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A Proof of the Exponentially Small Transversality of the Separatrices for the Standard Map

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Abstract: In 1984 V. F. Lazutkin [Laz84, LST89] obtained an asymptotic formula for the separatrix splitting angle for the standard map. The difficulty of this problem is related to the exponential smallness of the splitting with respect to a perturbation parameter. Lazutkin's proof was based on two conjectures. Probably, the original form of those conjectures was incorrect, but Lazutkin's method was very efficient and inspired a large number of studies on the exponentially small splitting of separatrices. The consequent works [Laz91, Laz92, GLS94] and [Gel96] prepared the base to fill all the gaps of the original proof. The present paper contains a complete and self-contained proof of a refined version of the original formula (formula (1.7) of the present paper). In this form the formula was obtained in [GLS94]. The proof is inspired by the ideas of Lazutkin's original paper [Laz84].

1. Standard Map

The standard map is a popular model for the motion near a nonlinear resonance [Chi79, Sin94]. The standard map is an area-preserving diffeomorphism of the two dimensional torus, $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$, defined by

$$SM : (x, y) \mapsto (x + y + \varepsilon \sin x,\ y + \varepsilon \sin x). \tag{1.1}$$

In the following we always assume ε to be a small positive parameter. If $\varepsilon = 0$, the transformation (1.1) is integrable, and the phase space is foliated by invariant circles $y =$ const. The circle $y = 0$ is formed by SM fixed points. An arbitrarily small perturbation breaks this line, and for $\varepsilon > 0$ only two fixed points survive, namely, $(0, 0)$ and $(0, \pi)$. The first one is hyperbolic and the other one is elliptic. Indeed, the matrix of the linear part at the origin is

$$\begin{pmatrix} 1+\varepsilon & 1 \\ \varepsilon & 1 \end{pmatrix},$$

and its eigenvalues are λ and λ^{-1}, where

$$\lambda = 1 + \varepsilon/2 + \sqrt{\varepsilon + \varepsilon^2/4}. \tag{1.2}$$

The stable, W^s, and the unstable, W^u, manifolds of this fixed point are analytic curves passing through $(0,0)$, the eigenvectors of the matrix being tangent vectors to these curves at $(0,0)$. The origin breaks each separatrix into two parts. We denote by W_1^s (W_1^u) the upper part of the stable (unstable) separatrix.

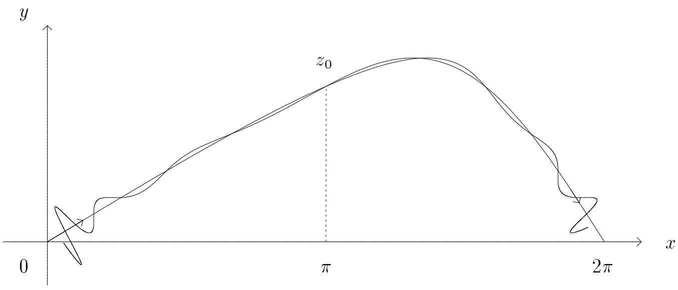

Fig. 1. Separatrices of the standard map

It is more convenient to use the parameter $h = \log\lambda$ as a small parameter instead of ε. It is easy to see that $\varepsilon \approx h^2$ since

$$\varepsilon = 4\sinh^2\frac{h}{2}.$$

It is convenient to represent the unstable separatrix, W_1^u, in a parametric form using a solution $(x, y) = (x^-(t), y^-(t))$ of the finite-difference system

$$\begin{aligned} x(t+h) &= x(t) + y(t+h), \\ y(t+h) &= y(t) + \varepsilon \sin x(t). \end{aligned} \tag{1.3}$$

We impose the following boundary conditions on the function $x^-(t)$:

$$\lim_{t\to-\infty} x^-(t) = 0, \qquad x^-(0) = \pi. \tag{1.4}$$

The solution of Eq. (1.3) is not defined uniquely by the boundary conditions (1.4). We study the solution, whose analytic continuation is entire and has a purely imaginary period 2πi. We assume that $t = 0$ corresponds to the first intersection of W_1^u with the line $x = \pi$ (if the intersection of the stable and unstable separatrices is transversal, then there are infinitely many such intersections). Under these additional assumptions the solution of the problem (1.3), (1.4) is unique. There are several ways to check the existence and uniqueness of such a solution. In particular, this follows from the convergence of an iteration procedure described in Sect. 10.

More geometrical arguments may be found in [GLS94]: the separatrix is one dimensional and the restriction of the map on the local separatrix is conjugated with the multiplication $\xi \mapsto \lambda\xi$, $\xi \in (\mathbb{C}, 0)$, then a solution of Eq. (1.3) is obtained after a substitution of e^t instead of ξ into the conjugating function. These arguments are quite general and the corresponding solution is defined up to a substitution $t \mapsto t + \text{const}$. The constant may be obtained from the second condition of (1.4).

Originally, the solution of (1.3) is only defined in a complex half-plane $\Re t < -R$ and represents the local separatrix. Since the sine function is entire, iterations of Eq. (1.3) allow to continue the solution up to an entire function.

To shorten the notation we omit the explicit dependence of the functions x^-, y^-, x^+, and y^+ on ε. We define the parameterization of W_1^s by

$$(x^+(t),\, y^+(t)) = \left(2\pi - x^-(-t),\, y^-(-t) + \varepsilon \sin x^-(-t)\right).$$

Direct substitution shows that these functions satisfy the system (1.3) as well as the boundary conditions

$$\lim_{t\to+\infty} x^+(t) = 0, \qquad x^+(0) = \pi. \tag{1.5}$$

Since $x^-(0) = \pi$ we have $x^+(0) = \pi$ and $y^+(0) = y^-(0)$, that is $t = 0$ corresponds to a homoclinic point. The splitting angle is not a natural measure for the separatrices splitting. Lazutkin proposed to study the *homoclinic invariant* defined by

$$\omega = \det \begin{pmatrix} \dot x^-(0) & \dot x^+(0) \\ \dot y^-(0) & \dot y^+(0) \end{pmatrix}. \tag{1.6}$$

The homoclinic invariant is equal to the value of the symplectic form $dx \wedge dy$ on a pair of vectors, tangent to the separatrices at the homoclinic point. The coordinate-independent definition of the homoclinic invariant for a symplectic map on a symplectic two-dimensional manifold may be found in [GLS94]. The homoclinic invariant has two remarkable properties: (i) it has the same value for all points of one homoclinic trajectory; (ii) it is invariant with respect to symplectic coordinate changes.

Theorem 1.1 (Main Theorem). *The homoclinic invariant ω of the homoclinic point $z_0 = (x^-(0), y^-(0))$ has the following asymptotic expansion:*

$$\omega \overset{\text{as}}{=} \frac{4\pi}{h^2} \mathrm{e}^{-\pi^2/h} \left(\sum_{n=0}^{\infty} h^{2n} \omega_n \right). \tag{1.7}$$

The sign $\overset{\text{as}}{=}$ means that the series on the right-hand side is asymptotic, that is if one retains a finite number of the first successive terms, the error is of the order of the first missing term: the absolute value of the error can be estimated from above by $O\left(h^{2N-2}\mathrm{e}^{-\pi^2/h}\right)$, where N is the number of the first missing term.

The coefficients in (1.7) are real numbers. The first values are

$$\omega_0 = 1118.827706\ldots,\ \omega_1 = 18.59891\ldots,\ \omega_2 = -4.34411\ldots/2!,$$
$$\omega_3 = -4.1829\ldots/3!,\ \ \omega_4 = -4.88\ldots/4!.$$

The value of ω_0 was computed in [LST89] as a solution of an ε-independent problem. In [Sur94] it was shown that this coefficient is not zero, since it may be obtained as a limit of an increasing sequence with positive first term. The other values were obtained in [GLS94]. Although the first coefficients are decreasing, the high-precision computations performed by C. Simó give numerical evidence of the divergent character of the series.

There are several independent ways for computing the constant ω_0, see e.g. [Tre96].

Corollary 1.2. *For all sufficiently small $\varepsilon > 0$ the stable and unstable separatrices of the standard map intersect transversally at the homoclinic point z_0 (the first intersection of the separatrices with the line $x = \pi$), and the splitting angle is given by*

$$\alpha \overset{\text{as}}{=} \frac{\pi}{h^2} \mathrm{e}^{-\pi^2/h} \left(\sum_{n=0}^{\infty} h^{2n} c_n \right),$$

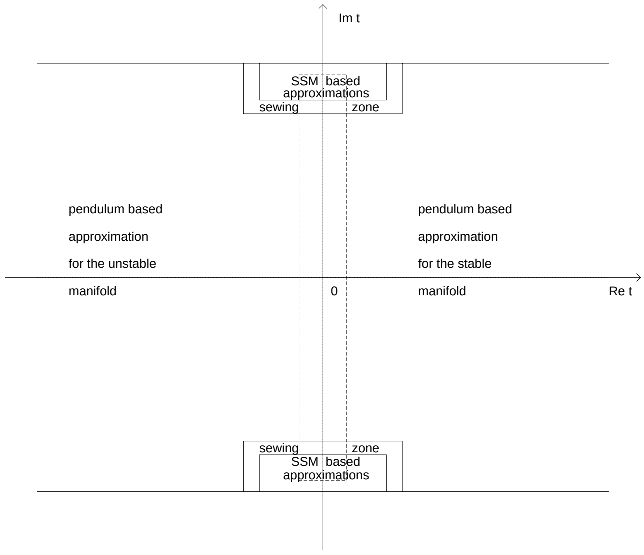

Fig. 2. The strip $|\Im t| < \pi/2$ is broken into several zones. Approximations for the unstable and/or stable manifold(s) are constructed for the corresponding values of t. The dashed line bounds the projection on the first coordinate of the domain of the time-energy coordinates (t, E)

where the coefficients c_n may be expressed in terms of the coefficients ω_n and a_{nk} defined in (2.10). In particular,

$$\omega_0 = c_0, \qquad \omega_1 = c_1 + \frac{c_0}{4}, \qquad \omega_2 = c_2 + \frac{c_1}{4} + \frac{25}{72}\, c_0.$$

To get these relations one should use the relation $\omega = \|e^u\| \cdot \|e^s\| \sin\alpha$, where $e^- = (\dot{x}^-(0), \dot{y}^-(0))$, $e^+ = (\dot{x}^+(0), \dot{y}^+(0))$ and $\|\cdot\|$ stands for the Euclidean norm. Then Proposition 3.1 provides the asymptotic for $\dot{x}^-$ and consequently for e^-.

Corollary 1.3. *The lobe area is given by*

$$S \overset{\text{as}}{=} 2\pi^{-1} \mathrm{e}^{-\pi^2/h} \left(\sum_{n=0}^{\infty} h^{2n} \omega_n \right).$$

Now we give an informal description of the proof. It is based on the detailed study of the analytical continuation of the stable and unstable manifolds. We make the parameter t in (1.3) complex and study the stable and unstable solutions $x^-(t)$ and $x^+(t)$, respectively. We construct approximations for these functions in complex domains, the unions of which include the half-planes $\Re t \le 0$ and $\Re t \ge 0$, respectively for x^- and x^+, as well as the rectangle, bounded by the dashed line in Fig. 2. Later this rectangle will be referred to as $\mathcal{D}(\sigma)$, where the parameter σ describes the distance to $\mathrm{i}\pi/2$ in a properly chosen scale (see (7.2)). The symmetries allow us to restrict our attention to the strip $|\Im t| \le \pi/2$ (see Fig. 2), and even to its upper part, $0 \le \Im t \le \pi/2$, due to the real-analyticity of the functions.

In the domains, marked as pendulum based approximation domains, we construct the asymptotic series for x^- and x^+. This series starts with the homoclinic solution $x_0(t) =$

$4 \arctan \mathrm{e}^t$ of the pendulum equation. A single series corresponds to both the stable and unstable manifolds. The series is asymptotic to both manifolds in the intersection of the corresponding domains with $\mathcal{D}(\sigma)$. This implies that for these values of t the difference $x^+(t) - x^-(t)$ is less than any power of the small parameter h.

The pendulum based approximation fails for the values of t close to the pendulum separatrix singularity $t = \mathrm{i}\pi/2$. In that region we construct another approximation. It starts with a term which arises from the study of the semistandard map

$$SSM : (u, v) \mapsto (u + v + \mathrm{e}^u,\ v + \mathrm{e}^u).$$

We use a sewing condition on the intermediate domain to ensure that the new series approximate the same invariant curve as the initial ones. This may be considered as a kind of complex time matching method. The accuracy of the SSM based approximation affords us to show that the difference $x^+(t) - x^-(t)$ does not vanish on the top edge of $\mathcal{D}(\sigma)$. In fact that difference is of the order of h^σ there.

How can we use this to get the exponentially fine estimates for real values of t? There is a coordinate system (t, E), such that the standard map takes the form of a shift $(t, E) \mapsto (t + h, E)$ and the unstable manifold is given by the equation $E = 0$. We show that the stable manifold is a graph of a h-periodic function $\Theta(t)$. The zeros of that function correspond to homoclinic points and the derivative of that function at zero is the homoclinic invariant. The SSM based approximation provides an estimate for that function with $O(h^N)$ error for some N. Originally, this approximation is not periodic, but the difference between approximations for the stable and unstable separatrices may be described by a periodic function. We use this function to approximate $\Theta(t)$. Then we apply the following simple lemma to the error term and get the exponentially fine estimate.

Lemma 1.4 ([Laz84]**).** *Let a function $R(t)$ be periodic with positive period h, analytic in the strip $|\Im t| < b$, and continuous in the closure of the strip. Then*

$$\left|\dot{R}(t)\right| \le 4 \max_{|\Im z| \le b} |R(z)| \,\frac{2\pi}{h} \exp(-2\pi b/h), \qquad t \in \mathbb{R},$$

provided $\exp(-2\pi b/h) \le 1/2$. *Moreover, if the mean value $\int_0^h R(t)\,dt = 0$, then*

$$|R(t)| \le 2 \max_{|\Im z| \le b} |R(z)|\ \exp(-2\pi b/h), \qquad t \in \mathbb{R}.$$

The rest of the paper contains the complete proof of the asymptotic formula (1.7). It is organized in the following way. In Sect. 2 we describe the formal series for the unstable separatrix and in Sect. 3 we give the exact statement about how the series approximates the unstable separatrix. In Sect. 4 we describe the basic facts about the invariant curves of the semistandard map, which we use in Sect. 5 to construct the SSM based approximation for the unstable separatrix. In Sect. 6 we estimate the difference $x^+(t) - x^-(t)$ near $t = \mathrm{i}\pi/2$. This difference is small (but not exponentially in that domain) and may be approximated by a linear combination of solutions for a variational equation near x^-. In Sect. 7 we formulate the existence theorem for the coordinate system (t, E) and derive the asymptotic formula (1.7) from that theorem.

In Sect. 9 we develop the theory of finite-difference equations, which will be widely used through the next sections. This theory may be of independent interest. Other sections are devoted to proving one by one all the theorems and propositions formulated in Sects. 2 to 7.

Remark I. Lazutkin's proof [Laz84] of the asymptotic formula for the splitting angle was based on two conjectures, called respectively Conjecture A and B. Conjecture A stated that the semistandard map can be conjugated to the shift $(\tau, \mathcal{E}) \mapsto (\tau + 1, \mathcal{E})$ by an analytic coordinate change defined in a neighborhood of complex segments of the SSM separatrices. Conjecture B contained a similar statement for the standard map. This part of the conjectures is correct. In addition Lazutkin's conjectures contained some requirements on the size of the domains and upper bounds for derivatives of the coordinate changes, which were essential for the proof.

The proof of Conjecture A published in [Laz90] contained an error, and it was pointed out by V. F. Lazutkin on p. 111 of [Laz92], that Conjecture A was *probably incorrect*. In the corrected version the domain of the coordinates $(\tau, \mathcal{E})$ was much smaller than in Conjecture A. In [GLS94] Conjecture A was replaced by Conjecture I, which is proved here in a slightly modified form (Proposition 6.2 of the present paper).

Theorem 7.1 of the present paper is quite similar to Conjecture B (and Conjecture II of [GLS94]), but it provides a smaller size for the domain of the coordinates (t, E) and larger upper bounds for derivatives of the coordinate change. This is compensated by finer approximations of the SM separatrices. Of course, the estimates of Theorem 7.1 are not optimal, but the estimates of Conjecture B are probably too "optimistic".

Remark II. E. Fontich and C. Simó [FS90] used Birkhoff normal form near a hyperbolic fixed point to construct the coordinates (t, E) and to obtain an exponentially small upper bound for the splitting. This idea was later used by Delshams and Seara [DS92] in the proof of a theorem, which states that Melnikov method provides a correct asymptotic formula for the splitting of the pendulum separatrix under a small fast periodic perturbation. In a recent paper [DGJS97] it was shown that a similar method may be used to study the case of fast quasiperiodic perturbations. The last problem leads to a very delicate analysis due to the presence of small denominators in the Melnikov function, which represents the leading term of the asymptotic formula for the splitting. This problem may be considered as a step towards higher dimensional hamiltonian systems. On the other hand, in the case of the standard map the Melnikov method may not be applied directly. This case is closer to a large fast periodic perturbation considered in [Gel97b]. All these papers, as well as [Gel97a] and the present one, develop (quite nontrivially, of course) Lazutkin's original ideas [Laz84].

2. Formal Separatrix

In the present section we construct the expansion for W_1^u and W_1^s in power series of the parameter h and show that a single expansion corresponds to both separatrices.

The first equation of the system (1.3) enables us to express the second component in terms of the first one. So the system is equivalent to a single finite-difference equation of second order:

$$\Delta_h^2 x = \varepsilon \sin x, \tag{2.1}$$

where

$$\Delta_h^2 x = x(t + h) - 2x(t) + x(t - h).$$

First, we solve the equation in the class of power series of the form

$$X \sim \sum_{n=0}^{\infty} h^{2n} x_n(t), \tag{2.2}$$

where the sign $\sim$ stresses that the series in the right-hand side of the equation is divergent, and X is considered as a generating function for this series. In the class of power series $X(t \pm h) = \exp\left(\pm h \frac{\partial}{\partial t}\right) X(t)$, and Eq. (2.1) reads

$$4 \sinh^2 \left(\frac{h}{2} \frac{\partial}{\partial t} \right) X = 4 \sinh^2 \left(\frac{h}{2} \right) \sin X. \tag{2.3}$$

Taking into account that $2 \sinh^2 \frac{a}{2} = \sum_{k=1}^{\infty} \frac{a^{2k}}{(2k)!}$ for any a and equating the terms of the order of h^{2n}, we get from (2.3),

$$\sum_{k=1}^{n} \frac{1}{(2k)!} \left(\frac{\partial}{\partial t} \right)^{2k} x_{n-k} = \sum_{k=1}^{n} \frac{1}{(2k)!} G_{n-k}(x_0, \ldots, x_{n-k}), \tag{2.4}$$

where G_n are defined by the following recurrent rule. Let $G_0 = \sin x_0$, $H_0 = \cos x_0$ and

$$G_n = \frac{1}{n} \sum_{k=1}^{n} k x_k H_{n-k}, \qquad H_n = -\frac{1}{n} \sum_{k=1}^{n} k x_k G_{n-k} \tag{2.5}$$

for $n \geq 1$. It is not difficult to check that

$$\sin \left(\sum_{n=0}^{\infty} h^{2n} x_n \right) = \sum_{n=0}^{\infty} h^{2n} G_n, \qquad \cos \left(\sum_{n=0}^{\infty} h^{2n} x_n \right) = \sum_{n=0}^{\infty} h^{2n} H_n$$

(differentiate the equalities with respect to h^2 and compare the result with (2.5)).

Equations (2.4) must be supplemented with the following boundary conditions:

$$\lim_{t \to -\infty} x_n(t) = 0, \quad n = 0, 1, 2, \ldots, \tag{2.6}$$

$$x_0(0) = \pi, \qquad x_n(0) = 0, \quad n = 1,\ 2, \ldots, \tag{2.7}$$

which arise from the expansion of (1.4) in power series. Let us dwell on the determination of the first function x_0. We have to solve Eq. (2.4) with $n = 1$, which reads

$$\frac{d^2}{dt^2} x_0 = \sin x_0, \tag{2.8}$$

subjected to the boundary conditions (2.6) and (2.7). The unique solution is

$$x_0(t) = 4 \arctan \mathrm{e}^t. \tag{2.9}$$

Equation (2.8) is the "pendulum equation" and (2.9) is the homoclinic solution to the unstable equilibrium point.

Proposition 2.1 ([GLS94]). *Equation* (2.4) *has a unique solution* $x_0, x_1, x_2, \ldots$ *satisfying the boundary conditions* (2.6) *and* (2.7)*. The leading term is given by* (2.9)*. The subsequent terms have the form*

$$x_n(t) = \sum_{k=1}^{n} a_{nk} \frac{\sinh t}{(\cosh t)^{2k}}, \tag{2.10}$$

where a_{nk} *are real numbers. In particular,*

$$x_1(t) = \frac{1}{4} \frac{\sinh t}{(\cosh t)^2}, \qquad x_2(t) = -\frac{41}{1728} \frac{\sinh t}{(\cosh t)^2} + \frac{91}{864} \frac{\sinh t}{(\cosh t)^4}. \tag{2.11}$$

The proof of that proposition, which is similar to [GLS94], is in Sect. 8.

The functions x_n have the following obvious properties:

1. $x_n(t) \to 0$ as $t \to \pm\infty$ for $n \geq 1$;
2. $x_0 - \pi$ and x_n, $n \geq 1$, are odd;
3. the functions x_n, $n \geq 0$, are $\mathrm{i}\pi$-antiperiodic;
4. x_n, $n \geq 0$, are analytic on the entire complex plane except the singular points $t = \mathrm{i}\pi/2 + \mathrm{i}\pi k$, $k \in \mathbb{Z}$; all singularities of x_0 are logarithmic branching points; all singularities of x_n, $n \geq 1$, are poles of the order $2n$;
5. the functions x_n, $n \geq 1$, are singlevalued; x_0 is singlevalued on the complex plane cut along straight segments, $(\mathrm{i}\pi/2 + \mathrm{i}\pi 2k, \mathrm{i}\pi/2 + \mathrm{i}\pi(2k+1))$, $k \in \mathbb{Z}$.

Any partial sum of (2.2), after restoring the y-component, represents a line, which connects the fixed point $(0,0)$ with its copy $(2\pi, 0)$. In this sense the formal series (2.2) represents the formal separatrix. The series provides a formal solution for Eq. (2.1), which satisfies the boundary conditions (1.4) and (1.5). In particular, this implies that the classical perturbation theory, based on the expansion in powers of a small parameter, cannot reveal the splitting of separatrices.

3. First Approximation Theorem

The functions x_n defined in the previous section have singularities, while x^- is an entire function of the variable t. This shows that the series (2.2) cannot approximate $x^-(t)$ in a neighborhood of the mentioned singularities. It is important to know where (2.2) does approximate our function. Since $x^-(t)$ is real on the real axis and so are the coefficients x_n, it is sufficient to consider them for $\Im t \geq 0$ only. Moreover, both the function and the formal series are $\mathrm{i}\pi$-antiperiodic. Using these symmetries we may restrict our attention to the following domain. Fix $\delta_0 \in (0, \pi/2)$ and let

$$\mathcal{D} = \left\{ t \in \mathbb{C} : 0 \leq \Im t \leq \frac{\pi}{2},\ \Re t \leq 20h,\ \arg\left(t - \mathrm{i}\frac{\pi}{2}\right) \leq -\delta_0 \right\}. \tag{3.1}$$

Proposition 3.1. *For any positive integer N the following estimate holds in the domain $\mathcal{D}$:*

$$\left| x^-(t) - \sum_{n=0}^{N-1} h^{2n} x_n(t) \right| \leq \mathrm{const}\, h^{2N} \left(1 + \frac{1}{|t - \mathrm{i}\pi/2|^{2N}} \right), \tag{3.2}$$

where const *depends only on N and δ_0. Moreover, a similar estimate is valid for the derivative with respect to t of the expression in the left-hand side, the exponent of $|t - \mathrm{i}\pi/2|$ being changed to $2N+1$.*

The proof of the proposition is in Sect. 10.

The stable separatrix, represented by the function $x^+(t) = 2\pi - x^-(-t)$ is approximated by the same series in the domain $-\overline{\mathcal{D}}$, which is a reflection of $\mathcal{D}$ with respect to the imaginary axis.

