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Exact Absorption Probabilities for the D3-Brane

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Abstract: We consider a minimal scalar in the presence of a three-brane in ten dimensions. The linearized equation of motion, which is just the wave equation in the three-brane metric, can be solved in terms of associated Mathieu functions. An exact expression for the reflection and absorption probabilities can be obtained in terms of the characteristic exponent of Mathieu's equation. We describe an algorithm for obtaining the low-energy behavior as a series expansion, and discuss the implications for the world-volume theory of D3-branes.

1. Introduction

One of the intriguing aspects of Ramond–Ramond solitons in string theory is the existence of two alternative descriptions, one in terms of supergravity [1] and the other in terms of Dirichlet branes (D-branes) [2]. The description in terms of D-branes is essentially perturbative in nature: each boundary picks up a factor of gN, which is the square of the open string coupling times a Chan–Paton factor. As realized in [3], the low-energy dynamics of N coincident D-branes is dictated by maximally supersymmetric gauge theory with gauge group $U(N)$, and gN is recognized as the 't Hooft parameter.

That the gauge theory and supergravity descriptions should be related was implicit in much early work on absorption and Hawking emission (see for example [4, 5]). A precise formulation of the duality between the two descriptions was conjectured recently in [6] by taking the so-called "decoupling limit". The simplest example comes from considering D3-branes in the type IIB theory. In the decoupling limit one obtains a duality between $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in four dimensions and string theory on the near horizon $AdS_5 \times S^5$ background [6]. The AdS_5 and the S^5 have the same radius of curvature R, where $R^4 = 4\pi g N \alpha'^2$.

It is difficult to find non-trivial checks of the duality because it relates two things that are rather poorly understood away from certain limits. On the AdS side, it is widely felt that the supergravity description should be capable of being elevated to a full closed string theory description, similar to non-linear sigma models; but it is not understood how to include Ramond–Ramond backgrounds in a non-linear sigma model.[1] We must for the present content ourselves with the supergravity limit. The validity of this limit relies on having a large number N of coincident branes, with a small closed string coupling g, but large gN. Large gN is exactly where the gauge theory is difficult to deal with: after 't Hooft scaling, the Feynman rules associate a factor gN with each vertex, so for generic amplitudes one must consider large graphs.

How then can we study the relation between the two dual descriptions concretely? Aside from the calculation of entropy [9], one of the simplest quantities that can be computed on both sides of the correspondence is the absorption cross-section of scalar fields incident on the branes. Suppression of stringy correction on the supergravity side relies on having $\omega\sqrt{\alpha'} \ll 1$ and $\sqrt{\alpha'}/R \ll 1$; but ωR can be arbitrary, suggesting the existence of a double scaling limit [5, 10]. Indeed, the wave equation for the fields propagating in the supergravity background of branes depend on gN only in the combination ωR. Remarkably, the leading order behavior in small ωR of the semi-classical cross-section is reproduced by a tree level gauge theory calculation (leading order in gN) [5, 10]. The relevant gauge theory amplitude apparently suffers no radiative corrections. An argument for why this is so was advanced in [11] for graviton absorption, and other examples have emerged in [12, 13].

A natural question which arises at this point is whether this pattern persists to higher order in ωR [14]. In order to address this question, one must examine higher order corrections in both D-brane and supergravity computations. On the supergravity side, a first step in this direction was taken in [15] where terms subleading by order $(\omega R)^4$ were examined. The coefficient of the $(\omega R)^4$ correction turns out to have a piece which depends logarithmically in ωR:

$$\sigma = \frac{\kappa^2 N^2 \omega^3}{32\pi}\left[1 + c_1'(\omega R)^4 \log \omega R + c_1(\omega R)^4 + \mathcal{O}((\omega R)^8)\right] \tag{1}$$

and the numerical value of c_1' was found to be $-1/6$.

The goal of this paper is to describe an algorithm for computing the absorption cross-section as a power series expansion in ωR to all orders. The absorption cross-section is determined by comparing the flux of incident partial waves at the asymptotic region and the near horizon region. We are therefore interested in finding the solution to the wave equation of scalar fields in the background of the D3-brane metric. It turns out that the wave equation in question is equivalent to Mathieu's modified differential equation[2]

$$\left[\frac{\partial^2}{\partial z^2} + 2q\cosh 2z - a\right]\psi(z) = 0 \tag{2}$$

under appropriate change of variables and field redefinitions. The exact solution of Mathieu's modified differential equation is known in the form of power series expansion with respect to q. From this, we can read off the absorption cross-section. For reviews of Mathieu functions see [16–20]. In view of the relative obscurity of these functions, most of the relevant details will be included in our exposition.

[1] See however [7, 8] for interesting recent work on including Ramond–Ramond fields in a world-sheet formulation.

[2] The usual form of Mathieu's equation is obtained from 2 via the replacement $z \to iz$.

First, let us see how Mathieu's modified equation arises from the wave equation of scalar fields. The supergravity background for the D3-brane has the simple form [1]

$$ds^2 = H^{-1/2}(-dt^2 + dx_{\parallel}^2) + H^{1/2}dx_{\perp}^2$$

as well as some RR 4-form background, where

$$H = 1 + \frac{R^4}{r^4}, \quad R^4 = 4\pi g N\alpha'^2 = \frac{N\kappa}{2\pi^{5/2}}.$$

For scalar fields decoupled from the RR 4-form (the example we will always have in mind is the dilaton), the equation of motion is simply

$$\frac{1}{\sqrt{g}}\partial_\mu \sqrt{g} g^{\mu\nu}\partial_\nu \phi = 0.$$

The radial wave equation for the l^{th} partial wave of energy ω which follows from this equation is

$$\left[\frac{\partial^2}{\partial r^2} + \frac{5}{r}\frac{\partial}{\partial r} - \frac{l(l+4)}{r^2} + \omega^2\left(1 + \frac{R^4}{r^4}\right)\right]\phi^{(l)}(r) = 0. \tag{3}$$

