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# Nodal Sets for Groundstates of Schrödinger Operators with Zero Magnetic Field in Non Simply Connected Domains ⋆

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**Abstract:** We investigate nodal sets of magnetic Schrödinger operators with zero magnetic field, acting on a non simply connected domain in $\mathbb{R}^2$. For the case of circulation $1/2$ of the magnetic vector potential around each hole in the region, we obtain a characterisation of the nodal set, and use this to obtain bounds on the multiplicity of the groundstate. For the case of one hole and a fixed electric potential, we show that the first eigenvalue takes its highest value for circulation $1/2$.

## 1. Introduction and Statement of Results

Let $\Omega \subset \mathbb{R}^2$ be a region with smooth ($C^\infty$) boundary, which is homeomorphic to a disk with $k$ holes, and consider the magnetic Schrödinger operator

$$H_{A,V} := (i\nabla + A)^2 + V \tag{1.1}$$

acting on $L^2(\Omega)$ with Neumann boundary conditions. The potential $V$ is assumed to be smooth, and we consider a smooth magnetic vector potential $A$ which corresponds to a zero magnetic field. That is,

$$B := \text{ curl } A = 0 \tag{1.2}$$

in $\Omega$. Assumption (1.2) implies that in any simply connected, open subset of $\Omega$, there exists a gauge function $\phi$ such that

$$\nabla\phi = A. \tag{1.3}$$

We shall see that the operator $H_{A,V}$ is unitarily equivalent to the non-magnetic Schrödinger operator $H_{O,V}$ if and only if one can extend this local gauge $e^{i\phi}$ to a globally

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defined function such that $\phi$ (which might not be a singlevalued function) satisfies (1.3). We shall see that this can be done precisely when each of the circulations

$$\Phi_i = \frac{1}{2\pi} \oint_{\sigma_i} A \cdot \mathrm{d}\mathbf{x}, \tag{1.4}$$

of $A$ round the $i^{\text{th}}$ hole ($i = 1, \dots, k$) takes an integer value. Here $\sigma_i$ is a closed path[1] which parametrises the boundary $\Sigma_i$ of the $i^{\text{th}}$ hole and turns once in an anti-clockwise direction.

Furthermore, if the circulations $\Phi = (\Phi_1, \dots, \Phi_k)$ of two distinct vector potentials $A$ and $A'$ are equal modulo $\mathbb{Z}^k$ then the corresponding operators $H_{A,V}$ and $H_{A',V}$ are unitarily equivalent under a gauge transformation.

**Theorem 1.1.** *Let $\Omega \subset \mathbb{R}^2$ be a region with smooth boundary, which is homeomorphic to a disk with $k$ holes. For a given smooth potential $V$, the first eigenvalue $\lambda_1$ of the magnetic Schrödinger operator $H_{A,V}$, where $A$ satisfies* (1.2), depends only on the circulations $\Phi = (\Phi_1, \dots, \Phi_k)$ of $A$. The function $\lambda_1(\Phi)$ has the following properties (in which $l \in \mathbb{Z}^k$ is arbitrary):

$$\lambda_1(\Phi + l) = \lambda_1(\Phi), \tag{1.5}$$

$$\lambda_1(l/2 + \Phi) = \lambda_1(l/2 - \Phi), \tag{1.6}$$

$$\lambda_1(\Phi) > \lambda_1(0, \dots, 0) \quad \text{for } \Phi \notin \mathbb{Z}^k. \tag{1.7}$$

For the case $k = 1$, we have in addition to Eq. (1.7) that

$$\lambda_1(\Phi) < \lambda_1(1/2) \tag{1.8}$$

for $\Phi \notin 1/2 + \mathbb{Z}$.

Equations (1.5), (1.6) and inequality (1.7) are straightforward, and are proved in Sect. 2 (see also Remark 2.2). In this context we should also mention the recent very interesting results [HN97] by Herbst and Nakamura concerning large magnetic fields. We choose Neumann boundary conditions on $H_{A,V}$ in this article because we were motivated by questions arising in the Ginzburg model of super-conductivity. Our results are also valid for the case of Dirichlet boundary conditions (see Remark 1.5 (vi)). Dirichlet boundary conditions are related to the Aharonov-Bohm effect for bound states. See [LO77, Hel88a, Hel88b, Hel94]. Such models also arise in the description of the Little-Parks experiment [LP62].

Inequality (1.8) appears, to the best of our knowledge, for the first time. Our proof of this result (see Sect. 4), uses a connection between the maximality of the first eigenvalue for flux $1/2$ and the structure of the nodal set of groundstates. The nodal sets for the single hole case with flux $1/2$ were recently investigated by Berger and Rubinstein [BR97]. Part of our work is motivated by their preprint.

Using semiclassical arguments as in [Hel88a], we can show that in general the first eigenvalue is not necessarily maximised for circulation $(1/2, \dots, 1/2)$.

**Definition 1.2.** *The nodal set $\mathcal{N}(u)$ of an eigenfunction $u$ of a magnetic Schrödinger operator on a manifold $\Omega$ with smooth boundary is defined in $\overline{\Omega}$ by*

$$\mathcal{N}(u) := \overline{\{x \in \Omega : u(x) = 0\}}. \tag{1.9}$$

[1] A piecewise smooth mapping $\gamma : [0, 1] \to X$ is called a path in $X$. The point $\gamma(0)$ is called the initial point and $\gamma(1)$ is called the final point. The image $\Gamma = \gamma([0, 1])$ of the path is called a curve.

Some useful information on nodal sets of real valued eigenfunctions of non-magnetic Schrödinger equations in two dimensions is given in Proposition 4.1. In particular we see that such nodal sets consist of the finite union of smoothly immersed circles and lines. It is "generically" the case that the nodal set of every complex eigenfunction of a magnetic Schrödinger operator consists of isolated points of intersection of the lines of zeros of the real and imaginary parts of the function. See [EMQ94].

The local properties of the nodal sets of eigenfunctions of the operator $H_{A,V}$ are the same as the local properties of complex solutions of non-magnetic Schrödinger equations. More precisely, since we may find at every point a local gauge $e^{i\phi}$ satisfying (1.3), we may multiply any eigenfunction of $H_{A,V}$ by a local gauge so that the product solves a non-magnetic Schrödinger equation. The nodal set is invariant under local gauge transformations.

We shall see in what follows that although the local properties of nodal sets of eigenfunctions of our magnetic Schrödinger operator are the same as the properties of a non-magnetic Schrödinger operator, the global properties differ in the case where $\Phi = (1/2, \dots, 1/2)$. In particular, in the non-magnetic case we see that (since a real eigenfunction must change sign at the nodal set) an even number of nodal lines (or perhaps no nodal lines) of an eigenfunction emerges from each boundary component of the region. In Theorem 1.4 we show that for $\Phi = (1/2, \dots, 1/2)$, an odd number of nodal lines of the groundstate emerge from each component.

**Definition 1.3.** *We say that a (nodal) set $\mathcal{N}$* **slits** $\overline{\Omega}$ *if it is the union of a collection of piecewise smooth, immersed lines such that*

(i) *each line starts and finishes at the boundary $\partial\Omega$ and leaves the boundary transversally;*
(ii) *internal intersections between lines are transversal;*
(iii) *the complement $\Omega \setminus \mathcal{N}$ is connected;*
(iv) *an odd number of nodal lines leaves each interior boundary component.*

*We shall say that a collection of paths slits $\overline{\Omega}$ if the union of the images of the paths slits $\overline{\Omega}$.*

See Fig. 1 for some examples of regions which are slit. Note that part (iii) of the above definition is the reason why a nodal set which slits $\overline{\Omega}$ contains no immersed circles, and also implies that each line of a slitting set links together a unique pair $\{\Sigma_i, \Sigma_j\}$ of distinct (i.e. $i \neq j$) boundary components. Note also that for the single hole case, a set which slits $\overline{\Omega}$ consists of one line which joins the outer boundary of $\Omega$ to the inner boundary.

In Corollary 4.3 we show that if a collection of paths slits a region then no sub- or supercollection of these paths can also slit the region. In Proposition 5.1 we show that the number $n$ of paths of such a collection must satisfy $k/2 \leq n \leq k$.

