Commun. Math. Phys. 204, 17 – 38 (1999)

Communications in
Mathematical Physics

Separation of Variables in the Elliptic Gaudin Model

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Received: 21 August 1998 / Accepted: 12 January 1999

Abstract: For the elliptic Gaudin model (a degenerate case of the XYZ integrable spin chain) a separation of variables is constructed in the classical case. The corresponding separated coordinates are obtained as the poles of a suitably normalized Baker-Akhiezer function. The classical results are generalized to the quantum case where the kernel of the separating integral operator is constructed. The simplest one-degree-of-freedom case is studied in detail.

1. Introduction

The quantum elliptic (or XYZ) Gaudin model was introduced in [1], see also [2], as a limiting case of the integrable XYZ spin chain [3]. The commuting Hamiltonians H_n of the model are expressed as quadratic combinations of sl_2 spin operators. Determining the spectrum of H_n turned out to be a difficult problem like the original XYZ spin chain. Let us list the known facts related to this problem.

- A solution by means of the Algebraic Bethe Ansatz has been obtained only recently [4]. See also [5].
- As shown in [6], in the SU(2)-invariant, or XXX, or rational, case the spectrum and the eigenfunctions of the model can be found via an alternative method, Separation of Variables, see also the survey [7].
- In [8] the separation of variables in the rational Gaudin model [6] was interpreted as a geometric Langlands correspondence.
- In [9] a separation of variables was constructed for the elliptic Gaudin–Calogero model which is closely related to the XYZ Gaudin model, though the separation of variables for the former one is much simpler.
- The results of [8] and [9] are based on the interpretation of the corresponding Gaudin models as conformal field theoretical models (Wess–Zumino–Witten models). The

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corresponding interpretation of the XYZ Gaudin model was obtained in [10], but the conformal field theoretical model corresponding to the XYZ Gaudin model turned out to be so complicated that writing down the geometric Langlands correspondence for this system, following [8], is not easy.

The main task of the present paper is to present a construction of separated variables for the XYZ Gaudin model both in the classical and quantum cases. The paper is organized as follows. After giving a detailed description of the XYZ Gaudin model in Sect. 2, we proceed, in Sect. 3, with the classical case and, following the general philosophy of [7], construct the separated coordinates as the poles of an appropriately normalized Baker-Akhiezer function. The corresponding eigenvalues of the Lax matrix are then shown to provide the canonically conjugated momenta. The whole construction is a simplified version of the one used in [11].

The quantum case is considered in Sect. 4. The separating classical canonical transformation is replaced by an integral operator K. We write down a system of differential equations for the kernel of K and show that it is integrable. The resulting integral operator K intertwines the original and the separated variables and provides, respectively, a Radon–Penrose transformation of the corresponding D-modules. The quantization constructed is a formal one, since we do not study the transformations of the functional spaces of quantum states, leaving it for a further study. A detailed study of the spectral problem is given in the simplest case only: $N = 1$ (Sect. 5). We show that the corresponding separated equation is none other than the (generalized) Lamé equation. Two appendices contain, respectively, a list of properties of elliptic functions, and the formulas describing a realization of finite-dimensional representations of $sl_2(\mathbb{C})$ on the elliptic curve which are used throughout the paper.

2. Description of the Model

Let us recall the definition of the XYZ Gaudin model, following [4]. The elementary Lax operator $L(u)$ of the model depending on a complex parameter u (spectral parameter) is given by

$$L(u) = \frac{1}{2}\sum_{a=1}^{3} w_a(u) S^a \otimes \sigma^a = \begin{pmatrix} \mathcal{A}(u) & \mathcal{B}(u) \\ \mathcal{C}(u) & -\mathcal{A}(u) \end{pmatrix}. \tag{2.1}$$

Here σ^a are the Pauli matrices,

$$w_1(u) = \frac{\theta_{11}'}{\theta_{10}}\frac{\theta_{10}(u)}{\theta_{11}(u)}, \quad w_2(u) = \frac{\theta_{11}'}{\theta_{00}}\frac{\theta_{00}(u)}{\theta_{11}(u)}, \quad w_3(u) = \frac{\theta_{11}'}{\theta_{01}}\frac{\theta_{01}(u)}{\theta_{11}(u)}, \tag{2.2}$$

where $\theta_{\alpha\beta}(u) = \theta_{\alpha\beta}(u;\tau)$, $\theta_{\alpha\beta} = \theta_{\alpha\beta}(0)$, $\theta_{11}' = d/du(\theta_{11}(u))|_{u=0}$, (see Appendix A) and S^a are generators of the Lie algebra $sl_2(\mathbb{C})$:

$$[S^a, S^b] = i S^c.$$

Hereafter (a, b, c) denotes a cyclic permutation of $(1, 2, 3)$. Note that $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are holomorphic except at $u \in \mathbb{Z} + \tau\mathbb{Z}$, where these operators have poles of first order.

Introducing the notation $\overset{1}{L} := L \otimes \mathbb{1}_2$ and $\overset{2}{L} := \mathbb{1}_2 \otimes L$, where $\mathbb{1}_2$ is the unit operator in $\mathbb{C}^2$, one can establish the commutation relation

$$[\overset{1}{L}(u), \overset{2}{L}(v)] = [r(u-v), \overset{1}{L}(u) + \overset{2}{L}(v)], \tag{2.3}$$

where $r(u)$ is a classical r matrix defined by

$$r(u) = -\frac{1}{2}\sum_{a=1}^{3} w_a(u)\sigma^a \otimes \sigma^a. \tag{2.4}$$

The r matrix behaves as $-\frac{1}{u}(\mathcal{P} - \frac{1}{2}) + O(u^{-3})$ when $u \to 0$. Here $\mathcal{P}$ is the permutation operator: $\mathcal{P}(x \otimes y) = y \otimes x$. Explicitly, in the natural basis in $\mathbb{C}^2 \otimes \mathbb{C}^2$,

$$r(u) = \begin{pmatrix} a(u) & 0 & 0 & d(u) \\ 0 & b(u) & c(u) & 0 \\ 0 & c(u) & b(u) & 0 \\ d(u) & 0 & 0 & a(u) \end{pmatrix}, \tag{2.5}$$

where

$$a(u) = -b(u) = -\frac{w_3(u)}{2}, \quad c(u) = -\frac{w_1(u) + w_2(u)}{2}, \quad d(u) = -\frac{w_1(u) - w_2(u)}{2}.$$

Since $w_a(u)$ are quasiperiodic in u because of (A.3):

$$\begin{array}{rcrcr} w_1(u) & = & w_1(u+1) & = & -w_1(u+\tau), \\ w_2(u) & = & -w_2(u+1) & = & -w_2(u+\tau), \\ w_3(u) & = & -w_3(u+1) & = & w_3(u+\tau), \end{array} \tag{2.6}$$

the L operator (2.1) has the following quasiperiodicity:

$$L(u+1) = \sigma^1 L(u)\sigma^1, \qquad L(u+\tau) = \sigma^3 L(u)\sigma^3. \tag{2.7}$$

Let ℓ_n $(n = 1, \dots, N)$ be half integers. The total Hilbert space of the model is $V = \bigotimes_{n=1}^N V_n$, where $V_n \simeq V^{(\ell_n)}$ and $V^{(\ell)}$ is a spin ℓ representation space of sl_2:

$$\rho^\ell : sl_2(\mathbb{C}) \to \mathrm{End}_{\mathbb{C}}(V^{(\ell)}), \qquad V^{(\ell)} \simeq \mathbb{C}^{2\ell+1}. \tag{2.8}$$

The generating function of the integrals of motion is

$$\hat{\tau}(u) = \frac{1}{2}\,\mathrm{Tr}\, T^2(u), \tag{2.9}$$

where the matrix $T(u)$ is constructed as the sum of elementary Lax operators (2.1),

$$T(u) = \sum_{n=1}^{N} L_n(u - z_n) = \begin{pmatrix} A(u) & B(u) \\ C(u) & -A(u) \end{pmatrix}. \tag{2.10}$$

Here z_n are mutually distinct complex parameters,

$$L_n(u) = \frac{1}{2}\sum_{a=1}^{3} w_a(u) S_n^a \otimes \sigma^a \tag{2.11}$$

$$S_n^a = \mathbb{1}_{V_1} \otimes \dots \otimes \mathbb{1}_{V_{n-1}} \otimes \rho^{\ell_n}(S^a) \otimes \mathbb{1}_{V_{n+1}} \otimes \dots \otimes \mathbb{1}_{V_N}. \tag{2.12}$$

By virtue of the commutation relations (2.3) the operator $T(u)$ satisfies the same commutation relations

$$[\overset{1}{T}(u), \overset{2}{T}(v)] = [r(u-v), \overset{1}{T}(u) + \overset{2}{T}(v)], \tag{2.13}$$

which implies the commutativity of $\hat{\tau}(u)$:

$$[\hat{\tau}(u), \hat{\tau}(v)] = 0. \tag{2.14}$$

Operator $\hat{\tau}(u)$ is explicitly written down as follows:

$$\hat{\tau}(u) = \sum_{n=1}^{N} \wp_{11}(u-z_n)\ell_n(\ell_n+1) + \sum_{n=1}^{N} H_n \zeta_{11}(u-z_n) + H_0. \tag{2.15}$$

Here $\wp_{11}$, ζ_{11} are normalized Weierstraß functions defined by (A.5) and

$$\begin{aligned} H_n &= \frac{1}{2}\sum_{m\neq n}\sum_{a=1}^{3} w_a(z_n-z_m)S_n^a S_m^a, \\ H_0 &= \sum_{n,m=1}^{N}\sum_{a=1}^{3} Z_a(z_n-z_m)S_n^a S_m^a \end{aligned} \tag{2.16}$$

are integrals of motion, where

$$Z_1(t) = \frac{\theta'_{11}}{4\theta_{10}}\frac{\theta'_{10}(t)}{\theta_{11}(t)}, \qquad Z_2(t) = \frac{\theta'_{11}}{4\theta_{00}}\frac{\theta'_{00}(t)}{\theta_{11}(t)}, \qquad Z_3(t) = \frac{\theta'_{11}}{4\theta_{01}}\frac{\theta'_{01}(t)}{\theta_{11}(t)}. \tag{2.17}$$

Note that the integrals of motion H_n $(n = 0, \ldots, N)$ appear as coefficients of the elliptic Knizhnik-Zamolodchikov equations in [10]. Our expression (2.16) for H_0 differs from that given in [4] because of different normalization of the $\wp$ and ζ functions.