4. Semistandard Map

The semistandard map, SSM,

$$SSM : (u, v) \mapsto (u + v + e^u,\ v + e^u) \tag{4.1}$$

was introduced by Greene and Percival [GP81] and was studied by many authors. It is convenient to define this map as a selfmap of $\mathbb{C}^2$. The semistandard map preserves the standard symplectic structure $du \wedge dv$. The second important property of SSM is *reversibility*. This means that there exists a map $R : \mathbb{C}^2 \to \mathbb{C}^2$ which satisfies the equations

$$R^2 = \mathrm{Id}, \qquad R \circ SSM \circ R = SSM^{-1}.$$

A concrete example of such a map is $R = R_0$, where

$$R_0 : (u, v) \mapsto (u, -v - \exp u). \tag{4.2}$$

Theorem 4.1 ([Laz84, Laz92]). *There exists a unique analytical injective map $\Gamma_- : \mathbb{C} \to \mathbb{C}^2$ such that $\Gamma_-(\tau + 1) = SSM(\Gamma_-(\tau))$ and the following normalizing condition holds. Let $\Gamma_-(\tau) = (u_-(\tau), v_-(\tau))$, then*

$$u_-(\tau) = -\log \frac{\tau^2}{2} + O\left(\frac{1}{\tau^2}\right)$$

as $\tau \to -\infty$ along the negative real semiaxis. The branch of log *in the last formula is fixed to be real at negative τ.*

The following asymptotic expansion for $\tau \to \infty$ is valid uniformly in a sector $\delta_0 \le \arg \tau \le 2\pi - \delta_0$, $\delta_0 \in]0, \pi/2[$ being an arbitrary fixed number,

$$u_-(\tau) \stackrel{\mathrm{as}}{=} -\log \frac{\tau^2}{2} + \sum_{k=1}^{\infty} a_k \tau^{-2k}, \tag{4.3}$$

where a_k are real numbers. The first three values of a_k are $a_1 = -\frac{1}{4}$, $a_2 = \frac{91}{864}$, $a_3 = -\frac{319}{2880}$.

The image of Γ_- is an invariant curve, the "unstable" manifold for the "$-$ infinity". In the next section we use it as an approximation to the standard map separatrix. The curve $\Gamma_+ = R_0(\Gamma_-)$ plays the role of the stable manifold.

Theorem 4.2 ([Laz92]). *Let $(u_+, v_+)(\tau) = R_0((u_-, v_-)(-\tau))$. The following estimates:*

$$|u_+(\tau) - u_-(\tau) \pmod{2\pi i}| \le \mathrm{const}\, |\tau|^2 \exp(-2\pi |\Im \tau|),$$

$$|v_+(\tau) - v_-(\tau)| \le \mathrm{const}\, |\tau| \exp(-2\pi |\Im \tau|),$$

are valid in the sector $-\pi + \delta_0 \le \arg \tau \le -\delta_0$, the constants depend only on the choice of δ_0.

For the sake of completeness we include the proofs of the last two theorems in Sects. 11 and 12. The proofs follow the lines of [Laz92].

5. Second Approximation Theorem

Notice first that the imaginary part of $x_0(t)$, the first term of (2.2), becomes large and positive if t approaches $\mathrm{i}\frac{\pi}{2}$ from the real axis. A simple use of Euler's formula

$$\sin x = \frac{\mathrm{e}^{\mathrm{i}x} - \mathrm{e}^{-\mathrm{i}x}}{2\mathrm{i}} = \frac{\mathrm{i}}{2}\mathrm{e}^{-\mathrm{i}x} + O\left(\mathrm{e}^{\mathrm{i}x}\right)$$

shows that the sine function can be replaced by an exponential term, the error being small. This suggests the following change of variables:

$$\begin{aligned} x &= -\mathrm{i}\log\tfrac{h^2}{2} + \mathrm{i}u, \\ y &= \mathrm{i}v. \end{aligned} \tag{5.1}$$

The standard map reads in these variables $(u, v) \longmapsto (u_1, v_1)$,

$$\begin{aligned} u_1 &= u + v_1, \\ v_1 &= v + \exp u + (\varepsilon/h^2 - 1)\exp u - (\varepsilon h^2/4)\ \exp(-u). \end{aligned} \tag{5.2}$$

Note that $\varepsilon/h^2 - 1 = O(h^2)$. If we cancel the last two terms in (5.2), we get the semistandard map SSM. One may expect that some segments of trajectories of SM are close, after the change of variables (5.1), to those of SSM.

In addition to the change of variables (5.1) we make the following change of the parameter along the unstable manifold in order to place the origin at the singularity and to change the step to one unit:

$$t = \mathrm{i}\frac{\pi}{2} + h\tau. \tag{5.3}$$

So, instead of $x^-(t)$ we will consider the function $u^-(\tau)$, the link being

$$x^-\left(i\frac{\pi}{2} + h\tau\right) = -\mathrm{i}\log\frac{h^2}{2} + \mathrm{i}u^-(\tau). \tag{5.4}$$

Equation (2.1) converts into

$$\Delta^2 u^- = (\varepsilon/h^2)\exp(u^-) - (\varepsilon h^2/4)\exp(-u^-), \tag{5.5}$$

where Δ^2 is a second order finite-difference operator, $\Delta^2 f(\tau) = f(\tau+1) - 2f(\tau) + f(\tau-1)$. We relate with u^- the formal series of the form

$$U^-(\tau,\varepsilon) \sim \sum_{n=0}^{\infty} h^{2n} u_n^-(\tau), \tag{5.6}$$

where $u_n^-(\tau)$ depend only on τ. Substituting the series into Eq. (5.5) and collecting the terms of the same order in h it is not difficult to write down the equations for $u_n^-(\tau)$:

$$\Delta^2 u_0^- = \exp u_0^-, \tag{5.7}$$

$$\begin{aligned} \Delta^2 u_n^- = \sum_{k=1}^{n+1} \frac{2}{(2k)!}\Big(&\mathcal{Y}_{n+1-k}(u_1,\ldots,u_{n+1-k})\exp(u_0^-) \\ &-\frac{1}{4}\mathcal{Y}_{n-1-k}(-u_1,\ldots,-u_{n-1-k})\exp(-u_0^-)\Big), \end{aligned} \tag{5.8}$$

where $n \geq 1$. The auxiliary polynomials $\mathcal{Y}_k$, similar to the Bell polynomials, are defined by the following recurrent rule. Let $\mathcal{Y}_0 = 1$ and

$$\mathcal{Y}_n(u_1, \ldots, u_n) = \frac{1}{n} \sum_{k=1}^{n} k u_k \mathcal{Y}_{n-k}(u_1, \ldots, u_{n-k}) \tag{5.9}$$

for $n \geq 1$. Differentiating with respect to h^2 it is not too difficult to check that

$$\exp\left(\sum_{n=1}^{\infty} h^{2n} u_n\right) = \sum_{n=0}^{\infty} h^{2n} \mathcal{Y}_n(u_1, \ldots, u_n).$$

The polynomials $\mathcal{Y}_k$ with $k < 0$ are assumed to be identically zero. We have to solve the system (5.7), (5.8) subjected to a *sewing condition* which can be expressed as a formal coincidence of the right- and left-hand sides of (5.4) after substituting there the expansions (2.2) and (5.6), and reexpansion of both in the double series in h^2 and τ^2.

Proposition 5.1. *There exists a unique sequence of entire functions, $u_n^-(\tau)$, $n = 0, 1, 2, \ldots$, which satisfy Eqs. (5.7), (5.8) and the sewing condition (5.4). In any sector $\delta_0 \leq \arg \tau \leq 2\pi - \delta_0$, $\delta_0 > 0$, the functions $u_n^-(\tau)$ have the following asymptotic expansions:*

$$u_0^-(\tau) \overset{\text{as}}{=} -\log \frac{\tau^2}{2} + \sum_{k=1}^{\infty} \frac{p_{0k}}{\tau^{2k}}, \tag{5.10}$$

$$u_n^-(\tau) \overset{\text{as}}{=} \sum_{k=-n}^{\infty} \frac{p_{nk}}{\tau^{2k}}. \tag{5.11}$$

The branch of log *is fixed to be real at negative τ. In (5.10) and (5.11) the coefficients p_{nk} are real numbers. The functions u_n^- are real-analytic. The asymptotic series may be differentiated with respect to τ.*

The proof of the proposition is in Sect. 13. Of course, we have $u_0^-(\tau) = u_-(\tau)$ due to the uniqueness (compare with Sect. 4).

How does the series in (5.6) approximate x^-? The following proposition gives the answer. Let $\mathcal{D}_2$ be the domain in the t-plane which is the intersection of $\mathcal{D}$ (defined by (3.1)) with the rectangle $|\Re t| \leq \sqrt{h}$, $|\Im t - \pi/2| \leq \sqrt{h}$. The corresponding domain $\tilde{\mathcal{D}}_2$ in the τ-plane is defined as

$$\tilde{\mathcal{D}}_2 = \{\tau \in \mathbb{C} : -h^{-1/2} \leq \Re\tau \leq 20,\ -h^{-1/2} \leq \Im\tau \leq 0,\ \pi \leq \arg\tau \leq 2\pi - \delta_0\}. \tag{5.12}$$

Proposition 5.2. *For any positive integer N there exists a positive constant, C_N, such that, if $\tau \in \tilde{\mathcal{D}}_2$, then*

$$\left| x^-\left(i\frac{\pi}{2} + h\tau\right) + i \log \frac{h^2}{2} - \mathrm{i} \sum_{n=0}^{N-1} h^{2n} u_n^-(\tau) \right| \leq C_N h^{2N} \left(1 + |\tau|^{2N}\right). \tag{5.13}$$

A similar estimate is valid for the derivative of the expression in the left-hand side.

The proof is in Sect. 14.

6. Comparing the Stable and Unstable Manifolds

In this section we will compare the stable and the unstable manifolds near $t = \mathrm{i}\pi/2$. We assume that the parameterization of the stable manifold is chosen to be

$$x^+(t) = 2\pi - x^-(-t). \tag{6.1}$$

Applying the change of variables (5.1), we introduce the function u^+ by

$$x^+\left(i\frac{\pi}{2} + h\tau\right) = -\mathrm{i}\log\frac{h^2}{2} + iu^+(\tau). \tag{6.2}$$

The equality (6.1) and the fact that all functions are analytical continuations of real analytic functions give the following chain of equalities:

$$\begin{aligned} x^+\left(\mathrm{i}\tfrac{\pi}{2} + h\tau\right) &= 2\pi - x^-\left(-\mathrm{i}\tfrac{\pi}{2} - h\tau\right) = 2\pi - \overline{x^-\left(\mathrm{i}\tfrac{\pi}{2} - h\overline{\tau}\right)} \\ &= 2\pi - \overline{-\mathrm{i}\log\tfrac{h^2}{2} + \mathrm{i}u^-(-\overline{\tau})} \\ &= -\mathrm{i}\log\tfrac{h^2}{2} + \mathrm{i}(-2\pi\mathrm{i} + u^-(-\tau)). \end{aligned}$$

Comparing with (6.2) we obtain

$$u^+(\tau) = -2\pi\mathrm{i} + u^-(-\tau). \tag{6.3}$$

Let us study the difference,

$$x^+(t) - x^-(t) = i(u^+(\tau) - u^-(\tau)). \tag{6.4}$$

We may expand u^+ in formal power series of h^2,

$$U^+(\tau,\varepsilon) \sim \sum_{n=0}^{\infty} h^{2n} u_n^+(\tau), \tag{6.5}$$

similar to (5.6), where u_n^+ obey the same equations (5.7), (5.8) with u_n^- being replaced by u_n^+. From (6.3) we obtain

$$u_0^+(\tau) = -2\pi\mathrm{i} + u_0^-(-\tau), \qquad u_n^+(\tau) = u_n^-(-\tau), \quad n \geq 1. \tag{6.6}$$

We study the formal series defined by

$$W(\tau,\varepsilon) \sim \sum_{n=0}^{\infty} h^{2n} w_n(\tau), \tag{6.7}$$

where

$$w_n(\tau) = u_n^+(\tau) - u_n^-(\tau). \tag{6.8}$$

We are going to estimate the differences (6.8) in the domain D_A, with $A > 1$,

$$D_A = \{\tau \in \mathbb{C} : \Im\tau \leq -A,\ -\pi + \delta_0 \leq \arg\tau \leq -\delta_0\}. \tag{6.9}$$

It turns out that in that sector the functions $u_n^+(\tau)$ and $u_n^-(\tau)$ have the same asymptotic expansions (5.10), (5.11). In the case of $u_0^{\pm}(\tau)$ one should take into account the choice of the branch of the logarithm. So each $w_n(\tau)$ tends to zero faster than any negative power of τ, when τ tends to infinity in $\mathcal{D}_A$. In fact, $w_n(\tau)$ are exponentially decreasing in that sector. For $n = 0$ it is Theorem 4.2. For $n \geq 1$ we will get this as a consequence of the propositions formulated later in the present section.

In order to formulate a more definite statement about the asymptotic behavior of $w_n(\tau)$ we need some preliminaries. Consider the first derivative of $U^-(\tau, \varepsilon)$ with respect to the variable τ:

$$\Phi_1^-(\tau, \varepsilon) = \frac{dU^-}{d\tau}(\tau, \varepsilon) \sim \sum_{n=0}^{\infty} h^{2n} \varphi_{1,n}^-(\tau), \tag{6.10}$$

where

$$\varphi_{1,n}^-(\tau) = \frac{du_n^-}{d\tau}(\tau). \tag{6.11}$$

It satisfies the equation

$$\Delta^2 \Phi = \left((\varepsilon/h^2) \mathrm{e}^{U^-} + (\varepsilon h^2/4) \mathrm{e}^{-U^-} \right) \Phi. \tag{6.12}$$

We shall consider Eq. (6.12) in the class of the formal series of the form $\sum_{n=0}^{\infty} h^{2n} \varphi_n$, where φ_n are entire functions of one variable τ.

We seek a second formal solution Φ_2^- of (6.12), which is linearly independent of (6.10) and satisfies the normalizing condition:

$$\mathcal{W}[\Phi_1^-; \Phi_2^-] = \Phi_1^- \overline{\Delta} \Phi_2^- - \Phi_2^- \overline{\Delta} \Phi_1^- = 1, \tag{6.13}$$

where $\mathcal{W}$ is a finite-difference Wronskian, its role for the theory of finite-difference equations is similar to the role of the classical Wronskian in the theory of ordinary differential equations. We will discuss it in Sect. 9.5. We used the notation $\overline{\Delta} F = F(\tau) - F(\tau - 1)$. The formal equality (6.13) is equivalent to a system

$$\sum_{k=0}^{n} \mathcal{W}[\varphi_{1,k}; \varphi_{2,n-k}] = \delta_n, \qquad n = 0, 1, 2, \ldots\ ; \tag{6.14}$$

where δ_n is Kronecker symbol, $\delta_0 = 1$ and $\delta_n = 0$ for $n \neq 0$. In fact, this system may be considered as a definition of the functions $\varphi_{2,k}$.

Proposition 6.1. *There exists a unique formal series, Φ_2^-, which satisfies Eq. (6.12), the normalizing condition (6.13) and has the form*

$$\Phi_2^-(\tau, \varepsilon) \sim \sum_{n=0}^{\infty} h^{2n} \varphi_{2,n}^-(\tau), \tag{6.15}$$

where $\varphi_{2,n}^-(\tau)$ are entire functions, which are real on the real axis and admit the following asymptotic expansion:

$$\varphi_{2,n}^-(\tau) \overset{\text{as}}{=} \sum_{k=-n-1}^{\infty} \frac{\varphi_{nk}}{\tau^{2k}}, \qquad \varphi_{nk} \in \mathbb{R}, \tag{6.16}$$

as τ goes to infinity in the sector D_A.

The proposition is proved in Sect. 15.

Proposition 6.2. *There exist two sequences of complex numbers,*

$$\{\theta_n\}_{n=0}^{\infty} \quad \textit{and} \quad \{\mu_n\}_{n=0}^{\infty},$$

such that for any positive constants A and δ,

$$w_n(\tau) = \mathrm{e}^{-\mathrm{i}2\pi\tau} \sum_{k=0}^{n} \left(\mu_{n-k}\varphi_{1,k}^{-}(\tau) + \theta_{n-k}\varphi_{2,k}^{-}(\tau) \right) + O\left(\mathrm{e}^{-\mathrm{i}4\pi(1-\delta)\tau}\right) \tag{6.17}$$

uniformly in D_A, the constant in the error term depends on δ, A and n.

The proposition is proved in Sect. 16. The following corollary follows directly from the proposition and the equalities (6.14).

Corollary 6.3.

$$\sum_{k=0}^{n} \mathcal{W}[\varphi_{1,k}^{-}; w_{n-k}](\tau) = \mathrm{e}^{-\mathrm{i}2\pi\tau}\theta_n + O\left(\mathrm{e}^{-\mathrm{i}4\pi(1-\delta)\tau}\right). \tag{6.18}$$

Now we can get an upper bound for the difference $x^{+}(t) - x^{-}(t)$ on the strip $|\Re t| < 20h$. Define the following rectangles:

$$\Pi_1 = \{\, 0 \le \Im t \le \pi/2 - \sqrt{h},\ |\Re t| < 20h \,\},$$

$$\Pi_2 = \{\, \pi/2 - \sqrt{h} \le \Im t \le \pi/2 - (2N/2\pi)h \log h^{-1},\ |\Re t| < 20h \,\},$$

$$\Pi_3 = \{\, \pi/2 - (2N/2\pi)h \log h^{-1} \le \Im t \le \pi/2 - Ah\},\ |\Re t| < 20h \,\}.$$

For $t \in \Pi_1$ we have from Proposition 3.1

$$x^{+}(t) - x^{-}(t) = O(h^N).$$

The same estimate holds in Π_2 due to the estimates of Propositions 5.2 and 6.2 and the fact that in this region $\mathrm{e}^{-\mathrm{i}2\pi\tau} = O(h^{2N})$. In Π_3 the exponent is no longer so small and we get

$$x^{+}(t) - x^{-}(t) = O\left(|\tau^2| \exp(-2\pi|\Im\tau|)\right).$$

Note that in Π_3 the right-hand side of the last estimate ranges from $O(h^N)$ to $O(1)$.

7. Analytic Integral and the Asymptotic Formula for the Homoclinic Invariant

To obtain the formula for the separatrix splitting we need an analytic integral along the unstable separatrix. Let

$$\mathcal{G}(R) = \left\{ (t, E) \in \mathbb{C}^2 : |\Re t| \le 10h,\ |\Im t| \le \pi/2 - Rh,\ |E| \le h^9 \right\}. \tag{7.1}$$

Here we denote by t and E coordinates in $\mathbb{C}^2$.

Theorem 7.1 (on the analytic integral). *There exist $R > 1$ and $h_0 > 0$ such that, if $0 < h < h_0$, then there exists a map $\boldsymbol{\Phi} : \mathcal{G}(R) \to \mathbb{C}^2$ such that*

(1) ***Φ** is an analytical diffeomorphism onto its image;*
(2) ***Φ** is symplectic:* $\boldsymbol{\Phi}^* dx \wedge dy = dt \wedge dE$*;*
(3) ***Φ** conjugates the standard map with the shift* $(t, E) \longmapsto (t + h, E)$*;*
(4) $\boldsymbol{\Phi}^{-1}\big(x^-(t), y^-(t)\big) = (t, 0)$*;*
(5) *the second projection of the inverse map* $E = pr_2 \circ \boldsymbol{\Phi}^{-1}$ *has the derivatives of the first order bounded by* const h^{-1} *and the derivatives of the second order bounded by* const h^{-10}*; the first derivatives of the first projection are bounded by* const h^{-2}*;*
(6) *the inverse map* $\boldsymbol{\Phi}^{-1}$ *is real at real values of its arguments.*

The proof of the theorem is in Sect. 17.

The second component E of the map $\boldsymbol{\Phi}^{-1}$ is a local analytic integral of the standard map, i.e., it is constant along trajectories. Of course, its domain is not invariant and it has no single-valued continuation: if a trajectory leaves the domain $\boldsymbol{\Phi}(\mathcal{G}(R))$ and comes back after several iterations, it may get a value of E different from the original one.

Given $\sigma > 0$, denote

$$\mathcal{D}(\sigma) = \left\{ t \in \mathbb{C} : |\Re t| \leq h, |\Im t| \leq \frac{\pi}{2} - \frac{\sigma}{2\pi}\, h \log \frac{1}{h} \right\}. \tag{7.2}$$

Define the map $\Theta : \mathcal{D}(\sigma) \to \mathbb{C}$ by

$$\Theta(t) = E\big(x^+(t), y^+(t)\big), \tag{7.3}$$

where E is the second component of the map $\boldsymbol{\Phi}^{-1}$.

Proposition 7.2. *For any* $\sigma > 9$ *the function* Θ *is analytic in* $\mathcal{D}(\sigma)$ *and has the following properties:*

1. $\Theta(t)$ *is real-analytic* h*-periodic function;*
2. $\Theta(0) = 0$ *and the homoclinic invariant (1.6) is given by* $\omega = \dot{\Theta}(0)$*;*
3. $\int_0^h \Theta(t)\, dt = O(h^{-3} \exp(-2\pi^2/h))$*;*
4. *On the upper edge of* $\mathcal{D}(\sigma)$ *with* $\sigma = N + 5$*,*

$$\Theta(t) = -\mathrm{e}^{-\mathrm{i}2\pi\tau} \sum_{n=0}^{N-1} \theta_n h^{2n-1} + O(h^{2N-2}),$$

where $\tau = (t - \mathrm{i}\pi/2)/h$*;*

5. *For any real* t*,*

$$\Theta(t) = 2\mathrm{e}^{-\pi^2/h} \sin\left(\frac{2\pi t}{h}\right) \sum_{n=0}^{N-1} \omega_n h^{2n-1} + O(h^{2N-1}\mathrm{e}^{-\pi^2/h}),$$

$$\dot{\Theta}(t) = 4\pi\mathrm{e}^{-\pi^2/h} \cos\left(\frac{2\pi t}{h}\right) \sum_{n=0}^{N-1} \omega_n h^{2n-2} + O(h^{2N-2}\mathrm{e}^{-\pi^2/h}),$$

where $\omega_n = \Im\theta_n$ *are real numbers.*

The main theorem (Theorem 1.1) immediately follows from 2 and 5.

Proof of the proposition. The estimates at the end of the last section show that in $\mathcal{D}(\sigma)$,

$$x^+(t) - x^-(t) = O\left(h^\sigma (\log h^{-1})^2\right). \tag{7.4}$$

It follows directly that the function $\Theta(t)$ is analytic in $\mathcal{D}(\sigma)$ and from assertion (6) of Theorem 7.1 it follows that it is real-analytic.

It follows from assertion (3) and Eq. (1.3) that $\Theta(t)$ is periodic with period h. Indeed, if a point (x,y) and its image $SM(x,y)$ belong to the domain of E, we have $E(x,y) = E\big(SM(x,y)\big)$. On the other hand Eq. (1.3) reads:

$$\left(x^+, y^+\right)(t+h) = SM\left((x^+,y^+)(t)\right).$$

Taking into account the definition (7.3) we obtain the periodicity: $\Theta(t+h) = \Theta(t)$.

Since $t = 0$ is a homoclinic point we have $(x^+,y^+)(0) = (x^-,y^-)(0)$ and $\Theta(0) = 0$.

Differentiating (7.3) we get

$$\dot\Theta(0) = \frac{\partial E}{\partial x}\dot x^+(0) + \frac{\partial E}{\partial y}\dot y^+(0),$$

where the derivatives of E are taken at the point $(x^+(0), y^+(0))$. It follows from (2) and (4) that

$$\left.\frac{\partial E}{\partial x}\right|_{(x^-,y^-)} = -\dot y^-, \qquad \left.\frac{\partial E}{\partial y}\right|_{(x^-,y^-)} = \dot x^-$$

at the points of the unstable separatrix $(x^-(t), y^-(t))$.

Since $(x^+(0), y^+(0)) = (x^-(0), y^-(0))$ we have

$$\dot\Theta(0) = \dot x^-(0)\,\dot y^+(0) - \dot y^-(0)\,\dot x^+(0),$$

the last expression coincides with the definition of the homoclinic invariant (1.6). This completes the proof of assertion 2.