**Theorem 1.4.** *Let $\Omega$ be a region with smooth boundary, which is homeomorphic to a disk with $k$ holes. Let $V$ be a smooth potential and let $A$ be a smooth magnetic vector potential satisfying Eq.* (1.2)*, such that the value of the circulations around each hole lie in $1/2 + \mathbb{Z}$ (that is $\Phi = (1/2, \dots, 1/2)$, modulo $\mathbb{Z}^k$).*

(i) *If the first eigenvalue of $H_{A,V}$ is simple then the nodal set of the corresponding eigenfunction slits $\overline{\Omega}$. Otherwise there exists an orthonormal basis $\{u_1, \dots, u_m\}$ of the groundstate eigenspace such that the nodal set of any non-zero combination $\sum_{i=1}^{m} a_i u_i$, with $a_i \overline{a}_j \in \mathbb{R}$ for each $1 \leq i, j \leq m$, slits $\overline{\Omega}$.*

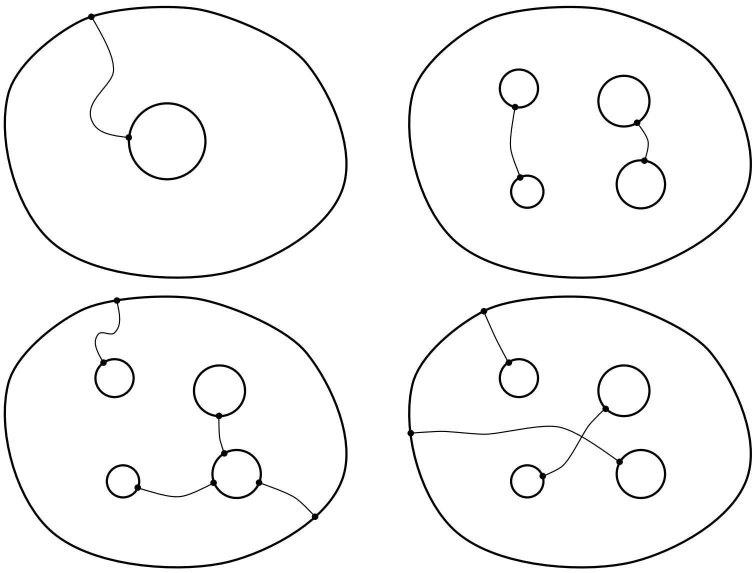

**Fig. 1.** Examples of some sets which slit $\overline{\Omega}$

(ii) *The multiplicity $m$ of the first eigenvalue of $H_{A,V}$ satisfies*

$$m \leq \begin{cases} 2, & k = 1, 2; \\ k, & k \text{ odd}, k \geq 3; \\ k-1, & k \text{ even}, k \geq 4. \end{cases} \tag{1.10}$$

(iii) *For $k = 1, 2$ with groundstate multiplicity two, the nodal sets of two linearly independent groundstates do not intersect. It follows that the nodal set of a combination $a_1 u_1 + a_2 u_2$ is empty whenever $a_1 \overline{a}_2 \notin \mathbb{R}$.*

Here we make some remarks connected to the above theorem.

*Remarks 1.5.* (i) The above bound on the multiplicity of the first eigenvalue is sharp in the case of one hole (see Example 5.3), but it is not expected to be sharp for many holes. It would be interesting to know an asymptotic result about the growth of the maximum multiplicity with the number of holes.

(ii) We prove the bound by taking advantage of topological obstructions to nodal sets caused by the holes. These obstructions prevent the existence of high dimensional groundstate eigenspaces. Our type of method was first discovered in [Che76] and has since been taken up and used by others, e.g. [Nad88, HOHON98, HOMN]. See also [Col93] for explicit constructions of examples with high multiplicity.

(iii) Our result bears similarities to bounds on multiplicities of higher eigenvalues of non-magnetic Schrödinger operators on surfaces with boundary. Some related literature on this topic is given in [Col93, Nad88, HOHON98, HOMN].

(iv) It has been shown in [BCC98] that no upper bound on the multiplicity exists when one adds a general magnetic field, even on the sphere.

(v) For the cases $k \geq 3$ we expect that there could be intersection of nodal sets of two independent groundstates, and correspondingly that the nodal set of a combination $a_1 u_1 + a_2 u_2$ will not in general be empty when $a_1 \overline{a}_2 \notin \mathbb{R}$.

(vi) If we assume that $H_{A,V}$ has Dirichlet boundary conditions then Theorems 1.1 and 1.4 hold with suitable changes to the proofs. More precisely, in Proposition 4.1 the Taylor expansion (4.2) for a zero of order $l$ at a point $x \in \partial\Omega$ becomes

$$f(x) = ar^l \sin l\omega + O(r^{l+1}),$$

and from Lemma 4.5 through to the proof of Theorem 1.4 (ii), all arguments which involve a function which has a zero of order $l = k$ (for example) should be replaced by the same argument involving a function with a zero of order $l = k + 1$.

## 2. Some Basic Results

The quadratic form corresponding to the operator $H_{A,V}$ is

$$Q_{A,V}(u) = \int_\Omega \left( |(i\nabla + A)u|^2 + V|u|^2 \right) \mathrm{d}^2 x, \tag{2.1}$$

with domain $\mathcal{Q}^{\mathrm{Neu}} = W^{1,2}(\Omega) = H^1(\Omega)$. This choice of quadratic form domain corresponds to Neumann boundary conditions for $H_{A,V}$. For the case of Dirichlet boundary conditions (see Remark 1.5 (vi)) the relevant quadratic form domain is $\mathcal{Q}^{\mathrm{Dir}} = W_0^{1,2}(\Omega)$.

*Remark 2.1.* Neumann boundary conditions for a magnetic Schrödinger operator mean that functions in the domain of the operator satisfy

$$i\frac{\partial u}{\partial n} = -A \cdot n\, u \tag{2.2}$$

on $\partial\Omega$, where $n$ is normal to $\partial\Omega$.

One can always assume that the vector potential satisfies the additional properties

$$\nabla \cdot A = 0 \text{ in } \Omega, \qquad A \cdot n = 0 \text{ on } \partial\Omega. \tag{2.3}$$

The reason is as follows: There is a solution $\phi$ (unique up to a constant) to the oblique derivative problem

$$\Delta\phi = -\nabla \cdot A \text{ in } \Omega, \qquad \nabla\phi \cdot n = -A \cdot n \text{ on } \partial\Omega. \tag{2.4}$$

See [GT83, Theorem 6.31 and the following remark]. Setting $A' = A + \nabla\phi$, the operator $H_{A',V}$ is unitarily equivalent to $H_{A,V}$ under the gauge transformation $e^{i\phi}$, and $A'$ satisfies the properties (2.3).

*Proof of Eq.* (1.5). Let $A$ and $A'$ be magnetic vector potentials with circulations that differ by an element of $\mathbb{Z}^k$. For any closed path $\sigma$,

$$\frac{1}{2\pi} \oint_\sigma (A' - A) \cdot \mathrm{d}\mathbf{x} \in \mathbb{Z},$$

and hence there exists a smooth, multivalued function $\phi$ such that $e^{i\phi}$ is univalued and $\nabla\phi = A' - A$. For $u \in H^1(\Omega)$ we have

$$(i\nabla + A')e^{i\phi}u = e^{i\phi}(i\nabla + A)u,$$

and therefore the operators $H_{A,V}$ and $H_{A',V}$ are unitarily equivalent. $\square$

*Remark 2.2.* For any magnetic vector potential $A$ satisfying (1.2) there exists a gauge function $\phi$ such that

$$A(x,y) - \sum_{i=1}^{k} \frac{\Phi_i}{2\pi r_i^2} \begin{pmatrix} -y + y_i \\ x - x_i \end{pmatrix} = (\nabla\phi)(x,y),$$

where $(x_i, y_i)$ is a fixed point in the $i^{\text{th}}$ hole, $r_i^2 = (x - x_i)^2 + (y - y_i)^2$ and $\Phi_i$ is the circulation of $A$ round the $i^{\text{th}}$ hole. Defining

$$A'(x,y) = \sum_{i=1}^{k} \frac{\Phi_i}{2\pi r_i^2} \begin{pmatrix} -y + y_i \\ x - x_i \end{pmatrix},$$

we see, for a fixed $V$, that

$$H_{A',V} = e^{-i\phi} H_{A,V} e^{i\phi}$$

and thus $H_{A,V}$ is unitarily equivalent to $H_{A',V}$. This means that the magnetic vector potential is determined up to a gauge transformation by its circulations $\Phi$, and verifies that the spectrum of $H_{A,V}$ is determined by $\Phi$.

*Proof of Eq.* (1.6). Let $A$ be a magnetic vector potential with circulation $\Phi$, and let $u$ be a groundstate of $H_{A,V}$. It is easy to show that $\overline{u}$ is a groundstate of $H_{-A,V}$ with the same eigenvalue, and hence

$$\lambda_1(-\Phi) = \lambda_1(\Phi). \tag{2.5}$$

We obtain Eq. (1.6) by combining (2.5) and (1.5) as follows:

$$\lambda_1(l/2 + \Phi) = \lambda_1(-l/2 - \Phi) = \lambda_1(l/2 - \Phi). \qquad \square$$

*Proof of Inequality* (1.7). Suppose for a contradiction that $\Phi \notin \mathbb{Z}^k$ and that $\lambda_1(\Phi) \leq \lambda_1(0)$, where $\Phi$ is the circulation vector of some magnetic vector potential $A$. Let $u_0$ denote the unique normalised positive groundstate of the operator $H_{0,V}$ and let $u_A$ be a normalised groundstate of the operator $H_{A,V}$. Using the diamagnetic inequality [Sim79] we have

$$Q_{0,V}(|u_A|) \leq Q_{A,V}(u_A) = \lambda_1(\Phi) \leq \lambda_1(0) = Q_{0,V}(u_0), \tag{2.6}$$

and thus $|u_A| = u_0$. It follows that $u_A = e^{i\phi}u_0$ for some smooth, real valued, multivalued function $\phi$, and hence

$$\begin{aligned}
\int_\Omega |A - \nabla\phi|^2 |u_0|^2 \mathrm{d}^2x &= \int_\Omega |(i\nabla + A - \nabla\phi)u_0|^2 \mathrm{d}^2x - \int_\Omega |\nabla u_0|^2 \mathrm{d}^2x \\
&= \int_\Omega |(i\nabla + A)u_A|^2 \mathrm{d}^2x - \int_\Omega |\nabla u_0|^2 \mathrm{d}^2x \\
&= Q_{A,V}(u_A) - Q_{0,V}(u_0) \\
&= 0,
\end{aligned}$$

and therefore $A = \nabla\phi$ in $\Omega$. Thus for each $i = 1, \ldots, k$ we have

$$\Phi_i = \frac{1}{2\pi} \oint_{\sigma_i} A \cdot \mathrm{d}\mathbf{x} = \frac{1}{2\pi} \oint_{\sigma_i} \mathrm{d}\phi \in \mathbb{Z},$$

where $\sigma_i$ is a closed path which parametrises the boundary $\Sigma_i$ of the $i^{\text{th}}$ hole and turns once in an anticlockwise direction. This contradicts our assumption that $\Phi \notin \mathbb{Z}^k$. $\square$

The proof of inequality (1.7) is an alternative to the proofs given in [LO77] and [Hel88a]. It has the advantage of being simpler and being independent of whether the boundary conditions are Neumann or Dirichlet. See also [HN97].