The classical Gaudin model is obtained if we replace all the commutators with the Poisson brackets, e.g.

$$\{\overset{1}{T}(u), \overset{2}{T}(v)\} = [r(u-v), \overset{1}{T}(u) + \overset{2}{T}(v)], \tag{2.18}$$

instead of (2.13). The spin variables S^a satisfy, respectively, the Poisson commutation relations $[S^a, S^b] = iS^c$ and are subject to the constraint $\sum_{a=1}^{3}(S^a)^2 = \ell^2$.

3. Classical Separation of Variables

According to the recipe in [7], the separated coordinates x_n should be constructed as the poles of a suitably normalized Baker-Akhiezer function (eigenvector of Lax matrix $T(u)$). The corresponding canonically conjugated variables should appear then as the corresponding eigenvalues of $T(x_n)$. Instead of choosing a normalization, we shall rather speak of a choice of a gauge transformation M of $T(u)$. The separated coordinates x_n will be obtained then as the zeros of the off-diagonal element $\tilde{B}(u)$ of the twisted matrix $\tilde{T} = M^{-1}TM$.

The classical XYZ Gaudin model is a degenerate case of the classical lattice Landau-Lifshits equation for which a separation of variables has been constructed in [11], see also a discussion in [7]. Here we use essentially the same gauge transformation $M(u)$ as in [11], and our calculations represent a revised and simplified version of those in [11].

3.1. Gauge transformation. Let $M(u; \tilde{u})$ be the following 2×2 matrix

$$M(u;\tilde{u}) := \begin{pmatrix} -\theta_{01}\left(\frac{u-\tilde{u}}{2};\frac{\tau}{2}\right) & -\theta_{01}\left(\frac{u+\tilde{u}}{2};\frac{\tau}{2}\right) \\ \theta_{00}\left(\frac{u-\tilde{u}}{2};\frac{\tau}{2}\right) & \theta_{00}\left(\frac{u+\tilde{u}}{2};\frac{\tau}{2}\right) \end{pmatrix}, \tag{3.1}$$

where u and $\tilde{u}$ are (possibly dynamical) parameters. (This matrix appears also in the context of the algebraic Bethe Ansatz. See [12,4].) A twisted L-operator $\tilde{L}(u, v; \tilde{u})$ depending on a parameter $\tilde{u}$ is defined by

$$\tilde{L}(u,v;\tilde{u}) = \begin{pmatrix} \tilde{\mathcal{A}}(u,v;\tilde{u}) & \tilde{\mathcal{B}}(u,v;\tilde{u}) \\ \tilde{\mathcal{C}}(u,v;\tilde{u}) & -\tilde{\mathcal{A}}(u,v;\tilde{u}) \end{pmatrix} := M^{-1}(u;\tilde{u})L(u-v)M(u;\tilde{u}). \tag{3.2}$$

Likewise we define the twisted Lax matrix by

$$\tilde{T}(u;\tilde{u}) = \begin{pmatrix} \tilde{A}(u;\tilde{u}) & \tilde{B}(u;\tilde{u}) \\ \tilde{C}(u;\tilde{u}) & -\tilde{A}(u;\tilde{u}) \end{pmatrix} := M^{-1}(u;\tilde{u})T(u)M(u;\tilde{u}). \tag{3.3}$$

Note that $M(u; \tilde{u})$ has the quasiperiodicity because of (A.3):

$$\begin{aligned} M(u+1;\tilde{u}) &= -\sigma_1 M(u;\tilde{u}), \\ M(u+\tau;\tilde{u}) &= e^{-\pi i(u+\tau/2)}\sigma_3 M(u;\tilde{u})\exp(\pi i\tilde{u}\sigma_3). \end{aligned} \tag{3.4}$$

These formulae together with (2.7) imply that the function $\tilde{B}(u, v; \tilde{u})$ has the following quasiperiodicity properties:

$$\tilde{B}(u+1;\tilde{u}) = \tilde{B}(u;\tilde{u}), \qquad \tilde{B}(u+\tau;\tilde{u}) = e^{-2\pi i\tilde{u}}\tilde{B}(u;\tilde{u}). \tag{3.5}$$

Hence by a standard argument in the theory of elliptic functions (see [13]), we have

$$\deg(\mathrm{div}(\tilde{B}(u))) = 0, \qquad -\tilde{u} + \sum(\mathrm{mult}_y\,\mathrm{div}(\tilde{B}(u)))\,y \in \mathbb{Z}+\tau\mathbb{Z}, \tag{3.6}$$

where $\mathrm{mult}_y\,\mathrm{div}(\tilde{B}(u))$ is the multiplicity of a divisor $[y]$ in the divisor $\mathrm{div}(\tilde{B}(u))$. By the definition (3.3), operator $\tilde{B}(u; \tilde{u})$ is holomorphic except at poles of $A(u)$, $B(u)$, $C(u)$, i.e., $u = z_n$ ($n = 1, \dots, N$), and at zeros of $\det M(u; \tilde{u})$, i.e., $u = 0$ modulo $\mathbb{Z} + \tau\mathbb{Z}$:

$$\mathrm{div}(\tilde{B}(u)) \geq -\left(\sum_{n=1}^{N}[z_n] + [0]\right) \quad (\mathrm{mod}\ \mathbb{Z}+\tau\mathbb{Z}). \tag{3.7}$$

Thus (3.7) and (3.6) imply that there are $(N + 1)$ points $x_0, \dots, x_N$ such that

$$\mathrm{div}(\tilde{B}(u)) \equiv \sum_{j=0}^{N}[x_j] - \left(\sum_{n=1}^{N}[z_n] + [0]\right) \quad (\mathrm{mod}\ \mathbb{Z}+\tau\mathbb{Z}), \tag{3.8}$$

and

$$\sum_{j=0}^{N} x_j \equiv \sum_{n=1}^{N} z_n - \tilde{u} \quad (\mathrm{mod}\ \mathbb{Z}+\tau\mathbb{Z}). \tag{3.9}$$

Let us fix the parameter $\tilde{u}$ by the condition that one of x_j, for example x_0, is a constant ξ. Note that $\tilde{u}$ becomes then a dynamical variable. Thus we have

$$\tilde{B}(x_j;\tilde{u}) = \tilde{B}(u=\xi;\tilde{u}) = 0. \tag{3.10}$$

Dynamical variables $x_1, \dots, x_N$ are (classically) separated coordinates of the system as we will see below.

3.2. Poisson commutation relations and classical separation of variables. The main purpose of this subsection is to prove the following commutation relations.

Theorem 3.1. *Generically the dynamical variables x_j and $-\tilde{A}(x_j)$ have the canonical Poisson brackets:*

(i) $\{x_i, x_j\} = 0$ *for all* $i, j = 1, \dots, N$.
(ii) $\{-\tilde{A}(x_i), -\tilde{A}(x_j)\} = 0$ *for all* $i, j = 1, \dots, N$.
(iii) $\{-\tilde{A}(x_i), x_j\} = \delta_{i,j}$ *for all* $i, j = 1, \dots, N$.

To prove the theorem, we follow the argument of [11]. First let us introduce several notations. Define the matrices $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$ as

$$\hat{A} := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \hat{B} := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \hat{C} := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \hat{D} := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.11}$$

Gauge transformation of them are defined as follows:

$$\begin{aligned} \hat{A}(u;\tilde{u}) &:= M(u;\tilde{u})\hat{A}M(u;\tilde{u})^{-1}, & \hat{B}(u;\tilde{u}) &:= M(u;\tilde{u})\hat{B}M(u;\tilde{u})^{-1}, \\ \hat{C}(u;\tilde{u}) &:= M(u;\tilde{u})\hat{C}M(u;\tilde{u})^{-1}, & \hat{D}(u;\tilde{u}) &:= M(u;\tilde{u})\hat{D}M(u;\tilde{u})^{-1}. \end{aligned} \tag{3.12}$$

Bracket $\langle , \rangle$ is the standard inner product of the 2×2 matrices:

$$\langle X, Y \rangle = \operatorname{tr} XY. \tag{3.13}$$

When $X(u)$ is a variable depending on the spectral parameter u, we will denote $X(x_i)$ by X_i for brevity. For example,

$$(\partial_u \langle \hat{C} T \rangle)_i = \left. \frac{\partial}{\partial u} \right|_{u=x_i} \operatorname{tr}(\hat{C}(u;\tilde{u})T(u)).$$

The following statement is proved by the same argument as in the proof of the Theorem in §2 of [11].

Lemma 3.2. *For any dynamical variable X,*

$$\{X, \tilde{u}\} = -\frac{\langle \hat{C}_0 \{X, T\}_0 \rangle}{\langle \partial_{\tilde{u}} \hat{C}_0 T_0 \rangle}, \tag{3.14}$$

and

$$\{X, x_j\} = \frac{\langle \hat{C}_0 \{X, T\}_0 \rangle \langle \partial_{\tilde{u}} \hat{C}_j T_j \rangle - \langle \hat{C}_j \{X, T\}_j \rangle \langle \partial_{\tilde{u}} \hat{C}_0 T_0 \rangle}{(\partial_u \langle \hat{C}, T \rangle)_j \langle \partial_{\tilde{u}} \hat{C}_0 T_0 \rangle}. \tag{3.15}$$

We also need the formula for the twisted r matrix.

