long correlation length. We prove that these two notions do not necessarily coincide. They differ when at the dual temperature the interface is pinned. The relevance of these results for the surface tension at the dual temperature is discussed at the beginning of Sect. 2.3. In Sect. 8 we justify the macroscopic theory of Sect. 4 starting from the microscopic theory, and we show how the interface emerges in the statistical description of the model, as a deterministic object in a coarse-grained description of the microscopic configurations. We add one appendix, Sect. 9, where we show that our method is robust. We apply it to a generic case with N interfaces.

In this paper we derive results by very different technical tools like exact computation, correlation inequalities and high-temperature representation. We can treat mathematically various interesting physical situations for the 2D Ising model. Each of the approaches just mentioned has its own strengths and weaknesses. It is certainly advantageous to combine these methods as we do in this paper. It is evident that the method of the high-temperature representation, combined with duality, is appropriate for studying interfaces for the 2D Ising model at a scale L^α, $\alpha > 1/2$. On the other hand we also show that we need few, but very precise results about specific quantities, like two-point correlation function, surface tension, wall free energy, values of the boundary magnetic field where the wetting transition takes place. These results depend on finer properties of the model at scales $L^\alpha, \alpha \leq 1/2$. Here exact computations are appropriate; moreover, some of these results can be obtained only by exact computations.

[A2] is a good review about exact results on interface problems in general. We also mention the work by Fisher [F] where deep insight about wetting and pinning problems and other phenomena in $2D$ is provided by analysing these questions in terms of random walks. Some of the results presented here are taken from [V] (see Chapter 6).

2. Definitions and Notations

We introduce the notation used in the paper, which follows essentially that of [PV1]. We recall the notions of duality, phase-separation line, surface tension and wall free energy. We also state some fundamental estimates for the two-point correlation function of the model. A large part of this material is standard; references are given in the text.

Throughout the paper we use the following convention: $O(x)$ denotes a non-negative function of $x \in \mathbb{R}^+$, such that there exists a constant C with $O(x) \leq Cx$; the function $O(x)$ may be different at different places.

2.1. Phase-separation line. As explained in the introduction, we study some macroscopic features of the 2D Ising model starting from the microscopic description of the model. It is therefore natural to start by fixing some macroscopic box $Q \subset \mathbb{R}^2$, which we choose in an asymmetric way for latter purposes,

$$Q := \{ x = (x_1, x_2) \in \mathbb{R}^2 : |x_1| \leq 1 , \ 0 \leq x_2 \leq 2 \}. \tag{2.1}$$

Let L be an integer and $\Lambda_L \subset \mathbb{Z}^2$,

$$\Lambda_L := \{ x = (x_1, x_2) \in \mathbb{Z}^2 : |x_1| \leq L \,,\ 0 \leq x_2 \leq 2L \}. \tag{2.2}$$

Notice that after scaling by $1/L$, $\Lambda_L \subset Q$. Spin configurations are denoted by $\omega \in \{-1,+1\}^{\Lambda_L}$; the spin variable at $x \in \mathbb{Z}^2$ is $\sigma(x)$, $\sigma(x)(\omega) = \omega(x) = \pm 1$. Phase-separation lines are stochastic lines (see below), whose positions are fixed on the boundary of Λ_L by boundary conditions. The boundary $\partial\Lambda_L$ of Λ_L is the subset

$$\partial\Lambda_L := \{ x \in \Lambda_L : \exists\, y \notin \Lambda_L \ \max_{i=1,2} |y_i - x_i| = 1 \}. \tag{2.3}$$

Boundary conditions (b.c.) for Λ_L consists in prescribing the value of the spin at $x \in \partial\Lambda_L$. For example, the $-$ b.c. means that $\omega(x) = -1\ \forall x \in \partial\Lambda_L$. In the general case boundary conditions are specified by $\eta \in \{-1,+1\}^{\partial\Lambda_L}$, so that for all configurations ω, $\omega(x) := \eta(x)\ \forall x \in \partial\Lambda_L$; we refer to that boundary condition as the η b.c.. Free boundary conditions means absence of boundary conditions.

The Hamiltonian of the model in Λ_L with η b.c. is the function on $\{-1,+1\}^{\Lambda_L}$

$$H^\eta_{\Lambda_L}(\omega) := \begin{cases} -\sum_{\langle t,t'\rangle \subset \Lambda_L} J(t,t')\sigma(t)(\omega)\sigma(t')(\omega) & \text{if } \omega(x) = \eta(x)\ \forall x \in \partial\Lambda_L; \\ +\infty & \text{otherwise.} \end{cases} \tag{2.4}$$

Here $\langle t,t'\rangle$ is the standard notation for a pair of nearest neighbour points of the lattice $\mathbb{Z}^2$, called bond. The coupling constants $J(t,t')$ are positive; we specify them later on. The Gibbs measure on $\{-1,+1\}^{\Lambda_L}$ with η b.c. is

$$\frac{\exp\{-\beta H^\eta_{\Lambda_L}(\omega)\}}{\Theta^\eta(\Lambda_L)}; \tag{2.5}$$

β is the inverse temperature and $\Theta^\eta(\Lambda_L)$, the partition function, is the normalization constant in (2.5). Expectation values are written $\langle\cdot\rangle^\eta_{\Lambda_L}$.

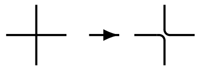
The dual lattice to $\mathbb{Z}^2$ is

$$\mathbb{Z}^{2*} := \{ x = (x_1, x_2) \in \mathbb{R}^2 : x + (1/2, 1/2) \in \mathbb{Z}^2 \}, \tag{2.6}$$

and the dual box $\Lambda^*_L \subset \mathbb{Z}^{2*}$ is

$$\Lambda^*_L := \{ x = (x_1, x_2) \in \mathbb{Z}^{2*} : |x_1| \leq L - 1/2 \,,\ 1/2 \leq x_2 \leq 2L - 1/2 \}. \tag{2.7}$$

Each bond $\langle t,t'\rangle$ defines a unit segment $e(t,t') \subset \mathbb{R}^2$ with end-points t, t'; to each bond $\langle t,t'\rangle$ such that $\langle t,t'\rangle \cap \Lambda_L\backslash\partial\Lambda_L \neq \emptyset$, there corresponds a unique dual bond $\langle t^*, t'^*\rangle \subset \Lambda^*_L$, which is defined by the condition that $e(t,t') \cap e(t^*, t'^*) \neq \emptyset$. Given boundary conditions η, each configuration ω, which is compatible with the η b.c., can be uniquely specified by giving all segments $e(t,t')$ such that $\sigma(t)(\omega)\sigma(t')(\omega) = -1$ and $\{t,t'\} \cap \Lambda_L\backslash\partial\Lambda_L \neq \emptyset$; this is equivalent to specify all dual segments $e(t^*, t'^*)$, or the corresponding dual bonds of Λ^*_L. The union of these dual segments forms a set of lines in $\mathbb{R}^2$, which we decompose into connected components. Whenever $\exists\, t \in \Lambda^*_L$, which belongs to four segments, we apply the deformation rule defined in the picture below, so that each configuration ω, compatible with the η b.c., is uniquely specified by a finite set of disjoint simple lines called **contours** of the configuration.



Let B be a set of dual bonds; the boundary δB of B is the set of $x \in \mathbb{Z}^{2*}$ such that there is an odd number of bonds of B adjacent to x. B is **closed** if $\delta B = \emptyset$ and **open** if $\delta B \neq \emptyset$. The contours of a configuration are either closed, or open with end-points on the boundary of Λ_L^*. The set $V_L(\eta) \subset \Lambda_L^*$ of the end-points of the open contours is uniquely determined by the η b.c.; its cardinality is even if $V_L(\eta) \neq \emptyset$. The set of closed contours is written $\underline{\gamma} = \{\gamma_1, \gamma_2, \ldots, \}$ and the set of open contours $\underline{\lambda} = \{\lambda_1, \lambda_2, \ldots\}$. We call the open contours the **phase-separation lines** of the configuration. Conversely, a family of contours $(\underline{\gamma}', \underline{\lambda}')$ is called η **compatible** if there exists ω compatible with the η b.c. such that $\underline{\gamma}(\omega) = \underline{\gamma}'$ and $\underline{\lambda}(\omega) = \underline{\lambda}'$.

The probability of $\underline{\lambda}$ can be computed with the Gibbs measure (2.5). It is however more convenient to introduce a non-normalized measure on the set of phase-separation lines, in order to exploit the duality property of the model. The **length** $|\gamma|$ of a closed contour γ is $\sum_{e\in\gamma} J(e)$. The sum of the lengths of the contours of a family $\underline{\gamma}$ is written $|\underline{\gamma}|$. Similar notations hold for open contours. Next we define two (normalized) partition functions, $Z^\eta(\Lambda_L)$ and $Z^\eta(\Lambda_L|\underline{\lambda})$, where $\underline{\lambda}$ and η are compatible,

$$Z^\eta(\Lambda_L) := \sum_{\omega:\ \eta \text{ comp.}} \exp\{-2\beta|\underline{\gamma}(\omega)|\} \exp\{-2\beta|\underline{\lambda}(\omega)|\}; \tag{2.8}$$

and

$$Z^\eta(\Lambda_L|\underline{\lambda}) := \sum_{\substack{\omega:\ \eta \text{ comp.} \\ \underline{\lambda}(\omega)=\underline{\lambda}}} \exp\{-2\beta|\underline{\gamma}(\omega)|\}. \tag{2.9}$$

We define a weight $q^\eta_{\Lambda_L}(\underline{\lambda})$ by setting

$$q^\eta_{\Lambda_L}(\underline{\lambda}) := \begin{cases} \exp\{-2\beta|\underline{\lambda}|\} \dfrac{Z^\eta(\Lambda_L|\underline{\lambda})}{Z^-(\Lambda_L)} & \text{if } \underline{\lambda} \text{ and } \eta \text{ are compatible,} \\ 0 & \text{otherwise.} \end{cases} \tag{2.10}$$

The weight $q^\eta_{\Lambda_L}(\underline{\lambda})$ does not define a probability measure on the set of η compatible phase-separation lines, since in (2.10) we divide by $Z^-(\Lambda_L)$ and not $Z^\eta(\Lambda_L)$.

2.2. Duality. A basic property of the 2D Ising model is self-duality. As a consequence of that property many questions about the model below the critical temperature can be translated into dual questions for the dual model above the critical temperature. For example, questions about the surface tension are translated into questions about the correlation length.

We define the dual objects to Λ_L, β and $J(t, t')$. The dual box Λ_L^* is defined in (2.7). The dual inverse temperature β^* is defined by

$$\tanh \beta^* := \exp\{-2\beta\}. \tag{2.11}$$

We recall that the critical inverse temperature β_c of the Ising model with coupling constants $J(t,t') \equiv 1$ is the fixed point of Eq. (2.11). Let $\langle t,t'\rangle$ be a bond of $\mathbb{Z}^2$ and $\langle t^*, t'^*\rangle$ its dual bond; the dual coupling constant $J^*(t^*, t'^*)$ is defined by

$$\tanh \beta^* J^*(t^*, t'^*) := \exp\{-2\beta J(t,t')\}. \tag{2.12}$$

Let

$$H_{\Lambda_L^*} := -\sum_{\langle t,t'\rangle\subset\Lambda_L^*} J^*(t,t')\sigma(t)\sigma(t') \tag{2.13}$$

be the Hamiltonian in the dual box Λ_L^* with free boundary conditions and dual coupling constants. The expectation value with respect to the corresponding Gibbs measure at the dual temperature β^* is written $\langle\,\cdot\,\rangle_{\Lambda_L^*}$.

A key dual statement is the following one. Let $\underline{\lambda}$ be a family of phase-separation lines, which are η compatible with a given η b.c. for Λ_L. Then

$$\sum_{\underline{\lambda}} q^{\eta}_{\Lambda_L}(\underline{\lambda}) = \langle \prod_{t\in V_L(\eta)} \sigma(t)\,\rangle_{\Lambda_L^*}. \tag{2.14}$$

Formula (2.14) is our starting point for analysing the interfaces of the model. It is proven in Sect. 6. In that section we identify the weight $q^{\eta}_{\Lambda_L}(\underline{\lambda})$ with the weight $q_{\Lambda_L^*}(\underline{\lambda})$ of $\underline{\lambda}$ in the high-temperature representation of the model defined in the dual box Λ_L^* with free boundary conditions (see Lemma 6.2).

2.3. Surface tension and wall free energy. We recall the definition of surface tension as given for example in the review paper [A2] formula (2.14a) (see also [Pf1]), since this is the definition which is mostly used. In Sect. II.D of [A2] other definitions of surface tension are reviewed and compared. The heuristic grounds given on p.10 of [A2] (see also note 12 in [Pf1]) lead to a definition of the surface tension as the logarithm of the ratio of two partition functions with different boundary conditions. The results of Sect. 7 show that this may lead to a wrong definition of the surface tension for an Ising model with modified coupling constants on one part of the boundary. The heuristic grounds give a correct definition only if the walls of the box are in the complete wetting regime, a crucial physical condition, which has been so far overlooked in the literature. See Sect. 7 where we consider explicitly the dual question of equivalence of short and long correlation length, but the results apply to the definition of the surface tension. As explained in the Introduction our working definition of the surface tension is different. The fact that we get the same quantity is a consequence of Proposition 2.2. The ultimate justification for the definition of the surface tension is that it should be equal to the quantity, which enters into the formulation of the variational problem describing the behaviour of the interface at the macroscopic level. This is the subject of this paper.

2.3.1. Surface tension. Consider the model defined in Λ'_L,

$$\Lambda'_L := \{\, x \in \mathbb{Z}^2 \,:\, |x_i| \le L\,,\ i = 1,2\,\}, \tag{2.15}$$

with coupling constants $J(t,t') \equiv 1$ and inverse temperature β. Let $n \in \mathbb{R}^2$ be a unit vector; denote by $\mathcal{D}_n$ the straight line perpendicular to n and passing through the origin of $\mathbb{R}^2$. The η_n b.c. for Λ'_L is defined by

$$\eta_n(x) := \begin{cases} -1 & \text{if } x \in \partial\Lambda'_L \text{ is below or on } \mathcal{D}_n, \\ +1 & \text{if } x \in \partial\Lambda'_L \text{ is above } \mathcal{D}_n. \end{cases} \tag{2.16}$$

Let D_n be the Euclidean length of the segment $\{x \in \mathbb{R}^2 : |x_i| \leq 1\} \cap \mathcal{D}_n$. If ω is compatible with the η_n b.c. there is a unique phase-separation line $\lambda(\omega)$. The limit $\hat{\tau}(n;\beta)$

$$\hat{\tau}(n;\beta) := -\lim_{L\to\infty} \frac{1}{LD_n} \ln \frac{Z^{\eta_n}(\Lambda'_L)}{Z^{-}(\Lambda'_L)} \tag{2.17}$$

exists and is called the **surface tension** at inverse temperature β. By symmetry of the model we have ($n = (n_1, n_2)$)

$$\hat{\tau}(n_1,n_2;\beta) = \hat{\tau}(-n_1,-n_2;\beta) = \hat{\tau}(n_2,-n_1;\beta) = \hat{\tau}(n_2,n_1;\beta). \tag{2.18}$$

We extend the definition of $\hat{\tau}(n;\beta)$ to $\mathbb{R}^2$ by homogeneity ($\|\cdot\|$ is the Euclidean norm),

$$\hat{\tau}(x;\beta) := \|x\|\hat{\tau}(x/\|x\|;\beta). \tag{2.19}$$

Proposition 2.1. *Let $J(t,t') \equiv 1$. The surface tension is a uniformly Lipschitz convex function on $\mathbb{R}^2$ such that $\hat{\tau}(x;\beta) = \hat{\tau}(-x;\beta)$. It is identically zero if $\beta \leq \beta_c$, and strictly positive for all $x \neq 0$ if $\beta > \beta_c$. In the latter case $\hat{\tau}$ defines a norm on $\mathbb{R}^2$. The main property of $\hat{\tau}$ is the sharp triangle inequality. For all $\beta > \beta_c$ there exists a strictly positive constant $\kappa = \kappa(\beta)$ such that for any x, $y \in \mathbb{R}^2$,*

$$\hat{\tau}(x;\beta) + \hat{\tau}(y;\beta) - \hat{\tau}(x+y;\beta) \geq \kappa(\|x\| + \|y\| - \|x+y\|). \tag{2.20}$$

Let $x(\theta) := (\cos\theta, \sin\theta)$ and $\hat{\tau}(\theta;\beta) := \hat{\tau}(x(\theta);\beta)$. Then the best constant κ is

$$\kappa := \inf_\theta \left(\frac{d^2}{d\theta^2}\hat{\tau}(\theta;\beta) + \hat{\tau}(\theta;\beta) \right) > 0. \tag{2.21}$$

The first part of the proposition is proved in [LP] and [Pf2] (Lemma 6.4). The arguments are not restricted to the 2D Ising model. Ioffe [I1] proved an equivalent form of inequality (2.20), but with a non-optimal value of κ. Inequality (2.20) as stated here first appeared in [V]. The strict positivity of the optimal constant κ given in (2.21) follows from the exact expression of $\hat{\tau}(\theta;\beta)$ [AA]; it is called the **positive stiffness property**.

Remark. Geometrically (2.21) means that the curvature of the Wulff shape is bounded above by $1/\kappa$. It is well-known that the surface tension is the support function of the Wulff crystal. The following result of Convex Theory is interesting, and appears to be new as far as we know [V]. It characterizes the compact convex bodies W in $\mathbb{R}^2$ which have a support function $\hat{\tau}$,

$$\hat{\tau}(x) := \sup_{y^* \in W} \langle y^*, x \rangle, \tag{2.22}$$

satisfying the sharp triangle inequality

$$\hat{\tau}(x) + \hat{\tau}(y) - \hat{\tau}(x+y) \geq K'(\|x\| + \|y\| - \|x+y\|). \tag{2.23}$$

In (2.22) $\langle \cdot, \cdot \rangle$ is the Euclidean scalar product. No smoothness of the boundary of W is assumed. Let W_1 and W_2 be two convex bodies; we say that ∂W_1 is tangent to ∂W_2 at x^* if W_1 and W_2 have a common support plane at x^*. We recall the notion of radius of curvature of W at x^*. Let U be an open neighborhood of x^*. Let $\mathcal{T}_i(x^*, U)$ be the family of discs $\mathcal{D}$ with the following properties:

1. $\partial\mathcal{D}$ is tangent to ∂W at x^*;
2. $W \cap U \supset \mathcal{D} \cap U$.