Let us calculate the function $\Theta(t) = E(x^+(t), y^+(t))$ by the Taylor formula taking $(x^-(t), y^-(t))$ as a center for the expansion:

$$\Theta(t) = E(x^-,y^-) + \frac{\partial E}{\partial x}\cdot(x^+ - x^-) + \frac{\partial E}{\partial y}\cdot(y^+ - y^-) + O_2, \tag{7.5}$$

where $\frac{\partial E}{\partial x}$ and $\frac{\partial E}{\partial y}$ are taken at (x^-,y^-), and we skipped the argument t on the functions x^-, y^-, x^+, y^+. It follows from (4) that $E(x^-,y^-) = 0$ and it follows from (5) that $O_2 = O(h^{-10}(x^+ - x^-)^2)$. Taking into account the above expression for the first derivatives of E we may rewrite Eq. (7.5) as

$$\Theta(t) = \mathcal{W}\left[\dot x^-; x^+ - x^-\right](t) + O(h^{-10}(x^+ - x^-)^2), \tag{7.6}$$

where $\mathcal{W}$ is a Wronskian

$$\mathcal{W}[f;g](t) = \det\begin{pmatrix} f & g \\ \underline{\Delta}_h f & \underline{\Delta}_h g\end{pmatrix}(t) = f(t-h)g(t) - g(t-h)f(t).$$

As in many other places we used the first equation of the system (1.3) written in the form $y(t) = x(t) - x(t-h) = \underline{\Delta}_h x$ to exclude the y-component from consideration.

We evaluate the right-hand side of (7.6) on the segment

$$\Im t = \frac{\pi}{2} - \frac{\sigma}{2\pi} h \log h^{-1}, \qquad |\Re t| < 10h.$$

It is convenient to use the variable $\tau = (t - i\pi/2)/h$ instead of t. In terms of this variable the segment takes the form:

$$\Im\tau = -\frac{\sigma}{2\pi} \log h^{-1}, \qquad |\Re\tau| < 10.$$

From (7.4) we get the upper bound for the quadratic term:

$$\Theta(t) = \mathcal{W}\left[\dot{x}^-; x^+ - x^-\right](t) + O(h^{2\sigma-10}(\log h^{-1})^4). \tag{7.7}$$

Let us evaluate the Wronskian. From (5.13) we have

$$\begin{aligned}\dot{x}^-(t) &= i\sum_{m=0}^{N-1} h^{2m-1}\frac{du_m^-}{d\tau} + O(h^{2N-1}(\log h^{-1})^{2N}) \\ &= i\sum_{m=0}^{N-1} h^{2m-1}\varphi_{1,m}(\tau) + O(h^{2N-1}(\log h^{-1})^{2N}),\end{aligned} \tag{7.8}$$

where we used the definition (6.11) for $\varphi_{1,m}$. Taking into account the last equality we have

$$\begin{aligned}\mathcal{W}\left[\dot{x}^-; x^+ - x^-\right](t) = i\sum_{m=0}^{N-1} & h^{2m-1}\mathcal{W}[\varphi_{1,m}; x^+ - x^-] \\ & + O(h^{2N-1+\sigma}(\log h^{-1})^{2N+2}).\end{aligned} \tag{7.9}$$

From (5.13) and (6.2), (6.3) we get that

$$\begin{aligned}x^+(t) - x^-(t) &= i\sum_{n=0}^{N-1} h^{2n}(u_n^+(\tau) - u_n^-(\tau)) + O\left(h^{2N}(\log h^{-1})^{2N}\right) \\ &= i\sum_{n=0}^{N-1} h^{2n} w_n(\tau) + O\left(h^{2N}(\log h^{-1})^{2N}\right),\end{aligned} \tag{7.10}$$

where we used the definition (6.8) for w_n to obtain the second equality.

Using (7.10) with N replaced by $N - m$ we get

$$\mathcal{W}[\varphi_{1,m}; x^+ - x^-] = i\sum_{k=0}^{N-1-m} h^{2k}\mathcal{W}[\varphi_{1,m}; w_k] + O(h^{2(N-m)}(\log h^{-1})^{2(N-m)}).$$

Substituting this into (7.9) we have

$$\begin{aligned}\mathcal{W}\left[\dot{x}^-; x^+ - x^-\right](t) = -\sum_{m=0}^{N-1}\sum_{k=0}^{N-m-1} & h^{2m-1+2k}\mathcal{W}[\varphi_{1,m}; w_k] \\ & + O(h^{2N-1}(\log h^{-1})^{2N}) + O(h^{2N-1+\sigma}(\log h^{-1})^{2N+2}).\end{aligned}$$

Introducing the new index $n = m + k$ and changing the order of the terms we rewrite the last equality as

$$\mathcal{W}\left[\dot{x}^-; x^+ - x^-\right](t) = -\sum_{n=0}^{N-1} h^{2n-1} \sum_{k=0}^{n} \mathcal{W}[\varphi_{1,n-k}; w_k] + O(h^{2N-1}(\log h^{-1})^{2N}).$$

Taking into account the relation (6.18) we get

$$\mathcal{W}\left[\dot{x}^-; x^+ - x^-\right](t) = -\mathrm{e}^{-2\mathrm{i}\pi\tau} \sum_{n=0}^{N-1} h^{2n-1}\theta_n$$
$$+O(h^{2\sigma(1-\delta)}) + O(h^{2N-1}(\log h^{-1})^{2N}).$$

Substitution of the last formula into (7.7) gives

$$\Theta(t) = -\mathrm{e}^{-2\mathrm{i}\pi\tau} \sum_{n=0}^{N-1} h^{2n-1}\theta_n$$
$$+O(h^{2\sigma(1-\delta)}) + O(h^{2N}(\log h^{-1})^{2N-1}) + O(h^{2\sigma-10}(\log h^{-1})^4).$$

Choosing $0 < \delta < 4/(N+5)$, $\sigma = N+5$ we see that all the error terms in the last formula become essentially of the same order:

$$\Theta(t) = -\mathrm{e}^{-2\mathrm{i}\pi\tau} \sum_{n=0}^{N-1} h^{2n-1}\theta_n + O(h^{2N-2}),$$

where we used that $(\log h^{-1})^{2N} = o(h^{-1})$. This finishes the proof of assertion 4.

Let us suppose that θ_n are purely imaginary and define real numbers ω_n by

$$\theta_n = \mathrm{i}\omega_n. \tag{7.11}$$

We justify this supposition at the end of the proof. Coming back to the variable t we obtain

$$\Theta(t) = -\mathrm{i}\mathrm{e}^{-\pi^2/h}\mathrm{e}^{-2\mathrm{i}\pi t/h} \sum_{n=0}^{N-1} h^{2n-1}\omega_n + O(h^{2N-2}).$$

Since $\Theta(t)$ is real-analytic we have on the complex conjugate segment ($\Im t = -\pi/2 + (\sigma/2\pi)h\log h^{-1}$):

$$\Theta(t) = \overline{\Theta(\overline{t})} = i\mathrm{e}^{-\pi^2/h}\mathrm{e}^{2\mathrm{i}\pi t/h} \sum_{n=0}^{N-1} h^{2n-1}\omega_n + O(h^{2N-2}).$$

Combining the last two formulas we obtain that

$$\Theta(t) = \mathrm{e}^{-\pi^2/h} 2\sin(2\pi t/h) \sum_{n=0}^{N-1} h^{2n-1}\omega_n + O(h^{2N-2})$$

on the union of two segments. A maximum modulus theorem implies that an analytic periodic function in a strip takes its maximum on the boundary of the strip. Applying this to the error term in the formula above we see that the same estimate is valid inside the strip $|\Im t| < \pi/2 - (\sigma/2\pi)h\log h^{-1}$.

Now we get the upper bound for $\Theta_0 = h^{-1}\int_0^h \Theta(t)\,dt$. The standard arguments based either on the symmetries of the standard map or on its area-preserving properties show that the algebraic value of the area of the domain, bounded by the segments of W_1^s and W_1^u ending at the main homoclinic point z_0 and at its image $SM(z_0)$, equals zero. We may calculate this area in the coordinates of Theorem 7.1:

$$\int_0^h \Theta(t)d(t + \Psi(t)) = 0,$$

where we used that the unstable manifold is represented by $E = 0$ and the stable one may be represented in the parametric form as the image of $\boldsymbol{\Phi}^{-1}(x^+(t), y^+(t))$. The second component of this function is $\Theta(t)$ and we denote the first component by $t+\Psi(t)$. Obviously, $\Psi(t)$ is an analytic h-periodic function in $\mathcal{D}(\sigma)$. Moreover

$$\Theta(t) = O(h^{\sigma-1}) \qquad \text{and} \qquad \Psi(t) = O(h^{\sigma-2}).$$

Let $\tilde{\Theta}(t) = \Theta(t) - \Theta_0$. We may apply Lemma 1.4 with $b = \pi/2 - (\sigma/2\pi)h \log h^{-1}$ to get the following estimates:

$$\tilde{\Theta}(t) = O(h^{-1}\mathrm{e}^{-\pi^2/h}) \qquad \text{and} \qquad \dot{\Psi}(t) = O(h^{-2}\mathrm{e}^{-\pi^2/h}) \qquad \text{for} \qquad t \in \mathbb{R}.$$

The integral may be rewritten as

$$\int_0^h (\Theta_0 + \tilde{\Theta}(t))(1 + \dot{\Psi}(t))\, dt = h\Theta_0 + \int_0^h \tilde{\Theta}(t)\dot{\Psi}(t)\, dt,$$

where we used that the mean values of $\tilde{\Theta}$ and $\dot{\Psi}$ equal zero. Since the integral in the left-hand side is equal to zero we have

$$\Theta_0 = -\frac{1}{h}\int_0^h \tilde{\Theta}(t)\dot{\Psi}(t)\, dt.$$

This results in the estimate

$$\Theta_0 = O(h^{-3}\mathrm{e}^{-2\pi^2/h}),$$

that is the constant Θ_0 is exponentially small value of the second order (the constant in the exponent is twice the constant from (1.7)).

The mean value of the function Θ may be nonzero but the estimate above shows that it is neglegible and we still may apply Lemma 1.4 to get

$$\Theta(t) = \mathrm{e}^{-\pi^2/h} 2\sin(2\pi t/h) \sum_{n=0}^{N-1} h^{2n-1}\omega_n + O(h^{N-5}\mathrm{e}^{-\pi^2/h}),$$

$$\dot{\Theta}(t) = \mathrm{e}^{-\pi^2/h}(4\pi/h)\cos(2\pi t/h) \sum_{n=0}^{N-1} h^{2n-1}\omega_n + O(h^{N-6}\mathrm{e}^{-\pi^2/h}).$$

Since N is an arbitrary integer we may use this formula with N replaced by $2N + 4$ in order to get assertion 5.

Now we can make a posteriori justification for considering θ_n as purely imaginary constants. If this was not true, the formulas would lead to the same estimate for $\Theta(t)$ but with shifted phase in the sin function. This would be in contradiction with $\Theta(0) = 0$ for all h. □

8. Proof of the Existence of the Formal Separatrix (Proposition 2.1)

As we have already mentioned, (2.9) gives the unique solution to the first equation (2.4) which satisfies the boundary conditions. Let us write down the equations for x_n with $n > 1$. Note first that, as it follows from (2.9) and (2.5),

$$\begin{aligned} H_0(x_0)(t) &= \cos x_0(t) = 1 - \frac{2}{\cosh^2 t}, \\ G_0(x_0)(t) &= \sin x_0(t) = -\frac{2\sinh t}{\cosh^2 t}. \end{aligned} \tag{8.1}$$

Using the recurrent equations (2.5), one easily obtains for $n \geq 1$,

$$\begin{aligned} &G_n\big(x_0(t), x_1(t), \dots, x_n(t)\big) \\ &= \left(1 - \frac{2}{\cosh^2 t}\right) x_n(t) + \tilde{G}_n\big(x_0(t), x_1(t), \dots, x_{n-1}(t)\big), \end{aligned} \tag{8.2}$$

where

$$\tilde{G}_n\big(x_0, x_1, \dots, x_{n-1}\big) = n^{-1} \sum_{k=1}^{n-1} k x_k H_{n-k}(x_0, x_1, \dots, x_{n-k}). \tag{8.3}$$

Taking into account (8.2), we rewrite Eq. (2.4) for $n = m + 1 \geq 2$ in the form

$$\begin{aligned} &\frac{d^2}{dt^2} x_m(t) - \left(1 - \frac{2}{\cosh^2 t}\right) x_m(t) = \tilde{G}_m\big(x_0(t), \dots, x_{m-1}(t)\big) \\ &+ \sum_{k=2}^{m+1} \frac{1}{(2k)!} \left\{ G_{m+1-k}(x_0(t), \dots, x_{m+1-k}(t)) - \frac{d^{2k}}{dt^{2k}} x_{m+1-k}(t) \right\}. \end{aligned} \tag{8.4}$$

We prove the proposition by induction. Let x_n, $0 < n \leq m - 1$, be the unique solution of Eq. (2.4) with $1 \leq p \leq m$, which satisfy the boundary conditions (2.6), (2.7) and have the form (2.10).

Equation (8.4) for x_m is a linear nonhomogeneous equation of the form

$$\frac{d^2}{dt^2} x(t) - \left(1 - \frac{2}{\cosh^2 t}\right) x(t) = G(t). \tag{8.5}$$

The corresponding homogeneous equation

$$\frac{d^2}{dt^2} x(t) - \left(1 - \frac{2}{\cosh^2 t}\right) x(t) = 0$$

has two basic solutions

$$\frac{2}{\cosh t} \quad \text{and} \quad \sinh t + \frac{t}{\cosh t}.$$

Neither of them satisfies both (2.6) and (2.7). This proves the uniqueness of x_m.

The proof of the following lemma is straightforward.

Lemma 8.1. *Let the right-hand side of* (8.5) *be of the form*

$$G(t) = \sum_{k=2}^{m+1} \frac{c_k}{(\cosh t)^{2k}} \sinh t. \tag{8.6}$$

Then Eq. (8.5) *has a solution*

$$X(t) = \sum_{k=1}^{m} \frac{a_k}{(\cosh t)^{2k}} \sinh t,$$

where a_k*,* $1 \le k \le m$*, are the unique solution of the system*

$$4k(k-1)a_k - (4k^2 - 6k)a_{k-1} = c_k, \qquad 2 \le k \le m+1,$$
$$a_{m+1} = 0.$$

In view of this lemma it is sufficient to check that the right-hand side of (8.4) is of the form (8.6).

Lemma 8.2. *Let* x_n*,* $1 \le n \le m-1$*, be of the form* (2.10)*. Then for* $n = 1, \ldots, m-1$*,*

$$G_n(x_0(t), x_1(t), \ldots, x_n(t)) = \sum_{k=1}^{n+1} \frac{g_{nk}}{(\cosh t)^{2k}} \sinh t,$$

$$H_n(x_0(t), x_1(t), \ldots, x_n(t)) = \sum_{k=1}^{n+1} \frac{a_{nk}}{(\cosh t)^{2k}},$$

and

$$\tilde{G}_m(x_0(t), x_1(t), \ldots, x_{m-1}(t)) = \sum_{k=2}^{m+1} \frac{\tilde{g}_{mk}}{(\cosh t)^{2k}} \sinh t.$$

Proof. A straightforward calculation, which uses (8.1), (2.5), and (8.3), gives explicitly

$$G_1 = a_{11} \left(\frac{\sinh t}{(\cosh t)^2} - 2\frac{\sinh t}{(\cosh t)^4} \right),$$
$$H_1 = 2a_{11} \left(\frac{1}{(\cosh t)^2} - \frac{1}{(\cosh t)^4} \right),$$
$$\tilde{G}_1 = 0.$$

The formulae (8.1) show that the assertion is true for $n = 1$. Then the assertion for $n > 1$ follows by induction which again uses (8.1), (2.5), and (8.3), if one takes into account the identity

$$\frac{\sinh t}{(\cosh t)^{2k}} \frac{\sinh t}{(\cosh t)^{2l}} = \frac{1}{(\cosh t)^{2(k+l-1)}} - \frac{1}{(\cosh t)^{2(k+l)}}. \qquad \square$$

It follows from Lemma 8.2 that the first term in the right-hand side of (8.4) is of the required form. Consider the expression in the curly braces in the second one. Applying (8.2) and the assertion of Lemma 8.2 concerning G_n, we find that the unique suspicious terms are

$$x_n(t) - \frac{d^{2k}}{dt^{2k}} x_n(t). \tag{8.7}$$

The following formula is obvious:

$$\frac{d^2}{dt^2} \sum_{k=1}^{n} \frac{a_k \sinh t}{(\cosh t)^{2k}} = \sum_{k=1}^{n+1} \left[a_k(4k^2 - 4k + 1) - a_{k-1}(2k-1)(2k-2) \right] \frac{\sinh t}{(\cosh t)^{2k}}.$$

It follows from this formula that the double differentiation does not change the first coefficient in the expression of the form (2.10) and increases the n by 1. So the expression (8.7) has the desired form (8.6). This finishes the proof of Proposition 2.1. One checks the explicit formulae (2.11) by direct substitution into the equations. □

9. Solutions of Linear Finite-Difference Equations

9.1. Solutions of the equation $\Delta a = g$. In this section we consider the way of solving the finite-difference equation

$$\Delta a = g, \tag{9.1}$$

where $\Delta a(x) = a(x+1) - a(x)$ is the first order difference operator, x denotes the variable which ranges over a domain $D \subset \mathbb{C}$. The function $a(x)$ defined by

$$a(x) = -\sum_{k=0}^{\infty} g(x+k) \quad \text{or} \quad a(x) = \sum_{k=1}^{\infty} g(x-k)$$

solves the equation provided the series on the right-hand side is well defined and convergent. Unfortunately, this will not be the case for the most parts of the present paper.

Following [Laz91] we will describe a special class of domains which are most convenient for solving Eq. (9.1). We fix a real number $\delta_0 \in]0, \pi/2[$. All constants in the estimates which follow depend on the choice of δ_0. Let A be a positive number. We assume that $A > \max\{1, 4\tan\delta_0\}$. We say a non-void subset $D \subset \mathbb{C}$ is of the type $(A, +)$ if the following is true:

(1) D is closed;
(2) D does not intersect with the open disk $\{x \in \mathbb{C} : |x| < A\}$;
(3) if $x \in D$ then the positive ray $\{z \in \mathbb{C} : z = x + t, t > 0\} \subset D$;
(4) D does not intersect with the negative δ_0-sector $|\arg x - \pi| < \delta_0$.

The definition of a domain of type $(A, -)$ is similar. One has to replace the last two conditions by the following:

(3) if $x \in D$ then the negative ray $\{z \in \mathbb{C} : z = x + t, t < 0\} \subset D$;
(4) D does not intersect with the positive δ_0-sector $|\arg x| < \delta_0$.

Let D be a closed domain in $\mathbb{C}$ such that $0 \notin D$. Given non-negative μ, denote by $\mathcal{X}_\mu(D)$ the space of all complex valued continuous functions defined in D, analytic in interior points of D, and possessing the finite norm

$$\|a\|_\mu = \sup_{x \in D} |x^\mu a(x)|. \tag{9.2}$$

Evidently, $\mathcal{X}_\mu(D)$ supplied with the norm (9.2) is a Banach space. If D satisfies the above condition (2), these norms are subordinated:

$$\|a\|_\mu \leq \frac{1}{A^\nu}\|a\|_{\mu+\nu}. \tag{9.3}$$

We will use the following definition of the norms of linear and bilinear maps. Given a linear map $\phi : \mathcal{X}_1 \to \mathcal{X}_2$ between two Banach spaces $(\mathcal{X}_i, \|\cdot\|)$, $i = 1, 2$, we define the norm of ϕ as

$$\|\phi\| = \sup_{a\in\mathcal{X}_1, a\neq 0} \frac{\|\phi(a)\|}{\|a\|}.$$

Analogously, if $\theta : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{X}_3$ is a bilinear map, $(\mathcal{X}_i, \|\cdot\|)$, $i = 1, 2, 3$, being Banach spaces, the norm of θ is defined by the equality

$$\|\theta\| = \sup_{a_1\in\mathcal{X}_1, a_2\in\mathcal{X}_2, a_1\neq 0, a_2\neq 0} \frac{\|\theta(a_1, a_2)\|}{\|a_1\|\,\|a_2\|}.$$

Proposition 9.1 ([Laz91]). *Let D be a domain of type $(A, +)$, $\mu > 0$. Then the formula*

$$\Delta_+^{-1} g(x) = -\sum_{k=0}^{\infty} g(x + k) \tag{9.4}$$

defines a linear map $\Delta_+^{-1} : \mathcal{X}_{\mu+1}(D) \to \mathcal{X}_\mu(D)$ with the norm bounded from above by a constant depending only on δ_0 and μ.

Analogously, if D is of type $(A, -)$, the formula

$$\Delta_-^{-1} g(x) = \sum_{k=1}^{\infty} g(x - k) \tag{9.5}$$

defines a linear map $\Delta_-^{-1} : \mathcal{X}_{\mu+1}(D) \to \mathcal{X}_\mu(D)$ with the norm bounded from above by a constant depending only on δ_0 and μ.

Proof. Evidently the series in (9.4), (9.5) converge and give analytic functions in the corresponding domains satisfying (9.1). It remains to estimate the norms of the right sides of (9.4), (9.5). We have

$$\|\Delta_\pm^{-1} g(x)\|_\mu \leq \|g\|_{\mu+1} \sup_{x\in D} \sum_{k=0,1}^{\infty} \frac{|x|^\mu}{|x \pm k|^{\mu+1}}. \tag{9.6}$$

Consider for definiteness the case + in (9.6), that is the case of a domain of the type $(A, +)$. The opposite case can be considered in the analogous way. Let us associate with the k^{th} term of the sum

$$\sum_{k=0}^{\infty} \frac{|x|^\mu}{|x + k|^{\mu+1}} \tag{9.7}$$

the rectangle with height equal to the value of this term and with the base $[\Re x + k, \Re x + k + 1]$ if $\Re x + k < 0$ and with the base $[\Re x + k - 1, \Re x + k]$ if $\Re x + k \geq 0$. Note that the area of the rectangle is equal to the value of the term. If we place the rectangles in $\mathbb{R}^2$ with bases on the first axis, then all but one rectangle will be situated under the graph of the function

$$h(t) = \frac{|x|^\mu}{|x + t|^{\mu+1}}.$$

The excepting term corresponds to the $x + k$ nearest to the imaginary axis. If $\Re x \geq 0$, the corresponding term can be bounded as follows:

$$\frac{|x|^\mu}{|x + k|^{\mu+1}} \leq \frac{|x|^\mu}{|x|^{\mu+1}} \leq \frac{1}{A} < 1.$$

If $\Re x < 0$, we have

$$\frac{|x|^\mu}{|x + k|^{\mu+1}} \leq \frac{|x|^\mu}{|\Im x|^{\mu+1}} \leq \frac{1}{A} \frac{1}{(\sin \delta_0)^\mu} < \frac{1}{(\sin \delta_0)^\mu},$$

since in this case $|\Im x| \geq A$ due to condition (2) in the definition of the type $(A, +)$, and $|x|/|\Im x| \leq 1/\sin \delta_0$ due to condition (4).

The remaining terms are evaluated from above by

$$\int_0^\infty h(t)\, dt = \int_0^\infty \frac{d\tau}{|\xi + \tau|^{\mu+1}},$$

where $\xi = x/|x|$. The last integral is bounded by a constant depending only on δ_0 as it follows again from condition (4). $\square$

9.2. Lemma on Cauchy integral. For our purposes it is necessary to resolve Eq. (9.1) in some domains which do not satisfy condition (3) in the definition of an admissible domain. The main idea is to represent the domain in question in the form

$$D = D^+ \cap D^-, \tag{9.8}$$

where D^+ and D^- are respectively of the type $(A, +)$ and $(A, -)$. If we could find appropriate "projections" which represent a function defined in D as a sum of those defined in $D^\pm$, the problem would be reduced to that already solved.

In this section we formulate and prove the lemma, which provides the desired representation. We assume that the set $D \subset \mathbb{C}$ can be represented in the form (9.8) and the sets $D^\pm$ satisfy the conditions (1) and (3) from the definition of the types $(A, \pm)$, respectively. Moreover, let $S = \{x \in \mathbb{C} : |\Re x| \leq 2\}$, the following intersections have to be equal and connected:

$$D^+ \cap S = D^- \cap S.$$

It follows directly that $D \cap S = D^\pm \cap S$. If $D \cap S$ is compact, then it is a rectangle with sides parallel to the real and imaginary axes. We also assume that the height of the rectangle is not less than 2. If the intersection is not compact, it is either the whole strip S, or one-half of S, obtained from S by cutting it by a line $\Im x = \text{const}$.