Lemma 3.3. *Define*

$$\tilde{r}(u, v; \tilde{u}) := \overset{1}{M}(u;\tilde{u})^{-1} \overset{2}{M}(v;\tilde{u})^{-1} r(u - v) \overset{1}{M}(u;\tilde{u}) \overset{2}{M}(v;\tilde{u}), \tag{3.16}$$

which we call the **twisted** r **matrix** *and let* $\tilde r_{ij}(u,v;\tilde u)$ *be its* (i,j) *element. Then it has the following form:*

$$\begin{aligned}
\tilde r_{11}(u,v;\tilde u) = -\tilde r_{22}(u,v;\tilde u) = -\tilde r_{33}(u,v;\tilde u) = \tilde r_{44}(u,v;\tilde u) = \\
= -\frac{1}{2}\left(\frac{\theta'_{11}(u-v)}{\theta_{11}(u-v)} - \frac{\theta'_{11}(u)}{\theta_{11}(u)} + \frac{\theta'_{11}(v)}{\theta_{11}(v)}\right),
\end{aligned} \tag{3.17}$$

$$\begin{aligned}
\tilde r_{12}(u,v;\tilde u) = -\tilde r_{13}(v,u;\tilde u) = -\tilde r_{21}(u,v;-\tilde u) = \tilde r_{31}(v,u;-\tilde u) = \\
= \tilde r_{24}(v,u;\tilde u) = -\tilde r_{34}(u,v;\tilde u) = -\tilde r_{42}(v,u;-\tilde u) = \tilde r_{43}(u,v;-\tilde u) = \\
= \frac{-\theta'_{11}\theta_{11}(v+\tilde u)}{2\theta_{11}(\tilde u)\theta_{11}(v)},
\end{aligned} \tag{3.18}$$

$$\tilde r_{14}(u,v:\tilde u) = \tilde r_{41}(u,v;\tilde u) = 0, \tag{3.19}$$

$$\tilde r_{23}(u,v;\tilde u) = \tilde r_{32}(u,v;-\tilde u) = \frac{-\theta'_{11}\theta_{11}(u-v+\tilde u)}{\theta_{11}(u-v)\theta_{11}(\tilde u)}. \tag{3.20}$$

Proof. The proof is given by a direct computation. For example, we have formulae like

$$\begin{aligned}
M(u,\tilde u)^{-1}\sigma_1 M(u,\tilde u) &= \frac{1}{\theta_{11}(u)\theta_{11}(\tilde u)}\begin{pmatrix}\theta_{10}(u)\theta_{10}(\tilde u) & \theta_{10}\theta_{10}(u+\tilde u)\\ -\theta_{10}\theta_{10}(u-\tilde u) & -\theta_{10}(u)\theta_{10}(\tilde u)\end{pmatrix},\\
M(u,\tilde u)^{-1}(i\sigma_2) M(u,\tilde u) &= \frac{1}{\theta_{11}(u)\theta_{11}(\tilde u)}\begin{pmatrix}\theta_{00}(u)\theta_{00}(\tilde u) & \theta_{00}\theta_{00}(u+\tilde u)\\ -\theta_{00}\theta_{00}(u-\tilde u) & -\theta_{00}(u)\theta_{00}(\tilde u)\end{pmatrix},\\
M(u,\tilde u)^{-1}\sigma_3 M(u,\tilde u) &= \frac{1}{\theta_{11}(u)\theta_{11}(\tilde u)}\begin{pmatrix}-\theta_{01}(u)\theta_{01}(\tilde u) & -\theta_{01}\theta_{01}(u+\tilde u)\\ \theta_{01}\theta_{01}(u-\tilde u) & \theta_{01}(u)\theta_{01}(\tilde u)\end{pmatrix},
\end{aligned}$$

which follow from the addition theorems (cf. [14, pp. 20, 22]) and the Landen transformation (cf. [13, §21.52]) of theta functions. Substituting them in the definition of $\tilde r$ (3.16) and using the addition theorems again, we can prove the lemma. □

Proof of Theorem 3.1. Using the formulae (3.14) and (3.15), we have

$$\begin{aligned}
\{x_j,x_k\} = \frac{1}{(\partial_u\langle\hat C T\rangle)_j(\partial_u\langle\hat C T\rangle)_k}\times \\
\times\Big[\frac{\langle\partial_{\tilde u}\hat C_j T_j\rangle\langle\partial_{\tilde u}\hat C_k T_k\rangle}{\langle\partial_{\tilde u}\hat C_0 T_0\rangle^2}\langle\overset{1}{\hat C}_0\overset{2}{\hat C}_0\{\overset{1}{T},\overset{2}{T}\}_{00}\rangle - \frac{\langle\partial_{\tilde u}\hat C_k T_k\rangle}{\langle\partial_{\tilde u}\hat C_0 T_0\rangle}\langle\overset{1}{\hat C}_j\overset{2}{\hat C}_0\{\overset{1}{T},\overset{2}{T}\}_{j0}\rangle \\
-\frac{\langle\partial_{\tilde u}\hat C_j T_j\rangle}{\langle\partial_{\tilde u}\hat C_0 T_0\rangle}\langle\overset{1}{\hat C}_0\overset{2}{\hat C}_k\{\overset{1}{T},\overset{2}{T}\}_{0k}\rangle + \langle\overset{1}{\hat C}_j\overset{2}{\hat C}_k\{\overset{1}{T},\overset{2}{T}\}_{jk}\rangle\Big].
\end{aligned} \tag{3.21}$$

Therefore computation of $\{x_j,x_k\}$ reduces to computation of $\langle\overset{1}{\hat C}_j\overset{2}{\hat C}_k\{\overset{1}{T},\overset{2}{T}\}_{jk}\rangle$.

As in Appendix B of [11], we have

$$\langle\overset{1}{\hat\Phi}_j\overset{2}{\hat\Psi}_k\{\overset{1}{T},\overset{2}{T}\}_{jk}\rangle = \mathrm{tr}_1\,\mathrm{tr}_2([\overset{1}{\hat\Phi}\overset{2}{\hat\Psi},\tilde r(x_j,x_k)](\overset{1}{\tilde T}(x_j;\tilde u)+\overset{2}{\tilde T}(x_k;\tilde u))), \tag{3.22}$$

for any $\Phi,\Psi = A,B,C,D$. Substituting $\Phi=\Psi=C$ and using (3.19), we have $\langle\overset{1}{\hat C}_j\overset{2}{\hat C}_k\{\overset{1}{T},\overset{2}{T}\}_{jk}\rangle = 0$. Thus (3.21) implies that $\{x_j,x_k\}=0$.

A direct consequence of this is $\{x_j, \tilde{u}\} = 0$, which follows from (3.9). Using these results and Lemma 3.2, we have for $j \neq k$,

$$\{\tilde{A}(x_j), x_k\} = \frac{\langle \partial_{\tilde{u}} \hat{C}_k T_k \rangle \langle \overset{1}{\hat{A}}_j \overset{2}{\hat{C}}_0 \{\overset{1}{T}, \overset{2}{T}\}_{j0} \rangle - \langle \partial_{\tilde{u}} \hat{C}_0 T_0 \rangle \langle \overset{1}{\hat{A}}_j \overset{2}{\hat{C}}_k \{\overset{1}{T}, \overset{2}{T}\}_{jk} \rangle}{(\partial_u \langle \hat{C}, T \rangle)_k \langle \partial_{\tilde{u}} \hat{C}_0 T_0 \rangle}. \tag{3.23}$$

Hence we need to know $\langle \overset{1}{\hat{A}}_j \overset{2}{\hat{C}}_k \{\overset{1}{T}, \overset{2}{T}\}_{jk} \rangle$ and $\langle \partial_{\tilde{u}} \hat{C}_k T_k \rangle$. The former can be computed by (3.22) and (3.19) and we have

$$\langle \overset{1}{\hat{A}}_j \overset{2}{\hat{C}}_k \{\overset{1}{T}, \overset{2}{T}\}_{jk} \rangle = -2\tilde{r}_{12}(x_j, x_k; \tilde{u}) \tilde{A}_k. \tag{3.24}$$

The factor $\langle \partial_{\tilde{u}} \hat{C}_k T_k \rangle$ is computed as follows:

$$\begin{aligned} \langle \partial_{\tilde{u}} \hat{C}_k T_k \rangle = \langle [M(x_k; \tilde{u})^{-1} \partial_{\tilde{u}} M(x_k; \tilde{u}), \hat{C}] \tilde{T}(x_k; \tilde{u}) \rangle = \\ = -\frac{\theta'_{11} \theta_{11}(x_k + \tilde{u})}{\theta_{11}(x_k) \theta_{11}(\tilde{u})} \tilde{A}_k. \end{aligned} \tag{3.25}$$

Substituting (3.24), (3.25) and (3.18) into (3.23), we have $\{\tilde{A}(x_j), x_k\} = 0$ for $j \neq k$.

The proof of $\{\tilde{A}(x_j), \tilde{A}(x_k)\} = 0$ is done in a similar way. In addition to the formulae we have shown above, we need

$$\langle \overset{1}{\hat{A}}_j \overset{2}{\hat{A}}_k \{\overset{1}{T}, \overset{2}{T}\}_{jk} \rangle = \tilde{r}_{13}(x_j, x_k; \tilde{u}) \tilde{C}_j + \tilde{r}_{12}(x_j, x_k; \tilde{u}) \tilde{C}_k, \tag{3.26}$$

$$\langle \partial_{\tilde{u}} \hat{A}_k T_k \rangle = \frac{-\theta'_{11} \theta_{11}(x_k + \tilde{u})}{2\theta_{11}(x_k) \theta_{11}(\tilde{u})} \tilde{C}_k. \tag{3.27}$$

Proof of the remaining equation $\{\tilde{A}(x_j), x_j\} = -1$ requires special care, since the r matrix $r(u)$ diverges at $u = 0$. Instead of (3.24), we use

$$\begin{aligned} \langle \overset{1}{\hat{A}}_j \overset{2}{\hat{C}}_j \{\overset{1}{T}, \overset{2}{T}\}_{jj} \rangle &= -2\tilde{r}_{12}(x_j, x_j; \tilde{u}) \tilde{A}_j - \lim_{u \to x_j} \tilde{r}_{32}(u, x_j; \tilde{u}) \tilde{B}(u; \tilde{u}) \\ &= -2\tilde{r}_{12}(x_j, x_j; \tilde{u}) \tilde{A}_j + (\partial_u \tilde{B})_j. \end{aligned} \tag{3.28}$$

Noting $(\partial_u \langle \hat{C}, T \rangle)_j = (\partial_u \tilde{B})_j$ and substituting (3.28) and (3.25) into (3.23), we have $\{\tilde{A}(x_j), x_j\} = -1$. □

Since $\tilde{B}$ is zero at $u = x_j$, the dynamical variable $X_j := -\tilde{A}(x_j)$ is an eigenvalue of $T(x_j)$:

$$\tilde{T}(x_j) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = X_j \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{3.29}$$

$$T(x_j) \begin{pmatrix} -\theta_{01}\left(\frac{x_j + \tilde{u}}{2}; \frac{\tau}{2}\right) \\ \theta_{00}\left(\frac{x_j + \tilde{u}}{2}; \frac{\tau}{2}\right) \end{pmatrix} = X_j \begin{pmatrix} -\theta_{01}\left(\frac{x_j + \tilde{u}}{2}; \frac{\tau}{2}\right) \\ \theta_{00}\left(\frac{x_j + \tilde{u}}{2}; \frac{\tau}{2}\right) \end{pmatrix}. \tag{3.30}$$

Thus if we define the *characteristic polynomial* by

$$W(z, u) := \det(z - T(u)), \tag{3.31}$$

each pair of dynamical variables (x_j, X_j) satisfies an equation

$$W(X_j, x_j) = 0, \tag{3.32}$$

for $j = 1, \ldots, N$. Therefore, following the definition in [7], canonical variables $(x_1, \ldots, x_N; X_1, \ldots, X_N)$ are *separated variables* of the classical elliptic Gaudin model.