We allow the degenerate cases where the disc is a single point or a half-plane. Consequently $\mathcal{T}_i(x^*, U) \neq \emptyset$. We denote by $\rho(\mathcal{D})$ the radius of the disc $\mathcal{D}$ and set

$$\underline{\rho}(x^*, U) := \sup\{\rho(\mathcal{D}) : \mathcal{D} \in \mathcal{T}_i(x^*, U)\}. \tag{2.24}$$

Clearly $\underline{\rho}(x^*, U_1) \leq \underline{\rho}(x^*, U_2)$ if $U_1 \supset U_2$. The **lower radius of curvature** at x^* is defined as

$$\underline{\rho}(x^*) := \sup\{\underline{\rho}(x^*, U) : U \text{ open neighborhood of } x^*\}. \tag{2.25}$$

Theorem 2.1. *Let W be a convex compact body and $\hat{\tau}$ be its support function. Then the following statements are equivalent:*

1. *The lower radius of curvature of ∂W is uniformly bounded below by $K > 0$.*
2. *There exists a positive constant K' such that for any $x, y \in \mathbb{R}^2$,*

$$\hat{\tau}(x) + \hat{\tau}(y) - \hat{\tau}(x + y) \geq K'(\|x\| + \|y\| - \|x + y\|). \tag{2.26}$$

There is a well-known dual relation between the surface tension at inverse temperature β and the decay-rate of the two-point function at the dual temperature β^*, which we recall here. Consider the 2D Ising model on the dual lattice, with coupling constants $J^*(t, t') \equiv 1$, inverse temperature β^* and free b.c.. The two-point function, or covariance, is

$$\langle \sigma(t)\sigma(t')\rangle(\beta^*) \quad , \quad t, t' \in \mathbb{Z}^{2*}, \tag{2.27}$$

where $\langle \cdot \rangle(\beta^*)$ denotes expectation value with respect to the infinite volume free b.c. Gibbs measure at inverse temperature β^*. The **decay-rate** of the two-point function is defined for all $t, t' \in \mathbb{Z}^{2*}$ as

$$\tau(t - t'; \beta^*) := -\lim_{\substack{k\in\mathbb{N}\\ k\to\infty}} \frac{1}{k} \ln\langle \sigma(kt)\sigma(kt')\rangle(\beta^*). \tag{2.28}$$

Proposition 2.2. *Let $J(t, t') \equiv 1$. The surface tension $\hat{\tau}(x; \beta)$ of the 2D Ising model and the decay-rate $\tau(x; \beta^*)$ are equal,*

$$\hat{\tau}(x; \beta) = \tau(x; \beta^*) \quad \forall x. \tag{2.29}$$

Remark. Identity (2.29) has been noticed several times; we refer the reader to [ZA] where a brief historical account with references is given at the beginning of their paper. However, a proof of formula (2.29) does not follow from duality only. There is an exchange of limits, which must be justified (see e.g. [BLP2]). We show in Sect. 7 that there are cases where the exchange of limits is not valid and such a relation does not hold.

2.3.2. Wall free energy. There is another thermodynamical quantity, which enters into the description of the properties of the interface, the wall free energy. In the phase-coexistence regime it depends on the nature of the bulk phase. Only the difference of wall free energies when the bulk phase is either the $+$ phase or the $-$ phase has an intrinsic meaning. In order to have interesting surface phenomena we single out one part of the boundary of the box Λ_L, the bottom part. (This is the reason for our asymmetrical choice of Λ_L.) We choose here the coupling constants of the model as follows.

$$J(t,t') := \begin{cases} h > 0 & \text{if } t_2 = 0 \text{ or } t_2' = 0, \\ 1 & \text{otherwise.} \end{cases} \tag{2.30}$$

We compare the free energy for two different b.c., one being the $-$ b.c. and the other one the $\eta_\pm$ b.c., defined as

$$\eta_\pm(x) := \begin{cases} -1 & \text{if } x \in \partial\Lambda_L \text{ and } x_2 = 0, \\ 1 & \text{if } x \in \partial\Lambda_L \text{ and } x_2 > 0. \end{cases} \tag{2.31}$$

We set[1]

$$\hat{\tau}_{\mathrm{bd}}(\beta, h) := -\lim_{L\to\infty} \frac{1}{2L+1} \ln \frac{Z^{\eta_\pm}(\Lambda_L)}{Z^-(\Lambda_L)}. \tag{2.32}$$

The quantity $\hat{\tau}_{\mathrm{bd}}(\beta, h)$, which gives the difference of two free energies, verifies the fundamental inequalities (2.34) for any $D \geq 2$, [FP1] and [FP2]. Let $n_w := (0, 1)$ and set

$$\hat{\tau}(\beta) := \hat{\tau}(n_w; \beta); \tag{2.33}$$

for any β and any h,

$$|\hat{\tau}_{\mathrm{bd}}(\beta, h)| \leq \hat{\tau}(\beta). \tag{2.34}$$

If $\beta > \beta_c$ and $h > 0$, then

$$0 < \hat{\tau}_{\mathrm{bd}}(\beta, h) \leq \hat{\tau}(\beta). \tag{2.35}$$

Suppose that $\beta > \beta_c$. The difference between the two free energies, per unit length, is interpreted as the free energy, per unit length, of the horizontal interface produced by the boundary condition $\eta_\pm$. If $\hat{\tau}_{\mathrm{bd}}(\beta, h) = \hat{\tau}(\beta)$, then this free energy is equal to the surface tension of an horizontal interface. This indicates that the interface produced by the boundary condition $\eta_\pm$ b.c. is not pinned; or in other terms, we have complete wetting of the wall by the $-$ phase. On the other hand, if $\hat{\tau}_{\mathrm{bd}}(\beta, h) < \hat{\tau}(\beta)$, then this indicates that the interface is pinned, or in other words, that we have partial wetting. What we just described is Cahn's criterion for the wetting transition: when $h > 0$ there is partial wetting of the wall if and only if $\hat{\tau}_{\mathrm{bd}}(\beta, h) < \hat{\tau}(\beta)$. In terms of Gibbs states one can prove, [FP1] and [FP2], that near the wall all Gibbs states are identical if and only if $|\hat{\tau}_{\mathrm{bd}}(\beta, h)| = \hat{\tau}(\beta)$. Intuitively this is easy to understand: at the *microscopic* level the state of the system near the wall is always the state of the $-$ phase near the wall, since the wall is in the complete wetting regime. By contrast, in the partial wetting regime

[1] The definition of $\hat{\tau}_{\mathrm{bd}}$ differs from the analogous quantity used in [PV1] or [PV2], because in these papers the reference bulk phase is the $+$ phase and here it is the $-$ phase.

the state near the wall depends on the nature of the bulk phase. The behaviour of Gibbs states near the wall can be distinguished by the order parameter

$$\lim_{L\to\infty} \langle\, \sigma(0,1)\, \rangle^{\eta}_{\Lambda_L}(\beta,h). \tag{2.36}$$

Fröhlich and Pfister in [FP2] proved that there are several Gibbs states near the bottom wall if and only if

$$\lim_{L\to\infty} \langle\, \sigma(0,1)\, \rangle^{-}_{\Lambda_L}(\beta,h) \neq \lim_{L\to\infty} \langle\, \sigma(0,1)\, \rangle^{\eta^{\pm}}_{\Lambda_L}(\beta,h). \tag{2.37}$$

This occurs if and only if $h < h_w$, with $h_w = h_w(\beta)$, a temperature dependent coupling, which is defined by (see (2.27) in [FP2])

$$h_w(\beta) = \inf\{\, h \geq 0 \,:\, \lim_{L\to\infty} \langle\, \sigma(0,1)\, \rangle^{-}_{\Lambda_L}(\beta,h) = \lim_{L\to\infty} \langle\, \sigma(0,1)\, \rangle^{\eta^{\pm}}_{\Lambda_L}(\beta,h)\,\}. \tag{2.38}$$

Using the results of Fröhlich and Pfister [FP1] and [FP2], and those of Pfister and Penrose [PP] one can show that the surface magnetizations computed by McCoy and Wu (see Chapter VI in [MW]) can be identified with the above quantities

$$\lim_{L\to\infty} \langle\, \sigma(0,1)\, \rangle^{-}_{\Lambda_L}(\beta,h) \quad \text{and} \quad \lim_{L\to\infty} \langle\, \sigma(0,1)\, \rangle^{\eta^{\pm}}_{\Lambda_L}(\beta,h).$$

Therefore h_w can be computed from their work, h_w being given by formula (5.44), p.137 of [MW]; it is not difficult to show that an equivalent form of this expression is (2.39), which is the formula given by Abraham for the value of h_w, where the pinning-depinning transition occurs,

$$\exp\{2\beta\}\{\cosh 2\beta - \cosh 2\beta h_w(\beta)\} = \sinh 2\beta. \tag{2.39}$$

An equivalent computation of h_w based on Cahn's criterion is given in [AC].

Remark. At the time when McCoy and Wu discovered this surface phase transition nobody understood what was physically implied: the transition was interpreted as a boundary hysteresis phenomenon. This interpretation is, however, misleading, the transition is not related to any kind of metastability. The plot of the quantities corresponding to $\lim_{L\to\infty} \langle\, \sigma(0,1)\, \rangle^{-}_{\Lambda_L}(\beta,h)$ and $\lim_{L\to\infty} \langle\, \sigma(0,1)\, \rangle^{\eta^{\pm}}_{\Lambda_L}(\beta,h)$ is given in Fig. 6.6, Chapter VI of [MW].

Besides the extensive computations for the semi-infinite Ising model of McCoy and Wu, Abraham, Abraham and coworkers, we also mention [AY] and [AF]; this list is not exhaustive.

As for the surface tension there is a dual expression for $\hat{\tau}_{\mathrm{bd}}$. We first introduce the two-point function of the model on the half-infinite lattice

$$\mathbb{L}^* := \{\, x \in \mathbb{Z}^{2*} \,:\, x_2 \geq 1/2\,\}, \tag{2.40}$$

as

$$\langle \sigma(t)\sigma(t')\rangle_{\mathbb{L}^*}(\beta^*,h^*) := \lim_{L\to\infty} \langle \sigma(t)\sigma(t')\rangle_{\Lambda_L^*}(\beta^*,h^*). \tag{2.41}$$

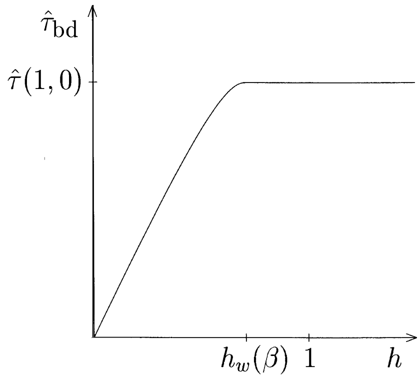


Fig. 1. $\hat{\tau}_{\rm bd}$ as a function of the magnetic field h, for $\beta = 1.4\beta_c$

Proposition 2.3. *Let the coupling constants be given by (2.30), $h > 0$, and let $\beta > \beta_c$. Let β^*, h^* be the dual coupling constants, $t, t' \in \Lambda_L^*$, $t(2) = t'(2) = 1/2$. Then the limit*

$$-\lim_{n\to\infty} \frac{1}{n} \ln\langle\sigma(nt)\sigma(nt')\rangle_{\mathbb{L}^*}(\beta^*, h^*) = |t_1 - t_1'| \cdot \tau_{\rm bd}(\beta^*, h^*) \tag{2.42}$$

exists and $\tau_{\rm bd}(\beta^*, h^*) = \hat{\tau}_{\rm bd}(\beta, h)$.

See [PV1] for a proof.

2.4. Two-point correlation function. There are close relations between surface tension, resp. wall free energy, and decay-rate of the two-point correlation function, resp. boundary two-point correlation function (Propositions 2.2 and 2.3). The next proposition states fundamental estimates about the two-point correlation functions, which we need later on. As in the previous section, see (2.33), $\hat{\tau}(\beta) := \hat{\tau}(n_w; \beta)$ and $\tau(\beta^*) = \hat{\tau}(\beta)$.

Proposition 2.4. *Let $J(e) \equiv 1$. Let $\beta^* < \beta_c$.*

1. *There exist positive constants K and a_b such that for all $x, y \in \mathbb{Z}^{2*}$,*

$$K \frac{\exp\{-\tau(y-x;\beta^*)\}}{\|x-y\|^{a_b}} \le \langle\sigma(x)\sigma(y)\rangle(\beta^*) \le \exp\{-\tau(y-x;\beta^*)\}. \tag{2.43}$$

2. *Let the coupling constants be given by (2.30), with $h = h^*$, $0 < h^* < \infty$. If $\tau_{\rm bd}(\beta^*, h^*) = \tau(\beta^*)$, then there exists a constant K' such that for all $x, y \in \mathbb{L}^*$, with $x_2 = y_2 = 1/2$,*

$$\begin{aligned} K' \frac{\exp\{-\tau(\beta^*)|x_1 - y_1|\}}{\|x-y\|^{3/2}} &\le \langle\sigma(x)\sigma(y)\rangle_{\mathbb{L}^*}(\beta^*, h^*) \\ &\le \exp\{-\tau(\beta^*)|x_1 - y_1|\}. \end{aligned} \tag{2.44}$$

3. *Let the coupling constants be given by (2.30), with $h = h^*$, $0 < h^* < \infty$. If $\tau_{\mathrm{bd}}(\beta^*, h^*) < \tau(\beta^*)$, then there exists a constant K'' such that for all $x, y \in \mathbb{L}^*$, with $x_2 = y_2 = 1/2$,*

$$\begin{aligned} K'' \exp\{-\tau_{\mathrm{bd}}(\beta^*, h^*)|x_1 - y_1|\} &\leq \langle\sigma(x)\sigma(y)\rangle_{\mathbb{L}^*}(\beta^*, h^*) \\ &\leq \exp\{-\tau_{\mathrm{bd}}(\beta^*, h^*)|x_1 - y_1|\}. \end{aligned} \tag{2.45}$$

Remarks. 1. The upper bounds are well-known consequences of sub-additivity and GKS inequalities, see e.g. [PV1].

2. The lower bound (2.43) has been proved recently by Alexander [Al]; his method is robust and can be applied to different models of statistical mechanics, e.g. percolation, Potts or random-cluster models. The value obtained by this method is not optimal (see the next remark).

3. The optimal value in (2.43) is $a_b = 1/2$. Notice that for our purpose the bound (2.43) derived by Alexander is sufficient. However, the determination of the asymptotic behaviour of the two-point function is an important theoretical question. A detailed asymptotic study of the two-point function of the Ising model when $D = 2$ is made in Chapter XII of [MW] (in particular (4.39) therein); see the very informative discussion of their results in Sect. 5 of the same chapter. For dimension $D \geq 2$ the expected behaviour is

$$\langle\sigma(x)\sigma(y)\rangle(\beta^*) = \varphi(n(y - x); \beta^*)\frac{\exp\{-\tau(y - x; \beta^*)\}}{\|x - y\|^{\frac{D-1}{2}}}, \tag{2.46}$$

with $n(y - x) = (x - y)/\|x - y\|$. Recently Ioffe [I2] proved such a formula for the simple self-avoiding walk on $\mathbb{Z}^D$, $D \geq 2$, with $\varphi : S^{D-1} \to \mathbb{R}^+$ an analytic function.

4. The lower bound (2.44) follows again from the work of [MW] when $h^* = 1$ (Chapter VII, in particular the discussion pp. 144–145). Using correlation inequalities, it can be extended to the general case as shown in [PV1, Prop. 7.1].

5. The lower bound in (2.45) is proven in [PV1, Prop. 7.1].

3. A Microscopic Model for the Pinning Transition

We define a microscopic model for a system with two coexisting phases, separated by an interface, where we have a reentrant pinning-depinning transition. Our model is inspired by the work of Patrick [Pa1], who showed that there is a reentrant pinning-depinning transition for the SOS model corresponding to our settings. In a recent work, Patrick and Upton [PU] studied in the Ising model questions similar to those investigated here. The interesting fact that we can have reentrant pinning-depinning transition for an Ising model with ferromagnetic coupling constants only is not new. This is for example proved in [ACD] for a different choice of the coupling constants; in our notations this corresponds to

$$J(t, t') := \begin{cases} c > 0 & \text{if } t_2 = 0 \text{ and } t'_2 = 1 \text{ or vice-versa,} \\ 0 & \text{if } t_2 = t'_2 = 0 \text{ or } t_2 = t'_2 = 1, \\ b > 0 & \text{if } t_2 = 1 \text{ and } t'_2 = 2 \text{ or vice-versa,} \\ 1 & \text{otherwise.} \end{cases} \tag{3.1}$$

In [ACD] the two boundary conditions $\eta_\pm$ are considered

$$\eta_\pm(x) := \begin{cases} \pm 1 & \text{if } x \in \Lambda_L,\, x_2 = 0, \\ 1 & \text{otherwise.} \end{cases} \tag{3.2}$$

This model differs from our model; if we integrate over the spins of the row $\{x \in \Lambda_L, x_2 = 1\}$, then the resulting Hamiltonian is equivalent to our Hamiltonian defined by the coupling constants (3.3), but with now an effective nonlinear temperature dependent coupling $h = h(T)$ (see formula (11) in [ACD]).

Our method proceeds in two steps. First, we derive a macroscopic variational problem characterizing the typical configurations. This part of the analysis is based on the probabilistic methods developed in Sect. 6 and following. The main advantage we gain is that these methods are robust (see for example the Appendix). In the second step, we solve explicitly the variational problem. It is at that point that we need the exact expressions of the surface tension and wall free energy.

Let Q be the macroscopic box (2.1) and denote by $W_Q := \{\, x \in Q : x(2) = 0 \,\}$ its bottom wall. We want to describe at the macroscopic level an interface going from the point $A := (-1, a)$, $0 < a < 2$, to the point $B := (1, b)$, $0 < b < 2$, which can be pinned by the bottom wall W_Q. The idea is to introduce a grid in Q with lattice spacing $1/L$, $L \in \mathbb{N}$, and to consider an Ising model on that grid. When L tends to infinity we hope to have a good microscopic description of the macroscopic physical situation. It is however more convenient to work with a fixed lattice with lattice spacing unity, when we investigate asymptotic properties of the model for L tending to infinity. Therefore we define the model in the box Λ_L (see (2.2)). We choose the coupling constants of the model as follows,

$$J(t, t') := \begin{cases} h > 0 & \text{if } t_2 = 0 \text{ or } t'_2 = 0, \\ 1 & \text{otherwise.} \end{cases} \tag{3.3}$$

The boundary conditions specify the end-points of one phase-separation line, which is the microscopic manifestation of the interface. The boundary conditions are η_{ab},

$$\eta_{ab}(x) := \begin{cases} +1 & \text{if } x \in \Lambda_L,\, x_2 = 2L, \\ +1 & \text{if } x_1 = -L \text{ and } aL \le x_2 \le 2L, \\ +1 & \text{if } x_1 = L \text{ and } bL \le x_2 \le 2L, \\ -1 & \text{otherwise.} \end{cases} \tag{3.4}$$

In each spin configuration compatible with η_{ab} there is a unique phase-separation line λ with end-points in $V_L(\eta_{ab}) := \{u^L, v^L\}$, $u_1^L = -L$ and $v_1^L = L$. The normalized partition function is denoted by $Z^{ab}(\Lambda_L) \equiv Z^{\eta_{ab}}(\Lambda_L)$.

Problem. Describe the statistical properties of the phase-separation line λ and show that there is reentrant pinning-depinning transition. Derive the macroscopic theory developed in Sect. 4 from the microscopic theory.

Remark. In [AK] the same model is studied, with similar, but different boundary conditions; the pinning of the interface is used in order to define the contact angle and give an exact derivation of the modified Young equation for partial wetting.

4. The Variational Problem

The interface is a macroscopic deterministic object, whose properties are described by a functional involving the surface tension or the wall free energy. The equilibrium state of the interface is given by the minimum of this functional.

In Q the interface is a simple rectifiable curve $\mathcal{C}$ with end-points $A = (-1, a)$, $0 < a < 2$, and $B = (1, b)$, $0 < b < 2$. We denote by $|\mathcal{C} \cap W_Q|$ the length of the portion of the interface in contact with the wall W_Q. Suppose that $[0, t] \to Q$, $s \mapsto \mathcal{C}(s) = (u(s), v(s))$, is a parameterization of the interface. The free energy of the interface $\mathcal{C}$ can be written

$$\mathsf{W}(\mathcal{C}) := \int_0^t \hat{\tau}(\dot{u}(s), \dot{v}(s))ds + |\mathcal{C} \cap W_Q| \cdot \Big[\hat{\tau}_{\text{bd}} - \hat{\tau}(1, 0)\Big], \tag{4.1}$$

because the function $\hat{\tau}(x_1, x_2)$ is positively homogeneous and $\hat{\tau}(x_1, x_2) = \hat{\tau}(-x_2, x_1)$. The interface at equilibrium is the minimum of this functional. Therefore we have to solve the

Variational problem. Find the minimum of the functional W among all simple rectifiable open curves in Q with extremities $A = (-1, a)$ and $B = (1, b)$.