In order to relate (3) to Mathieu's equation alluded to earlier, one performs the following change of variables:

$$r = Re^{-z}, \quad \phi(r) = e^{2z}\psi(z).$$

In terms of these new variables, Eq. 3) reads

$$\left[\frac{\partial^2}{\partial z^2} + 2(\omega R)^2 \cosh 2z - (l+2)^2\right]\psi(z) = 0, \tag{4}$$

which is precisely of the form (2) for $q = (\omega R)^2$ and $a = (l+2)^2$. Note that we have reduced the problem of particle absorption by three-branes to the computation of the tunneling S-matrix for a one-dimensional Schrödinger equation.[3]

The rest of the paper is organized as follows. In Sect. 2 we present the method for obtaining the absorption probability from Mathieu's equation. This method will be of primary interest to the mathematically oriented reader, but those concerned with the string theory implications may wish to skip directly to the final answer, (34). Section 3 is concerned with the world-volume interpretation of this probability. Section 4 concludes with a brief discussion. The appendix includes some formulas judged too cumbersome to include in the main text.

[3] As an aside we note that the equations of motion for supergravity fields other than minimal scalars generically do not lead to the Mathieu equation. For example, the fixed scalar considered in [15] experiences a "transmutation of angular momentum", in the sense that the low-energy radial function at infinity and near the horizon are Bessel functions of different orders. To put it differently, the potential function in the Schrödinger operator is asymmetric.

2. Cross-Sections from Mathieu Functions

Mathieu functions arise in the study of a variety of physical problems: for example, the solution of the flat-space Laplace equation in elliptical coordinates; Bloch waves for the potential $\cos 2x$; the Faraday instability; classical motion of a driven pendulum; the sine-Gordon model [21]; and, in the present context, as tunneling wavefunctions in the potential $-\cosh 2z$. Our analysis is an extension of [22], and our conventions will be a hybrid of those of [16] and [22].

The so-called Floquet solutions of (2) can be expressed in the form

$$J(\nu, z) = \sum_{n=-\infty}^{\infty} \phi\left(n + \tfrac{1}{2}\nu\right) e^{(2n+\nu)z}. \tag{5}$$

These solutions are analogous to Bloch waves because of the property

$$J(\nu, z + i\pi) = e^{i\pi\nu} J(\nu, z). \tag{6}$$

The quantity ν is termed the Floquet exponent and is determined in terms of a and q. Clearly, $J(\nu, -z)$ is also a solution of (2) . Since $J(\nu, -z)$ acquires a phase $e^{-i\pi\nu}$ under $z \to z + i\pi$, $J(\nu, -z)$ is also a Floquet solution with exponent $-\nu$. It follows that there is a proportionality relation

$$J(-\nu, z) \propto J(\nu, -z), \tag{7}$$

which will become useful in the later discussions.

It is straightforward to see that (5) solves (2) if

$$\phi(z+1) + \phi(z-1) = \frac{z^2 - r^2}{\lambda^2}\phi(z), \tag{8}$$

where we have defined $r = \frac{1}{2}\sqrt{a}$ and $\lambda = \frac{1}{2}\sqrt{q}$. A meromorphic function ϕ was found in [22] which satisfies the recursion relation (8) and in addition has the property $\phi \to 0$ as $\Re z \to \infty$. Explicitly,

$$\begin{aligned}
\phi(z) &= \frac{\lambda^{2z}}{\Gamma(z+r+1)\Gamma(z-r+1)} \upsilon(z), \\
\upsilon(z) &= \sum_{n=0}^{\infty} (-1)^n \lambda^{4n} A_z^{(n)}, \\
A_z^{(0)} &= 1, \\
A_z^{(q)} &= \sum_{p_1=0}^{\infty} \sum_{p_2=2}^{\infty} \cdots \sum_{p_q=2}^{\infty} a_{z+p_1} a_{z+p_1+p_2} \cdots a_{z+p_1+\ldots+p_q}, \\
a_z &= \frac{1}{(z+r+1)(z+r+2)(z-r+1)(z-r+2)}.
\end{aligned} \tag{9}$$

The value of $\nu = 2\mu$ is determined by relation (7), which implies

$$\frac{\phi(\mu)}{\phi(\mu-1)} \times \frac{\phi(-\mu+1)}{\phi(-\mu)} = 1. \tag{10}$$

The recursion relation (8) can be written in the form

$$V(z) = \frac{\phi(z+1)}{\phi(z)} + \frac{\phi(z-1)}{\phi(z)} = G_{z+1} + \frac{1}{G_z},$$

where we have defined $V(z) = (z^2 - r^2)/\lambda^2$ and $G_z = \phi(z)/\phi(z-1)$. Then, we can express the first factor of (10) as a continued fraction:

$$\frac{\phi(\mu)}{\phi(\mu-1)} = G_\mu = \frac{1}{V(\mu) - G_{\mu+1}} = \frac{1}{V(\mu)-} \frac{1}{V(\mu+1) - \cdots}.$$

Similarly, the recursion relation (8) can be written in yet another form

$$V(z) = \frac{\phi(-z+1)}{\phi(-z)} + \frac{\phi(-z-1)}{\phi(-z)} = H_{z-1} + \frac{1}{H_z},$$

where this time we have defined $H_z = \phi(-z)/\phi(-z-1)$. Now we can also express the second factor of (10) as a continued fraction:

$$\frac{\phi(-\mu+1)}{\phi(-\mu)} = H_{\mu-1} = \frac{1}{V(\mu-1) - H_{\mu-2}} = \frac{1}{V(\mu-1)-} \frac{1}{V(\mu-2) - \cdots}.$$

It is now straightforward to solve for μ order by order in λ. We simply substitute the ansatz

$$\nu = \nu_0 + \nu_1\lambda^4 + \nu_2\lambda^8 + \cdots$$

into (10) expressed in terms of the continued fractions. If we are only interested in the value of ν to some finite order in λ, we can truncate the continued fraction by finite iteration. In Eq. 46) of the appendix we give the first few terms of the series for the partial waves $l = 0$, $l = 1$, and $l = 2$.