We leave the proof of inequality (1.8) until Sect. 4 because it depends on Theorem 1.4 (i).

## 3. A Twofold Riemannian Covering Manifold

In this section we consider the case where the circulations of the magnetic vector potential $A$ satisfy

$$\Phi_i \in 1/2 + \mathbb{Z} \tag{3.1}$$

for each $1 \leq i \leq k$. The proofs of our results use a twofold Riemannian covering manifold $\tilde{\Omega}$ of the domain $\Omega$ (see Remark 3.4 however). For the case of more than one hole, there exists more than one twofold Riemannian covering manifold of $\Omega$. We shall take a particular choice of covering manifold on which the circulation of the lifted magnetic (1-form) potential $\tilde{A}$ along any closed curve is an integer. Before the precise definition, we introduce some basic notation. For further details see for example [Kos80] or [GHL90].

*Notation 3.1.* Let $\tilde{\Omega}$ be a covering manifold of $\Omega$, and let $\Pi$ be the associated covering map. We denote the lifts of various quantities as follows:

For a set $\mathcal{N}$ define $\tilde{\mathcal{N}} = \{x \in \tilde{\Omega} : \Pi(x) \in \mathcal{N}\}$. For a function $f : \Omega \to \mathbb{C}$, define $\tilde{f} : \tilde{\Omega} \to \mathbb{C}$ by $\tilde{f} = f \circ \Pi$. For a path $\sigma : [0,1] \to \Omega$ and a point $x \in \tilde{\Omega}$ such that $\Pi(x) = \sigma(0)$ let $\tilde{\sigma} : [0,1] \to \tilde{\Omega}$ denote the unique lifted path such that $\tilde{\sigma}(0) = x$ and $\Pi \circ \tilde{\sigma} = \sigma$.

We endow the covering manifold with the metric obtained by lifting the flat Euclidean metric of $\Omega$ to $\tilde{\Omega}$. This is the unique metric which makes $\Pi$ a local isometry, and therefore a Riemannian covering map. Let $\tilde{\Delta} = \operatorname{div}\operatorname{grad}$ denote the Laplace-Beltrami operator on $L^2(\tilde{\Omega})$ induced by the lifted metric on $\Omega$, and let $\tilde{A}$ be the 1-form on $\tilde{\Omega}$ obtained by lifting the 1-form associated with the smooth vector potential $A$ defined on $\Omega$.

Let $\tilde{\Omega}_\infty$ be the universal covering manifold of $\Omega$ and let $\Pi_\infty$ be the associated covering map. The universal covering of any manifold is simply connected.

Note that due to (3.1) if two points $x_\infty, y_\infty \in \tilde{\Omega}_\infty$ satisfy $\Pi_\infty(x_\infty) = \Pi_\infty(y_\infty)$ then for any path $\sigma$ joining $x_\infty$ to $y_\infty$, the integral

$$\frac{1}{2\pi} \oint_{\Pi_\infty \circ \sigma} A \cdot \mathrm{d}\mathbf{x} \tag{3.2}$$

lies either in $1/2 + \mathbb{Z}$ or in $\mathbb{Z}$. The value of (3.2) is independent of the path $\sigma$ because $\operatorname{curl} A = 0$ and because the universal covering manifold is simply connected. We therefore construct the twofold covering manifold (as a quotient of the universal covering manifold) as follows:

**Definition 3.2.** (i) *We define the twofold covering manifold* $\tilde{\Omega}$ *by identifying points* $x_\infty$, $y_\infty$ *in* $\tilde{\Omega}_\infty$ *according to the equivalence relation* $x_\infty \sim y_\infty$ *if and only if*

$$\Pi_\infty(x_\infty) = \Pi_\infty(y_\infty) \tag{3.3}$$

*and for each path* $\sigma$ *in* $\tilde{\Omega}_\infty$ *joining* $x_\infty$ *to* $y_\infty$ *we have*

$$\frac{1}{2\pi}\int_{\Pi_\infty \circ \sigma} A \cdot \mathrm{d}\mathbf{x} \in \mathbb{Z}. \tag{3.4}$$

*The covering map* $\Pi : \tilde{\Omega} \to \Omega$ *is defined by* $\Pi(x) = \Pi_\infty(x_\infty)$, *where* $x = [x_\infty]$ *is the equivalence class (under* $\sim$*) containing* $x_\infty$.

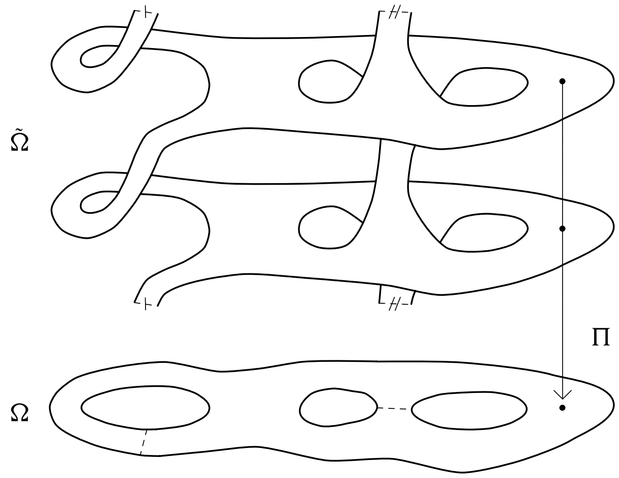


**Fig. 2.** Realization of a twofold covering manifold

(ii) *On our twofold covering manifold we define the symmetry map* $G : \tilde{\Omega} \to \tilde{\Omega}$ *by setting* $Gx$ *to be the other point in* $\tilde{\Omega}$ *which lies above* $\Pi(x) \in \Omega$. *Note that* $\Pi^{-1}(\Pi(x)) = \{x, Gx\}$.

(iii) *We say that a function* $f : \tilde{\Omega} \to \mathbb{C}$ *is symmetric if* $f(Gx) = f(x)$ *for all* $x \in \tilde{\Omega}$, *and antisymmetric if* $f(Gx) = -f(x)$ *for all* $x \in \tilde{\Omega}$.

Note that the identity map and $G$ form a group $\mathscr{G} = \{I, G\}$, with the composition $G^2 = I$, which acts freely on $\tilde{\Omega}$. The quotient of $\tilde{\Omega}$ by $\mathscr{G}$ is the original manifold $\Omega$. The lift $\tilde{f}$ of a function $f$ on $\Omega$ is symmetric.

Using Eq. (3.4) we have

$$\frac{1}{2\pi}\oint_\sigma \tilde{A} \cdot \mathrm{d}\tilde{\mathbf{x}} = \frac{1}{2\pi}\oint_{\Pi \circ \sigma} A \cdot \mathrm{d}\mathbf{x} \in \mathbb{Z}, \tag{3.5}$$

for any closed path $\sigma$ in $\tilde{\Omega}$. Hence there exists a smooth, multivalued function $\theta$ on $\tilde{\Omega}$ such that $\exp i\theta$ is univalued and

$$\operatorname{grad}\theta = \tilde{A}. \tag{3.6}$$

**Lemma 3.3.** *The operator $\mathscr{L} : L^2(\Omega) \to L^2(\tilde{\Omega})$ defined by*

$$\mathscr{L}u = \frac{1}{\sqrt{2}} e^{i\theta} \tilde{u} \tag{3.7}$$

*is a isometry onto the antisymmetric functions in $L^2(\tilde{\Omega})$, and maps eigenfunctions of $H_{A,V}$ onto antisymmetric eigenfunctions of the Schrödinger operator*

$$\tilde{H}_{0,V} = -\tilde{\Delta} + \tilde{V} \tag{3.8}$$

*acting on $L^2(\tilde{\Omega})$ with Neumann boundary conditions.*

*Proof.* We shall first show that the function $e^{i\theta}$ is antisymmetric (under $G$). For any point $x \in \tilde{\Omega}$, let $\sigma : [0,1] \to \tilde{\Omega}$ be a path which joins $x$ to $Gx$. Using the terminology of Definition 3.2 we have $\Pi(x) = \Pi(Gx)$ but $x \not\sim Gx$, and hence

$$\frac{1}{2\pi} \oint_{\Pi\circ\sigma} A \cdot \mathrm{d}\mathbf{x} = l + 1/2$$

for some $l \in \mathbb{Z}$. Keeping in mind that $\theta$ is multivalued, we get

$$\theta(Gx) - \theta(x) = \int_\sigma \mathrm{d}\theta = \int_\sigma \tilde{A} \cdot \mathrm{d}\tilde{\mathbf{x}} = \oint_{\Pi\circ\sigma} A \cdot \mathrm{d}\mathbf{x} = (2l+1)\pi.$$

Hence $\exp[i\theta(Gx)] = -\exp[i\theta(x)]$ as claimed.