Denote by $\mathcal{L}$ the space consisting of all complex valued Lipschitz functions defined on ∂D which take constant values for $\Re x \geq 1$ and for $\Re x \leq -1$ (given a function, left and right constant values are not necessarily equal). The norm in $\mathcal{L}$ is defined as

$$\|\chi\| = \max_x |\chi(x)| + \sup_{x \neq y} \frac{|\chi(x) - \chi(y)|}{|x - y|}.$$

Lemma 9.2 (on the Cauchy integral [Laz91]**).** *Let* $\chi \in \mathcal{L}$, $g \in \mathcal{X}_0(D)$ *and*

$$J_g = \frac{1}{2\pi} \int_{\partial D} |g(\xi)|\, |d\xi| < \infty. \tag{9.9}$$

Then the integral

$$h(x) = \frac{1}{2\pi \mathrm{i}} \int_{\partial D} \frac{\chi(\xi) g(\xi)}{\xi - x} d\xi \tag{9.10}$$

defines two functions h_{int} *and* h_{ext}*, in the interior of* D *and in the exterior of* D *respectively. Both the functions,* h_{int} *and* h_{ext}*, admit continuous prolongations onto the closures of their domains, belong to* $\mathcal{X}_0(D)$ *and* $\mathcal{X}_0\left(\overline{\mathbb{C} \setminus D}\right)$ *respectively, and*

$$|h_{\text{int,ext}}(x)| \le \|\chi\|(J_g + \sup |g|). \tag{9.11}$$

If $\operatorname{supp} \chi \not\supset \partial D$*, then* h_{int} *and* h_{ext} *define together a single analytical function on* $\mathbb{C} \setminus \operatorname{supp} \chi$.

Proof. Define an auxiliary function

$$\begin{aligned} \varphi_{x_0}(x) &= \frac{1}{2\pi \mathrm{i}} \int_{\partial D} \frac{\chi(\xi) - \chi(x_0)}{\xi - x} g(\xi)\, d\xi, \qquad x_0 \in \partial D, \quad x \in \mathbb{C} \setminus \partial D, \\ \varphi_{x_0}(x_0) &= \frac{1}{2\pi \mathrm{i}} \int_{\partial D} \frac{\chi(\xi) - \chi(x_0)}{\xi - x_0} g(\xi)\, d\xi, \qquad x_0 \in \partial D. \end{aligned} \tag{9.12}$$

This function is analytic in x, $x \notin \partial D$. Let

$$h_{\text{int}}(x) = \begin{cases} \varphi_{x_0}(x_0) + \chi(x_0) g(x_0), & x = x_0 \in \partial D, \\ h(x), & x \in \text{Interior of } D, \end{cases}$$

$$h_{\text{ext}}(x) = \begin{cases} \varphi_{x_0}(x_0), & x = x_0 \in \partial D, \\ h(x), & x \in \text{Exterior of } D. \end{cases}$$

These two functions are analytical inside their domains of definition and we only have to establish that they are continuous and to check the upper bound (9.11).

Denote

$$\hat{g}(x) = \begin{cases} g(x) \text{ if } x \in D, \\ 0 \quad\ \text{if } x \notin D, \end{cases}$$

then for all $x \in \mathbb{C} \setminus \partial D$ and $x_0 \in \partial D$,

$$h(x) = \varphi_{x_0}(x) + \chi(x_0) \hat{g}(x). \tag{9.13}$$

Since the function $\hat{g}$ is bounded and continuous inside D and outside D, it is sufficient to establish that the function $\varphi_{x_0}(x)$ is continuous at all points $x = x_0$, $x_0 \in \partial D$.

Given $\sigma > 0$, define the set

$$\Delta^\sigma = \{x \in \partial D : |\Re x| < 1 + \sigma\}.$$

If $\sigma \leq 1$, then Δ^σ is a union of two or less segments parallel to the real axis.

Consider $x_0 \in \partial D \backslash \Delta^0$, the function $\varphi_{x_0}(x)$ is not only continuous but even analytical in x in a small neighborhood of the point $x = x_0$, because $\chi(\xi)$ is a constant for $|\Re\xi| > 1$.

Consider $x_0 \in \Delta^{1/2} \supset \Delta^0$, and study the restriction of the function $\varphi_{x_0}(x)$ on the segment

$$l(x_0) = \{x \in \mathbb{C} : \Re x = \Re x_0, |\Im(x - x_0)| \leq 1/2\}$$

orthogonal to the rectilinear part of ∂D. The following assertions finish the proof of the continuity

$$\text{the map } x_0 \longmapsto \varphi_{x_0}(x_0) \text{ is continuous,} \tag{9.14}$$

$$\lim_{x \to x_0,\ x \in l(x_0)} \varphi_{x_0}(x) = \varphi_{x_0}(x_0). \tag{9.15}$$

Proof of the assertion (9.14). Denote i_ϵ the ϵ–neighborhood of x_0 in ∂D, ϵ being a sufficiently small positive number. We have

$$\varphi_{x_0}(x_0) - \varphi_{x_1}(x_1) = \frac{1}{2\pi \mathrm{i}} \int_{i_\epsilon} g(\xi) \left(\frac{\chi(\xi) - \chi(x_0)}{\xi - x_0} - \frac{\chi(\xi) - \chi(x_1)}{\xi - x_1} \right) d\xi$$
$$+ \frac{1}{2\pi \mathrm{i}} \int_{\partial D \backslash i_\epsilon} g(\xi) \left(\frac{\chi(\xi) - \chi(x_0)}{\xi - x_0} - \frac{\chi(\xi) - \chi(x_1)}{\xi - x_1} \right) d\xi.$$

Let us consider the first term in the right-hand side of the last formula. The expression in the parenthesis can be bounded from above by $2\|\chi\|$. So the module of the first term is evaluated from above by

$$\frac{2}{\pi} \sup_{i_\epsilon} |g| \, \|\chi\| \, \epsilon.$$

Given positive ϵ', take ϵ small enough for the first term to become less than $\epsilon'/2$.

Fix such an $\epsilon < 1$ and consider the second term. Let $|x_0 - x_1| \leq \epsilon/2$, then

$$\left| \frac{\chi(\xi) - \chi(x_0)}{\xi - x_0} - \frac{\chi(\xi) - \chi(x_1)}{\xi - x_1} \right| = \left| \frac{\chi(x_1) - \chi(x_0)}{\xi - x_0} + \frac{\chi(x_1) - \chi(\xi)}{x_1 - \xi} \, \frac{x_1 - x_0}{\xi - x_0} \right|$$

does not exceed $\frac{2}{\epsilon}\|\chi\| \, |x_1 - x_0|$. So the second term can be bounded from above by

$$\frac{4}{\epsilon} \|\chi\| \, |x_1 - x_0| \, J_g.$$

Taking x_1 close to x_0 so that the expression becomes less than $\epsilon'/2$, we obtain

$$|\varphi_{x_0}(x_0) - \varphi_{x_1}(x_1)| \leq \epsilon',$$

which completes the proof of the assertion (9.14). □

Proof of the assertion (9.15). We have

$$\varphi_{x_0}(x) - \varphi_{x_0}(x_0) = \frac{1}{2\pi \mathrm{i}} \int_{\partial D} g(\xi)\,(\chi(\xi) - \chi(x_0)) \left(\frac{1}{\xi - x} - \frac{1}{\xi - x_0} \right) d\xi. \quad (9.16)$$

Since

$$\frac{1}{\xi - x} - \frac{1}{\xi - x_0} = \frac{x - x_0}{(\xi - x)(\xi - x_0)} \quad \text{and} \quad \left| \frac{\chi(\xi) - \chi(x_0)}{\xi - x_0} \right| \le \|\chi\|,$$

the module of the right-hand side of (9.16) is bounded from above by

$$\|\chi\|\,|x - x_0| \frac{1}{2\pi} \int_{\partial D} \frac{|g(\xi)|}{|\xi - x|} |d\xi|. \quad (9.17)$$

The integral in (9.17) can be broken into two integrals: one distributed onto Δ^1 and another onto $\partial D \setminus \Delta^1$. The latter is bounded by $2J_g$ for $|\xi - x| \ge 1/2$ on the domain of integration. The former can be evaluated from above by

$$\frac{\sup|g|}{2\pi} \int_{\Delta^1} \frac{|d\xi|}{|\xi - x|} \le \text{const } \sup|g| \ \log|x - x_0|^{-1}.$$

Indeed,

$$\int_{\Delta^1} \frac{|d\xi|}{|\xi - x|} = \int_{-2}^{2} \frac{dt}{\sqrt{(t - \Re x_0)^2 + |x - x_0|^2}} < \int_{-4}^{4} \frac{d\tau}{\sqrt{\tau^2 + |x - x_0|^2}}$$
$$= \int_{-4/|x - x_0|}^{4/|x - x_0|} \frac{ds}{\sqrt{s^2 + 1}} \le \text{const } \log \frac{1}{|x - x_0|}.$$

Substituting these estimates into (9.17) and taking into account the obvious equality $\lim_{t \to 0} t \log t = 0$ we get the assertion (9.15). □

To obtain the upper estimates we note that an analytic function has no maximum of the module inside the domain of analyticity. Thus it is sufficient to estimate the function $\varphi_{x_0}(x_0)$ on ∂D and to obtain an upper bound for $|h(x)|$ at infinity. Obviously,

$$|\varphi_{x_0}(x_0)| \le \frac{1}{2\pi} \int_{\partial D} \left| \frac{\chi(\xi) - \chi(x_0)}{\xi - x_0} \right| \ |g(\xi)|\ |d\xi| \le \|\chi\|\ J_g. \quad (9.18)$$

To obtain the estimates for large values of $|x|$ choose x_0 to be an arbitrary point on ∂D such that $|\Re x_0| > 1$ and $\Re x_0 \Re x > 0$. Then we can use the estimate $|\xi - x| \ge 2$ to obtain the upper bound

$$|\varphi_{x_0}(x)| \le \frac{1}{2\pi} \int_{\partial D} \left| \frac{\chi(\xi) - \chi(x_0)}{\xi - x} \right| \ |g(\xi)|\ |d\xi| \le \|\chi\|\ J_g. \quad (9.19)$$

The estimate (9.11) follows from the last two estimates and the representation (9.13). □

In the following we will need an alternative estimate for the functions h_{int} and h_{ext}.

Lemma 9.3. *Let the assumptions of Lemma 9.2 be satisfied and let D be a subset of a square with the side R, $R > 2$, then*

$$|h_{int,ext}(x)| \le \text{const } \log R\,\|\chi\| \sup_D |g|. \quad (9.20)$$

Proof. Since the domain D is compact, $h_{ext}(x)$ is regular at infinity. So it is sufficient to estimate $h_{int,ext}$ on the boundary ∂D. Instead of the estimate (9.18) we decompose the integral in (9.12) into the sum of two integrals (on $\Delta^{1/2}$ and on $\partial D \setminus \Delta^{1/2}$). The second one can grow logarithmically as R goes to infinity. □

9.3. An example of nonadmissible domain. The domain D_A is defined as follows:

$$D_A = \{x \in \mathbb{C} : \Im x \leq -A,\ -\pi + \delta_0 \leq \arg x \leq -\delta_0\}, \tag{9.21}$$

where the parameter A satisfies the inequality $A > \max\{4\tan\delta_0, 1\}$. Any domain $D = D_A$ can be represented in the form (9.8) with $D^\pm = D_A^\pm$, where

$$D_A^+ = \{x \in \mathbb{C} : \Im x \leq -A,\ -\pi + \delta_0 \leq \arg x\},$$
$$D_A^- = \{x \in \mathbb{C} : \Im x \leq -A,\ \arg x \leq -\delta_0\}.$$

The following propositions enable us to reduce the problem of solving the equation $\Delta a = g$ in D_A to those in the domains $D_A^\pm$.

Proposition 9.4 ([Laz91]**).** *Given a positive constant δ, there exist two linear maps $P^\pm : \mathcal{X}_{1+\delta}(D_A) \to \mathcal{X}_0(D_A^\pm)$, such that*

(1) *for each $g \in \mathcal{X}_{1+\delta}(D_A)$ and for each $x \in D_A$ we have*

$$g(x) = (P^+g)(x) + (P^-g)(x);$$

(2) $\|P^\pm\| \leq \text{const}\, A^{-\delta}$*, the constant depends only on δ.*

Proof. It follows immediately from the lemma on the Cauchy integral that the bilinear operators

$$J_{\text{int}} : \mathcal{L} \times \mathcal{X}_{1+\delta}(D_A) \longrightarrow \mathcal{X}_0(D_A), \tag{9.22}$$

$$J_{\text{ext}} : \mathcal{L} \times \mathcal{X}_{1+\delta}(D_A) \longrightarrow \mathcal{X}_0\left(\overline{\mathbb{C} \setminus D_A}\right), \tag{9.23}$$

which assign h_{int} and h_{ext} to the pair (χ, g), have bounded norms obeying the estimate

$$\|J_{\text{int,ext}}\| \leq \frac{\text{const}}{A^\delta}.$$

Indeed, $\int_{\partial D_A} |g(\xi)|\,|d\xi| \leq \text{const}\,\|g\|_{1+\delta} A^{-\delta}$ and $\sup|g| \leq \|g\|_{1+\delta} A^{-1-\delta}$.

Take a C^∞-function $\chi_0 : \mathbb{R} \to [0,1]$ such that $\chi_0(t) = 0$ if $t \leq -1$ and $\chi_0(t) = 1$ if $t \geq 1$. Define two C^∞ functions $\chi_\pm \in \mathcal{L}$ as

$$\chi_+(x) = \chi_0(\Re x), \qquad \chi_-(x) = 1 - \chi_+(x), \tag{9.24}$$

and consider $J_{\text{int}}(\chi_\pm, g)$ and $J_{\text{ext}}(\chi_\pm, g)$. Since $\operatorname{supp}\chi_\pm \subset \partial D_A^\pm \cap \partial D_A$, both the integrals define a single function whose domain contains the interior of $D_A^\pm$. Being restricted onto the interior, this function, according to Lemma 9.2, has the continuous prolongation onto $D_A^\pm$. The latter can be taken as $P^\pm g$. All proclaimed properties follow from Lemma 9.2 and the identity $J_{\text{int}}(1, g) = g$. □

Proposition 9.5 ([Laz91]). *Given positive numbers δ, μ, and A, $A > 4\tan\delta_0$, there exists a linear map $\Delta^{-1} : \mathcal{X}_{\mu+2+\delta}(D_A) \to \mathcal{X}_\mu(D_A)$ such that*

- *given $g \in \mathcal{X}_{\mu+2+\delta}(D_A)$, $\Delta^{-1}(g)$ is a solution of Eq. (9.1) at all the values of the independent variable x for which it has meaning;*
- *$\|\Delta^{-1}\| \leq \text{const}\, A^{-\delta}$, where* const *depends only on δ_0, μ and δ.*

Proof. The way of solving the equation $\Delta a = g$ in D_A consists of the following steps: represent the function $x^{1+\mu}g$ as a sum of two functions, the first one has analytical continuation to the right and the other to the left; divide the result by $x^{1+\mu}$ and solve the equation for each of these functions separately; then a can be taken as a sum of these two solutions. The weight $x^{1+\mu}$ is chosen to provide convergence of the sums (9.4) and (9.5). More precisely, let us fix a branch of $x^{1+\mu}$ in the lower half-plane. Define the map

$$\mathcal{I} : \mathcal{X}_{2+\mu+\delta}(D_A) \longrightarrow \mathcal{X}_{1+\delta}(D_A)$$

by the equation

$$(\mathcal{I}g)(x) = x^{1+\mu}g(x).$$

Evidently, $\mathcal{I}$ is an isomorphism between Banach spaces. Define also by the same formula two isomorphisms $\mathcal{I}_\pm : \mathcal{X}_{1+\mu}(D_A^\pm) \longrightarrow \mathcal{X}_0(D_A^\pm)$. Set

$$\Delta^{-1} = \Delta_+^{-1}\mathcal{I}_+^{-1}P_+\mathcal{I} + \Delta_-^{-1}\mathcal{I}_-^{-1}P_-\mathcal{I}, \tag{9.25}$$

where $\Delta_\pm^{-1}$ and $P_\pm$ were defined in Propositions 9.1 and 9.4, respectively. The proclaimed properties of Δ^{-1} follow immediately from those of $\Delta_\pm^{-1}$ and $P_\pm$. □

9.4. The method of variation of parameters. In this section we develop a formal theory of systems of two finite-difference equations

$$\vec{u}(t+h) = \mathrm{A}(t)\vec{u}(t) + \vec{g}(t), \tag{9.26}$$

where $\mathrm{A}(t)$ is a given matrix function and the function $\vec{g}(t)$ is assumed to be known. The system can be reduced to a pair of first order linear difference equations described in the previous section in the following way. Let $\vec{u}_1$ and $\vec{u}_2$ be two linearly independent solutions of the homogeneous equation

$$\vec{u}_k(t+h) = \mathrm{A}(t)\vec{u}_k(t), \qquad k = 1, 2. \tag{9.27}$$

Then a solution of the nonhomogeneous equation can be represented in the form

$$\vec{u}(t) = c_1(t)\vec{u}_1(t) + c_2(t)\vec{u}_2(t) \tag{9.28}$$

with

$$\Delta_h c_1(t) = \frac{\det(\vec{g}(t); \vec{u}_2(t+h))}{W(t+h)}, \tag{9.29}$$

$$\Delta_h c_2(t) = \frac{\det(\vec{u}_1(t+h); \vec{g}(t))}{W(t+h)}, \tag{9.30}$$

where

$$W(t) = \det(\vec{u}_1(t); \vec{u}_2(t)). \tag{9.31}$$

Indeed, substituting (9.28) into Eq. (9.26) we get

$$\begin{aligned} c_1(t+h)\vec{u}_1(t+h) + c_2(t+h)\vec{u}_2(t+h) \\ &= \mathrm{A}(t)\left(c_1(t)\vec{u}_1(t) + c_2(t)\vec{u}_2(t)\right) + \vec{g}(t) \\ &= c_1(t)\vec{u}_1(t+h) + c_2(t)\vec{u}_2(t+h) + \vec{g}(t). \end{aligned}$$

We gather the terms containing c_k on the left-hand side:

$$(\vec{u}_1(t+h); \vec{u}_2(t+h)) \begin{pmatrix} \Delta_h c_1(t) \\ \Delta_h c_2(t) \end{pmatrix} = \vec{g}(t).$$

This system has the determinant equal to $W(t+h)$, and, provided the determinant is not zero, it is equivalent to (9.29) and (9.30). Inversely, given a solution $\vec{u}(t)$ of the system (9.26), we can represent it in the form (9.28) taking

$$c_1(t) = \frac{\det(\vec{u}(t); \vec{u}_2(t))}{W(t)}, \qquad c_2(t) = \frac{\det(\vec{u}_1(t); \vec{u}(t))}{W(t)}.$$

In general, it is not easy to find two linearly independent solutions of a system. But in many cases one can find one solution $\vec{u}_1(t)$, then the second solution can be easily constructed.

First we note that

$$(\vec{u}_1(t+h); \vec{u}_2(t+h)) = \mathrm{A}(t)\left(\vec{u}_1(t); \vec{u}_2(t)\right),$$

and we have

$$W(t+h) = \det(\mathrm{A}(t))W(t). \tag{9.32}$$

Provided $\det(\mathrm{A}(t)) \neq 0$ this equation can be replaced using the substitution

$$W(t) = \exp w(t) \tag{9.33}$$

by the standard first order finite-difference equation

$$\Delta_h w(t) = \log\det(\mathrm{A}(t)). \tag{9.34}$$

In particular, in the case of $\det(\mathrm{A}(t)) \equiv 1$ this equation implies that the Wronskian, $W(t)$, of two solutions of the homogeneous equation (9.27) is h-periodic. A particular solution of Eq. (9.32) is given by $W = 1$.

Using $W(t)$ we can construct the second solution of the homogeneous equation (9.27). The first equation of the system (9.27) with $k = 2$ reads

$$u_{21}(t+h) = A_{11}(t)u_{21}(t) + A_{12}(t)u_{22}(t). \tag{9.35}$$

The second subscript in $u_{kl}(t)$ refers to the component of a vector $\vec{u}_k(t)$, and $A_{ik}(t)$ denotes the ik-component of the matrix $\mathrm{A}(t)$. Using (9.31) in the form

$$u_{22}(t) = \frac{W(t) + u_{12}(t)u_{21}(t)}{u_{11}(t)} \tag{9.36}$$

we can exclude the second component of the vector $\vec{u}_2(t)$ and obtain a first order finite-difference equation on the first component:

$$u_{21}(t+h) = A_{11}(t)u_{21}(t) + A_{12}(t)\frac{W(t) + u_{12}(t)u_{21}(t)}{u_{11}(t)}.$$

Taking into account that $u_{11}(t)$ also satisfies Eq. (9.35) we can rewrite the last equation as

$$u_{21}(t+h) = \frac{u_{11}(t+h)}{u_{11}(t)}u_{21}(t) + \frac{A_{12}(t)W(t)}{u_{11}(t)}. \tag{9.37}$$

The corresponding homogeneous equation has a solution, $u_{11}(t)$, and we again use the variation of parameters looking for u_{21} in the following form:

$$u_{21}(t) = c_0(t)u_{11}(t). \tag{9.38}$$

Then

$$c_0(t+h)u_{11}(t+h) = u_{11}(t+h)c_0(t) + \frac{A_{12}(t)W(t)}{u_{11}(t)}$$

and we have

$$\Delta_h c_0(t) = \frac{A_{12}(t)W(t)}{u_{11}(t)u_{11}(t+h)}. \tag{9.39}$$

Thus we reduce the problem of construction of the second solution for the homogeneous system to the standard form of the single first order difference equation. The components of the vector $\vec{u}_2$ can be obtained by (9.38) and (9.36).

9.5. Solutions of second order difference equations. Our main object here is the second order linear operator L of the form

$$Lu(t) = \Delta_h^2 u(t) - q(t)u(t). \tag{9.40}$$

We will consider both the homogeneous

$$L\varphi = 0 \tag{9.41}$$

and nonhomogeneous

$$Lu = f \tag{9.42}$$

equations. The last equation is equivalent to a system of two equations

$$u(t+h) = u(t) + v(t+h), \qquad v(t+h) = v(t) + q(t)u(t) + f(t).$$

This system has the form (9.26) with

$$\mathrm{A}(t) = \begin{pmatrix} 1 + q(t) & 1 \\ q(t) & 1 \end{pmatrix}, \qquad \vec{g}(t) = \begin{pmatrix} f(t) \\ f(t) \end{pmatrix}.$$

Obviously, $\det(\mathrm{A}(t)) = 1$.