There is a remarkable resummation of the Bloch wave expansion (5) in terms of Bessel functions:[4]

$$J(\nu, z) = \sum_{n=-\infty}^{\infty} \frac{\phi\left(n + \frac{1}{2}\nu\right)}{\phi(\nu/2)} J_n(\sqrt{q}e^{-z}) J_{n+\nu}(\sqrt{q}e^{z}). \tag{11}$$

The expansion (11) is uniformly convergent everywhere and is convenient for extracting the asymptotic behavior for large $|z|$ [22, 19]. For $\nu \notin \mathbf{Z}$, the Floquet solutions $J(\pm\nu, z)$ are independent. It is useful, however, to consider other linear combinations, $N(\nu, z)$, $H^{(1)}(\nu, z)$, and $H^{(2)}(\nu, z)$, in analogy with Bessel functions:

$$\begin{aligned} N(\nu, z) &= \frac{\cos\pi\nu\, J(\nu, z) - J(-\nu, z)}{\sin\pi\nu}, \\ H^{(1)}(\nu, z) &= J(\nu, z) + iN(\nu, z) = \frac{J(-\nu, z) - e^{-i\pi\nu} J(\nu, z)}{i\sin\pi\nu}, \\ H^{(2)}(\nu, z) &= J(\nu, z) - iN(\nu, z) = \frac{J(-\nu, z) - e^{i\pi\nu} J(\nu, z)}{-i\sin\pi\nu}. \end{aligned} \tag{12}$$

[4] We use a notational convention where $J_\nu(z)$ with subscript ν denote Bessel functions whereas $J(\nu, z)$ with argument ν denote solutions to Mathieu's equation (2).

Some useful relations among the various solutions are

$$
\begin{aligned}
J(\nu, z) &= \frac{H^{(1)}(\nu, z) + H^{(2)}(\nu, z)}{2}, \\
J(-\nu, z) &= \frac{e^{i\pi\nu} H^{(1)}(\nu, z) + e^{-i\pi\nu} H^{(2)}(\nu, z)}{2}.
\end{aligned} \tag{13}
$$

Using (10) and the standard relation $J_{-n} = (-1)^n J_n$, it is straightforward to show that solutions (12) also admit expansions in terms of Bessel functions, generalizing (11):

$$
Z^{(j)}(\nu, z) = \sum_{n=-\infty}^{\infty} \frac{\phi\left(n + \frac{1}{2}\nu\right)}{\phi(\nu/2)} J_n(\sqrt{q} e^{-z}) Z^{(j)}_{n+\nu}(\sqrt{q} e^{z}). \tag{14}
$$

Here, $Z^{(j)}$ runs over J, N, $H^{(1)}$, and $H^{(2)}$. These solutions are termed associated Mathieu functions of the first, second, third, and fourth kinds.[5] We will primarily be interested in the third kind, since that is the one which describes tunneling from asymptotic infinity into the three-brane.

The asymptotic behavior for $\Re z \to \infty$ is manifest from the expansion (14):

$$
Z^{(j)}(\nu, z) \to Z^{(j)}_{\nu}(\sqrt{q} e^{z}) \quad \text{as } \Re z \to \infty. \tag{15}
$$

The behavior for $\Re z \to -\infty$ is more difficult to decipher. The first step is to use (10) to show that the constant of proportionality in (7) is precisely $\phi(-\nu/2)/\phi(\nu/2)$:

$$
J(-\nu, z) = \frac{\phi(-\nu/2)}{\phi(\nu/2)} J(\nu, -z). \tag{16}
$$

It is useful at this point to introduce the two quantities

$$
\eta = e^{i\pi\nu} \quad \chi = \frac{\phi(-\nu/2)}{\phi(\nu/2)}. \tag{17}
$$

Now the behavior of $H^{(1)}(z)$ as $\Re z \to -\infty$ can be investigated by using (12), (13) and (16):

$$
H^{(1)}(\nu, z) = \frac{1}{2i \sin \pi\nu} \left[\left(\chi - \frac{1}{\chi} \right) H^{(1)}(\nu, -z) + \left(\chi - \frac{e^{-2i\pi\nu}}{\chi} \right) H^{(2)}(\nu, -z) \right]. \tag{18}
$$

Recalling the asymptotics

$$
\left.
\begin{aligned}
H^{(1)}_{\nu}(\xi) &\to \sqrt{\frac{2}{\pi\xi}} e^{i\left(\xi - \frac{\pi}{2}\nu - \frac{\pi}{4}\right)} \\
H^{(2)}_{\nu}(\xi) &\to \sqrt{\frac{2}{\pi\xi}} e^{-i\left(\xi - \frac{\pi}{2}\nu - \frac{\pi}{4}\right)}
\end{aligned}
\right\} \quad \text{as } \Re\xi \to \infty, \tag{19}
$$

[5] We emphasize, however, that of these only $J(\nu, z)$ is a Floquet solution.

we obtain

$$\sqrt{\eta}\left(\eta-\frac{1}{\eta}\right)H^{(1)}(\nu,z)\to\begin{cases}\left(\eta-\frac{1}{\eta}\right)\sqrt{\frac{2}{\pi\sqrt{q}e^{z}}}e^{i\left(\sqrt{q}e^{z}-\frac{\pi}{4}\right)} & \text{for } \Re z\to\infty\\ \left(\chi-\frac{1}{\chi}\right)\sqrt{\frac{2}{\pi\sqrt{q}e^{-z}}}e^{i\left(\sqrt{q}e^{-z}-\frac{\pi}{4}\right)} & \\ \quad+\left(\eta\chi-\frac{1}{\eta\chi}\right)\sqrt{\frac{2}{\pi\sqrt{q}e^{-z}}}e^{-i\left(\sqrt{q}e^{-z}-\frac{\pi}{4}\right)} & \text{for } \Re z\to-\infty.\end{cases} \tag{20}$$

From (20) we read off the amplitudes $A=\chi-\frac{1}{\chi}$, $B=\chi\eta-\frac{1}{\chi\eta}$, and $C=\eta-\frac{1}{\eta}$ for the reflected, incident, and transmitted waves, respectively.