Let $\mathcal{D}$ be the straight line from A to B and $\mathcal{W}$ be the curve composed of three straight line segments: from A to a point $P_1 \in W_Q$, from P_1 to $P_2 \in W_Q$, and from P_2 to B. The points P_1 resp. P_2 are such that the angles between the first segment and the wall resp. between the last segment and the wall are equal to $\theta_Y \in [0, \pi/2]$, which is a solution of the Herring–Young equation (4.2)

$$\cos\theta_Y\ \hat{\tau}(\theta_Y) - \sin\theta_Y\ \hat{\tau}'(\theta_Y) = \hat{\tau}_{\text{bd}}. \tag{4.2}$$

$\mathcal{W}$ is a simple curve in Q if and only if

$$\theta_Y \in [\arctan\frac{a+b}{2}, \pi/2). \tag{4.3}$$

Remarks. 1. The choice, $\theta_Y \in [0, \pi/2]$, leads to a different sign at the right-hand side of the Herring–Young equation (4.2) than in [PV2] formulae (1.5) or (4.60); in these latter references we use $\pi - \theta$ instead of θ.

2. For the case under consideration the existence of θ_Y is an immediate consequence of the Winterbottom construction. In our case we have supposed that $h > 0$, so that $\hat{\tau}_{\text{bd}} > 0$. Since $\tau'(\pi/2) = 0$ the case $\theta_Y = \pi/2$ never occurs.

Proposition 4.1. *Let θ_Y be the solution of the Herring–Young equation (4.2).*

1. *If* $\tan\theta_Y \le \frac{a+b}{2}$*, then the minimum of the variational problem is given by the curve* $\mathcal{D}$.
2. *If* $\pi/2 > \theta_Y > \arctan(\frac{a+b}{2})$*, then the minimum of the variational problem is given by* $\mathcal{D}$ *if* $\mathsf{W}(\mathcal{D}) < \mathsf{W}(\mathcal{W})$*, by* $\mathcal{W}$ *if* $\mathsf{W}(\mathcal{D}) > \mathsf{W}(\mathcal{W})$ *and by both* $\mathcal{D}$ *and* $\mathcal{W}$ *if* $\mathsf{W}(\mathcal{D}) = \mathsf{W}(\mathcal{W})$.

Proof. The proof is an easy consequence of the two following lemmas. Lemma 4.1 states that the minimum is a polygonal line.

Lemma 4.1. *Let $\mathcal{C}$ be some simple rectifiable parameterized curve with initial point A and final point B. If $\mathcal{C}$ does not intersect the wall, then*

$$\mathtt{W}(\mathcal{C}) \geq \mathtt{W}(\mathcal{D}) \tag{4.4}$$

with equality if and only if $\mathcal{C}=\mathcal{D}$. If $\mathcal{C}$ intersects the wall, let t_1 be the first time $\mathcal{C}$ touches the wall and t_2 the last time $\mathcal{C}$ touches the wall. Let $\widehat{\mathcal{C}}$ be the curve defined by three segments from A to $\mathcal{C}(t_1)$, from $\mathcal{C}(t_1)$ to $\mathcal{C}(t_2)$ and from $\mathcal{C}(t_2)$ to B. Then

$$\mathtt{W}(\mathcal{C}) \geq \mathtt{W}(\widehat{\mathcal{C}}). \tag{4.5}$$

Equality holds if and only if $\mathcal{C} = \widehat{\mathcal{C}}$.

Proof. Since $\hat{\tau}$ is convex and homogeneous, we have in the first case by Jensen's inequality

$$\mathtt{W}(\mathcal{C}) = t\frac{1}{t}\int_0^t \hat{\tau}(\dot{u}(s), \dot{v}(s))ds \geq t\,\hat{\tau}(\frac{1}{t}\int_0^t \dot{u}(s)ds, \frac{1}{t}\int_0^t \dot{v}(s)ds) = \mathtt{W}(\mathcal{D}). \tag{4.6}$$

The inequality is strict if $\mathcal{C} \neq \mathcal{D}$ as is seen using the sharp triangle inequality (2.20).

In the second case we apply Jensen's inequality to the part of $\mathcal{C}$ between A and $\mathcal{C}(t_1)$ and between $\mathcal{C}(t_2)$ and B to compare with the corresponding straight segments of $\widehat{\mathcal{C}}$. Combining Jensen's inequality and the fact that $\hat{\tau}_{\mathrm{bd}} \leq \hat{\tau}$, we can also compare the part of $\mathcal{C}$ between $\mathcal{C}(t_1)$ and $\mathcal{C}(t_2)$ with the corresponding straight segment of $\widehat{\mathcal{C}}$. □

Lemma 4.2. *Let $\widehat{\mathcal{C}}$ be a polygonal line from A to $\hat{P}_1 \in W_Q$, then from $\hat{P}_1$ to $\hat{P}_2 \in W_Q$, and finally from $\hat{P}_2$ to B. Let θ_Y be the solution of the Herring–Young equation (4.2). If $\pi/2 > \theta_Y > \arctan(\frac{a+b}{2})$ then*

$$\mathtt{W}(\widehat{\mathcal{C}}) \geq \mathtt{W}(\mathcal{W}), \tag{4.7}$$

with equality if and only if $\widehat{\mathcal{C}} = \mathcal{W}$. If $\arctan(\frac{a+b}{2}) \geq \theta_Y$,

$$\mathtt{W}(\widehat{\mathcal{C}}) > \mathtt{W}(\mathcal{D}). \tag{4.8}$$

Proof. Let θ_1 be the angle of the segment of $\widehat{\mathcal{C}}$ from A to $\hat{P}_1$ with the wall W_Q, and θ_2 be the angle of segment from $\hat{P}_2$ to B with the wall W_Q. A necessary and sufficient condition, that the polygonal line $\widehat{\mathcal{C}}$ is a simple polygonal line, is

$$\frac{a}{\tan\theta_1} + \frac{b}{\tan\theta_2} \leq 2. \tag{4.9}$$

In particular, we certainly have $\theta_1 \geq \theta_a$, where $\theta_a := \arctan a/2$, and $\theta_2 \geq \theta_b$, where $\theta_b := \arctan b/2$. Since we suppose that $a > 0$ and $b > 0$ we have $\theta_a > 0$ and $\theta_b > 0$. We suppose that $\theta_Y \in (0, \pi/2)$, since $\theta_Y = 0$ occurs only if $\hat{\tau}(0) = \hat{\tau}_{\mathrm{bd}}$, and in that case by Lemma 4.1 $\mathtt{W}(\widehat{\mathcal{C}}) > \mathtt{W}(\mathcal{D})$. We compute

$$\begin{aligned}\mathtt{W}(\widehat{\mathcal{C}}) &= \hat{\tau}(\theta_1)\frac{a}{\sin\theta_1} + \hat{\tau}_{\mathrm{bd}}(2 - \frac{a}{\tan\theta_1} - \frac{b}{\tan\theta_2}) + \hat{\tau}(\theta_2)\frac{b}{\sin\theta_2} \\ &= g(\theta_1, a) + g(\theta_2, b),\end{aligned} \tag{4.10}$$

where

$$g(\theta,x):=\hat\tau(\theta)\frac{x}{\sin\theta}+\hat\tau_{\mathrm{bd}}(1-\frac{x}{\tan\theta}). \tag{4.11}$$

Since θ_Y is a solution of (4.2),

$$\frac{\partial}{\partial\theta}g(\theta_Y,x)=\frac{x}{\sin^2\theta_Y}\Big(\sin\theta_Y\,\hat\tau'(\theta_Y)-\cos\theta_Y\,\hat\tau(\theta_Y)+\hat\tau_{\mathrm{bd}}\Big)=0. \tag{4.12}$$

The second derivative of $g(\theta,x)$ is

$$\frac{\partial^2}{\partial\theta^2}g(\theta,x)=\frac{x(\hat\tau(\theta)+\hat\tau''(\theta))}{\sin\theta}-\frac{2}{\tan\theta}\frac{\partial}{\partial\theta}g(\theta,x). \tag{4.13}$$

Therefore, for $\theta\in(0,\pi/2]$, we have

$$\frac{\partial}{\partial\theta}g(\theta,x)=x\int_{\theta_Y}^{\theta}\exp\{-\int_{\gamma}^{\theta}\frac{2}{\tan\alpha}\,d\alpha\}\,\frac{\hat\tau(\gamma)+\hat\tau''(\gamma)}{\sin\gamma}\,d\gamma. \tag{4.14}$$

Since $\hat\tau$ has positive stiffness, i.e. $\hat\tau(\theta)+\hat\tau''(\theta)>0$, (4.14) implies that θ_Y is an absolute minimum of $g(\theta,x)$ over the interval $(0,\pi/2]$, and that g is strictly monotonous over the intervals $(\theta_Y,\pi/2]$ and $(0,\theta_Y)$. Therefore

$$\mathtt{W}(\widehat{\mathcal{C}})\geq g(\theta_Y,a)+g(\theta_Y,b). \tag{4.15}$$

If (4.3) holds, then (4.15) implies $\mathtt{W}(\widehat{\mathcal{C}})\geq\mathtt{W}(\mathcal{W})$, because in that case

$$g(\theta_Y,a)+g(\theta_Y,b)=\mathtt{W}(\mathcal{W}). \tag{4.16}$$

If (4.14) does not hold, $\mathcal{W}$ is not a simple line and is not even necessarily inside Q. The two segments from A to the wall and from B to the wall intersect at some point $P\in Q$. Let $\widehat{\mathcal{W}}$ be the simple polygonal curve going from A to P, then from P to B. A simple application of Lemma 4.1, using the fact that $\hat\tau(1,0)\geq\hat\tau_{\mathrm{bd}}$, gives

$$g(\theta_Y,a)+g(\theta_Y,b)\geq\mathtt{W}(\widehat{\mathcal{W}}). \tag{4.17}$$

Applying again Lemma 4.1 we get

$$\mathtt{W}(\widehat{\mathcal{W}})>\mathtt{W}(\mathcal{D}). \tag{4.18}$$

□

5. Reentrance and Pinning Transition

The results of Sect. 4 show that, when the parameters a and b are well-chosen, the system under consideration can undergo a phase transition from a phase in which the interface is pinned to the wall on a macroscopic distance to a phase in which it does not touch the wall. It is interesting to consider the corresponding phase diagram, which is obtained using the explicit expressions for the mass gap of the 2-point function and the mass gap of the boundary 2-point function (by duality this provides exact expressions for the surface tension and wall free energy). The expressions we use are the following:

$$\begin{aligned} \hat{\tau}(\theta;\beta) &= |\cos\theta|\sinh^{-1}(\alpha|\cos\theta|) + |\sin\theta|\sinh^{-1}(\alpha|\sin\theta|), \\ \alpha &= \frac{2}{b}\big((1-b^2)/(1+\sqrt{\sin^2 2\theta + b^2\cos^2 2\theta})\big)^{1/2}, \\ b &= 2\sinh 2\beta\cosh^{-2} 2\beta, \end{aligned} \tag{5.1}$$

and for $0 \le h < h_w(\beta)$, with β^* and h^* the dual coupling constants to β and h,

$$\cosh\hat{\tau}_{\mathrm{bd}}(\beta,h) = \cosh^2(\beta^*)\coth(2\beta^* h^*) - \sinh^2(\beta^*)\coth[2\beta^*(h^*-1)]. \tag{5.2}$$

They can be found, for example, in [MW] [Eq. (4.39) of Chap. XII and Eq. (5.29) of Chap. VII]. Figure 2 shows a set of phase-transition lines, depending on the parameters a and b, in the T-h plane ($T = 1/k\beta$ being the temperature). The shaded area corresponds to the set of parameters

$$\{(T,h)\ :\ \hat{\tau}_{\mathrm{bd}}(\beta,h) < \hat{\tau}((1,0);\beta)\}. \tag{5.3}$$

In other words, the boundary of that region is the wetting transition line: If we set $a = b = 0$, then for values of the temperature and boundary magnetic field inside this set, the phase-separation line is pinned to the wall microscopically (partial wetting),

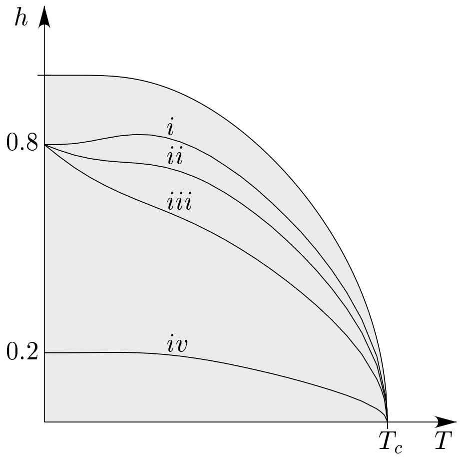


Fig. 2. A sequence of phase-transition lines, separating the phase in which the interface is a straight line and the phase in which it is pinned to the wall. The *shaded area* corresponds to the values of (T,h) so that $\hat{\tau}_{\mathrm{bd}}(\beta,h) < \hat{\tau}((1,0);\beta)$. The four curves correspond to: i) $a = 0.1$, $b = 0.1$; ii) $a = 0.1$, $b = 0.2$; iii) $a = 0.1$, $b = 0.4$; iv) $a = 0.4$, $b = 0.4$. Observe that the system in case i) exhibits reentrance (see also Fig. 3)

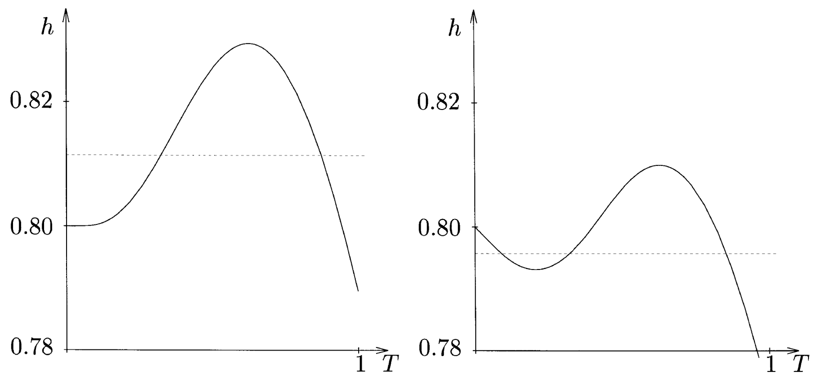


Fig. 3. This figure shows part of the phase-transition line for $a = 0.1, b = 0.1$ (*left*), and $a = 0.1, b = 0.12$ (*right*). For values of the parameters T and h below these curves the interface is pinned, while it is a straight line above these curves. Increasing the temperature along the dashed lines, we see that the system exhibits reentrance; this corresponds to the two situations discussed in the introduction

while for values of the parameters outside this set it takes off and fluctuates far from the wall (complete wetting). Notice that in the macroscopic limit, the interface lies always along the wall in this case. The four curves i) to iv) in Fig. 2 represent the phase-transition line for various values of the parameters $a > 0$ and $b > 0$. For any value of the parameters β and h above the phase-transition line, the system's interface is the straight line, while, for any value of these parameters below the curve, it is pinned. Clearly, since a and b are strictly positive, the phase-transition line must be inside the shaded region (when $\hat{\tau}_{\text{bd}}(\beta, h) = \hat{\tau}((1, 0); \beta)$, Jensen's inequality implies that the interface is always a straight line).
The phenomenon of reentrance described in the introduction can be seen in Figs. 2 and 3. Suppose $a = b = 0.2$ and h is slightly above 0.8 (this corresponds to the dashed line of the first picture in Fig. 3). At very low temperature, the interface does not touch the wall; if we increase the temperature, then there is a first transition and the interface becomes tied to the wall; if we increase further the temperature, then a second transition takes place and the interface is again the straight line; finally, at $T = T_c$, the system becomes disordered. In fact for slightly different values of a and b, there can even be one more transition, as shown in the second picture of Fig. 3.

6. High-Temperature Representation

We give the main results about the high-temperature representation of the Ising model. These results are not restricted to dimension 2, but for simplicity we consider only this case; we also use a definition of contour adapted to this particular case. We stress that the high-temperature representation is a non-perturbative approach; the basic objects in the high-temperature representation are defined for all positive β and we apply this representation for all $\beta < \beta_c$. The results are essential for the rest of our analysis, in particular Lemmas 6.9 and 6.11 about random-line representations of the two-point correlation function, and Lemmas 6.10 and 6.13, which characterize those random-lines, which give the main contribution to the two-point correlation function.

6.1. Ising model on a finite graph. We consider here the high-temperature representation of the Ising model with free boundary conditions, but we could treat + boundary conditions. The correct point of view is to define the model on a graph $\mathcal{G} = (V, B)$; to each vertex $t \in V$ of the graph we associate a spin variable $\sigma(t)$ and to each bond $e = \langle t, t' \rangle \in B$ a nonnegative coupling constant $K(e) = K(t, t')$, which takes into account the inverse temperature, so that in the applications $K(e) = \beta^* J^*(e)$. The Gibbs measure on $\mathcal{G}$ is

$$\frac{\exp\{\sum_{e=\langle t,t' \rangle \in B} K(e)\sigma(t)\sigma(t')\}}{\Xi(\mathcal{G})}. \tag{6.1}$$

The constant $\Xi(\mathcal{G})$ is the partition function,

$$\begin{aligned} \Xi(\mathcal{G}) : &= \sum_{\sigma(t)=\pm 1,\, t\in V} \exp\{\sum_{e=\langle t,t' \rangle \in B} K(e)\sigma(t)\sigma(t')\} \\ &= \sum_{\sigma(t)=\pm 1,\, t\in V} \prod_{e=\langle t,t' \rangle \in B} \cosh K(e)(1 + \sigma(t)\sigma(t') \tanh K(e)). \end{aligned} \tag{6.2}$$

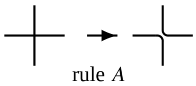
Expectation values with respect to the probability measure (6.1) are denoted by $\langle \cdot \rangle_{\mathcal{G}}$. All graphs are subgraphs of $(\mathbb{Z}^{2*}, \mathcal{E}^*)$, where $\mathbb{Z}^{2*}$ is the lattice

$$\mathbb{Z}^{2*} := \{ x = (x_1, x_2) \in \mathbb{R}^2 \,:\, x + (1/2, 1/2) \in \mathbb{Z}^2 \}; \tag{6.3}$$

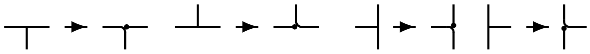
$\mathcal{E}^*$ the set of all bonds of $\mathbb{Z}^{2*}$, i.e. the set of all $e = \langle t, t' \rangle$, $\{t, t'\}$ a pair of nearest neighbours points of $\mathbb{Z}^{2*}$. We make the following convention. If $V \subset \mathbb{Z}^{2*}$, then $\mathcal{E}(V) := \{ \langle t, t' \rangle \in \mathcal{E}^* : t, t' \in V \}$ and the graph generated by V is $\mathcal{G}(V) := (V, \mathcal{E}(V))$. Similarly, if $B \subset \mathcal{E}^*$, then $V(B) := \{ t \in \mathbb{Z}^{2*} : \exists t',\ \langle t, t' \rangle \in B \}$ and the graph generated by B is $\mathcal{G}(B) := (V(B), B)$.