The action of $\mathscr{L}$ upon a function $u \in L^2(\Omega)$ consists of two steps. The first step is to lift $u$ to the symmetric function $\tilde{u}$. This is a bijection onto the space of symmetric functions of $L^2(\tilde{\Omega})$. The second step is to multiply $\tilde{u}$ by the antisymmetric function $e^{i\theta}$. This step is a bijection from the space of symmetric functions onto the space of antisymmetric functions in $L^2(\tilde{\Omega})$. To see that $\mathscr{L}$ is an isometry onto its range, we take two functions $u, v \in L^2(\Omega)$ and note that

$$\langle \mathscr{L}u, \mathscr{L}v \rangle_{L^2(\tilde{\Omega})} = \frac{1}{2} \int_{\tilde{\Omega}} e^{i\phi} \tilde{u}.e^{-i\phi} \overline{\tilde{v}} \mathrm{d}\tilde{x} = \int_\Omega u\overline{v} \mathrm{d}x = \langle u, v \rangle_{L^2(\Omega)}.$$

For every eigenfunction $u$ of $H_{A,V}$, the lift $\tilde{u}$ is an eigenfunction of the lifted magnetic Schrödinger operator

$$\tilde{H}_{A,V} = (i \operatorname{div} + \tilde{A})(i \operatorname{grad} + \tilde{A}) + \tilde{V} \tag{3.9}$$

on $\tilde{\Omega}$ where $\tilde{V}$ and $\tilde{A}$ are the lifts of $V$ and $A$ respectively. We now multiply by the gauge $e^{i\theta}$. Using Eq. (3.6), the function $e^{i\theta}\tilde{u}$ is an eigenfunction of the non-magnetic Schrödinger operator $\tilde{H}_{0,V}$. $\square$

The spectrum of $H_{A,V}$ consists of the eigenvalues corresponding to the antisymmetric eigenfunctions of $\tilde{H}_{0,V}$. It turns out to be useful (see Lemma 4.4) to single out the case where a function $u$ has the following property:

**Property P.** *The function $u$ is a groundstate of the operator $H_{A,V}$, and the corresponding eigenfunction $\mathscr{L}u$ of $\tilde{H}_{0,V}$ has a constant phase. In other words, there exists a constant $\alpha \in \mathbb{C} \setminus \{0\}$ such that $\mathscr{L}(\alpha u)$ is a real valued function.*

Due to the symmetry of $\tilde{\Omega}$, the groundstate of the operator $\tilde{H}_{0,V}$ is symmetric. In contrast, if $u$ has Property P then $\mathscr{L}(\alpha u)$ is an antisymmetric eigenfunction (and therefore an excited state) of $\tilde{H}_{0,V}$. Consequently both $\mathscr{L}(\alpha u)$ and $u$ have a nonempty nodal set.

*Remark 3.4.* It is not necessary to use the covering manifold to describe Property P. An alternative is to formulate the property in terms of an antilinear operator $K$. We define the operator below.

Since $\Phi_i \in 1/2 + \mathbb{Z}$ for each $i = 1, \dots, k$, we see that

$$\frac{1}{2\pi} \oint_\sigma 2A \cdot \mathrm{d}\mathbf{x} \in \mathbb{Z}$$

for all closed paths $\sigma$ in $\Omega$. It follows that there exists a smooth, multivalued function $\psi$ such that $e^{i\psi}$ is univalued and $\nabla\psi = 2A$. The multivalued function $\theta$ given in Eq. (3.6) is related to $\psi$ by the formula

$$\psi \circ \Pi = 2\theta + c$$

for some constant $c$. We define $K$ by the formula

$$K = e^{-i\psi}\Gamma, \tag{3.10}$$

where $\Gamma$ is the operator $\Gamma u = \overline{u}$. Then $K^2 = \mathrm{Id}$ and $K$ commutes with $H_{A,V}$. It turns out that a function $u \in L^2(\Omega)$ has Property P if and only if it is an eigenfunction of both $H_{A,V}$ and $K$.

One could in fact completely dispense with the covering manifold, but at the expense of a clear geometrical picture in the following sections.

## 4. Characterisation of the Nodal Set

We first collect some well known facts about eigenfunctions of non-magnetic Schrödinger operators acting on two dimensional Riemannian manifolds:

**Proposition 4.1 (Non-magnetic Schrödinger operators).** *Let $f$ be a real valued eigenfunction of a non-magnetic Schrödinger operator with smooth potential and Neumann boundary conditions, on a two dimensional locally flat Riemannian manifold $\Omega$ with smooth boundary. Then $f \in C^\infty(\overline{\Omega})$. Furthermore, $f$ has the following properties:*

(i) *If $f$ has a zero of order $l$ at a point $x_0 \in \overline{\Omega}$ then the Taylor expansion of $f$ is*

$$f(x) = p_l(x - x_0) + O(|x - x_0|^{l+1}), \tag{4.1}$$

*where $p_l$ is a real valued, non-zero, harmonic, homogeneous polynomial of degree $l$.*

*Moreover if $x_0 \in \partial\Omega$, the Neumann boundary conditions imply that*

$$f(x) = ar^l \cos l\omega + O(r^{l+1}) \tag{4.2}$$

*for some non-zero $a \in \mathbb{R}$, where $(r, \omega)$ are polar coordinates of $x$ around $x_0$. The angle $\omega$ is chosen so that the tangent to the boundary at $x_0$ is given by the equation* $\sin\omega = 0$.

(ii) *The nodal set $\mathcal{N}(f)$ is the union of finitely many, smoothly immersed circles in $\Omega$, and smoothly immersed lines which connect points of $\partial\Omega$. Each of these immersions is called a nodal line. Note that self-intersections are allowed. The connected components of $\Omega \setminus \mathcal{N}(f)$ are called nodal domains.*

(iii) *If $f$ has a zero of order $l$ at a point $x_0 \in \Omega$ then exactly $l$ segments of nodal lines pass through $x_0$. The tangents to the nodal lines at $x_0$ dissect the full circle into $2l$ equal angles.*
*If $f$ has a zero of order $l$ at a point $x \in \partial\Omega$ then exactly $l$ segments of nodal lines meet the boundary at $x_0$. The tangents to the nodal lines at $x_0$ are given by the equation* $\cos l\omega = 0$*, where $\omega$ is chosen as in* (4.2).

*Proof.* The proof that $f \in C^\infty(\overline{\Omega})$ can be found in [Wlo82, Theorem 20.4].

The proof of part (i) is trivial because $V$ and $f$ are smooth functions so the Taylor expansion (with remainder) exists. The properties of the first term of the expansion follow by substituting the Taylor expansion into the groundstate eigenvalue equation.

See [Ber55, Che76] for proofs of the other parts. □

Proposition 4.1 can be generalised to include eigenfunctions of magnetic Schrödinger operators with a smooth magnetic vector potential $A$. The eigenfunctions still lie in $C^\infty(\overline{\Omega})$ and the expansions (4.1) and (4.2) hold, except that the polynomial $p_l$ and the constant $a$ are allowed to be complex. However statements (ii) and (iii) about the nodal set do not carry over.

**Theorem 4.2.** *Let $\mathcal{N} \subset \Omega$ be the union of finitely many smoothly immersed circles and smoothly immersed lines which connect points of $\partial\Omega$. The following statements are equivalent:*

(i) *$\Omega \setminus \mathcal{N}$ is connected (therefore $\mathcal{N}$ contains no smoothly immersed circles), and an odd number of lines emanate from each hole.*

(ii) *In the twofold covering manifold, the open set $\tilde{\Omega} \setminus \tilde{\mathcal{N}}$ decomposes into two open path connected subsets $D_1, D_2$ such that $D_2 = GD_1$ and $\partial D_1 \cap \tilde{\Omega} = \partial D_2 \cap \tilde{\Omega} = \tilde{\mathcal{N}}$.*

*Proof.* (i) ⇒ (ii) Let $D_1$ be a connected component of $\tilde{\Omega} \setminus \tilde{\mathcal{N}}$. Suppose for a contradiction that this is the only component. Due to the symmetry of the manifold, $GD_1 = D_1$, and thus for any point $x \in D_1$ there exists a path $\sigma$ lying in $D_1$ (i.e. not intersecting $\tilde{\mathcal{N}}$), which joins $x$ and $Gx$. Using the terminology of Definition 3.2 we have $\Pi(x) = \Pi(Gx)$ but $x \not\sim Gx$, and hence

$$\frac{1}{2\pi} \oint_{\Pi\circ\sigma} A \cdot d\mathbf{x} \in 1/2 + \mathbb{Z}.$$

The closed path $\Pi \circ \sigma$ must therefore circulate an odd number of holes. Since an odd number of lines of $\mathcal{N}$ emanate from each hole, the path $\Pi \circ \sigma$ must intersect with one of them. This contradicts the fact that $\sigma$ does not intersect $\tilde{\mathcal{N}}$.