The Wronskian $\mathcal{W}_{f;g}$ of two functions f and g is defined by the formula

$$\mathcal{W}_{f;g}(t) = \det \begin{pmatrix} f(t) & g(t) \\ \bar{\Delta}_h f(t) & \bar{\Delta}_h g(t) \end{pmatrix} = f(t-h)g(t) - f(t)g(t-h). \tag{9.43}$$

The results of the previous section can be summarized in the following form:

- If φ_1 and φ_2 are two solutions of the homogeneous equation (9.41), then $\mathcal{W}_{\varphi_1;\varphi_2}$ is h-periodic.
- If φ_1 and φ_2 are two solutions of the homogeneous equation (9.41), such that $\mathcal{W}_{\varphi_1;\varphi_2} \equiv 1$, then the general solution to the homogeneous equation (9.41) has the form

$$\varphi(t) = \alpha_1(t)\varphi_1(t) + \alpha_2(t)\varphi_2(t), \tag{9.44}$$

  where $\alpha_1(t)$ and $\alpha_2(t)$ are h-periodic functions,

$$\alpha_1(t) = \mathcal{W}_{\varphi;\varphi_2}(t), \qquad \alpha_2(t) = \mathcal{W}_{\varphi_1;\varphi}(t).$$

- If φ_1 and φ_2 are two solutions of the homogeneous equation (9.41) and $\mathcal{W}_{\varphi_1;\varphi_2} \equiv 1$, then the general solution to the nonhomogeneous equation (9.42) has the form

$$u(t) = a_1(t)\varphi_1(t) + a_2(t)\varphi_2(t), \tag{9.45}$$

  where a_1 and a_2 obey the equations

$$\Delta_h a_1 = -\varphi_2 f, \qquad \Delta_h a_2 = \varphi_1 f. \tag{9.46}$$

- Conversely, if a_1 and a_2 satisfy (9.46), then the function u defined by Eq. (9.45) is a solution of Eq. (9.42).
- Let φ_1 be a solution of the homogeneous equation (9.41) and φ_2 satisfy the equality $\mathcal{W}_{\varphi_1;\varphi_2} \equiv 1$, then φ_2 is also a solution of Eq. (9.41). Application of the method of variation of the parameter shows that we can represent φ_2 in the form $\varphi_2(t) = C(t)\varphi_1(t)$, where $C(t)$ satisfies the equation:

$$\Delta_h C = \frac{1}{\varphi_1(t)\varphi_1(t+h)}.$$

9.6. Example: $q(x) = \frac{2}{x^2}$. We use the developed techniques to construct an inverse operator to

$$L_0 u = \Delta^2 u - \frac{2}{x^2} u.$$

This operator was studied in [Laz84, Laz91]. The homogeneous equation $L_0\varphi = 0$ has two solutions:

$$\varphi_{01}(x) = 6 + 12x + 12x^2 \sum_{k=1}^{\infty} (x-k)^{-2},$$

$$\varphi_{02}(x) = -\frac{x^2}{6}.$$

Indeed, substitution to the equation shows that the function x^2 is a solution of the homogeneous equation $L_0\varphi_{02} = 0$; it can be checked by a direct substitution that

$$\mathcal{W}_{\varphi_{01};\varphi_{02}} = 1.$$

Consequently, φ_{01} is a solution of the homogeneous equation. It is not difficult to establish that φ_{01} admits uniform asymptotic expansion in $D(A)$,

$$D(A) = \left\{ x \in \mathbb{C} : \left| \arg\left(x + \frac{A}{\sin\delta_0} \right) \right| \geq \delta_0 \right\},$$

of the form

$$\varphi_{01} = 12 \sum_{m=1}^{\infty} (-1)^m \frac{B_m}{x^{2m-1}},$$

where B_m are Bernoulli numbers $B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, $B_3 = \frac{1}{42}$, $B_4 = \frac{1}{30}$, $B_5 = \frac{5}{66}$, $\ldots$. The function φ_{01} has a meromorphic analytic continuation on $\mathbb{C}$ and

$$\varphi_{01}(x) = -\varphi_{01}(-x) + \frac{12\pi^2 x^2}{\sin^2 \pi x}.$$

Proposition 9.6. *Given $\gamma > 3$ and $A > 1$, the expression*

$$L_0^{-1} f = -\varphi_{01} \Delta_-^{-1}(\varphi_{02} f) + \varphi_{02} \Delta_-^{-1}(\varphi_{01} f), \tag{9.47}$$

where Δ_- was defined by (9.5), defines a continuous operator $L_0^{-1} : \mathcal{X}_\gamma(D(A)) \to \mathcal{X}_{\gamma-2}(D(A))$ with a norm bounded by a constant depending only on δ_0 and γ. If $f \in \mathcal{X}_\gamma(D(A))$, then $w = L_0^{-1} f$ is a solution to the equation $L_0 w = f$.

Proof. Since

$$|\varphi_{01}(x)| \le \text{const}\,|x|^{-1} \quad \text{and} \quad |\varphi_{02}(x)| \le \text{const}\,|x|^2, \tag{9.48}$$

we have $\varphi_{01} f \in \mathcal{X}_{\gamma+1}(D(A))$ and $\varphi_{02} f \in \mathcal{X}_{\gamma-2}(D(A))$. Proposition 9.1 implies the right side of (9.47) to be in $\mathcal{X}_{\gamma-2}(D(A))$, the corresponding norms depending only on δ_0 and γ. $\square$

10. Proof of the First Approximation Theorem (Proposition 3.1)

This proof is a modification of a similar proof from the work [Laz84]. Let

$$\hat{X}_N(t, h^2) = \sum_{n=0}^{N-1} h^{2n} x_n(t) \tag{10.1}$$

be the sum of the first N terms of the series (2.2). We write the solution of Eq. (2.1) in the form $x^-(t) = \hat{X}_N(t, h^2) + Z(t, h^2)$ and we prove the existence and upper bounds for the function Z, first, in a domain, which contains no points near $t = \mathrm{i}\pi/2$, and then in a domain near that point.

10.1. Far from the singularity. As a first step we study the solution of Eq. (2.1) in a domain, which contains no points close to the singularity, namely, in

$$\mathcal{D}_0 = \{t \in \mathbb{C} : \Re t \le -1/2\} \cup \{t \in \mathbb{C} : |\Re t| \le 1, |\Im t| \le 1.5\}.$$

Lemma 10.1. *In the domain $\mathcal{D}_0$,*

$$|x^-(t) - \hat{X}_N(t, h^2)| \le \text{const}\, \mathrm{e}^{\Re t} h^{2N}. \tag{10.2}$$

Proof of Lemma 10.1. The functions $x_n(t)$ were defined in such a way that

$$f_N(t,h^2) = \Delta_h^2 \hat X_N - \varepsilon \sin \hat X_N \qquad (10.3)$$

has all derivatives with respect to the second argument up to the order N equal to zero at $h = 0$ for all values of t. In D_0 we have

$$|\partial_2^{N+1} f_N(t,h^2)| \le \text{const}\, \mathrm{e}^{2\Re t}$$

for all $h \in (0, h_0)$. Then the Taylor formula implies

$$|f_N(t,h^2)| \le \max_{0\le \tilde h \le h} |\partial_2^{N+1} f_N(t,\tilde h^2)| \,\frac{h^{2(N+1)}}{(N+1)!} \le \text{const}\, \mathrm{e}^{2\Re t} h^{2N+2}.$$

We look for a solution of Eq. (2.1) in the form:

$$x^-(t) = \hat X_N(t,h^2) + Z(t,h^2).$$

Substituting into Eq. (2.1) and using (10.3) we write the equation on Z:

$$\Delta_h^2 Z = \varepsilon \sin(\hat X_N + Z) - \varepsilon \sin(\hat X_N) - f_N. \qquad (10.4)$$

It is convenient to rewrite this equation in the form

$$\Delta_h^2 Z - \varepsilon Z = \varepsilon \left(\sin(\hat X_N + Z) - \sin(\hat X_N) - Z\right) - f_N. \qquad (10.5)$$

Define the linear operator L, which acts on a function f of the complex variable t by the formula

$$Lf = \Delta_h^2 f - \varepsilon f. \qquad (10.6)$$

We consider this operator on the space $\mathcal{X}$ of all continuous functions in $\mathcal{D}_0$, analytical in internal points and having finite norm

$$\|f\| = \sup_{t\in\mathcal{D}_0} |\mathrm{e}^{-2\Re t} f(t)| < \infty.$$

The homogeneous equation $L\phi = 0$ has two linearly independent solutions e^{-t} and e^{t}, respectively. No one of them belongs to $\mathcal{X}$. Thus it is possible to define the inverse operator by the formula

$$L^{-1} f(t) = \frac{1}{\sinh(h)} \sum_{k=1}^{\infty} f(t-kh)\sinh(kh). \qquad (10.7)$$

Equation (10.5) may be rewritten in the form

$$LZ = \varepsilon \left(\sin(\hat X_N + Z) - \sin \hat X_N - Z\right) - f_N.$$

In the space $\mathcal{X}$ this equation is equivalent to the equation

$$Z = \varepsilon L^{-1}(\sin(\hat X_n + Z) - Z - \sin(\hat X_N)) - L^{-1}(f_N). \qquad (10.8)$$

To show that the last equation has a solution in $\mathcal{X}$ we use the convergent iteration scheme:

$$Z_0 \equiv 0, \qquad Z_n = \mathcal{F}(Z_{n-1}), \quad n \ge 1, \qquad (10.9)$$

where $\mathcal{F}$ is the nonlinear operator from the right-hand side of (10.8):

$$\mathcal{F}(Z) = \varepsilon L^{-1}\left(\sin(\hat{X}_n + Z) - Z - \sin(\hat{X}_N)\right) - L^{-1}(f_N).$$

Let us estimate the first iterate $Z_1 = L^{-1}(f_N)$:

$$\begin{aligned}|Z_1(t)| &\le \frac{1}{\sinh(h)}\sum_{k=1}^{\infty}|f_N(t-kh,h^2)|\sinh(kh)\\ &\le \frac{\text{const}\,h^{2N+2}}{\sinh(h)}\sum_{k=1}^{\infty}\mathrm{e}^{2(\Re t-kh)}\sinh(kh) = \frac{\text{const}\,h^{2N+2}}{4\sinh(h/2)\sinh(3h/2)}.\end{aligned}$$

Since $\sinh(x)\ge x$ for any positive x, we get

$$|Z_1| \le (C_0/2)h^{2N}\mathrm{e}^{2\Re t}, \tag{10.10}$$

where C_0 denotes a constant. We continue by induction. Suppose that

$$|Z_n - Z_{n-1}| \le \frac{C_0C_1^{n-1}h^{2N}\mathrm{e}^{2n\Re t}}{(2n)!}, \tag{10.11}$$

and prove that this estimate is also true for n replaced by $n+1$. Indeed,

$$\begin{aligned}Z_{n+1} - Z_n &= \mathcal{F}(Z_n) - \mathcal{F}(Z_{n-1})\\ &= \varepsilon L^{-1}\left((\sin(\hat{X}_N + Z_n) - Z_n) - (\sin(\hat{X}_N + Z_{n-1}) - Z_{n-1})\right).\end{aligned} \tag{10.12}$$

Note that

$$\begin{aligned}&(\sin(\hat{X}_N + Z_n) - Z_n) - (\sin(\hat{X}_N + Z_{n-1}) - Z_{n-1})\\ &= (Z_n - Z_{n-1})\int_0^1\left(\cos(\hat{X}_N + \xi Z_n + (1-\xi)Z_{n-1}) - 1\right)\,d\xi.\end{aligned}$$

Using the inductive assumption it is not difficult to see that the sequence of Z_n is bounded by const $\mathrm{e}^{2\Re t}$ and, consequently, the expression under the integral is bounded by

$$\text{const}\,|\hat{X}_N + \xi Z_n + (1-\xi)Z_{n-1}|^2,$$

and the last expression does not exceed $K\mathrm{e}^{2\Re t}$, where K is a constant. Using this estimate we get from (10.12),

$$|Z_{n+1} - Z_n| \le \frac{\varepsilon K}{\sinh(h)}\sum_{k=1}^{\infty}\mathrm{e}^{2\Re(t-kh)}|Z_n(t-kh) - Z_{n-1}(t-kh)|\sinh kh.$$

Now we use again the induction assumption:

$$\begin{aligned}|Z_{n+1} - Z_n| &\le \frac{\varepsilon K}{\sinh(h)}\frac{C_0C_1^{n-1}h^{2N}\mathrm{e}^{2\Re t+2n\Re t}}{(2n)!}\sum_{k=1}^{\infty}\mathrm{e}^{-2(n+1)kh}\sinh kh\\ &= \frac{\varepsilon KC_0C_1^{n-1}h^{2N}\mathrm{e}^{2(n+1)\Re t}}{\sinh(h)(2n)!}\cdot\frac{\sinh(h)}{4\sinh\left(\frac{(2n+1)h}{2}\right)\sinh\left(\frac{(2n+3)h}{2}\right)}.\end{aligned}$$

Using again that $\sinh(x) > x$ we get

$$|Z_{n+1} - Z_n| < \frac{\varepsilon K C_0 C_1^{n-1} h^{2N} \mathrm{e}^{2(n+1)\Re t}}{h^2(2n)!(2n+1)(2n+3)} < C_0 \frac{(\varepsilon/h^2) K C_1^{n-1} h^{2N} \mathrm{e}^{2(n+1)\Re t}}{(2n+2)!}.$$

For $h \le h_0$ we may assume that $(\varepsilon/h^2) \le 2$ and letting $C_1 = 2K$ we conclude by induction that the upper bound (10.11) is valid for all positive integers n. Consequently, the sequence Z_n converges to a solution of Eq. (10.8) and the limit, $\tilde{Z}$, is bounded in the following way:

$$\begin{aligned} |\tilde{Z}(t,h^2)| &\le \sum_{n=1}^{\infty} |Z_n(t,h^2) - Z_{n-1}(t,h^2)| \\ &\le \sum_{n=1}^{\infty} \frac{C_0 C_1^{n-1} h^{2N} \mathrm{e}^{2n\Re t}}{(2n)!} \le \mathrm{const}\, h^{2N} \mathrm{e}^{2\Re t}. \end{aligned}$$

The function $\tilde{x}^-(t) = \hat{X}_N(t,h^2) + \tilde{Z}(t,h^2)$ satisfies Eq. (2.1) and the first boundary condition (1.4), but the second boundary condition may be satisfied only approximately: $\tilde{x}^-(0) = \hat{X}_N(0,h^2) + \tilde{Z}(0,h^2) = \pi + O(h^{2N})$. Let us choose the constant $t_0(h)$ from the condition $\hat{X}_N(t_h,h^2) + \tilde{Z}(t_0(h),h^2) = 0$. By the implicit function theorem $t_0(h) = O(h^{2N})$. Let

$$Z(t,h^2) = \tilde{Z}(t + t_0(h), h^2) + \hat{X}_N(t + t_0(h)) - X_0(t).$$

Obviously, $\hat{X}_N(t + t_0(h), h^2) - \hat{X}_N(t,h^2) = O(\mathrm{e}^{2\Re t} h^{2N})$ in $\mathcal{D}_0$. This finishes the proof of Lemma 10.1. □

10.2. Near singularity. Now we are going to study Eq. (10.4) in the domain $\mathcal{D}_1 = \mathcal{D} \setminus \mathcal{D}_0$, which contains points close to the singularity of the functions $x_n(t)$. It is convenient to use the parameter on the separatrices defined by the formula

$$\tau = \frac{t - \mathrm{i}\frac{\pi}{2}}{h} \tag{10.13}$$

instead of t. Let $\tilde{\mathcal{D}}_1$ denote the set of τ such that $t = i\frac{\pi}{2} + h\tau \in \mathcal{D}_1$. Now we rewrite Eq. (10.4) in the form

$$\Delta^2 Z - \frac{2}{\tau^2} Z = F_0 + (F_{11} + F_{12})Z + F_2(Z), \tag{10.14}$$

where

$$F_0 = -f_N, \tag{10.15}$$

$$F_{11} = \varepsilon \cos x_0 - 2/\tau^2, \tag{10.16}$$

$$F_{12} = \varepsilon(\cos \hat{X}_N - \cos x_0), \tag{10.17}$$

$$F_2(Z) = \varepsilon(\sin(\hat{X}_N + Z) - \sin \hat{X}_N - \cos \hat{X}_N\, Z). \tag{10.18}$$

These functions can be bounded from above in all $\tilde{\mathcal{D}}_1$, except the unit disk centered at $\tau = 0$, in the following way:

$$|F_0| \le \text{const}\, \tau^{-2N-2}, \tag{10.19}$$
$$|F_{11}| \le \text{const}\, h^2, \tag{10.20}$$
$$|F_{12}| \le \text{const}\, \tau^{-4}, \tag{10.21}$$
$$|F_2(Z)| = \varepsilon |Z|^2 \left| \int_0^1 \sin(\hat{X}_N + \xi Z)\, d\xi \right| \le \text{const}\, \tau^{-2} |Z|^2, \tag{10.22}$$

the last estimate being valid provided $|Z| \le 1$.

From Eq. (10.14) we obtain that Z satisfies the equation

$$Z = L_0^{-1}(F_0 + (F_{11} + F_{12})Z + F_2(Z)) + Z_{in}, \tag{10.23}$$

where the operator L_0^{-1} is acting by the formula (9.47) on complex valued functions, defined in $\tilde{\mathcal{D}}_1$ and continued by zero to the left from this set. The term Z_{in} is a solution of the homogeneous equation, $L_0(Z_{in}) = 0$. We let

$$Z_{in}(\tau) = a_1(\tau)\varphi_{01}(\tau) + a_2(\tau)\varphi_{02}(\tau), \tag{10.24}$$

where

$$a_1(\tau) = \mathcal{W}_{Z;\varphi_{02}}(\tau - [\Re\tau + h^{-1}]), \tag{10.25}$$
$$a_2(\tau) = \mathcal{W}_{\varphi_{01};Z}(\tau - [\Re\tau + h^{-1}]), \tag{10.26}$$

where $[s]$ denotes the integer part of s. The functions a_k, $k = 0, 1$, are periodical complex valued functions. Probably, they are not continuous. Using the estimates (9.48) and Lemma 10.1 we obtain $|a_1(\tau)| \le \text{const}\, h^{2N+2}$ and $|a_2(\tau)| \le \text{const}\, h^{2N-1}$, then

$$|Z_{in}(\tau)| \le \text{const}\, \left(|\tau|^2 h^{2N+2} + |\tau|^{-1} h^{2N-1}\right). \tag{10.27}$$

We consider Eq. (10.23) on the sequence of closed intervals $l_n = [ir - h^{-1}, ir - h^{-1} + n] \cap \tilde{\mathcal{D}}_1$. The explicit expression for the operator L_0^{-1} shows that it expresses the value of a function $L_0^{-1}(g)$ at a point τ through the values of the function g at the points $\tau - k$, $k \ge 1$. Thus Eq. (10.23) provides an expression for the values of Z on l_n through the values of Z on l_{n-1}.

Let l be a closed interval in $\mathbb{C}$ parallel to the real axis. Denote by $\mathcal{X}_m(l)$ the space of all complex valued functions defined on l and continued by zero to the left of the interval. The norm of a function $a \in \mathcal{X}_m(l)$ is defined by

$$\|a\|_m = \sup_l |\tau^m a(\tau)|.$$

Lemma 10.2. *Given $m > 1$, $n > 0$, the formula (9.47) defines a continuous linear operator $L_0^{-1} : \mathcal{X}_{m+2}(l_n) \to \mathcal{X}_m(l_{n+1})$, $\|L_0^{-1}\| \le$ const. Given $g \in \mathcal{X}_0(l_n)$, the operator L_1, defined by $L_1 Z = Z - \varepsilon L_0^{-1}(g \cdot Z)$, has a bounded inverse, L_1^{-1}, in $\mathcal{X}_m(l_n)$.*

The proof of the lemma is similar to the proof of Proposition 9.6. One has to take into account that L_1 is a Volterra type operator.

Now we can rewrite Eq. (10.23) in the form

$$Z = L_1^{-1} L_0^{-1}(F_0 + F_{12} Z + F_2(Z)) + L_1^{-1} Z_{in} \; ; \tag{10.28}$$

we substitute $g = \varepsilon^{-1} F_{11}$ into the definition of L_1. The norms of the functions on the right-hand side can be estimated in the following way:

$$\begin{aligned} \left\| L_1^{-1} L_0^{-1}(F_0) \right\|_{2N} &\leq \text{const}, \\ \left\| L_!^{-1} L_0^{-1}(F_{12} Z) \right\|_{2N} &\leq \text{const} \sup_{l_n} |\tau|^{-2} \|Z\|_{2N}, \\ \left\| L_1^{-1} L_0^{-1}(F_2(Z)) \right\|_{2N} &\leq \text{const} \sup_{l_n} |\tau|^{-2N} \|Z\|_{2N}^2, \\ \left\| L_1^{-1} L_0^{-1}(Z_{in}) \right\|_{2N} &\leq \text{const}, \end{aligned}$$

where the norms in the left-hand side are in $\mathcal{X}_{2N}(l_{n+1})$ and the norms of Z in the right-hand side are in $\mathcal{X}_{2N}(l_n)$. Denoting the latter by $\|Z\|_{2N,n}$ we obtain from Eq. (10.28) the following set of estimates:

$$\|Z\|_{2N,n+1} \leq \text{const} + \text{const} \sup_{l_n} |\tau|^{-2} \|Z\|_{2N,n} + \text{const} \sup_{l_n} |\tau|^{-2N} \|Z\|_{2N,n}^2 . \tag{10.29}$$

Lemma 10.3. *Let y_n, $n \in \mathbb{N}$ be a sequence of nonnegative numbers, such that $y_{n+1} \leq a + b y_n + c y_n^2$ for some positive numbers a, b and c, $2b + 4ac < 1$. If $y_1 \leq 2a$, then $y_n \leq 2a$ for all $n \geq 1$.*

The lemma is almost trivial. Indeed, let $y_n \in (0, 2a)$, then

$$y_{n+1} \leq a + b y_n + c y_n^2 \leq a + b 2a + c(2a)^2 = a + a(2b + 4ac) \leq 2a.$$

Applying Lemma 10.3 to the sequence $\|Z\|_{2N,n}$ we obtain the upper bound $\|Z\|_{2N,n} \leq$ const for all l_n, such that $\sup_{l_n} |\tau|^{-2} \leq R^{-2}$ for a sufficiently large constant R. These l_n cover all $\tilde{\mathcal{D}}_1$ except the R-neighborhood of $\tau = 0$. To extend the estimate on this subset we note that we need no more than $[R] + 1$ steps in n.

Thus

$$\sup_{\tau \in \tilde{\mathcal{D}}_1} \left| \tau^{2N} Z(\tau, \varepsilon) \right| \leq \text{const}. \tag{10.30}$$

This estimate together with Lemma 10.1 imply Proposition 3.1. □

11. Proof of the Existence of the SSM Separatrix (Theorem 4.1)

The map $\Gamma_- : \mathbb{C} \to \mathbb{C}^2$ can be written as $\Gamma_-(\tau) = (u_-(\tau), v_-(\tau))$, $\tau \in \mathbb{C}$, where the second component can be expressed in terms of the first one:

$$v_-(\tau) = u_-(\tau) - u_-(\tau - 1).$$

So it is sufficient to find $u_-(\tau)$ by solving the equation

$$\Delta^2 u_-(\tau) = \exp(u_-(\tau)). \tag{11.1}$$

Given a number $A > 1$, denote by $D(A)$ the domain in $\mathbb{C}$ defined by the inequality

$$\left| \arg \left(\tau + \frac{A}{\sin \delta_0} \right) \right| \geq \delta_0. \tag{11.2}$$

We shall consider analytical functions defined in $D(A)$. Evidently the domain $D(A)$ is of a type $(A, -)$ in the sense of Sect. 9. Fix a branch of the log so that $\log \frac{\tau^2}{2}$ becomes real on the negative axis of the variable τ and introduce a new unknown function $w : D(A) \to \mathbb{C}$ by setting

$$u_-(\tau) = -\log \frac{\tau^2}{2} + w(\tau), \qquad \tau \in D(A). \tag{11.3}$$

Substituting (11.3) into Eq. (11.1) yields the equation

$$\Delta^2 w(\tau) - 2 \log \left(1 - \frac{1}{\tau^2} \right) = \frac{2}{\tau^2} \exp(w(\tau)).$$

The latter can be rewritten as follows:

$$L_0 w = w_0 + \mathcal{F}(w), \tag{11.4}$$

where

$$L_0 w(\tau) = \Delta^2 w(\tau) - \frac{2}{\tau^2} w(\tau), \tag{11.5}$$

$$w_0(\tau) = 2 \log \left(1 - \frac{1}{\tau^2} \right) + \frac{2}{\tau^2}, \tag{11.6}$$

$$\mathcal{F}(w)(\tau) = F(\tau, w(\tau)), \tag{11.7}$$

$$F(\tau, w) = \frac{2}{\tau^2} \left(\mathrm{e}^w - w - 1 \right). \tag{11.8}$$

We shall try to resolve Eq. (11.4) with respect to the unknown function w in the space $\mathcal{X}_\mu(D(A))$ with an appropriate μ. The operator L_0 was studied in Sect. 9.6.

The following proposition contains necessary estimates for the operator $\mathcal{F}$ defined by the formulae (11.7) and (11.8).

Proposition 11.1. *Let w, w_1, w_2 belong to $\mathcal{X}_\mu(D(A))$, $\mu > 0$, and let*

$$|w(\tau)| \leq 1, \quad |w_1(\tau)| \leq 1, \quad |w_2(\tau)| \leq 1 \qquad \forall \tau \in D(A).$$

Then, given real γ such that $\gamma < 2 + 2\mu$, the following estimates hold:

$$\|\mathcal{F}(w)\|_\gamma \leq \frac{\text{const}}{A^{2+2\mu-\gamma}} \|w\|_\mu, \tag{11.9}$$

$$\|\mathcal{F}(w_1) - \mathcal{F}(w_2)\|_\gamma \leq \frac{\text{const}}{A^{2+2\mu-\gamma}} \max \{\|w_1\|_\mu, \|w_2\|_\mu\} \|w_1 - w_2\|_\mu, \tag{11.10}$$

where the constants depend only on the choice of δ_0, μ and γ.