Let $\mathcal{G} = (V, B)$ be a graph. We need the following geometric notions. Let $B_1 \subset B$. The **index** of a site t **in** B_1 is the number of bonds of B_1, which are adjacent to t. The **boundary** of B_1 is the subset of V $\delta B := \{ t \in V :$ index of t in B_1 is odd$\}$. A **path** is an ordered sequence of sites and bonds, $t_0, e_0, t_1, e_1, \ldots, t_n$, where $t_i \in V$ for all $i = 0, \ldots n$, and $e_j = \langle t_j, t_{j+1} \rangle \in B$, $j = 0, \ldots, n-1$. By definition all bonds of a path are different, but not necessarily all sites of the path. The initial point of the path is t_0 and the final point is t_n. A path is **closed** if its final point coincides with its initial point; otherwise it is **open**. Unoriented paths are called **contours**. Given $B_1 \subset B$ we can decompose B_1 uniquely into a finite number of contours by the following procedure.

1. If $\delta B_1 = \emptyset$, then choose a bond $e = \langle t, t' \rangle$ in B_1 and set $t_0 := t$, $e_0 := e$ and $t_1 = t'$. The path is uniquely continued using rule A specified in the picture below and by requiring that it is maximal and that its final point is t_0. We have thus defined a closed path; forgetting the orientation this defines uniquely a closed contour. Repeat this construction until all bonds of B_1 belong to some contour.
2. If $\delta B_1 \neq \emptyset$, then choose first $t \in \delta B_1$, and set $t_0 := t$. Then choose e_0 among the adjacent bonds to t_0 according to rules A' specified in the picture below. Initial points are marked by dots in the picture specifying the rules A'. The path is uniquely continued using rules A and A' and by requiring that it is maximal and its final point $t_n \in \delta B_1$. We have thus defined an open path, since $t_0 \neq t_n$; forgetting the orientation this defines uniquely an open contour. Repeat this construction starting with a new point of δB_1 until all points of δB_1 belong to some open contours; if there are still bonds of B_1 which do not belong to some contours, then do Construction 1 above.



rule A

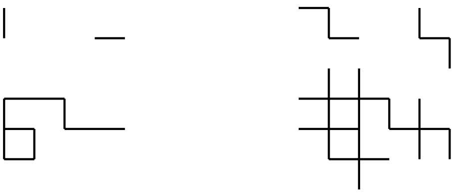


rules A'
the dots denote initial points of open paths

Let $\underline{\theta} = \{\theta_1, \dots, \theta_n\}$ be a family of contours; we denote by $\mathcal{E}(\theta_1, \dots, \theta_n)$ the set of all bonds of the contours $\theta_1, \dots, \theta_n$. We say that $\underline{\theta}$ is **compatible** if either $\mathcal{E}(\theta_1, \dots, \theta_n) = \emptyset$ or $\{\theta_1, \dots, \theta_n\}$ is the decomposition into contours of the set $\mathcal{E}(\theta_1, \dots, \theta_n)$. If we want to stress the condition that each contour is a contour of the graph $\mathcal{G}$, then we say that $\underline{\theta}$ is $\mathcal{G}$-**compatible**. Notice that the notion of compatibility introduced here is purely geometrical; it is different from the notion of compatibility defined in Subsect. 2.1.

Let e be a bond and $B(e)$ the set formed by e and all bonds of $\mathcal{E}^*$, which are adjacent to e. The **edge-boundary** of e is the set of bonds of the contour $\Delta(e) \ni e$ of the decomposition of $B(e)$ into contours. Let $B_1 \subset \mathcal{E}^*$; the **edge-boundary** $\Delta(B_1)$ of B_1 is $\Delta(B_1) := \cup_{e \in B_1} \Delta(e)$. The next lemma is proven in [PV1]; its proof is not difficult.

Lemma 6.1. *Let $\underline{\theta}$ be a family of compatible contours. Then a non-empty compatible family of n closed contours $\underline{\gamma} = \{\gamma_1, \dots, \gamma_n\}$ is compatible with $\underline{\theta}$, that is $\underline{\gamma} \cup \underline{\theta}$ is compatible, if and only if no bond of γ_i is a bond of $\Delta(\underline{\theta})$, $\forall i = 1, \dots, n$.*



Two bonds e, e' and a contour θ with their edge-boundaries $\Delta(e)$, $\Delta(e')$, $\Delta(\theta)$

We define the high-temperature representation of the model. The partition function $\Xi(\mathcal{G})$ is given in (6.2). We expand the product in (6.2). Each term of the expansion is labeled by a set of bonds $\langle t, t'\rangle$: we specify the bonds corresponding to the factors $\tanh K(e)$. Then we sum over $\sigma(t)$, $t \in V$; after summation only terms labeled by sets of bonds with empty boundary give a non-zero contribution. Any term of the expansion of (6.2), which gives a non-zero contribution, can be uniquely labeled by a $\mathcal{G}$-compatible family $\underline{\gamma}$ of closed contours. Let e be a bond, θ a contour and $\underline{\theta}$ a compatible family of contours; we set

$$w(e) := \tanh K(e) \quad , \quad w(\theta) := \prod_{e \in \theta} w(e) \quad , \quad w(\underline{\theta}) := \prod_{\theta \in \underline{\theta}} w(\theta). \tag{6.4}$$

If $\underline{\theta} = \emptyset$, then $w(\underline{\theta}) := 1$. $\Xi(\mathcal{G})$ can be written as ($|V|$ is the cardinality of V)

$$\Xi(\mathcal{G}) = 2^{|V|} \prod_{e \in B} \cosh K(e) \sum_{\substack{\underline{\gamma}:\, \delta\underline{\gamma}=\emptyset \\ \mathcal{G}-\text{comp.}}} w(\underline{\gamma}) \equiv 2^{|V|} \prod_{e \in B} \cosh K(e) \cdot Z(\mathcal{G}), \tag{6.5}$$

with $Z(\mathcal{G})$ the **normalized partition function**,

$$Z(\mathcal{G}) := \sum_{\substack{\underline{\gamma}:\, \delta\underline{\gamma}=\emptyset \\ \mathcal{G}-\text{comp.}}} w(\underline{\gamma}). \tag{6.6}$$

Notice that $Z(\mathcal{G}_1) = Z(\mathcal{G}_2)$ if the two graphs $\mathcal{G}_i = (V_i, B_i)$, $i = 1, 2$, have the same set of closed contours. More generally, given any $\mathcal{G}$-compatible family $\underline{\theta}$ of contours, we set

$$Z(\mathcal{G}|\underline{\theta}) := \sum_{\substack{\underline{\gamma}:\, \delta\underline{\gamma}=\emptyset \\ \underline{\gamma}\cup\underline{\theta}\ \mathcal{G}-\text{comp.}}} w(\underline{\gamma}). \tag{6.7}$$

We define a **weight** $q_{\mathcal{G}}(\underline{\theta})$ for an arbitrary family $\underline{\theta}$,

$$q_{\mathcal{G}}(\underline{\theta}) := \begin{cases} w(\underline{\theta}) \dfrac{Z(\mathcal{G}|\underline{\theta})}{Z(\mathcal{G})} & \text{if } \underline{\theta} \text{ is } \mathcal{G}\text{-compatible,} \\ 0 & \text{otherwise.} \end{cases} \tag{6.8}$$

The usefulness of the weights $q_{\mathcal{G}}(\underline{\theta})$ comes from the following representation of the correlation function $\langle \prod_{t \in A} \sigma(t) \rangle_{\mathcal{G}}$. If the cardinality of A is odd, then by symmetry $\langle \prod_{t \in A} \sigma(t) \rangle_{\mathcal{G}} = 0$. Suppose that $|A| = 2m$, $m \geq 1$. We expand the numerator of $\langle \prod_{t \in A} \sigma(t) \rangle_{\mathcal{G}}$ as above. The presence of the variables $\sigma(t)$, $t \in A$, implies that the only terms in the expansion of the numerator of $\langle \prod_{t \in A} \sigma(t) \rangle_{\mathcal{G}}$, which give non-zero contributions, are those labeled by compatible families of contours containing a subfamily $\underline{\lambda} = \{\lambda_1, \ldots, \lambda_m\}$ of m open contours such that $\delta\underline{\lambda} = A$. Summing over all closed contours for a given family of m open contours $\underline{\lambda}$, we get a contribution to the numerator equal to $w(\underline{\lambda})Z(\mathcal{G}|\underline{\lambda})$. We can therefore write the key-identity, a **random-line representation** for the even correlation function,

$$\langle \prod_{t \in A} \sigma(t) \rangle_{\mathcal{G}} = \sum_{\underline{\lambda}:\, \delta\underline{\lambda}=A} q_{\mathcal{G}}(\underline{\lambda}). \tag{6.9}$$

From now on, if we specify the graph $\mathcal{G}$ by its set of vertices $V \subset \mathbb{Z}^{2*}$, then we write $\langle \cdot \rangle_V$ and $q_V(\underline{\lambda})$ instead of $\langle \cdot \rangle_{\mathcal{G}(V)}$ and $q_{\mathcal{G}(V)}(\underline{\lambda})$. Our first application of (6.9) is

Lemma 6.2. *Let Λ_L be the square box (2.2) and Λ_L^* its dual box. Let η be boundary conditions for Λ_L and $V_L(\eta) \subset \mathbb{Z}^{2*}$ the set of end-points of the phase-separation lines of the configurations in Λ_L with η boundary conditions. Then*

$$\sum_{\underline{\lambda}} q^{\eta}_{\Lambda_L}(\underline{\lambda}) = \langle \prod_{t \in V_L(\eta)} \sigma(t) \rangle_{\Lambda_L^*}. \tag{6.10}$$

Proof. Since Λ_L is a square box, the set of η compatible families of contours in Λ_L coincides with the set of compatible families of contours $\underline{\theta}$ of the graph $\mathcal{G}(\Lambda_L^*)$ such that $\delta\underline{\theta} = V_L(\eta)$. By duality (compare (2.10) and (6.9)),

$$q^{\eta}_{\Lambda_L}(\underline{\lambda}) = q_{\Lambda_L^*}(\underline{\lambda}). \tag{6.11}$$

□

Lemma 6.3. *Let $\mathcal{G} = (V, B)$ be a graph and $\underline{\theta}$ a $\mathcal{G}$-compatible family of contours. Then $\frac{Z(\mathcal{G}|\underline{\theta})}{Z(\mathcal{G})}$ is a decreasing function of $K(e)$ for any $e \in B$. If $\mathcal{G}' = (V', B')$ and $V \subset V'$, $B \subset B'$, then $q_{\mathcal{G}}(\underline{\theta}) \geq q_{\mathcal{G}'}(\underline{\theta})$.*

Proof. Let $B_1 := B\backslash\Delta(\underline{\theta})$ and $\mathcal{G}(B_1)$ be the graph defined by this set B_1 of bonds. Let $V(B_1)$ be the set of vertices of $\mathcal{G}(B_1)$. By Lemma 6.1 we have,

$$Z(\mathcal{G}|\underline{\theta}) = Z(\mathcal{G}(B_1)). \tag{6.12}$$

Therefore

$$\ln \frac{Z(\mathcal{G}|\underline{\theta})}{Z(\mathcal{G})} = \ln \frac{\Xi(\mathcal{G}(B_1))}{\Xi(\mathcal{G})} + \ln \prod_{e\in\Delta(\underline{\theta})} \cosh K(e) + \ln 2\,(|V| - |V(B_1)|). \tag{6.13}$$

If $e = \langle t, t'\rangle \in B_1$, then

$$\frac{\partial}{\partial K(e)} \ln \frac{Z(\mathcal{G}|\underline{\theta})}{Z(\mathcal{G})} = \langle \sigma(t)\sigma(t')\rangle_{\mathcal{G}(B_1)} - \langle \sigma(t)\sigma(t')\rangle_{\mathcal{G}} \leq 0, \tag{6.14}$$

by GKS-inequalities, since $V(B_1) \subset V$. If $e = \langle t, t'\rangle \in \Delta(\underline{\theta})$, then

$$\frac{\partial}{\partial K(e)} \ln \frac{Z(\mathcal{G}|\underline{\theta})}{Z(\mathcal{G})} = -\langle \sigma(t)\sigma(t')\rangle_{\mathcal{G}} + \tanh K(e) \leq 0, \tag{6.15}$$

since by GKS-inequalities

$$\langle \sigma(t)\sigma(t')\rangle_{\mathcal{G}} \geq \langle \sigma(t)\sigma(t')\rangle_{\{t,t'\}} = \tanh K(e). \qquad \square \tag{6.16}$$

We make the following convention. If $\underline{\theta_1}$ and $\underline{\theta_2}$ are two compatible families of contours, such that $\mathcal{E}(\underline{\theta_1}) \cap \mathcal{E}(\underline{\theta_2}) = \emptyset$, then the decomposition of $\mathcal{E}(\underline{\theta_1}) \cup \mathcal{E}(\underline{\theta_2})$ into contours does not coincide necessarily with $\underline{\theta_1} \cup \underline{\theta_2}$. In such a situation we interpret $q_{\mathcal{G}}(\underline{\theta_1} \cup \underline{\theta_2})$ as the weight of the family of contours of the decomposition of $\mathcal{E}(\underline{\theta_1}) \cup \mathcal{E}(\underline{\theta_2})$ if necessary.

Lemma 6.4. *Let $\underline{\theta_1}$ and $\underline{\theta_2}$ be two compatible families of contours of the graph $\mathcal{G} = (V, B)$, such that $\mathcal{E}(\underline{\theta_1}) \cap \mathcal{E}(\underline{\theta_2}) = \emptyset$. Let $\mathcal{G}'$ be the graph defined by the set of bonds $\Big(B\backslash\Delta(\underline{\theta_2})\Big) \cup \Big(\Delta(\underline{\theta_2}) \cap \mathcal{E}(\underline{\theta_1})\Big)$. If $\Delta(\underline{\theta_2}) \cap \mathcal{E}(\underline{\theta_1}) = \emptyset$, then*

$$q_{\mathcal{G}}(\underline{\theta_1} \cup \underline{\theta_2}) = q_{\mathcal{G}'}(\underline{\theta_1})\, q_{\mathcal{G}}(\underline{\theta_2}). \tag{6.17}$$

If $\Delta(\underline{\theta_2}) \cap \mathcal{E}(\underline{\theta_1}) \neq \emptyset$, then

$$q_{\mathcal{G}}(\underline{\theta_1} \cup \underline{\theta_2}) \geq q_{\mathcal{G}'}(\underline{\theta_1})\, q_{\mathcal{G}}(\underline{\theta_2}). \tag{6.18}$$

In both cases

$$q_{\mathcal{G}}(\underline{\theta_1} \cup \underline{\theta_2}) \geq q_{\mathcal{G}}(\underline{\theta_1})\, q_{\mathcal{G}}(\underline{\theta_2}). \tag{6.19}$$

Proof. We have

$$q_{\mathcal{G}}(\underline{\theta_1} \cup \underline{\theta_2}) = w(\underline{\theta_1})w(\underline{\theta_2})\frac{Z(\mathcal{G}|\underline{\theta_1} \cup \underline{\theta_2})}{Z(\mathcal{G})} = w(\underline{\theta_1})\frac{Z(\mathcal{G}|\underline{\theta_1} \cup \underline{\theta_2})}{Z(\mathcal{G}')}\, w(\underline{\theta_2})\frac{Z(\mathcal{G}')}{Z(\mathcal{G})}. \quad (6.20)$$

A family of closed contours $\underline{\gamma}$ of $\mathcal{G}$ contributes to $Z(\mathcal{G}|\underline{\theta_1} \cup \underline{\theta_2})$ if and only if

$$\underline{\gamma} \cap \Big(\Delta(\underline{\theta_1}) \cup \Delta(\underline{\theta_2})\Big) = \Big(\underline{\gamma} \cap \Delta(\underline{\theta_1})\Big) \cup \Big(\underline{\gamma} \cap \Delta(\underline{\theta_2})\Big) = \emptyset. \quad (6.21)$$

This is equivalent to say that $\underline{\gamma}$ is a family of closed contours of the graph $\mathcal{G}'$ and $\underline{\gamma} \cap \Delta(\underline{\theta_1}) = \emptyset$. Therefore (see Lemma 6.1)

$$Z(\mathcal{G}|\underline{\theta_1} \cup \underline{\theta_2}) = Z(\mathcal{G}'|\underline{\theta_1}). \quad (6.22)$$

If $\Delta(\underline{\theta_2}) \cap \mathcal{E}(\underline{\theta_1}) = \emptyset$, then $\mathcal{G}'$ is the graph defined by the set of bonds $B\backslash\Delta(\underline{\theta_2})$; hence

$$Z(\mathcal{G}') = Z(\mathcal{G}|\underline{\theta_2}). \quad (6.23)$$

If $\Delta(\underline{\theta_2}) \cap \mathcal{E}(\underline{\theta_1}) \neq \emptyset$, then

$$Z(\mathcal{G}') \geq Z(\mathcal{G}|\underline{\theta_2}), \quad (6.24)$$

since the graph $\mathcal{G}'$ contains some bonds of $\Delta(\underline{\theta_2})$. The last affirmation follows from the above results and Lemma 6.3. □

Let λ_1 and λ_2 be two open contours such that $\delta\lambda_1 = \{x, y\}$ and $\delta\lambda_2 = \{u, v\}$. We say that λ_1 and λ_2 are **disjoint** if either they are compatible or $\mathcal{E}(\lambda_1) \cap \mathcal{E}(\lambda_2) = \emptyset$ and the decomposition of $\mathcal{E}(\lambda_1) \cup \mathcal{E}(\lambda_2)$ into contours is a single contour. If λ_1 and λ_2 are disjoint, then we write $\lambda_1 \amalg \lambda_2$ the family $\{\lambda_1, \lambda_2\}$ or the single contour of the decomposition into contours of $\mathcal{E}(\lambda_1) \cup \mathcal{E}(\lambda_2)$. Notice that when $\lambda_1 \amalg \lambda_2 = \lambda$ is a single contour, then $\{x, y\} \cap \{u, v\} \neq \emptyset$.

Lemma 6.5. *Let λ_1 and λ_2 be two open contours such that $\delta\lambda_1 = \{x, y\}$ and $\delta\lambda_2 = \{u, v\}$. Then*

$$\sum_{\substack{\lambda\,:\,\lambda=\lambda_1\amalg\lambda_2 \\ \delta\lambda_1=\{x,y\},\,\delta\lambda_2=\{u,v\}}} q_{\mathcal{G}}(\lambda) \leq \sum_{\substack{\lambda_1: \\ \delta\lambda_1=\{x,y\}}} q_{\mathcal{G}}(\lambda_1) \sum_{\substack{\lambda_2: \\ \delta\lambda_2=\{u,v\}}} q_{\mathcal{G}}(\lambda_2). \quad (6.25)$$

Proof. The proof is easy if $\lambda_1 \amalg \lambda_2 = \{\lambda_1, \lambda_2\}$. Indeed, from Lemma 6.4, since $\mathcal{E}(\lambda_1) \cap \Delta(\lambda_2) = \emptyset$,

$$q_{\mathcal{G}}(\lambda) = q_{\mathcal{G}'}(\lambda_1)\, q_{\mathcal{G}}(\lambda_2). \quad (6.26)$$

Summing over λ_1, keeping λ_2 fixed, we get from the basic formula (6.9) and GKS inequalities

$$\begin{aligned}\sum_{\substack{\lambda_1: \\ \lambda=\lambda_1\amalg\lambda_2}} q_{\mathcal{G}}(\lambda) &\leq \langle\, \sigma(x)\sigma(y)\,\rangle_{\mathcal{G}'}\, q_{\mathcal{G}}(\lambda_2) \quad (6.27)\\ &\leq \langle\, \sigma(x)\sigma(y)\,\rangle_{\mathcal{G}}\, q_{\mathcal{G}}(\lambda_2)\\ &= \sum_{\substack{\lambda_1: \\ \delta\lambda_1=\{x,y\}}} q_{\mathcal{G}}(\lambda_1)\, q_{\mathcal{G}}(\lambda_2).\end{aligned}$$

We can now sum over λ_2. When $\lambda_1 \amalg \lambda_2$ is a single contour λ, then the proof is more delicate, since the second case in Lemma 6.4 occurs. However, the proof is similar. For details we refer to the proof of Lemma 5.4 in [PV1]. □

Lemma 6.6. *Let $\mathcal{G} = (V, B)$ and $B_1 \subset B$. Let $\mathcal{G}' = (V_1, B_1)$ be the graph generated by B_1. Let $x, y \in V_1$. Then*

$$\sum_{\substack{\lambda\,:\,\delta\lambda=\{x,y\} \\ \mathcal{E}(\lambda)\subset B_1}} q_{\mathcal{G}}(\lambda) \leq \sum_{\lambda\,:\,\delta\lambda=\{x,y\}} q_{\mathcal{G}'}(\lambda) = \langle\, \sigma(x)\sigma(y)\,\rangle_{\mathcal{G}'}. \tag{6.28}$$

Proof. The result follows directly from Lemma 6.3. □

The next lemma gives a concentration result for the random-line representation (6.9). Let $\mathcal{G} = (V, B)$ and $V_1 \subset V$. We define

$$\partial_{\mathrm{ext}} V_1 := \{\, t \in V\backslash V_1 : \exists t' \in V_1,\ \langle\, t, t'\,\rangle \in B\,\}. \tag{6.29}$$

Similarly, if $B_1 \subset B$, then we set

$$\partial_{\mathrm{ext}} B_1 := \partial_{\mathrm{ext}} V(B_1). \tag{6.30}$$

We say that B_1 is **connected** if for any pair of sites $x, y \in V(B_1)$, there is a path from x to y with all its bonds in B_1.