A consistency check on (20) is the unitarity relation, $|B|^2=|A|^2+|C|^2$. One way to prove this relation is to send $z\to z+i\pi/2$ so that the $-\cosh$ potential is inverted to $+\cosh$. Clearly there are wavefunctions in this potential which are everywhere real and exponentially decaying on one side (but not the other unless a is an eigen-energy).[6] In fact, $H^{(1)}(z+i\pi/2)$ is just such a solution, up to a constant overall phase. Hence A/C is pure imaginary. Now, $2\cos\pi\nu=\eta+\frac{1}{\eta}$ is always real for real q (a consequence of Hill's equation). Hence η is always either real or of unit modulus. The statement that A/C is imaginary means that the same is true of χ, and moreover χ is real when η is of unit modulus and vice versa. The verification of unitarity,

$$\left|\eta\chi-\frac{1}{\eta\chi}\right|^2=\left|\eta-\frac{1}{\eta}\right|^2+\left|\chi-\frac{1}{\chi}\right|^2, \tag{21}$$

is now straightforward. It proves easiest in practice to compute the absorption probability from

$$P=\frac{\left|\eta-\frac{1}{\eta}\right|^2}{\left|\eta-\frac{1}{\eta}\right|^2+\left|\chi-\frac{1}{\chi}\right|^2}, \tag{22}$$

but of course there are several equivalent alternative forms.

Following the methods of [22], it is straightforward though tedious to obtain a series expansion of χ in q. The first observation is that any formal sum

$$A_q=\sum_{p_1=-\infty}^{\infty}\sum_{p_2=2}^{\infty}\cdots\sum_{p_q=2}^{\infty}t_{p_1}t_{p_1+p_2}\cdots t_{p_1+\ldots+p_q}, \tag{23}$$

where the t_n are regarded as independent variables, can be written in terms of products of single sums of products of the t_n. A recursion relation is derived in [22] to demonstrate this fact:

$$\begin{aligned} qA_q&=\sum_{n=-\infty}^{\infty}\left(t_n\frac{\partial A_q}{\partial t_n}\right)\\ &=\sum_{n=-\infty}^{\infty}\left[t_n-t_n(t_{n-1}+t_n+t_{n+1})\frac{\partial}{\partial t_n}+t_{n-1}t_nt_{n+1}\frac{\partial^2}{\partial t_{n-1}\partial t_{n+1}}\right]A_{q-1}.\end{aligned} \tag{24}$$

[6] Incidentally, (20) provides an implicit equation for the eigen-energies of the $+\cosh$ potential: namely $\chi=\pm1$ for even/odd wavefunctions.

Let us introduce the notation

$$S[\alpha_0, \alpha_1, \ldots, \alpha_k] = \sum_{n=-\infty}^{\infty} \prod_{j=0}^{k} t_{n+j}^{\alpha_j}, \tag{25}$$

where the α_j are natural numbers with α_0 and α_k nonzero.[7] Then the map $H : A_{q-1} \to qA_q$ defined in (24) can be viewed formally as a linear operator on the infinite-dimensional vector space whose basis is 1 together with all possible products of the $S[\alpha_0, \alpha_1, \ldots, \alpha_k]$.[8] We have

$$A_q = \frac{1}{q!} H^q(1), \tag{26}$$

where $H^q(1)$ is the operator H acting q times on unity. Amusingly, the problem of computing the generalization of the function v in (9) to arbitrary t_n at finite λ is formally identical to Euclidean evolution by the Hamiltonian H:

$$v \equiv \sum_{q=0}^{\infty} (-\lambda^4)^q A_q = e^{-\lambda^4 H}(1). \tag{27}$$

In Eq. (47) of the appendix we write out the first four A_q in terms of the $S[\alpha_0, \alpha_1, \ldots, \alpha_k]$.

Now let us specialize to $A_q = A_z^{(q)}$ by setting

$$t_n = \begin{cases} a_{z+n} & \text{for } n \geq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{28}$$

The sums $S[\alpha_0, \alpha_1, \ldots, \alpha_q]$ then have the general form $\sum_{n=0}^{\infty} \frac{1}{f(z+n)}$, where $f(z)$ is a polynomial of degree $4 \sum_{i=0}^{q} \alpha_i$. Such sums can be performed explicitly in terms of the function $\psi(z) = \Gamma'(z)/\Gamma(z)$ and its derivatives. The first step is to make a partial fraction decomposition:

$$\frac{1}{f(z)} = \sum_{f(y)=0} \sum_{\ell=1}^{\infty} \frac{c_y^{(\ell)}}{(z-y)^{\ell}}. \tag{29}$$

The first sum is over the roots of $f(z)$. For a root y of multiplicity k, only the first k of the constants $c_y^{(\ell)}$ can be nonzero. Each term in the partial fraction decomposition makes a contribution to the sum over n which can be read off from

$$\begin{aligned} \psi(z) &= -\mathbf{C} - \sum_{n=0}^{\infty} \left[\frac{1}{z+n} - \frac{1}{n+1} \right], \\ \psi^{(k)}(z) &= (-1)^{k+1} k! \sum_{n=0}^{\infty} \frac{1}{(z+n)^{k+1}}. \end{aligned} \tag{30}$$

7 Note that only questions of convergence stand in the way of extending the following discussion to arbitrary real sequences $\{\alpha_j\}_{j=-\infty}^{\infty}$ modulo the equivalence relation $\{\alpha_j\}_{j=-\infty}^{\infty} \sim \{\alpha_{j+k}\}_{j=-\infty}^{\infty}$ for integer k.