4. Quantum System: General Case

We return now to the quantum elliptic Gaudin model and construct the quantum separation of variables. The special case $N = 1$ is considered in the next section, Sect. 5.

4.1. Kernel function. Suppose that the representation space $V_n = V^{\ell_n}$ (2.8) is realized as a space of functions on a certain space with coordinate y_n and that the operators S^a are differential operators on, e.g., polynomials or elliptic functions. The separating operator K is expressed as an integral operator

$$K\, f(x_1, \ldots, x_N) = \int dy_1 \cdots dy_N\, \Phi(x_1, \ldots, x_N | y_1, \ldots, y_N) f(y_1, \ldots, y_N), \tag{4.1}$$

which maps a function of $(y_1, \ldots, y_N)$ in $V_1 \otimes \cdots \otimes V_N$ to a function of N-variables x_i on the elliptic curve $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$.

Let us define the operator X_i as follows:

$$X_i := \frac{\partial}{\partial x_i} - \Lambda(x_i), \qquad \Lambda(x) = \sum_{n=1}^{N} \ell_n \frac{\theta_{11}'(x - z_n)}{\theta_{11}(x - z_n)}. \tag{4.2}$$

Lemma 4.1. *The following system of partial differential equations satisfies the Frobenius integrability condition:*

$$\tilde{B}^*(x_i; \tilde{u})\Phi = 0, \qquad i = 1, \ldots, N, \tag{4.3}$$

$$(X_i + \tilde{A}^*(x_i; \tilde{u}))\Phi = 0, \qquad i = 1, \ldots, N, \tag{4.4}$$

where P^* is the (formal) adjoint of a differential operator P with respect to $(y_1, \ldots, y_N)$ and we set

$$\tilde{u} = \sum_{n=1}^{N} z_n - \sum_{j=0}^{N} x_j \tag{4.5}$$

for a certain constant $x_0 = \xi$.

Proof. This is a consequence of the commutation relation (2.13). By multiplying $\overset{1}{M}(u; \tilde{u})\overset{2}{M}(v; \tilde{u})$ from the right and its inverse from the left, we have

$$[\overset{1}{\tilde{T}}(u; \tilde{u}), \overset{2}{\tilde{T}}(v; \tilde{u})] = [\tilde{r}(u, v; \tilde{u}), \overset{1}{\tilde{T}}(u; \tilde{u}) + \overset{2}{\tilde{T}}(v; \tilde{u})]. \tag{4.6}$$

Note that $\tilde{u}$ is *not* a dynamical variable in contrast to that in Sect. 3.

In order to show the consistency of Eqs. (4.3) for i and for j, we prove that $[\tilde{B}^*(x_i;\tilde{u}), \tilde{B}^*(x_j;\tilde{u})]$ is expressed as a linear combination of $\tilde{B}^*(x_i;\tilde{u})$ and $\tilde{B}^*(x_j;\tilde{u})$.

Since the formal adjoint is an algebra anti-isomorphism, $(PQ)^* = Q^*P^*$, we have

$$[\tilde{B}^*(x_i;\tilde{u}), \tilde{B}^*(x_j;\tilde{u})] = [\tilde{B}(x_j;\tilde{u}), \tilde{B}(x_i;\tilde{u})]^*. \tag{4.7}$$

The (1,4)-element of (4.6) gives

$$[\tilde{B}(u;\tilde{u}), \tilde{B}(v;\tilde{u})] = 2(\tilde{r}_{12}(u,v;\tilde{u})\tilde{B}(u) - \tilde{r}_{12}(v,u;\tilde{u})\tilde{B}(v)) \tag{4.8}$$

by virtue of (3.19) and (3.18). Replacing u and v in (4.8) by x_i and x_j respectively which are not dynamical, we obtain

$$[\tilde{B}^*(x_i;\tilde{u}), \tilde{B}^*(x_j;\tilde{u})] = \frac{\theta_{11}'\theta_{11}(x_j+\tilde{u})}{\theta_{11}(x_j)\theta_{11}(\tilde{u})}\tilde{B}^*(x_i) - \frac{\theta_{11}'\theta_{11}(x_i+\tilde{u})}{\theta_{11}(x_i)\theta_{11}(\tilde{u})}\tilde{B}^*(x_j), \tag{4.9}$$

which means that Eq. (4.3) for i and for j are compatible.

Next we show the compatibility condition

$$[X_i + \tilde{A}^*(x_i;\tilde{u}), X_j + \tilde{A}^*(x_j;\tilde{u})] = 0, \tag{4.10}$$

which implies the consistency of Eqs. (4.4) for i and for j $(i \neq j)$. It is obvious from (4.2) that

$$[X_i, X_j] = 0. \tag{4.11}$$

Because of (4.5), we have

$$[X_i, \tilde{A}^*(x_j;\tilde{u})] = -\left(\frac{\partial}{\partial\tilde{u}}\tilde{A}(x_j;\tilde{u})\right)^*.$$

By the same argument as that for (3.27) the right-hand side is rewritten as

$$[X_i, \tilde{A}^*(x_j;\tilde{u})] = -\frac{\theta_{11}'\theta_{11}(\tilde{u}+x_j)}{2\theta_{11}(\tilde{u})\theta_{11}(x_j)}\tilde{C}^*(x_j;\tilde{u}) - \frac{\theta_{11}'\theta_{11}(\tilde{u}-x_j)}{2\theta_{11}(\tilde{u})\theta_{11}(x_j)}\tilde{B}^*(x_j;\tilde{u}). \tag{4.12}$$

Exchanging i and j, we have

$$[X_j, \tilde{A}^*(x_i;\tilde{u})] = -\frac{\theta_{11}'\theta_{11}(\tilde{u}+x_i)}{2\theta_{11}(\tilde{u})\theta_{11}(x_i)}\tilde{C}^*(x_i;\tilde{u}) - \frac{\theta_{11}'\theta_{11}(\tilde{u}-x_i)}{2\theta_{11}(\tilde{u})\theta_{11}(x_i)}\tilde{B}^*(x_i;\tilde{u}). \tag{4.13}$$

The (1,1)-element of (4.6) means

$$\begin{aligned}[\tilde{A}^*(x_i;\tilde{u}), \tilde{A}^*(x_j;\tilde{u})] &= -\tilde{r}_{13}(x_i,x_j;\tilde{u})\tilde{C}^*(x_i;\tilde{u}) - \tilde{r}_{12}(x_i,x_j;\tilde{u})\tilde{C}^*(x_j;\tilde{u}) \\ &\quad + \tilde{r}_{31}(x_i,x_j;\tilde{u})\tilde{B}^*(x_i;\tilde{u}) + \tilde{r}_{21}(x_i,x_j;\tilde{u})\tilde{B}^*(x_j;\tilde{u}).\end{aligned} \tag{4.14}$$

Summing up (4.11), (4.12), (4.13) and (4.14), we have proved (4.10) because of (3.18).

The consistency of (4.4) for i and (4.3) for j is shown as follows. First assume $i \neq j$. Then the same computation as above gives

$$\begin{aligned}[X_i + \tilde{A}^*(x_i;\tilde{u}), \tilde{B}^*(x_j;\tilde{u})] &= -\left(\frac{\partial}{\partial\tilde{u}}\tilde{B}(x_j;\tilde{u})\right)^* + [\tilde{A}^*(x_i;\tilde{u}), \tilde{B}^*(x_j;\tilde{u})] \\ &= \left(\frac{\theta_{11}'(x_i-x_j)}{\theta_{11}(x_i-x_j)} - \frac{\theta_{11}'(x_i)}{\theta_{11}(x_i)}\right)\tilde{B}^*(x_j;\tilde{u}) - \frac{\theta_{11}'\theta(x_i-x_j-\tilde{u})}{\theta_{11}(x_i-x_j)\theta_{11}(\tilde{u})}\tilde{B}^*(x_i;\tilde{u}).\end{aligned} \tag{4.15}$$

Thus we have proved the compatibility of (4.4) for i and (4.3) for j. Here we used

$$\frac{\partial}{\partial\tilde{u}}\tilde{B}(x_j;\tilde{u}) = -\frac{\theta'_{11}\theta_{11}(x_j+\tilde{u})}{\theta_{11}(x_j)\theta_{11}(\tilde{u})}\tilde{A}(x_j;\tilde{u}) + \frac{\theta'_{11}(x_j)}{\theta_{11}(x_j)}\tilde{B}(x_j;\tilde{u}), \tag{4.16}$$

and the (1,2)-element of (4.6).

The case $i = j$ is almost the same, but there is another term coming from $[X_i, \tilde{B}^*(x_i;\tilde{u})]$:

$$\begin{aligned}&[X_i + \tilde{A}^*(x_i;\tilde{u}), \tilde{B}^*(x_i;\tilde{u})] = \\ &= \left.\frac{\partial}{\partial u}\right|_{u=x_i}\tilde{B}^*(u;\tilde{u}) - \left(\frac{\partial}{\partial\tilde{u}}\tilde{B}(x_i;\tilde{u})\right)^* + [\tilde{A}^*(x_i;\tilde{u}), \tilde{B}^*(x_i;\tilde{u})].\end{aligned} \tag{4.17}$$

By the same computation as (3.28), it follows from the (1,2)-element of (4.6) that

$$\begin{aligned}&[\tilde{A}^*(x_i;\tilde{u}), \tilde{B}^*(x_i;\tilde{u})] = \\ &= \frac{\theta'_{11}(\tilde{u})}{\theta_{11}(\tilde{u})}\tilde{B}^*(x_i;\tilde{u}) - \frac{\theta'_{11}\theta_{11}(x_i+\tilde{u})}{\theta_{11}(x_i)\theta_{11}(\tilde{u})}\tilde{A}^*(x_i;\tilde{u}) - \left.\frac{\partial}{\partial u}\right|_{u=x_i}\tilde{B}^*(u;\tilde{u}).\end{aligned} \tag{4.18}$$

Substituting (4.18) and (4.16) for $j = i$ into (4.17), we obtain

$$[X_i + \tilde{A}^*(x_i;\tilde{u}), \tilde{B}^*(x_i;\tilde{u})] = \left(\frac{\theta'_{11}(\tilde{u})}{\theta_{11}(\tilde{u})} - \frac{\theta'_{11}(x_i)}{\theta_{11}(x_i)}\right)\tilde{B}^*(x_i;\tilde{u}), \tag{4.19}$$

which proves the consistency of (4.4) for i and (4.3) for i. □

4.2. Separating operator. The separating integral operator K is defined by (4.1) with the kernel function $\Phi(x|y)$ satisfying Eqs. (4.3) and (4.4).