Proof. It follows immediately from the definition of the norms and from inequalities

$$|F(\tau, w)| \le \frac{\text{const}}{|\tau|^2}|w|^2, \tag{11.11}$$

$$|F(\tau, w_1) - F(\tau, w_2)| \le \frac{\text{const}}{|\tau|^2} \max\{|w_1|, |w_2|\}\, |w_1 - w_2|, \tag{11.12}$$

which are obviously valid provided $|w|$, $|w_1|$ and $|w_2| \le 1$. □

Let $\mu = 2$, $\gamma = 4$, and consider our main equation (11.4) in the space $\mathcal{X}_2(D(A))$, A being sufficiently large.

Proposition 11.2. *In the space $\mathcal{X}_2(D(A))$ Eq. (11.4) is equivalent to*

$$w = \mathcal{G}(w), \tag{11.13}$$

where

$$\mathcal{G}(w) = L_0^{-1} w_0 + L_0^{-1} \mathcal{F}(w). \tag{11.14}$$

Proof. Let $w \in \mathcal{X}_2(D(A))$ satisfy (11.13). Then, due to Proposition 9.6, it is a solution of (11.4). Conversely, let $w \in \mathcal{X}_2(D(A))$ be a solution of (11.4). Note that w_0 and $\mathcal{F}(w)$ belong to $\mathcal{X}_4(D(A))$. So the function

$$w_1 = L_0^{-1} w_0 + L_0^{-1} \mathcal{F}(w) \tag{11.15}$$

belongs to $\mathcal{X}_2(D(A))$ and satisfies the equation

$$L_0 w_1 = w_0 + \mathcal{F}(w).$$

Hence $w_1 - w$ is a solution of the homogeneous equation and has the form

$$w_1 - w = \alpha_1 \varphi_{01} + \alpha_2 \varphi_{02},$$

where α_1 and α_2 are periodic functions:

$$\alpha_1 = \mathcal{W}_{w_1 - w; \varphi_{02}}, \qquad \alpha_2 = -\mathcal{W}_{w_1 - w; \varphi_{01}}.$$

Since $w_1 - w \in \mathcal{X}_2(D(A))$ and φ_{01}, φ_{02} obey the estimates (9.48), it follows that α_1, $\alpha_2 \in \mathcal{X}_1(D(A))$. In this case periodicity implies that $\alpha_1 = \alpha_2 = 0$. We have $w_1 = w$, and (11.15) converts to (11.13). □

So we may consider (11.13) instead of (11.1). We will consider the former in a closed ball $B(R)$ of radius R satisfying the inequality

$$R > R_0 = 2 \sup_{A \ge 1} \left\| L_0^{-1} w_0 \right\|_2, \tag{11.16}$$

with the center at the origin. The supremum is finite since the norm on the right-hand side is a nonincreasing function of A.

Proposition 11.3. *If A is sufficiently large (larger than a constant depending only on δ_0 and R), then the map $\mathcal{G}$ defined by (11.14) maps $B(R)$ into itself and it is a contraction map.*

Proof. The estimate (11.9) and Proposition 9.6 yield that

$$\|\mathcal{G}(w)\|_2 \leq \left\|L_0^{-1} w_0\right\|_2 + \frac{\text{const}}{A^2} \|w\|_2^2 \leq R,$$

provided A is sufficiently large. Also, due to (11.10),

$$\begin{aligned}\|\mathcal{G}(w_1) - \mathcal{G}(w_2)\|_2 &\leq \left\|L_0^{-1} \left(\mathcal{F}(w_1) - \mathcal{F}(w_2)\right)\right\|_2 \leq \text{const} \, \|\mathcal{F}(w_1) - \mathcal{F}(w_2)\|_4 \\ &\leq \frac{\text{const}}{A^2} \max\{\|w_1\|_2, \|w_2\|_2\} \|w_1 - w_2\|_2 \\ &\leq \frac{\text{const}\, R}{A^2} \|w_1 - w_2\|_2 \leq \frac{1}{2} \|w_1 - w_2\|_2.\end{aligned}$$

These two estimates involve the proposition. □

Corollary 11.4. *Let R satisfy (11.16). There exists a unique solution $u_-(\tau) = -\log(\tau^2/2) + w(\tau)$ to Eq. (11.1) with w belonging to $B(R)$.*

Let us return to the function u_- connected with w by the equality (11.3). Note that u_- can be prolonged onto the entire complex plane as an entire function by means of Eq. (11.1).

Equation 11.13 may be solved by the iteration method starting from the zero function. Using the explicit formulas it is easy to see that the iterations preserve the property to be real-analytic. Consequently, the function w, and then u_-, are real-analytic.

It remains to obtain the asymptotic expansion for $u_-(\tau)$.

Proposition 11.5. *There exists a sequence $\{a_k\}_{k=1}^{\infty}$ such that the function u_- has the asymptotic expansion (4.3) uniform in each sector $\delta_0 \leq \arg\tau \leq 2\pi - \delta_0$, $\delta_0 \in]0, \pi/2[$.*

Proof. Fix a positive integer N and introduce a function $\eta(\tau)$ as

$$u_-(\tau) = -\log\frac{\tau^2}{2} + \sum_{k=1}^{N} a_k \tau^{-2k} + \eta(\tau). \tag{11.17}$$

It is clear from Corollary 11.4 that

$$|\eta(\tau)| \leq 1 \quad \text{if} \quad \tau \in D(A), \tag{11.18}$$

provided A is sufficiently large.

Substituting (11.17) into Eq. (11.1) yields the following equation for the function η:

$$L_0\eta = \eta_0 + \mathcal{F}_N(\eta), \tag{11.19}$$

where

$$\eta_0 = 2\log\left(1 - \frac{1}{\tau^2}\right) + 2\sum_{k=1}^{N+1} \frac{1}{k}\frac{1}{\tau^{2k}} = O\left(\frac{1}{\tau^{2N+4}}\right), \tag{11.20}$$

$$\mathcal{F}_N(\eta)(\tau) = F_N(\tau, \eta(\tau)), \tag{11.21}$$

$$F_N(\tau,\eta) = \frac{2}{\tau^2}\exp\left(\eta + \sum_{k=1}^{N}\frac{a_k}{\tau^{2k}}\right) - \sum_{k=1}^{N+1}\frac{2}{k\tau^{2k}} - \sum_{k=1}^{N} a_k \Delta^2 \frac{1}{\tau^{2k}} - \frac{2}{\tau^2}\eta. \tag{11.22}$$

Let us choose the numbers a_k so that

$$\exp\left(\sum_{k=1}^{N}\frac{a_k}{\tau^{2k}}\right)-\sum_{k=1}^{N+1}\frac{1}{k}\,\frac{1}{\tau^{2k-2}}-\frac{\tau^2}{2}\sum_{k=1}^{N}a_k\Delta^2\frac{1}{\tau^{2k}}=O\left(\frac{1}{\tau^{2N+4}}\right). \tag{11.23}$$

It is not difficult to check that this determines the coefficients a_k uniquely, and they do not depend on the choice of N. Then we have, taking into account (11.18),

$$F_N(\tau,\eta)=O\left(\frac{1}{\tau^2}\eta^2\right)+O\left(\frac{1}{\tau^4}\eta\right)+O\left(\frac{1}{\tau^{2N+4}}\right) \tag{11.24}$$

and

$$\frac{\partial}{\partial\eta}F_N(\tau,\eta)=O\left(\frac{1}{\tau^2}\eta\right)+O\left(\frac{1}{\tau^4}\right) \tag{11.25}$$

as τ tends to infinity.

We consider Eq. (11.19) in the space $\mathcal{X}_{2N+2}(D(A))$. It follows from (11.20), (11.24), (11.25), and Proposition 9.6 with $\gamma = 2N + 4$ that

$$\left\|L_0^{-1}\eta_0\right\|_{2N+2} \text{ is bounded}, \tag{11.26}$$

$$\left\|L_0^{-1}\mathcal{F}_N(\eta)\right\|_{2N+2}\le \text{const}\left(\frac{1}{A^{2+2N}}\|\eta\|^2_{2N+2}+\frac{1}{A^2}\|\eta\|_{2N+2}+1\right), \tag{11.27}$$

$$\begin{aligned}&\left\|L_0^{-1}(\mathcal{F}_N(\eta_1)-\mathcal{F}_N(\eta_2))\right\|_{2N+2}\\ &\le \text{const}\left(\frac{1}{A^{2+2N}}\max\{\|\eta_1\|_{2N+2},\|\eta_2\|_{2N+2}\}\,\|\eta_1-\eta_2\|_{2N+2}\right)\\ &\quad+\text{const}\left(\frac{1}{A^4}\|\eta_1-\eta_2\|_{2N+2}\right).\end{aligned} \tag{11.28}$$

Applying the contraction principle in an appropriately chosen closed ball we obtain the existence of a fixed point $\eta\in\mathcal{X}_{2N+2}(D(A))$, A being sufficiently large, which proves (in view of uniqueness of u_-) the asymptotic expansion up to the order τ^{-2N}. □

12. Proof of the Exponential Closeness of the Separatrices for the SSM (Theorem 4.2)

In this section we will prove an exponential estimate for the distance between separatrices of the semistandard map. All functions are considered in the domain D_A, A being sufficiently large. Recall that the domain D_A was defined by the formula (9.21).

First of all, let us notice that, since $u_+(\tau) = u_-(-\tau) - 2\pi\mathrm{i}$ and $u_-(\tau)$ have the same asymptotics in the domain D_A, their difference

$$w(\tau) = u_+(\tau) - u_-(\tau) \tag{12.1}$$

admits the estimate

$$|w(\tau)| \leq \text{const}\,\frac{1}{|\tau|^N}, \qquad \forall \tau \in D_A, \tag{12.2}$$

where the positive integer N can be chosen arbitrary and the constant depends only on δ_0 and N.

The function w obeys the following equation which is a consequence of (11.1) for u_+ and u_-:

$$\Delta^2 w - \mathrm{e}^{u_-} w = \mathrm{e}^{u_-}\left(\mathrm{e}^w - 1 - w\right). \tag{12.3}$$

Denote

$$F(\tau, w) = \mathrm{e}^{u_-(\tau)}\left(\mathrm{e}^w - 1 - w\right), \qquad (\tau, w) \in \mathbb{C}^2, \tag{12.4}$$

$$\mathcal{F}(w)(\tau) = F(\tau, w(\tau)), \tag{12.5}$$

and

$$Lw = \Delta^2 w - \mathrm{e}^{u_-} w. \tag{12.6}$$

Then Eq. (12.3) can be rewritten as follows:

$$Lw = \mathcal{F}(w). \tag{12.7}$$

First we will deal with the problem of reversing the operator L in the spaces $\mathcal{X}_\mu(D_A)$.

12.1. Construction of L^{-1}. By means of simple differentiation of the equation, one finds that the homogeneous equation

$$L\varphi = 0 \tag{12.8}$$

has a solution

$$\varphi_1(\tau) = \frac{du_-}{d\tau}(\tau). \tag{12.9}$$

It follows from Theorem 4.1 that

$$\varphi_1(\tau) = -\frac{2}{\tau} + \frac{1}{2}\,\frac{1}{\tau^3} + O\left(\frac{1}{\tau^5}\right), \tag{12.10}$$

if τ tends to infinity outside a sector $|\arg\tau| < \delta_0$.

Lemma 12.1. *There is a solution of Eq. (12.8), such that*

$$\mathcal{W}_{\varphi_1;\varphi_2} = 1, \tag{12.11}$$

$$\varphi_2(\tau) \stackrel{\text{as}}{=} \sum_{k=-1}^{\infty} \frac{\varphi_{0k}}{\tau^{2k}} \tag{12.12}$$

uniformly in any sector $|\arg\tau| > \delta$.

Proof. The solution $\varphi_2(\tau)$ can be determined by solving Eq. (12.11), which can be solved explicitly with respect to φ_2. The function φ_2 may be represented in the form

$$\varphi_2(\tau) = C(\tau)\varphi_1(\tau), \tag{12.13}$$

where

$$\Delta C = \frac{1}{\varphi_1(\tau)\varphi_1(\tau+1)}. \tag{12.14}$$

The asymptotic expansion (12.10) results in

$$\frac{1}{\varphi_1(\tau)\varphi_1(\tau+1)} = \frac{1}{4}\tau^2 + \frac{1}{4}\tau + \frac{1}{8} + O\left(\frac{1}{\tau^2}\right).$$

Let us introduce a new unknown function $a(\tau)$ by setting

$$C(\tau) = \frac{1}{12}\tau^3 + \frac{1}{24}\tau + a(\tau).$$

Then Eq. (12.14) reads:

$$\Delta a(\tau) = \frac{1}{\varphi_1(\tau)\varphi_1(\tau+1)} - \frac{1}{4}\tau^2 - \frac{1}{4}\tau - \frac{1}{8} = O\left(\frac{1}{\tau^2}\right).$$

The latter can be solved by means of the operator Δ_-^{-1} (see (9.5)) and

$$a(\tau) = \sum_{k=1}^{\infty}\left(\frac{1}{\varphi_1(\tau-k)\varphi_1(\tau-k+1)} - \frac{1}{4}(\tau-k)^2 - \frac{1}{4}(\tau-k) - \frac{1}{8}\right) \tag{12.15}$$

does not exceed $O\left(1/\tau\right)$. It follows from (12.10), (12.13), (12.14), and (12.15) that

$$\varphi_2(\tau) = -\frac{1}{6}\tau^2 - \frac{1}{24} + O\left(\frac{1}{\tau^2}\right). \tag{12.16}$$

The solution of such a form is unique and it follows from the symmetries that its expansion contains only odd powers. We will discuss the corresponding arguments for a more general situation in Sect. 15. □

Now we can build L^{-1} in the same manner as L_0^{-1} in (9.47):

$$L^{-1}f = -\varphi_1\,\Delta^{-1}(\varphi_2 f) + \varphi_2\,\Delta^{-1}(\varphi_1 f). \tag{12.17}$$

Here Δ^{-1} is the operator defined in Proposition 9.5.

Proposition 12.2. *Given $\gamma > 4$, $\delta > 0$ such that $\gamma - \delta > 4$, and $A > 1$, the expression (12.17) defines a bounded operator $L^{-1} : \mathcal{X}_\gamma(D_A) \longrightarrow \mathcal{X}_{\gamma-3-\delta}(D_A)$ with the norm satisfying the estimate*

$$\|L^{-1}\| \le \frac{\text{const}}{A^\delta}, \tag{12.18}$$

where the constant depends only on δ_0, γ, and δ. If $f \in \mathcal{X}_\gamma(D_A)$, then $w = L^{-1}f$ is a solution to the equation $Lw = f$.

Proof. Let $f \in \mathcal{X}_\gamma(D_A)$. Then, as it follows from (12.10) and (12.16),

$$\varphi_2 f \in \mathcal{X}_{\gamma-2}(D_A), \qquad \varphi_1 f \in \mathcal{X}_{\gamma+1}(D_A).$$

Proposition 9.5 yields the first assertion and the estimate (12.18). The last assertion is a consequence of (12.11). □

Applying the operator L^{-1}, we can rewrite (12.7) as

$$w = \varphi + L^{-1}\mathcal{F}(w), \tag{12.19}$$

where φ is a solution of the homogeneous equation $L\varphi = 0$. We are going to resolve (12.19) in an appropriate space $\mathcal{X}_\mu(D_A)$.

12.2. The application of the contraction principle. First, let us estimate the norms of w and $\mathcal{F}(w)$. The inequality (12.2) gives us

$$|w(\tau)| < 1 \qquad \forall \tau \in D_A \tag{12.20}$$

if A is sufficiently large, and

$$\|w\|_\mu \leq \frac{\text{const}}{A^\beta} \tag{12.21}$$

for arbitrary chosen positive μ and β, const depending on μ and β. Similarly, like in Sect. 11, if $\max\{|w|, |w_1|, |w_2|\} < 1$, then

$$|F(\tau, w)| \leq \frac{\text{const}}{|\tau|^2}\, |w|^2, \tag{12.22}$$

$$|F(\tau, w_1) - F(\tau, w_2)| \leq \frac{\text{const}}{|\tau|^2} \max\{|w_1|, |w_2|\}\, |w_1 - w_2|, \tag{12.23}$$

which yield

$$\|\mathcal{F}(w)\|_\gamma \leq \frac{\text{const}}{A^{2+2\mu-\gamma}}\, \|w\|_\mu^2, \tag{12.24}$$

$$\|\mathcal{F}(w_1) - \mathcal{F}(w_2)\|_\gamma \leq \frac{\text{const}}{A^{2+2\mu-\gamma}} \max\{\|w_1\|_\mu, \|w_2\|_\mu\}\, \|w_1 - w_2\|_\mu, \tag{12.25}$$

provided

$$\gamma < 2 + 2\mu, \tag{12.26}$$

const depending only on the choice of δ_0, μ and γ.

It follows from the estimates (12.21), (12.24) that φ from Eq. (12.19) admits an estimate:

$$\|\varphi\|_\mu \leq \|w\|_\mu + \frac{\text{const}}{A^\delta}\,\frac{1}{A^{2+2\mu-\gamma}}\,\|w\|_\mu^2 \leq \frac{\text{const}}{A^\beta} + \frac{\text{const}}{A^{\mu-1+2\beta}} \leq \frac{\text{const}}{A^\nu}, \tag{12.27}$$

where positive $\nu = \min\{\beta, \mu - 1 + 2\beta\}$ can be made arbitrary large, the constants depending on that choice.

Let us fix $\delta > 0$, and consider Eq. (12.19) in the unit closed ball B with the center at the origin in the space $\mathcal{X}_\mu(D_A)$, $\mu > 1 + \delta$. Take $\gamma = 3 + \mu + \delta$ (note that (12.26) is fulfilled). Then (12.24) gives us

$$\left\|L^{-1}\mathcal{F}(w)\right\|_\mu \leq \frac{\text{const}}{A^{\mu-1-\delta}} < \frac{1}{2}, \qquad w \in B,$$

provided A is sufficiently large. Also (12.27) ensures that $\|\varphi\|_\mu < 1/2$ for large A. Hence the nonlinear operator $\mathcal{G}$ defined as

$$\mathcal{G}(w) = \varphi + L^{-1}\mathcal{F}(w) \tag{12.28}$$

maps B into itself. Proposition 12.2 and the inequality (12.25) prove the contractibility of $\mathcal{G}$. So $\mathcal{G}$ has a unique fixed point in B which necessarily coincides with w defined by (12.1), since the latter belongs to B too. We have also that

$$w = \lim_{n\to\infty} w_n \quad \text{in} \quad \mathcal{X}_\mu(D_A), \tag{12.29}$$

where

$$w_0 = \varphi \tag{12.30}$$

and

$$w_{n+1} = \mathcal{G}\big(w_n\big), \qquad n \geq 0. \tag{12.31}$$

The last thing is crucial because it enables us to prove an exponential estimate for w by establishing such estimates for the members of the iterated sequence. First we establish it for w_0.

12.3. The estimate of a solution of homogeneous equation. In the situation met in the preceding section it appeared that the summand φ (a solution of the homogeneous equation) did vanish. This was so because such a solution belonging to $\mathcal{X}_\mu(D(A))$ with $\mu > 1$ is necessarily zero. The situation considered here is analogous but, since the geometry of D_A differs from that of $D(A)$, it does not vanish. Instead an exponential estimate for such a solution is possible.

Proposition 12.3. *Let $\mu \geq 2$ and let $\varphi \in \mathcal{X}_\mu(D_A)$ be a solution of the homogeneous equation (12.8). Then*

$$|\varphi(\tau)| \leq \text{const}\,\|\varphi\|_\mu \frac{|\tau|^2}{A^{\mu+1}}\,\mathrm{e}^{2\pi(A-|\Im\tau|)}, \tag{12.32}$$

where the constant depends only on δ_0 and μ.

Proof. A general solution of the homogeneous equation can be represented in the form

$$\varphi = \alpha_1\varphi_1 + \alpha_2\varphi_2, \tag{12.33}$$

where the functions

$$\alpha_1 = \mathcal{W}_{\varphi;\varphi_2}, \qquad \alpha_2 = -\mathcal{W}_{\varphi;\varphi_1}$$

are periodic. The estimates (12.10) and (12.16) involve

$$|\alpha_1(\tau)| \le \text{const}\, \|\varphi\|_\mu\, |\tau|^{2-\mu}, \qquad |\alpha_2(\tau)| \le \text{const}\, \|\varphi\|_\mu |\tau|^{-1-\mu}.$$

Since $\alpha_{1,2}$ are periodic, they define functions $\beta_{1,2}(z)$ for $|z| \le \mathrm{e}^{-2\pi A}$ by the equalities

$$\beta_{1,2}\left(\mathrm{e}^{-\mathrm{i}2\pi\tau}\right) = \alpha_{1,2}(\tau),$$

which tend to zero as $z \to 0$. So $\beta_{1,2}$ have analytic continuations to $z = 0$, and $\beta_{1,2}(0) = 0$. Hence, denoting $r = \mathrm{e}^{-2\pi A}$, we have

$$\left|\frac{\beta_1(z)}{z}\right| \le \max_{|z|=r}\left|\frac{\beta_1(z)}{z}\right| = \mathrm{e}^{2\pi A} \max_{\Im\tau=-A} |\alpha_1(\tau)| \le \text{const}\, \|\varphi\|_\mu A^{2-\mu}\mathrm{e}^{2\pi A},$$

$$\left|\frac{\beta_2(z)}{z}\right| \le \max_{|z|=r}\left|\frac{\beta_2(z)}{z}\right| = \mathrm{e}^{2\pi A} \max_{\Im\tau=-A} |\alpha_2(\tau)| \le \text{const}\, \|\varphi\|_\mu A^{-1-\mu}\mathrm{e}^{2\pi A}.$$

Hence

$$\begin{aligned} |\alpha_1(\tau)| &\le \text{const}\, \|\varphi\|_\mu A^{2-\mu}\mathrm{e}^{2\pi(A-|\Im\tau|)}, \\ |\alpha_2(\tau)| &\le \text{const}\, \|\varphi\|_\mu A^{-1-\mu}\mathrm{e}^{2\pi(A-|\Im\tau|)}. \end{aligned} \tag{12.34}$$

Applying (12.34), (12.10), and (12.16) to Eq. (12.33), we obtain the desired estimate (12.32). □

12.4. Proof of the exponential estimate. Fix $\mu > 3$, and let A be so large that $\|\varphi\|_\mu < \frac{1}{2}$. By using (12.28) and (12.29), we will prove by induction that if A is sufficiently large, the functions w_n satisfy the estimates:

$$|w_n(\tau)| \le C_n|\tau|^2 \frac{1}{A^{\mu+1}}\, \mathrm{e}^{2\pi(A-|\Im\tau|)}, \tag{12.35}$$

and the sequence of the constants C_n is bounded. In view of (12.29) we shall obtain the desired estimate for the limiting function w.