8 This space is reminiscent of the loop spaces encountered, for instance, in the $c = 0$ matrix model [23, 24]. In this analogy, H plays the role of the Fokker–Planck Hamiltonian.

where $\mathbf{C} = \log \gamma \approx 0.5772$ is Euler's constant. This leads to the sum

$$\sum_{n=0}^{\infty} \frac{1}{f(z+n)} = \sum_{f(y)=0} \sum_{\ell=1}^{\infty} c_y^{(\ell)} \frac{(-1)^\ell}{(\ell-1)!} \psi^{(\ell-1)}(z-y). \tag{31}$$

The coefficients $c_y^{(1)}$ satisfy the relation

$$\sum_{f(y)=0} c_y^{(1)} = \frac{1}{2\pi i} \oint_\gamma \frac{dz}{f(z)} = 0, \tag{32}$$

where γ is a contour that encloses all the roots of $f(z)$, ensuring that the divergences from the various $1/(z-y)$ terms in the partial fraction decomposition cancel. In effect this allows us to use the second line of (30) even at $k = 0$. Explicit expressions for the first few $S[\{\alpha_i\}]$'s are included in Eq. (48) of the appendix.

To complete the task of computing the absorption cross-section, we need to determine the value of $\chi = \phi(-\nu/2)/\phi(\nu/2)$. All that remains to be done now is to substitute the expansion for ν given in (46) into (9) and collect terms of given order in λ. Because ν is an integer plus powers of λ, the ψ functions can all be Taylor expanded around integers or half-integers. To simplify the final expressions, it is useful to recall the relation of ψ to the Riemann zeta function $\zeta(s)$ and its generalizations $\zeta(s,z)$:

$$\begin{aligned} \psi(1) = -\mathbf{C}, \quad \psi^{(k)}(z) &= (-1)^{k+1} k! \zeta(k+1, z), \\ \zeta(s, z+1) &= \zeta(s,z) - \frac{1}{z^s}, \\ \zeta(s,1) = \zeta(s), \quad \zeta\left(s, \tfrac{1}{2}\right) &= (2^s - 1)\zeta(s). \end{aligned} \tag{33}$$

The final expressions for the absorption probability of the l^{th} partial wave have the form

$$P_l = \frac{4\pi^2}{(l+1)!^4 (l+2)^2} (\omega R/2)^{8+4l} \sum_{n=0}^{\infty} \sum_{k=0}^{n} b_{n,k} (\omega R)^{4n} \left(\log \omega \bar{R}\right)^k, \tag{34}$$

where $\bar{R} = e^{\mathbf{C}} R/2$. The overall normalization has been chosen so that $b_{0,0} = 1$. We have computed the values of the first few $b_{n,k}$'s for $l = 0$, $l = 1$, and $l = 2$ which we summarize in Table 1. We find that $b_{n,k}$ is rational for $n - k < 2$, whereas for $n - k \geq 2$ it is a linear combination of $\zeta(2), \zeta(3), \ldots, \zeta(n-k)$ with rational coefficients.

The absorption cross-section for the l^{th} partial wave can now be computed from a version of the Optical Theorem:

$$\sigma_l = \frac{8\pi^2/3}{\omega^5} (l+1)(l+2)^2(l+3) P_l. \tag{35}$$

The generalization of this formula to arbitrary dimensions was derived in [25].

Table 1. Leading coefficients $b_{n,k}$ for the expansion with respect to ωR for the absorption cross-section (34) of $l = 0$, $l = 1$, and $l = 2$ partial waves

	$l = 0$	$l = 1$	$l = 2$
$b_{1,1}$	$-\frac{1}{6}$	$-\frac{1}{24}$	$-\frac{1}{60}$
$b_{1,0}$	$\frac{7}{72}$	$\frac{53}{1152}$	$\frac{19}{800}$
$b_{2,2}$	$\frac{17}{576}$	$\frac{1}{1152}$	$\frac{1}{7200}$
$b_{2,1}$	$-\frac{161}{4608}$	$-\frac{757}{276480}$	$-\frac{821}{1728000}$
$b_{2,0}$	$\frac{5561}{663552} - \frac{11\zeta(2)}{576}$	$\frac{261343}{132710400} - \frac{\zeta(2)}{4608}$	$\frac{44071}{103680000} - \frac{\zeta(2)}{28800}$
$b_{3,3}$	$-\frac{11}{2592}$	$-\frac{1}{82944}$	$-\frac{1}{1296000}$
$b_{3,2}$	$\frac{623}{82944}$	$\frac{7}{69120}$	$\frac{479}{103680000}$
$b_{3,1}$	$-\frac{39037}{9953280} + \frac{49\zeta(2)}{6912}$	$-\frac{554911}{3185049600} + \frac{\zeta(2)}{110592}$	$-\frac{1731599}{174182400000} + \frac{\zeta(2)}{1728000}$
$b_{3,0}$	$\frac{1093099}{2388787200} - \frac{1379\zeta(2)}{331776} + \frac{5\zeta(3)}{41472}$	$\frac{65129557}{764411904000} - \frac{101\zeta(2)}{2211840} - \frac{\zeta(3)}{663552}$	$\frac{1148018521}{167215104000000} - \frac{479\zeta(2)}{414720000} - \frac{\zeta(3)}{10368000}$