Since $\Omega \setminus \mathcal{N}$ is connected there can only be two connected components $D_1, D_2$ of $\tilde{\Omega} \setminus \tilde{\mathcal{N}}$. As above, we see that $GD_1 \neq D_1$, and therefore $D_2 = GD_1$.

Suppose now for a contradiction that $\partial D_1 \cap \tilde{\Omega} \neq \tilde{\mathcal{N}}$. Then there exists a point $x \in \partial D_1 \cap \tilde{\Omega}$ such that $x \notin \partial D_2 \cap \tilde{\Omega}$. The set $D_1$ borders with itself at $x$, and since $D_1$ is path connected there exists a closed path $\sigma$ such that $\sigma(0) = \sigma(1) = x$, which intersects $\tilde{\mathcal{N}}$ transversally at $x$ and which does not intersect $\tilde{\mathcal{N}}$ anywhere else. Since $\sigma$ is closed,

$$\oint_{\Pi\circ\sigma} A \cdot d\mathbf{x} \in \mathbb{Z},$$

and therefore $\Pi \circ \sigma$ circulates an even number of holes. Since an odd number of lines emanate from each hole, $\Pi \circ \sigma$ intersects $\mathcal{N}$ an even number of times. This contradicts the fact that $\sigma$ intersects $\tilde{\mathcal{N}}$ only once.

(ii) ⇒ (i) Since $D_2 = GD_1$ we see that $\Pi D_1 = \Pi D_2$, and hence $\Omega \setminus \mathcal{N} = \Pi(D_1 \cup D_2) = \Pi D_1 \cup \Pi D_2 = \Pi D_1$. Since $\Pi$ is continuous, $\Omega \setminus \mathcal{N}$ is connected. Let $\sigma \subset \Omega$ be a closed path which circulates the $i^{\text{th}}$ hole. Due to the construction of $\tilde{\Omega}$, $\sigma$ may be lifted to a path $\tilde{\sigma}$ in $\tilde{\Omega}$ which begins at a point $x \in D_1$ and ends at $Gx \in D_2$. Since $D_1$ and $D_2$ coborder, the path $\tilde{\sigma}$ crosses $\tilde{\mathcal{N}}$ an odd number of times and therefore $\sigma$ crosses $\mathcal{N}$ an odd number of times. By choosing $\sigma \subset \Omega$ sufficiently close to $\sigma_i$ we see that an odd number of segments of lines leave the $i^{\text{th}}$ boundary component. Since $\Omega \setminus \mathcal{N}$ is connected, each of these line endings belongs to a distinct line, and hence an odd number of lines leaves each boundary component. □

**Corollary 4.3.** *Suppose that a collection of paths slits a region. Then no subcollection of these paths can slit the region. Also, no supercollection of these paths (i.e. a collection of paths which contain the original collection) can slit the region.*

*Proof.* Suppose that the union $\mathcal{N}$ of a collection of lines $\{\Gamma_1, \dots, \Gamma_n\}$ slits $\overline{\Omega}$. Using Theorem 4.2, we see that in the twofold covering manifold the open set $\tilde{\Omega}\setminus\tilde{\mathcal{N}}$ decomposes into two cobordering, open, path connected subsets $D_1, D_2$. Let $S$ be the union of a strict subcollection of the lines. The non-empty set $\tilde{\mathcal{N}} \setminus \tilde{S}$ connects together the two regions $D_1$ and $D_2$ and thus $\tilde{\Omega} \setminus \tilde{S} = D_1 \cup D_2 \cup (\tilde{\mathcal{N}} \setminus \tilde{S})$ is connected. Using Theorem 4.2 in the reverse direction, we see that $S$ does not slit $\overline{\Omega}$.

It follows easily that no supercollection of $\mathcal{N}$ can slit because then $\mathcal{N}$ would be a strict subset of $S$ which slits $\overline{\Omega}$, and this is not possible by the above paragraph. □

**Lemma 4.4.** *If a groundstate $u$ of $H_{A,V}$ has Property P then the nodal set of $u$ slits $\overline{\Omega}$.*

*Proof.* By multiplying the function $u$ by a non-zero complex constant we may assume that the eigenfunction $\mathscr{L}u$ of $\tilde{H}_{0,V}$ is real valued. Since $\mathscr{L}u$ is an antisymmetric function on the covering manifold $\tilde{\Omega}$, the nodal domains $D_1, \dots, D_l$ of $\mathscr{L}u$ have the property that for each $i = 1, \dots, l$, we have $GD_i = D_j$ for some $j \neq i$. Suppose for a contradiction that $l > 2$. Then there exist two cobordering domains $D_1, D_2$ such that $GD_1 \neq D_2$. Define $D = \text{Interior}(\overline{D_1 \cup D_2})$, so that $D$ is the union of $D_1$, $D_2$ and the border between them. Let $\tilde{Q}^D_{0,V}$ denote the quadratic form corresponding to the Schrödinger operator

$$\tilde{H}^D_{0,V} = -\tilde{\Delta} + \tilde{V}$$

on $D$ with Dirichlet boundary conditions on $\tilde{S} = \partial D \cap \tilde{\Omega}$ and Neumann boundary condition $\partial D \cap \partial\tilde{\Omega}$, and let $g$ denote the corresponding positive groundstate. Since the boundary of $D$ is piecewise smooth, the restriction $\mathscr{L}u|_D$ lies in the quadratic form domain of $\tilde{Q}^D_{0,V}$. Define the antisymmetric function $h$ on $\tilde{\Omega}$ by

$$h(y) = \begin{cases} g(y), & y \in D, \\ -g(Gy), & y \in GD, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\tilde{Q}_{0,V}$ denote the quadratic form of the operator $\tilde{H}_{0,V}$, which we define in Eq. (3.8). Since $\mathscr{L}u$ is an antisymmetric eigenfunction which corresponds to a groundstate of $H_{A,V}$, it has the least energy of all antisymmetric functions, and therefore

$$\frac{\tilde{Q}_{0,V}(\mathscr{L}u)}{\|\mathscr{L}u\|^2_{L^2(\tilde{\Omega})}} \le \frac{\tilde{Q}_{0,V}(h)}{\|h\|^2_{L^2(\tilde{\Omega})}} = \frac{\tilde{Q}^D_{0,V}(g)}{\|g\|^2_{L^2(D)}} \le \frac{\tilde{Q}^D_{0,V}(\mathscr{L}u|_D)}{\|\mathscr{L}u|_D\|^2_{L^2(D)}} = \frac{\tilde{Q}_{0,V}(\mathscr{L}u)}{\|\mathscr{L}u\|^2_{L^2(\tilde{\Omega})}}. \quad (4.3)$$

We have in fact equality in (4.3), and therefore, by uniqueness of the groundstate, we have that $\mathscr{L}u|_D = \lambda g$ for some $\lambda \neq 0$. This contradicts the fact that $\mathscr{L}u|_D$ is zero on $\partial D_1 \cap D$. Hence $l = 2$. This means that the nodal set $\mathcal{N}$ of $u$ satisfies statement (ii) in Theorem 4.2. Using the equivalence proved in Theorem 4.2 we see that parts (iii) and (iv) of the definition of slitting are satisfied. Parts (i) and (ii) follow from the fact that $u$ can be approximated locally by harmonic polynomials. See Proposition 4.1. $\square$

*Proof of Theorem 1.4 (i).* Let $U$ denote the groundstate eigenspace of $H_{A,V}$. For all $u \in U$ we have Re$[\mathscr{L}u]$, Im$[\mathscr{L}u] \in \mathscr{L}U$ are eigenfunctions of $\tilde{H}_{0,V}$, if they are not identically zero. It follows that we may find an orthonormal basis $\{f_1, \ldots, f_m\}$ of real valued functions for $\mathscr{L}U$. Since $\mathscr{L}$ is an isometry, the functions $\{u_1, \ldots, u_m\}$ defined by $u_i = \mathscr{L}^{-1} f_i$ are an orthonormal basis of $U$.

Now let $u = \sum_{i=1}^m \alpha_i u_i$, where $\alpha_i \overline{\alpha}_j \in \mathbb{R}$ for each $1 \le i, j \le m$. Take some $\alpha_j \neq 0$. Then

$$\mathscr{L}(\overline{\alpha_j} u) = \sum_{i=1}^m \alpha_i \overline{\alpha_j} f_i$$

is a real valued function, and so $u$ has Property P. The result now follows from Lemma 4.4. $\square$

**Lemma 4.5.** *If a groundstate $u$ of $H_{A,V}$ has a zero of order $l$ at a point $x \in \partial\Omega$ then $l \le k$. Moreover, if $k$ is even and $x$ lies on an interior boundary component ($\Sigma_1$, say) then $l \le k - 1$.*

*Proof.* Assume first that $u$ has Property P, and suppose for a contradiction that $l \ge k+1$. Let $\Sigma_i$ denote the boundary component on which $x$ lies, where $i \in \{0, 1, \ldots, k\}$. At least $k + 1$ distinct nodal lines emerge from $\Sigma_i$. Since there are only $k$ boundary components distinct from $\Sigma_i$ there must exist two nodal lines which both start at $\Sigma_i$ and finish at $\Sigma_j$ for some $j \neq i$. In both cases, such a nodal set would split $\overline{\Omega}$ into more than one nodal domain, thus contradicting the assumption that $\mathcal{N}(u)$ slits $\overline{\Omega}$. Hence $l \le k$.