Lemma 6.7. *Let $\mathcal{G} = (V, B)$, $B_1 \subset B$ be a connected subset and x, y two sites of the bonds of B_1. Suppose that all bonds incident to x and y belong to B_1. Then*

$$\begin{aligned} 0 &\leq \langle\, \sigma(x)\sigma(y)\,\rangle_{\mathcal{G}} - \sum_{\substack{\lambda:\delta\lambda=\{x,y\} \\ \mathcal{E}(\lambda)\subset B_1}} q_{\mathcal{G}}(\lambda) \\ &\leq \sum_{z\in\partial_{\mathrm{ext}} B_1} \sum_{\substack{\lambda:\delta\lambda_1=\{x,y\} \\ \lambda\ni z}} q_{\mathcal{G}}(\lambda) \\ &\leq \sum_{z\in\partial_{\mathrm{ext}} B_1} \langle\, \sigma(x)\sigma(z)\,\rangle_{\mathcal{G}}\, \langle\, \sigma(z)\sigma(y)\,\rangle_{\mathcal{G}}. \end{aligned} \tag{6.31}$$

Proof. Equations (6.9) gives

$$\langle\, \sigma(x)\sigma(y)\,\rangle_{\mathcal{G}} = \sum_{\substack{\lambda:\delta\lambda=\{x,y\} \\ \mathcal{E}(\lambda)\subset B_1}} q_{\mathcal{G}}(\lambda) + \sum_{\substack{\lambda:\delta\lambda=\{x,y\} \\ \mathcal{E}(\lambda)\not\subset B_1}} q_{\mathcal{G}}(\lambda). \tag{6.32}$$

We estimate the second sum. For any λ contributing to this sum, let $z(\lambda)$ be the first point of $\partial_{\mathrm{ext}} B_1$ of the path from x to y defined by the contour λ. Any such a path can be decomposed into λ_1 such that $\delta\lambda_1 = \{x, z\}$ and λ_2 such that $\delta\lambda_2 = \{z, y\}$ so that $\lambda = \lambda_1 \amalg \lambda_2$. The result then follows from Lemma 6.5 and (6.9). □

There is a useful formula for the weight $q_{\mathcal{G}}(\underline{\theta})$, which is a consequence of the following elementary remarks. Let K denote the function $e \in V \mapsto K(e) \in \mathbb{R}$. Given a compatible family of contours, we introduce a new function K_s, $0 \leq s \leq 1$,

$$K_s := \begin{cases} K(e) & \text{if } e \notin \Delta(\underline{\theta}), \\ sK(e) & \text{if } e \in \Delta(\underline{\theta}). \end{cases} \tag{6.33}$$

Then $Z(\mathcal{G}|\underline{\theta})(K) = Z(\mathcal{G})(K_s)|_{s=0}$. On the other hand we have

$$\ln \Xi(\mathcal{G})(K) - \ln \Xi(\mathcal{G})(K_0) = \int_0^1 \frac{d}{ds} \ln \Xi(\mathcal{G})(K_s)\, ds \tag{6.34}$$
$$= \sum_{e=\langle t,t'\rangle \in \Delta(\underline{\theta})} K(e) \int_0^1 \langle\, \sigma(t)\sigma(t')\,\rangle_{\mathcal{G}}(K_s)\, ds.$$

Therefore, for a compatible family of contours $\underline{\theta}$,

$$q_{\mathcal{G}}(\underline{\theta}) = w(\underline{\theta}) \prod_{e\in\Delta(\underline{\theta})} \cosh K(e) \exp\Big(- \sum_{e=\langle t,t'\rangle\in\Delta(\underline{\theta})} K(e) \int_0^1 \langle\, \sigma(t)\sigma(t')\,\rangle_{\mathcal{G}}(K_s)\, ds\Big). \tag{6.35}$$

Formula (6.35) allows to compare $q_{\mathcal{G}}(\underline{\theta})(K)$ for different functions K or different graphs $\mathcal{G}'$. For example we get immediately the lower bound

$$q_{\mathcal{G}}(\underline{\theta}) \geq w(\underline{\theta}) \prod_{e\in\Delta(\underline{\theta})} \frac{1}{2}\Big(1 + \mathrm{e}^{-2K(e)}\Big). \tag{6.36}$$

Lemma 6.8. *Let $\mathcal{G} = (V, B)$, $V_1 \subset V$ and $\mathcal{G}'$ be the graph generated by $V\backslash V_1$. Let $\underline{\theta}$ be a compatible family of contours of $\mathcal{G}$ such that no site of $\underline{\theta}$ belongs to $\partial_{\text{ext}} V_1$. We set for all $t \in \partial_{\text{ext}} V_1$,*

$$K(t) := \sum_{\substack{t'\in V_1:\\ \langle t,t'\rangle\in B}} K(\langle t, t'\rangle). \tag{6.37}$$

Then

$$| \ln q_{\mathcal{G}'}(\underline{\theta}) - \ln q_{\mathcal{G}}(\underline{\theta}) | \leq \tag{6.38}$$
$$\sum_{e=\langle t,t'\rangle\in\Delta(\underline{\theta})} K(e) \sum_{t''\in\partial_{\text{ext}} V_1} K(t'')\Big(\langle\, \sigma(t)\sigma(t'')\,\rangle_{\mathcal{G}'} + \langle\, \sigma(t')\sigma(t'')\,\rangle_{\mathcal{G}'}\Big).$$

Proof. Formula (6.35) gives

$$\ln \frac{q_{\mathcal{G}'}(\underline{\theta})}{q_{\mathcal{G}}(\underline{\theta})} = \sum_{e=\langle t,t'\rangle\in\Delta(\underline{\theta})} K(e) \int_0^1 \Big(\langle\, \sigma(t)\sigma(t')\,\rangle_{\mathcal{G}}(K_s) - \langle\, \sigma(t)\sigma(t')\,\rangle_{\mathcal{G}'}(K_s)\Big)\, ds. \tag{6.39}$$

We put a magnetic field h' on each $t \in V_1$ and let $h' \to \infty$. We have

$$\langle\, \sigma(t)\sigma(t')\,\rangle_{\mathcal{G}}(K_s) \leq \langle\, \sigma(t)\sigma(t')\,\rangle^{+}_{\mathcal{G}'}(K_s), \tag{6.40}$$

where $\langle\, \sigma(t)\sigma(t')\,\rangle^{+}_{\mathcal{G}'}(K_s)$ is the expectation with respect to a Gibbs measure on $\mathcal{G}'$ with coupling constants given by K_s on the bonds of $\mathcal{G}'$ and magnetic field $K(t)$ for $t \in \partial_{\text{ext}} V_1$.

Since $-\sigma(t)\sigma(t')+\sigma(t)+\sigma(t')$ is an increasing function we get by FKG inequalities

$$\begin{aligned}&\langle\,\sigma(t)\sigma(t')\,\rangle^{+}_{\mathcal{G}'}(K_s)-\langle\,\sigma(t)\sigma(t')\,\rangle_{\mathcal{G}'}(K_s)\\&\quad\le\langle\,\sigma(t)\,\rangle^{+}_{\mathcal{G}'}(K_s)-\langle\,\sigma(t)\,\rangle_{\mathcal{G}'}(K_s)+\langle\,\sigma(t')\,\rangle^{+}_{\mathcal{G}'}(K_s)-\langle\,\sigma(t')\,\rangle_{\mathcal{G}'}(K_s).\end{aligned}\tag{6.41}$$

We define an interpolating magnetic field for $t\in\partial_{\text{ext}}V_1$,

$$K_a(t):=aK(t)\quad,\quad 0\le a\le 1.\tag{6.42}$$

Let $\langle\,\cdot\,\rangle^{+}_{\mathcal{G}'}(K_s;a)$ be the expectation value with respect to this new measure and set

$$\begin{aligned}\langle\,\sigma(t);\;\sigma(t')\,\rangle^{+}_{\mathcal{G}'}(K_s;a):=\;&\langle\,\sigma(t)\sigma(t')\,\rangle^{+}_{\mathcal{G}'}(K_s;a)\\&-\langle\,\sigma(t)\,\rangle^{+}_{\mathcal{G}'}(K_s;a)\,\langle\,\sigma(t')\,\rangle^{+}_{\mathcal{G}'}(K_s;a).\end{aligned}\tag{6.43}$$

We have $\langle\,\sigma(t)\,\rangle_{\mathcal{G}'}(K_s)=\langle\,\sigma(t)\,\rangle^{+}_{\mathcal{G}'}(K_s;0)$ and $\langle\,\sigma(t)\,\rangle^{+}_{\mathcal{G}'}(K_s)=\langle\,\sigma(t)\,\rangle^{+}_{\mathcal{G}'}(K_s;1)$; therefore

$$\langle\,\sigma(t)\,\rangle^{+}_{\mathcal{G}'}(K_s)-\langle\,\sigma(t)\,\rangle_{\mathcal{G}'}(K_s)=\sum_{t''\in\partial_{\text{ext}}V_1}K(t'')\int_0^1\langle\,\sigma(t);\;\sigma(t'')\,\rangle^{+}_{\mathcal{G}'}(K_s;a)\,da.\tag{6.44}$$

GHS inequalities imply that $\langle\,\sigma(t);\;\sigma(t'')\,\rangle^{+}_{\mathcal{G}'}(K_s;a)$ is decreasing in a; thus

$$\langle\,\sigma(t);\;\sigma(t'')\,\rangle^{+}_{\mathcal{G}'}(K_s;a)\le\langle\,\sigma(t);\;\sigma(t'')\,\rangle^{+}_{\mathcal{G}'}(K_s;0)=\langle\,\sigma(t)\sigma(t'')\,\rangle_{\mathcal{G}'}(K_s),\tag{6.45}$$

since by symmetry $\langle\,\sigma(t)\,\rangle^{+}_{\mathcal{G}'}(K_s)=0$. The lemma follows from (6.39), (6.41), (6.44) and (6.45). □

6.2. Ising model on $\mathbb{Z}^{2*}$ above T_c. We consider the model on $(\mathbb{Z}^{2*},\mathcal{E}^*)$ and choose as coupling constants $K(e):=\beta^*\ \forall e$, with $\beta^*<\beta_c$. We recall that the decay-rate $\tau(y-x)=\tau(y-x;\beta^*)$ is strictly positive for such β^* and that for any $\Lambda\subset\mathbb{Z}^{2*}$ (see Proposition 2.4)

$$\langle\,\sigma(x)\sigma(y)\,\rangle_\Lambda(\beta^*)\le\exp\{-\tau(y-x;\beta^*)\}.\tag{6.46}$$

Given any $\Lambda\subset\mathbb{Z}^{2*}$ and a family of compatible contours $\underline{\theta}$ in $\mathcal{G}(\Lambda)$, we define weights $q_\Lambda(\underline{\theta})$ (see Lemma 6.3),

$$q_\Lambda(\underline{\theta}):=\lim_{\Lambda_n\uparrow\Lambda}q_{\Lambda_n}(\underline{\theta}),\tag{6.47}$$

where Λ_n is an increasing sequence of finite subsets Λ_n of Λ, such that eventually every site of Λ is contained in some Λ_n. When $\Lambda=\mathbb{Z}^{2*}$ we write $q(\underline{\theta})$ instead of $q_{\mathbb{Z}^{2*}}(\underline{\theta})$. Lemmas 6.3 to 6.8 are still valid for the weights $q_\Lambda(\underline{\theta})$. On the other hand the random-line representation does not extend automatically in the infinite case.

Lemma 6.9. *Let $K(e) := \beta^* \ \forall e$ and $\beta^* < \beta_c$. Then the two-point correlation function of the Ising model has a random-line representation,*

$$\langle \sigma(t)\sigma(t') \rangle = \sum_{\lambda:\delta\lambda=\{t,t'\}} q(\lambda). \tag{6.48}$$

A formula similar to (6.9) is true for even correlation functions.

Proof. The hypothesis $\beta^* < \beta_c$ is equivalent to

$$\sum_{t\in\mathbb{Z}^{2*}} \langle \sigma(0)\sigma(t) \rangle < \infty. \tag{6.49}$$

Let $\Lambda_1 \subset \Lambda_2$ be two finite subsets and suppose that $t, t' \in \Lambda_1$. Let B_1 be the set of bonds between sites of Λ_1; suppose furthermore that B_1 is connected. Then formula (6.9) and Lemma 6.7 give

$$\begin{aligned} 0 &\le \langle \sigma(t)\sigma(t') \rangle_{\Lambda_2} - \sum_{\substack{\lambda:\delta\lambda=\{t,t'\} \\ \mathcal{E}(\lambda)\subset B_1}} q_{\Lambda_2}(\lambda) \\ &\le \sum_{s\in\partial_{\text{ext}} B_1} \langle \sigma(t)\sigma(s) \rangle \langle \sigma(s)\sigma(t') \rangle. \end{aligned} \tag{6.50}$$

Given $\varepsilon > 0$, we can find Λ_1 so that the last sum in (6.50) is smaller than ε. Letting $\Lambda_2 \uparrow \mathbb{Z}^{2*}$ we get

$$0 \le \langle \sigma(t)\sigma(t') \rangle - \sum_{\substack{\lambda:\delta\lambda=\{t,t'\} \\ \mathcal{E}(\lambda)\subset B_1}} q(\lambda) \le \varepsilon. \tag{6.51}$$

The result now follows by letting $\Lambda_1 \uparrow \mathbb{Z}^{2*}$. □

Lemma 6.10. *Let $K(e) := \beta^* \ \forall e$ and $\beta^* < \beta_c$. Set*

$$\mathcal{S}(x, y; \rho) := \{ t \in \mathbb{Z}^{2*} : \|x - t\| + \|y - t\| \le \|x - y\| + \rho\}, \tag{6.52}$$

with $\|\cdot\|$ the Euclidean norm. Then

$$\sum_{\substack{\lambda:\delta\lambda=\{x,y\} \\ \mathcal{E}(\lambda)\not\subset\mathcal{E}(\mathcal{S}(x,y;\rho))}} q(\lambda) \le \frac{|\partial_{\text{ext}}\mathcal{S}(x, y; \rho)|}{K} \|x - y\|^{1/2} e^{-\kappa\rho} \langle \sigma(x)\sigma(y) \rangle. \tag{6.53}$$

K is the constant of Proposition 2.4.

Proof. By Lemma 6.7,

$$\begin{aligned} \sum_{\substack{\lambda:\delta\lambda=\{x,y\} \\ \mathcal{E}(\lambda)\not\subset\mathcal{E}(\mathcal{S}(x,y;\rho))}} q(\lambda) &\le \sum_{t\in\partial_{\text{ext}}\mathcal{S}(x,y;\rho)} \langle \sigma(x)\sigma(t) \rangle \langle \sigma(t)\sigma(y) \rangle \\ &= \sum_{t\in\partial_{\text{ext}}\mathcal{S}(x,y;\rho)} \frac{\langle \sigma(x)\sigma(t) \rangle \langle \sigma(t)\sigma(y) \rangle}{\langle \sigma(x)\sigma(y) \rangle} \langle \sigma(x)\sigma(y) \rangle. \end{aligned} \tag{6.54}$$

We apply the sharp triangle inequality to the numerator of the last expression,

$$\begin{aligned}\langle\,\sigma(x)\sigma(t)\,\rangle\,\langle\,\sigma(t)\sigma(y)\,\rangle &\leq \mathrm{e}^{-\tau(x-t)-\tau(y-t)+\tau(x-y)}\mathrm{e}^{-\tau(x-y)}\\ &\leq \mathrm{e}^{-\tau(x-y)-\kappa\rho}.\end{aligned} \tag{6.55}$$

Finally we apply Proposition 2.4 to the denominator,

$$\mathrm{e}^{-\tau(x-y)} \leq \frac{\|x-y\|^{1/2}}{K}\,\langle\,\sigma(x)\sigma(y)\,\rangle. \tag{6.56}$$

□

Lemma 6.10 characterizes those random-lines, which give the main contribution to the two-point correlation function. If $\rho \geq C \ln \|x-y\|$, with C large enough, then the coefficient in front of $\langle\,\sigma(x)\sigma(y)\,\rangle$ in (6.53) tends to zero when $\|x-y\|$ diverges. The result is sharp.

6.3. Ising model on $\mathbb{L}^*$ above T_c. Let $\beta^* < \beta_c$ and $h^* > 0$. We consider the model on subsets $\Lambda_L^* \subset \mathbb{L}^*$ and choose as coupling constants

$$K(e) := \begin{cases} h^*\beta^* & \forall\, e=\langle\, t,t'\,\rangle, \text{ with } t_2 = t_2' = 1/2,\\ \beta^* & \text{otherwise.}\end{cases} \tag{6.57}$$

We set

$$\Sigma_L^* := \{\, t \in \Lambda_L^* : t_2 = 1/2\,\} \quad , \quad \Sigma^* := \{\, t \in \mathbb{L}^* : t_2 = 1/2\,\}. \tag{6.58}$$

The weight $q_{\mathbb{L}^*}(\underline{\theta})$ is defined by (6.47). Lemma 6.11 establishes the random-line representation for the two-point function, its proof is similar to that of Lemma 6.9.

Lemma 6.11. *Let $\beta^* < \beta_c$, $h^* > 0$ and the coupling constants be given by (6.57). Then the two-point correlation function of the Ising model on $\mathbb{L}^*$ has a random-line representation,*

$$\langle\,\sigma(t)\sigma(t')\,\rangle_{\mathbb{L}^*} = \sum_{\lambda:\delta\lambda=\{t,t'\}} q_{\mathbb{L}^*}(\lambda). \tag{6.59}$$

A formula similar to (6.59) is true for even correlation functions.