First, Proposition 12.3 gives us (12.35) for $n = 0$. Let (12.35) be true for a given n. Substituting (12.35) into the right side of (12.22) yields

$$|\mathcal{F}(w_n)(\tau)| \le \text{const}\, |C_n|^2|\tau|^2 \frac{1}{A^{2\mu+2}} \mathrm{e}^{4\pi(A-|\Im\tau|)}. \tag{12.36}$$

Multiplying (12.36) by $|\tau|^\gamma$ and taking the maximum we obtain

$$\|\mathcal{F}(w_n)\|_\gamma \le \text{const}\, |C_n|^2 \frac{1}{A^{2\mu+2}}\, \max |\tau|^{\gamma+2}\mathrm{e}^{4\pi(A-|\Im\tau|)}. \tag{12.37}$$

Here we used the inequality $|\tau| \le \text{const}\, |\Im\tau|$. If $A > (\gamma + 2)/(4\pi)$ then the maximum on the right-hand side of (12.37) is reached at $\Im\tau = -A$. So

$$\|\mathcal{F}(w_n)\|_\gamma \le \text{const}\,|C_n|^2 \frac{1}{A^{2\mu-\gamma}} = \text{const}\,|C_n|^2 \frac{1}{A^{\mu-3-\delta}} \tag{12.38}$$

(recall that $\gamma = 3 + \mu + \delta$). Using Proposition 12.2 we have

$$\left\|L^{-1}\mathcal{F}(w_n)\right\|_\mu \le \text{const}\,|C_n|^2 \frac{1}{A^{\mu-3}}. \tag{12.39}$$

Consider another inverse operator $\left(L^{-1}\right)_1$ defined by the formula

$$\left(L^{-1}\right)_1 f(\tau) = \mathrm{e}^{-2\pi \mathrm{i}\tau}\left(L^{-1}f_1\right)(\tau),$$

where

$$f_1(\tau) = \mathrm{e}^{2\pi\mathrm{i}\tau} f(\tau).$$

The additional factors $\mathrm{e}^{\pm 2\pi\mathrm{i}\tau}$ do not change the way of obtaining (12.39) because of the presence of a more strongly decreasing exponent $\mathrm{e}^{-4\pi|\Im\tau|}$. So we have

$$\left\|\left(L^{-1}\right)_1\mathcal{F}(w_n)\right\|_\mu \le \text{const}\,|C_n|^2 \frac{1}{A^{\mu-3}}. \tag{12.40}$$

Since L commutes with the multiplication by $\mathrm{e}^{\pm 2\pi\mathrm{i}\tau}$, the expression $\left(L^{-1}\right)_1 f$ represents another solution to the nonhomogeneous equation $Lu = f$. Hence the difference between $\left(L^{-1}\right)_1\mathcal{F}(w_n)$ and $L^{-1}\mathcal{F}(w_n)$ satisfies the homogeneous equation and admits the estimate

$$\left\|\left(L^{-1}\right)_1\mathcal{F}(w_n) - L^{-1}\mathcal{F}(w_n)\right\|_\mu \le \text{const}\,|C_n|^2 \frac{1}{A^{\mu-3}}, \tag{12.41}$$

which is a consequence of (12.39) and (12.40). We may apply Proposition 12.3 to the difference taking into account (12.41). The result is

$$\left|\left(L^{-1}\right)_1\mathcal{F}(w_n)(\tau) - L^{-1}\mathcal{F}(w_n)(\tau)\right| \le \text{const}\,\frac{|C_n|^2|\tau|^2}{A^{2\mu-2}}\,\mathrm{e}^{2\pi(A-|\Im\tau|)}. \tag{12.42}$$

The expression $\left(L^{-1}\right)_1\mathcal{F}(w_n)(\tau)$ admits a direct estimate. First, it follows from (12.36) that

$$\left|\mathrm{e}^{2\pi\mathrm{i}\tau}\mathcal{F}(w_n)(\tau)\right| \le \text{const}\,|C_n|^2|\tau|^2 \frac{1}{A^{2\mu+2}}\,\mathrm{e}^{4\pi A-2\pi|\Im\tau|}.$$

Denoting

$$f_1(\tau) = \mathrm{e}^{2\pi\mathrm{i}\tau}\mathcal{F}(w_n)(\tau),$$

we obtain consequently, as in (12.38) and (12.39),

$$\|f_1\|_\gamma \le \text{const}\,|C_n|^2 \frac{1}{A^{2\mu-\gamma}}\,\mathrm{e}^{2\pi A},$$

$$\left\|L^{-1}f_1\right\|_\mu \le \text{const}\,|C_n|^2 \frac{1}{A^{\mu-3}}\,\mathrm{e}^{2\pi A},$$

provided A is sufficiently large. Multiplying by $\mathrm{e}^{-2\pi\mathrm{i}\tau}$ gives finally

$$\begin{aligned}\left|\left(L^{-1}\right)_1\mathcal{F}(w_n)(\tau)\right| &\le \text{const}\,|C_n|^2 \frac{1}{A^{\mu-3}}\,\frac{1}{|\tau|^\mu}\,\mathrm{e}^{2\pi(A-|\Im\tau|)}\\ &\le \text{const}\,|C_n|^2 \frac{1}{A^{2\mu-1}}\,|\tau|^2\,\mathrm{e}^{2\pi(A-|\Im\tau|)}.\end{aligned} \tag{12.43}$$

Comparing (12.42) and (12.43) yields

$$\left|L^{-1}\mathcal{F}(w_n)(\tau)\right| \le \text{const}\,|C_n|^2 \frac{|\tau|^2}{A^{2\mu-1}}\, \mathrm{e}^{2\pi(A-|\Im\tau|)}.$$

Returning to our initial recurrent relation (12.31), where $\mathcal{G}$ is defined by (12.28), let us notice that if we take

$$C_{n+1} = C_0 + \frac{\text{const}}{A^{\mu-3}}\, C_n^2, \tag{12.44}$$

then the estimate (12.35) becomes valid for the next value of the index n.

The assertion of Theorem 4.2 follows from (12.44) and Lemma 10.3.

13. Existence of the SSM Based Expansion (Proposition 5.1)

In a neighborhood of $t = \mathrm{i}\pi/2$ the coefficients (2.10) of the formal series (2.2) may be expanded in convergent Laurent series. Passing to the new time $\tau = (t - \mathrm{i}\pi/2)/h$ we get the following chain of the equalities:

$$X = \mathrm{i}\log\frac{h^2}{2} + \frac{1}{\mathrm{i}}\sum_{n=0}^{\infty} h^{2n} \sum_{k=-n}^{\infty} \tilde{p}_{n,k}(t-\mathrm{i}\pi/2)^{2k} = \mathrm{i}\log\frac{h^2}{2} + \frac{1}{\mathrm{i}}\sum_{n=0}^{\infty} h^{2n} U_n,$$

where we introduced the notation

$$U_n \sim \sum_{k=-n}^{\infty} p_{n+k,-k}\tau^{-2k}.$$

From (2.3) we get the following equations:

$$4\sinh^2\left(\frac{1}{2}\frac{\partial}{\partial\tau}\right) U_0 = \exp(U_0),$$

$$4\sinh^2\left(\frac{1}{2}\frac{\partial}{\partial\tau}\right) U_n = \sum_{k=1}^{n+1} \frac{2}{(2k)!}\Big(\mathcal{Y}_{n+1-k}(U_1,\dots,U_{n+1-k})\exp(U_0)$$
$$-\frac{1}{4}\mathcal{Y}_{n-1-k}(-U_1,\dots,-U_{n+1-k})\exp(-U_0)\Big).$$

Consequently the formal series U_n are formal solutions for the finite-difference equations (5.7) and (5.8). These equations may be solved recurrently if we consider the equation number n as a equation for u_n^-. Considered in this way the equations (5.8) are linear; they can be rewritten in the form

$$\Delta^2 u_n^- - \mathrm{e}^{u_0^-} u_n^- = \mathrm{e}^{u_0^-} P_n(u_1^-,\dots,u_{n-1}^-) + \mathrm{e}^{-u_0^-} Q_n(u_1^-,\dots,u_{n-2}^-),$$

where P_n and Q_n are some polynomials. As in Proposition 12.2 the operator in the left-hand side has a bounded inverse acting from $\mathcal{X}_m(D(A)) \to \mathcal{X}_{m-4}(D(A))$ for $m > 4$. Define the function $v_{n,j} = u_n^- - \sum_{k=-n}^{j} p_{nk}\tau^{-2k}$. This function satisfies a similar equation, but with the right-hand side in $\mathcal{X}_{2j+4}$. Then there is a unique solution of the equation in $\mathcal{X}_{2j}$. That implies that the constructed formal series are asymptotic to analytic solutions in $D(A)$.

Equations (5.8) afford to obtain analytic continuations of the functions u_n^- from the sector to the entire complex plane. □

14. Proof of the Second Approximation Theorem (Proposition 5.2)

Denote by

$$\hat{U}_N(\tau, h^2) = \sum_{n=0}^{N-1} h^{2n} u_n^-(\tau) \tag{14.1}$$

a partial sum of (5.6) and consider

$$Z(\tau, h^2) = u^-(\tau) - \hat{U}_N(\tau, h^2). \tag{14.2}$$

Substitution to Eq. (5.5) provides an equation for Z. It is convenient to rewrite this equation in the form

$$\Delta^2 Z - e^{u_0^-} Z = F_0 + F_1 Z + F_2(Z), \tag{14.3}$$

where

$$F_0 = (\varepsilon/h^2)e^{\hat{U}_N} - (\varepsilon h^2/4)e^{-\hat{U}_N} - \Delta^2 \hat{U}_N,$$

$$F_1 Z = ((\varepsilon/h^2)e^{\hat{U}_N} + (\varepsilon h^2/4)e^{-\hat{U}_N} - e^{u_0^-})\, Z,$$

$$F_2(Z) = (\varepsilon/h^2)e^{\hat{U}_N}(e^Z - 1 - Z) - (\varepsilon h^2/4)e^{-\hat{U}_N}(e^{-Z} - 1 + Z).$$

Let $\tau \in \tilde{\mathcal{D}}_2$ and $z \in \mathbb{C}$, $|z| \le 1$, then

$$|F_0| \le \text{const}\, h^{2N} |\tau|^{2N-2}, \tag{14.4}$$

$$|F_1| \le \text{const}\, h^2, \tag{14.5}$$

$$|F_2(z)| \le \text{const}\, |\tau|^{-2} |z|^2. \tag{14.6}$$

These estimates follow directly from Eq. (5.7), (5.8) and the asymptotical formulae (5.10), (5.11).

Denoting the linear operator in the left-hand side of (14.3) by L we can write this equation as

$$L(Z) = F_0 + F_1 Z + F_2(Z).$$

We can apply the operator L^{-1}, defined by formula (12.17) with Δ^{-1} replaced by Δ_-^{-1} (see (9.5)), to both sides of the equation to obtain that Z satisfies the following equation:

$$Z = Z_{in} + L^{-1}(F_0 + F_1 Z + F_2(Z)), \tag{14.7}$$

where the operator L^{-1} is acting on complex valued functions, defined in $\tilde{\mathcal{D}}_2$ and continued by zero to the left from this set. The term Z_{in} is a solution of the homogeneous equation $L(Z_{in}) = 0$, and we have to choose it in such a way that the right-hand side of (14.7) would be equal to the difference (14.2) in the intersection of $\tilde{D}_2$ with the strip $h^{-1/2} \le \Re\tau \le h^{-1/2} + 1$. In this way we obtain the coincidence of the solution of Eq. (14.7) with Z. We let

$$Z_{in}(\tau) = a_1(\tau)\varphi_1(\tau) + a_2(\tau)\varphi_2(\tau), \tag{14.8}$$

where

$$a_1(\tau) = \mathcal{W}_{Z;\varphi_2}(\tau - [\Re\tau + h^{-1/2}]), \tag{14.9}$$

$$a_2(\tau) = \mathcal{W}_{Z;\varphi_1}(\tau - [\Re\tau + h^{-1/2}]), \tag{14.10}$$

where $[s]$ denotes the integer part of s. The functions a_k, $k = 0, 1$, are periodical complex valued functions. Probably, they are not continuous. In $\tilde{\mathcal{D}}_2$ they afford the following estimates:

$$|a_1| \leq \text{const}\, h^{N-1/2}, \qquad |a_2| \leq \text{const}\, h^{N+1}. \tag{14.11}$$

Indeed, let $\tau \in \tilde{\mathcal{D}}_2$, then

$$|Z| \leq \text{const}\, h^N, \qquad |\Delta Z| \leq \text{const}\, h^{N+1}$$

due to Proposition 3.1 and the sewing condition. Then the estimates (14.11) follow from (14.9), (14.10) and the estimates (12.10), (12.16).

We consider Eq. (14.7) on a sequence of closed intervals $l_n = [ir - h^{-1}, ir - h^{-1} + n] \cap \tilde{\mathcal{D}}_2$. The explicit expression for the operator L^{-1} shows that it expresses the value of a function $L^{-1}g$ at a point τ through the values of the function g at the points $\tau - k$, $k \geq 1$. Thus Eq. (14.7) provides an expression for the values of Z on l_n through the values of Z on l_{n-1}.

As in the proof of Proposition 3.1, let l be a closed interval in $\mathbb{C}$ parallel to the real axis. Denote by $\mathcal{X}_m(l)$ the space of all complex valued functions defined on l and continued by zero to the left from the interval. The norm of a function $a \in \mathcal{X}_m(l)$ is defined by

$$\|a\|_m = \sup_l |\tau^m a(\tau)|.$$

Lemma 14.1. *Given $m > 2$, $n \geq 0$, the formula*

$$L^{-1} f = -\varphi_1 \Delta_-^{-1}(\varphi_2 f) + \varphi_2 \Delta_-^{-1}(\varphi_1 f)$$

defines a continuous linear operator $L^{-1} : \mathcal{X}_m(l_n) \to \mathcal{X}_{m-2}(l_{n+1})$ with the norm bounded by a constant. The value of the constant can be chosen to depend only on m.

Applying this lemma to Eq. (14.7) and taking into account the estimates (14.4)–(14.6) we obtain

$$\|Z\|_{4,n+1} \leq \text{const}\, h^{N-3} + \text{const}\, h\, \|Z\|_{4,n} + \text{const} \sup_{l_n} |\tau|^{-2}\, \|Z\|_{4,n}^2,$$

where $\|Z\|_{4,n}$ denotes the norm Z in $\mathcal{X}_4(l_n)$. Applying Lemma 10.3 to the sequence of $\|Z\|_{4,n}$ we obtain the estimate

$$\|Z\|_4 \leq \text{const}\, h^{N-3}.$$

This estimate is valid in all $\tilde{\mathcal{D}}_2$ except a 1-neighborhood of $\tau = 0$. We can not use the above estimate to cover this subset, since τ^{-2} is not bounded there, but we can simply iterate twice the original equation for Z.

Since N is arbitrary we can increase it to obtain

$$|u^- - \hat{U}_N| \leq |u_- - \hat{U}_{2N+3}| + |\hat{U}_{2N+3} - \hat{U}_N| \leq \text{const}\, h^{2N} + \text{const}\, h^{2N} |\tau|^{2N}.$$

The last estimate is equivalent to the one from the assertion of the theorem. $\square$

15. Existence of the Second Solution for the Variational Equation (Proposition 6.1)

Since Φ_1^- is a nontrivial solution of the homogeneous equation we can find Φ_2^- from the normalizing condition (6.13), Eq. (6.12) being satisfied automatically. In the class of formal series (6.13) is equivalent to the system

$$\mathcal{W}(\varphi_{1,0}^-;\varphi_{2,0}^-) = 1, \tag{15.1}$$

$$\mathcal{W}(\varphi_{1,0}^-;\varphi_{2,n}^-) = -\sum_{k=1}^{n} \mathcal{W}(\varphi_{1,k}^-;\varphi_{2,n-k}^-), \qquad n \geq 1. \tag{15.2}$$

The solution of Eq. (15.1) was obtained in Lemma 12.1. But the following reasoning works also in the case of Eq. (15.1). We use the n^{th} equation to define $\varphi_{2,n}^-$. We use the induction in n. The induction step consists of three parts:

(1) There is a formal series of the form (6.16) satisfying (15.2);
(2) There is an analytical function satisfying (15.2) which has the series as asymptotic expansion;
(3) The solution is unique.

Step (3) is simple. Indeed, Eqs. (15.2) are first order linear equations. The general solution of the corresponding homogeneous equation is a product of $\varphi_{1,0}^-$ and a periodic function. Since the asymptotic of $\varphi_{1,0}^-$ contains odd powers of τ (its expansion (12.10) starts with $-2\tau^{-1}$ and contains only negative odd powers), then if there is a solution of the form (6.16), it is unique.

Then we look for the solution of Eq. (15.2) in the form

$$\varphi_{2,n}^-(\tau) = A_n(\tau)\varphi_{1,0}^-(\tau). \tag{15.3}$$

Substitution to Eq. (15.2) provides

$$\Delta A_n(\tau) = -\frac{1}{\varphi_{1,0}^-(\tau)\varphi_{1,0}^-(\tau+1)} \sum_{k=1}^{n} \mathcal{W}(\varphi_{1,k}^-;\varphi_{2,n-k}^-) \equiv f_n(\tau). \tag{15.4}$$

Since the series for the functions $\varphi_{1,k}^-(\tau)$ contain only odd powers of τ, and the series for $\varphi_{2,k}^-(\tau)$ contain only even powers, it is easy to check that $f_n(-1-\tau) = f_n(\tau)$ in the class of formal series. Then it follows from the following lemma, that Eq. (15.4) has a formal solution represented by a series which contains only odd powers of τ.

Lemma 15.1. *Let* $f(\tau) = \sum_{k=-2m}^{\infty} b_k\tau^{-k}$ *be a formal series in powers of* τ*, such that* $f(\tau-1) = f(-\tau)$*. Then there is a unique representation of* $f(\tau)$ *in the form* $f(\tau) = \sum_{k=-m}^{\infty} c_k\Delta\tau^{-2k+1}$.

Proof. The set of $p_j(\tau) = \Delta\tau^{-j} = (\tau+1)^{-j} - \tau^{-j}$, $j \geq -2m-1$, $j \neq 0$, forms a basis in the space of formal series of the form $\sum_{k=-2m}^{\infty} b_k\tau^{-k}$ with $b_1 = 0$. The series $f(\tau-1) - f(-\tau)$ contains the term

$$\frac{b_1}{\tau-1} - \frac{b_1}{-\tau} = \frac{b_1(2\tau-1)}{\tau(\tau-1)} = b_1\left(\frac{2}{\tau} + \frac{1}{\tau^2} + \ldots\right).$$

The expansion starts with $2b_1/\tau$. The other terms of f do not contribute to this order. Since $f(\tau-1) - f(-\tau) = 0$ we have $b_1 = 0$.

So $f(\tau)$ can be represented as a linear combination of $p_j(\tau)$. This combination contains only odd j because $p_j(\tau-1) = -(-1)^j p_j(-\tau)$. □

It follows from the expansions (6.16) and (6.11), (5.10), (5.12) that $f_m(\tau)$ satisfies the assumptions of Lemma 15.1 with $m = n+1$. Then the lemma implies that there is a formal solution of Eq. (15.4):

$$A_n(\tau) \sim \sum_{k=-n+1}^{\infty} \frac{a_k}{\tau^{2k-1}}.$$

To complete Step (2) we look for an analytic solution $A_n(\tau)$ in the form

$$A_n(\tau) = \sum_{k=-n-1}^{m} \frac{a_k}{\tau^{2k-1}} + r_{n,m}(\tau), \qquad m > 0. \tag{15.5}$$

Substitution to Eq. (15.4) gives

$$\Delta r_{n,m}(\tau) = f_n(\tau) - \Delta \sum_{k=-n-1}^{m} \frac{a_k}{\tau^{2k-1}}. \tag{15.6}$$

According to the construction the formal series on the right-hand side of the last equation starts with a term of the order of τ^{-2m-1}. Due to the induction assumption formal series provide asymptotic expansions. Thus the right-hand side is an analytic function in $D(A)$ (see (11.2)) and it is equal to $O(\tau^{-2m-1})$. We apply the operator Δ_-^{-1} defined by (9.5) to obtain the solution of Eq. (15.6) in $\mathcal{X}_{2m}(D(A))$. Then we can restore $\varphi_{2,n}^-(\tau)$ by (15.3). Since the solution is unique the function A_n obtained by this procedure does not depend on the choice of m. Thus the constructed formal series are asymptotic.

The function $\varphi_{2,n}^-(\tau)$ can be continued analytically onto the whole complex plane using Eq. (6.12).

To prove that the constructed functions are real on the real axis we can repeat the reasoning with τ on the real semiaxis $\tau < -A$, with some constant $A > 0$. The obtained functions are real and coincide with the restriction of the previously constructed functions on the real axis due to the uniqueness. □

16. Construction of the Asymptotic Expression for the Distance Between SM Separatrices (Proposition 6.2)

Since both U^+ and U^- satisfy Eq. (5.5), $W = U^+ - U^-$ satisfy

$$\Delta^2 W = \left((\varepsilon/h^2) \mathrm{e}^{U^-} + (\varepsilon h^2/4) \mathrm{e}^{-U^-} \right) W + F, \tag{16.1}$$

where

$$F = (\varepsilon/h^2) \mathrm{e}^{U^-} \left(\mathrm{e}^{W} - 1 - W \right) - (\varepsilon h^2/4) \mathrm{e}^{-U^-} \left(\mathrm{e}^{-W} - 1 + W \right). \tag{16.2}$$

The function W can be represented in the form

$$W = A_1 \Phi_1^- + A_2 \Phi_2^-, \tag{16.3}$$

where

$$\Delta A_1 = -\Phi_2^- F, \qquad \Delta A_2 = \Phi_1^- F. \tag{16.4}$$

We consider this equation in the class of the formal series

$$A_k(\tau,\varepsilon) \sim \sum_{n=0}^{\infty} h^{2n} a_{k,n}(\tau), \qquad k = 1, 2. \tag{16.5}$$

The coefficient of the series for $W(\tau,\varepsilon)$ is expressed in the following way:

$$w_n(\tau) = \sum_{k=0}^{n} (a_{1,k}(\tau)\varphi_{1,n-k}(\tau) + a_{2,k}(\tau)\varphi_{2,n-k}(\tau)). \tag{16.6}$$

To obtain the equations on $a_{k,n}$ we represent F as a formal series

$$F(\tau,\varepsilon) \sim \sum_{n=0}^{\infty} h^{2n} F_n(\tau). \tag{16.7}$$

It is not difficult to obtain an expression for F_n:

$$\begin{aligned} F_n = {} & \mathrm{e}^{u_0^-} \sum_{k+l+m=n+1} \frac{2}{(2k)!} \mathcal{Y}_l(u_1^-, \ldots, u_l^-)(\mathrm{e}^{w_0} \mathcal{Y}_m(w_1, \ldots, w_m) - \delta_m - w_m) \\ & - \mathrm{e}^{-u_0^-} \sum_{k+l+m=n-1} \frac{2}{(2k)!} \mathcal{Y}_l(-u_1^-, \ldots, -u_l^-) \\ & \qquad \times (\mathrm{e}^{-w_0} \mathcal{Y}_m(-w_1, \ldots, -w_m) - \delta_m + w_m) \,; \end{aligned}$$

here $k \geq 1, l, m \geq 0$, δ_n is the Kronecker symbol, $\mathcal{Y}_n$ are polynomials (5.9). In particular, we have

$$\begin{aligned} F_0 &= \mathrm{e}^{u_0^-} (\mathrm{e}^{w_0} - 1 - w_0), \\ F_1 &= \mathrm{e}^{u_0^-} u_1(\mathrm{e}^{w_0} - 1 - w_0) + \mathrm{e}^{u_0^-} w_1(\mathrm{e}^{w_0} - 1) + (2/4!)\mathrm{e}^{u_0^-} (\mathrm{e}^{w_0} - 1 - w_0). \end{aligned}$$

From (16.4) we obtain that

$$\Delta a_{1,n} = -\sum_{k=0}^{n} \varphi_{2,k} F_{n-k}, \qquad \Delta a_{2,n} = \sum_{k=0}^{n} k\varphi_{1,k} F_{n-k}. \tag{16.8}$$

First, we study the equation for $n = 0$, namely,

$$\Delta a_{1,0} = -\varphi_{2,0} F_0, \qquad \Delta a_{2,0} = -\varphi_{1,0} F_0. \tag{16.9}$$

The estimate $|w_0(\tau)| \leq \text{const}\, |\tau|^2 \exp(-2\pi|\Im\tau|)$ from Theorem 4.2 implies that the functions on the right-hand sides of Eqs. (16.9) do not exceed

$$\text{const}\, \exp(-4\pi(1-\delta)|\Im\tau|) \tag{16.10}$$

for any $\delta > 0$. Of course, the constant in the estimate depends on δ.

Let $\mathfrak{Y}_\delta(D)$ be the space of continuous complex valued functions defined in D, analytical in interior points of D and provided with the norm

$$\|g\| = \sup_{\tau\in D} |\exp(4\pi(1-\delta)\mathrm{i}\tau) g(\tau)|.$$

Lemma 16.1. *Let $\delta > 0$, and $a(\tau)$ be a solution of the equation $\Delta a = g$, $g \in \mathfrak{Y}_\delta(D_A)$, which goes to zero as $\Im\tau \to -\infty$. Then there is a complex number θ, such that*

$$a(\tau) = \theta \mathrm{e}^{-2\pi \mathrm{i}\tau} + O(\exp(4\pi(1-\delta-\mu)\mathrm{i}\tau)),$$

where $\mu > 0$ is an arbitrary small number. The constant in the O estimate depends on μ. If $\delta < 1/2$ then for each a the representation above is unique.