3. The World-Volume Dynamics

Let us now consider the world-volume interpretation for the case where the minimal scalar is the dilaton. In the 't Hooft limit $g \to 0$, $N \to \infty$ with gN fixed, quantum fluctuations of bulk fields decouple and the dynamics is strictly on the brane world-volume. The only sense in which bulk fields enter is as a source of world-volume fluctuations in the form of a local operator. The s-wave of the dilaton corresponds in the world-volume theory to the operator $\mathcal{O}$ which slides the gauge coupling. The absorption probability $P_{l=0}$ then translates directly into the discontinuity of the cut in the two-point function $\mathcal{O}$ through the formula [11]

$$
\begin{aligned}
P_{l=0} &= \frac{\pi^3 \omega^4 R^8}{8iN^2} \operatorname{Disc} \Pi(p^2), \\
\Pi(p^2) &= \int d^4x \, e^{ip\cdot x} \Pi(x^2),
\end{aligned}
\tag{36}
$$

where

$$
\Pi(x^2) = \langle \mathcal{O}(x)\mathcal{O}(0) \rangle. \tag{37}
$$

The dynamics of the world-volume theory at leading order in energy is captured by its superconformal limit in the infrared. To higher order in energy, however, one must account for the effect of irrelevant perturbations which takes the theory away from the fixed point. The correlator $\langle \dots \rangle$ is therefore taken with respect to some quantum effective action which we will describe later in this section.

In (36) it should be noted that the discontinuity is taken across the cut positioned along the positive real axis of the complex $s = -p^2$ plane, evaluated at $s = \omega^2$. Working backward, one can read off $\Pi(x^2)$ from $P_{l=0}$, with the result

$$
\Pi(x^2) = \frac{3N^2}{\pi^4 x^8} \sum_{n=0}^{\infty} \sum_{k=0}^{n} c_{n,k} \left(\frac{R^2}{x^2} \right)^{2n} \left(\log \frac{R^2}{x^2} \right)^k . \tag{38}
$$

To obtain $P_{l=0}$ from (38) we must specify a regularization scheme for the Fourier integrals. The minimal scheme, following [26, 27], is to analytically continue the formula

$$\int d^4x \frac{e^{ip\cdot x}}{x^{2h}} = \pi^2 \left(\frac{4}{p^2}\right)^{2-h} \frac{\Gamma(2-h)}{\Gamma(h)} \tag{39}$$

beyond its radius of convergence $|h-1| < 1$ to a meromorphic function on the entire complex h plane, and then read off the behavior near the poles at positive integer h by matching terms in the Taylor expansions in a of

$$\begin{aligned} \int d^4x \frac{e^{ip\cdot x}(\mu x)^{2a}}{x^{2n}} &= \pi^2 \left(\frac{4}{p^2}\right)^{2-n} \left(\frac{4\mu^2}{p^2}\right)^a \frac{\Gamma(2-n+a)}{\Gamma(n-a)}, \\ \text{Disc} \int d^4x \frac{e^{ip\cdot x}(\mu x)^{2a}}{x^{2n}} &= -\left(\frac{4}{\omega^2}\right)^{2-n} \left(\frac{4\mu^2}{\omega^2}\right)^a \frac{2\pi^3 i}{\Gamma(n-a)\Gamma(n-a-1)}. \end{aligned} \tag{40}$$

For the expansions in a one uses

$$\begin{aligned} (\mu x)^{2a} &= \sum_{n=0}^{\infty} \frac{a^n}{n!} (\log \mu^2 x^2)^n, \\ \log \Gamma(1+a) &= \tfrac{1}{2} \log \frac{\pi a}{\sin \pi a} - \mathbf{C} a - \sum_{n=1}^{\infty} \frac{a^{2n+1}}{2n+1} \zeta(2n+1). \end{aligned} \tag{41}$$

Upon setting the energy scale $\mu = 1/R$ one obtains the $c_{n,k}$ as numbers involving $\zeta(s)$ in the same way as the $b_{n,k}$: explicitly,

$$\begin{aligned} c_{0,0} = 1, \quad c_{1,1} &= -320, \quad c_{2,2} = 571200, \\ c_{1,0}, &= -1024, \quad c_{2,1} = 4408560, \\ & \qquad c_{2,0} = \frac{14}{3}(1422697 - 12000\pi^2). \end{aligned} \tag{42}$$

One can formally define a dimension Δ for the operator $\mathcal{O}$ in 37 through a version of the Callan-Symanzik equation:

$$\left[x \frac{\partial}{\partial x} + 2\Delta\right] \Pi(x^2) = 0. \tag{43}$$

For $R^4/x^4 \ll 1$, this results in a series of the same form as (38):

$$\begin{aligned} \Delta &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \Delta_{n,k} \left(\frac{R^2}{x^2}\right)^{2n} \left(\log \frac{R^2}{x^2}\right)^k \\ &= 4 - 64 \frac{R^4}{x^4} \left(37 + 10 \log \frac{R^2}{x^2}\right) + \dots . \end{aligned} \tag{44}$$

The challenge at this point is to reproduce (38) and its generalizations to higher partial waves through a quantum field theory analysis. As we mentioned earlier in this section, this requires a knowledge of the world-volume dynamics beyond the superconformal limit in the infrared. In principle, this theory is well defined as a low-energy effective action of the full string theory. At present, however, no concrete formulation of this effective theory is known. Therefore, instead of trying to reproduce (38), we can attempt to learn about this effective theory from the data provided by (38).

The leading term has precisely the form one expects in a conformal theory. The leading correction, $\frac{R^4}{x^4} \log \frac{R^2}{x^2}$, has the form one would obtain by perturbing the conformal field theory by a dimension eight operator. It was speculated in [15] that this correction and perhaps the full semi-classical cross-section would eventually find its world-volume explanation in the non-abelian Dirac–Born–Infeld (DBI) action, with the symmetrized trace prescription proposed in [28] to pick out the leading correction at dimension eight ($\mathrm{Tr}[F^4]$), rather than dimension six ($\mathrm{Tr}[F^3]$) as one would expect from other prescriptions.