Lemma 6.12. *Let $\beta^* < \beta_c$, $h^* > 0$, $\Lambda_L^* \subset \mathbb{L}^*$ and $\underline{\theta}$ be a family of compatible contours. Let $q_{\Lambda_L^*}(\underline{\theta})$ be the weight for the model defined on Λ_L^* with coupling constants (6.57). Let $q(\underline{\theta})$ be the weight for the model on $\mathbb{Z}^{2*}$ with coupling constants $K(e) \equiv \beta^*$.*

1. *If $h^* \leq 1$, then $q_{\Lambda_L^*}(\underline{\theta}) \geq q(\underline{\theta})$.*
2. *Let $d(\underline{\theta}) := \min\{\,|t_2 - 3/2| : t \in \Delta(\underline{\theta})\} \geq 1$. If $h^* \geq 1$, then*

$$\ln \frac{q_{\Lambda_L^*}(\underline{\theta})}{q(\underline{\theta})} \geq -O(L^2)\exp\{-O(d(\underline{\theta}))\}. \tag{6.60}$$

Proof. The first case follows directly from Lemma 6.3. The second case follows from Lemma 6.8. By Lemma 6.3 $q_{\Lambda_L^*\backslash\Sigma_L^*}(\underline{\theta}) \geq q(\underline{\theta})$. Since

$$q_{\Lambda_L^*}(\underline{\theta}) \geq \frac{q_{\Lambda_L^*}(\underline{\theta})}{q_{\Lambda_L^*\backslash\Sigma_L^*}(\underline{\theta})} q(\underline{\theta}), \tag{6.61}$$

we must compare $q_{\Lambda_L^*}(\underline{\theta})$ and $q_{\Lambda_L^*\backslash\Sigma_L^*}(\underline{\theta})$. We apply Lemma 6.8 with $\mathcal{G}$ the graph generated by Λ_L^* and $\mathcal{G}'$ the graph generated by $\Lambda_L^*\backslash\Sigma_L^*$. Notice that

$$\langle\, \sigma(t)\sigma(t')\,\rangle_{\mathcal{G}'} \leq \langle\, \sigma(t)\sigma(t')\,\rangle; \tag{6.62}$$

therefore, if $t \in \Delta(\underline{\theta})$,

$$\begin{aligned}
\sum_{t'\in\Lambda_L^*:t_2'=3/2} \langle\, \sigma(t)\sigma(t')\,\rangle_{\mathcal{G}'} &\leq \sum_{t':\,t_2'=3/2} \langle\, \sigma(t)\sigma(t')\,\rangle \\
&\leq \sum_{t':\,t_2'=3/2} \sum_{\substack{\lambda: \\ \delta\lambda=\{t,t'\}}} q(\lambda) \\
&\leq \sum_{t':\,t_2'=3/2} \sum_{s:\,s_2=3/2} \sum_{\substack{\lambda:\,z(\lambda)=s \\ \delta\lambda=\{t,t'\}}} q(\lambda),
\end{aligned} \tag{6.63}$$

with $z(\lambda)$ the first site z of the path defined by λ with initial point t, such that $z_2 = 3/2$. To estimate the last sums we use Lemma 6.5. We have

$$\sum_{t':\,t_2'=3/2} \langle\, \sigma(t)\sigma(t')\,\rangle \leq \sum_{t':t_2'=3/2} \sum_{s:s_2=3/2} \exp\{-\tau(t-s) - \tau(s-t')\}. \tag{6.64}$$

We sum over t' and get a finite contribution independent of s; then the sum over s gives a contribution $\exp\{-O(d(\underline{\theta}))\}$. Since $|\Delta(\underline{\theta})| \leq O(L^2)$ we get (6.60). □

The next lemma characterizes those random-lines, which give the main contribution to the boundary two-point correlation function. We consider the case $\beta^* < \beta_c$ and $h^* > h_w(\beta)^*$, when the random-lines stick to Σ^*. In the other cases there is a result similar to that of Lemma 6.10.

Lemma 6.13. *Let $\beta^* < \beta_c$, $h^* > h_w(\beta)^*$ and the coupling constants given by (6.57). Let $x, y \in \Sigma^*$, $x_1 < y_1$ and $\rho_i \in \mathbb{N}$, $i = 1, 2$; we set*

$$\mathcal{B}(x, y; \rho_1, \rho_2) := \{\, t \in \mathbb{L}^* : x_1 - \rho_1 \leq t_1 \leq y_1 + \rho_1,\ 1/2 \leq t_2 \leq 1/2 + \rho_2\,\}. \tag{6.65}$$

Then

$$\begin{aligned}
\sum_{\substack{\lambda:\delta\lambda=\{x,y\} \\ \mathcal{E}(\lambda)\not\subset\mathcal{E}(\mathcal{B})}} q_{\mathbb{L}^*}(\lambda) \leq \frac{\langle\, \sigma(x)\sigma(y)\,\rangle_{\mathbb{L}^*}}{K''} \Big(&2\rho_2 \exp\{-2\rho_1\hat{\tau}_{\mathrm{bd}}\} \\
&+ O\big(\rho_2|x_1 - y_1|\big) \exp\{-\kappa\rho_2\}\Big).
\end{aligned} \tag{6.66}$$

K'' is the constant of Proposition 2.4; $\hat{\tau}_{\mathrm{bd}} = \hat{\tau}_{\mathrm{bd}}(\beta, h)$ with β and h the dual values of β^* and h^*; κ the constant in the sharp triangle inequality and $C := \hat{\tau}((1, 0)) - \hat{\tau}_{\mathrm{bd}} > 0$.

Proof. We decompose $\partial_{\mathrm{ext}}\mathcal{B}$ into two parts:

$$V_1 := \{ t \in \partial_{\mathrm{ext}}\mathcal{B} : t_1 = x_1 - \rho_1 - 1 \text{ or } t_1 = y_1 + \rho_1 + 1 \} , \ V_2 := \partial_{\mathrm{ext}}\mathcal{B}\backslash V_1. \quad (6.67)$$

We consider λ as a unit-speed parametrized curve, $s \in [0, |\lambda|] \mapsto \lambda(s)$, with initial point $\lambda(0) = x$; we suppose that s^* is the first time such that $\lambda \in \partial_{\mathrm{ext}}\mathcal{B}$; we set $t = \lambda(s^*)$. We have

$$\sum_{\substack{\lambda:\delta\lambda=\{x,y\}\\ \mathcal{E}(\lambda)\not\subset\mathcal{E}(\mathcal{B})}} q_{\mathbb{L}^*}(\lambda) \leq \sum_{\substack{t\in\partial_{\mathrm{ext}}\mathcal{B}:\\ t\in V_1}} \sum_{\substack{\lambda:\delta\lambda=\{x,y\}\\ \lambda\ni t}} q_{\mathbb{L}^*}(\lambda) + \sum_{\substack{t\in\partial_{\mathrm{ext}}\mathcal{B}:\\ t\in V_2}} \sum_{\substack{\lambda:\delta\lambda=\{x,y\}\\ \lambda\ni t}} q_{\mathbb{L}^*}(\lambda). \quad (6.68)$$

We treat these two sums separately. By Lemma 6.7, symmetry and GKS inequalities,

$$\begin{aligned}\sum_{\substack{t\in\partial_{\mathrm{ext}}\mathcal{B}:\\ t\in V_1}} \sum_{\substack{\lambda:\delta\lambda=\{x,y\}\\ \lambda\ni t}} q_{\mathbb{L}^*}(\lambda) &\leq 2 \sum_{\substack{t\in\partial_{\mathrm{ext}}\mathcal{B}\\ t_1=x_1-\rho_1-1}} \langle\, \sigma(x)\sigma(t)\,\rangle_{\mathbb{L}^*} \langle\, \sigma(t)\sigma(y)\,\rangle_{\mathbb{L}^*} \quad (6.69)\\ &= 2 \sum_{\substack{t\in\partial_{\mathrm{ext}}\mathcal{B}\\ t_1=x_1-\rho_1-1}} \langle\, \sigma(\overline{x})\sigma(t)\,\rangle_{\mathbb{L}^*} \langle\, \sigma(t)\sigma(y)\rangle_{\mathbb{L}^*}\\ &\leq 2 \sum_{\substack{t\in\partial_{\mathrm{ext}}\mathcal{B}\\ t_1=x_1-\rho_1-1}} \langle\, \sigma(\overline{x})\sigma(y)\,\rangle_{\mathbb{L}^*}\\ &\leq \frac{2\rho_2}{K''} \exp\{-2\rho_1\hat{\tau}_{\mathrm{bd}}\}\langle\, \sigma(x)\sigma(y)\,\rangle_{\mathbb{L}^*},\end{aligned}$$

where $\overline{x}$ is the image of x under a reflection of axis $\{u \,:\, u_1 = x_1 - \rho_1 - 1\}$.

Let $t \in V_2$, with $t = \lambda(s^*)$. Let s_1 be the last time before s^* such that $\lambda(s_1) \in \Sigma^*$ and s_2 the first time after s^* such that $\lambda(s_2) \in \Sigma^*$. We set $u := \lambda(s_1)$ and $v := \lambda(s_2)$; we have $x_1 - \rho_1 \leq u_1 \leq y_1 + \rho_1$. By definition no bond of λ between times s_1 and s^* belong to $\mathcal{E}(\Sigma^*)$. Therefore Lemma 6.6 and GKS inequalities give

$$\sum_{\substack{\lambda':\delta\lambda'=\{u,t\}\\ \mathcal{E}(\lambda')\cap\mathcal{E}(\Sigma^*)=\emptyset}} q_{\mathbb{L}^*}(\lambda') \leq \langle\, \sigma(u)\sigma(t)\,\rangle. \quad (6.70)$$

The hypothesis $h^* > h_w^*$ implies that $C := \hat{\tau}((1,0)) - \hat{\tau}_{\mathrm{bd}} > 0$. Using Lemma 6.5, (6.70) and the sharp triangle inequality we get

$$\begin{aligned}\sum_{\substack{\lambda:\lambda\ni t\\ \delta\lambda=\{x,y\}}} q_{\mathbb{L}^*}(\lambda) &\leq \sum_{u,v} \langle\, \sigma(x)\sigma(u)\,\rangle_{\mathbb{L}^*} \langle\, \sigma(u)\sigma(t)\,\rangle \langle\, \sigma(t)\sigma(v)\,\rangle \langle\, \sigma(v)\sigma(y)\,\rangle_{\mathbb{L}^*} \quad (6.71)\\ &\leq \sum_{u,v} \exp\{-\hat{\tau}_{\mathrm{bd}}(|u_1 - x_1| + |y_1 - v_1|)\}\\ &\quad \cdot \exp\{-\hat{\tau}(t-u) - \hat{\tau}(v-t)\}\\ &\leq \sum_{u,v} \exp\{-\hat{\tau}_{\mathrm{bd}}(|u_1 - x_1| + |y_1 - v_1|)\} \exp\{-\hat{\tau}(u-v)\}\\ &\quad \cdot \exp\{-\kappa(\|u-t\| + \|t-v\| - \|u-v\|)\}.\end{aligned}$$

We have $\hat{\tau}(u-v) = C|u_1 - v_1| + \hat{\tau}_{\rm bd}|u_1 - v_1|$. Therefore

$$\exp\Big(-\hat{\tau}_{\rm bd}(|u_1-x_1|+|y_1-v_1|)-\hat{\tau}(u-v)\Big) \le \frac{\langle\, \sigma_x\sigma_y\,\rangle_{\mathbb{L}^*}}{K''} \tag{6.72}$$

$$\cdot\exp\Big(-\hat{\tau}_{\rm bd}(|u_1-x_1|+|y_1-v_1|+|u_1-v_1|-|x_1-y_1|)-C|u_1-v_1|\Big).$$

We sum over u, v and t, which are sums over u_1, v_1 and t_1. We set for $s \in \mathbb{R}$ and $[a,b] \subset \mathbb{R}$,

$$d(s,[a,b]) := \min\{\,|t-s| : t \in [a,b]\,\}. \tag{6.73}$$

First notice that

$$|u_1-x_1|+|y_1-v_1|+|u_1-v_1|-|x_1-y_1| \ge 2d(v_1,[x_1,y_1]) \quad \text{if } v_1 \notin [x_1,y_1], \tag{6.74}$$

and

$$|u_1-x_1|+|y_1-v_1|+|u_1-v_1|-|x_1-y_1| \ge 2d(u_1,[x_1,y_1]) \quad \text{if } u_1 \notin [x_1,y_1]. \tag{6.75}$$

Let $\alpha := \kappa/(C+\kappa)$; if $|u_1 - v_1| \le \alpha\rho_2$, then

$$\exp\{-\kappa(\|u-t\|+\|t-v\|-\|u-v\|)\} \le \exp\{-\kappa(2-\alpha)\rho_2\}. \tag{6.76}$$

If $t_1 \notin [u_1, v_1]$ or $t_1 \notin [v_1, u_1]$, then

$$\|u-t\|+\|t-v\|-\|u-v\| \ge \rho_2 + \min\{|u_1-t_1|, |v_1-t_1|\}. \tag{6.77}$$

Let $v_1 \notin [x_1, y_1]$. We consider two cases. First suppose that $|u_1 - v_1| \ge \alpha\rho_2$. We sum over t_1 using (6.77), getting at most a contribution $O(|u_1 - v_1|)$; then we sum over u_1, such that $|u_1 - v_1| \ge \alpha\rho_2$, using the factor $\exp\{-C|u_1 - v_1|\}$; finally we sum over v_1 using (6.74). Thus we get a contribution

$$O\Big(\exp\{-\kappa(2-\alpha)\rho_2\}\Big). \tag{6.78}$$

Suppose that $|u_1 - v_1| \le \alpha\rho_2$. We sum over t_1, using now (6.77) and (6.76), getting at most a contribution

$$O(\rho_2|u_1-v_1|)\exp\{-\kappa(2-\alpha)\rho_2\}; \tag{6.79}$$

then we sum over u_1 using the factor $\exp\{-C|u_1 - v_1|\}$; finally we sum over v_1 using (6.74), getting a contribution (6.78). The case $u_1 \notin [x_1, y_1]$ is similar. It remains to consider the case where $x_1 < u_1 < v_1 < y_1$. We proceed in the same manner, but this time the last sum gives a factor $|x_1 - y_1|$ since in this case

$$|u_1-x_1|+|y_1-v_1|+|u_1-v_1|-|x_1-y_1| = 0. \tag{6.80}$$

Therefore we get a contribution

$$O\big(\rho_2|x_1-y_1|\big)\exp\{-\kappa\rho_2\}. \tag{6.81}$$

□

7. On the Correlation Length Above T_c

Let $\beta^* < \beta_c$ and $0 < h^* < \infty$. The model is defined in the box Λ_L^* with free boundary conditions and coupling constants (6.57). We study the influence of the boundary effect on the correlation length due to the coupling constants $K(e) = h^*\beta^*$, $e \in \mathcal{E}(\Sigma_L^*)$. We consider two definitions, which we call short correlation length and long correlation length, following a similar terminology introduced in [SML] about the long range-order.

The **short correlation length** is the standard correlation length. Let $t, t' \in \mathbb{Z}^{2*}$; we define

$$\frac{1}{\xi_{\mathrm{sh}}(t, t'; \beta^*)} := -\lim_{\substack{k\to\infty \\ k\in\mathbb{N}}} \frac{1}{k\|t-t'\|} \ln\langle\, \sigma(kt)\sigma(kt')\,\rangle(\beta^*). \tag{7.1}$$

In (7.1) we compute the expectation value with respect to the infinite volume Gibbs state on $\mathbb{Z}^{2*}$, which is unique. Then we take the limit $k \to \infty$. We have $\xi_{\mathrm{sh}}(t, t'; \beta^*) = \xi_{\mathrm{sh}}(s, s'; \beta^*)$ if $s - s'$ is a multiple of $t - t'$. In the case of the long correlation length we perform the thermodynamical limit and the limit $k \to \infty$ simultaneously. Let $t, t' \in \Lambda_L^*$; the **long correlation length** is defined by

$$\frac{1}{\xi_{\mathrm{lg}}(t, t'; \beta^*, h^*)} := -\lim_{\substack{k\to\infty \\ k\in\mathbb{N}}} \frac{1}{k\|t-t'\|} \ln\langle\, \sigma(kt)\sigma(kt')\,\rangle_{\Lambda_{kL}^*}(\beta^*, h^*). \tag{7.2}$$

$\xi_{\mathrm{lg}}(t, t'; \beta^*, h^*)$ depends on the position of the sites t and t' in the box Λ_L^*.

The next lemma contains one of the main estimate of the paper, which we shall use later on, when discussing phase-separation lines.

Lemma 7.1. *Let $\beta^* < \beta_c$ and $0 < h^* < \infty$.*

(1) *There exist constants c_1, c_2, c', c'' with the following property. Let $t, t' \in \Lambda_L^*$; suppose that there exist $p, p' \in \Lambda_L^*$ such that*
   1. $\|p - t\| \leq c_1 \ln L$ *and* $\|p' - t'\| \leq c_2 \ln L$,
   2. $\mathcal{S}(p, p'; c' \ln L) \subset \Lambda_L^* \cap \{t \in \mathbb{L}^* : t_2 \geq c'' \ln L\}$ *(see (6.53)).*

   Then there exist C and L_0 such that $\forall L \geq L_0$ and $\forall\, t, t'$ as above,

$$\langle\, \sigma(t)\sigma(t')\,\rangle_{\Lambda_L^*}(\beta^*, h^*) \geq \frac{1}{L^C} e^{-\tau(p-p';\beta^*)}. \tag{7.3}$$

(2) *Let $h^* > h_w(\beta)^*$. There exist c_3, c_4 with the following property. Let $m = (m_1, 1/2) \in \Lambda_L^*$ and $n = (n_1, 1/2) \in \Lambda_L^*$; suppose that (see (6.65))*

$$\mathcal{B}(m, n; c_3 \ln L, c_4 \ln L) \subset \Lambda_L^*. \tag{7.4}$$

   Then there exist C and L_0 such that $\forall L \geq L_0$ and $\forall\, m, n$ as above,

$$\langle\, \sigma(m)\sigma(n)\,\rangle_{\Lambda_L^*}(\beta^*, h^*) \geq C e^{-\hat{\tau}_{\mathrm{bd}}(\beta^*, h^*)|n_1 - m_1|}. \tag{7.5}$$

Proof. By GKS inequalities

$$\langle\, \sigma(t)\sigma(t')\,\rangle_{\Lambda_L^*} \geq \langle\, \sigma(t)\sigma(p)\,\rangle_{\Lambda_L^*} \langle\, \sigma(p)\sigma(p')\,\rangle_{\Lambda_L^*} \langle\, \sigma(p')\sigma(t')\,\rangle_{\Lambda_L^*}. \tag{7.6}$$

From (6.36) we have

$$\langle\, \sigma(t)\sigma(p)\,\rangle_{\Lambda_L^*} \geq \exp\{-O(\ln L)\} \tag{7.7}$$

and

$$\langle \sigma(p')\sigma(t') \rangle_{\Lambda_L^*} \geq \exp\{-O(\ln L)\}. \tag{7.8}$$

Let $\mathcal{S}_L := \mathcal{S}(p, p'; c' \ln L)$; by Lemmas 6.12, 6.10, Proposition 2.4 and taking c' and c'' large enough, there exists L_0 such that $\forall L \geq L_0$,

$$\begin{aligned} \langle \sigma(p)\sigma(p') \rangle_{\Lambda_L^*} &\geq \sum_{\substack{\lambda:\mathcal{E}(\lambda)\subset\mathcal{E}(\mathcal{S}_L) \\ \delta\lambda=\{p,p'\}}} q_{\Lambda_L^*}(\lambda) \\ &\geq \frac{1}{2} \sum_{\substack{\lambda:\mathcal{E}(\lambda)\subset\mathcal{E}(\mathcal{S}_L) \\ \delta\lambda=\{p,p'\}}} q(\lambda) \\ &= \frac{1}{2} \sum_{\lambda:\delta\lambda=\{p,p'\}} q(\lambda) - \frac{1}{2} \sum_{\substack{\lambda:\mathcal{E}(\lambda)\not\subset\mathcal{E}(\mathcal{S}_L) \\ \delta\lambda=\{p,p'\}}} q(\lambda) \\ &\geq \frac{1}{4} \langle \sigma(p)\sigma(p') \rangle \\ &\geq \frac{K}{4\|p-p'\|^{1/2}} \mathrm{e}^{-\tau(p-p')}. \end{aligned} \tag{7.9}$$

This proves (1), since $\|p - p'\| \leq O(L)$.