Proof. First, we construct a suitable solution for the nonhomogeneous equation. Then the solution a differs from the obtained one by a periodic function. The first Fourier coefficient of the difference will play the role of θ.

There exists a linear map $\Delta^{-1} : \mathfrak{Y}_\delta(D_A) \to \mathfrak{Y}_{\delta+\mu}(D_A)$ such that

– for any $g \in \mathfrak{Y}_\delta(D_A)$, $\Delta^{-1}(g)$ is a solution of the equation $\Delta a = g$ at all the values of the independent variable τ for which it has meaning;
– $\|\Delta^{-1}\| \le$ const, where const depends only on δ_0, δ and μ.

The proof of these facts follows the lines of the proof of Proposition 9.5; the operators $\mathcal{I}$, $\mathcal{I}_+$ and $\mathcal{I}_-$ have to be replaced by the operator of multiplication $g(\tau) \mapsto \tau^2 \exp((4\pi(1-\delta-\mu/2)\mathrm{i}\tau)g(\tau)$. Then the operator Δ^{-1} is defined by the same formula (9.25). Obviously, $P_\pm \mathcal{I} g \in \mathcal{X}_0(D_A^\pm)$ and in D_A the following identity holds: $g = \mathcal{I}_-^{-1} P_- \mathcal{I} g + \mathcal{I}_+^{-1} P_+ \mathcal{I} g$. The estimates of the norms of the functions obtained after application of the operators $\Delta_\pm^{-1}$ follow directly from the definitions (9.4), (9.5) of these operators.

Now consider the function $b = a - \Delta^{-1} g$. It is a solution of the homogeneous equation $\Delta b = 0$ and $b \to 0$ as $\Im\tau \to -\infty$. Then b can be represented as a Fourier series $b = \sum_{k=1}^\infty b_k \mathrm{e}^{-2\pi \mathrm{i} k\tau}$. Let $\theta = b_1$. The Fourier expansion without the first term is bounded by $O(\exp(4\pi \mathrm{i}\tau))$ in D_A. □

Applying Lemma 16.1 to Eqs. (16.9) we obtain the desired representation for w_0 taking into account (16.6) with $n = 0$.

Then we follow by induction in n. Suppose that we checked the estimates (6.17) up to $n-1$. The right-hand side of Eq. (16.8) contains F_k, $k = 0, 1, 2, \ldots, n$. All the functions F_k are polynomial in $w_1, w_2, \ldots, w_k$, $\mathrm{e}^{w_0} - 1$, $\mathrm{e}^{w_0} - 1 - w_0$, $\mathrm{e}^{-w_0} - 1$ and $\mathrm{e}^{-w_0} - 1 + w_0$, with coefficients being some analytic functions of τ, which grow no faster than τ^N for some $N = N(k)$. These polynomials contain no zero and first order terms with respect to $w_0, \ldots, w_k$. Thus they can be bounded from above by (16.10), except the function F_n because of its dependence on w_n. This dependence goes through one term of the sums (16.8) only, namely,

$$\mathrm{e}^{u_0^-} w_n(\mathrm{e}^{w_0} - 1).$$

We know a priori that $w_n = O(\tau^{-m})$ for all $m > 0$, thus the expression above is bounded by

$$\text{const}\ \exp(-(2\pi - \delta')|\Im\tau|) \tag{16.11}$$

with arbitrary $\delta' > 0$. Applying to Eqs. (16.9) Lemma 16.1 with $\delta = \delta' + 1/2$, we obtain that $a_{1,n}$ and $a_{2,n}$ are exponentially small. Then from (16.6) it follows that w_n does not exceed (16.11). Now it is possible to improve the estimate for F_n up to the form (16.10) and again apply Lemma 16.1, but now with arbitrary small positive δ.

Denote the constants provided by Lemma 16.1 to $a_{2,n}$ and $a_{1,n}$ by θ_n and μ_n, respectively, and restore w_n by (16.6). Proposition 6.2 is proved. □

17. Proof of the Theorem on the Analytic Integral (Theorem 7.1)

In the coordinate form the map $\boldsymbol{\Phi}(t,E) = \big(x(t,E), y(t,E)\big)$. Assertion (3) reads:

$$x(t+h,E) = x(t,E) + y(t+h,E), \qquad y(t+h,E) = y(t,E) + \varepsilon \sin x(t,E).$$

This system is equivalent to a single equation

$$\Delta_h^2 x(t,E) = \varepsilon \sin x(t,E).$$

The normalizing condition (4) is equivalent to $x(t,0) = x^-(t)$. Assertion (2) is equivalent to

$$J = \det \begin{pmatrix} \frac{\partial x}{\partial t} & \frac{\partial x}{\partial E} \\ \frac{\partial y}{\partial t} & \frac{\partial y}{\partial E} \end{pmatrix} \equiv 1$$

or, equivalently,

$$\mathcal{W}\left[\frac{\partial x}{\partial t}; \frac{\partial x}{\partial E}\right] \equiv 1,$$

where $\mathcal{W}$ is the finite-difference Wronskian. We construct the desired solution using a convergent iteration procedure. First, we have to study the variational equation in a neighborhood of the unstable separatrix.

17.1. Variational equation. Let D be a closed subset of the strip $|\Im t| < \pi/2$. Given nonnegative μ, denote by $\mathfrak{X}_\mu(D)$ the Banach space of all complex valued continuous functions defined in D, analytic in interior points of D and possessing the finite norm

$$\|a\|_\mu = \sup_{t\in D} |\cosh^\mu(t) a(t)|. \tag{17.1}$$

The following lemma provides an instrument for solving first order linear equations in $\mathfrak{X}_\mu(D(R))$,

$$D(R) = \{t \in \mathbb{C} : |\Im t| \le \pi/2 - Rh,\ |\Re t| \le 10h\}. \tag{17.2}$$

Lemma 17.1. *Let $\mu > 0$, $R > 1$. There is a linear operator $\Delta_h^{-1} : \mathfrak{X}_\mu(D(R)) \to \mathfrak{X}_\mu(D(R))$, such that for any $g \in \mathfrak{X}_\mu(D(R))$ the function $a = \Delta_h^{-1} g$ is a solution of the equation $\Delta_h a = g$ and*

$$\|\Delta_h^{-1}\| \le \text{const}\, h^{-1} \log h^{-1}. \tag{17.3}$$

The constant in the estimate depends only on μ. Moreover, if g is an analytic continuation of a real-analytic function defined on the intersection of $D(R)$ with the real axis, the same is true about $a = \Delta_h^{-1} g$.

Proof. We use the method described in Sect. 9 (Propositions 9.5 and 9.6). Let

$$D^\pm(R) = \{t \in \mathbb{C} : |\Im t| \le \pi/2 - Rh,\ \pm\Re t \ge -10h\}.$$

It is clear that $D(R) = D^+(R) \cap D^-(R)$. The operators

$$(\Delta_{h,+}^{-1} g)(t) = \sum_{k=0}^{\infty} g(t+kh),$$

$$(\Delta_{h,-}^{-1} g)(t) = \sum_{k=1}^{\infty} g(t - kh)$$

solve the equation $\Delta_h a = g$ in $\mathfrak{X}_\mu(D^\pm(R))$, respectively. We note that

$$\begin{aligned} \|\Delta_{h,\pm} g\|_\mu &= \sup_{D(R)} \left| \cosh^\mu(t) \sum_{k=0,1}^{\infty} g(t \pm kh) \right| \\ &\leq \sup_{D(R)} |\cosh^\mu(t)| \sum_{k=0,1}^{\infty} \frac{\|g\|_\mu}{|\cosh^\mu(t \pm kh)|} \\ &\leq \text{const}\, h^{-1} \|g\|_\mu. \end{aligned} \tag{17.4}$$

To reduce the problem in $\mathfrak{X}_\mu(D(R))$ to the already solved ones we define two operators

$$P_\pm : \mathfrak{X}_0(D(R)) \to \mathfrak{X}_0(D^\pm(R))$$

using the Cauchy type integral (9.10) with χ replaced by $\chi_\pm$ (see (9.24)), and x by t/h:

$$(P_\pm g)(t) = \frac{1}{2\pi \mathrm{i}} \int_{\partial D} \frac{\chi_\pm(\xi h) g(\xi)}{\xi - t} \, d\xi.$$

The estimate (9.20) implies that

$$\|P_\pm\| \leq \text{const } \log h^{-1}.$$

Obviously, $g(t) = (P_+ g)(t) + (P_- g)(t)$ for $t \in D(R)$. We also define three isomorphisms $\mathcal{I}$, $\mathcal{I}_+$, and $\mathcal{I}_-$ acting by the same formula

$$g(t) \mapsto \cosh^\mu(t) g(t)$$

from $\mathfrak{X}_\mu(D)$ to $\mathfrak{X}_0(D)$ with $D = D(R)$, $D^+(R)$, and $D^-(R)$, respectively. Finally, we set

$$\Delta_h^{-1} = \Delta_+^{-1} \mathcal{I}_+^{-1} P_+ \mathcal{I} + \Delta_-^{-1} \mathcal{I}_-^{-1} P_- \mathcal{I}. \qquad \square \tag{17.5}$$

Lemma 17.2. *Given $\mu > 0$, $R > R_0$, where R_0 is a sufficiently large constant, there are two solutions ϕ_1 and $\phi_2 \in \mathfrak{X}_1(D(R))$ of the homogeneous equation*

$$L\phi \equiv \Delta_h^2 \phi - \varepsilon \cos(x^-(t))\phi = 0, \tag{17.6}$$

such that

$$\mathcal{W}[\phi_1; \phi_2] \equiv \phi_1 \bar{\Delta}_h \phi_2 - \phi_2 \bar{\Delta}_h \phi_1 = 1, \tag{17.7}$$

and

$$\|\phi_1\|_1 \leq \text{const}, \qquad \|\phi_2\|_1 \leq \text{const}\, h^{-1-\mu}. \tag{17.8}$$

The constants in the estimates do not depend on R, but the constant in the second estimate depends on μ. Moreover, $\phi_1(t, \varepsilon)$ and $\phi_2(t, \varepsilon)$ are real on real values of t.

Proof. We define the first solution by simple differentiation

$$\phi_1(t,\varepsilon) = \frac{dx^-(t)}{dt}.$$

Its properties follow from Proposition 3.1 (with $N = 1$) and the explicit formula (2.9) for the principal term in the approximation of $x^-(t)$. We look for ϕ_2 in the form

$$\phi_2(t,\varepsilon) = a(t,\varepsilon)\,\phi_1(t,\varepsilon).$$

Substituting to the normalizing condition we get the following equation:

$$\Delta_h a(t,\varepsilon) = \frac{1}{\phi_1(t,\varepsilon)\phi_1(t+h,\varepsilon)}.$$

The norm in $\mathfrak{X}_{\mu'}(D(R))$ of the right-hand side is bounded by a constant for any $\mu' \geq 0$ since the function $\phi_1(t,\varepsilon)$ is separated from zero by a constant in $D(R)$ provided $R > R_0$. Now we can obtain the function $a(t,\varepsilon)$ applying the operator Δ_h^{-1} from Lemma 17.1 (and using μ' instead of μ), then

$$\begin{aligned}\|\phi_2\|_1 &\leq \|\phi_1\|_1\,\|a\|_0 \leq \text{const}\,h^{-\mu'}\|\phi_1\|_1\,\|a\|_{\mu'}\\ &\leq \text{const}\,h^{-1-\mu'}\log h^{-1} \leq \text{const}\,h^{-1-2\mu'}.\end{aligned}$$

We obtain the desired estimate if $\mu' = \mu/2$. □

Lemma 17.3. *Given μ, $\nu > 0$, and $R > R_0$, there is a linear operator L^{-1} : $\mathfrak{X}_\nu(D(R)) \to \mathfrak{X}_\nu(D(R))$, such that for each $g \in \mathfrak{X}_\nu(D(R))$ the function $a = L^{-1}g$ is a solution of the equation $La = g$ in $D(R)$, and*

$$\|L^{-1}\| \leq \text{const}\,h^{-4-\mu}, \tag{17.9}$$

where the constant depends only on μ, R_0 and ν.

Proof. Define the operator L^{-1} by the formulas

$$L^{-1}g = -\phi_1\Delta_h^{-1}(\phi_2 g) + \phi_2\Delta_h^{-1}(\phi_1 g).$$

In Sect. 9.5 it was shown that such a formula provides a solution for the equation $La = g$. To estimate the norm we note that

$$\begin{aligned}\left\|\phi_1\Delta_h^{-1}(\phi_2 g)\right\|_{\nu+2} &\leq \|\phi_1\|_1\,\left\|\Delta_h^{-1}(\phi_2 g)\right\|_{\nu+1}\\ &\leq \|\phi_1\|_1\,\left\|\Delta_h^{-1}\right\|\,\|\phi_2\|_1\,\|g\|_\nu \leq \text{const}\,h^{-2-\mu}\|g\|_\nu.\end{aligned}$$

The similar estimate is valid for the second term. To obtain the final estimate we note that

$$\|L^{-1}g\|_\nu \leq \text{const}\,h^{-2}\|L^{-1}g\|_{\nu+2}$$

since $1/|\cosh(t)| \leq \text{const}\,h^{-1}$ in $D(R)$. □

17.2. Iterative method. We are looking for the first component of the map $\boldsymbol{\Phi}$, namely $x(t,E,\varepsilon)$, in the form

$$x(t,E,\varepsilon) = x^{-}(t) + Z(t,E,\varepsilon). \tag{17.10}$$

Since the map $\boldsymbol{\Phi}$ conjugates the standard map with the shift we have the equation

$$\Delta_h^2 x(t,E,\varepsilon) = \varepsilon \sin(x(t,E,\varepsilon)). \tag{17.11}$$

This equation may be rewritten in the form

$$LZ = \varepsilon \mathcal{F}(Z), \tag{17.12}$$

where

$$(LZ)(t,E,\varepsilon) = \Delta_h^2 Z(t,E,\varepsilon) - \varepsilon \cos(x^{-}(t)) Z(t,E,\varepsilon), \tag{17.13}$$

$$(\mathcal{F}(Z))(t,E,\varepsilon) = F(x^{-}(t), Z(t,E,\varepsilon)), \tag{17.14}$$

$$F(x,Z) = \sin(x+Z) - \sin(x) - \cos(x)Z. \tag{17.15}$$

Applying Lemma 17.3 we obtain that every solution of the equation

$$Z(t,E,\varepsilon) = E\phi_2(t,\varepsilon) + \varepsilon L^{-1}(\mathcal{F}(Z))(t,E,\varepsilon) \tag{17.16}$$

satisfies also Eq. (17.12). We will prove that the nonlinear operator on the right-hand side of the last equation has a unique fixed point in a small ball in $\mathfrak{X}_1(D(R))$ provided E is sufficiently small.

Lemma 17.4. *Given $\mu > 0$, if h is sufficiently small, Z, Z_1, $Z_2 \in \mathfrak{X}_1$, $\|Z\|_1$, $\|Z_1\|_1$, $\|Z_2\|_1 \le h^{\mu}$, then*

$$\left\|L^{-1}(\mathcal{F}(Z))\right\|_1 \le \text{const}\, h^{-7-\mu} \|Z\|_1^2, \tag{17.17}$$

$$\left\|L^{-1}(\mathcal{F}(Z_1) - F(Z_2))\right\|_1 \le \text{const}\, h^{-7-\mu} \max\{\|Z_1\|_1, \|Z_2\|_1\}\, \|Z_1 - Z_2\|_1. \tag{17.18}$$

Proof. Let z, z_1, z_2 be complex numbers lying inside the unit disk and $t \in D(R)$, $R > 1$, then the following estimates hold:

$$|F(x^{-}(t), z)| \le \text{const}\, h^{-2} |z|^2, \tag{17.19}$$

$$|F(x^{-}(t), z_1) - F(x^{-}(t), z_2)| \le \text{const}\, h^{-2} \max\{|z_1|, |z_2|\}\, |z_1 - z_2|, \tag{17.20}$$

since in $D(R)$ we have the estimate

$$|\cos x^{-}(t)| \le \text{const}/|\cosh^2(t)| \le \text{const}\, h^{-2}.$$

Using Lemma 17.3 we have

$$\begin{aligned} \left\|L^{-1}(\mathcal{F}(Z))\right\|_1 &\le \text{const}\, h^{-4-\mu} \|\mathcal{F}(Z)\|_1 \\ &\le \text{const}\, h^{-6-\mu} \|Z^2\|_1 \le \text{const}\, h^{-7-\mu} \|Z\|_1^2. \end{aligned}$$

A similar calculation leads to the second estimate. □

Proposition 17.5. *Given $\mu > 0$, $R > R_0$, let $E \in \mathbb{C}$, $|E| \le 3h^{8+3\mu}$. Then there is a unique solution $Z(t, E, \varepsilon)$ of Eq. (17.16), such that $Z \in \mathfrak{X}_1(D(R))$, $\|Z\|_1 \le h^{8+2\mu}$. Moreover, $Z(t, E, \varepsilon)$ depends analytically on E, it is real provided its arguments are real, and $Z(t, 0, \varepsilon) = 0$.*

Proof. It follows directly from Lemma 17.4 that the nonlinear operator on the right-hand side of (17.16) leaves invariant the ball in $\mathfrak{X}_1(D(R))$ centered at zero and with the radius equal to $h^{8+2\mu}$. The restriction of the operator on the ball is a contraction. That proves the existence and uniqueness of Z. The solution can be obtained as a limit of the sequence

$$Z_0 \equiv 0, \qquad Z_{n+1} = \phi_2 E + \varepsilon L^{-1}(\mathcal{F}(Z_n)), \quad n \ge 0.$$

This sequence converges uniformly with respect to E and all Z_n are real on real values of the arguments. Thus the same is true about the limit. If $E = 0$ the equation has a trivial solution, thus $Z(t, 0, \varepsilon) \equiv 0$ due to the uniqueness. □

We need the estimates of the derivatives of the functions $x(t, E, \varepsilon)$ and $Z(t, E, \varepsilon)$.

Lemma 17.6. *Given $\mu > 0$, $R > R_0 + 1$, then for $t \in D(R)$, $E \in \mathbb{C}$, $|E| \le 2h^{8+2\mu}$, the following estimates are valid:*

$$\left|\frac{\partial x}{\partial t}\right| \le \text{const}\, h^{-1}, \qquad \left|\frac{\partial x}{\partial E}\right| = \left|\frac{\partial Z}{\partial E}\right| \le \text{const}\, h^{-1-\mu},$$

$$\left|\frac{\partial^2 x}{\partial t^2}\right| \le \text{const}\, h^{-2}, \qquad \left|\frac{\partial^2 x}{\partial t \partial E}\right| \le \text{const}\, h^{-2-\mu},$$

$$\left|\frac{\partial^2 x}{\partial E^2}\right| \le \text{const}\, h^{-9-4\mu}, \qquad \left|\frac{\partial Z}{\partial t}\right| \le \text{const}\, h^{6+2\mu},$$

$$\left|\frac{\partial \varepsilon L^{-1}(\mathcal{F}(Z))}{\partial E}\right| \le \text{const}\, h^2.$$

Proof. We apply Cauchy type estimates to obtain the desired estimates for the derivatives from the upper bounds for the functions. Since the domain of validity of the estimates for functions is slightly wider than the domain in the assumptions of the current lemma, the differentiation $\frac{\partial}{\partial t}$ acts on the estimates as a multiplication by const h^{-1}, and $\frac{\partial}{\partial E}$ acts as a multiplication by const $h^{-8-3\mu}$.

We also use the inequality $|\cdot| \le \text{const}\, h^{-1} \|\cdot\|_1$ to bound the module of a function by its norm in $\mathfrak{X}_1(D(R))$. It follows from the estimate (17.8) and Proposition 17.5 that

$$|\phi_1| \le \text{const}\, h^{-1}, \qquad |\phi_2| \le \text{const}\, h^{-2-\mu}, \qquad |Z| \le \text{const}\, h^{7+2\mu}.$$

Since $x(t, E, \varepsilon) = x^-(t) + Z(t, E, \varepsilon)$ and $\phi_1 = \frac{\partial x^-}{\partial t}$, we obtain the desired estimates for the derivatives of x and Z using a Cauchy type estimates.

To obtain the last estimate we note that (17.17) implies

$$|L^{-1}(\mathcal{F}(Z))| \le \text{const}\, h^{-8-\mu} \|Z\|_1^2 \le \text{const}\, h^{8+3\mu}$$

and we again use a Cauchy type estimate. Finally,

$$x = x^- + E\phi_2 + \varepsilon L^{-1}(\mathcal{F}(Z)),$$

and $\frac{\partial^2 x^-}{\partial E^2} = \frac{\partial^2}{\partial E^2}\left(\varepsilon L^{-1}(\mathcal{F}(Z))\right)$ follows directly. The estimate for the second derivative follows immediately. □

17.3. Construction of the symplectic map $\boldsymbol{\Phi}$. Define the map by the formula

$$\tilde{\boldsymbol{\Phi}}(t,E) = \left(x(t,E,\varepsilon), \overline{\Delta}_h x(t,E,\varepsilon)\right).$$

This map satisfies all the properties of the map $\boldsymbol{\Phi}$ from the theorem on the analytic integral except (2), since it may be not symplectic. Indeed, the determinant of the Jacobian of this map

$$J = \frac{\partial x}{\partial t}\frac{\partial \overline{\Delta}_h x}{\partial E} - \frac{\partial x}{\partial E}\frac{\partial \overline{\Delta}_h x}{\partial t} = \mathcal{W}\left[\frac{\partial x}{\partial t};\frac{\partial x}{\partial E}\right]$$

is h-periodic in t since $\phi = \frac{\partial x}{\partial t}$ and $\phi = \frac{\partial x}{\partial E}$ are two solutions of the homogeneous equation, which may be obtained by simple differentiation of (17.11) with respect to t or E. To estimate the value of J we note that

$$\frac{\partial x}{\partial t} = \phi_1 + \frac{\partial Z}{\partial t} \quad \text{and} \quad \frac{\partial x}{\partial E} = \phi_2 + \frac{\partial \varepsilon L^{-1}(\mathcal{F}(Z))}{\partial E}.$$

An application of the estimates of Lemma 17.6 and the equality $\mathcal{W}[\phi_1;\phi_2] = 1$ give us

$$J = 1 + O(h).$$

An application of the implicit function theorem provides the existence of the inverse map and the estimates on the derivatives. To estimate the derivatives of the inverse map we use the following simple and rather general fact. Let f be a diffeomorphism of a subset of $\mathbb{C}^n$ onto its image and g be its inverse, then $Dg = (Df)^{-1}$, where $Dg = \{\partial_i g_k\}$ and $Df = \{\partial_i f_k\}$, and for the second derivatives we have $\partial^2_{kl} g_i = -\partial^2_{qr} f_p\, \partial_p g_i\, \partial_k g_q\, \partial_l g_r$, where the summation on the repeated indices is assumed.

To obtain a symplectic map we use the substitution $S : (t,E) \mapsto (t,\tilde{E})$, where

$$\tilde{E}(t,E,\varepsilon) = \int_0^E J(t,s,\varepsilon)\,ds.$$

A similar change was used in [Laz92] in the study of the semistandard map. Obviously, the Jacobian of the map S equals J. Moreover, S commutes with the translation $(t,E) \mapsto (t+h,E)$. Due to the chain rule the map

$$\boldsymbol{\Phi} = \tilde{\boldsymbol{\Phi}} \circ S^{-1}$$

is symplectic. The other properties of this map are preserved due to the following estimates of the derivatives of the map S.

Lemma 17.7. *Given $R > R_0 + 3$, $\mu > 0$, then for $t \in D(R)$, $E \in \mathbb{C}$, $|E| \le h^{8+3\mu}$, the following estimates hold:*

$$\frac{\partial \tilde{E}}{\partial E} = J = 1 + O(h), \quad \frac{\partial \tilde{E}}{\partial t} = O(h^{8+3\mu}), \tag{17.21}$$

$$\frac{\partial^2 \tilde{E}}{\partial t^2} = O(h^{7+3\mu}), \quad \frac{\partial^2 \tilde{E}}{\partial t \partial E} = O(1), \quad \frac{\partial^2 \tilde{E}}{\partial E^2} = O(h^{-7-3\mu}). \tag{17.22}$$

Proof. It follows directly from the definition of $\tilde{E}$ and the equality $J = 1 + O(h^\mu)$ by Cauchy type estimates. □

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