Proposition 4.2. (i) *For any function f of $(y_1,\ldots,y_N)$ in $V_1\otimes\cdots\otimes V_N$, we have*

$$K(\tilde{B}(x_i;\tilde{u})f) = 0, \tag{4.20}$$

$$K(-\tilde{A}(x_i;\tilde{u})f) = X_i f. \tag{4.21}$$

(ii) *The elliptic Gaudin Hamiltonian $\hat{\tau}(u)$ with the spectral parameter fixed to $u = x_i$ is transformed as follows.*

$$K(\hat{\tau}(x_i)f)(x) = X_j^2\,K(f)(x), \tag{4.22}$$

where X_i is defined by (4.2).

Proof. (i) is a direct consequence of (4.3) and (4.4) respectively.

(ii) By Definition (2.9),

$$\begin{aligned}&K(\hat{\tau}(x_i)f)(x) = \\ &\quad\frac{1}{2}\int\Phi(x|y)\left((2\tilde{A}(x_i;\tilde{u})^2 + \tilde{B}(x_i;\tilde{u})\tilde{C}(x_i;\tilde{u}) + \tilde{C}(x_i;\tilde{u})\tilde{B}(x_i;\tilde{u}))f(y)\right)dy \\ &= \int(\tilde{A}^*(x_i;\tilde{u}))^2\Phi(x|y)\,f(y)\,dy + \int\tilde{C}^*(x_i;\tilde{u})\tilde{B}^*(x_i;\tilde{u})\Phi(x|y)\,f(y)\,dy \\ &\quad+\frac{1}{2}\int[\tilde{B}^*(x_i;\tilde{u}), \tilde{C}^*(x_i;\tilde{u})]\Phi(x|y)\,f(y)\,dy.\end{aligned} \tag{4.23}$$

The first term in the right-hand side of (4.23) is rewritten by the following formula:

$$\begin{aligned}(\tilde{A}^*(x_i;\tilde{u}))^2\Phi(x|y) &= -\tilde{A}^*(x_i)X_i\Phi(x|y)\\ &= X_i^2\Phi(x|y)+[X_i,\tilde{A}^*(x_i)]\Phi(x|y),\end{aligned}\tag{4.24}$$

where we used (4.4). The last term of (4.24) is

$$[X_i,\tilde{A}^*(x_i;\tilde{u})] = \left.\frac{\partial}{\partial u}\right|_{u=x_i}\tilde{A}^*(u;\tilde{u}) - \left.\frac{\partial}{\partial \tilde{u}}\right|_{u=x_i}\tilde{A}^*(u;\tilde{u})\tag{4.25}$$

because $\tilde{u} = \sum z_n - \sum x_i$. Hence, similarly to the derivation of (4.12), we can prove that

$$\begin{aligned}[X_i,\tilde{A}^*(x_i;\tilde{u})] &= \left.\frac{\partial}{\partial u}\right|_{u=x_i}\tilde{A}^*(u;\tilde{u}) - \frac{\theta'_{11}\theta_{11}(\tilde{u}+x_i)}{2\theta_{11}(\tilde{u})\theta_{11}(x_i)}\tilde{C}^*(x_i;\tilde{u})\\ &- \frac{\theta'_{11}\theta_{11}(\tilde{u}-x_i)}{2\theta_{11}(\tilde{u})\theta_{11}(x_i)}\tilde{B}^*(x_i;\tilde{u}).\end{aligned}\tag{4.26}$$

The $(2,3)$-element of the commutation relation (4.6) gives

$$\begin{aligned}[\tilde{B}(u;\tilde{u}),\tilde{C}(u;\tilde{u})] &= 2\tilde{A}'(u;\tilde{u}) - \frac{\theta'_{11}\theta_{11}(u-\tilde{u})}{\theta_{11}(\tilde{u})\theta_{11}(u)}\tilde{B}(u;\tilde{u})\\ &- \frac{\theta'_{11}\theta_{11}(u+\tilde{u})}{\theta_{11}(\tilde{u})\theta_{11}(u)}\tilde{C}(u;\tilde{u})\end{aligned}\tag{4.27}$$

in the limit $v \to u$. Substituting (4.24), (4.26) and (4.27) into (4.23) and using (4.3), we obtain (4.22). □

Equation (4.20) is a quantum version of (3.10) and Eq. (4.21) together with the canonical commutation relation $[X_i, x_j] = \delta_{ij}$ means that operators $(x_1,\dots,x_N; X_1,\dots,X_N)$ are the quantization of the classical separated variables in Sect. 3.2.

The second statement of Proposition 4.2 provides a formal separation of variables for the quantum elliptic Gaudin model. Using the language of [8] and [9], the kernel $\Phi(x|y)$ provides a Radon–Penrose transformation of the corresponding D-modules (cf. [15]).

In principle, the quantum separation of variables should result in a one dimensional spectral problem for the separated equation (4.22) which is equivalent to the spectral problem for the original Hamiltonians (2.16). To achieve this goal one needs to specify an integration contour in (4.1) to study in detail the action of the integral operator K on the functional space V. Here we examine only the simplest case $N = 1$, leaving the general case for further study.

5. Quantum System: Case $N = 1$

In this section we examine the special case of $N = 1$. In this case, everything can be computed explicitly and we shall see that the separated equation is nothing but the classical Lamé equation and its generalization.

We adopt the realization of the representation ρ^ℓ of sl_2 on the space of elliptic functions reviewed in Appendix B. We could use the standard realization on the space of sections of a line bundle over $\mathbb{P}^1$, but the result is essentially the same up to coordinate transformation and gauge transformation. We omit the suffix n of z_n and S_n^a for brevity.

5.1. Separated variables. The quantum twisted B operator $\tilde{B}(u;\tilde{u}) = \tilde{\mathcal{B}}(u;\tilde{u})$ is defined as in the classical case (3.3) or (3.2). Substituting (2.1) we obtain

$$\begin{aligned}\tilde{B}(u;\tilde{u}) &= \frac{\theta'_{11}}{2\theta_{11}(u)\theta_{11}(\tilde{u})\theta_{11}(u-z)}\Big(\theta_{10}(u-z)\theta_{10}(u+\tilde{u})S^1 \\ &\quad - \theta_{00}(u-z)\theta_{00}(u+\tilde{u})iS^2 - \theta_{01}(u-z)\theta_{01}(u+\tilde{u})S^3\Big). \quad (5.1)\end{aligned}$$

The realization of the representation (B.2) gives the following expression:

$$\tilde{B}(u;\tilde{u}) = \tilde{B}^{(1)}(u;\tilde{u})\frac{d}{dy} + \tilde{B}^{(0)}(u;\tilde{u}), \quad (5.2)$$

where

$$\begin{aligned}\tilde{B}^{(1)}(u;\tilde{u}) &= \theta_{11}(u)^{-1}\theta_{11}(\tilde{u})^{-1}\theta_{11}(u-z)^{-1}\theta_{11}(2y)^{-1} \\ &\quad \times \theta_{10}\left(y+u-\frac{z}{2}+\frac{\tilde{u}}{2}\right)\theta_{10}\left(y-u+\frac{z}{2}-\frac{\tilde{u}}{2}\right) \\ &\quad \times \theta_{10}\left(-y-\frac{z}{2}-\frac{\tilde{u}}{2}\right)\theta_{10}\left(-y+\frac{z}{2}+\frac{\tilde{u}}{2}\right), \quad (5.3)\end{aligned}$$

$$\begin{aligned}\tilde{B}^{(0)}(u;\tilde{u}) &= \frac{2\ell\theta'_{11}}{2\theta_{11}(u)\theta_{11}(\tilde{u})\theta_{11}(u-z)\theta_{11}(y)^2}\Bigg(\theta_{11}(u+\tilde{u})\theta_{11}(u-z)\theta_{11}(y)^2 \\ &\quad + 2\theta_{10}\left(y+u-\frac{z}{2}+\frac{\tilde{u}}{2}\right)\theta_{10}\left(-y+u-\frac{z}{2}+\frac{\tilde{u}}{2}\right)\theta_{10}\left(-\frac{z}{2}-\frac{\tilde{u}}{2}\right)^2\Bigg). \quad (5.4)\end{aligned}$$

A special point in the case $N=1$ is that we can make use of the freedom of $\tilde{u}$ so that $\tilde{B}(u;\tilde{u})$ is a multiplication operator with the divisor of the form,

$$\operatorname{div}(\tilde{B}(u;\tilde{u})) = [x]+[z]-[z]-[0] = [x]-[0] \quad (\mathrm{mod}\ \mathbb{Z}+\tau\mathbb{Z}), \quad (5.5)$$

as in the classical case, (3.8), (3.9). In fact, if we put $\tilde{u} = -z \pm 2y + 1$, $\tilde{B}^{(1)}(u;\tilde{u}) = 0$ by virtue of (5.3), and then (5.4) implies

$$\tilde{B}(u;\tilde{u})|_{\tilde{u}=-z\pm 2y+1} = \frac{2\ell\theta'_{11}\theta_{11}(u-z\pm 2y)}{-2\theta_{11}(u)\theta_{11}(-z\pm 2y)}. \quad (5.6)$$

(We substitute the variable "from the left", namely we define

$$\tilde{B}(u;\tilde{u})|_{\tilde{u}=-z\pm 2y+1} = \tilde{B}^{(1)}(u;-z\pm 2y+1)\frac{d}{dy} + \tilde{B}^{(0)}(u;-z\pm 2y+1).$$

Hereafter we always follow this normal ordering convention.) Therefore we can take $x = z \mp 2y$ in (5.5). This is the one of the "separated variables" in this case.