We estimate $\langle \sigma(m)\sigma(n) \rangle_{\Lambda_L^*}$ by Lemma 6.3, 6.13 and Proposition 2.4. Let $\mathcal{B}_L := \mathcal{B}(m, n; c_3 \ln L, c_4 \ln L)$; by taking c_3 and c_4 large enough, there exists L_0 such that $\forall L \geq L_0$,

$$\begin{aligned} \langle \sigma(m)\sigma(n) \rangle_{\Lambda_L^*} &\geq \sum_{\substack{\lambda:\mathcal{E}(\lambda)\subset\mathcal{E}(\mathcal{B}_L) \\ \delta\lambda:=\{m,n\}}} q_{\Lambda_L^*}(\lambda) \\ &\geq \sum_{\substack{\lambda:\mathcal{E}(\lambda)\subset\mathcal{E}(\mathcal{B}_L) \\ \delta\lambda:=\{m,n\}}} q_{\mathbb{L}^*}(\lambda) \\ &= \frac{1}{2} \sum_{\lambda:\delta\lambda=\{m,n\}} q_{\mathbb{L}^*}(\lambda) - \frac{1}{2} \sum_{\substack{\lambda:\mathcal{E}(\lambda)\not\subset\mathcal{E}(\mathcal{B}_L) \\ \delta\lambda=\{m,n\}}} q_{\mathbb{L}^*}(\lambda) \\ &\geq \frac{1}{4} \langle \sigma(m)\sigma(n) \rangle_{\mathbb{L}^*} \\ &\geq \frac{K''}{4} \mathrm{e}^{-\hat{\tau}_{\mathrm{bd}}|n_1 - m_1|}. \end{aligned} \tag{7.10}$$

This proves (2). □

Let $t, t' \in \Lambda_L^*$. Suppose that $h^* > h_w(\beta)^*$. We apply Lemma 7.1 to show that we may have (depending on the choice of t and t')

$$\xi_{\mathrm{lg}}(t, t'; \beta^*, h^*) > \xi_{\mathrm{sh}}(t, t'; \beta^*). \tag{7.11}$$

We assume that $t_1 < t_1'$ and choose $m = (m_1, 1/2)$ and $n = (n_1, 1/2)$, $m_1 < n_1$, $m, n \in \Lambda_L^*$. By GKS inequalities

$$\langle \sigma(kt)\sigma(kt') \rangle_{\Lambda_{kL}^*} \geq \langle \sigma(kt)\sigma(km) \rangle_{\Lambda_{kL}^*} \langle \sigma(km)\sigma(kn) \rangle_{\Lambda_{kL}^*} \langle \sigma(kn)\sigma(kt') \rangle_{\Lambda_{kL}^*}. \tag{7.12}$$

If k is large enough, then we can use Lemma 7.1 to estimate (7.12). There exists $\widetilde{C}$ such that

$$\langle \sigma(kt)\sigma(kt') \rangle_{\Lambda_{kL}^*} \geq \frac{1}{O(k^{\widetilde{C}})} \mathrm{e}^{-k(\tau(t-m) + \tau(n-t'))} \mathrm{e}^{-k\hat{\tau}_{\mathrm{bd}}|n_1 - m_1|}. \tag{7.13}$$

Therefore

$$\frac{1}{\xi_{\mathrm{lg}}(t, t'; \beta^*, h^*)} \leq \frac{\tau(t-m) + \tau(n-t') + \hat{\tau}_{\mathrm{bd}}|n_1 - m_1|}{\|t - t'\|}. \tag{7.14}$$

We can optimize this upper bound by taking the minimum over m and n. On the other hand

$$\frac{1}{\xi_{\mathrm{sh}}(t, t'; \beta^*)} = \frac{\tau(t - t')}{\|t - t'\|}. \tag{7.15}$$

The results of Sect. 4 show that there exist t, t', when $h^* > h_w(\beta)^*$, such that for suitable m and n,

$$\tau(t-m) + \tau(n-t') + \hat{\tau}_{\mathrm{bd}}|n_1 - m_1| < \tau(t - t'), \tag{7.16}$$

and so

$$\xi_{\mathrm{lg}}(t, t'; \beta^*, h^*) > \xi_{\mathrm{sh}}(t, t'; \beta^*). \tag{7.17}$$

8. From Microscopic to Macroscopic Theory

We show that the phase-separation line λ is concentrated in a neighborhood of the solution of the variational problem of Sect. 4, scaled by L, with probability tending to 1 when $L \to \infty$. The thickness of the neighborhood is at most $O((L \ln L)^{1/2})$. Consequently, if we do a coarse-grained description of the configurations, using cells of linear size L^α, $1/2 < \alpha < 1$, then we see the emergence of an interface, which coincides with the solution of the variational problem. This justifies the macroscopic theory, starting from the microscopic theory. It is possible to consider even a more general situation. Suppose that we prescribe a curve $\mathcal{C} \subset Q$ from A to B. We can estimate the probability that the phase-separation line is in a neighborhood of this curve scaled by L, the thickness of the neighborhood being at most $O((L \ln L)^{1/2})$. Using the method developed fully in [PV1], this probability is roughly equal to

$$\exp\Big(- L(\mathtt{W}(\mathcal{C}) - \mathtt{W}^*)\Big), \tag{8.1}$$

where $\mathtt{W}^*$ is the minimum of the variational problem. We shall not give the details of that estimate here.

8.1. Main result. The weight of a separation line λ in Λ_L^*, going from u^L to v^L, is given by $q_{\Lambda_L^*}(\lambda)$. These weights define a measure on the set of the phase-separation lines, such that the total mass is

$$\sum_{\substack{\mathcal{E}(\lambda)\subset\mathcal{E}(\Lambda_L^*):\\ \delta\lambda=\{u^L,v^L\}}} q_{\Lambda_L^*}(\lambda) = \langle\sigma(u^L)\sigma(v^L)\rangle_{\Lambda_L^*}. \tag{8.2}$$

Consequently we can introduce the following probability measure:

$$P_L^{ab}[\lambda] = \frac{q_{\Lambda_L^*}(\lambda)}{\langle\sigma(u^L)\sigma(v^L)\rangle_{\Lambda_L^*}}. \tag{8.3}$$

Let $\mathcal{D}$ and $\mathcal{W}$ be the curves in Q introduced in Sect. 4. We set

$$I_i^L := \{\, x \in \Sigma_L^* : \|x - w_i^L\| \le (ML\log L)^{1/2}\,\} \quad , \quad i = 1, 2, \tag{8.4}$$

with $w_i^L = (LP_i, 1/2)$ and $[P_1, P_2] = \mathcal{W} \cap W_Q$. We set

$$\rho_L := M \ln L. \tag{8.5}$$

We define two sets of phase-separation lines. The set $\mathcal{T}_{\mathcal{D}}$ contains all λ, $\mathcal{E}(\lambda) \subset \mathcal{E}(\Lambda_L^*)$, such that

a_1. $\delta\lambda = \{u^L, v^L\}$;
a_2. $\mathcal{E}(\lambda)$ is inside $\mathcal{E}(\mathcal{S}(u^L, v^L; \rho_L))$.

The set $\mathcal{T}_{\mathcal{W}}$ contains all λ, $\mathcal{E}(\lambda) \subset \mathcal{E}(\Lambda_L^*)$, considered as parameterized curves $s \mapsto \lambda(s)$, such that

b_1. $\delta\lambda = \{u^L, v^L\}$, $\lambda(0) := u^L$;
b_2. $\exists s_1$ such that $\lambda(s_1) \in I_1^L$ and for all $s < s_1$, $\lambda(s) \cap \Sigma_L^* = \emptyset$;
b_3. $\lambda_1 := \{\lambda(s) : s \le s_1\}$ is inside $\mathcal{S}(u^L, \lambda(s_1); \rho_L)$;
b_4. $\exists s_2$ such that $\lambda(s_2) \in I_2^L$ and for all $s_2 < s$, $\lambda(s) \cap \Sigma_L^* = \emptyset$;
b_5. $\lambda_3 := \{\lambda(s) : s_2 \le s\}$ is inside $\mathcal{S}(\lambda(s_2), v^L; \rho_L)$;
b_6. $\lambda_2 := \{\lambda(s) : s_1 \le s \le s_2\}$ is inside

$$\{x \in \Lambda_L^* : x(2) \le \rho_L\ ,\ \lambda(s_1)(1) - \rho_L \le x(1) \le \lambda(s_2)(1) + \rho_L\}.$$

Theorem 8.1. *Let $\beta > \beta_c$, $h > 0$, $0 < a < 1$, $0 < b < 1$. There exist $M > 0$ and $L_0 = L_0(h, \beta, M)$ such that, for all $L \ge L_0$, the following statements are true.*

1. *Suppose that the solution of the variational problem in Q is the curve $\mathcal{D}$. Then*

$$P_L^{ab}[\mathcal{T}_{\mathcal{D}}] \ge 1 - L^{-O(M)}. \tag{8.6}$$

2. *Suppose that the solution of the variational problem in Q is the curve $\mathcal{W}$. Then*

$$P_L^{ab}[\mathcal{T}_{\mathcal{W}}] \ge 1 - L^{-O(M)}. \tag{8.7}$$

3. *Suppose that the solution of the variational problem in Q is either the curve $\mathcal{D}$ or the curve $\mathcal{W}$. Then*

$$P_L^{ab}[\mathcal{T}_{\mathcal{D}} \cup \mathcal{T}_{\mathcal{W}}] \geq 1 - L^{-O(M)}. \tag{8.8}$$

Comment. The results of Theorem 8.1 are optimal in the following sense: At a finer scale we do not expect the phase-separation line to converge to some non-random set, but rather to some random process. It is known that fluctuations of a phase-separation line of length $O(L)$, which is not in contact with the wall, are $O(L^{1/2})$ (see [Hi2] and [DH]). On the other hand, if the phase-separation line is attracted by the wall on a length $O(L)$, then we expect that its excursions away from the wall have a size typically bounded by $O(\log L)$.

Proof. 1. Suppose that the minimum of the variational problem is given by $\mathcal{D}$, $\mathtt{W}(\mathcal{D}) = \mathtt{W}^*$. Let $\mathtt{W}^{**}$ be the minimum of the functional over all simple curves in Q, with end-points A and B, and which touch the wall W_Q. By hypothesis there exists $\delta > 0$ with $\mathtt{W}^{**} = \mathtt{W}^* + \delta$.

We set $\mathcal{S}_1 := \mathcal{S}(u^L, v^L; \rho_L)$; for L large enough $\mathcal{S}_1 \cap \Sigma_L^* = \emptyset$, since $a > 0$ and $b > 0$. We apply Lemma 7.1. We have

$$\begin{aligned} P_L^{ab}[\{\lambda \notin \mathcal{T}_{\mathcal{D}}\}] &= \frac{1}{\langle \sigma(u^L)\sigma(v^L)\rangle_{\Lambda_L^*}} \sum_{\lambda \notin \mathcal{T}_{\mathcal{D}}} q_{\Lambda_L^*}(\lambda) \\ &\leq L^C \exp\{\mathtt{W}^* L\} \sum_{\lambda \notin \mathcal{T}_{\mathcal{D}}} q_L^*(\lambda). \end{aligned} \tag{8.9}$$

We estimate the numerator of $P_L^{ab}[\{\lambda \notin \mathcal{T}_{\mathcal{D}}\}]$. There are two cases, either $\lambda \cap \Sigma_L^* \neq \emptyset$ or $\lambda \cap \Sigma_L^* = \emptyset$. The first case is easy to estimate. Consider λ as a unit-speed parametrized curve from u^L to v^L and suppose that $z_1(\lambda)$, resp. $z_2(\lambda)$, is the first, resp. last, point of $\lambda \cap \Sigma_L^* \neq \emptyset$. Then by Lemmas 6.5 and 6.6,

$$\sum_{\lambda \cap \Sigma_L^* \neq \emptyset} q_{\Lambda_L^*}(\lambda) \leq \sum_{z_1, z_2 \in \Sigma_L^*} \mathrm{e}^{-\hat{\tau}(z_1 - u^L)} \mathrm{e}^{-\hat{\tau}_{\mathrm{bd}}(z_2 - z_1)} \mathrm{e}^{-\hat{\tau}(v^L - z_2)}. \tag{8.10}$$

We can bound above this sum by $O(L^2)\exp\{-L\mathtt{W}^{**}\}$. In the second case we have $\lambda \cap \Sigma_L^* = \emptyset$. Using Lemmas 6.7, 6.6, GKS inequalities and Lemma 6.10,

$$\begin{aligned} \sum_{\substack{\lambda \notin \mathcal{T}_{\mathcal{D}} \\ \lambda \cap \Sigma_L^* = \emptyset}} q_{\Lambda_L^*}(\lambda) &\leq \sum_{z \in \partial_{\mathrm{ext}} \mathcal{S}_1} \sum_{\substack{z \in \lambda,\, \lambda \cap \Sigma_L^* = \emptyset \\ \delta\lambda = \{u^L, v^L\}}} q_{\Lambda_L^*}(\lambda) \\ &\leq \sum_{z \in \partial_{\mathrm{ext}} \mathcal{S}_1} \langle\, \sigma(u^L)\sigma(z)\,\rangle_{\Lambda_L^* \setminus \Sigma_L^*} \langle\, \sigma(z)\sigma(v^L)\,\rangle_{\Lambda_L^* \setminus \Sigma_L^*} \\ &\leq \sum_{z \in \partial_{\mathrm{ext}} \mathcal{S}_1} \langle\, \sigma(u^L)\sigma(z)\,\rangle \langle\, \sigma(z)\sigma(v^L)\,\rangle \\ &\leq O(L^{3/2 - \kappa M}) \langle\, \sigma(u^L)\sigma(v^L)\,\rangle \\ &\leq O(L^{3/2 - \kappa M}) \exp\{-\mathtt{W}^* L\}. \end{aligned} \tag{8.11}$$

This proves the first statement.

2. Suppose that the minimum of the variational problem is given by $\mathcal{W}$, $\mathtt{W}(\mathcal{W}) = \mathtt{W}^*$. Then there exists $\delta > 0$ such that $\mathtt{W}(\mathcal{D}) = \mathtt{W}^* + \delta$. We estimate $P_L^{ab}[\{\lambda \notin \mathcal{T}_{\mathcal{W}}\}]$ in several steps. Notice that condition b_1 is always satisfied.
1. The probability that condition b_2 is satisfied, but not b_3, can be estimated as in (8.11) using Lemma 6.6; it is smaller than $O(L^{C+1})/L^{\Delta M}$.
2. The probability that condition b_4 is satisfied, but not b_5, is estimated in the same way; it is smaller than $O(L^{C+1})/L^{\Delta M}$.
3. The probability that conditions b_2 and b_4 are satisfied, but not b_6, can be estimated by Lemma 6.13; it is smaller than $L^{-O(M)}$.
4. We estimate the probability that condition b_2 is not satisfied. The case with condition b_5 is similar. If λ does not intersect Σ_L^*, then this probability is smaller than $O(L^C)\exp\{-\delta L\}$, since $\mathtt{W}(\mathcal{D}) = \mathtt{W}^* + \delta$. Suppose that there exist s_1 and s_2, with $\lambda(s_i) \in \Sigma_L^*$, $\lambda(s) \cap \Sigma_L^* = \emptyset$ for all $s < s_1$ and $\lambda(s) \cap \Sigma_L^* = \emptyset$ for all $s_2 < s$. Let $p_i^L := \lambda(s_i)$, $i = 1, 2$. Under these conditions, b_2 is not satisfied if and only if $p_1^L \notin I_1^L$. Let $\mathcal{C}(p_1^L, p_2^L)$ be the polygonal curve from u^L to p_1^L, then from p_1^L to p_2^L and finally from p_2^L to v^L. Then the probability of this event is bounded above by

$$\sum_{\substack{p_1^L \in \Sigma_L^*:\\ p_1^L \notin I_1^L}} \sum_{p_2^L \in \Sigma_L^*} \exp\{-\mathtt{W}(\mathcal{C}(p_1^L, p_2^L))\} \leq \tag{8.12}$$

$$O(L^2)\max\{\exp\{-\mathtt{W}(\mathcal{C}(p_1^L, p_2^L))\} \mid p_1^L \in \Sigma_L^* \setminus I_1^L,\ p_2^L \in \Sigma_L^*\}.$$

Suppose that $\mathcal{C}$ denotes the polygonal line giving the maximum; scaled by $1/L$ we get a polygonal line in Q, denoted by $\mathcal{C}^*$, from A to some point P_1^*, then from P_1^* to P_2^* and finally from P_2^* to B. Let θ^* be the angle between the straight line from A to P_1^* with the wall. We have

$$\mathtt{W}(\mathcal{C}) = L\mathtt{W}(\mathcal{C}^*) \geq L(g(\theta^*, a) + g(\theta_Y, b)). \tag{8.13}$$

By hypothesis

$$|\theta^* - \theta_Y| \geq \frac{1}{L^{1/2}} O((M \log L)^{1/2}). \tag{8.14}$$

Therefore (use a Taylor expansion of g around θ_Y and the monotonicity of $g(\theta, x)$ on $[0, \theta_Y]$, respectively $[\theta_Y, \pi/2]$) there exists a positive constant α such that

$$\begin{aligned} \mathtt{W}(\mathcal{C}^*) &\geq g(\theta_Y, a) + g(\theta_Y, b) + \frac{\alpha M \log L}{L} \\ &= \mathtt{W}^* + \frac{\alpha M \log L}{L}. \end{aligned} \tag{8.15}$$

We conclude that the probability, that condition b_2 is not satisfied, is bounded above by $O(L^{C+2})/L^{\alpha M}$. If M is large enough, the second statement of the theorem is true.

3. The proof of the third statement of the theorem is similar. □

9. Appendix: N Phase-Separation Lines

In this appendix we indicate how we can treat problems with N phase-separation lines. We consider the simplest case, in order to illustrate the basic ideas. We reduce the question of finding typical configurations to a similar questions for a single phase separation line.

We assume in this section that all coupling constants are equal, $K(e) = \beta$, $\beta > \beta_c$.

We fix $2N$ points $A_i, i = 1, \ldots, 2N$, on the boundary of Q. Then we scale the box Q by $L \in \mathbb{N}$ and get $2N$ points A_i^L, $i = 1, \ldots, 2N$. We assume that A_i^L, $i = 1, \ldots, 2N$, are at the middle of bonds of the lattice $\mathbb{Z}^2$. Consequently, these points give naturally a partition of $\partial\Lambda_L$ into $2N$ subsets (see Fig. 4), which we denote by $[A_i^L, A_{i+1}^L]$, $i = 1, \ldots, 2N$, with $A_{2N+1}^L \equiv A_1^L$. Let η be the boundary conditions for Λ_L,

$$\eta(x) = \begin{cases} +1 & \text{if } x \in [A_i^L, A_{i+1}^L] \text{ and } i \text{ is odd,} \\ -1 & \text{if } x \in [A_i^L, A_{i+1}^L] \text{ and } i \text{ is even.} \end{cases} \tag{9.1}$$

This boundary condition defines N phase-separation lines $\lambda_i(\omega)$ $i = 1, \ldots, N$, in any configuration ω compatible with η. The set $V_L(\eta) := \{a_i^L : i = 1, \ldots, 2N\}$ of end-points of these phase-separation lines is uniquely determined by the points A_i^L. Given ω compatible with η, the N phase-separation lines $\lambda_i(\omega)$ give a partition of $V_L(\eta)$ into two-point subsets $\delta\lambda_j(\omega) = \{a_{j_1}^L, a_{j_2}^L\}$. The set of all possible partitions of $V_L(\eta)$ compatible with N phase-separation lines is denoted by $\mathcal{P}(V_L(\eta))$ and an element of $\mathcal{P}(V_L(\eta))$ by $\underline{a}^L = (a_{1_1}^L, a_{1_2}^L; \ldots; a_{N_1}^L, a_{N_2}^L)$.