If $u$ does not have Property P then we can obtain a contradiction using the same methods above on the function $\mathscr{L}^{-1}$[Re$[\mathscr{L}u]$]. This function is a groundstate of $H_{A,V}$, has a zero of order at least $l$ at $x$, and does have Property P.

Suppose that $k$ is even, that $x \in \Sigma_i$ (with $i \in \{1, \ldots, k\}$) and that $l = k$. Since $\mathcal{N}(u)$ slits $\overline{\Omega}$ there must be an odd number of nodal lines leaving $\Sigma_i$. Therefore at least $k + 1$ nodal lines leave $\Sigma_i$, and we obtain a contradiction as before. $\square$

**Lemma 4.6.** *Suppose that the groundstate eigenspace $U$ of $H_{A,V}$ is $m$ dimensional.*

(i) *For each point $x \in \partial\Omega$ there exists a function $u_x \in U$ which has Property P and which has a zero of order at least $m - 1$ at $x$.*
(ii) *If $m = k+1$ then for each point $x$ lying on the outer boundary $\Sigma_0$ of $\overline{\Omega}$ there exists a unique $u_x \in U$ (up to multiplication by a complex constant) which has a zero of order $k$ at $x$. The function $u_x$ has Property P. The nodal set of $u_x$ consists of $k$ lines which emanate from $x$ (which is the only point of intersection of lines), and which end at each of the $k$ distinct interior boundary components of $\Omega$. Each nodal line depends smoothly on $x$.*

(iii) *If $k$ is even and $m = k$ then for each point $x$ lying on an interior component of the boundary of $\overline{\Omega}$ there exists a unique $u_x \in U$ (up to multiplication by a complex constant) which has a zero of order $k-1$ at $x$. The function $u_x$ has Property P.*

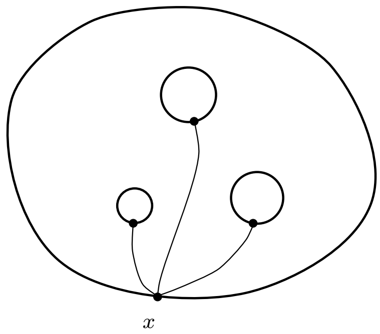


**Fig. 3.**

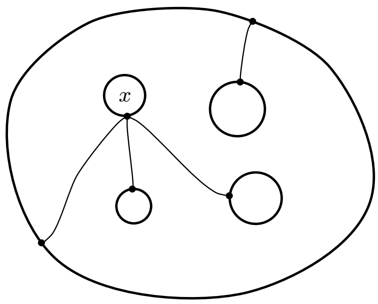


**Fig. 4.**

For pictorial representations of cases (ii) and (iii), see Figs. 3 and 4 respectively.

*Proof.* (i) We shall first prove by induction the following statement: If $U_m$ is an $m$ dimensional vector space of groundstates of $H_{A,V}$ then for each point $x \in \partial\Omega$ there exists a function $f \in U_m$ which has a zero of order at least $m-1$ at $x$.

The first step of the induction, for $m = 1$, is trivial. Assume now that the above statement is true for some general $m$. Suppose that $U_{m+1}$ is an $m+1$ dimensional vector space of groundstates of $H_{A,V}$. Let $U_m$ be any $m$ dimensional subspace of $U_{m+1}$. Then there exists a function $f_1 \in U_m$ which has a zero of order at least $m-1$ at $x$. We can assume that the order of the zero is exactly $m-1$, otherwise we have found a function with a zero of order at least $m$, and the argument for the induction step would finish. Now take

$$U'_m = \{f \in U_{m+1} : f \perp f_1\}.$$

By the same argument, there exists a function $f_2 \in U'_m$ which has a zero of order $m-1$ at $x$. Using the Taylor expansions

$$f_i(r,\omega) = a_i r^{m-1} \cos(m-1)\omega + O(r^m) \qquad i = 1, 2,$$

(written in polar coordinates based at $x$, with $a_i \in \mathbb{C} \setminus \{0\}$), we see that the function $f = a_2 f_1 - a_1 f_2$ is not identically zero, and has a zero of order at least $m$ at $x$. This finishes the induction step.

If $f$ has Property P then we choose $u = f$. Otherwise, if $f$ does not have Property P then $\mathrm{Re}[\mathscr{L} f]$ is not identically zero, and has a zero of order at least $m-1$ at points $y \in \tilde{\Omega}$ such that $\Pi(y) = x$. Using Lemma 3.3 we see that $u := \mathscr{L}^{-1}(\mathrm{Re}[\mathscr{L} f])$ has Property P, and has a zero of order at least $m-1$ at $x$.

(ii) For this part we consider the case $m = k+1$ and take any point $x \in \Sigma_0$. Part (c) shows that there exists a function $u_x \in U$ with Property P and which has a zero of order at least $k$ at $x$. Lemma 4.5 shows that the zero is of order $k$, and therefore $k$ nodal lines emanate from $x$. To prove uniqueness, suppose that $v_x$ is a linearly independent function which also has a zero of order $k$ at $x$. As above, using the Taylor expansions of $u_x$ and $v_x$ at $x$, we may find a linear combination of $u_x$ and $v_x$ which is not identically zero and which has a zero of order at least $k+1$ at $x$. This contradicts Lemma 4.5.

Due to Lemma 4.4, $\Omega \setminus \mathcal{N}$ is connected, and therefore each pair of nodal lines only intersect at $x$. The nodal lines must also end at distinct interior boundary components.

Since zeros of order larger than 1 only occur at points of intersection of nodal lines, there can only occur zeros of order 1 away from $x$. At such zeros, the gradient of $u_x$ is non-zero. We may multiply $u_x$ by the local gauge $e^{i\phi}$, where $\phi$ is given in Eq. (1.3) to make it a real valued function. The function $w_x = e^{i\phi}u_x$ has locally the same nodal set as $u_x$. Note that $w_x$ depends smoothly on $x$. In order to see this, one should note that a linear combination of eigenfunctions with a zero of order $m-1$ at $x$ can be found by solving a system of linear equations which, by uniqueness (see above), has full rank. Since the gradient of $w_x$ is non-zero at the nodal set away from $x$, the nodal lines depend smoothly on $x$.

(iii) The proof of this part is similar. $\square$

*Proof of Theorem* 1.4 (ii). Let $m$ denote the multiplicity of the first eigenvalue of $H_{A,V}$. Lemma 4.6 (i) shows that for any point $x \in \partial\Omega$ there exists a groundstate of $H_{A,V}$ which has a zero of order $l \geq m-1$ at $x$. Lemma 4.5 shows that $l \leq k$. This gives the universal bound $m \leq k+1$, and in particular shows that for $k = 1$ we have $m \leq 2$.

We consider now the case when $k \geq 2$ and suppose for a contradiction that $m = k+1$. Lemma 4.6 (ii) shows that for each point $x$ lying on $\Sigma_0$ there exists a unique eigenfunction $u_x$ which has a zero of order $k$ at $x$. Since each $u_x$ has Property P, the nodal set of each $u_x$ slits $\overline{\Omega}$. The nodal set of each individual $u_x$ has $k$ nodal lines $\{\Gamma_{x,1}, \ldots, \Gamma_{x,k}\}$, emanating from $x$, and each line ends at a distinct interior boundary component. We may parametrise each line $\Gamma_{x,i}$ by a path $\gamma_{x,i}$ chosen so that $\gamma_{x,i}(0) = x$ and $\gamma_{x,i}(1) \in \Sigma_i$ for each $i$. Each path $\gamma_{x,i}$ varies smoothly with $x$.

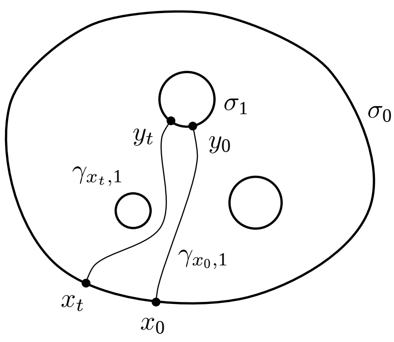


**Fig. 5.**

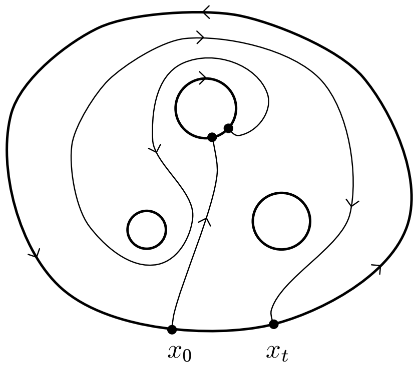


**Fig. 6.**

We shall see that if we move $x$ round the boundary $\Sigma_0$, the nodal sets of the corresponding functions wind round the holes. After one complete turn, we cannot obtain the original nodal set, thus contradicting uniqueness of the original eigenfunction. We obtain the contradiction formally as follows:

Let $\sigma_0$ be a closed path which parametrises the outer boundary component $\Sigma_0$ of $\Omega$, and which turns once in a clockwise direction. For $s \in [0,1]$, let $x_s = \sigma_0(s)$ and let $y_s = \gamma_{x_s,1}(1)$. Since $\sigma_0$ is closed, $x_0 = x_1$. Also, since $\gamma_{x_s,1}$ depends smoothly on $x_s$, which in turn depends smoothly on $s$, the point $y_s$ moves smoothly round the inner boundary component $\Sigma_1$. For a fixed $t \in [0,1]$ define

$$\sigma_{0,t}(s) = \sigma_0(st) = x_{st},$$
$$\sigma_{1,t}(s) = y_{st}.$$

The paths $\sigma_{0,t}$ and $\sigma_{1,t}$ are parametrisations of segments of $\Sigma_0$ and $\Sigma_1$ respectively. Note that $\sigma_{0,1} = \sigma_0$ and $\sigma_{1,1} = \sigma_1^p$ for some $p \in \mathbb{Z}$, where $\sigma_1^p$ means running $p$ times around the closed path $\sigma_1$. For all $t \in [0, 1]$ we have

$$\sigma_{0,t}^{-1} \circ \gamma_{x_t,1}^{-1} \circ \sigma_{1,t} \circ \gamma_{x_0,1} \sim 0, \tag{4.4}$$

where $\circ$ denotes gluing of paths and $\sim$ denotes homotopy. This means that the left hand side of (4.4) is a closed path that does not enclose any holes. See Fig. 5. Setting $t = 1$ we get

$$\sigma_0^{-1} \circ \gamma_{x_0,1}^{-1} \circ \sigma_1^p \circ \gamma_{x_0,1} \sim 0,$$

and therefore

$$\sigma_1^p \sim \gamma_{x_0,1}^{-1} \circ \sigma_1^p \circ \gamma_{x_0,1} \sim \sigma_0.$$

This gives us a contradiction because the path $\sigma_1^p$ is not homotopic to $\sigma_0$. Hence $m \leq k$.

Finally we consider the case where $k$ is even and $k \geq 4$. Let $\overline{\Omega}'$ denote the closure $\overline{\Omega}$ of our region with the points of the outer boundary identified. Let $D_{k-1} \subset \mathbb{R}^2$ denote an open disk with $k - 1$ smaller, disjoint, closed disks removed. There exists a homeomorphism

$$X : \overline{\Omega}' \to \overline{D}_{k-1} \tag{4.5}$$

such that $X$ restricted to $\Omega$ is smooth, and such that the boundary component $\Sigma_1$ maps

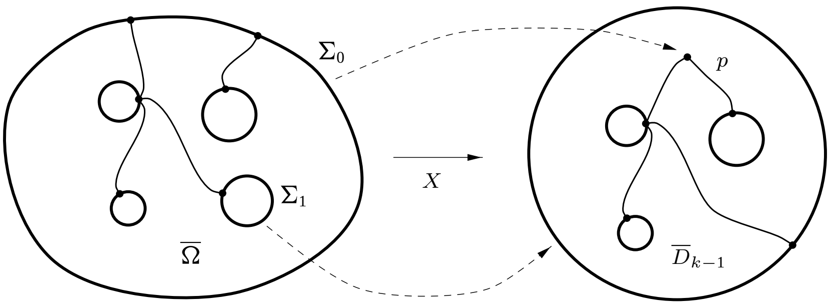


**Fig. 7.**

to the outer boundary of $\overline{D}_{k-1}$. See Fig. 7. One can imagine $X$ as a composition of mapping $\overline{\Omega}$ onto the surface of a sphere, deforming it so that $\Sigma_1$ becomes very large and $\Sigma_0$ very small, and then finally pulling off the sphere. Let $p := X(\Sigma_0) \in D_{k-1}$, so that $X(\Omega) = D_{k-1} \setminus \{p\}$.

Let $\mathcal{N}$ be a set which slits $\overline{\Omega}$. We claim that $X(\mathcal{N})$ slits $\overline{D}_{k-1}$. For since $k$ is even, the number of nodal lines hitting the outer boundary component $\Sigma_0$ is even (possibly zero). This corresponds to an even number of paths in $X(\mathcal{N})$ starting or finishing at $p$. These paths can be paired together to link distinct boundary components. Since $X^{-1}$ is a smooth bijection away from $p$, the resulting paths are still piecewise smooth. It is easy to verify that all the other slitting conditions are satisfied.

Suppose for a contradiction that $m = k$. For $s \in [0,1]$, let $x_s = \sigma_1(s)$ be a point on the interior boundary component $\Sigma_1$ of $\Omega$. Lemma 4.6 (iii) shows that there exists a unique $u_{x_s} \in U$ (up to multiplication by a complex constant) which has a zero of order $k-1$ at $x_s$. The nodal set $\mathcal{N}(u_{x_s})$ consists of $k-1$ nodal lines emanating from $x_s$. As shown above, the set $S_s := X(\mathcal{N}(u_{x_s}))$ slits $\overline{D}_{k-1}$ and consists of $k-1$ lines emanating from the point $y_s = X(x_s)$ on the outer boundary of $\overline{D}_{k-1}$.

We have thus constructed a family of slitting sets $S_s$ which depends continuously on the parameter $s \in [0,1]$, and such that $S_0 = S_1$. By moving the point $y_s$ round the outer boundary of $\overline{D}_{k-1}$ and using the homotopy argument above, we obtain a similar contradiction. Hence $m \leq k-1$. □

*Proof of Theorem* 1.4 (iii). Suppose that $k = 1$ and that the multiplicity of the first eigenvalue is two. Suppose for a contradiction that there exist two linearly independent groundstates $v_1$ and $v_2$ such that the set $S = \mathcal{N}(v_1) \cap \mathcal{N}(v_2)$ is non-empty, and let $z$ be any point in $S$. Since $\{v_1, v_2\}$ is a basis of the groundstate eigenspace $U$ of $H_{A,V}$, the nodal set of every function $u \in U$ contains the point $z$.

From Lemma 4.6 (ii) we see that for each point $x$ on the outer boundary $\Sigma_0$ of $\Omega$ there exists a unique eigenfunction $u_x \in U$ such that $x \in \mathcal{N}(u_x)$. If we start $x$ at the point $x_0 = \sigma_0(0)$ and then move $x$ continuously round the outer boundary $\Sigma_0$ once in a clockwise direction then the segment of the nodal line joining $x$ to $z$ deforms continuously and winds around the inner boundary $\Sigma_1$ (see Fig. 8). The resulting nodal line is different from the original, thus contradicting uniqueness of the eigenfunction $u_{x_0}$. This argument can be formalised using a homotopy argument similar to that found in the proof of part (ii).

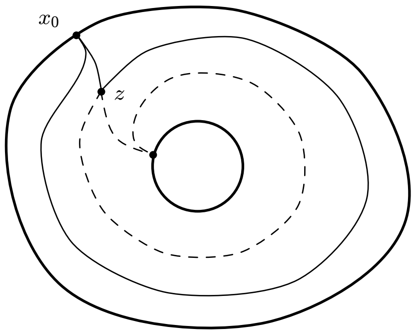


**Fig. 8.**

Suppose that $u = \alpha_1 u_1 + \alpha_2 u_2$, where $\alpha_1 \overline{\alpha_2} \notin \mathbb{R}$. Since each function $\mathscr{L} u_i$ is real valued (see the construction of the $u_i$ in the proof of Theorem 1.4 (i)), we have

$$\mathcal{N}(\mathscr{L}(\overline{\alpha_2} u)) = \mathcal{N}(\alpha_1 \overline{\alpha_2} \mathscr{L} u_1 + |\alpha_2|^2 \mathscr{L} u_2) = \mathcal{N}(\mathscr{L} u_1) \cap \mathcal{N}(\mathscr{L} u_2).$$

Since the nodal sets of $u_1$ and $u_2$ do not intersect, we have

$$\mathcal{N}(u) = \Pi(\mathcal{N}(\mathscr{L} u)) \subseteq \Pi(\mathcal{N}(\mathscr{L} u_1)) \cap \Pi(\mathcal{N}(\mathscr{L} u_2)) = \mathcal{N}(u_1) \cap \mathcal{N}(u_2) = \emptyset.$$

For the case $k = 2$, the proof uses the map $X : \overline{\Omega}' \to \overline{D}_1$ (see Eq. (4.5)) to essentially reduce the region with two holes to the single hole case. □

*Proof of Inequality* (1.8) *from Theorem* 1.1. Suppose that $k = 1$, and let $A_1$ and $A_2$ be magnetic vector potentials, where $A_1$ has circulation $1/2$. Let $\Phi$ denote the circulation of $A_2$. Suppose for a contradiction that $\Phi \notin 1/2 + \mathbb{Z}$ and that $\lambda_1(H_{A_2,V}) \geq \lambda_1(H_{A_1,V})$. Using Theorem 1.4 (i), there exists a groundstate $u_1$ of $H_{A_1,V}$ which has a nodal set $\mathcal{N}$ which slits $\overline{\Omega}$. As we are in the single hole case, the nodal set consists of a single line $\Gamma$ which joins the outer boundary to the inner boundary.