However, the DBI action arises from summing disc diagrams, so it defines a classical field theory, and in no way captures the effect of a resummation of infinite insertions of boundaries in the large gN limit. Furthermore, the non-renormalizability of the action makes it impossible to proceed to the quantum theory from a knowledge of the tree-level amplitudes alone, as was the standard strategy in deriving low-energy renormalizable quantum field theories from string theory. We require some further input from the string theory.

It was conjectured in [29, 30] that all operators in the gauge theory except those in short multiplets acquire large anomalous dimensions in the strong 't Hooft coupling limit, and perhaps even decouple from the operator algebra.[9] The supergravity fields corresponding to the operators in short multiplets have been tabulated in [31]. Inspection of this table reveals that the only scalar $SO(6)$ singlet operators are the renormalizable lagrangian $\mathcal{O}_4$ (coupling to the s-wave of the dilaton) and a dimension eight operator $\mathcal{O}_8$ which couples to uniform dilations of the S^5 part of the near-horizon geometry. There is also a dimension four pseudo-scalar which couples to the axion, which we shall ignore in the following.

On the grounds of group theory and large anomalous scaling dimensions, we are then led to the tentative conclusion that the effective lagrangian for the low-energy dynamics at large gN is

$$\mathcal{L} = \mathcal{O}_4 + R^4 \mathcal{O}_8. \tag{45}$$

The relation to DBI is merely that the low-energy effective lagrangian of the same system at small gN is the DBI action. On this view, the phrase "DBI action" must be interpreted in [15] (and in the many other papers in the literature, e.g. [32], where it was invoked in the context of an effective world-volume theory of D-brane black holes) as a metonym for its strong-coupling relative. Equation (45) is a fantastic simplification over the still incompletely known non-abelian DBI action. But in a way it is no less problematical as a specification of a quantum theory. The natural interpretation of (45) is as the Wilsonian effective action with cutoff on the order R.[10] The difficulties with this approach include pinning down the normalization of $\mathcal{O}_8$ at a given cutoff, defining an appropriate regularization scheme which allows one to recover maximal supersymmetry, and the apparent vanishing of $\langle \mathcal{O}_4 \mathcal{O}_4 \mathcal{O}_8 \rangle$ in the AdS/CFT prescription to leading order in large gN.

Nevertheless, let us try to argue that (45) at least has the potential to reproduce all the correction terms in (38). Following [15], we can consider as a toy model free $U(1)$

[9] We thank T. Banks for a discussion on this point.

[10] If the cutoff Λ is made arbitrary, then one must introduce a coupling $\lambda(\Lambda)$ in front of $\mathcal{O}_8$ which runs precisely in order to keep the physical observables, e.g. correlation functions, invariant with respect to the change in the choice of the cut-off.

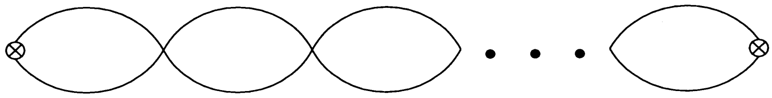

Fig. 1. A diagram with n quartic vertices contributing at order $O(R^{4n})$

gauge theory with an F^4 interaction. From graphs such as the one in Fig. 1, one indeed obtains a $(R^4/x^4)^n(\log R^2/x^2)^n$ correction to the two-point function. It is fascinating that the final forms (34) and (38) of the absorption probability and two-point function are so simple and suggestive of Feynman integrals, regulated at the scale $\mu = 1/R$. For small ωR, it seems that the perturbative expansion around the conformal limit may be better defined than we have any right to expect based on previous experience with non-renormalizable divergences in quantum field theories. Quite remarkably, one type of interaction alone is sufficient to reproduce the form of (38). This might indeed be a consequence of superconformal invariance and the decoupling of non-chiral operators in the large gN limit severely restricting the dynamics away from the infrared fixed point. We regretfully leave a more detailed study for future work.

4. Discussion

The biggest obstacle to finding evidence for the conjectured throat-brane equivalence [6, 29, 30] between $\mathcal{N} = 4$ super Yang-Mills theory and supergravity on $AdS_5 \times S^5$ is that supergravity's validity is restricted to the region of strong 't Hooft coupling, where gauge theory calculations are difficult. Let us adopt units where the radius of S^5 is 1. Briefly, since $1/\alpha' \sim g_{YM}\sqrt{N}$ in these units, the α' corrections to the supergravity action are important except in the limit of large $g_{YM}\sqrt{N}$. For example, the supergravity fields on AdS_5 (with Kaluza Klein masses on the order $1/R$) are much lighter than massive string states (with masses on the order $1/\sqrt{\alpha'}$) only in this limit. The corresponding non-chiral fields in the gauge theory "freeze out" on account of an anomalous dimension on the order $(g_{YM}\sqrt{N})^{1/2}$ [29]. Large N can be regarded as a separate requirement: since powers of $\kappa \sim 1/N$ suppress quantum loop corrections to supergravity, the identification of the classical supergravity action with the generator of connected Green's functions can only capture the leading large N asymptotics.