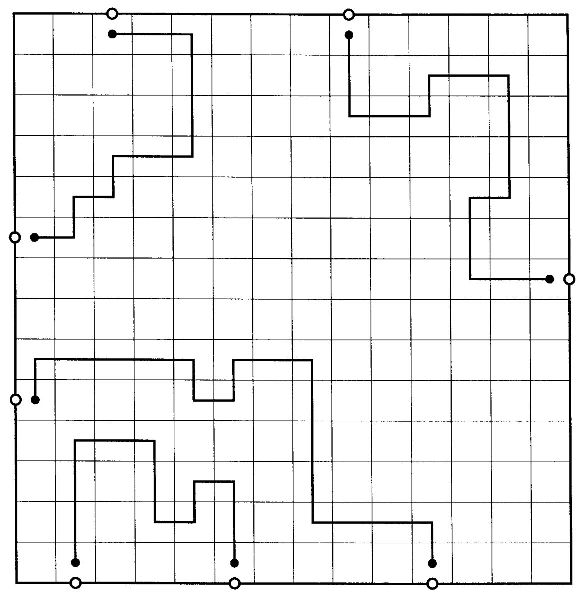


Fig. 4. The box Λ_L, the points A_i^L (*white dots*) and the points a_i^L (*black dots*). A family of phase-separation lines is also drawn

Lemma 9.1. *Let η be a b.c. with N phase-separation lines for Λ_L. Let $\underline{b}^L = (b^L_{1_1}, b^L_{1_2}; \ldots; b^L_{N_1}, b^L_{N_2}) \in \mathcal{P}(V_L(\eta))$. Then*

$$\left\langle \left\{ \underline{\lambda} : \delta\lambda_j = \{b^L_{j_1}, b^L_{j_2}\},\ j = 1, \ldots, N \right\} \right\rangle^{\eta}_{\Lambda_L} \leq \frac{\prod_{j\geq 1} \langle\, \sigma(b^L_{j1})\sigma(b^L_{j2})\,\rangle^*_{\Lambda_L}}{\max_{\underline{a}^L \in \mathcal{P}(V_L(\eta))} \prod_{j\geq 1} \langle\, \sigma(a^L_{j1})\sigma(a^L_{j2})\,\rangle^*_{\Lambda_L}}. \tag{9.2}$$

Proof. Let $q^{\eta}_{\Lambda_L}(\underline{\lambda})$ be the weight of the compatible family $\underline{\lambda}$ of N phase-separation lines. We estimate the denominator of the left-hand side of (9.2). Let

$$\underline{a}^L = (a^L_{1_1}, a^L_{1_2}; \ldots; a^L_{N_1}, a^L_{N_2}) \in \mathcal{P}(V_L(\eta)).$$

By Lemma 6.2 and GKS inequalities

$$\begin{aligned} \sum_{\underline{\lambda}} q^{\eta}_{\Lambda_L}(\underline{\lambda}) &= \langle \prod_{t\in V_L(\eta)} \sigma(t)\,\rangle^*_{\Lambda_L} \\ &\geq \prod_{j\geq 1} \langle\, \sigma(a^L_{j1})\sigma(a^L_{j2})\,\rangle^*_{\Lambda_L}. \end{aligned} \tag{9.3}$$

We estimate the numerator of the left-hand side of (9.2). By Lemma 6.5,

$$\sum_{\substack{\underline{\lambda}: \\ \delta\lambda_j = \{b^L_{j_1}, b^L_{j_2}\}}} q^{\eta}_{\Lambda_L}(\underline{\lambda}) \leq \prod_{j\geq 1} \langle\, \sigma(b^L_{j1})\sigma(b^L_{j2})\,\rangle^*_{\Lambda_L}. \tag{9.4}$$

□

When $J(e) \equiv \beta$ it is easy to analyze the right-hand side of (9.2). Let

$$\underline{a}^L = (a^L_{1_1}, a^L_{1_2}; \ldots; a^L_{N_1}, a^L_{N_2}) \in \mathcal{P}(V_L(\eta));$$

we set

$$\mathtt{W}(\underline{a}^L) := \frac{1}{L} \sum_{j=1}^{N} \tau(a^L_{j_2} - a^L_{j_1}), \tag{9.5}$$

and

$$\mathtt{W}_\eta := \min\{\mathtt{W}(\underline{a}^L) : \underline{a}^L \in \mathcal{P}(V_L(\eta))\}. \tag{9.6}$$

Then by Proposition 2.4 and Lemma 7.1,

$$\left\langle \left\{ \underline{\lambda} : \delta\lambda_j = \{b^L_{j_1}, b^L_{j_2}\},\ j = 1, \ldots, N \right\} \right\rangle^{\eta}_{\Lambda_L} \leq L^{O(N)} \exp\{-L(\mathtt{W}(\underline{b}^L) - \mathtt{W}_\eta)\}. \tag{9.7}$$

In the generic case the minimum in (9.6) is attained at a single $\underline{b}^L \in \mathcal{P}(V_L(\eta))$; there exists $\varepsilon > 0$ such that

$$\mathtt{W}(\underline{a}^L) \geq \mathtt{W}_\eta + \varepsilon \quad , \quad \underline{a}^L \neq \underline{b}^L. \tag{9.8}$$

We can use Lemma 6.5 to bound above the denominator of the left-hand side of (9.2),

$$\sum_{\underline{\lambda}} q^{\eta}_{\Lambda_L}(\underline{\lambda}) \leq \sum_{\underline{p}\in\mathcal{P}(V_L(\eta))} \prod_{j\geq 1} \langle\, \sigma(a^L_{p_{j_1}})\sigma(a^L_{p_{j_2}})\,\rangle^*_{\Lambda_L}. \tag{9.9}$$

Notice that this is slightly better than what we would have obtained using the Gaussian inequality. For L large enough only a single term dominates in (9.9), namely the term given by the partition $\underline{p}$ such that

$$\underline{b}^L = (a^L_{p_{1_1}}, a^L_{p_{1_2}}; \ldots ; a^L_{p_{N_1}}, a^L_{p_{N_2}}). \tag{9.10}$$

Therefore in the generic case, for fixed N and large L,

$$\prod_{j\geq 1} \langle\, \sigma(b^L_{j_1})\sigma(b^L_{j_2})\,\rangle^*_{\Lambda_L} \leq \sum_{\underline{\lambda}} q^{\eta}_{\Lambda_L}(\underline{\lambda}) \leq (1+O(\mathrm{e}^{-\varepsilon L})) \prod_{j\geq 1} \langle\, \sigma(b^L_{j_1})\sigma(b^L_{j_2})\,\rangle^*_{\Lambda_L}. \tag{9.11}$$

Let $\underline{\lambda}$ be a family of compatible phase-separation lines, such that $\delta\lambda_j = \{b^L_{j_1}, b^L_{j_2}\}$, $j = 1, \ldots, N$. Formula (6.11) and Lemma 6.4 imply that

$$q^{\eta}_{\Lambda_L}(\underline{\lambda}) = q_{\Lambda^*_L}(\underline{\lambda}) \geq \prod_{j\geq 1} q_{\Lambda^*_L}(\lambda_j). \tag{9.12}$$

Notice that the factor $\langle\, \sigma(b^L_{j_1})\sigma(b^L_{j_2})\,\rangle^*_{\Lambda_L}$ in (9.11) is equal to

$$\langle\, \sigma(b^L_{j_1})\sigma(b^L_{j_2})\,\rangle^*_{\Lambda_L} = \sum_{\substack{\lambda:\\ \delta\lambda=\{b^L_{j_1},b^L_{j_2}\}}} q_{\Lambda^*_L}(\lambda). \tag{9.13}$$

We summarize the results obtained so far.

1. In the generic situation described above the typical phase-separation lines $\underline{\lambda}$ compatible with the b.c. η are those such that $\delta\lambda_j = \{b^L_{j_1}, b^L_{j_2}\}$, $j = 1, \ldots, N$, where $\underline{b}^L = (b^L_{1_1}, b^L_{1_2}; \ldots ; b^L_{N_1}, b^L_{N_2})$ is the element of $\mathcal{P}(V_L(\eta))$, which minimizes $\mathtt{W}(\underline{a}^L) := \frac{1}{L}\sum_{j=1}^N \tau(a^L_{j_2} - a^L_{j_1})$.
2. The probability of the occurrence of $\underline{\lambda}$ compatible with the b.c. η, assuming that $\delta\lambda_j = \{b^L_{j_1}, b^L_{j_2}\}$, $j = 1, \ldots, N$, is bounded below by

$$\prod_{j\geq 1} \frac{q_{\Lambda^*_L}(\lambda_j)}{\sum_{\substack{\lambda:\\ \delta\lambda=\{b^L_{j_1},b^L_{j_2}\}}} q_{\Lambda^*_L}(\lambda)} \geq \prod_{j\geq 1} \frac{q(\lambda_j)}{\sum_{\substack{\lambda:\\ \delta\lambda=\{b^L_{j_1},b^L_{j_2}\}}} q(\lambda)}. \tag{9.14}$$

We suppose that we are in the generic case. Then there are N segments with total length minimal, which do not intersect. Therefore the distance between two segments is at least δL, $\delta > 0$. We also suppose that for each pair of points $\{b^L_{j_1}, b^L_{j_2}\}$ we can apply Case 1 of Lemma 7.1. If L is large enough, then the ellipses $\mathcal{S}_j := \mathcal{S}(b^L_{j_1}, b^L_{j_2}; c' \ln L)$, $j = 1, \ldots, N$, are disjoint two by two. Let

$$\{\,\underline{\lambda} : \delta\lambda_j = \{b^L_{j_1}, b^L_{j_2}\},\ \lambda_j \subset \mathcal{S}_j,\ j = 1, \ldots, N\,\}. \tag{9.15}$$

We can easily estimate the probability of the event (9.15) using (9.14). Indeed, we can reduce the estimate to an estimate for an event concerning a single interface,

$$\{\,\lambda : \delta\lambda = \{b^L_{j_1}, b^L_{j_2}\},\ \lambda \subset \mathcal{S}_j\,\}. \tag{9.16}$$

We have, using Lemma 6.7, GKS inequalities and Lemma 6.10,

$$\begin{aligned}\sum_{\substack{\mathcal{E}(\lambda)\not\subset\mathcal{E}(\mathcal{S}_j):\\ \delta\lambda=\{b^L_{j_1},b^L_{j_2}\}}} q_{\Lambda^*_L}(\lambda) &\le \sum_{z\in\partial_{\text{ext}}\mathcal{S}_j}\ \sum_{\substack{\lambda\ni z:\\ \delta\lambda=\{b^L_{j_1},b^L_{j_2}\}}} q_{\Lambda^*_L}(\lambda)\\ &\le \sum_{z\in\partial_{\text{ext}}\mathcal{S}_j} \langle\,\sigma(b^L_{j_1})\sigma(z)\,\rangle_{\Lambda^*_L}\,\langle\,\sigma(z)\sigma(b^L_{j_2})\,\rangle_{\Lambda^*_L}\\ &\le \sum_{z\in\partial_{\text{ext}}\mathcal{S}_j} \langle\,\sigma(b^L_{j_1})\sigma(z)\,\rangle\,\langle\,\sigma(z)\sigma(b^L_{j_2})\,\rangle\\ &\le O(L^{3/2-\kappa c'})\langle\,\sigma(b^L_{j_1})\sigma(b^L_{j_2})\,\rangle.\end{aligned}$$

On the other hand, by Lemma 7.1 and Proposition 2.4,

$$\begin{aligned}\sum_{\substack{\mathcal{E}(\lambda)\subset\mathcal{E}(\Lambda^*_L):\\ \delta\lambda=\{b^L_{j_1},b^L_{j_2}\}}} q_{\Lambda^*_L}(\lambda) &= \langle\,\sigma(b^L_{j_1})\sigma(b^L_{j_2})\,\rangle_{\Lambda^*_L}\\ &\ge L^{-C}\,\mathrm{e}^{-\tau(b^L_{j_1}-b^L_{j_2})}\\ &\ge L^{-C-1/2}\langle\,\sigma(b^L_{j_1})\sigma(b^L_{j_2})\,\rangle.\end{aligned} \tag{9.17}$$

Choosing c' so large that $3/2-\kappa c'+C+1/2=\alpha<0$, the probability of the event (9.16) is larger than $1-O(L^{-\alpha})$. Therefore, the probability of the event (9.15) is also larger than $1-O(L^{-\alpha})$.

Acknowledgements. We thank A. Patrick for communicating to us the exact expressions of the mass gaps.

References

[A1] Abraham, D.B.: Solvable model with a roughening transition for a planar Ising ferromagnet. Phys. Rev. Lett. **44**, 1165–1168 (1980)

[A2] Abraham, D.B.: Surface Structures and Phase Transitions – Exact Results. In: *Phase Transitions and Critical Phenomena*, Vol **10**, , Eds. C. Domb, J.L. Lebowitz, London: Academic Press, 1986, pp. 1–74

[Al] Alexander, K.S.: Power-law corrections to exponential decay of connectivities and correlations in lattice models. To appear in Annals of Probability

[Az1] Aizenman, M.: Translation invariance and Instability of phase coexistence in the two dimensional Ising model. Commun. Math. Phys. **73**, 83–94 (1980)

[Az2] Aizenman, M.: Geometric Analysis of ϕ^4 Fields and Ising Models. Parts I and II. Commun. Math. Phys. **86**, 1–48 (1982)

[AY] Au Yang, H.: Thermodynamics of an anisotropic boundary of a two-dimensional Ising model. J. Math. Phys. **14**, 937–946 (1973)

[AC] Abraham,D.B., de Coninck, J.: Description of phases in a film-thickening transition. J. Phys. **16** A, L333–L337 (1983)

[AK] Abraham, D.B., Ko, L.-F. Exact derivation of the modified Young equation for partial wetting. Phys. Rev. Lett. **63**, 275–278 (1989)

[AR] Abraham, D.B., Reed, P.: Phase separation in the two-dimensional ferromagnet. Phys. Rev. Lett. **33**, 377–379 (1974); Interface profile of the Ising ferromagnet in two dimensions Commun. Math. Phys. **49**, 35–46 (1974)

[ACD] Abraham D.B., de Coninck, J., Dunlop, F.: Contact angle for 2D Ising ferromagnet. Phys. Rev. B **39**, 4708–4710 (1989)

[AA] Akutsu, N., Akutsu, Y.: Relationship between the anisotropic interface tension, the scaled interface width and the equilibrium shape in two dimensions. J. Phys. **19** A, 2813–2820 (1986)

[AF] Au-Yang, H., Fisher, M.E.: Bounded and inhomogeneous Ising models II. Specific-heat scaling function for a strip. Phys. Rev. B **11**, 3469–3487 (1975)

[ABCP] Alberti, G., Bellettini, G., Cassandro, M., Presutti, E.: Surface tension in Ising systems with Kac potentials. J. Stat. Phys. **82**, 743–796 (1996)

[BLP1] Bricmont, J., Lebowitz, J.L., Pfister, C.-E.: On the local structure of the phase separation line in the two-dimensional Ising system. J. Stat. Phys. **26**, 313–332 (1981)

[BLP2] Bricmon,t J., Lebowitz, J.L., Pfister, C.-E.: On the surface tension of lattice systems. Annals of the New York Academy of Sciences **337**, 214–223 (1980)

[D] Dobrushin, R.L.: A statistical behaviour of shapes of boundaries of phases. In: R. Kotecký (ed.) *Phase Transitions: Mathematics, Physics, Biology...*, Singapore: World Scientific, 1993, pp. 60–70

[DH] Dobrushin, R.L., Hryniv, O.: Fluctuations of the phase boundary in the 2D Ising ferromagnet. Commun. Math. Phys. **189**, 395–445 (1997)

[F] Fisher, M.E.: Walks, Walls, Wetting, and Melting. J. Stat. Phys. **34**, 667–729 (1984)

[FP1] Fröhlich, J., Pfister, C.-E.: Semi–infinite Ising model I. Thermodynamic functions and phase diagram in absence of magnetic field. Commun. Math. Phys. **109**, 493–523 (1987)

[FP2] Fröhlich, J., Pfister, C.-E.: Semi–infinite Ising model II. The wetting and layering transitions. Commun. Math. Phys. **112**, 51–74 (1987)

[G] Gallavotti, G.: The phase separation line in the two–dimensional Ising model. Commun. Math. Phys. **27**, 103–136 (1972)

[Hi1] Higuchi, Y.: On the absence of non-translation invariant Gibbs states for the two-dimensional Ising model. In: J. Fritz, J.L. Lebowitz and D. Szász (eds) *Random Fields*, Esztergom, Amsterdam: North-Holland, Vol I, 1979, pp. 517–534

[Hi2] Higuchi, Y.: On some limit theorems related to the phase separation line in the two-dimensional Ising model. Z. Wahrsch. verw. Geb. **50**, 287–315 (1979)

[I1] Ioffe, D.: Large deviations for the 2D Ising model: A lower bound without cluster expansions. J. Stat. Phys. **74**, 411–432 (1994)

[I2] Ioffe, D.: Ornstein–Zernike Behaviour and Analyticity of Shapes for Self-Avoiding Walks on $\mathbb{Z}^d$. To appear in Markov Processes and Related Fields

[LP] Lebowitz, J.L., Pfister, C.-E.: Surface tension and phase coexistence. Phys. Rev. Lett. **46**, 1031–1033 (1981)

[MW] McCoy, B.M., Wu, T.T.: The Two-dimensional Ising Model. Cambridge, MA: Harvard University Press, 1973

[Pa1] Patrick, A.: The influence of boundary conditions on Solid-On-Solid models. J. Stat. Phys. **90**, 389–433 (1998)

[Pf1] Pfister, C.-E.: Interface and surface tension in Ising model. In: *Scaling and self–similarity in physics*, ed. J. Fröhlich, Basel: Birkhäuser, 1983 pp. 139–161

[Pf2] Pfister, C.-E.: Large deviations and phase separation in the two–dimensional Ising model. Helv. Phys. Acta **64**, 953–1054 (1991)

[PU] Patrick, A., Upton, P.J.: Surface phase transitions in two dimensions: Metastability and re–entrance. Dublin Institute for Advanced Studies, Preprint DIAS-STP-96-08 (1996)

[PP] Pfister, C.-E., Penrose, O.: Analyticity properties of the surface free energy of the Ising model. Commun. Math. Phys. **115**, 691–699 (1988)

[PV1] Pfister, C.-E., Velenik, Y.: Large deviations and continuum limit in the 2D Ising model. Prob. Th. Rel. Fields **109**, 435–506 (1997)

[PV2] Pfister C.-E., Velenik Y.: Mathematical theory of the wetting phenomenon in the 2D Ising model. Helv. Phys. Acta **69**, 949–973 (1996)

[SML] Schultz, T.D., Mattis, D.C., Lieb, E.H.: Two-dimensional Ising model as a soluble problem of many fermions. Rev. Mod. Phys. **36**, 856–871 (1964)

[T] Talagrand, M.: A new look at independence. Ann. Prob. **24**, 1–34 (1996)

[V] Velenik, Y.: Phase separation as a large deviations problem: A microscopic derivation of surface thermodynamics for some 2D spin systems. Thèse **1712** EPF-L (1997); available electronically from the author

[ZA] Zia, R.K.P., Avron, J.E.: Total surface energy and equilibrium shapes: Exact results for the d=2 Ising crystal. Phys. Rev. B **25**, 2042–2045 (1982)

Communicated by M. E. Fisher