In the classical model, Theorem 3.1, $-\tilde{A}(x;\tilde{u})$ is a dynamical variable canonically conjugate to x. This is also the case in the quantum model. The definition of $\tilde{A}$, (3.3), is rewritten in the form

$$\begin{aligned}\tilde{A}(u;\tilde{u}) &= \frac{\theta'_{11}}{2\theta_{11}(u)\theta_{11}(\tilde{u})\theta_{11}(u-z)}\Big(\theta_{10}^{-1}\theta_{10}(u-z)\theta_{10}(u)\theta_{10}(\tilde{u})S^1 \\ &\quad - \theta_{00}^{-1}\theta_{00}(u-z)\theta_{00}(u)\theta_{00}(\tilde{u})iS^2 - \theta_{01}^{-1}\theta_{01}(u-z)\theta_{01}(u)\theta_{01}(\tilde{u})S^3\Big) \quad (5.7)\end{aligned}$$

by (2.1). Substituting $\tilde{u} = -z \pm 2y + 1$ and $u = x = z \mp 2y$ (from the left), we obtain

$$\begin{aligned}
&\tilde{A}(u;\tilde{u})|_{u=z\mp 2y,\tilde{u}=-z\pm 2y+1} = \\
&\qquad = \pm\frac{1}{2}\left(\frac{d}{dy} + 2\ell\left(\frac{\theta_{11}'\theta_{10}(2y)}{\theta_{10}\theta_{11}(2y)} + \frac{\theta_{11}'\theta_{00}(2y)}{\theta_{00}\theta_{11}(2y)} + \frac{\theta_{11}'\theta_{01}(2y)}{\theta_{01}\theta_{11}(2y)}\right)\right) \\
&\qquad = \pm\frac{1}{2}\left(\frac{d}{dy} - \ell\frac{\wp''(y)}{\wp'(y)}\right).
\end{aligned} \tag{5.8}$$

The last equality can be proved by comparing the poles of both sides. Therefore $x = z \mp 2y$ and $X := -\tilde{A}(u;\tilde{u})|_{u=z\mp 2y,\tilde{u}=-z\pm 2y+1}$ are the canonical conjugate variables satisfying,

$$[X,\ x] = 1. \tag{5.9}$$

We did not make use of the formulation in the previous sections explicitly. In fact, thanks to the special choice of $\tilde{u}$, Φ in (4.1) is a δ-function type kernel, which reduces the integral operator K to a coordinate transformation operator from y to x.

5.2. Solving the spectral problem. For the case $N = 1$, the generating function of the quantum integrals of motion $\hat{\tau}(u)$ (u is the spectral parameter)

$$\hat{\tau}(u) = \frac{1}{2}\sum_{a=1}^{3} w_a(u)^2(\rho^{(\ell)}(S^a))^2 \tag{5.10}$$

is explicitly written down. Here we shift the spectral parameter in the original definition (2.9) as $u \mapsto u_z$ and set $z = 0$ for the sake of simplicity. Using (B.2) or (B.6) and various identities of elliptic functions in [13], we can expand the right hand side of (5.10):

$$\hat{\tau}\left(y, \frac{d}{dy}, \frac{d^2}{dy^2}; u\right) = \frac{1}{4}\left(\frac{d^2}{dy^2} - 2\ell\frac{\wp''(y)}{\wp'(y)}\frac{d}{dy} + 4\ell(2\ell-1)\wp(y) + 4\ell(\ell+1)\wp(u)\right) \tag{5.11}$$

or

$$\begin{aligned}
\hat{\tau}\left(\eta, \frac{d}{d\eta}, \frac{d^2}{d\eta^2}; \lambda\right) \;=\;& (\eta-e_1)(\eta-e_2)(\eta-e_3) \\
&\times\left(\frac{d^2}{d\eta^2} + \frac{1-2\ell}{2}\left(\frac{1}{\eta-e_1} + \frac{1}{\eta-e_2} + \frac{1}{\eta-e_3}\right)\frac{d}{d\eta} + \right. \\
&\left. + \frac{\ell(2\ell-1)\eta + \ell(\ell+1)\lambda}{(\eta-e_1)(\eta-e_2)(\eta-e_3)}\right),
\end{aligned} \tag{5.12}$$

where $\lambda = \wp(u)$.

As is expected from Proposition 4.2 and the result for the rational Gaudin model in [6], operator $\hat{\tau}(u)$ is factorized as follows when the spectral parameter u is fixed to a separated variable $x_1 = 2y$. (We may also take $x_1 = -2y$.):

$$\hat{\tau}\left(y, \frac{d}{dy}, \frac{d^2}{dy^2}; u\right)\bigg|_{u=2y} = \left(-\tilde{A}(u;\tilde{u})|_{u=2y,\tilde{u}=2y+1}\right)^2 = X^2, \tag{5.13}$$

which immediately follows from (5.7). (The operator X is defined before (5.9).) This is consistent with the general result (4.22).

Equations (5.11) or (5.12) show that the spectral problem of the elliptic Gaudin model with $N = 1$ is an ordinary differential equation of second order on the elliptic curve $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$:

$$\hat{\tau}\left(y, \frac{d}{dy}, \frac{d^2}{dy^2}; u\right)\psi(y) = t(u)\psi(y), \tag{5.14}$$

or on the projective line $\mathbb{P}^1(\mathbb{C})$:

$$\hat{\tau}\left(\eta, \frac{d}{d\eta}, \frac{d^2}{d\eta^2}; \lambda\right)\psi(\eta) = t(\lambda)\psi(\eta). \tag{5.15}$$

Here $t(u)$, $t(\lambda)$ are eigenvalues of $\hat{\tau}$, $\psi \in V^{(\ell)}$ is an eigenvalue corresponding to this eigenvalue. Since operators $\hat{\tau}(u)$ and U_α commute with each other by virtue of (B.12, B.13) and (5.10), we can decompose each eigenspace of $\hat{\tau}(u)$ into those of U_α.

Equation (5.14) has regular singularities: $u = 0 \pmod{\Gamma}$ with exponents -4ℓ, $-2\ell + 1$, and $u = \omega_\alpha$ ($\alpha = 1, 2, 3$, $\omega_1 = 1$, $\omega_2 = \tau$, $\omega_3 = 1 + \tau$) with exponents 0, $2\ell + 1$. Equation (5.15) has regular singularities: $\eta = e_\alpha$ with exponents 0, $(2\ell + 1)/2$, and $\eta = \infty$ with $\frac{1}{2} - \ell$, -2ℓ.

If ℓ is an integer, these equations are ordinary Lamé equations, while for $\ell \in \frac{1}{2} + \mathbb{Z}$ they are generalized Lamé equations studied by Brioschi, Halphen and Crawford. Following the classical theory of Lamé functions (see [13, Chap. XXIII]), we can solve the spectral problem (5.14), (5.15) in $V^{(\ell)}$ as follows.

5.2.1. Case $\ell \in \mathbb{Z}$. We want a solution $\psi(\eta)$ of (5.15) such that $\psi(\eta) \in V^{(\ell)}$. Let us assume that $\psi(\eta)$ is expanded around the singular point e_α as

$$\psi(\eta) = \sum_{r=0}^{\infty} a_r^\alpha (\eta - e_\alpha)^{2\ell - r}, \tag{5.16}$$

a_0 being 1. The condition $\psi(\eta) \in V^{(\ell)}$ means that $a_r^\alpha = 0$ for $r > 2\ell$. Substituting (5.16) into (5.15), we obtain the following recursion relation:

$$\begin{aligned} r(\ell + \tfrac{1}{2} - r)a_r^\alpha \;&=\; \Big(\big(\ell(2\ell - 1) - 3(r - 1)(2\ell - r + 1)\big)e_\alpha + E\Big)a_{r-1}^\alpha \\ &\quad + (2\ell - r + 2)\left(\ell - r + \frac{3}{2}\right)(e_\alpha - e_\beta)(e_\alpha - e_\gamma)a_{r-2}^\alpha \end{aligned} \tag{5.17}$$

for $r > 0$ where $E = \ell(\ell + 1)\lambda - t(\lambda)$. (Undefined coefficients a_r^α for $r < 0$ are 0.) Hence, as a function of E, $a_r^\alpha = a_r^\alpha(E)$ is a polynomial of degree r of the form

$$a_r^\alpha(E) = A_r E^r + O(E^{r-1}), \qquad A_r = \left(r! \prod_{j=1}^{r}\left(\ell - r - \frac{1}{2} + j\right)\right)^{-1}. \tag{5.18}$$

Let us denote the roots of $a_{2\ell+1}^\alpha(E) = 0$ by E_i^α ($i = 1, \ldots, 2\ell + 1$). The recursion relation (5.17) implies $a_r^\alpha(E_i^\alpha) = 0$ for $r \geq 2\ell + 1$. Hence we obtain a polynomial solution $\psi(\eta) = \psi(\eta; E_i^\alpha)$ of (5.15) of the form (5.16) for each $i = 1, \dots, 2\ell + 1$, provided that

$$t(\lambda) = \ell(\ell+1)\lambda - E_i^\alpha. \tag{5.19}$$

Conversely, if $\psi(\eta) \in V^{(\ell)}$ is a solution of the spectral problem (5.15), then there exists certain i for each $\alpha = 1, 2, 3$ such that $\psi(\eta) = \psi(\eta; E_i^\alpha)$. This is proved by expanding the polynomial $\psi(\eta)$ as in (5.16) and tracing back the above argument.

Proposition 5.1. *Assume that $\omega_2 = \tau$ is pure imaginary and that parameters z_n are all real numbers. Then all E_i^α are real and the spectral problem (5.15) is non-degenerate. Namely $E_i^\alpha \neq E_j^\alpha$ for distinct i, j and the solutions $\psi(\eta; E_i^\alpha)$ span the space $V^{(\ell)}$. In particular E_i^α ($i = 1, \dots, 2\ell+1$) for $\alpha = 1, 2, 3$ coincide up to order, and $a^1_{2\ell+1}(E) = a^2_{2\ell+1}(E) = a^3_{2\ell+1}(E)$. Hence we can omit the index α for E_i^α and $a^\alpha_{2\ell+1}(E)$.*

Vector $\psi(\eta; E_i)$ is an eigenvector of U_α with eigenvalue $(-1)^\ell$ if $a^\alpha_\ell(E_i) \neq 0$ and $(-1)^{\ell+1}$ if $a^\alpha_\ell(E_i) = 0$.

Proof. Under the assumption $\tau \in i\mathbb{R}$, operator $\hat{\tau}(u)$ ($u \in \mathbb{R}$) is an hermitian operator because of (B.11), and hence it is obvious that E_i^α are real and that $\psi(\eta; E_i^\alpha)$ span $V^{(\ell)}$.