To proceed to finite or small $g_{YM}\sqrt{N}$ seems difficult without some profound new insight into the description of string theory in Ramond–Ramond backgrounds. Any hope of systematic perturbative field theory evidence in favor of the throat-brane conjecture would seem to depend on finding some other small coupling parameter. The only candidate seems to be ωR, where ω is the energy of a given process (i.e. absorption). As a first step in investigating a possible perturbation expansion in ωR, we have given an algorithm, which can be readily implemented on a computer, for extracting the absorption cross-section of a minimal scalar in an arbitrary partial wave. The notion [15, 33] that the DBI action of D3-branes can in any meaningful way "holograph" supergravity or string theory on the full extremal three-brane geometry must be viewed with skepticism. It is perhaps more reasonable to hope that a quantum field theoretic derivation of at least the leading log terms in the ωR series expansion might be achieved (in part because these terms have a simpler cutoff dependence than terms with fewer powers of logarithms). In geometrical terms, the hope would be to see the r/R corrections to the near-horizon geometry (where r is the usual radial variable entering into the harmonic function $H = 1 + R^4/r^4$) reflected order by order in the non-renormalizable contributions to the Green's functions for some quantum effective world volume theory.

While the motivation for this work was primarily our hope to achieve a better understanding of the double scaling limit described in [5, 10], our main technical results can be stated in the more prosaic setting of Schrödinger operators in one dimension. For a particle moving in a potential $V(z) = -2q \cosh 2z$, we have found a simple expression (22) for the transmission coefficient in terms of the Floquet exponent ν and a quantity χ related to the transformation properties of Floquet solutions under parity. The computation of the Floquet is well understood in terms of partial fractions. We implement the methods of [22] to give a method for computing χ as well. The Hamiltonian form of (27), and the surprising symmetry in the transmission probability between $\eta = e^{i\pi\nu}$ and χ, tantalizes us with the hope that one might be able to give a treatment of Mathieu functions which puts η and χ on an equal footing.

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Appendix A. Explicit Formulas

In this appendix we present explicit forms for some results which were considered too lengthy to write out in the main text. Most of the computations were done with Mathematica.

First, the Floquet exponent for $r = 1$, $r = 3/2$, and $r = 2$ (corresponding to $l = 0$, $l = 1$, and $l = 2$) can be expanded as a power series in λ as follows:

$$\begin{aligned} r = 1 : \nu &= 2 - \frac{i}{3}\sqrt{5}\,\lambda^4 + \frac{7\,i}{108\sqrt{5}}\,\lambda^8 + \frac{11851\,i}{31104\sqrt{5}}\,\lambda^{12} + \cdots , \\ r = \frac{3}{2} : \nu &= 3 - \frac{1}{6}\,\lambda^4 + \frac{133}{4320}\,\lambda^8 + \frac{311}{1555200}\,\lambda^{12} + \cdots , \\ r = 2 : \nu &= 4 - \frac{1}{15}\,\lambda^4 - \frac{137}{27000}\,\lambda^8 + \frac{305843}{680400000}\,\lambda^{12} + \cdots . \end{aligned} \tag{46}$$

By iterating (26) one can obtain expressions for the formal series A_q defined in (23) in terms of the "loop variables" $S[\alpha_0, \alpha_1, \ldots, \alpha_k]$. These grow in size very rapidly:

$$\begin{aligned} A_1 &= S[1], \\ A_2 &= \frac{S[1]^2}{2} - \frac{S[2]}{2} - S[1,1], \\ A_3 &= \frac{S[1]^3}{6} - \frac{S[1]S[2]}{2} + \frac{S[3]}{3} - S[1]S[1,1] + S[1,2] + S[2,1] + S[1,1,1], \\ A_4 &= \frac{S[1]^4}{24} - \frac{S[1]^2 S[2]}{4} + \frac{S[2]^2}{8} + \\ &\quad + \frac{S[1]S[3]}{3} - \frac{S[4]}{4} - \frac{S[1]^2 S[1,1]}{2} + \frac{S[2]S[1,1]}{2} + \frac{S[1,1]^2}{2} + \end{aligned} \tag{47}$$

$$+S[1]S[1,2] - S[1,3] + S[1]S[2,1] - \frac{3S[2,2]}{2} - S[3,1] + S[1]S[1,1,1]$$
$$-S[1,1,2] - 2S[1,2,1] - S[2,1,1] - S[1,1,1,1],$$

and so on.

After making the identification (28), the formal sums $S[\{\alpha_i\}]$ may be evaluated explicitly in the manner indicated in the paragraph following (28).

$$\begin{aligned} S[1] = {} & \frac{-3-2z}{(-1+2r)(1+2r)(-1+r-z)(1+r+z)} \\ & + \frac{\psi(1-r+z) - \psi(1+r+z)}{-r+4r^3}, \\ S[2] = {} & \frac{35+84z+70z^2+20z^3+8r^4(1+2z) - 2r^2\left(35+50z+28z^2+8z^3\right)}{\left(-1+4r^2\right)^3(1-r+z)^2(1+r+z)^2} \\ & + \frac{\left(-1+20r^2\right)(\psi(1-r+z)-\psi(1+r+z))}{2r^3\left(-1+4r^2\right)^3} \\ & + \frac{\left(1+4r^2\right)\left(\psi^{(1)}(1-r+z)+\psi^{(1)}(1+r+z)\right)}{2r^2\left(1-4r^2\right)^2}, \end{aligned} \tag{48}$$

$$\begin{aligned} & S[1,1] = \\ & -\frac{4r^6(3+2z)+r^4\left(35-26z-36z^2-8z^3\right)-r^2\left(109+143z+65z^2+10z^3\right)+2(2+z)^2}{4(-1+r)(1+r)\left(r-4r^3\right)^2(-2+r-z)(-1+r-z)(1+r+z)(2+r+z)} \\ & + \frac{\left(-1+10r^2\right)(\psi(1-r+z)-\psi(1+r+z))}{4r^3\left(1-4r^2\right)^2\left(-1+r^2\right)} + \frac{\psi^{(1)}(2-r+z)+\psi^{(1)}(2+r+z)}{4r^2-16r^4}. \end{aligned}$$

These formulas also become very lengthy, and they have many different forms because of the various identities for the ψ function.

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