We shall need an operator $H_{\Gamma,A_2,V}$, which has extra Dirichlet boundary conditions imposed along the line $\Gamma$. This is defined formally as the self-adjoint operator corresponding to the restriction of the closed quadratic form $Q_{A_2,V}$ (defined in (2.1)) to the domain

$$\mathcal{Q}_\Gamma^{\mathrm{Neu}} = \{u \in \mathcal{Q}^{\mathrm{Neu}} = W^{1,2}(\Omega) : u|_\Gamma = 0\}.$$

Using our supposition, and the fact that the nodal set of $u_1$ consists of the line $\Gamma$, we have

$$\lambda_1(H_{A_2,V}) \geq \lambda_1(H_{A_1,V}) = \lambda_1(H_{\Gamma,A_1,V}). \tag{4.6}$$

Since $\Omega \setminus \Gamma$ is simply connected, $H_{\Gamma,A_1,V}$ is unitarily equivalent to $H_{\Gamma,A_2,V}$, and therefore

$$\lambda_1(H_{\Gamma,A_1,V}) = \lambda_1(H_{\Gamma,A_2,V}) = \inf_{u \in \mathcal{Q}_\Gamma^{\mathrm{Neu}}(\Omega)} Q_{A_2,V}(u) \geq \lambda_1(H_{A_2,V}). \tag{4.7}$$

We have equality in (4.6) and (4.7), and therefore the groundstate $u_2 \in \mathcal{Q}^{\mathrm{Neu}}$ of $H_{\Gamma,A_2,V}$ is also a groundstate of $H_{A_2,V}$ The nodal sets of $u_1$ and $u_2$ both contain $\Gamma$.

Since curl $A_1 =$ curl $A_2 = 0$ in the connected set $\Omega \setminus \Gamma$, there exist smooth functions $\phi_1, \phi_2 : \Omega \setminus \Gamma \to \mathbb{R}$ such that $\nabla\phi_i = A_i$. The functions $\phi_1$ and $\phi_2$ supply us with gauge transformations $e^{i\phi_1}$ and $e^{i\phi_2}$, from which we see both $e^{i\phi_1}u_1$ and $e^{i\phi_2}u_2$ are groundstates of $H_{\Gamma,0,V}$. By uniqueness of the groundstate of a non magnetic Schrödinger operator, we have

$$u_2 = \lambda e^{i(\phi_2 - \phi_1)} u_1$$

for some constant $\lambda \in \mathbb{C} \setminus \{0\}$. Let $\phi_3 = \phi_2 - \phi_1$. Since both $u_1$ and $u_2$ are smooth functions on $\Omega$ we may extend $\phi_3$ to a $C^1$ multivalued function on $\Omega$. The values that $\phi_3$ takes at a point differ by multiples of $2\pi$. Hence for a path $\sigma$ which circulates $\Omega$ once

$$\begin{aligned}\frac{1}{2\pi}\int_\sigma A_2 \cdot \mathrm{d}\mathbf{x} &= \frac{1}{2\pi}\int_\sigma A_1 \cdot \mathrm{d}\mathbf{x} + \frac{1}{2\pi}\int_\sigma (A_2 - A_1)\cdot \mathrm{d}\mathbf{x}\\ &= \frac{1}{2} + \frac{1}{2\pi}\int_\sigma \mathrm{d}\phi_3 = \frac{1}{2} + l.\end{aligned}$$

This contradicts our assumption that $\Phi \notin 1/2 + \mathbb{Z}$. $\square$

*Remark 4.7.* Using semiclassical arguments as in [Hel88a], we can show that for $k \geq 2$, the first eigenvalue is not necessarily maximised for circulation $(1/2, \ldots, 1/2)$. However, we may use methods similar to those in the above proof to show that

$$\lambda_1(1/2, \ldots, 1/2) = \inf_{S \in \mathscr{S}} \lambda_1(H_{S,0,V}), \tag{4.8}$$

where $\mathscr{S}$ is the collection of all sets $S$ which slit $\Omega$, and where $H_{S,0,V}$ is defined (as in the above proof) to have extra Dirichlet boundary conditions along $S \in \mathscr{S}$.

## 5. Additional Results and Examples

**Proposition 5.1.** *If a collection of paths $\{\gamma_1, \ldots, \gamma_n\}$ slits a region $\Omega$ with $k$ holes then $k/2 \leq n \leq k$.*

*Proof.* The lower bound on $n$ is elementary because there are an odd number of lines (i.e. at least one) leaving each of the $k$ holes. There must therefore be at least $k/2$ lines.

We finally prove the upper bound on $n$. Let $\sigma_0$ be a closed path which parametrises the outer boundary $\Sigma_0$ of $\Omega$, and let $\sigma_1, \ldots, \sigma_k$ be closed paths which parametrise the $k$ other boundary components $\Sigma_1, \ldots, \Sigma_k$. Define

$$S_0 = \bigcup_{i=0}^{k} \sigma_i(0), \tag{5.1}$$

$$S_1 = \left( \bigcup_{i=0}^{k} \{\sigma_i\big((0,1)\big)\} \right) \cup \left( \bigcup_{j=1}^{n} \{\gamma_j\big([0,1]\big)\} \right), \tag{5.2}$$

$$S_2 = \{\Omega \setminus \mathcal{N}\}. \tag{5.3}$$

Let $(N_0, N_1, N_2) = (k+1, k+1+n, 1)$ be the triple of integers associated to this decomposition, in which $N_i$ is the number of elements in the collection $S_i$. The decomposition $D$ is not a standard CW decomposition of $\overline{\Omega}$, and therefore the number $N := N_0 - N_1 + N_2$ will not yield the Euler number $\chi(\Omega) = -k+1$. It is however possible to modify the decomposition to make it into a proper CW decomposition in two steps:

(i) We first add vertices where intersections of elements of $S_1$ occur at points which are not in $S_0$. This step will decompose some elements of $S_1$ into smaller parts but leaves the element $\Omega \setminus \mathcal{N}$ of $S_2$ unaltered. Let $S_0'$ denote the new collection of vertices.

(ii) If $\Omega \setminus \mathcal{N}$ is not simply connected then the second step is to add some extra lines, which begin and end at already existing vertices in $S_0'$ in order to break up (without disconnecting) the region into a single simply connected 2-cell.

Note that after each step, $S_2'$ still consists of just one connected open set, so $N_2' = 1$, whilst the number $N_0' - N_1'$ of vertices minus lines does not increase. It follows that

$$N_0 - N_1 + N_2 \geq N_0' - N_1' + N_2' = \chi(\Omega).$$

Substituting in $N_0 = k+1$, $N_1 = k+1+n$, $N_2 = 1$, and $\chi(\Omega) = -k+1$, we obtain

$$n \leq k. \qquad \square$$

*Example 5.2.* The example of the circle $S^1$ is interesting to analyse. Consider the operator

$$P_\alpha = -(\partial_\phi - i\alpha)^2$$

on $L^2(S^1)$. The spectrum can be easily seen to be

$$\sigma(P_\alpha) = \{(n-\alpha)^2 : n \in \mathbb{Z}\},$$

and therefore

$$\lambda_1(P_\alpha) = \min_{n \in \mathbb{Z}} (n-\alpha)^2.$$

When $\alpha$ is an integer, the first eigenvalue is 0 and is simple and the corresponding eigenfunction is $\exp i\alpha\phi$. The first eigenvalue is actually simple whenever $\alpha$ is not a half-integer.

On the other hand, if $\alpha$ is a half-integer, the first eigenvalue is $1/4$, with multiplicity two. If, for example $\alpha = 1/2$, the corresponding eigenspace is spanned by the functions 1 and $\exp(i\phi)$ (or alternatively by the functions $\exp(i\phi/2)\cos(\phi/2)$ and $\exp(i\phi/2)\sin(\phi/2)$), and one can parametrise all the resulting eigenfunctions, in terms of a parameter $\phi_0$, by $\exp(i\phi/2)\sin((\phi-\phi_0)/2)$.

It is easy to see how the degeneracy of the first eigenvalue disappears when considering

$$P_{\alpha,\epsilon,v} = -(\partial_\phi - i\alpha)^2 + \epsilon v(\phi),$$

perturbatively as $\epsilon \neq 0$ is small, provided $v(\phi)$ satisfies the condition

$$\int_0^{2\pi} v(\phi)e^{i\phi} \neq 0.$$

*Example 5.3.* In [Hel88b, Subsect. 7.3], an example is given in which the multiplicity of the first eigenvalue is two. The domain $\Omega$ and potential $V$ are symmetric under the map $S : z \mapsto -z$, and the magnetic potential is given explicitly by

$$A = \frac{\Phi}{2\pi r^2}\begin{pmatrix} -y \\ x \end{pmatrix}.$$

If we take the case when the flux is an half-integer and we compose the operator $K$ (see Remark 3.4) with the operator $S$ defined by

$$(Su)(z) = u(Sz),$$

the operator

$$M = SK$$

commutes with $P_{A,V}$ and satisfies

$$M^2 = -I.$$

Kramer's theorem shows that the multiplicity is at least two. One can indeed show that $u$ and $Mu$ are linearly independent.

An alternative proof is simply to say that $Su$ is also an eigenvector with nodal set $S\gamma$, where $\gamma$ is the nodal set of $u$. Since $S\gamma$ is not equal to $\gamma$, the function $Su$ is linearly independent of $u$.

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