In order to show non-degeneracy of the spectral problem (5.15) we have only to prove that E_i^2 are distinct with each other. Define

$$\tilde{a}_r^2(E) := \begin{cases} a_r^2(E), & r < \ell+1, \\ (-1)^{r-l} a_r^2(E), & \ell+1 \le r \le 2\ell+1. \end{cases}$$

Then the leading coefficient of $\tilde{a}_r^2$ is

$$\tilde{a}_r^2(E) = \tilde{A}_r E^r + O(E^{r-1}), \qquad \tilde{A}_r = |A_r|. \tag{5.20}$$

The recursion relation (5.17) is rewritten as

$$c_r \tilde{a}_r^2(E) = q_r \tilde{a}^2_{r-1}(E) - k_{r-2}\tilde{a}^2_{r-2}(E), \tag{5.21}$$

where

$$\begin{aligned} c_r &= r|\ell + \tfrac{1}{2} - r|, \\ q_r &= \big(\ell(2\ell-1) - 3(r-1)(2\ell-r+1)\big)e_\alpha + E, \\ k_r &= \left|\ell - r + \frac{3}{2}\right| (2\ell - r + 2)(e_1 - e_2)(e_2 - e_3). \end{aligned} \tag{5.22}$$

Since $e_1 > e_2 > e_3$ under the assumption of the proposition, we have $c_r > 0$ and $k_r > 0$. This fact together with $\tilde{A}_r > 0$ (see (5.20)) implies that all the roots of $\tilde{a}_r^2(E)$ are real and distinct by Sturm's theorem (see, e.g., Chap. IX, §§4–5, [16]). This proves the first statement of the proposition.

The operators U_α and $\hat{\tau}$ commute and each eigenspace of $\hat{\tau}$ is one-dimensional. Hence $\psi(\eta; E_i)$ is an eigenvector of U_α. Recall that U_α has eigenvalues $(-1)^\ell$ with multiplicity $\ell+1$ and $(-1)^{\ell+1}$ with multiplicity ℓ. (See §B.2.) If $a^\alpha_\ell(E_i) \neq 0$, then

$$U_\alpha \psi(\eta; E_i) = (-1)^\ell \psi(\eta; E_i)$$

because of (B.15). Hence there are at most $\ell+1$ of E_i's such that $a^\alpha_\ell(E_i) \neq 0$. In other words, at least ℓ of E_i's satisfy $a^\alpha_\ell(E_i) = 0$. Since $a^\alpha_\ell(E)$ is a polynomial of degree ℓ, this proves the second statement of the proposition. $\square$

5.2.2. Case $\ell \in \frac{1}{2}+\mathbb{Z}$. As in the case $\ell \in \mathbb{Z}$, we consider an expansion (5.16) of a solution $\psi(\eta)$ of the spectral problem (5.15), but this time we consider the series which terminate at $r = \ell - \frac{1}{2}$:

$$\psi(\eta) = \sum_{r=0}^{\ell-1/2} a_r^\alpha (\eta - e_\alpha)^{2\ell - r}. \tag{5.23}$$

They are parametrized by zeros of the polynomial $a^\alpha_{\ell+\frac{1}{2}}(E)$, $\{E_i^\alpha\}_{i=1,\dots,\ell+\frac{1}{2}}$ as in the previous case: $\psi(\eta) = \psi(\eta; E_i^\alpha)$.

Another set of solutions are obtained from this set by applying the operator U_α:

$$U_\alpha \psi(\eta; E_i^\alpha) = \sum_{r=0}^{\ell-\frac{1}{2}} a^{\alpha\prime}{}_r(E_i^\alpha)(\eta - e_\alpha)^r, \tag{5.24}$$

since U_α and $\hat{\tau}(u)$ commute.

The following proposition is proved in the same manner as Proposition 5.1.

Proposition 5.2. *Assume that $\omega_2 = \tau$ is pure imaginary and that parameters z_n are all real numbers.*

Then all E_i^α are real and $E_i^\alpha \neq E_j^\alpha$ for distinct i, j.

The solutions $\psi(\eta; E_i^\alpha)$ and $U_\alpha \psi(\eta; E_i^\alpha)$ span the space $V^{(\ell)}$. In particular E_i^α ($i = 1, \dots, \ell+\frac{1}{2}$) for $\alpha = 1, 2, 3$ coincide up to order, and $a^1_{\ell+\frac{1}{2}}(E) = a^2_{\ell+\frac{1}{2}}(E) = a^3_{\ell+\frac{1}{2}}(E)$. Hence we can omit the index α for E_i^α and $a^\alpha_{\ell+\frac{1}{2}}(E)$.

Vectors $\psi(\eta; E_i) \pm U_\alpha \psi(\eta; E_i)$ are eigenvectors of U_α with eigenvalues $\mp i$.

This proposition means that each eigenvalue E_i degenerates with multiplicity two. It was Crawford [17] who first found the relation of these two solutions (one is obtained from the other by operating U_2) by the explicit expansions of type (5.23), (5.24). See also p.578 of [13].

A. Notations

We use the notation for the theta functions with characteristics as follows (see [14]): for $a, b = 0, 1$,

$$\theta_{ab}(u;\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i (n+a/2)^2 \tau + 2\pi i (n+a/2)(u+b/2)}. \tag{A.1}$$

Unless otherwise specified, $\theta_{ab}(u) = \theta_{ab}(u;\tau)$. We also use abbreviations

$$\theta_{ab} = \theta_{ab}(0), \qquad \theta'_{ab} = \left.\frac{d}{du}\right|_{u=0} \theta_{ab}(u). \tag{A.2}$$

Quasi-periodicity properties of theta functions:

$$\theta_{ab}(u) = (-1)^a \theta_{ab}(u+1) = e^{\pi i \tau + 2\pi i u} \theta_{ab}(u+\tau). \tag{A.3}$$

Parity of thetas:

$$\theta_{00}(-u) = \theta_{00}(u), \quad \theta_{01}(-u) = \theta_{01}(u), \quad \theta_{10}(-u) = \theta_{10}(u), \quad \theta_{11}(-u) = -\theta_{11}(u).$$

A.1. Weierstrass functions. Below we fix $\omega_1 = 1$ and $\omega_2 = \tau$,

$$\sigma(u) = u \prod_{m,n\neq 0} \left(1 - \frac{u}{\omega_{mn}}\right) \exp\left[\frac{u}{\omega_{mn}} + \frac{1}{2}\left(\frac{u}{\omega_{mn}}\right)^2\right], \tag{A.4}$$

where $\omega_{mn} = m\omega_1 + n\omega_2$,

$$\zeta(u) = \frac{\sigma'(u)}{\sigma(u)}, \qquad \wp(u) = -\zeta'(u),$$

$$\begin{aligned}
\sigma(u + \omega_l) &= -\sigma(u) e^{\eta_l(2u+\omega_l)}, \\
\zeta(u + \omega_l) &= \zeta(u) + 2\eta_l, \\
\wp(u + \omega_l) &= \wp(u),
\end{aligned}$$

where $\eta_l = \zeta\ (\omega_l/2)$, which satisfy

$$\eta_1\omega_2 - \eta_2\omega_1 = \pi i.$$

Sigma function is expressed by theta functions as follows:

$$\sigma(u) = \omega_1 e^{\eta_1 u^2/\omega_1} \frac{\theta_{11}(u/\omega_1)}{\theta'_{11}},$$

$$\sigma(-u) = -\sigma(u), \qquad \zeta(-u) = -\zeta(u), \qquad \wp(-z) = \wp(u),$$

$$u \sim 0: \qquad \sigma(u) = u + O(u^5), \qquad \zeta(u) = u^{-1} + O(u^3), \qquad \wp(u) = u^{-2} + O(u^2).$$

Other sigma functions are defined as follows:

$$\begin{aligned}
\sigma_{00}(u) &= e^{-(\eta_1+\eta_2)u} \frac{\sigma\left(u + \frac{\omega_1+\omega_2}{2}\right)}{\sigma\left(\frac{\omega_1+\omega_2}{2}\right)} = e^{\frac{\eta_1}{\omega_1}u^2} \frac{\theta_{00}(u/\omega_1)}{\theta_{00}(0)}, \\
\sigma_{10}(u) &= e^{-\eta_1 u} \frac{\sigma\left(u + \frac{\omega_1}{2}\right)}{\sigma\left(\frac{\omega_1}{2}\right)} = e^{\frac{\eta_1}{\omega_1}u^2} \frac{\theta_{10}(u/\omega_1)}{\theta_{10}(0)}, \\
\sigma_{01}(u) &= e^{-\eta_2 u} \frac{\sigma\left(u + \frac{\omega_2}{2}\right)}{\sigma\left(\frac{\omega_2}{2}\right)} = e^{\frac{\eta_1}{\omega_1}u^2} \frac{\theta_{01}(u/\omega_1)}{\theta_{01}(0)},
\end{aligned}$$

which satisfy

$$\sigma_{g_1g_2}(u + \omega_l) = (-1)^{g_l} e^{\eta_l(2u+\omega_l)} \sigma_{g_1g_2}(u),$$

$$\sigma_{g_1g_2}(-u) = \sigma_{g_1g_2}(u), \qquad \sigma_{g_1g_2}(0) = 1.$$

Defining $e_1 = \wp(\omega_1/2)$, $e_2 = \wp((\omega_1 + \omega_2)/2)$, $e_3 = \wp(\omega_2/2)$, we have

$$\frac{\sigma_{10}^2(u)}{\sigma^2(u)} + e_1 = \frac{\sigma_{00}^2(u)}{\sigma^2(u)} + e_2 = \frac{\sigma_{01}^2(u)}{\sigma^2(u)} + e_3 = \wp(u),$$

$$e_1 + e_2 + e_3 = 0,$$

$$e_1 - e_2 = \left(\frac{\pi}{\omega_1}\right)^2 \theta_{01}(0)^4, \quad e_1 - e_3 = \left(\frac{\pi}{\omega_1}\right)^2 \theta_{00}(0)^4, \quad e_2 - e_3 = \left(\frac{\pi}{\omega_1}\right)^2 \theta_{10}(0)^4.$$

We also use normalized Weierstraß functions:

$$\zeta_{11}(u) = \frac{d}{du}\theta_{11}(u), \qquad \wp_{11}(u) = -\frac{d}{du}\zeta_{11}(u). \tag{A.5}$$

B. Realization of Spin ℓ Representations on an Elliptic Curve

We recall here the following realization of the spin ℓ representation of the Lie algebra $sl_2(\mathbb{C})$. Let e, f, h be the Chevalley generators and define $S^1 = e+f$, $S^2 = -ie+if$ and $S^3 = h$. They satisfy the relation $[S^a, S^b] = 2iS^c$ for any cyclic permutation (a, b, c) of $(1, 2, 3)$ and represented by the Pauli matrices σ^a.

B.1. Spin ℓ representations. The representation space $V^{(\ell)}$ is realized by

$$\begin{aligned} V^{(\ell)} &= \bigoplus_{k=0}^{2\ell} \mathbb{C}\wp(y)^k \\ &= \{\text{ even elliptic function } f(y) \mid \operatorname{div}(f) \geq -4\ell(\mathbb{Z} + \tau\mathbb{Z})\}. \end{aligned} \tag{B.1}$$

The generators S^a act on this space as differential operators of first order:

$$\begin{aligned} \rho^{(\ell)}(S^1) &= \frac{\theta_{10}\theta_{10}(2y)}{\theta'_{11}\theta_{11}(2y)}\frac{d}{dy} + 2\ell\frac{\theta_{10}(y)^2}{\theta_{11}(y)^2}, \\ \frac{1}{i}\rho^{(\ell)}(S^2) &= \frac{\theta_{00}\theta_{00}(2y)}{\theta'_{11}\theta_{11}(2y)}\frac{d}{dy} + 2\ell\frac{\theta_{00}(y)^2}{\theta_{11}(y)^2}, \\ \rho^{(\ell)}(S^3) &= \frac{\theta_{01}\theta_{01}(2y)}{\theta'_{11}\theta_{11}(2y)}\frac{d}{dy} + 2\ell\frac{\theta_{01}(y)^2}{\theta_{11}(y)^2}, \end{aligned} \tag{B.2}$$

or in terms of usual Weierstraß functions,

$$\begin{aligned} \rho^{(\ell)}(S^1) &= a_1\left(\frac{\sigma_{10}(2y)}{\sigma(2y)}\frac{d}{dy} + 2\ell(\wp(y) - e_1)\right), \\ \rho^{(\ell)}(S^2) &= a_2\left(\frac{\sigma_{00}(2y)}{\sigma(2y)}\frac{d}{dy} + 2\ell(\wp(y) - e_2)\right), \\ \rho^{(\ell)}(S^3) &= a_3\left(\frac{\sigma_{01}(2y)}{\sigma(2y)}\frac{d}{dy} + 2\ell(\wp(y) - e_3)\right), \end{aligned} \tag{B.3}$$

where $e_a = \wp(\omega_{\bar{a}}/2)$ ($\bar{a} = 1, 3, 2$, $\omega_1 = 1$, $\omega_2 = \tau$, $\omega_3 = 1 + \tau$) for $a = 1, 2, 3$ respectively and

$$a_1 = \frac{1}{\sqrt{e_1 - e_2}\sqrt{e_1 - e_3}}, \quad a_2 = \frac{i}{\sqrt{e_1 - e_2}\sqrt{e_2 - e_3}}, \quad a_3 = \frac{1}{\sqrt{e_2 - e_3}\sqrt{e_1 - e_3}}. \tag{B.4}$$

This realization is equivalent to the realization on the space of polynomials of degree $\leq 2\ell$ (or, sections of a line bundle on $\mathbb{P}^1(\mathbb{C})$),

$$e = x^2\frac{d}{dx} - 2\ell x, \qquad f = -\frac{d}{dx}, \qquad h = 2x\frac{d}{dx} - 2\ell x,$$

via a coordinate transformation, $x = -\theta_{01}(y; \tau/2)/\theta_{00}(y; \tau/2)$, and a gauge transformation:

$$\{\text{polynomials in } x\} \ni \varphi(x) \mapsto \left(\frac{\theta_{00}(y; \tau/2)}{\theta_{11}(y; \tau)^2}\right)^n \varphi(x(y)) \in V^{(\ell)}.$$

Note that this is also obtained by a gauge transformation from a quasi-classical limit of the representation of the Sklyanin algebra on theta functions [18].

The following expression is obtained from the coordinate transformation $\eta = \wp(y)$:

$$V^{(\ell)} = \bigoplus_{k=0}^{2\ell} \mathbb{C}\eta^k, \tag{B.5}$$

and S^α acts on $V^{(\ell)}$ as

$$\rho^{(\ell)}(S^\alpha) = a_\alpha \left(((e_\alpha - e_\beta)(e_\alpha - e_\gamma) - (\eta - e_\alpha)^2)\frac{d}{d\eta} + 2\ell(\eta - e_\alpha) \right). \tag{B.6}$$

Let us assume that τ is a pure imaginary number. Then, as is well known (see, e.g., [13]), e_a are real numbers and $e_1 > e_2 > e_3$. This implies that a_1 and a_3 are real, while a_2 is purely imaginary.

We introduce the following hermitian form in this representation space: for elliptic functions $f(y)$, $g(y)$ belonging to $V^{(\ell)}$ defined by (B.1), we define

$$\langle f, g\rangle := \int_C \overline{f(\bar{y}_2)}\, g(y_1)\, \mu(y_1, y_2), \tag{B.7}$$

where the 2-cycle C is defined by

$$C := \{(y_1, y_2) \in (\mathbb{C}/\Gamma)^2,\ y_2 = \bar{y}_1\},$$

and the 2-form $\mu(y_1, y_2)$ is defined by

$$\begin{aligned} \mu(y_1, y_2) &:= (e_1 - e_2)^{2(\ell+1)}(e_2 - e_3)^{2(\ell+1)} \\ &\quad\times \frac{\sigma(2y_2)\sigma(y_2)^{4\ell}\sigma(2y_1)\sigma(y_1)^{4\ell}}{\sigma_{00}(y_2 - y_1)^{2(\ell+1)}\sigma_{00}(y_2 + y_1)^{2(\ell+1)}} \frac{dy_2 \wedge dy_1}{4i} \\ &= \left(1 + \frac{(\wp(y_2) - e_2)(\wp(y_1) - e_2)}{(e_1 - e_2)(e_2 - e_3)}\right)^{-2(\ell+1)} \frac{\wp'(y_2)\wp'(y_1)dy_2 \wedge dy_1}{4i}. \end{aligned} \tag{B.8}$$

This is nothing but a twisted version of the inner product introduced in [18]. If we take the description of $V^{(\ell)}$ of the form (B.5), this hermitian form is expressed as follows:

$$\langle f, g\rangle := \int_{\mathbb{C}} \overline{f(\bar{\eta})}\, g(\eta)\, \mu(\eta, \bar{\eta}), \tag{B.9}$$

where the 2-form $\mu(\eta, \bar{\eta})$ is defined by

$$\mu(\eta, \bar{\eta}) := \left(1 + \frac{(\bar{\eta} - e_2)(\eta - e_2)}{(e_1 - e_2)(e_2 - e_3)}\right)^{-2(\ell+1)} \frac{d\bar{\eta} \wedge d\eta}{2i}.$$

An orthogonal basis with respect to this inner product is given by $\{(\eta - e_2)^j\}_{j=0,\dots,2\ell}$:

$$\langle(\eta - e_2)^j, (\eta - e_2)^k\rangle = 2\pi \frac{(2j)!!(4\ell - 2j)!!}{(4\ell + 2)!!}(e_1 - e_2)^{j+1}(e_2 - e_3)^{j+1}\delta_{jk}. \tag{B.10}$$

The generators S^a of the Lie algebra sl_2 act on the space $V^{(\ell)}$ as self-adjoint operators:

$$\langle\rho^{(\ell)}(S^a)f, g\rangle = \langle f, \rho^{(\ell)}(S^a)g\rangle. \tag{B.11}$$

This was first proved in [18], but we can check it directly by using formula (B.10).

Hence, if u and z_n are real numbers, the operator $\hat{\tau}(u)$ defined by (2.9) and the integrals of motion H_n defined by (2.16) are hermitian operators on the Hilbert space V with respect to $\langle\cdot, \cdot\rangle$.

B.2. Involutions. There are involutive automorphisms of the Lie algebra sl_2 defined by

$$X_a(S^b) = (-1)^{1-\delta_{ab}} S^b. \tag{B.12}$$

These automorphisms are induced on the spin ℓ representations as

$$X_a(S^b) = U_a^{-1} S^b U_a, \tag{B.13}$$

where operators $U_a : V^{(\ell)} \to V^{(\ell)}$ are defined by

$$\begin{aligned}
(U_1 f)(y) &= e^{\pi i \ell} \left(\frac{\wp(y) - e_1}{\sqrt{e_1 - e_2}\sqrt{e_1 - e_3}} \right)^{2\ell} f\left(y + \frac{\omega_1}{2} \right), \\
(U_2 f)(y) &= e^{2\pi i \ell} \left(\frac{\wp(y) - e_2}{\sqrt{e_1 - e_2}\sqrt{e_2 - e_3}} \right)^{2\ell} f\left(y + \frac{\omega_1 + \omega_2}{2} \right), \\
(U_3 f)(y) &= e^{-\pi i \ell} \left(\frac{\wp(y) - e_3}{\sqrt{e_1 - e_3}\sqrt{e_2 - e_3}} \right)^{2\ell} f\left(y + \frac{\omega_2}{2} \right),
\end{aligned} \tag{B.14}$$

for a elliptic function $f(y) \in V^{(\ell)}$ (cf. [18]). They satisfy commutation relations

$$U_\alpha^2 = (-1)^{2\ell}, \qquad U_\alpha U_\beta = (-1)^{2\ell} U_\beta U_\alpha = U_\gamma$$

for any cyclic permutation (α, β, γ) of $(1, 2, 3)$. The action of these operators on the bases $\{(\eta - e_\alpha)^j\}_{j=0,\ldots,2\ell}$ is:

$$\begin{aligned}
U_1(\eta - e_1)^j &= e^{\pi i \ell}(e_1 - e_2)^{j-\ell}(e_1 - e_3)^{j-\ell}(\eta - e_1)^{2\ell - j}, \\
U_2(\eta - e_2)^j &= e^{\pi i (2\ell - j)}(e_1 - e_2)^{j-\ell}(e_2 - e_3)^{j-\ell}(\eta - e_2)^{2\ell - j}, \\
U_3(\eta - e_3)^j &= e^{-\pi i \ell}(e_1 - e_3)^{j-\ell}(e_2 - e_3)^{j-\ell}(\eta - e_3)^{2\ell - j}.
\end{aligned} \tag{B.15}$$

Hence eigenvalues of U_a are $(-1)^\ell$ with multiplicity $\ell + 1$ and $(-1)^{\ell+1}$ with multiplicity ℓ if ℓ is an integer, and $\pm i$ both with multiplicity $\ell + \frac{1}{2}$ if ℓ is a half of an odd integer.

When $\omega_1 = 1$ and ω_2 is a pure imaginary number, these operators are unitary with respect to the hermitian form (B.7).

Acknowledgements. One of the authors (E.S.) is grateful to the Department of Applied Mathematics, University of Leeds, UK, where the main part of the paper was written, for hospitality, and acknowledges the support of EPSRC. The other one (T.T.) is grateful to the Department of Mathematics, University of California at Berkeley, USA, where some part of the work was done, for hospitality. He also acknowledges the support of Postdoctoral Fellowship for Research abroad of JSPS. Special thanks are given to Benjamin Enriquez and Vladimir Rubtsov who gave us their article [9] before publication and explained details.

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Communicated by G. Felder